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A Comparison of the Deck Group and the Fundamental Group on Uniform Spaces Obtained by Gluing

Raymond David Phillippi
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To the Graduate Council:

I am submitting herewith a dissertation written by Raymond David Phillippi entitled "A Comparison of the Deck Group and the Fundamental Group on Uniform Spaces Obtained by Gluing." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Conrad Plaut, Major Professor

We have read this dissertation and recommend its acceptance:

Jurek Dydak, Nikolay Brodskiy, Joseph Macek

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Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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A COMPARISON OF THE DECK GROUP AND THE FUNDAMENTAL GROUP ON UNIFORM SPACES OBTAINED BY GLUING.

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Raymond David Phillippi
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Dedication

This dissertation is dedicated to my wife Julia,
and my children Nancy-Kate, Sarah, and Benjamin
without whom the beauties of mathematics would be very plain indeed.

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and encouragement, and the remainder of my committee:
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Abstract

We define a uniformity on a glued space under uniformly continuous attachment maps. If the component spaces are uniform coverable then the resulting glued space is uniform coverable. We consider examples including the glued uniformity on a finite dimensional CW complex which is shown to be uniformly coverable. For one dimensional CW complexes, the resulting deck group is equivalent to the fundamental group. Other properties of the deck group are explored.

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1 Introduction

The fundamental group is a powerful tool for determining when two topological spaces are not homeomorphic. It works well for topological spaces which are locally nice in the sense that they are locally path connected and semi-locally simply connected. For more complicated spaces, variations of the fundamental group have been found and used to explore topological properties. In 2001, Sormani and Wei [12] defined a "revised" fundamental group on the Gromov-Hausdorff limit of sequences of compact manifolds with a uniform lower Ricci curvature bound and a uniform upper diameter bound. In 2000, Cannon and Conner [5] defined a "Big" fundamental group for use in studying spaces similar to the Hawaiian Earring (which is not a semi-locally simply connected space).

More recently, Berestovski and Plaut [2] have defined a "deck" group in the category of uniform spaces, which allows for a type of covering space theory in this category. In essence, paths are replaced with inverse limits of sequences allowing one to suspend path connected requirements. The deck group is equivalent to the fundamental group in many instances, although there are examples in which they differ. Many questions arise concerning the extent to which traditional theory corresponds to the uniform covering space theory. The present paper attempts to address some of these questions. In particular we will show that some of the lifting properties associated with a traditional cover extend to uniform covers (see theorem 38). We will prove an analog of the Van Kampen Theorem for deck groups (theorem 61). We also define a uniformity on a space formed by attaching uniform spaces via uniformly continuous maps (definition 42). If the spaces involved are uniform coverable then so is the glued uniform space (56). In particular we will show that this uniformity on a one-dimensional CW complex has a deck group which is equivalent to the fundamental group 66).

2 Background

The following is a brief introduction to uniform spaces. We will follow the construction in [11]. [8] and [3] are also helpful resources for uniform spaces. If the reader is already familiar with uniform spaces, it is important to note that, following [11], our use of the term "entourage" will mean a symmetric entourage.

Intuitively, a uniform space is a topological space with enough structure to be able to define a notion of a uniformly continuous function. In metric spaces a function $f : X \to Y$ is uniformly continuous if the inverse image of an ε-ball centered at any point $f(x)$ in the image of f contains a δ-ball centered at x. ε and δ are fixed in this definition, but the criterion is independent of the points x in X. In essence, the metric provides, for each $\varepsilon > 0$, a way of specifying a collection of neighborhoods, one for each point in the space. Informally speaking, the inverse image of each ε collection must contain a δ collection. Uniform Spaces are generalizations of metric spaces in which collections of neighborhoods are specified. Just as a metric is defined between each pair of points, a uniform structure is defined on the product space $X \times X$.

Definition 1 *A filter on a set X is defined to be a collection $\mathcal{F}$ of subsets of X which satisfy the following properties:*

1. $\emptyset \notin \mathcal{F}$.
2. *If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ (intersections of elements of $\mathcal{F}$ are also elements of $\mathcal{F}$).*
3. *If $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$ (supersets of elements of $\mathcal{F}$ are also elements of $\mathcal{F}$).*

Let X be a set and let $\Delta = \{(a, a) \in X \times X \mid a \in X\}$ (the diagonal). A subset E of $X \times X$ is said to be symmetric if $(a, b) \in E \Longrightarrow (b, a) \in E$. We define a "product" on the collection of subsets of E in the following way.

Definition 2 *If E, F are any two subsets of $X \times X$ then $E \circ F := \{(a, c) \mid$ there exists $b \in X$ such that $(a, b) \in E$ and $(b, c) \in F\}$.*

Thus a pair (a, c) is in the product of E and F if there exists a sequence with three elements a, b, c such that the first pair is an element of E while the second pair is an element of F. In particular, if one forms E^n (the nth product of E with itself) then $(a, b) \in E^n$ implies the existence of a sequence $a, x_1, x_2, ...x_{n-1}, b$ such that each sequential pair is an element of E.

Definition 3 *Let Ω be a collection of subsets of $X \times X$ which satisfy the following properties:*

1. *For all $E \in \Omega$, $\Delta \subset E$. (Every subset in the collection contains the diagonal).*
2. *For all $E \in \Omega$, E is symmetric.*
3. *If $E, F \in \Omega$ then there exists $D \in \Omega$ such that $D \subset E \cap F$.*

4. For every $E \in \Omega$ there exists an $F \in \Omega$ such that $F^2 \subset E$

Then the collection $\mathcal{F} := \{K \subset X \times X \mid$ there exists $E \in \Omega$ such that $E \subset K\}$ forms a filter on $X \times X$. $\mathcal{F}$ is called a uniformity on X while Ω is called a basis for the uniformity $\mathcal{F}$. Notice that if Ω is a basis for a uniformity then the collection $\Omega \cup \{X \times X\}$ is also a basis for the same uniformity. Hence we may add the set $X \times X$ to any basis, and will do so automatically. A set X together with a uniformity will be called a uniform space. If Ω also satisfies: 5) $\cap_\Omega E = \Delta$ (The intersection of all E in Ω is the diagonal) then the uniform space will be called Hausdorff (see 8).

Definition 4 *An entourage is defined to be any symmetric element of a uniformity.*

Proposition 5 *If X is a uniform space with basis Ω, and uniformity $\mathcal{F}$ then the set Σ of all entourages (symmetric elements of $\mathcal{F}$) forms a basis for the uniformity. If Ω satisfies condition 5 then so does the set of all entourages.*

Proof. 1) For any $G \in \mathcal{F}$ we have, by definition, that G is a superset for some $F \in \Omega$. Thus $\Delta \subset F \subset G$.

2) This statement is true by definition.

3) Let G_1, G_2 be entourages, and $F_1, F_2 \in \Omega$ such that $F_1 \subset G_1$ and $F_2 \subset G_2$. We let $K \in \Omega$ such that $K \subset F_1 \cap F_2 \subset G_1 \cap G_2$. Then, since K is an entourage, K satisfies property 3 for G_1, G_2.

4) Let G be an entourage and $F \in \Omega$ such that $F \subset G$. Let $K \in \Omega$ be such that $K^2 \subset F \subset G$. Then, since K is an entourage, K satisfies property 4 for G.

5) We assume that $\cap_\Omega E = \Delta$. Since the collection of entourages is larger than Ω we have $\cap_{F \in \Sigma} F \subset \cap_{E \in \Omega} E = \Delta$. However, since $\Delta \subset E$ for all entouages $\Delta \subset \cap_{F \in \Sigma}$ and property 5 holds. ■

Definition 6 *The basis for a uniformity consisting of all entourages will be called a full basis.*

We note that in most references conditions 2 is not required for a basis of a uniformity, or for the definition of an entourage. However, given a uniformity, it is always possible to form a basis consisting of symmetric subsets of $X \times X$ (see [2], or [3]). Since we will be working exclusively with symmetric elements of a uniformity, it will be convenient to assume this condition from the start. It is also important to note that a given set X can have more than one distinct uniformity, just as a given set can have multiple topologies defined on it. Every entourage E in a uniform space determines a collection of subsets of X, one for each point $x \in X$, by considering the projections of E onto X. Following the notation for metric spaces, we have the following definition.

Definition 7 *Let E be an entourage of a uniform space X. The E-ball centered at $x \in X$ is $B(x, E) := \{y \in X \mid (x, y) \in E\}$.*

Proposition 8 *Declare a set $U \subset X$ to be open if for all $x \in U$ there exists $E \in \Omega$ such that $B(x, E) \subset U$. Then the collection of all open subsets of X forms a topology on X called the topology induced by Ω. For each $x \in X$ and $E \in \Omega$ we can find an open set U such that $U \subset B(x, E)$. Further, condition 5 in the definition of a uniformity is equivalent to the topology being Hausdorff (see propostions 1 and 3 of II.1.2 of [3].*

Proof. Clearly $\emptyset$ is open since it is vacuous and X is open since the condition is satisfied for every E in Ω. Suppose U and V are open. Then for each x in $U \cap V$ there exists $E_x, F_x \in \Omega$ such that $B(x, E_x) \subset U$ and $B(x, F_x) \subset V$. By property 3 we may find $D_x \in \Omega$ such that $D_x \subset E_x \cap F_x$. By definition $B(x, D_x) \subset B(x, E_x) \cap B(x, F_x) \subset U \cap V$ and hence $U \cap V$ is open. Finally, if $x \in \cup_\alpha U_\alpha$ where each U_α is open, then $x \in U_\beta$ for some β. Hence there exists an $E \in \Omega$ such that $B(x, E) \subset U_\beta \subset \cup_\alpha U_\alpha$ and $\cup_\alpha U_\alpha$ is open. We conclude that the collection of open subsets forms a topology. To prove the second statement, let $U = \{y \in X \mid \text{for some } F \in \Omega,\ B(y, F) \subset B(x, E)\}$. Then, for each $y \in U$ we have that $y \in B(y, F) \subset B(x, E)$ and hence $U \subset B(x, E)$. Further, $x \in U$ since, in particular $B(x, E) \subset B(x, E)$. We must show that U is open. Let $y \in U$ and $F \in \Omega$ such that $B(y, F) \subset B(x, E)$. We must find a K such that $B(y, K) \subset U$. We choose K to be any entourage such that $K^2 \subset F$. The problem is now to show that if $a \in B(y, K)$ then $a \in U$. In fact, if $a \in B(y, K)$ and $b \in B(a, K)$ then $(y, a), (a, b) \in K$ so that $(y, b) \in K^2 \subset F$. But then, since $B(y, F) \subset B(x, E)$ we must have that $b \in B(x, E)$. Thus we have shown that $B(a, K) \subset B(x, E)$ so that $a \in U$. ■

To see that this topology is Hausdorff under condition 5, let x, y be distinct points of X. We may find an entourage E such that $(x, y) \notin E$. Further, we find an entourage F such that $F^2 \subset E$. We claim that $B(x, F) \cap B(y, F)$ is empty. Then, since $B(x, F)$ and $B(y, F)$ contain open sets containing x and y respectively, the Hausdorff condition would be satisfied. Now, for the purpose of contradiction, suppose that $a \in B(x, F) \cap B(y, F)$. Then, since F is symmetric we have that $(x, a) \in F$ and $(a, y) \in F$. But then $(x, y) \in E$ which is a contradiction. Thus condition 5 implies Hausdorff. To see that Hausdorff implies condition 5 (assuming conditions 1-4) let $(x, y) \in \Delta^c$. Then we may find open sets U and V such that $x \in U$, $y \in V$ and $U \cap V$ is empty. Further we may find entourages E, F such that $B(x, E) \subset U$ and $B(y, F) \subset V$. Choosing $D \subset E \cap F$ we have that $B(x, D) \subset U$, $B(y, D) \subset V$ and $B(x, D) \cap B(y, D)$ is empty. Then, in particular $y \notin B(x, D)$ and $(x, y) \notin D$. Thus (using condition 1) we have that $\cap_\Omega = \Delta$.

Definition 9 *The topology on X determined by 8 is called the topology induced by the uniformity base Ω.*

Uniform spaces generalize both metric spaces and topological groups as the following examples show. For background and relevant definitions for these examples, one may consult [3].

Example 10 *(Metric Spaces) Let M be a metric space (see [4]). For each $\delta > 0$ we define $E(\delta) = \{(x, y) \mid d(x, y) < \delta\}$. We then define a uniformity base on $M \times M$*

as the collection $\Omega = \{E(\delta) \mid \delta > 0\}$. *Then the properties of metric spaces translate into properties of uniformity bases in the following way.*

1) Since $d(x,x) = 0$ *for all* x *we have that* $(x,x) < \delta$ *and hence* $(x,x) \in E(\delta)$ *for all* $x \in X$ *and* $\delta > 0$. *Thus* $\Delta \subset E(\delta)$ *for all* $E(\delta) \in \Omega$.

2) Since $d(x,y) = d(y,x)$ *we have that* $(x,y) \in E(\delta) \Longrightarrow (y,x) \in E(\delta)$. *Hence* $E(\delta)$ *is symmetric.*

3) $(x,y) \in E(\delta_1) \cap E(\delta_2) \Longrightarrow d(x,y) < \delta_1$ *and* δ_2 *and hence* $(x,y) \in E(\min(\delta_1, \delta_2))$. *Thus, in fact* $E(\delta_1) \cap E(\delta_2) = E(\min(\delta_1, \delta_2))$.

4) If $(x,y) \in E(\delta/2)^2$ *then by definition there exists* $z \in M$ *such that* $(x,z), (z,y) \in E(\delta/2)$. *But then, by the triangle inequality,* $d(x,y) \leq d(x,z) + d(z,y) < \delta/2 + \delta/2 = \delta$ *so that* $(x,y) \in E(\delta)$. *This shows that for each* $\delta > 0$, $E(\delta/2)^2 \subset E(\delta)$.

Thus Ω *forms a uniformity base on the metric space* M. *Notice that:*

$$B(x, E(\delta)) = \{y | (x,y) \in E(\delta)\} = \{y | d(x,y) < \delta\} = B(x, \delta)$$

Hence the topology induced by the uniformity base is equivalent to the metric topology.

Example 11 *(Topological Groups) Let* G *be a topological group (see chapter III of [3]) and* U *a symmetric open subset of the identity (i.e.* $e \in U$ *and* $g \in U \Longrightarrow g^{-1} \in U$*). Then we define* $E(U) = \{(g,h) \mid gh^{-1} \in U\}$. *The collection* $\Omega = \{E(U) \mid U$ *is a symmetric subset of the identity*$\}$ *forms a uniformity base.*

1) Since $gg^{-1} = e \in U$ *we have that* $(g,g) \in E(U)$ *for all* $g \in G$ *and hence* $\Delta \subset E(U)$ *for all* U.

2) $(g,h) \in U \Longrightarrow gh^{-1} \in U \Longrightarrow (gh^{-1})^{-1} = hg^{-1} \in U$ *(since* U *is symmetric)* $\Longrightarrow (h,g) \in U$. *Hence* $E(U)$ *is symmetric.*

3) By the properties of the topology we have that if U, V *are open subsets of the identity then so is* $U \cap V$. *In fact,* $U \cap V$ *is also symmetric since* $g \in U \cap V \Longrightarrow g^{-1} \in U$ *and* $g^{-1} \in V$ *(since* U *and* V *are symmetric) and hence* $g^{-1} \in U \cap V$. *We show that in fact* $E(U \cap V) = E(U) \cap E(V)$. *Let* $(g,h) \in E(U \cap V)$. *Then* $gh^{-1} \in U \cap V$ *and hence* $gh^{-1} \in U$ *and* $gh^{-1} \in V$. *Thus* $(g,h) \in E(U)$ *and* $E(V)$.

4) In a topological group $WV = \{g_1 g_2 \mid g_1 \in W, g_2 \in V\}$. *By the properties of a topological group, it is possible to find, for each* U *(a symmetric open subset of the identity) a* V *(a symmetric open subset of the identity) such that* $V^2 \subset U$. *We show that* $E(V)^2 \subset E(U)$. *Let* $(g_1, g_2) \in E(V)^2$. *Then there exists an* $h \in G$ *such that* $(g_1, h) \in E(V)$ *and* $(h, g_2) \in E(V)$. *Thus* $g_1 h^{-1}, h g_2^{-1} \in V$. *But then* $g_1 g_2^{-1} = (g_1 h^{-1})(h g_2^{-1}) \in V^2 \subset U$ *and we have the result.*

Let X and Y be uniform spaces, and $f : X \to Y$. We wish to consider the properties of entourages (subsets of $X \times X$ and $Y \times Y$) under the mapping $f \times f$. Following [11], we adopt the following convention.

Remark 12 *Let* X *and* Y *be uniform spaces,* $f : X \to Y$ *,* E *an entourage of* $X \times X$ *and* F *an entourage of* $Y \times Y$. *We define* $f(E) = \{(f(x), f(y)) \mid (x,y) \in E\}$. *Thus* $f(E)$ *really means* $(f \times f)(E)$. *Similarly* $f^{-1}(F) = \{(x,y) \in X \times X \mid (f(x), f(y)) \in F\}$ *or in other words* $f^{-1}(F) = (f \times f)^{-1}(F)$. *Further, we will state that* E *is an entourage of* X *rather that of* $X \times X$.

Definition 13 *A function $f : X \to Y$ is defined to be uniformly continuous if for each entourage F of Y, $f^{-1}(F)$ is an entourage. If f is surjective and, in addition, there exists for each entourage E of X an entourage F of Y such that $F \subset f(E)$ then f will be called bi-uniformly continuous. Finally, if a bi-uniformly continuous function is also one-to-one, then it is termed a uniform homeomorphism.*

Proposition 14 *A uniformly continuous function is a continuous function between the topological spaces induced on X and Y by the uniformities (see proposition 1 of section II.2.1 in [3]).*

Proof. Let U be open in Y, and $x \in f^{-1}(U)$. By definition, we may find an entourage F in Y such that $B(f(x), F) \subset U$. Let $E \subset f^{-1}(F)$ and consider $B(x, E)$. If $a \in B(x, E)$ then $(x, a) \in E$ so that, by the choice of E we have $(f(x), f(a)) \in F$. But then $f(a) \in B(f(x), F) \subset U$. Hence $B(x, E) \subset f^{-1}(U)$ which implies that $f^{-1}(U)$ is open. ■

We note that it is possible for a bi-uniformly continuous function between uniform spaces to fail to be a uniform homeomorphism. For example, let $\mathbb{R}$ be given the standard metric, and let S^1 be the unit circle under the length metric (the distance between points is the length of the shortest curve between them). If $\mathbb{R}$ and S^1 are given the metric uniformities outlined in 10 and $f : \mathbb{R} \to S^1$ is the covering map given by $f(x) = e^{2\pi ix}$ then f is bi-uniformly continuous, but not one-to-one. To see that f is bi-uniformly continuous, let $F(\delta)$ be the entourage in S^1 determined by δ and $E(\delta)$ the entourage in $\mathbb{R}$ determined by δ. Then, for $\delta < \frac{1}{2}$, $E(\delta) \subset f^{-1}(F(\delta))$ while $f(E(\delta)) = F(\delta)$.

There is, then, a category whose objects are uniform spaces and whose morphisms are uniformly continuous functions. In [2] a covering space theory is developed for this category, which allows one to define a "deck group" which will be labeled $\delta_1(X)$. In many cases, the deck group is the traditional fundamental group. In cases where it is not, however, this theory has the advantage of extending many algebraic topological results to spaces for which the traditional fundamental group is less useful (see example). It is worth emphasizing that the deck group is a functor from the uniform space category, and not the more general topological space category. Thus it is possible for homeomorphic spaces to have different uniform fundamental groups.

We will outline Plaut, Berestovski theory below. Let X be a uniform space. We begin by constructing, for each entourage E in X, a uniform space X_E, which is a traditional covering space of X. Let E be a fixed entourage of X which means, in particular that E is symmetric (See 4 above.) We consider the following collection of sequences $S_E := \{ x_0, x_1, ...x_n \mid (x_{i-1}, x_i) \in E \text{ for } 1 \leq i \leq n\}$. If X is a metric space and $E = E(\delta)$ then $S_{E(\delta)}$ would consist of those sequences such that two consecutive elements in the sequence are no farther apart than δ. An element of S_E is called an E-chain. In order to avoid confusion, we will denote an E-chain using the notation $\gamma = \{x_0, x_1, ...x_n\}$. If $\gamma = \{x_0, x_1, ...x_n\}$,and $\eta = \{y_0, y_1, ...y_m\}$ are two E-chains such that $x_n = y_0$ then the E-chain $\gamma\eta = \{x_0, ...x_n, y_1...y_m\}$, and the E-chain $\gamma^{-1} = \{x_n, x_{n-1}, ...x_0\}$ (which is an E-chain since E is symmetric).

Definition 15 *Let $\gamma = \{x_0, x_1, ...x_n\} \in S_E$ and suppose that for some $1 \leq i \leq n$, there exists $z \in X$ such that $(x_{i-1}, z) \in E$ and $(z, x_i) \in E$. Then the E-chain $\eta = \{x_0, x_1, ...x_{i-1}, z, x_i, ...x_n\}$ is said to be an expansion of γ whereas γ is a contraction of η. A finite sequence of E-chains $\eta_0, \eta_1, ...\eta_m \in S_E$ such that η_i is an expansion or contraction of η_{i-1} for $1 \leq i \leq m$ is called an E-homotopy from η_0 to η_m. Notice that in an E-homotopy, the beginning and ending points in the sequences are all the same. An E-homotopy is essentially a "fixed-endpoint" homotopy. Two E-chains $\gamma, \gamma' \in S_E$ are $E-$equivalent if there exists an E-homotopy between them. To form X_E we fix a basepoint $* \in X$, and let S_E^* be the set of E-chains such that $x_0 = *$, i.e. the set of E-chains which begin at $*$. It is easy to see that E-equivalence is an equivalence relation on S_E^*. We define X_E to be S_E^* modulo E-equivalence. Elements of X_E will be denoted $[\gamma]_E$. The common initial point and end point of the E-chains in $[\gamma]_E$ will be called the initial point and end point of $[\gamma]_E$.*

An E-chain in the uniform structure of X is analogous to a path in the topological structure of X. We now define a uniform structure on X_E in a manner which, for length spaces, is roughly parallel to the δ cover in [12].

Definition 16 *For each $D \subset E$ we define an entourage D^* of X_E to be the set of all pairs $([\gamma]_E, [\eta]_E)$ such that $[\gamma]_E = [\{* = x_0, x_1, ...x_n, a\}]_E$, $[\eta]_E = [\{* = x_0, x_1, ...x_n, b\}]_E$ and $(a, b) \in D$. Thus D^* is the collection of all pairs in X_E whose endpoints form a pair in D and which are E-homotopic to E-chains which differ only in their endpoints. The uniformity properties of D^* follow easily from the uniformity properties of D, as we now show.*

Proposition 17 *The collection $\{D^* \subset X_E \times X_E \mid D \subset E\}$ is a basis for a uniformity on X_E (see proposition 16 of [2]). If X is Hausdorff, then so is X_E.*

Proof. 1) Let $\gamma = \{* = x_0, x_1, ...x_n\}$ be any E-chain beginning at $*$. Then $([\gamma]_E, [\gamma]_E) \in D^*$ since $(x_n, x_n) \in D$ and $\gamma = \gamma$.

2) Let $[\gamma]_E = [\{* = x_0, x_1, ...x_n, a\}]_E$, $[\eta]_E = [\{* = x_0, x_1, ...x_n, b\}]_E$ be such that $([\gamma]_E, [\eta]_E) \in D^*$. Then $(a, b) \in D$ and the symmetry of D implies that $(b, a) \in D$. Hence $([\eta]_E, [\gamma]_E) \in D^*$.

3) Let $D_1, D_2 \subset E$. Choose an entourage $K \subset D_1 \cap D_2$. Then, if $[\gamma]_E = [\{* = x_0, x_1, ...x_n, a\}]_E$, $[\eta]_E = [\{* = x_0, x_1, ...x_n, b\}]_E$ are such that $(a, b) \in K$ then $(a, b) \in D_1 \cap D_2$. Hence $K^* \subset D_1^* \cap D_2^*$.

4) Let $D \subset E$ and find an entourage K such that $K^2 \subset D$. Suppose that $([\gamma]_E, [\eta]_E) \in (K^*)^2$. Then there exists $[\lambda]_E$ such that $([\gamma]_E, [\lambda]_E), ([\lambda]_E, [\eta]_E) \in K^*$. Then there exists $\gamma \in [\gamma]_E, \lambda_1, \lambda_2 \in [\lambda]_E$,and $\eta \in [\eta]_E$ such that $\gamma = \{* = x_0, x_1, ...x_{n-1}, a\}, \lambda_1 = \{* = x_0, x_1, ...x_{n-1}, c\}, \lambda_2 = \{* = y_0, y_1, ...y_{m-1}, c\}$, and $\eta = \{* = y_0, y_1, ...y_{m-1}, b\}$. In particular we have that (a, c) and (c, b) are elements of $K \subset E$ and hence $\bar{\eta} = \{* = y_0, y_1, ...y_{m-1}, c, b\}$ is an E-chain in the equivalence class $[\eta]_E$ whereas $\bar{\gamma} = \{* = x_0, x_1, ...x_{n-1}, c, a\}$ is an E-chain in the equivalence class of $[\gamma]_E$. Using the equivalence $\lambda_1 \sim \lambda_2$ we see that $\bar{\gamma} \sim \{* = y_0, y_1, ...y_{m-1}, c, a\}$ which differs from $\bar{\eta}$ only in the final element. Since (a, c) and (c, b) are elements of K and

$K^2 \subset D$ we have $(a,b) \in D$. Thus, by definition, $([\gamma]_E, [\eta]_E) \in D^*$ and we obtain that $(K^*)^2 \subset D^*$

We now assume that X satisfies condition 5) in the definition of a basis for a uniform space. Suppose $[\gamma]_E$, and $[\eta]_E$ are distinct elements of X_E. We must find some D^* such that $([\gamma]_E, [\eta]_E) \notin D^*$. In general, we may not assume that the endpoints of E-chains in $[\gamma]_E$,and $[\eta]_E$ are distinct even if $[\gamma]_E$,and $[\eta]_E$ are distinct. Suppose for the sake of contradiction, however, that the endpoints are the same. Then for any D^*, if $([\gamma]_E, [\eta]_E) \in D^*$ then $[\gamma]_E = [\{* = x_0, x_1, ...x_n, a\}]_E$ and $[\eta]_E = [\{* = x_0, x_1, ...x_n, b\}]_E$. Since the endpoints are the same $a = b$ and $\{* = x_0, x_1, ...x_n, b\} = \{* = x_0, x_1, ...x_n, a\}$ which implies that $[\gamma]_E = [\eta]_E$ a contradiction. Hence we may assume that the endpoints are distinct. Now, we may find an entourage K_1 in X such that $(a,b) \notin K_1$, and then an entourage $K_2 \subset K_1 \cap E$. If $([\gamma]_E, [\eta]_E) \in K_2^*$ however, then $(a,b) \in K_2 \subset K_1$ which is a contradiction. ■

Thus X_E is in fact a uniform space. Notice that if $E = X \times X$ then $(a,b) \in E$ for all $a, b \in X$. Hence for all $c \in X$ we have that $\{a, c, b\}$ is an expansion of $\{a, b\}$.This implies that $[\gamma]_{X\times X}$ is uniquely determined by its end point. Thus we have the following proposition.

Proposition 18 *If X is a uniform space then X is uniformly homeomorphic to $X_{X\times X}$ (see notation 28 in [2]).*

Proof. We simply define $f : X_{X\times X} \to X$ as $f([\{* = x_0, x_1, ...x_n\}]_{X\times X}) = x_n$ (the end point map). The map is surjective since given $a \in X$ we have $\{*, a\}$ is an $X \times X$ chain and hence $f([\{*, a\}]_{X\times X}) = a$. It is also injective since $[\gamma]_{X\times X}$ is uniquely determined by its end point. Let E be an entourage of X and E^* the associated entourage of $X_{X\times X}$. If $([\{*, a\}]_{X\times X}, [\{*, b\}]_{X\times X}) \in E^*$ then $(a,b) \in E$ and hence $(f([\{*, a\}]_{X\times X}), f([\{*, b\}]_{X\times X})) = (a,b) \in E$. If $(f([\{*, a\}]_{X\times X}), f([\{*, b\}]_{X\times X})) \in E$ then by necessity $(a,b) \in E$ and hence $([\{*, a\}]_{X\times X}, [\{*, b\}]_{X\times X}) \in E^*$. Thus $f(E^*) = E$ and f is a uniform homeomorphism. ■

For each entourage E of X there is an associated group in X_E. Define any E-chain whose initial and end points are the same to be an E-loop. If we let $\delta_E(X) = \{[\gamma]_E \mid \gamma \text{ is an } E-\text{loop}\}$ then $\delta_E(X)$ forms a group under the operation induced by concatenation. More specifically, if $[\gamma]_E, [\eta]_E \in \delta_E(X)$ then the product $[\gamma]_E * [\eta]_E = [\gamma\eta]_E$.

Proposition 19 *$\delta_E(X)$ under the operation $*$ forms a group (see definition 36 of [2]).*

Proof. We first establish that the operation is well defined on the equivalence classes of X_E. Let $\gamma' \in [\gamma]_E$ and $\eta' \in [\eta]_E$. Let $\gamma = \gamma_0, \gamma_1, \gamma_2, ...\gamma_s = \gamma'$ and $\eta = \eta_0, \eta_1, \eta_2, ...\eta_t = \eta'$ be corresponding E-homotopies. Then

$$\gamma\eta = \gamma_0\eta_0, \gamma_1\eta_0, \gamma_2\eta_0, ...\gamma_s\eta_0, \gamma_s\eta_1, \gamma_s\eta_2, ...\gamma_s\eta_t = \gamma'\eta'$$

is an E-homotopy from $\gamma\eta$ to $\gamma'\eta'$. The operation is also associative since $[\gamma]_E * ([\eta]_E * [\lambda]_E) = [\gamma\eta\lambda]_E = ([\gamma]_E * [\eta]_E) * [\lambda]_E$. The element $[\{*, *\}]_E$ is an identity element. To

see this, let $[\gamma]_E \in \delta_E(X)$ where $\gamma = \{*, x_1, ...x_{n-1}, *\}$. Then, in particular, by the definition of an E-chain, $(x_{n-1}, *) \in E$. Hence $[\gamma]_E * [\{*, *\}]_E = [\{x_0, x_1, ...x_{n-1}, *, *\}]_E = [\gamma]_E$ since $(x_{n-1}, *)$ and $(*, *) \in E$ implies that $\{* = x_0, x_1, ...x_{n-1}, *, *\}$ is an expansion of γ. A similar result holds for $[\{*, *\}]_E * [\gamma]_E$. Finally, we show that $[\gamma^{-1}] = [\gamma]^{-1}$. Notice that:

$$[\gamma]_E * [\gamma^{-1}]_E = [\gamma\gamma^{-1}]_E = [\{*, x_1, ...x_{n-1}, *, x_{n-1}, x_{n-2}, ...x_1, *\}]_E$$

Now, since $(x_{n-1}, x_{n-1}), (x_{n-1}, *)$ and $(*, x_n) \in E$ we have that $\{x_{n-1}, x_{n-1}\}$ is a contraction of $\{x_{n-1}, *, x_{n-1}\}$. Further, since (x_{n-2}, x_{n-1}) and $(x_{n-1}, x_{n-1}) \in E$ we have that $\{x_{n-2}, x_{n-1}\}$ is a contraction of $\{x_{n-2}, x_{n-1}, x_{n-1}\}$. Similarly, $\{x_{n-2}, x_{n-2}\}$ is a contraction of $\{x_{n-2}, x_{n-1}, x_{n-2}\}$. Thus we have established that

$$[\{*, x_1, ...x_{n-1}, *, x_{n-1}, x_{n-2}, ...x_1, *\}]_E = [\{*, x_1, ...x_{n-2}, x_{n-2}, ...x_1, *\}]_E$$

We may proceed inductively to show that in fact $[\gamma]_E * [\gamma^{-1}]_E = [\{*, *\}]_E$. Since $(\gamma^{-1})^{-1} = \gamma$ we have that $[\gamma^{-1}]_E * [\gamma]_E = [\gamma^{-1}]_E * [(\gamma^{-1})^{-1}]_E = [\{*, *\}]_E$ and hence $[\gamma^{-1}] = [\gamma]^{-1}$. ■

Definition 20 *The group $\delta_E(X)$ is called the E-deck group of the entourage E.*

Proposition 21 *If F and E are entourages in X such that $F \subset E$ then any F-chain is automatically an E-chain. We may therefore define a function $\phi_{EF} : X_F \to X_E$ by setting $\phi_{EF}([\gamma]_F) = [\gamma]_E$. ϕ_{EF} is uniformly continuous (see lemma 22 of [2]). Further, $\phi_{EF}|_{\delta_F(X)} : \delta_F(X) \to \delta_E(X)$ is a group homomorphism (theorem 39 in [2]).*

Proof. To see that this function is well defined on the equivalence classes of X_F suppose $\gamma = \eta_0, \eta_1, ...\eta_s = \gamma'$ is an F-homotopy. Then each η_i is an E-chain and hence γ' is E-equivalent to γ, and $[\gamma]_E = [\gamma']_E$. We show further that each ϕ_{EF} is uniformly continuous. Let D^* be an entourage in X_E, so that $D \subset E$. If we choose $K \subset D \cap F$ then $K \subset F$ so we can define K_F^* to be the entourage of X_F defined by K . If $([\gamma]_F, [\eta]_F) \in K_F^*$ then $[\gamma]_F = [\{* = x_0, x_1, ...x_n, a\}]_F$, and $[\eta]_F = [\{* = x_0, x_1, ...x_n, b\}]_F$ for some F-chain $\{* = x_0, x_1, ...x_n\}$ and $(a, b) \in K$. Hence $\phi_{EF}([\gamma]_F) = [\{* = x_0, x_1, ...x_n, a\}]_E$ and $\phi_{EF}([\eta]_F) = [\{* = x_0, x_1, ...x_n, b\}]_E$ where $(a, b) \in K \subset D$. Hence $(\phi_{EF}([\gamma]_F), \phi_{EF}([\eta]_F)) = ([\gamma]_E, [\eta]_E) \in D^*$. This shows that $K_F^* \subset \phi_{EF}^{-1}(D^*)$ and ϕ_{EF} is uniformly continuous. To see that $\phi_{EF}|_{\delta_F(X)}$ is a group homomorphism, we note first that the image of $\phi_{EF}|_{\delta_F(X)}$ lies in $\delta_E(X)$ since end points are preserved under this mapping. Further, $\phi_{EF}|_{\delta_F(X)}([\gamma]_F * [\eta]_F) = \phi_{EF}|_{\delta_F(X)}([\gamma\eta]_F) = [\gamma\eta]_E = [\gamma]_E * [\eta]_E$. ■

Proposition 22 *Let Ω be the full basis for X consisting of all entourages. If $E, F \in \Omega$, then we set $E \leq F$ if $F \subset E$. This is a partial ordering on the collection of all entourages which is directed, since given E, F we may find $D \subset E \cap F$ and hence $E \leq D$ and $F \leq D$. The collection $\{X_E\}_{E \in \Omega}$ together with the mappings ϕ_{EF} form an inverse system in the category of uniform spaces. Further, the collection $\{\delta_E(X)\}$ together with the homomorphisms $\phi_{EF}|_{\delta_F(X)}$ form an inverse system in the category of groups (see the paragraph following lemma 26 in [2]).*

Proof. We must show that if $E \leq F \leq D$ then $\phi_{EF} \circ \phi_{FD} = \phi_{ED}$. This follows from $(\phi_{EF} \circ \phi_{FD})([\gamma]_D) = \phi_{EF}([\gamma]_F) = [\gamma]_E = \phi_{ED}([\gamma]_D)$. ■

By the preceding proposition, we may form, for each uniform space X, the inverse limits $\tilde{X} = \varprojlim X_E$ and $\delta_1(X) = \varprojlim \delta_E(X)$. Notice that if $(\gamma) \in \tilde{X}$ then for each entourage E the projection mapping $\psi_E(\gamma) = [\gamma]_E$ has the same initial and end points. In general, we focus our attention on uniform spaces of the following type.

Definition 23 *$\tilde{X}$ is called the fundamental inverse system of X. $\delta_1(X)$ is called the deck group. A uniform space is said to be uniform coverable if there exists a basis Ω for the uniformity (which includes the entourage $X \times X$) such that the projection maps $\psi_E : \tilde{X} \to X_E$ are surjective. The basis Ω is called a covering basis.*

Let X be a uniform space. In [11] Plaut considers the group H_X of all uniform homeomorphisms of X with the product operation given by composition. The topology on H_X is the topology of uniform convergence. If G is a subgroup of H_X then G is said to act discretely on X if there exists an entourage E of X such that $(g(x), x) \in E \Longrightarrow g = id$.The term comes from the fact that if G acts discretely on X then G is a discrete subgroup of H_X (in the subspace topology). Let $\{X_\alpha, f_{\alpha\beta}\}_{\alpha \in A}$ be an inverse system of sets over an index A, and $\{G_\alpha, h_{\alpha\beta}\}_{\alpha \in A}$ be an inverse system of groups over the same index such that G_α acts on X_α. If the systems are compatible in the sense that $f_{\alpha\beta}(gx) = h_{\alpha\beta}(g) f_{\alpha\beta}(x)$ then we have an inverse system of actions. In this case the group $\varprojlim G_\alpha$ acts on $\varprojlim X_\alpha$ via the relation $(g_\alpha)(x_\alpha) = (g_\alpha x_\alpha)$ and such an action is termed a prodiscrete action. In particular, every discrete action is prodiscrete. If, given a prodiscrete action, there exists a basis Ω for the uniformity on X such that $g(E) = E$ for all $g \in G$ and $E \in \Omega$ then the projection $\pi : X \to X/G$ is called a uniform cover. That X/G forms a uniform space is a non-trivial result in [11]. In [2] it is proved that if a uniform space is uniform coverable then the projection $\psi_{X \times X} : \tilde{X} \to X_{X \times X}$ ($= X$ by 18) is a uniform cover and $\delta_1(X)$ is the deck group.

The set of uniform coverable uniform spaces is large and includes path connected, locally compact topological groups [1], geodesic metric spaces, connected locally path connected compact topological spaces, Peano continua [2] and finite dimensional CW complexes (see 64 of this work).

Definition 24 *For a uniform coverable space X, $\tilde{X}$ will be called the uniform universal cover and $\delta_1(X)$ the uniform fundamental group.*

Definition 25 *A uniform space is called chain connected if, for each entourage E and pair $(a, b) \in X \times X$ there exists an E-chain from a to b. Thus, between every two points in X there exist "finer and finer" E-chains between them.*

Proposition 26 *A uniform coverable uniform space is chain connected (see lemma 44 in [2]).*

Proof. Let $(a,b) \in X \times X$. Then, in particular $\{*,a\}$ and $\{*,b\}$ are $X \times X$ chains. Since $X \times X$ is an entourage in the covering basis of X we have that $\psi_{X\times X}$ is surjective. Hence there exists $(\gamma_1),(\gamma_2) \in \tilde{X}$ such that $\psi_{X\times X}(\gamma_1) = [\{*,a\}]_{X\times X}$ and $\psi_{X\times X}(\gamma_2) = [\{*,b\}]_{X\times X}$. Let E be any entourage of the covering basis of X and consider $\psi_E(\gamma_1) = [\eta]_E = [\{* = x_0, x_1, ...x_n = a\}]_E$ and $\psi_E(\gamma_2) = [\lambda]_E = [\{* = y_0, y_1, ...y_m = b\}]_E$. Then $\eta^{-1}\lambda$ is an E-chain from a to b and the result follows. ■

Proposition 27 *Let X and Y be uniform spaces and $f : X \to Y$ a uniformly continuous function. Let F be an entourage in Y and E and entourage in X such that $f(E) \subset F$. If $\gamma = \{* = x_0, x_1, ...x_n\}$ is an E-chain, we define $f(\gamma) = \{f(*) = f(x_0), f(x_1), ...f(x_n)\}$. We let $*$ be the basepoint in X and $f(*)$ be the basepoint in Y. Define $f_{EF} : X_E \to Y_F$ by $f_{EF}([\gamma]_E) = [f(\gamma)]_F$. Then f_{EF} is a well defined uniformly continuous function (see theorem 30 of [2]). In particular, $\phi_{EF} = id_{FE}$.*

Proof. Notice that $f(\gamma)$ is an F-chain by the assumption on E. Further, if $\gamma = \eta_0, \eta_1, ...\eta_s = \gamma'$ is an E-homotopy then $f(\gamma) = f(\eta_0), f(\eta_1), ...f(\eta_s) = f(\gamma')$ is an F-homotopy and hence the function f_{EF} is well defined. To see that the function is uniformly continuous, let $D \subset F$ and $K \subset E \cap f^{-1}(D)$. If $\gamma = \{* = x_0, x_1, ...x_n, a\}$ and $\eta = \{* = x_0, x_1, ...x_n, b\}$ are E-chains such that $([\gamma]_E, [\eta]_E) \in K^*$ then $(a,b) \in K$ and hence $(f(a), f(b)) \in D$ (since $K \subset f^{-1}(D)$). Thus $(f([\gamma]_E), f([\eta]_E)) \in D^*$ and $K^* \subset f_{EF}^{-1}(D^*)$. ■

3 Basic Results

In the work that follows, it will frequently be necessary to switch basepoints in the consideration of E chains. We will need to see, for example, that if E is a covering entourage from the vantage of one basepoint, then it is a covering entourage from the vantage of any other basepoint. Independence of basepoint for path connected spaces is an important feature of traditional covering space theory. The fact that, for a chain connected uniform space X, the same is true, i.e. it is possible to substitute any other point of X for the basepoint is established in [2]. Lemmas 28 and 29 along with proposition 30 provide a more explicit justification.

For the following lemmas, let $z \in X$ where X is a uniform space. Denote by $\tilde{X}^z$ the inverse limit of the fundamental system of X using z as basepoint and by X_E^z, ϕ_{EF}^z, and ψ_E^z the elements, bonding maps, and projection maps of this system. The equivalence classes of X_E^z will be denoted as $[x]_E^z$.

Lemma 28 *Suppose that* $([\gamma_F]_F) \in \tilde{X}^{b_1}$ *is such that* $\psi_E^{b_1}[([\gamma_F]_F)] = [\gamma_E]_E^{b_1}$ *where* $\gamma_E := \{b_1 = x_0, x_1, ..., x_n = b_2\}$. *In other words, the* E*th representative of* $([\gamma_F]_F)$ *is the equivalence class determined by the chain* γ_E *from* b_1 *to* b_2. *Then there exists* $(\gamma^{-1}) \in \tilde{X}^{b_2}$ *such that* $\psi_E^{b_2}[(\gamma^{-1})] = [\gamma_E^{-1}]_E^{b_2}$ *where* $u^{-1} := \{b_2 = x_n, x_{n-1}, ..., x_0 = b_1\}$ *(the* E*th representative of* (γ^{-1}) *is the equivalence class determined by* u^{-1} *from* b_2 *to* b_1*).*

Proof. For each entourage F we define the Fth element of (γ^{-1}) to be $[\gamma_F^{-1}]_F^{b_2}$ where γ_F is the Fth element of $([\gamma_F]_F)$. γ_F^{-1} exists since F is symmetric. Then, if $F \subset K$ we have that $\phi_{KF}^{b_1}([\gamma_F]_F^{b_1}) = [\gamma_K]_K^{b_1}$ so there exists a K homotopy $\gamma_F = \zeta_0, \zeta_1, ...\zeta_n = \gamma_K$. Since K is symmetric, $\gamma_F^{-1} = \zeta_n^{-1}, \zeta_{n-1}^{-1}...\zeta_0^{-1} = \gamma_K^{-1}$ is a K homotopy from γ_F^{-1} to γ_K^{-1}. Hence $\phi_{KF}^{b_2}([\gamma_F^{-1}]_F^{b_2}) = [\gamma_K^{-1}]_K^{b_2}$ and hence $(\gamma^{-1}) \in \tilde{X}^{b_2}$ is well defined. Further, we have that $\psi_E^{b_2}[(\gamma^{-1})] = [\gamma_E^{-1}]_E$ by definition, so that the Eth representative of (γ^{-1}) is $[\gamma_E^{-1}]_E^{b_2}$. ■

Lemma 29 *Let* $([\gamma_F]_F) \in \tilde{X}^{b_1}$ *be such that the endpoint of each* γ_F *is* b_2 *and let* $([\lambda_F]_F) \in \tilde{X}^{b_2}$. *Define* $[\mu]_F^{b_1} = [\gamma_F \lambda_F]_F^{b_1}$ *Then* $([\mu_F]_F) \in \tilde{X}^{b_1}$.

Proof. Let $F \subset E$. Since $\phi_{EF}^{b_1}([\gamma_F]_F^{b_1}) = [\gamma_E]_E^{b_1}$ there must be an E homotopy $\gamma_F = \gamma_0, \gamma_1, ...\gamma_n = \gamma_E$ from γ_F to γ_E. Similarly there exists an E homotopy $\lambda_F = \lambda_0, \lambda_1, ...\lambda_m = \lambda_E$ from λ_F to λ_E. Then we have that

$$\gamma_F \lambda_F = \gamma_0\lambda_0, \gamma_1\lambda_0, ..., \gamma_{n-1}\lambda_0, \gamma_n\lambda_0, \gamma_n\lambda_1, ...\gamma_n\lambda_m = \gamma_E\lambda_E$$

is an E homotopy from $\gamma_F\lambda_F$ to $\gamma_E\lambda_E$. Hence, $\phi_{EF}^{b_1}([\gamma_F\lambda_F]_F^{b_1}) = [\gamma_E\lambda_E]_E^{b_1}$ and (μ) is well defined. ■

Proposition 30 *Let* X *be a chain connected uniform space and* $b_1, b_2 \in X$. *If* E *is a covering entourage with respect to the basepoint* b_1 *then it is a covering entourage with respect to* b_2. *In particular a uniform coverable uniform space is uniform coverable with respect to any element of* X.

Proof. Since X is chain connected, there exists an E chain γ_E from b_1 to b_2 and since E is a covering entourage with respect to b_1 there exists $([\gamma_F]_F) \in \tilde{X}^{b_1}$ such that $\psi_E^{b_1}([\gamma_F]_F) = [\gamma_E]_E^{b_1}$, i.e. $([\gamma_F]_F)$ is an element in the fundamental inverse system of X with respect to b_1 whose Eth representative is the equivalence class of the E-chain γ_E. We must show that $\psi_E^{b_2} : \tilde{X}^{b_2} \to X_E^{b_2}$ is surjective. Let $[\lambda_E]_E^{b_2} \in X_E^{b_2}$ be such that the endpoint of λ_E is b_3. Notice that $\gamma_E\lambda_E$ is an E chain with initial point b_1 and hence $[\gamma_E\lambda_E]_E^{b_1} \in X_E^{b_1}$. Since E is a covering entourage with respect to the basepoint b_1 there exists $([\eta_F]_F) \in \tilde{X}^{b_1}$ such that $\psi_E^{b_1}([\eta_F]_F) = [\gamma_E\lambda_E]_E^{b_1}$, i.e. the Eth representative of $[\eta_E]_E = [\gamma_E\lambda_E]_E^{b_1}$. We wish to define an element of $\tilde{X}^{b_2}$ such that for each F the Fth representative is the equivalence class determined by the chain which travels along γ_F^{-1} to b_1 and then along η_F to the endpoint b_3. This is possible since $([\gamma_F^{-1}]_F) \in \tilde{X}^{b_2}$ by 28 and then by 29 the element $([\mu_F]_F) \in \tilde{X}^{b_2}$ whose Fth element is $[\gamma_F^{-1}\eta_F]_F^{b_2}$ is well defined. Further, we have that $\psi_E^{b_2}(\mu) = [\gamma_E^{-1}\eta_E]_E^{b_2} = [\gamma_E^{-1}\gamma_E\lambda_E]_E^{b_2} = [\gamma_E]_E^{b_2}$ and the result follows. If X is a uniform coverable uniform space with respect to the basepoint b_1 then by definition there exists a basis for the uniformity (including the set $X \times X$) which consists of covering entourages with respect to b_1. In particular, it is chain connected and thus, by the preceeding the same collection of entourages forms a basis of covering with respect to any point of X. ■

Since we have established that for any chain connected uniform space the notion of coverability is independent of the choice of basepoint, we will simply denote the basepoint by $*$. Then $[*]_E = [\{*,*\}]_E$ will always be the choice of basepoint for X_E. Further, $(*) \in \tilde{X}$ will designate the point whose Eth representative is $[*]_E$ and will always be the choice of basepoint for $\tilde{X}$. If X is a uniform coverable uniform space then by definition there exists a basis for the uniformity consisting of covering entourages. We will show (see proposition 33) that in fact it is possible to find a basis consisting of open entourages. The following lemmas are needed for this proposition. These lemmas may be known, but the author has not found a direct reference for them.

Lemma 31 *If F,G are open entourages and E is any entourage then $F \circ E \circ G$ is an open entourage.*

Proof. Let $(a,b) \in F \circ E \circ G$. Then there exists $(x,y) \in E$ such that $(a,x) \in F, (x,y) \in E, (y,b) \in G$. Since F, G are open entourages, $B(x,F)$ and $B(y,G)$ are open subsets of X. Hence, we may find entourages K_1 and K_2 such that $B(a,K_1) \subset B(x,F)$ and $B(b,K_2) \subset B(y,G)$. Let $(s,t) \in B(a,K_1) \times B(b,K_2)$. Then, by the choice of K_1 and K_2 we have that $s \in B(x,F)$ and $t \in B(y,G)$ which implies that $(s,x) \in F$ and $(y,t) \in G$. Since $(x,y) \in E$ we have that $(s,t) \in F \circ E \circ G$. Thus $B(a,K_1) \times B(b,K_2) \subset F \circ E \circ G$. By 8 there must exist an open set U of $X \times X$ such that $(a,b) \in U \subset B(a,K_1) \times B(b,K_2) \subset F \circ E \circ G$ and hence $F \circ E \circ G$ is open in $X \times X$. ■

Lemma 32 *If E is an entourage then $int(E)$ is an entourage and $int(E) \circ E \circ int(E)$ is an open entourage.*

Proof. The fact that $int(E)$ is an element of the uniformity is a result of Corollary 2 in section II.1.2 of [3]. To see that $int(E)$ is symmetric, let $(a,b) \in E$, and U, V open

subsets of X such that $(a,b) \in U \times V \subset int(E)$. Since E is symmetric we have that $V \times U \subset E$ and hence $V \times U \subset int(E)$ (since $V \times U$ is an open subset of E). Since $(b,a) \in V \times U$ the result follows. The second claim then follows from lemma31 since $int(E)$ is open. ■

Proposition 33 *If X is a uniform coverable uniform space then there exists a basis for the uniformity consisting of open covering entourages.*

Proof. Let E be a covering entourage, and set $E' = int(E) \circ E \circ int(E)$. Then E' is an open entourage by 31. We must show that it is a covering entourage. Let $(a,b) \in E'$, so that there exists $(x,y) \in E$ such that $(a,x),(y,b) \in int(E) \subset E$. Then, $\{a,x\},\{x,y\}$,and $\{y,b\}$ are all E-chains and since E is a covering entourage with respect to any basepoint by 30, we may find $(\gamma_1) \in \tilde{X}^a, (\gamma_2) \in \tilde{X}^x, (\gamma_3) \in \tilde{X}^y$ such that $\psi^a_E(\gamma_1) = [\{a,x\}]^a_E$, $\psi^x_E(\gamma_2) = [\{x,y\}]^x_E$,and $\psi^y_E(\gamma_3) = [\{y,b\}]^y_E$. By 29 there exists $(\gamma) \in \tilde{X}^a$ such that $\psi^a_E(\gamma) = [\{a,x,y,b\}]^a_E$. Since $E \subset E'$ we have: $\psi^a_{E'}(\gamma) = \phi_{(E')E}([\{a,x,y,b\}]^a_E) = [\{a,x,y,b\}]^a_{E'}$. Notice that $(x,x) \in int(E)$ implies that $(x,b) \in E'$ and thus $\{a,x,y,b\}$ is E'-homotopic to $\{a,b\}$. In other words, $\psi^a_{E'}(\gamma) = [\{a,b\}]^a_{E'}$. Now, let $\eta = \{* = x_0, x_1, ...x_n\}$ be any E' chain. By the preceeding, we may find for $1 \leq i \leq n$, $(\gamma_i) \in \tilde{X}^{x_{i-1}}$ such that $\psi_{E'}(\gamma_i) = [\{x_{i-1}, x_i\}]^{x_{i-1}}_{E'.}$ Again, by 29 we can find a chain $(\gamma) \in \tilde{X}^*$ such that $\psi_{E'}(\gamma) = [\eta]^*_E$ and hence E' is a covering entourage. To see that the collection of all such entourages forms a basis for the uniformity, let E be an arbitrary entourage and choose F such that $F^3 \subset E$. Then $E' \subset F^3 \subset E$. Finally, $X \times X$ is an open entourage and is a covering entourage by definition, since X is assumed to be uniform coverable. ■

For the remainder of this section we will consider X, Y to be uniform coverable uniform spaces with associated uniform universal covers $\tilde{X}, \tilde{Y}$ and projection maps $\psi^X : \tilde{X} \to X$ and $\psi^Y : \tilde{Y} \to Y$. If $f : X \to Y$ is a uniformly continuous function such that $f(*) = *$ then from [2] we know that f induces a uniformly continuous function $\tilde{f} : \tilde{X} \to \tilde{Y}$. In fact $\tilde{f}$ is the unique uniformly continuous function such that $\tilde{f}(*) = *$ and $f \circ \psi^X = \psi^Y \circ \tilde{f}$. We wish to prove some lifting properties of covers, but to do so it will be necessary to characterize the mapping $\tilde{f}$. Consider the mapping $f_{EF} : X_E \to Y_F$ defined in 27 which sends the Eth equivalence class of an E-chain γ to the Fth equivalence class of $f(\gamma)$. In other words $f_{EF}([\gamma]_E) = [f(\gamma)]_F$. Then by Theorem 30 in [2] this is a unique uniformly continuous function such that $f_{EF}([*]_E) = [*]_F$ and $\phi^Y_{YF} \circ f_{EF} = f \circ \phi^X_{XE}$ ($\phi_{XE} = \phi_{(X\times X)E}$ and $\phi_{YF} = \phi_{(Y\times Y)F}$ by definition). Given an element $([\gamma_E]_E) \in \tilde{X}$ we associate an element $([\eta_F]_F) \in \tilde{Y}$ in the following way. We may use the fact that f is uniformly continuous to find an element E of the covering basis of X such that $f(E) \subset F$. Then we set $[\eta_F]_F = [f(\gamma_E)]_F$. The following proposition shows that $([\eta_F]_F)$ is a well defined element of $\tilde{Y}$ and that the above association is equivalent to the function $\tilde{f}$.

Proposition 34 *The above association defines a function $\chi : \tilde{X} \to \tilde{Y}$ and $\chi = \tilde{f}$.*

Proof. This mapping does not depend on the choice of covering entourage. To see this we first let K be a covering entourage in X such that $K \subset E$. We then have $f(K) \subset f(E) \subset F$ so that K can be used to define $[\eta_F]_F$. Since $([\gamma_E]_E) \in \tilde{X}$ we

have that $[\gamma_E]_E = \phi^X_{EK}([\gamma_K]_K))$. Thus, in particular, γ_K is an E-chain in $[\gamma_E]_E$ and $f_{EF}([\gamma_E]_E) = f_{EF}([\gamma_K]_E) = [f(\gamma_K)]_F$. However, we also have, by definition that $f_{KF}([\gamma_K]_K) = [f(\gamma_K)]_F = f_{EF}([\gamma_E]_E)$. Thus any covering entouage contained in E can be used to define the mapping. Now, let M be any covering entourage such that $f(M) \subset F$ and choose a covering entourage $K \subset M \cap E$. Then, by what we have just shown, $f_{MF}([\gamma_M]_M) = f_{KF}([\gamma_K]_K) = f_{EF}([\gamma_E]_E)$. We conclude that the mapping is independent of the choice of covering entourage E in X with $f(E) \subset F$. To see that $([\eta_F]_F)$ is a well defined element in $\tilde{Y}$, suppose H is an entourage in Y such that $H \subset F$. Let K be a covering entourage in X such that $f(K) \subset H$ and choose a covering entourage $M \subset K \cap E$. Then, in fact $f(K) \subset F \cap H$. Hence we may use γ_K to define the Fth and Hth equivalence classes in $([\eta_F]_F)$ i.e. the Fth and Hth equivalence classes are defined to be $[f(\gamma_K)]_F$ and $[f(\gamma_K)]_H$ respectively. Since $f(\gamma_K)$ is both an F and an H-chain, we have $\phi_{FH}([f(\gamma_K)]_F) = [f(\gamma_K)]_H$ and hence (η) is well defined.

$\chi(*) = *$ since $f_{EF}([*]_E) = [f(*)]_F = [*]_F$ for all E such that $f(E) \subset F$. We now show that χ is uniformly continuous. Let D^* be a basis entourage in Y_F ($D \subset F$) and consider $(\psi^Y_F)^{-1}(D^*)$. This is the set of all pairs $((\eta_1),(\eta_2)) \in \tilde{Y} \times \tilde{Y}$ such that the Fth equivalence classes, $\psi^Y_F(\eta_1), \psi^Y_F(\eta_2)$ contain elements $\eta_{1F} = \{* = y_0, y_1, ...y_n, a\}$ and $\eta_{2F} = \{* = y_0, y_1, ...y_n, b\}$ respectively with $(a,b) \in D$. Let E be a covering entourage in X such that $f(E) \subset F$. Let K be a covering entourage in X such that $K \subset E$ and $f(K) \subset D$. In particular, K^* is a basis element in X_E. Let $((\gamma_1),(\gamma_2)) \in (\psi^X_E)^{-1}(K^*)$. Then, by definition, the Eth equivalence classes $\psi^X_E(\gamma_1), \psi^X_E(\gamma_2)$ contain elements $\gamma_{1E} = \{* = x_0, x_1, ...x_m, c\}$ and $\gamma_{2E} = \{* = x_0, x_1, ...x_m, d\}$ respectively with $(c,d) \in K$. But then, by the choice of K we have that $(f(c), f(d)) \in D$. Consider $f_{EF}(\psi^X_E(\gamma_1), \psi^X_E(\gamma_2)) = ([f(\gamma_{1E})]_F, [f(\gamma_{2E})]_F)$. Since $f(\gamma_{1E}) = \{* = f(x_0), f(x_1), ...f(x_m), f(c)\}$ and $f(\gamma_{2E}) = \{* = f(x_0), f(x_1), ...f(x_m), f(d)\}$ we have that $\chi((\gamma_1),(\gamma_2))$ is an element of D^* and hence $\chi((\psi^X_E)^{-1}(K^*)) \subset (\psi^Y_F)^{-1}(D^*)$. Thus χ is uniformly continuous. By the definition of χ it is clear that if a is the endpoint of $([\gamma_E]_E)$ then the endpoint of $\chi([\gamma_E]_E)$ is $f(a)$. Hence $\psi^Y \circ \chi = f \circ \psi^X$ and by uniqueness in theorem 54 of [2] we have $\chi = \tilde{f}$. ■

The map $\tilde{f}$ induces a homomorphism $f_* : \delta_1(X) \to \delta_1(Y)$ given by $f_* = \tilde{f}|_{\pi_1(X)}$ [2]. We have the following corollary.

Corollary 35 *Let X, Y and Z be uniform coverable spaces $f : X \to Y$ uniformly continuous and $g : Z \to Y$ bi-uniformly continuous such that $f_*(\delta_1(X)) \subset g_*(\delta_1(Z))$ (see diagram 1). Then for each covering entourage F in Z there is a covering entourage E in X such that $f_{Eg(F)}(\delta_E(X)) \subset g_{Fg(F)}(\delta_F(Z))$ where $f_{Eg(F)} : X_E \to Y_{g(F)}$ and $g_{Fg(F)} : Z_F \to Y_{g(F)}$ are the induced maps (diagram 2).*

1)
$$\begin{array}{ccc} & & Z \\ & & \downarrow g \\ X & \underset{f}{\to} & Y \end{array}$$

2)
$$\begin{array}{ccc} & & Z_F \\ & & \downarrow g_{Fg(F)} \\ X_E & \underset{f_{Eg(F)}}{\to} & Y_{g(F)} \end{array}$$

Proof. $g(F)$ is an entourage of Y since g is bi-uniformly continuous. We choose E to be any covering entourage in X such that $f(E) \subset g(F)$. Let l_E be an E-loop

so that $[l_E]_E \in \delta_E(X)$. Since E is a covering entourage we know that $[l_E]_E$ is the Eth element of some $([l_K]_K) \in \delta_1(X)$. Using the characterization of $\tilde{f}$ above and the fact that $f_* = \tilde{f} \mid_{\delta_1(X)}$ we have that the $g(F)$th element in $f_*([l_K]_K)$ is $[f(l_E)]_{g(F)}$. Now, using the property $f_*(\delta_1(X)) \subset g_*(\delta_1(Z))$ we know that there is an element $([m_J]_J) \in \delta_1(Z)$ such that $\tilde{g}([m_J]_J) = \tilde{f}([l_K]_K)$. We have that m_F is an F-loop in the Fth equivalence class of $([m_J]_J)$ i.e. $[m_F]_F \in \delta_F(Z)$. Again using the above characterization, and the fact that F is a covering entourage whose image under g is contained in (in fact is equal to) the entourage $g(F)$, we must have that $[g(m_F)]_{g(F)}$ is the $g(F)$ equivalence class in $\tilde{g}([m_J]_J) = \tilde{f}([l_K]_K)$ and combining this with above we have $[g(m_F)]_{g(F)} = [f(l_E)]_{g(F)}$. Hence $f_{Eg(F)}([l_E]_E) = g_{Fg(F)}([m_F]_F)$. Since $[l_E]_E$ was an arbitrary element of $\delta_E(X)$, we have demonstrated that for each $[l_E]_E \in \delta_E(X)$ there exists an $[m_F]_F \in \delta_F(Z)$ such that $f_{Eg(F)}([l_E]_E) = g_{Fg(F)}([m_F]_F)$ and the result now follows. ■

The next lemma establishes a "unique chain lifting property" for discrete covers (defined in the paragraph following 23). It is a result (theorem 39 in [2]) that if X is chain connected then the mapping $\phi_{XE} : X_E \to X$ is a discrete cover.

Lemma 36 *Suppose $f : Y \to X$ is a discrete cover under the action of a group G, $b \in X$ and $p \in f^{-1}(b)$. Then there is a sufficiently small entourage E such that if $F = f(E)$ then every F-chain with initial point at b has a unique lift to an E-chain with initial point at p. In other words, for every F-chain $\{b = x_0, x_1, ...x_n\}$ there is a unique E-chain $\{p = y_0, y_1 ... y_n\}$ such that $f(y_i) = x_i$. If $\{b = x_0, x_1, ...x_n = c\} = \gamma_0, \gamma_1, ...\gamma_k = \{b = t_0, t_1, ...t_m = c\}$ is an $f(E)$ homotopy from γ_0 to γ_k then $\gamma'_0, \gamma'_1, ...\gamma'_k$ is an E-homotopy where each γ'_i is the unique lift of γ_i which starts at p.*

Proof. The proof of Proposition 55 in [2] essentially proves this lemma. In the interest of completeness, the following more direct proof is offered. Let D be an entourage of Y such that $(a, g(a)) \in D \Rightarrow g = e$, and let E be an invariant entourage such that $E^3 \subset D$. Since f is bi-uniformly continuous (see [11]), $F = f(E)$ is an entourage. We have $f(p) = b$ by assumption. For the purpose of induction suppose that $\{p = y_1, y_2, ...y_i\}$ is a unique lift of $\{b = x_0, x_1 ... x_i\}$. We have that $(x_i, x_{i+1}) \in F$. There is then, since $F = f(E)$, an $(a, b) \in E$ such that $f(a) = x_i$ and $f(b) = x_{i+1}$. Since $a \in f^{-1}(x_i)$, and f is the quotient defined by the action of G, there must be a $g \in G$ such that $g(a) = y_i$. Let $y_{i+1} = g(b)$. Then $(y_i, y_{i+1}) = (g(a), g(b)) \in E$ since E is G invariant and $\{p = y_1, y_2, ...y_i, y_{i+1}\}$ is a lift of $\{b = x_0, x_1 ... x_i\}$. To prove uniqueness, suppose there exists another y such that $(y_i, y) \in E$ and $f(y) = x_{i+1}$. Then $y \in f^{-1}(x_{i+1})$ and there must be an $h \in G$ such that $h(y) = y_{i+1}$. But then $(y, y_i) \in E$ by symmetry and $(y_i, h(y)) = (y_i, y_{i+1}) \in E$ which implies that $(y, h(y)) \in E^2 \subset D$. By the assumption on D, $h = e$ and $y = y_{i+1}$.

For the second statement let γ'_i be the unique lift of γ_i which starts at p. We need only verify that $\gamma'_0, \gamma'_1, ...\gamma'_k$ is an E-homotopy. Notice that each γ'_i is an E-chain. Suppose γ_i is obtained from γ_{i-1} by the insertion of an element a between x_{j-1} and x_j. Denote γ'_{i-1} by $\{p = y_0, y_1, ...y_n\}$ and γ'_i by $\{p = z_0, z_1, ...z_{j-1}, A, z_j, ...z_n\}$, so that $f(y_k) = x_k = f(z_k)$ for all $0 \leq k \leq n$ and $f(A) = a$. By the uniqueness of chain lifting applied to the first $j-1$ elements we must have that $y_k = z_k$ for $0 \leq k \leq j-1$. Now, $(y_{j-1}, y_j) \in E$ implies that $(y_j, z_{j-1}) \in E$ by symmetry and the fact that $y_{j-1} = z_{j-1}$.

We also know that $(z_{j-1}, A), (A, z_j) \in E$ hence $(y_j, z_j) \in E^3 \subset D$. Since $f(y_j) = f(z_j)$ we must have that $z_j = g(y_j)$ for some $g \in G$ so that $(y_j, z_j) \in D \Longrightarrow (y_j, g(y_j)) \in D$ and by the choice of D we have that $g = e$. Hence $y_j = z_j$. Then, the uniqueness of chain lifting implies that $y_k = z_k$ for $j \leq k \leq n$. Hence γ'_i is an expansion of γ'_{i-1}. If γ_i is obtained from γ_{i-1} by the removal of an element then the argument above implies that γ'_{i-1} is an expansion of γ'_i, i.e. γ'_i is a contraction of γ'_{i-1}. Thus $\gamma'_0, \gamma'_1, ...\gamma'_k$ is a well defined E-homotopy. In particular we have that each γ'_i ends at the same point in $f^{-1}(c)$. ■

Lemma 37 *Suppose X, Y, Z are uniform coverable spaces $f : X \to Y$ is uniformly continuous and $g : Z \to Y$ is a discrete cover with covering group G and assume that $f(*) = * = g(*)$ (see diagram). Then the following are equivalent:*

1. For every entourage F in Z there exists an entourage $E \subset f^{-1}(g(F))$ in X such that $f_{Eg(F)}(\delta_E) \subset g_{Fg(F)}(\delta_F)$.
2. There exists a unique uniformly continuous lift $L_f : X \to Z$ such that $L_f(*) = *$ and $g \circ L_f = f$.

$$\begin{array}{ccc} & & Z \\ & \nearrow^{L_f} & \downarrow g \\ X & \underset{f}{\rightarrow} & Y = Z\backslash G \end{array}$$

Proof. (1$\Longrightarrow$2). We define $L_f : X \to Z$ in the following way. Choose an entourage F in Z small enough to satisfy the conditions of 36. Since g is a discrete cover it is bi-uniformly continuous [11] and hence $g(F)$ is an entourage in X. We use 1, to choose an entourage E in X such that $f_{Eg(F)}(\delta_E) \subset g_{Fg(F)}(\delta_F)$. Let $x \in X$. Since X is uniform coverable it is chain connected, so choose an E-chain c from $*$ to x. Then $f(c)$ is a $g(F)$ chain in Y. By 36 there is a unique lift $\{* = z_0, z_1, ...z_n\}$ of $f(c)$ beginning at $*$. We set $L_f(x) = z_n$. To see that this is well defined, let c' be a second E-chain in X from $*$ to x with lift $\{* = k_0, k_1, ...k_m\}$ in Z. Then $c(c')^{-1}$ is an E-loop in X. Hence $f(c(c')^{-1})$ is a $g(F)$-loop in Y. By 36 there is a unique lift of $f(c(c')^{-1})$ to an F chain in Z. By uniqueness of lifts beginning at $*$, the first n elements in the lift of $f(c(c')^{-1})$ must be $z_0, z_1, ...z_n$. Denote the lift of $f(c(c')^{-1})$ by $\{* = z_0, z_1, ...z_n, s_{m-1}, s_{m-2}, ...s_0\}$. Since $[c(c')^{-1}]_E \in \delta_E(X)$ and since $f_{Eg(F)}(\delta_E(X)) \subset g_{Fg(F)}(\delta_F(Z))$ there must be an F-loop l in Z whose image $g(l)$ is in the same equivalence class as $f(c(c')^{-1})$. However, by the second part of 36 above, the $g(F)$ homotopy between $f(c(c')^{-1})$ and $g(l)$ must lift to an F-homotopy between the lift of $f(c(c')^{-1})$ and l. In particular, this implies that the lift of $f(c(c')^{-1})$ is a loop with base point $*$, i.e. $s_0 = *\}$. Then, by the uniqueness of the lift of c' beginning at $*$ we then have that the lift of $f(c(c')^{-1})$ is $\{* = z_0, z_1, ..., z_n = k_m, k_{m-1}, ...k_0 = *\}$ and hence $z_n = k_m$. Thus L_f is well defined. By construction we have $g \circ L_f = f$. To see that L_f is uniformly continuous, let M be an entourage of Z and choose $D \subset M \cap F$ so that D satisfies the conditions of 36. Let $N \subset E \cap f^{-1}(g(D)$ be such that $f_{Ng(D)}(\delta_N) \subset g_{Dg(D)}(\delta_D)$. If $(x_1, x_2) \in N$ then, applying the above process, we find N-chains c_{x_1}, c_{x_2} from $*$ to x_1, x_2 respectively. Let $c_{x_1}(c_{x_2})^{-1}$ denote the N-chain

which travels along c_{x_1} from $*$ to x_1, jumps to x_2 and travels along $c_{x_2}^{-1}$ back to $*$. This is an N-loop, and we may apply the preceeding to see that $f(c_{x_1}(c_{x_2})^{-1})$ lifts to a unique D-loop in Z. This then implies that the unique lift of the chains $f(c_{x_1})$ and $f(c_{x_2})$ end in elements $z_1, z_2 \in Z$ such that $(z_1, z_2) \in D \subset M$. Since $N \subset E$, c_{x_1}, c_{x_2} are also E-chains and hence by above we must have $z_1 = L_f(x_1)$, $z_2 = L_f(x_2)$. Hence $L_f(N) \subset M$ and L_f is uniformly continuous.

To prove uniqueness of L_f, suppose $L : X \to Z$ is another uniformly continuous functions such that $L(*) = *$ and $g \circ L = f$. Let $x \in X$. Since L_f and L are uniformly continuous we may choose $K \subset L^{-1}(F) \cap L_f^{-1}(F) \cap E$. Then K is an entourage such that $L(K) \subset F$ and $L_f(K) \subset F$. Choose a K chain (which is also an E chain by the choice of K) $c = \{* = x_0, x_1, ...x_n = x\}$ in X from $*$ to x, and let $\{* = z_0, z_1, ...z_n\}$ be the unique lift of $f(c)$ to an F chain. By the choice of K we have that $L(c)$ and $L_f(c)$ are F chains in Z. Since $g(L(c)) = g(L_f(c)) = f(c)$ by assumption, we must have, by uniqueness, that $L(c) = L_f(c) = \{* = z_0, z_1, ...z_n\}$. In particular $L(x) = L_f(x)$ and the result follows.

($2 \Longrightarrow 1$). Let F be an entourage in Z and choose an entourage K in X such that $L_f(K) \subset F$. Let $[l]_K \in \delta_K$. Then $f(l) = g \circ L_f(l)$. Since $L_f(l)$ is an F-loop (by the choice of K) we have that $[L_f(l)]_F \in \delta_F(Z)$. Hence $f_{Kg(F)}([l]_K) \in g_{Fg(F)}(\delta_F)$. ■

Theorem 38 *Suppose X, Y, Z are uniform coverable, $f : X \to Y$ is a uniformly continuous function, $g : Z \to Y$ is a uniform cover. Then f lifts to a function $L_f : X \to Z$ such that $g \circ L_f = f$ if and only if $f_*(\delta_1(X)) \subset g_*(\delta_1(Z))$.*

Proof. Since g is a uniform cover, we have by theorem 48 in [11] that $Z = \varprojlim\{Z_\alpha, \phi_{\alpha\beta}\}$ such that $Y = Z_1$ for some minimal element in the index, $g = \psi_1$ and each $\phi_{1\beta} : Z_\beta \to Y$ is a discrete cover. We thus have the following diagram:

$$\begin{array}{ccc} & & Z = \varprojlim\{Z_\alpha, \phi_{\alpha\beta}\} \\ & & \downarrow g = \psi_1 \\ X & \underset{f}{\longrightarrow} & Y = Z_1 \end{array}$$

Suppose $f_*(\delta_1(X)) \subset g_*(\delta_1(Z))$. We wish to show that for each index β we may apply the previous lemma to obtain a lift $L_f^\beta : X \to Z_\beta$ and that L_f^β satisfy the universal property of inverse limits, defining a function $L_f : X \to Z$. Note that a basis for the uniformity on Z is given by the sets $\psi_\beta^{-1}(D_\beta)$ where β is arbitrary and D_β is a basis element of Z_β. Let D_β be an entourage in Z_β and let F be a covering entourage contained in $\psi_\beta^{-1}(D_\beta)$. By 35, there is an entourage E in X such that $f_{E\psi_1(F)}(\delta_E(X)) \subset (\psi_1)_{F\psi_1(F)}(\delta_F(Z))$. Thus the image under f of any E-loop is $\psi_1(F)$-homotopic to the projection ψ_1 of some F-loop in Z. However, to be able to apply the previous lemma, we need to know that the image of any E-loop is $\phi_{1\beta}(D_\beta)$-homotopic to the image under $\phi_{1\beta}$ of some D_β-loop. We use the fact that $\phi_{\alpha\beta}$ form an inverse system and hence $\psi_1 = \phi_{1,\beta} \circ \psi_\beta$. Let γ be an E-loop in X, and let l be an F-loop in Z such that $f(\gamma)$ is $\psi_1(F)$-homotopic to $\psi_1(l)$. Then $\psi_\beta(l)$ is a D_β-loop in Z_β and $\psi_1(l) = \phi_{1,\beta}(\psi_\beta(l))$. Since $\psi_1(F) \subset \phi_{1,\beta}(D_\beta)$ the $\psi_1(F)$-homotopy from $\psi_1(l)$ to $f(\gamma)$ is also a $\phi_{1\beta}(D_\beta)$-homotopy. Hence we may apply 37 to obtain a unique lift $L_f^\beta : X \to Z_\beta$ such that $\phi_{1\beta} \circ L_f^\beta = f$. Notice that $\phi_{1\alpha} \circ (\phi_{\alpha\beta} \circ L_f^\beta) = \phi_{1\beta} \circ L_f^\beta = f$

and hence by uniqueness, $\phi_{\alpha\beta} \circ L_f^\beta = L_f^\alpha$. By the universal property of inverse limits we have a function $L_f : X \to Z$ and $g \circ L_f = \psi_1 \circ L_f = f$.

For the reverse, suppose we have such a lift L_f such that $g \circ L_f = f$. If $(l) \in \delta_1(X)$. Then for each entourage K in Y, we choose an entourage D in Y so that the Kth element of $g_*(m)$ is determined by $g_{DK}([m_D]_D)$ for any loop (m) in $\delta_1(Z)$. Then we choose an E in X so that both the Dth element in $(L_f)_*(l)$ and the Kth element in $f_*(l)$ are determined by $(L_f)_{ED}([l_E]_E)$ and $f_{EK}([l_E]_E)$ respectively. Then the Kth element in $f_*(l)$ is $[f(l_E)]_K = [g(L_f(l_E))]_K$. The Dth element of $(L_f)_*(l)$ is $[L_f(l_E)]_D$. Since K was arbitrary, we have that $f_*(l) = g_*((L_f)_*(l))$ and hence $f_*(\delta_1(X)) \subset g_*(\delta_1(Z))$. ■

Proposition 39 *Let X be a uniform coverable uniform space. Let D be a dense subspace of X with $* \in D$. Then D is uniform coverable in the subspace uniformity. If $i : D \to X$ is the inclusion mapping and $\tilde{\imath} : \tilde{D} \to \tilde{X}$ the induced map from 34 then $\tilde{\imath}$ is injective, and $\tilde{\imath}(\tilde{D})$ is dense in $\tilde{X}$. Further, $i_* : \delta_1(D) \to \delta_1(X)$ is an isomorphism.*

Proof. By 33 we may choose Ω to be a basis for X consisting of open covering entourages. For each E in Ω we set $E_D = E \cap (D \times D)$ and the collection Ω_D of all such elements forms a basis for the subspace uniformity on D. It suffices to show that each E_D is a covering entourage. Let $\gamma = \{* = d_0, d_1, ..., d_n\}$ be an E_D chain in D. Then γ is also an E-chain in X and hence $[\gamma]_E \in X_E$.Then, since E is a covering entourage for X, there exists $([u_S]_S) \in \tilde{X}$ such that $\phi_E([u_S]_S) = [u_E]_E = [\gamma]_E$. For each entourage $F \subset E$, u_F is an F chain in X which is E-equivalent to γ. We find an F_D chain in $[u_F]_F$ by slightly moving the element of u_F onto elements of D. We must find an entourage K which is small enough so that our choice of elements in D actually form a F_D chain. We will then show below that these chains in D form a well defined element of $\tilde{D}$.

Let $u_F = \{* = x_0, x_1, ...x_m = d_n\}$. Since F is open and $(x_i, x_{i+1}) \in F$ for $i = 0, 1, ...m-1$, there is a basis entourage K_i such that $B(x_i, K_i) \times B(x_{i+1}, K_i) \subset F$. This implies that if $d_i \in B(x_i, K_i)$ and $d_{i+1} \in B(x_{i+1}, K)$ then in fact $(d_i, d_{i+1}) \in F$. Let $K \in \Omega$ be such that $K \subset \cap_{i=0}^{m-1} F_i \cap F$ so that, for all i, $B(x_i, K) \times B(x_{i+1}, K) \subset F$ and $K \subset F$. Using the density of D, and the fact that K and hence the balls of K are open, we can find a $y_i \in B(x_i, K) \cap D$. If $x_i \in D$ then we will specifically choose $y_i = x_i$, so that, in particular, $y_0 = *$ and $y_m = x_m = d_n$. By the choice of K we have that $(y_i, y_{i+1}) \in F$ for all i which implies that $v_F = \{* = y_0, y_1, y_2 ... y_m = d_n\}$ is an F chain. To see that it is F equivalent to u_F notice that, again by the choice of K, for each $* \leq i \leq m-2$, we have that $(x_i, y_{i+1}) \in B(x_i, K) \times B(x_{i+1}, K) \subset F$ and $y_{i+1} \in B(x_{i+1}, K)$ implies $(y_{i+1}, x_{i+1}) \in K \subset F$. Thus y_{i+1} may be inserted into $x_i x_{i+1}$. We form an equivalence between the chains by inserting the y elements and then removing the x elements. Specifically, we have the following equivalences:

$$*x_1 ... x_{m-1} d_n \sim *y_1 x_1 y_2 x_2 ... y_{m-1} x_{m-1} d_n \sim *y_1 ... y_{m-1} d_n$$

We wish to show that $([v_F]_{F_D})$ is a well defined element of $\tilde{D}$, i.e. that if $M \subset F$ the bonding map ϕ_{FM} takes $[v_M]_{M_D}$ to $[v_F]_{F_D}$. In particular, $\phi_{E_D}([v_F]_{F_D}) = [v_E]_{E_D}$and, $v_E = \gamma_E$ since each element of γ_E is an element of D and hence by convention the elements of v_E are simply taken to be the elements of γ_E. Hence it suffices to show

that if $M_D \subset F_D \subset E_D$ then v_M is F_D-equivalent to v_F. Since $v_M \in [u_M]_M$ and $v_F \in [u_F]_F$ and $\phi_{FM}[u_M]_M = [u_F]_F$, we know that v_M is F-equivalent to v_F i.e. there exists an F-homotopy $v_M = \gamma_0, \gamma_1 ... \gamma_k = v_F$. The γ_i however may contain elements of X which are not in D. We need to show that it is possible to move any such elements slightly and still maintain the conditions of an F-homotopy. We will first find an entourage N small enough that if the elements of chains γ_i are replaced with elements of D no more than a distance N away then the resulting chains will form an F_D-homotopy. Toward this end, we note that if γ_i is obtained from γ_{i-1} by the deletion of an element x_j then $(x_{j-1}, x_{j+1}) \in F$ and, since F is open, we can find an entourage N_i of X such that $B(x_{j-1}, N_i) \times B(x_{j+1}, N_i) \subset F$. Similarly, if γ_i is obtained from γ_{i-1} by the insertion of an element a between x_j and x_{j+1} then both (x_j, a) and $(a, x_{j+1}) \in F$ and we can find an entourage N_i of X, such that $B(x_j, N_i) \times B(a, N_i) \subset F$ and $B(a, N_i) \times B(x_{j+1}, N_i) \subset F$. Thus if e is the number of expansions and r the number of deletions, we have chosen $2e+r$ such N'. We index them as $N_1, N_2, ... N_{2e+r}$. We then choose N to be an element of Ω contained in the (finite) intersection $\cap_{i=1}^{2e+r} N_i \cap F$.

We will use induction to build an F_D homotopy $\eta_M = \gamma'_0, \gamma'_1 ... \gamma'_k = \eta_F$. We begin by assigning $\gamma'_0 = \gamma_0$. It is then trivial to verify the following for $i = 0$.

1. γ'_i has the same length as γ_i
2. If d_j is the jth element of γ'_i and x_j is the jth element of γ_i then $(x_j, d_j) \in N$.
3. If $x_j \in D$ then $x_j = d_j$.

We assume that γ'_{i-1} has been chosen and satisfies the three properties above. To choose γ'_i, first assume that γ_i is obtained from γ_{i-1} by the deletion of an element x_j. Then, by property 2, we know that $(x_{j-1}, d_{j-1}), (x_{j+1}, d_{j+1}) \in N \Longrightarrow d_{j-1} \in B(x_{j-1}, N)$ and $d_{j+1} \in B(x_{j+1}, N)$. Also, N was chosen so that $B(x_{j-1}, N) \times B(x_{j+1}, N) \subset F$ which then implies that $(d_{j-1}, d_{j+1}) \in F$ and d_j can be removed from γ'_{i-1} obtaining an F_D-chain which we set equal to γ'_i. γ_i and γ'_i obey properties 1-3 since γ_{i-1} and γ'_{i-1} do. On the other hand, if γ_i is obtained from γ_{i-1} by the insertion of an element a between x_j and x_{j+1} then we can use the density of D to find an element $d \in B(a, N)$. Using property 2 we have that $d_j \in B(x_j, N), d_{j+1} \in B(x_{j+1}, N)$. Then, again by the choice of N we have that $(d_j, d), (d, d_{j+1}) \in N$ which implies that $(d_j, d), (d, d_{j+1}) \in F$. We then set γ'_i to be the F_D-chain obtained from γ'_{i-1} by the insertion of the element d. If $a \in D$ then we take $d = a$ so that property 3 is satisfied. The first two properties are satisfied as well. Proceeding inductively, we define γ'_i for all i. By property 3 we must have that $\gamma'_k = \gamma_k = \eta_F$ and thus $\eta_M = \gamma'_0, \gamma'_1 ... \gamma'_k = \eta_F$ is an F_D homotopy from η_M to η_F and (η) is well defined.

We now consider the mapping $\tilde{\imath}$. To see injectivity, suppose $\tilde{\imath}(([v_{S_D}]_{S_D})) = \tilde{\imath}(([w_{S_D}]_{S_D}))$. Then v_{S_D} is S homotopic to w_{S_D} for each entourage S. By what we have just shown, v_{S_D} must then be S_D homotopic to w_{S_D} for each entourage S_D and thus $([v_{S_D}]_{S_D}) = ([w_{S_D}]_{S_D})$. To see that $\tilde{\imath}(\tilde{D})$ is dense in $\tilde{X}$, let $F \subset E$ be open covering entourages in X, F^* the corresponding entourage in X_E and $L = \phi_E^{-1}(F^*)$ a basis entourage in $\tilde{X}$. If $([u_S]_S) \in \tilde{X}$ then we need to find a $([v_{S_D}]_{S_D}) \in \tilde{D}$ such that $\tilde{\imath}([v_{S_D}]_{S_D}) \in B(([u_S]_S), L)$. $B(([u_S]_S), L)$ is the set of all $([w_S]_S) \in \tilde{X}$ such

that w_E is E-homotopic to an E-chain of the form $\{* = x_0, x_1, ...x_{n-1}, a\}$ and u_E is E-homotopic to an E-chain of the form $\{* = x_0, x_1, ...x_{n-1}, b\}$ where $(a, b) \in F$. Since $B(b, F)$ is open in X we can find $d \in D$ such that $(b, d) \in F$. Consider the chain $v_E = \{* = x_0, x_1, ...x_{n-1}, b, d\}$. By what we have shown, there is an element $([v_{S_D}]_{S_D}) \in \tilde{D}$ whose E_Dth element is E-homotopic to v_E. By 34 and the fact that $i(E_D) \subset E$ we know that $\tilde{\imath}(([v_{S_D}]_{S_D})) = ([v_{S_D}]_S)$ which is an element in $\tilde{X}$ whose Eth element is E-homotopic to v_E. Since $\{* = x_0, x_1, ...x_{n-1}, b, b\}$ is an expansion of $\{* = x_0, x_1, ...x_{n-1}, b\}$, u_E is E-homotopic to a chain which differs from v_E only in the endpoint b. Since $(b, d) \in F, ([u_E]_E, [v_E]_E) \in F^*$ and we see that $\tilde{\imath}(([v_{S_D}]_{S_D})) \in B(([u_S]_S), L)$. Hence $\tilde{\imath}(\tilde{D})$ is dense in $\tilde{X}$. We have already shown that $\tilde{\imath}$ is injective. Thus to show that the induced homomorphism i_* is an isomorphism we only need show that it is surjective. However, if $([u_S]_S)$ is in $\delta_1(X)$ then by assumption the endpoint of each u_S is $*$ which lies in D. Hence, by what we have shown above, there is a loop $([u_{S_D}]_{S_D}) \in \delta_1(D)$ such that each u_{S_D} is S homotopic to u_S. Thus $i_*(([u_{S_D}]_{S_D})) = ([u_S]_S)$ and i_* is surjective. ■

4 A Uniformity for Glued Uniform Spaces

Our goal is to show that connected finite dimensional CW complexes have a uniformity, compatible with thier topology, which is coverable. We present the basic definitions here (in the context of uniform spaces). The reader is referred to section I.2.4 of [3] for the basics of the quotient topology and to [6] and [13] for more details on gluing and CW complexes. Let X be a uniform space. Let $\{Y_\alpha\}_{\alpha\in A}$ be a collection of uniform spaces and for each α let Z_α be a subspace of Y_α and $f_\alpha : Z_\alpha \to X$ be uniformly continuous. We wish to define an equivalence relation on the disjoint union $X \amalg \{Y_\alpha\}$. First, we set $y_1 \sim y_2$ if $f_\alpha(y_1) = f_\beta(y_2)$ for any two indices $\alpha, \beta \in A$. Then, assign $x \sim y$ if $f(y) = x$. This clearly partitions $Z \amalg \{\mathrm{Im}(f_\alpha)\}$. The remainder of points are given their own class. We denote by $X\hat{}\{Y_\alpha\}$ the collection of equivalence classes. We note that $[a]$ contains more that one element if and only if it contains for some $\alpha \in A$ one element of $\mathrm{Im}(f_\alpha)$, together with all of its pre-images under any f_β. This implies that for any equivalence class $[a]$ in $X \hat{} \{Y_\alpha\}$ we must have that either $X \cap [a]$ is empty or it contains a unique element. $X \hat{} \{Y_\alpha\}$ has a quotient topology in which a set is open if and only if the inverse image of the set under the inclusion maps $i_{Y_\alpha} : Y_\alpha \to X \hat{} \{Y_\alpha\}$ and $i_X : X \to X \hat{} \{Y_\alpha\}$ are open. We will define a uniformity on $X \hat{} \{Y_\alpha\}$ compatible with this topology in such a way that if all the spaces involved are uniform coverable, then $X \hat{} \{Y_\alpha\}$ is uniform coverable.

CW complexes are obtained by gluing special uniform spaces called n-cells. An n-cell is the subspace of $\mathbb{R}^n$ consisting of the unit ball B^n together with its boundary S^{n-1}. We choose $* \in S^{n-1}$ as the basepoint of B^n. Let $f : S^{n-1} \to X$ be a continuous function. It can be shown (see Theorems 1 and 2 of section II.4.1 in [3]) that every Compact Hausdorff space has a unique uniform structure compatible with its topology and every continuous function f defined on it is uniformly continuous. This implies that f is uniformly continuous for the metric uniformity on S^1, which is a uniform subspace of B^n under the metric uniformity. A CW complex is formed inductively by dimension. We begin with X_0, a discrete set of points (the 0-skeleton). The discrete topology is induced by a uniform structure whose uniformity base consists of the single set Δ (see example 2 in II.1.1 of [3]). A collection of 1-cells $\{B^1_\alpha\}$ are glued to X_0 by continuous (hence uniformly continuous) f_α from $\{S^0_\alpha\}$. The resulting quotient space is denoted by X_1 (the 1-skeleton). Then, a collection of 2-cells $\{B^2_\beta\}$ are attached to X_1 via continuous maps f_β from $\{S^1_\beta\}$ to form the 2-skeleton X_2. This process either ends at some dimension n in which case the complex is said to be n-dimensional, or it can continue inductively, with the resulting infinite dimensional complex given the weak topology. All B^n and S^{n-1} for $n \geq 2$ are uniform coverable by [2].

We turn now to the task of defining an appropriate uniform structure on a glued space, $X\hat{}\{Y_\alpha\}$. Let X and $\{Y_\alpha\}_{\alpha\in A}$ be Hausdorff uniform spaces and for each α let Z_α be a subspace of Y_α under the subspace uniformity and $f_\alpha : Z_\alpha \longrightarrow X$ a uniformly continuous function. Let E be any entourage in X , F_α an entourage in Y_α, for each α and $([a],[b]) \in X\hat{}\{Y_\alpha\} \times X\hat{}\{Y_\alpha\}$.

Definition 40 *We will abuse terminology slightly and say that* $([a],[b])$ *is "in* E*" if there exists* $x_1 \in X \cap [a], x_2 \in X \cap [b]$ *such that* $(x_1, x_2) \in E$*. Similarly,* $([a],[b])$ *is*

"in F_α" if there exists $y_1 \in Y_\alpha \cap [a], y_2 \in Y_\alpha \cap [b]$ such that $(y_1, y_2) \in F_\alpha$. Now, let F_α be an entourage in Y_α such that $F_\alpha \cap Z_\alpha \times Z_\alpha \subset f_\alpha^{-1}(E)$. We will call any pair (E, F_α) acceptable if it meets this condition.

Lemma 41 *For each α and entourage E of X, there is an entourage F_α of Y_α such that (E, F_α) is acceptable. If (E, F_α) is acceptable then $(E, F_\alpha \cap K)$ is acceptable for all entourages K in Y_α. If (E, F_α) is acceptable and $(y_1, y_2) \in F_\alpha \cap Z_\alpha \times Z_\alpha$ then $([y_1], [y_2])$ is in E.*

Proof. If E is any entourage of X then $f_\alpha^{-1}(E)$ is an entourage of Z_α in the subspace uniformity by the uniform continuity of f_α. Therefore, there must be an entourage in Y_α whose intersection with $Z_\alpha \times Z_\alpha$ is $f_\alpha^{-1}(E)$. For any entourage K in Y, $(K \cap F_\alpha) \cap Z_\alpha \times Z_\alpha \subset F_\alpha \cap Z_\alpha \times Z_\alpha \subset f_\alpha^{-1}(E)$. For the last statement, $(y_1, y_2) \in F_\alpha \cap Z_\alpha \times Z_\alpha \subset f_\alpha^{-1}(E)$ implies that $(f_\alpha(y_1), f_\alpha(y_2)) \in E$. Since $f(y_1) \in X \cap [y_1]$ and $f(y_2) \in X \cap [y_2]$ we have that $([y_1], [y_2])$ is in E. ■

Note: If the pairs $([a], [b]), ([b], [c])$ are in an entourage E of X then $([a], [c])$ is in E^2. If fact, if we let x_1, x_2, x_3 be the unique elements in $X \cap [a], X \cap [b], X \cap [c]$ respectively then by definition we have $(x_1, x_2), (x_2, x_3) \in E$ which implies that $(x_1, x_3) \in E^2$ and hence $([a], [c])$ is in E^2. Notice however that if f_α is not injective then the same may not apply if the pairs are in an entourage of some Y_α. In this case, two distant elements $y_1, y_2$of Y_α could get mapped to the same x by f_α which implies that $[y_1] = [y_2]$. If y_3 is close to y_1 and y_4 is close to y_2 then it could happen that $([y_3], [y_1]) = ([y_3], [y_2])$ and $([y_2], [y_4])$ are both in F_α and yet $([y_3], [y_4])$ not an element of F_α^2.

For the following definition, choose for each $\alpha \in A$ an entourage F_α of Y_α such that (E, F_α) is acceptable. Let E' be an entourage in X such that $(E')^4 \subset E$ and then choose for each α, F'_α such that $(F'_\alpha)^4 \subset F_\alpha$ and (E', F'_α) is acceptable. It is possible to do this by first choosing an entourage E' such that $(E')^4 \subset E$. We can then find entourages H_α, K_α in Y_α such that (E', H_α) is acceptable and $K_\alpha^4 \subset F_\alpha$. Letting $F'_\alpha \subset H_\alpha \cap K_\alpha$ we can use the above lemma to see that F'_α has both of these properties.

Definition 42 *Let E, E' be as in the preceeding paragraph and F_α and F'_α be chosen for each α as in the preceeding paragraph. We define $\langle E, E', F_\alpha, F'_\alpha \rangle$ (a glued entourage) to be the set of all $([a], [b])$ such that either:*

1. *$([a], [b])$ is in E or F_α for some $\alpha \in A$.*
2. *There exists $(x_1, x_2) \in E'$ such that $([a], [x_1]), ([x_2], [d])$ are in E' or F'_α for some $\alpha \in A$.*

Condition 2 provides a way of measuring the closeness of two points which originate from distinct spaces prior to gluing and are not in Z_α or $\mathrm{Im}(f_\alpha)$ for some α. In effect, two points in $X \hat{} \{Y_\alpha\}$ are close as measured by $\langle E, E', F_\alpha, F'_\alpha \rangle$, if they are sufficiently close (as measured by E', F'_α) to points which are (or whose images are)

sufficiently close together in X. We have abused notation slightly since F_α, F'_α represents collections of entourages. To illustrate this somewhat complicated definition we present an example.

Example 43 *Let X be the subset of $\mathbb{R}^2$ consisting of the union of the unit circle centered at the origin and the line segment between the points (0,1) and (0,2). Let Y be the closed unit ball in $\mathbb{R}^2$. Both X and Y are given the metric uniformity induced by the metric in $\mathbb{R}^2$. We glue the boundary of Y to the unit circle in X by the mapping which loops around the circle twice, i.e. $f(e^{2\pi i\theta}) = e^{4\pi i\theta}$. Thus each point on the unit circle in X has precisely two pre-images on diametrically opposite sides of Y. The boundary of Y must be "stretched" to twice its size to accomplish this gluing. We let $E(1.1) = \{(x_1, x_2) \in X | d(x_1, x_2) < 1.1\}$ and note that $E(1/4)^4 \subset E(1.1)$. We let δ_1, δ_2 be small enough that $f(F(\delta_1) \cap S^1 \times S^1) \subset E(1.1)$, $f(F(\delta_2) \cap S^1 \times S^1) \subset E(1/4)$ and $F(\delta_2)^4 \subset F(\delta_1)$.*

We consider two $\langle E(1.1), E(1/4), F(\delta_1), F(\delta_2)\rangle$ balls in $X \hat{} Y$. Let $x = (0, 2)$ and consider $B([x], \langle E(1.1), E(1/4), F(\delta_1), F(\delta_2)\rangle)$. In this case x is the unique element in $[x]$ and $([x], [a])$ cannot be in $F(\delta_1)$ or $F(\delta_2)$. Thus $([x], [a])$ can be in this ball by condition 1 only if there exists $x' \in X \cap [a]$ such that $d(x, x') < 1.1$. Thus the set of points in $X \hat{} Y$ which meet condition 1 is the set of points within a distance of 1.1 from the point $(0, 2)$. This consists of the equivalence classes of all the points which lie on the line segment in X, together with a small arc of the unit circle. We claim that condition 2 adds no additional points. If there is an $x' \in X \cap [b]$ which meets the criteria then there would be $x_1, x_2 \in X$ such that no two consecutive elements in the sequence x, x_1, x_2, x' are more than a distance $1/4$ apart which means that $(x, x') \in E(1.1)$ and $[b]$ already meets condition 1. On the other hand, if there was a $y \in Y \cap [b]$ which meets the criteria then there would be $x_1, x_2 \in X$ such that $([x], [x_1])$ is in $E(1/4)$ (it can't be in $F(\delta_2)$), $([x_1], [x_2])$ is in $E(1/4)$ and $([x_2], [y])$ is in $F(\delta_2)$. This implies that x_2 lies on the boundary. But then x_1 would have to be within $1/4$ of both $(0, 2)$ and a point on the boundary which is not possible. We note that $B([x], \langle E(1.1), E(1/4), F(\delta_1), F(\delta_2)\rangle)$ is not an open set in $X \hat{} Y$ under the quotient topology.

Now let $x = (0, 5/4)$. $[x'] \in B([x], \langle E(1.1), E(1/4), F(\delta_1), F(\delta_2)\rangle)$ by condition 1 for those x' which lie on the line segment, as well as a (somewhat larger) arc of the circle which we will label A_1. This time, however, there are additional points of Y which do not lie on the boundary, but which meet condition 2. If we let A_2 be the (smaller) arc of the circle in X whose points are within a distance of $1/2$ from x, then for all $x_1 \in A_2$ we can find an x_2 on the line segment such that the distance between x_1 and x_2 is less than $1/4$ while the distance between x_2 and x is also less than $1/4$. The pre-image of this arc, $f^{-1}(A_2)$ will consist of two arcs on the boundary of Y. Then any y which lies in the δ_2 neighborhood of $f^{-1}(A_2)$ will satisfy $[y] \in B([x], \langle E(1.1), E(1/4), F(\delta_1), F(\delta_2)\rangle)$.

We wish to show that the collection of all $\langle E, E', F_\alpha, F'_\alpha\rangle$ forms a basis of a uniform space on $X \hat{} \{Y_\alpha\}$. To do this we will need to investigate the properties of certain sequences of elements in $X \hat{} \{Y_\alpha\}$. We fix, for the moment, $\langle E, E', F_\alpha, F'_\alpha\rangle$ and consider a glued entourage $\langle H, H', K_\alpha, K'_\alpha\rangle$ such that for each α we have $(H)^4 \subset E'$

and $(K_\alpha)^4 \subset F'_\alpha$. The following lemmas will be useful.

Lemma 44 *Let $[a], [x_1], [x_2], [b], [x_3], [x_4], [c]$ be a sequence of elements of $X \hat{} \{Y_\alpha\}$ such that x_1, x_2, x_3, x_4 are elements of X and we have that:*

1. *$([a], [x_1]), ([x_4], [c])$ are in either F'_α for some α or E'.*
2. *$([x_1], [x_2]), ([x_3], [x_4])$ are in H.*
3. *$([x_2], [b]), ([b], [x_3])$ are in either K'_α for some α or H'*

Then $([a], [c]) \in \langle E, E', F_\alpha, F'_\alpha \rangle$.

Proof. We will show that $([x_2], [x_3])$ is in H. Then condition 2 would imply that $([x_1], [x_4])$ is in $(H)^3 \subset E'$. This, together with condition 1 would imply that $([a], [c])$ meets condition 2 of the definition of $\langle E, E', F_\alpha, F'_\alpha \rangle$.

We now assume $([x_2], [b]), ([b], [x_3])$ are in either K'_α for some α or H' and show that $([x_2], [x_3])$ is in H. First assume that $X \cap [b]$ is empty. Then we would have $b \in Y_\alpha$ for some α and b the unique element of $[b]$. Then neither of $([x_2], [b])$ or $([b], [x_3])$ can be in $(H)'$ Since both of these pairs must satisfy condition 3 they must then be in $(F^*_\alpha)'$ where α is the index such that $b \in Y_\alpha$. Thus we have a $y_2, y_3 \in Y_\alpha \cap [x_2], Y_\alpha \cap [x_3]$ such that $(y_2, b), (b, y_3) \in K'_\alpha$. Thus $(y_2, y_3) \in (K'_\alpha)^2 \subset K_\alpha$. Since $y_2 \in [x_2], y_3 \in [x_3]$ we also have that $f_\alpha(y_2) = x_2$ and $f_\alpha(y_3) = x_3$. Thus $(y_2, y_3) \in K_\alpha \cap Z_\alpha \times Z_\alpha$ and by 41 above $([y_2], [y_3])$ is in H. But $([y_2], [y_3]) = ([x_2], [x_3])$ and we are done in this case.

For the second case let $x \in X \cap [b]$. We show in this case that $([x_2], [b])$ and $([b], [x_3])$ are both necessarily in H' If $([x_2], [b])$ were in K'_α for some α, then let $y_2 \in Y_\alpha \cap [x_2], y \in Y_\alpha \cap [b]$ such that $(y_2, y) \in K'_\alpha$. Since $f_\alpha(y_2) = x_2, f_\alpha(y) = x$ we have $(y_2, y) \in K'_\alpha \cap Z_\alpha \times Z_\alpha$ and we can again use 41 to show that $([x_2], [x]) = ([x_2], [b])$ is in H'. Similarly $([b], [x_3])$ is in H'. This then implies that $(x_2, x), (x, x_3) \in H'$ so that $(x_2, x_3) \in (H')^2 \subset H$ and the result follows. ■

Lemma 45 *Let $[a], [b], [x_3], [x_4], [c]$ be a sequence of elements of $X \hat{} \{Y_\alpha\}$ such that $x_3, x_4 \in X$ and :*

1. *$([a], [b]), ([x_4], [c])$ are in either K_α for some α or H.*
2. *$([b], [x_3])$ is in either K'_α for some α or H'*
3. *$([x_3], [x_4])$ is in H*

Then $([a], [c]) \in \langle E, E', F_\alpha, F'_\alpha \rangle$.

Proof. First assume that $X \cap [b]$ is empty. Then we would have $b \in Y_\alpha$ for some α and b the unique element of $[b]$. We must then have that $([a], [b])$ is in K_α and $([b], [x_3])$ in K'_α where α is the index such that $b \in Y_\alpha$. Thus there is a $y \in Y_\alpha \cap [a]$ and $y_1 \in Y_\alpha \cap [x_3]$ such that $(y, b) \in K_\alpha$ and $(b, y_1) \in K'_\alpha \subset K_\alpha$. Then $(y, y_1) \in (K_\alpha)^2 \subset F'_\alpha$ which in turn implies that $([a], [x_3])$ is in F'_α since $[a] = [y]$ and $[y_1] = [y_3]$. Then conditions 2 and 3 imply that $([a], [c])$ meets condition 2 of the definition of $\langle E, E', F_\alpha, F'_\alpha \rangle$.

Now, assume that $x \in X \cap [b]$. Then $[a], [x], [x], [b], [x_1], [x_2], [c]$ meets the conditions of lemma 2 (note that $H \subset E'$ and $K_\alpha \subset F'_\alpha$) and the result follows. ■

Proposition 46 *The collection* $\Omega = \{\langle E, E', F_\alpha, F'_\alpha\rangle\}$ *is a basis for a uniformity on* $X \hat{} \{Y_\alpha\}$.

Proof. 1. $\Delta_{X \hat{} \{Y_\alpha\}} \subset \langle E, E', F_\alpha, F'_\alpha\rangle$

Notice that if $[a]$ contains an $x \in X$ then $([a],[a]) \in \langle E, E', F_\alpha, F'_\alpha\rangle$ by condition 1 of the definition since $(x,x) \in E$. If $[a]$ does not contain an element of X then by necessity, $a \in Y_\alpha$ for some α and a is the unique element in $[a]$. Since $(a,a) \in F_\alpha$ in this case, we have $([a],[a]) \in \langle E, E', F_\alpha, F'_\alpha\rangle$ by condition 1 of the definition.

2. $\langle E, E', F_\alpha, F'_\alpha\rangle$ is symmetric.

Clearly if $([a],[b])$ is in any entourage K then the symmetry of K implies $([b],[a])$ is also in K. Thus $([b],[a])$ meets conditions 1 or 2 of the definition of $\langle E, E', F_\alpha, F'_\alpha\rangle$ if $([a],[b])$ does.

3. Let $\langle E, E', F_\alpha, F'_\alpha\rangle, \langle H, H', K_\alpha, K'_\alpha\rangle \in \Omega$. Then there exists

$$\langle L, L', M_\alpha, M'_\alpha\rangle \subset \langle E, E', F_\alpha, F'_\alpha\rangle \cap \langle H, H', K_\alpha, K'_\alpha\rangle$$

We first show that it is possible to choose $L \subset E \cap H, L' \subset E' \cap H'$ and for each α, $M_\alpha \subset F_\alpha \cap K_\alpha, M'_\alpha \subset F'_\alpha \cap K'_\alpha$. We first note that within a given uniform space, the properties of being acceptable with an entourage of X, of having power contained in another entourage, and of being contained in an intersection are all preserved under intersections with other entourages. We first use the uniformity of X to find an L such that $L \subset E \cap H$ and then by intersecting entourages we can find an L' such that $(L')^4 \subset L$ and $L' \subset E' \cap H'$. Then, again by intersecting entourages, we can find for each α, an M_α entourage in Y_α so that (L, M_α) is acceptable and $M_\alpha \subset F_\alpha \cap K_\alpha$. Finally, we can choose for each α an M'_α such that (L', M'_α) is acceptable, $(M'_\alpha)^4 \subset M_\alpha$ and $M'_\alpha \subset F'_\alpha \cap K'_\alpha$.

Now, suppose $([a],[b])$ is in L. Then, since $L \subset E \cap H$ we have that $([a],[b])$ is both in E and in H and hence $([a],[b]) \in \langle E, E', F_\alpha, F'_\alpha\rangle \cap \langle H, H', K_\alpha, K'_\alpha\rangle$. A similar argument works if $([a],[b])$ is in M_α for some α. Thus, if $([a],[b])$ meets condition 1 in the definition of $\langle L, L', M_\alpha, M'_\alpha\rangle$ then it meets condition 1 in the definition of both $\langle E, E', F_\alpha, F'_\alpha\rangle$ and $\langle H, H', K_\alpha, K'_\alpha\rangle$. Now, suppose there is a $([c],[d])$ in L' such that $([a],[c]),([d],[b])$ are in either L' or M'_α for some α. Then, as above, $([c],[d])$ is in $E' \cap H'$ while $([a],[c]),([c],[b])$ are each in either $E' \cap H'$ or $F'_\alpha \cap K'_\alpha$ for some α and again $([a],[b]) \in \langle E, E', F_\alpha, F'_\alpha\rangle \cap \langle H, H', K_\alpha, K'_\alpha\rangle$

4. For each $\langle E, E', F_\alpha, F'_\alpha\rangle \in \Omega$ there is an $\langle H, H', K_\alpha, K'_\alpha\rangle$ such that

$$(\langle H, H', K_\alpha, K'_\alpha\rangle)^2 \subset \langle E, E', F_\alpha, F'_\alpha\rangle$$

We choose H, K_α as in the paragraph preceding 44, so that $H^4 \subset E'$ and $K_\alpha^4 \subset F'_\alpha$ for each α. Suppose that $([a],[c]) \in (\langle H, H', K_\alpha, K'_\alpha\rangle)^2$. Then there exists a $[b] \in X \hat{} \{Y_\alpha\}$ such that $([a],[b]),([b],[c]) \in \langle H, H', K_\alpha, K'_\alpha\rangle$. Each of these pairs must satisfy one of the two conditions in the definition of $\langle H, H', K_\alpha, K'_\alpha\rangle$. Thus there are 4 cases to consider.

Case 1: Both $([a],[b])$ and $([b],[c])$ satisfy condition 1 of definition 42.

Suppose that $X \cap [b]$ is empty. Then $b \in Y_\alpha$ for some α and b is the unique element of $[b]$. Hence there must exist a $y_1 \in Y_\alpha \cap [a]$ and a $y_2 \in Y_\alpha \cap [c]$ such that $(y_1, b), (b, y_2) \in K_\alpha$. Then $(y_1, y_2) \in (K_\alpha)^2 \subset F'_\alpha \subset F_\alpha$ and we have that

$([a],[c]) \in \langle E, E', F_\alpha, F'_\alpha \rangle$. On the other hand, let $x \in X \cap [b]$. Then clearly $(x,x) \in E'$ so that $([b],[b])$ is in E' and hence $([a],[c])$ satisfies condition 2 of the definition of $\langle E, E', F_\alpha, F'_\alpha \rangle$.

Case 2: $([a],[b])$ satisfies condition 1 of the definition of $\langle H, H', K_\alpha, K'_\alpha \rangle$ while $([b],[c])$ satisfies condition 2.

We have in this case that $([a],[b])$ is in either H or K_α for some α and a pair $([x_3],[x_4])$ in H' such that $([b],[x_3]),([x_4],[c])$ are in either H' or K'_α. Then the sequence $[a],[b],[x_3],[x_4],[c]$ satisfies the conditions of 45 above and we have that $([a],[c]) \in \langle E, E', F_\alpha, F'_\alpha \rangle$.

Case 3: $([a],[b])$ satisfies condition 2 of the definition of $\langle H, H', K_\alpha, K'_\alpha \rangle$ while $([b],[c])$ satisfies condition 1.

We have by the symmetry of the conditions that the pairs $([c],[b]),([b],[a])$ fall into the category of case 2. Hence $([c],[a]) \in \langle E, E', F_\alpha, F'_\alpha \rangle$ and hence by the symmetry of $\langle E, E', F_\alpha, F'_\alpha \rangle$, $([a],[c]) \in \langle E, E', F_\alpha, F'_\alpha \rangle$.

Case 4: Both $([a],[b])$ and $([b],[c])$ satisfy condition 2 of the definition.

We have the existence of pairs $([x_1],[x_2])$ and $([x_3],[x_4])$ in H' such that

$$([a],[x_1]),([x_2],[b]),([b],[x_3]),([x_4],[c])$$

are all in either H' or K'_α for some α. Then the sequence $[a],[x_1],[x_2],[b],[x_3],[x_4],[c]$ satisfies the conditions of 44 and hence $([a],[c]) \in \langle E, E', F_\alpha, F'_\alpha \rangle$. ■

We will call the uniform structure on the space $X \hat{} \{Y_\alpha\}$ as the glued uniformity. We have not verified condition 5 (the Hausdorff condition) for the space $X \hat{} \{Y_\alpha\}$. If the component spaces are Hausdorff, it is unknow to the author whether $X \hat{} \{Y_\alpha\}$ is Hausdorff. However, in the special case that the subspaces Z_α are closed, we can show that $X \hat{} \{Y_\alpha\}$ meets condition 5.

Proposition 47 *Let X and $\{Y_\alpha\}$ be as in the previous proposition, and suppose further that Z_α is a closed subspace of Y_α for each α. Then $X \hat{} \{Y_\alpha\}$ is Hausdorff.*

Proof. We will show that for fixed $([a],[b]) \in \triangle^C \subset X \hat{} \{Y_\alpha\} \times X \hat{} \{Y_\alpha\}$, there exists an entourage $\langle E, E', F_\alpha, F'_\alpha \rangle$ such that $([a],[b]) \notin \langle E, E', F_\alpha, F'_\alpha \rangle$. We will consider the following cases.

Case 1: At least one of $X \cap [a]$ and $X \cap [b]$ is empty.

Since each $\langle E, E', F_\alpha, F'_\alpha \rangle$ is symmetric, we may assume without loss of generality, that $X \cap [a]$ is empty. Then there exists an index β such that $a \in Y_\beta$ is the unique element of $[a]$. We will choose an entourage F_β in Y_β in such a way that if $\langle E, E', F_\alpha, F'_\alpha \rangle$ is formed using F_β then $B([a], \langle E, E', F_\alpha, F'_\alpha \rangle)$ contains only equivalence classes of $i_{Y_\beta}(Y_\beta)$. Then we may use the Hausdorff condition in Y_β to imply the condition in $X \hat{} \{Y_\alpha\}$. Since Z_β is closed, we can then find an entourage F_β such that $B(a, F_\beta) \subset Z_\beta^C$ (the complement of Z_β in Y_β). Letting E be an entourage in X we may assume (by intersecting F_β with an acceptable entourage if necessary) that (E, F_β) is acceptable. We then form a glued entourage $\langle E, E', F_\alpha, F'_\alpha \rangle$. By the choice of F_β the pair $([a],[b])$ cannot satisfy condition 2 in the definition of $\langle E, E', F_\alpha, F'_\alpha \rangle$, and can satisfy condition 1 only if $b \in Y_\beta$ such that $(a,b) \in F_\beta$. If it should happen that $b \in Y_\beta$ such that $(a,b) \in F_\beta$ then we may then use the Hausdorff condition on

Y_β to choose a smaller entourage H_β so that $B(a, H_\beta) \cap B(b, H_\beta)$ is empty. Since (E, H_β) is acceptable by 41 we may form an entoruage of the form $\langle E, E', H_\alpha, H'_\alpha\rangle$. Then $([a], [b])$ cannot meet either condition 1 or 2 in the definition of $\langle E, E', H_\alpha, H'_\alpha\rangle$ and hence $([a], [b]) \notin \langle E, E', H_\alpha, H'_\alpha\rangle$

Case 2: There exists $x \in X \cap [a]$ and $x' \in X \cap [b]$.

We use the Hausdorff condition on X to choose an entourage E such that $B(x, E) \cap B(x', E)$ is empty. Consider any $\langle E, E', F_\alpha, F'_\alpha\rangle$. By the choice of E we cannot have $([a], [b])$ $(= ([x], [x']))$ in E. Suppose there exists an index β and $y \in Y_\beta \cap [a], y' \in Y_\beta \cap [b]$ such that $(y, y') \in F_\beta$. Then by 41 we would have $([a], [b])$ in E which is a contradiction. Finally, assume there exists $([x_1], [x_2])$ in E' such that $([x], [x_1]), ([x_2], [x'])$ are in E' or F'_α for some α. Then, again applying [acceptable properties], we have that $([x], [x_1]), ([x_2], [x'])$ are by necessity in E' which implies that $(x, x') \in (E')^3 \subset E$ a contradiction. ■

Our ultimate goal is to show that the glued uniform space of any finite dimensional CW complex is uniform coverable. The uniform structure on $X \hat{} \{Y_\alpha\}$ certainly allows us to form $\varprojlim (X \hat{} \{Y_\alpha\})_{\langle E, E', F_\alpha, F'_\alpha\rangle}$ and we will show that if the original spaces are uniform coverable so is $X \hat{} \{Y_\alpha\}$. One difficulty, however, is that the uniform structure provides a topology on the glued space which may be distinctly smaller than the quotient topology (see 52 below). Since the topology of a CW complex is the quotient topology, we must show that the uniform topology on it is equivalent to the quotient topology. In fact, we will show that if each of the subspaces Z_α is compact then equivalence of the topologies follows (see 51). Since the boundary of each n-cell is a compact subspace of the cell, the result would apply to CW complexes. To accomplish this we will use the following facts. 48 is proved as corollary 8.15 in [8] whereas 49 follows from corollary 2 of proposition 2 of section II.1.2 of [3].

Remark 48 *Let U be an open subset of a uniform space Y and let K be a compact subset of Y which is contained in U. Then there exists an entourage F such that the F neighborhood of K, $F(K) \subset U$, or in other words, $k \in K$, $a \in Y$ and $(k, a) \in F$ implies that $a \in U$.*

Remark 49 *Let U be an open set of a uniform space Y and $x \in U$. Then there exists an entourage F such that $cl(B(x, F)) \subset U$.*

Proposition 50 *The functions $i_X : X \to X \hat{} \{Y_\alpha\}$ is a uniform homeomorphism onto its image, and $i_{Y_\alpha} : Y_\alpha \to X \hat{} \{Y_\alpha\}$ is uniformly continuous.*

Proof. Let $\langle E, E', F_\alpha, F'_\alpha\rangle$ be a basis entourage of $X \hat{} \{Y_\alpha\}$. If $(x_1, x_2) \in E$ then $i_X(x_1, x_2) = ([x_1], [x_2]) \in \langle E, E', F_\alpha, F'_\alpha\rangle$. On the other hand, if $(y_1, y_2) \in F_\beta$ for some β, then $i_{Y_\beta}(y_1, y_2) = ([y_1], [y_2]) \in \langle E, E', F_\alpha, F'_\alpha\rangle$. Hence $E \subset i_X^{-1}(\langle E, E', F_\alpha, F'_\alpha\rangle)$ and $F_\beta \subset i_{Y_\beta}^{-1}(\langle E, E', F_\alpha, F'_\alpha\rangle)$ which implies that i_X and i_Y are uniformly continuous. i_X is injective since the elements in $X \cap [a]$ are unique (if they exist). Further, let $x_1, x_2 \in X$ such that $([x_1], [x_2]) \in \langle E, E', F_\alpha, F'_\alpha\rangle$. Then, if $([x_1], [x_2])$ meets condition 1 in the definition of $\langle E, E', F_\alpha, F'_\alpha\rangle$ then $(x_1, x_2) \in E$. If $([x_1], [x_2])$ meets condition 2 in the definition of $\langle E, E', F_\alpha, F'_\alpha\rangle$ then there exists $x_3, x_4 \in X$ such that $([x_3], [x_4])$

is in E' whereas $([x_1],[x_3])$ and $([x_4],[x_2])$ are in either F'_β for some β or E'. However, since (E', F'_α) is acceptable for all α by 41 we know that, $([x_1],[x_3]),([x_4],[x_2])$ must be in E'. Then $(x_1,x_3),(x_3,x_4),(x_4,x_2) \in E'$ implies that $(x_1,x_2) \in (E')^3 \subset E$. This shows that $([x_1],[x_2]) \in \langle E, E', F_\alpha, F'_\alpha\rangle$ implies that $(x_1,x_2) \in E$ and hence $i_X(E) = \langle E, E', F_\alpha, F'_\alpha\rangle \cap i_X(X) \times i_X(X)$. This implies that the image of E under i_X is an entourage in the subspace uniformity and thus i_X is uniformly homeomorphic to its image. ■

Proposition 51 *The uniform topology on $X \hat{} \{Y_\alpha\}$ is a subset of the quotient topology. If Z_α is a compact subspace of Y_α for each α then the topologies are equivalent.*

Proof. The quotient topology is induced by the maps i_X and $\{i_{Y_\alpha}\}_{\alpha\in A}$. By proposition 2, these maps are uniformly continuous (hence continuous) maps to the space $X \hat{} \{Y_\alpha\}$ endowed with the uniform topology. Therefore, if a set U is open in $X \hat{} \{Y_\alpha\}$ under the uniform topology, its inverse image must be open in each of the spaces X and Y_α. Hence U is open in the quotient topology by definition and the first statement is a consequence of 50. Now, suppose that Z_α is a compact subspace of Y_α and let U be a subset of $X \hat{} \{Y_\alpha\}$ which is open in the quotient topology. We will show that if $[a] \in U$ then there exists an $\langle E, E', F_\alpha, F'_\alpha\rangle$ such that $B([a], \langle E, E', F_\alpha, F'_\alpha\rangle) \subset U$. We will consider the following 2 cases.

Case 1: There exists $x \in X \cap [a]$.

Since U is open in the quotient topology, $i_X^{-1}(U)$ is open in X and hence there exists by 49, an entourage E such that $cl(B(x,E)) \subset i_X^{-1}(U)$. By the continuity of each f_α we have that $f_\alpha^{-1}(cl(B(x,E))$ is a closed subset of Z_α and hence is compact. Further, if $y \in f_\alpha^{-1}(cl(B(x,E))$ then $f_\alpha(y) = x'$ for some x' in $cl(B(x,E))$ which implies that $[y] = [x'] \in U$. Hence $f_\alpha^{-1}(cl(B(x,E)) \subset i_{y_\alpha}^{-1}(U)$ which is open in Y. We can now apply 48 to find an entourage H_α in Y_α such that the H_α neighborhood of $f_\alpha^{-1}(cl(B(x,E))$ is a subset of $i_{y_\alpha}^{-1}(U)$. For each α, we then choose an entourage F_α in Y_α such that (E, F_α) is acceptable and, by intersecting F_α with H_α we may assume that the F_α neighborhood of $f_\alpha^{-1}(cl(B(x,E))$ is contained in $i_{y_\alpha}^{-1}(U)$.

We now show that $B([a], \langle E, E', F_\alpha, F'_\alpha\rangle) \subset U$. Suppose $[b]$ is an element in $B([a], \langle E, E', F_\alpha, F'_\alpha\rangle)$. If $([a],[b])$ is in E then $(x,x') \in E$ for $x' \in X \cap [b]$. Then, by the choice of E, we have that x' is in $i_X^{-1}(U)$ which implies that $[b]$ is in U. If $([a],[b])$ is in F_α for some α then $(y,y') \in F_\alpha$ for some $y \in Y_\alpha \cap [a]$ and $y' \in Y_\alpha \cap [b]$. Further, since $f_\alpha(y) = x$, we must have that $y \in f_\alpha^{-1}(cl(B(x,E))$. Since the F_α neighborhood of $f_\alpha^{-1}(cl(B(x,E)) \subset i_{Y_\alpha}^{-1}(U)$ we have that $y' \in i_{Y_\alpha}^{-1}(U)$.Hence $[b] \in U$.

On the other hand, assume that there exists a pair $([x_1],[x_2])$ in E' such that $([a],[x_1]),([x_2],[b])$ are in either F'_α for some α, or E'. If $y \in Y_\alpha \cap [a]$ and $y' \in Y_\alpha \cap [x_1]$ are such that $(y,y') \in F_\alpha$ then since $f_\alpha(y) = x$ and $f_\alpha(y') = x_1$ we have that $(y,y') \in F_\alpha \cap Z_\alpha \times Z_\alpha$. By 41 from the beginning of this section, $([a],[x_1])$ is then in E' by necessity. Thus $(x,x_1) \in E'$ and $(x_1,x_2) \in E'$ which implies that $(x,x_2) \in (E')^2$. We first suppose that $([x_2],[b])$ is also in E'. Then there exists x_3 in $X \cap [b]$ and $(x_2,x_3) \in E'$. Hence $(x,x_3) \in (E')^3 \subset E$. Then, using the fact that $cl(B(x,E)) \subset i_X^{-1}(U)$ by the choice of E, we have $x_3 \in i_X^{-1}(U)$ and hence $[b] \in U$. On the other hand, if $([x_2],[b])$ is in F'_β then $(y,y') \in F_\beta$ for some $y \in Y_\beta \cap [x_2]$ and $y' \in Y_\beta \cap [b]$. In particular, we know that $f_\beta(y) = x_2$. Since $(x,x_2) \in (E')^2 \subset E$ we

must have $y \in f_\beta^{-1}(cl(B(x,E))$. Then y' is in the F_β neighborhood of $f_\beta^{-1}(cl(B(x,E))$ and must therefore be in $i_{Y_\beta}^{-1}(U)$. Since $[y'] = [b]$ we have that $[b] \in U$.

Case 2: $X \cap [a]$ is empty.

Notice that $y \in Z_\alpha \cap [a]$ for any α then $f_\alpha(y) \in [a]$ which contradicts the assumption in this case. Hence $a \in Z_\beta^C \subset Y_\beta$ is the unique element of $[a]$. Since Z_β is a compact subset of a Hausdorff space it is closed and hence we can find an entourage G in Y_β such that $B(a,G) \subset Z_\beta^C$. We can also find an entourage H such that $B(a,H) \subset i_{Y_\beta}^{-1}(U)$. Let $F_\beta \subset G \cap H$. Given any entourage E in X we may assume (by intersecting with an acceptable entourage) that F_β is acceptable with E. For each index $\alpha \neq \beta$ we choose an acceptable entourage F_α. Suppose that $[b] \in B([a], \langle E, E', F_\alpha, F'_\alpha \rangle)$. We will show that, by necessity, $([a],[b])$ is in F_β. We cannot have $([a],[b])$ in any entourage of X or F_α for $\alpha \neq \beta$ since $[a]$ contains no elements of X or Y_α. Suppose there exists a pair $([x_1],[x_2])$ in E' such that $([a],[x_1])$ is in F'_β and $([x_2],[b])$ is in either E' or F'_α for some α. Then $([a],[x_1])$ in F'_β implies that $(a,y) \in F'_\beta \subset F_\beta$ for some $y \in Y_\beta \cap [x_1]$. Hence, by the choice of F_β we must have $y \in Z_\beta^C$ and y the unique element of $[x_1]$. This is a contradiction since $x_1 \in X \cap [x_1]$. Hence $([a],[b])$ can be an element of $\langle E, E', F_\alpha, F'_\alpha \rangle$ only if $([a],[b])$ is in F_β. Then, again by the choice of F_β we have that $b \in i_{Y_\beta}^{-1}(U)$ and hence $[b] \in U$. This proves the proposition. ■

Example 52 *The above proposition fails in general, i.e. there exists* $X \hat{} \{Y_\alpha\}$ *and* $U \subset X \hat{} \{Y_\alpha\}$ *such that* U *is open in the quotient topology but not in the uniform topology.*

Proof. Let Y be the square in $\mathbb{R}^2$ with corners at $(1,1),(1,-1),(-1,1)$ and $(-1,-1)$ (include the interior of the square). Let Z be the set $\{(x,y)|y < x^2\}$. Let X be the interval $[-1,1]$ and define the function $f : Z \to X$ by the projection $f((x,y)) = x$. To see that this function is uniformly continuous, notice that if $(x_1,y_1),(x_2,y_2)$ are less than a distance ε apart, then $|x_2 - x_1| < \varepsilon$. Suppose $F(\varepsilon)$ is the entourage in Y defined by the distance ε and $E(\varepsilon)$ is the entourage in X defined by the distance ε. Then $((x_1,y_1),(x_2,y_2)) \in F(\varepsilon)$ implies $|x_2 - x_1| < \varepsilon$ which implies that $(f(x_1,y_1), f(x_2,y_2)) = (x_1,x_2) \in E(\varepsilon)$. Thus $F(\varepsilon) \subset f^{-1}(E(\varepsilon))$ and f is uniformly continuous.

Consider the subset U of $X \hat{} Y$ equal to $\{[x]|x \in X\}$. U is the image of X under $i_X : X \to X \hat{} Y$ and hence $i_X^{-1}(U)$ is open in X. Similarly $i_Y^{-1}(U) = Z$ which is an open subset of Y. Thus U is an open subset of $X \hat{} Y$ in the quotient topology. Consider the point $i_X(0) = [0]$. Let $\langle E(\varepsilon), E(\varepsilon'), F(\delta), F(\delta') \rangle$ be an arbitrary entourage of $X \hat{} Y$. Let $y < \delta$. Then the point $(0,y) \in Y$ is not in Z and hence $(0,y)$ is the unique element in the equivalence class $[(0,y)]$. In particular $[(0,y)] \notin U$. On the other hand, since $f(0,0) = 0$ we have that $(0,0) \in [0] \in U$. However, the distance between $(0,0)$ and $(0,y)$ is $y < \delta$. Hence $([(0,0)],[(0,y)]) = ([0],[(0,y)])$ is in $F(\delta)$. But then $[(0,y)] \in B([0], \langle E(\varepsilon), E(\varepsilon'), F(\delta), F(\delta') \rangle)$ but $[(0,y)] \notin U$. Since $\langle E(\varepsilon), E(\varepsilon'), F(\delta), F(\delta') \rangle$ was arbitrary, U cannot be open in the uniform topology on $X \hat{} Y$. This proves the result. ■

We will now consider the question of uniform coverability of $X \hat{} \{Y_\alpha\}$. We will show (see proposition 5 below) that if the original uniform spaces are uniform coverable then $X \hat{} \{Y_\alpha\}$ is uniform coverable as well. A natural candidate for a covering

basis on $X \hat{} \{Y_\alpha\}$ would be the set of entourages $\langle E, E', F_\alpha, F'_\alpha\rangle$ where E and F_α are elements of the covering basis of X and Y_α respectively. Before we proceed with the proof that such entourages of $X \hat{} \{Y_\alpha\}$ are indeed covering entourages, we must establish first of all that such entourages exist, and that they form a basis. Toward that end we consider the following more general lemma.

Proposition 53 *Let Σ be a basis for the uniformity on X and Φ_α be a basis for the uniformity on Y_α. Then the set of all entourages $\langle H, H', K_\alpha, K'_\alpha\rangle$ such that $H, H' \in \Sigma$ and $K_\alpha, K'_\alpha \in \Phi_\alpha$ forms a basis for the uniformity on $X \hat{} \{Y_\alpha\}$.*

Proof. Let $\langle E, E', F_\alpha, F'_\alpha\rangle$ be a basis entourage in $X \hat{} \{Y_\alpha\}$. Choose $H \in \Sigma$ such that $H \subset E$. For each α we choose an (arbitrary) entourage D_α in Y_α such that (H, D_α) is acceptable and then we find a basis entourage $K_\alpha \in \Phi_\alpha$ such that $K_\alpha \subset D_\alpha$. By 41 (H, K_α) is acceptable. We then choose any H' and K'_α which meet the conditions in the definition of $\langle H, H', K_\alpha, K'_\alpha\rangle$. By choosing basis elements H'', F''_α contained in $H' \cap E'$ and $K'_\alpha \cap F'_\alpha$ we may assume that the following hold:

1) $H \subset E, H'' \subset E', K_\alpha \subset F_\alpha$, and $K''_\alpha \subset F'_\alpha$

Notice that if $([a], [b])$ meets either condition 1 or 2 in definition 42 for the entourage $\langle H, H'', K_\alpha, K''_\alpha\rangle$ then by 1) it must also satisfy the same condition in the definition of $\langle E, E', F_\alpha, F'_\alpha\rangle$. Thus $\langle H, H'', K_\alpha, K''_\alpha\rangle \subset \langle E, E', F_\alpha, F'_\alpha\rangle$ and the result follows. ■

Lemma 54 *Let $\langle E, E', F_\alpha, F'_\alpha\rangle$ be an entourage of $X \hat{} \{Y_\alpha\}$. If*

$\gamma := \{a_0, a_1, ...a_n\}$, $\xi := \{b_0, b_1, ...b_m\}$ are E-chains of X, then the chains $\bar{\gamma} := \{[a_0], [a_1]...[a_n]\}$ and $\bar{\xi} := \{[b_0], [b_1]...[b_m]\}$ are $\langle E, E', F_\alpha, F'_\alpha\rangle$ chains. Further, if there exists an E homotopy from γ to ξ then there exists a $\langle E, E', F_\alpha, F'_\alpha\rangle$ homotopy from $\bar{\gamma}$ to $\bar{\xi}$. A similar result holds for F_α-chains.

Proof. Since i_X, i_{Y_α} are uniformly continuous by 50 and $i_X(E), i_{Y_\alpha}(F_\alpha) \subset \langle E, E', F_\alpha, F'_\alpha\rangle$ we have from Definition 19 and Theorem 27 in [2] that $\bar{\gamma}, \bar{\xi}$ are $\langle E, E', F_\alpha, F'_\alpha\rangle$-chains and that there exists uniformly continuous functions $(i_X)_{E\langle E,E',F_\alpha,F'_\alpha\rangle}, (i_{Y_\alpha})_{F_\alpha\langle E,E',F_\alpha,F'_\alpha\rangle}$ defined by $(i_X)_{E\langle E,E',F_\alpha,F'_\alpha\rangle}([\gamma]_E) = [i_X(\gamma)]_{\langle E,E',F_\alpha,F'_\alpha\rangle} = [\bar{\gamma}]_{\langle E,E',F_\alpha,F'_\alpha\rangle}$. The well definedness of these maps gives the second statement. ■

Let $x \in X$ and $(X \hat{} \{Y_\alpha\})^{\sim[x]}$ be the fundamental inverse system of $X \hat{} \{Y_\alpha\}$ with $[x]$ as basepoint. By the previous lemma it is natural to consider the map which takes each element $([u_E]_E)$ in $\tilde{X}^x$ to the element of $(X \hat{} \{Y_\alpha\})^{\sim[x]}$ which has as its $\langle E, E', F_\alpha, F'_\alpha\rangle$th representative, the barred version of the Eth representative of (γ), i.e. $[\bar{u}_E]_{\langle E,E',F_\alpha,F'_\alpha\rangle}$. We must show that such an element is well defined.

Lemma 55 *There are mappings $\tilde{i}^x_X : \tilde{X}^x \to (X \hat{} \{Y_\alpha\})^{\sim[x]}$ and $\tilde{i}^y_{Y_\alpha} : \tilde{Y}^y_\alpha \to (X \hat{} \{Y_\alpha\})^{\sim[y]}$ defined by:*

$$\tilde{i}^x_X([u_E]_E) = ([\bar{u}_E]_{\langle E,E',F_\alpha,F'_\alpha\rangle}), \tilde{i}^y_{Y_\alpha}([u_{F_\alpha}]_{F_\alpha}) = ([\bar{u}_{F_\alpha}]_{\langle E,E',F_\alpha,F'_\alpha\rangle})$$

Proof. Consider $[\bar{u}_E]_{\langle E,E',F_\alpha,F'_\alpha\rangle}$. Suppose $\langle H, H', K_\alpha, K'_\alpha\rangle \subset \langle E, E', F_\alpha, F'_\alpha\rangle$. We must show that there is an $\langle E, E', F_\alpha, F'_\alpha\rangle$-homotopy between $\bar{u}_H$ and $\bar{u}_E$. We choose an entourage $M \subset H \cap E$. Since $([u_E]_E) \in \tilde{X}^x$ we have that u_M is H-homotopic to u_H

and E-homotopic to u_E. By what we have just shown, $\bar{u}_M$ is then $\langle H, H', K_\alpha, K'_\alpha\rangle$-homotopic to $\bar{u}_H$. Then, since $\langle H, H', K_\alpha, K'_\alpha\rangle \subset \langle E, E', F_\alpha, F'_\alpha\rangle$ we know that every $\langle H, H', K_\alpha, K'_\alpha\rangle$-homotopy is, in particular, an $\langle E, E', F_\alpha, F'_\alpha\rangle$-homotopy. Thus, in fact $\bar{u}_M$ is $\langle E, E', F_\alpha, F'_\alpha\rangle$-homotopic to $\bar{u}_H$. The E-homotopy between u_M and u_E implies an $\langle E, E', F_\alpha, F'_\alpha\rangle$-homotopy between $\bar{u}_M$ and $\bar{u}_E$. Thus $\bar{u}_H$ is $\langle E, E', F_\alpha, F'_\alpha\rangle$-homotopic to $\bar{u}_M$ which is in turn $\langle E, E', F_\alpha, F'_\alpha\rangle$-homotopic to $\bar{u}_E$. Thus there is an $\langle E, E', F_\alpha, F'_\alpha\rangle$-homotopy between $\bar{u}_H$ and $\bar{u}_E$ and $([\bar{u}_E]_{\langle E,E',F_\alpha,F'_\alpha\rangle})$ is well defined. The same proof works for each $\tilde{\imath}_{Y_\alpha}$. ■

Theorem 56 *If X and Y_α are uniform coverable uniform spaces for each α then $X \hat{} \{Y_\alpha\}$ is uniform coverable.*

Proof. Let $*$ be a basepoint of X and choose $[*]$ as the basepoint of $X \hat{} \{Y_\alpha\}$. Let $\langle H, H', K_\alpha, K'_\alpha\rangle$ be a glued entourage such that H, H' are covering entourages of X and K_α, K'_α are covering entourages of Y_α. We will show that such an entourage is a covering entourage in $X \hat{} \{Y_\alpha\}$. By 53 we know that such entourages form a basis for the uniformity on $X\hat{}\{Y_\alpha\}$ and thus $X\hat{}\{Y_\alpha\}$ will be uniform coverable. We let $\gamma := \{[*] = [a_0], [a_1], ...[a_K]\}$ be an $\langle H, H', K_\alpha, K'_\alpha\rangle$ chain and we must then show that there exists

$$([\gamma_{\langle E,E',F_\alpha,F'_\alpha\rangle}]_{\langle E,E',F_\alpha,F'_\alpha\rangle}) \in (X\hat{}\{Y_\alpha\})^{\sim[*]}$$

such that the $\langle H, H', K_\alpha, K'_\alpha\rangle$th element is $[\gamma]_{\langle H,H',K_\alpha,K'_\alpha\rangle}$.

Our strategy is as follows. First, we show that it is possible to assume that each of the pairs $([a_i], [a_{i+1}])$ satisfies condition 1 in the definition of $\langle H, H', K_\alpha, K'_\alpha\rangle$. This means that we may assume that $(a_i, a_{i+1}) \in H$ or K_α. Then we use the fact that H, K_α are covering entourages to find elements of $\tilde{X}^{a_i}$ or $\tilde{Y}_\alpha^{a_i}$ whose Hth or K_αth element is $[\{a_i, a_{i+1}\}]$. We then use the mappings $\tilde{\imath}_X, \tilde{\imath}_Y$ to find an element in $(X\hat{}\{Y_\alpha\})^{\sim[a_i]}$ whose $\langle H, H', K_\alpha, K'_\alpha\rangle$th element is $([a_i], [a_{i+1}])$. Then by 29 we can combine these to form an element of $(X\hat{}\{Y_\alpha\})^{\sim[*]}$ whose $\langle H, H', K_\alpha, K'_\alpha\rangle$th element is $[\gamma]_{\langle H,H',K_\alpha,K'_\alpha\rangle}$

To begin, we show that it is possible to assume that each of the pairs $([a_i], [a_{i+1}])$ satisfies condition 1 in the definition of $\langle H, H', K_\alpha, K'_\alpha\rangle$. Suppose that in γ, the pair $([a_i], [a_{i+1}])$ satisfies condition 2 in the definition of $\langle H, H', K_\alpha, K'_\alpha\rangle$. Then there exists a pair $([x_1], [x_2])$ in H' (and hence in H) such that $([a_i], [x_1]), ([x_2], [a_{i+1}])$ are in H' or K'_α (and hence are in H or K_α) for some α. We will show that $([a_i], [a_{i+1}])$ is $\langle H, H', K_\alpha, K'_\alpha\rangle$ homotopic to $([a_i], [x_1], [x_2], [a_{i+1}])$. First, we claim that $[x_1]$ may be inserted into $[a_i], [a_{i+1}]$. This is because first of all the pair $([a_i], [x_1])$ is in H' or K'_α (and hence is in H or K_α) and satisfies condition 1 of the definition of $\langle H, H', K_\alpha, K'_\alpha\rangle$. Further, $([x_1], [a_{i+1}])$ satisfies condition 2 in the definition of $\langle H, H', K_\alpha, K'_\alpha\rangle$ since $([x_2], [x_2])$ is in H' while $([x_1], [x_2])$ and $([x_2], [a_{i+1}])$ are in either H' or K'_α. Also, note that $[x_2]$ can be inserted into $[x_1], [a_{i+1}]$ since both $([x_1], [x_2])$ and $([x_2], [a_{i+1}])$ are in H' or K'_α (and hence is in H or K_α) and satisfy condition 1 of the definition of $\langle H, H', K_\alpha, K'_\alpha\rangle$. Applying these homotopies to each of the pairs $([a_i], [a_{i+1}])$ in γ, it follows that γ is $\langle H, H', K_\alpha, K'_\alpha\rangle$ related to an $\langle H, H', K_\alpha, K'_\alpha\rangle$ chain in which each of the pairs $([a_i], [a_{i+1}])$ are in H or K_α for some α.

Let β be fixed and suppose that $([a_i], [a_{i+1}])$ is in K_β. Then there exists $y_i \in Y_\beta \cap [a_i], y_{i+1} \in Y_\beta \cap [a_{i+1}]$ such that $(y_i, y_{i+1}) \in K_\beta$. Since K_β is a covering entourage with respect to the basepoint of Y_β, we have by 30 that it is a covering

entourage with respect to y_i. Hence it is possible to find $([u_{i,i+1}]_{F_\beta}) \in \tilde{Y}_\beta^{y_i}$ such that the K_βth representative $[u_{i,i+1}]_{k_\beta}$ is the equivalence class $[\{y_i, y_{i+1}\}]_{K_\beta}$. Then $\tilde{i}_{Y_\beta}^{y_i}([u_{i,i+1}]_{F_\beta}) = ([\bar{u}_{i,i+1}]_{E\hat{}\{F_\beta\}})$ is a well defined element of $(X\hat{}\{Y_\alpha\})^{\sim[a_i]}$ such that the $\langle H, H', K_\alpha, K'_\alpha\rangle$th element is the equivalence class $[([a_i],[a_{i+1}])]_{\langle H,H',K_\alpha,K'_\alpha\rangle}$. A similar argument works if $([a_i],[a_{i+1}])$ is in X. We note that in particular, there is an element $([\bar{u}_{0,1}]_{\langle E,E',F_\alpha,F'_\alpha\rangle} \in (X\hat{}\{Y_\alpha\})^{\sim[*]}$ whose $\langle H, H', K_\alpha, K'_\alpha\rangle$th element is $[\{[*],[a_1]\}]_{\langle H,H',K_\alpha,K'_\alpha\rangle}$. We then use 29 to obtain an element $([u_{\langle E,E',F_\alpha,F'_\alpha\rangle}]_{\langle E,E',F_\alpha,F'_\alpha\rangle}) \in (X\hat{}\{Y_\alpha\})^{\sim[*]}$ whose $\langle H, H', K_\alpha, K'_\alpha\rangle$th element is

$$[\{[*],[a_1],...[a_n]\}]_{\langle H,H',K_\alpha,K'_\alpha\rangle} = [\gamma]_{\langle H,H',K_\alpha,K'_\alpha\rangle}$$

To finish the proof we note that in particular, $X \times X$ and $Y_\alpha \times Y_\alpha$ are covering entourages in their respective uniform spaces (by the definition of a covering basis). We note that $X\times X$ and $Y_\alpha \times Y_\alpha$ are covering entourages such that $(X\times X)^4 \subset X\times X$ and $(Y_\alpha \times Y_\alpha)^4 \subset Y_\alpha \times Y_\alpha$ and hence the above process works for the particular entourage $\langle X\times X, X\times X, Y_\alpha\times Y_\alpha, Y_\alpha\times Y_\alpha\rangle$ where $(X\times X)'$ and $(Y_\alpha\times Y_\alpha)'$ are taken to be $X\times X$ and $Y_\alpha\times Y_\alpha$ respectively. Thus if $(X\hat{}\{Y_\alpha\})\times(X\hat{}\{Y_\alpha\}) = \langle X\times X, X\times X, Y_\alpha\times Y_\alpha, Y_\alpha\times Y_\alpha\rangle$, we have that $(X\hat{}\{Y_\alpha\})\times(X\hat{}\{Y_\alpha\})$ is a covering entourage and $X\hat{}\{Y_\alpha\}$ would then be uniform coverable. To see that $(X\hat{}\{Y_\alpha\})\times(X\hat{}\{Y_\alpha\}) = \langle X\times X, X\times X, Y_\alpha\times Y_\alpha, Y_\alpha\times Y_\alpha\rangle$, we first let $([a],[b]) \in (X\hat{}\{Y_\alpha\})\times(X\hat{}\{Y_\alpha\})$ such that $a \in Y_\alpha$ and $b \in Y_\beta$. We let $z_1 \in Z_\alpha$ and $z_2 \in Z_\beta$. Then $([z_1],[z_2]) = ([f_\alpha(z_1)],[f_\beta(z_2)])$ is in $X\times X$, $([a],[z_1])$ is in $Y_\alpha\times Y_\alpha$ and $([z_2],[b])$ is in $Y_\beta\times Y_\beta$ hence $([a],[b])$ satisfies condition 2 in the definition of $\langle X\times X, X\times X, Y_\alpha\times Y_\alpha, Y_\alpha\times Y_\alpha\rangle$. A similar line of reasoning works if a or b (or both) are elements of X. Hence $(X\hat{}\{Y_\alpha\})\times(X\hat{}\{Y_\alpha\}) \subset \langle X\times X, X\times X, Y_\alpha\times Y_\alpha, Y_\alpha\times Y_\alpha\rangle$. The other containment follows by definition so that $(X\hat{}\{Y_\alpha\})\times(X\hat{}\{Y_\alpha\}) = \langle X\times X, X\times X, Y_\alpha\times Y_\alpha, Y_\alpha\times Y_\alpha\rangle$. ■

Proposition 57 *Let $X, \{Y_\alpha\}$ be uniform spaces and for each α, Z_α a compact subset of Y_α. Suppose the balls of the entourages $E, E', F_\alpha, F'_\alpha$ are path connected in their respective spaces. Then the glued entourage $\langle E, E', F_\alpha, F'_\alpha\rangle$ has path connected balls.*

Proof. By 51 we may assume that the topology is the quotient topology. We consider the following cases.

Case 1) $X \cap [a]$ is non-empty.

In this case, let $x \in X\cap[a]$ and suppose $([a],[b]) \in \langle E, E', F_\alpha, F'_\alpha\rangle$. From the proof of 50 we know that $i_X^{-1}(B([a],\langle E, E', F_\alpha, F'_\alpha\rangle) = B(x, E)$. This is a path connected subset of X by assumption. Thus, if $X\cap[b]$ is non-empty we let $x' \in X\cap[b]$. If p is a path from x to x' which lies in $B(x,E)$ then $i_X \circ p$ is a path from $[a]$ to $[b]$ which lies in $B([a],\langle E, E', F_\alpha, F'_\alpha\rangle)$. On the other hand, suppose $X\cap[b]$ is empty. Then $b \in Y_\alpha$ is the unique element in $[b]$. Consequently, $([a],[b]) \in \langle E, E', F_\alpha, F'_\alpha\rangle$ implies that $([a],[b])$ satisfies condition 2 in the definition of $\langle E, E', F_\alpha, F'_\alpha\rangle$. It follows that $b \in B(z, F'_\alpha)$ for some $z \in Z_\alpha$ and we note that for any $y_\alpha \in B(z, F'_\alpha)$ we have $[y_\alpha] \in B([a], \langle E, E', F_\alpha, F'_\alpha\rangle)$. Since $f_\alpha(z) \in B(x, E)$ we have, by what we have already proved, a path $i_X \circ p_1$ from $[a]$ to $[z]$ which lies in $B([a],\langle E, E', F_\alpha, F'_\alpha\rangle)$. Since $B(z, F'_\alpha)$ is path connected we find a path p_2 in Y_α from z to b which lies in

$B(z, F'_\alpha)$ and then $i_{Y_\alpha} \circ p_2$ is a path from $[z]$ to $[b]$. Since $im(p_2) \subset B(z, F'_\alpha)$ we know that $im(i_{Y_\alpha} \circ p_2) \subset B([a], \langle E, E', F_\alpha, F'_\alpha \rangle)$ and thus $B([a], \langle E, E', F_\alpha, F'_\alpha \rangle)$ is path connected in this case.

Case 2) $X \cap [a]$ is empty.

In this case we may assume that $a \in Y_\alpha$ is the unique element in $[a]$. We first suppose that $([a], [b])$ meet condition 1 in the definition of $\langle E, E', F_\alpha, F'_\alpha \rangle$. This can only happen if there exists a $y \in Y_\alpha \cap [b]$ such that $(a, y) \in F_\alpha$. Since $B(a, F_\alpha)$ is path connected by assumption, we choose a path p from a to y which lies in $B(a, F_\alpha)$ and then $i_{Y_\alpha} \circ p$ is a path from $[a]$ to $[b]$ which lies in $B([a], \langle E, E', F_\alpha, F'_\alpha \rangle$. Now suppose $([a], [b])$ meet condition 1 in the definition of $\langle E, E', F_\alpha, F'_\alpha \rangle$. Then there exists $z_\beta \in Z_\beta$ and $z_\alpha \in Z_\alpha$ such that $([a], [f_\alpha(z_\alpha)])$ is in F'_α, $([f_\alpha(z_\alpha)], [f_\beta(z_\beta)])$ is in E' and $([f_\beta], [b])$ is in F'_β. Then, since $B(y_\beta, F'_\beta)$ is path connected we find a path p_1 lying in $B(z_\alpha, F'_\alpha)$ between a and z_α. Any point lying in the image of p_1 is within F'_α of a and hence $i_{Y_\alpha} \circ p_1$ is a path between $[a]$ and $[z_\alpha] = [f_\alpha(z_\alpha)]$ which lies in $B([a], \langle E, E', F_\alpha, F'_\alpha \rangle)$. We then note that $B(f_\alpha(z_\alpha), E')$ is path connected and find a path p_2 between $f_\alpha(z_\alpha)$ and $f_\beta(z_\beta)$. Any point which lies on p_2 is within E' of $f_\alpha(z_\alpha)$ and hence points on $i_X \circ p_2$ are in $B([a], \langle E, E', F_\alpha, F'_\alpha \rangle)$ by condition 2. $i_X \circ p_2$ is a path in $X \hat{} \{Y_\alpha\}$ connecting $[f_\alpha(z_\alpha)]$ to $[f_\beta(z_\beta)]$. Finally we choose a path p_3 between z_β and b which lies in $B(z_\beta, F'_\beta)$. Points on p_3 are within F'_β of z_β and hence points on $i_{Y_\beta} \circ p_3$ lie in $B([a], \langle E, E', F_\alpha, F'_\alpha \rangle)$ by condition 2. Since $i_X \circ p_2$ is a path connecting $[z_\beta] = [f_\beta(z_\beta)]$ to $[b]$ we connect $i_{Y_\alpha} \circ p_1, i_X \circ p_2$ and $i_X \circ p_2$ to obtain a path between $[a]$ and $[b]$ which lies in $B([a], \langle E, E', F_\alpha, F'_\alpha \rangle)$. Thus $B([a], \langle E, E', F_\alpha, F'_\alpha \rangle)$ is path connected in this case as well. ■

5 The Van Kampen Theorem for Deck Groups

The Van Kampen Theorem is a highly useful tool for calculating the fundamental group of topological spaces. If a larger topological space is composed of simpler "pieces" whose fundamental groups are known, and whose intersections obey some basic path connected requirements, then the Van Kampen Theorem provides a presentation for the fundamental group of the larger space. The fundamental groups of the component spaces are first combined into a larger group (the free product). Since the generators of these groups are eqivalence classes of loops, any loop which lies in more than one component space will appear more than once in the free product. Thus, to complete the construction, generators that lie in more than one component space are identified.

A "Van Kampen" theorem for uniform fundamental groups would be an important tool for calculating such groups on more complicated uniform spaces. The propositions below provide such a tool, but it is important to note that the theorem works on two levels. First, 59, we have a "Van Kampen" theorem for spaces at the deck group level (in which a given entourage is fixed). Then we consider inverse limits of such deck groups in proposition 2. To use the traditional Van Kampen Theorem, one considers a topological space as a union of open sets whose fundamental groups are known. The requirement that the sets be open cannot be dropped, as the example of the Hawaiian Earring shows (see [9]). One main feature of 61 is that it does not require the openness of subsets. Further, the lack of a requirement of openness allows 61 to be applicable to more complicated spaces like those considered in [9]. This is because individual deck groups will "miss" small holes and are unencumbered with the complications that arise in considering an infinite number of smaller and smaller generators. These generators are picked up in the inverse limit. We note, however, that δ_1 may not equal the fundamental group in these cases (see [Example]).

To begin, we will provide some definitions and notation for free products. See [7], p. 68 for more details. Let $\{G_\alpha\}_{\alpha\in A}$ be a collection of groups, indexed by A. We may consider the set $S = \{(x_1)_{\alpha_1}(x_2)_{\alpha_2}...(x_n)_{\alpha_n}|x_i \in G_{\alpha_i}\}$. This is the collection of all finite sequences of elements from $\cup_\alpha G_\alpha$. We wish to define an equivalence relation on S. First, we declare all of the identity elements to be equivalent to each other. Suppose $(x_k)_{\alpha_k} = (y)_{\alpha_k}(z)_{\alpha_k}$ or in other words, x_k equals the product yz in G_{α_k}. Then, the sequence $(x_1)_{\alpha_1}(x_2)_{\alpha_2}...(x_{k-1})_{\alpha_{k-1}}(x_k)_{\alpha_k}...(x_n)_{\alpha_n}$ will be called a contraction of $(x_1)_{\alpha_1}(x_2)_{\alpha_2}...(x_{k-1})_{\alpha_{k-1}}(y)_{\alpha_k}(z)_{\alpha_k}...(x_n)_{\alpha_n}$ while the second sequence will be called an expansion of the first. Then any two sequences $a_1, a_2 \in S$ are equivalent ($a_1 \sim a_2$) if there exists sequences $a_1 = s_0, s_1, ...s_m = a_2$ such that each s_k is an expansion or contraction of s_{k-1}. We then form the quotient set $S/\sim$.

We define a group operation on the set $S/\sim$ by concatenating. In other words:

$$\begin{aligned} & [(x_1)_{\alpha_1}(x_2)_{\alpha_2}...(x_n)_{\alpha_n}] * [(y_1)_{\beta_1}(y_2)_{\beta_2}...(y_m)_{\beta_n}] \\ = \; & [(x_1)_{\alpha_1}(x_2)_{\alpha_2}...(x_n)_{\alpha_n}(y_1)_{\beta_1}(y_2)_{\beta_2}...(y_m)_{\beta_n}] \end{aligned}$$

The well definedness of the operation $*$, as well as the fact that it is associative follows as in the proof of the similar assertions for $\delta_E(X)$ in 19. It is easy to see that the

equivalence class containing all the identity elements is the identity for this group, and that an inverse to $[\ (x_1)_{\alpha_1}(x_2)_{\alpha_2}...(x_n)_{\alpha_n}]$ is $[(x_n)^{-1}_{\alpha_n}(x_{n-1})^{-1}_{\alpha_{n-1}}...(x_1)^{-1}_{\alpha_1}]$.This group is called the free product of the groups G_α and will be denoted $*G_\alpha$.

Lemma 58 *Let E be an entourage in a uniform space and suppose $\gamma = \gamma_1 x \gamma_2$ is an E chain,where $\gamma_1 = \{a_0, a_1, ...a_{i-1}\}$ and $\gamma_2 = \{a_{i+1}, a_{i+2}, ..., a_n\}$. Let $\tau = \{x = x_0, x_1, ...x_m\}$ be an E chain from x to some point x_m. Then $\gamma \sim \gamma_1 \tau \tau^{-1} \gamma_2$.*

Proof. We begin by noting that (x, x) and (x, a_{i+1}) are both elements of E since γ is an E chain. Hence x may be inserted into x, a_{i+1} to obtain x, x, a_{i+1}. Then the following equivalences follow from the fact that τ is an E chain:

$$\begin{aligned} x, x &\sim x, x_1, x \sim x, x_1, x_1, x \sim x, x_1, x_2, x_1, x \sim x, x_1, x_2, x_2, x_1, x \\ &\sim ... \sim x, x_1, ...x_{m-1}, x_m, x_{m-1}, ...x_1, x = \tau\tau^{-1} \end{aligned}$$

.

These equivalences imply that $\gamma = \gamma_1 x \gamma_2 \sim \gamma_1 \tau \tau^{-1} \gamma_2$ ■

Proposition 59 ***(Van Kampen Theorem for $\delta_E(X)$):*** *Let $\{X_\alpha\}$ be a collection of subsets of a uniform space X whose union is X. Let $*$ be a base point in X and suppose $* \in X_\alpha$ for all α. Let E be an entourage. Suppose that for each triple $X_\alpha, X_\beta, X_\gamma$ we have that $E(X_\alpha) \cap E(X_\beta) \cap E(X_\gamma)$ is E chain connected. If λ is an E-loop in $E(X_\alpha) \cap E(X_\beta)$ let $[\lambda]_\alpha, [\lambda]_\beta$ be the E-equivalence classes of λ in the subspaces $E(X_\alpha)$ and $E(X_\beta)$ respectively. Let $*\delta_E(E_\alpha)$ be the free product of all $\delta_E(X_\alpha)$, and let N be the normal subgroup generated by elements of the form $[\gamma]_\alpha[\gamma]^{-1}_\beta$. Then $\delta_E(X) \tilde{=} * \delta_E(E(X_\alpha))/N$.*

Proof. We begin by showing that if $\gamma := \{* = x_0, x_1, ...x_n = *\}$ is an $E-$loop in X, then there exists E-loops $g_1, g_2, ...g_n$, each of which is contained in some $E(X_\alpha)$, such that $[\gamma]_E = [g_1 g_2 ... g_n]_E = [g_1]_E * [g_2]_E * ... * [g_n]_E$ in $\delta_E(X)$. To see this, we choose for $1 \leq s \leq n-1$, an element $X_s \in \{X_\alpha\}$ such that $x_s \in X_s$. For convenience we set $X_n = X_{n-1}$. Notice that since $(x_s, x_{s+1}) \in E$ and $x_{s+1} \in X_{s+1}$ we have by the symmetry of E that $x_s \in E(X_s) \cap E(X_{s+1})$. Using the fact that $E(X_s) \cap E(X_{s+1})$ is E-chain connected and contains $*$ we may find, for each $1 \leq s \leq n-1$ an E-chain $\tau_s \subset E(X_s) \cap E(X_{s+1})$ from x_s to $*$. For convenience we set $\tau_0 = \tau_n = *$. Then, using 58 we have that $\gamma \sim *\tau_1\tau_1^{-1}\tau_2\tau_2^{-1}...\tau_{n-1}^{-1}* = \tau_0^{-1}\tau_1\tau_1^{-1}\tau_2\tau_2^{-1}...\tau_{n-1}^{-1}\tau_n$. Defining $g_s = \tau_{s-1}^{-1}\tau_s$ for $1 \leq s \leq n$, we have that each g_s is an E-loop contained in $E(X_s)$ and the result follows. Notice that the factorization $[\gamma]_E = [g_1]_E * [g_2]_E * ... * [g_n]_E$ resulting from 58 depends on the choice of $X_s \in \{X_\alpha\}$ and on the choice of the chains τ_s. Any such factorization corresponds to an element of the group $*\delta_E(E(X_\alpha)$, namely the equivalence class of the sequence $[g_1]_1 * [g_2]_2 * ... * [g_n]_n$ where $[g_s]_s \in \delta_E(X_s)$.

We define a map $\chi : *\delta_E(E(X_\alpha)) \to \delta_E(X)$ by mapping $[g_0]_0 * [g_1]_1 * ... * [g_l]_l$ to $[g_0 g_1 ... g_l]_E$. To see that the mapping χ defined above is well defined on $*\delta_E(E(X_\alpha))$, notice that if $[g_k]_k = [h]_k * [h']_k = [hh']_k$ in $\delta_E(E(X_k))$ then there exists an E-homotopy from g_k to hh' in the subspace $E(X_k)$. This implies that there is an E-homotopy in X from g_k to hh' and hence $[g_0 g_1 ... g_k ... g_l]_E = [g_0 g_1 ... hh' ... g_l]_E$. Thus any expansion or contraction of $[g_0]_0 * [g_1]_1 * ... * [g_l]_l$ has the same image under χ and since all the

elements in the equivalence class of $[g_0]_0 * [g_1]_1 * ... * [g_l]_l$ can be obtained through a finite number of expansions or contractions, χ is well defined. Clearly χ is a homomorphism since the operation in both $*\delta_E(E(X_\alpha))$ and $\delta_E(X)$ is concatenation. Further, we have shown that any $[\gamma]_E$ has a factorization, and hence χ is surjective. For any generator $[\gamma]_\alpha * [\gamma]_\beta^{-1}$ of N we have $\chi([\gamma]_\alpha * [\gamma]_\beta^{-1}) = [\gamma * \gamma^{-1}]_E = [*]_E$ so that N is a subgroup of $\ker(\chi)$. Thus there is a well defined surjective function $\chi : *\delta_E(E(X_\alpha)/N \to \delta_E(X)$ defined by $\chi(gN) = \chi(g)$ (see [7]). The remainder of the proof will establish that χ is injective. This is accomplished by showing that any two factorizations of $[\gamma]_E$ are in the same equivalence class of $*\delta_E(E(X_\alpha)/N$

We first establish that any choice of the chains τ_s results in factorizations whose representatives in $*\delta_E(E(X_\alpha))/N$ are equal. For each $1 \leq s \leq n-1$, let ν_s be an E-chain in $E(X_s) \cap E(X_{s+1})$ from x_{i_s} to $*$, and set $\nu_0 = \nu_n = *$. Let $[\gamma]_E = [h_1]_E * [h_2]_E * ... * [h_n]_E$ be the resulting factorization (i.e. $h_s = \nu_{s-1}^{-1}\nu_s$), then $[g_1]_1 * [g_2]_2 * ... * [g_n]_n$ is in the same equivalence class as $[h_1]_1 * [h_2]_2 * ... * [h_n]_n$ in $*\delta_E(E(X_\alpha)/N$. For some fixed s it suffices to consider the case that $\nu_k = \tau_k$ for all $k \neq s$. Then applying this case $n-1$ times would obtain the result. We note that $h_k (= \nu_{k-1}^{-1}\nu_k)$ is equal to $g_k (= \tau_{k-1}^{-1}\tau_k)$ for all $k \neq s$ or $s+1$ while $h_s = \tau_{s-1}^{-1}\nu_s$ and $h_{s+1} = \nu_s^{-1}\tau_{s+1}$. We form the loop $\nu_s^{-1}\tau_s$ and note that it is an E-loop which lies in both $\delta_E(E(X_s))$ and $\delta_E(E(X_{s+1}))$. Hence by the definition of N we have that $[\nu_s^{-1}\tau_s]_s = [\nu_s^{-1}\tau_s]_{s+1}$ in $*\delta_E(E(X_\alpha)/N$. By the definition of g_s we have that $g_s = \tau_{s-1}^{-1}\tau_s \sim \tau_{s-1}^{-1}x_s\tau_s$ (since τ_{s-1}^{-1} ends at x_{s-1} while τ_s begins at x_s and x_s can be inserted between x_{s-1} and x_s). Further, 58 then gives $\tau_{s-1}^{-1}x_s\tau_s \sim \tau_{s-1}^{-1}\nu_s\nu_s^{-1}\tau_s$ as E-chains. These equivalences take place in the subspace $E(X_s)$ so, if we consider $[g_s]_s$ as an element of $\delta_E(X_s)$ then we can write $[g_s]_s = [\tau_{s-1}^{-1}\tau_s] = [\tau_{s-1}^{-1}\nu_s\nu_s^{-1}\tau_s] = [\tau_{s-1}^{-1}\nu_s]_s * [\nu_s^{-1}\tau_s]_s = [h_s]_s * [\nu_s^{-1}\tau_s]_s$. Now, as an element of $*\delta_E(E(X_\alpha)/N$ we have that $[g_s]_s * [g_{s+1}]_{s+1} = [h_s]_s * [\nu_s^{-1}\tau_s]_s * [g_{s+1}]_{s+1}$ (since $[g_s]_s = [h_s]_s * [\nu_s^{-1}\tau_s]_s$ in $\delta_E(X_s)) = [h_s]_s * [\nu_s^{-1}\tau_s]_{s+1} * [g_{s+1}]_{s+1}$ (by the definition of N)$= [h_s]_s * [\nu_s^{-1}\tau_s g_{s+1}]_{s+1} = [h_s]_s * [\nu_s^{-1}\tau_s\tau_s^{-1}\tau_{s+1}]_{s+1}$. Since $\nu_s^{-1}\tau_s\tau_s^{-1}\tau_{s+1} \sim \nu_s^{-1}x_s\tau_{s+1} \sim \nu_s^{-1}\tau_{s+1}$ as above, we have that $[g_s]_s * [g_{s+1}]_{s+1} = [h_s]_s * [\nu_s^{-1}\tau_s\tau_s^{-1}\tau_{s+1}]_{s+1} = [h_s]_s * [h_{s+1}]_{s+1}$ and the result follows.

Now we show that the representative in $*\delta_E(E(X_\alpha)/N$ of the factorization of $[\gamma]_E$ does not depend on the choice of subspace X_s containing x_s. Suppose that for each $1 \leq s \leq n-1$ we have that $x_s \in X_{\alpha_s}$ for some $X_{\alpha_s} \in \{X_\alpha\}$. Let ν_s be an E-chain in $E(X_{\alpha_s}) \cap E(X_{\alpha_{s+1}})$ and set $\nu_0 = \nu_n = *$. Let $[\gamma]_E = [h_1]_E * [h_2]_E * ... * [h_n]_E$ be the resulting factorization (i.e. $h_s = \nu_{s-1}^{-1}\nu_s$), then $[g_1]_1 * [g_2]_2 * ... * [g_n]_n$ is in the same equivalence class as $[h_1]_{\alpha_1} * [h_2]_{\alpha_2} * ... * [h_n]_{\alpha_n}$ in $*\delta_E(E(X_\alpha)/N$. We will use the previous paragraph, and the definition of N repeatedly to replace the chains ν_s with τ_s. To begin, notice that since ν_1 can be replaced with any E-chain in $E(X_{\alpha_1}) \cap E(X_{\alpha_2})$, we may assume that ν_1 lies in $E(X_{\alpha_1}) \cap E(X_{\alpha_2}) \cap E(X_1)$ (which is E-chain connected by the requirements of the proposition). Then, in particular, the E-chain $*\nu_1$ lies in both $E(X_{\alpha_1})$ and $E(X_1)$ so by the definition of N, $[h_1]_{\alpha_1} = [h_1]_1$ and we have that $[h_1]_{\alpha_1} * [h_2]_{\alpha_2} * ... * [h_n]_{\alpha_n} = [h_1]_1 * [h_2]_{\alpha_2} * ... * [h_n]_{\alpha_n}$. This is a factorization based on the assumption that $x_1 \in X_1$ instead of X_{α_1}. Then, ν_1 can be replaced with any other E-chain in $E(X_1) \cap E(X_{\alpha_2})$. This time, using the assumption that $E(X_1) \cap E(X_{\alpha_2}) \cap E(X_2)$ is E-chain connected we may replace ν_1 with some E-chain ν_1' which lies in $E(X_1) \cap E(X_{\alpha_2}) \cap E(X_2)$. This gives us an

$h'_1 = *\nu'_1$ and $h'_2 = (\nu'_1)^{-1}\nu_2$ and by the results of the previous paragraph, we have that $[h_1]_1 * [h_2]_{\alpha_2} * ... * [h_n]_{\alpha_n} = [h'_1]_1 * [h'_2]_{\alpha_2} * ... * [h_n]_{\alpha_n}$. Then, since $h'_2 = (\nu'_1)^{-1}\nu_2$ lies in both $E(X_2)$ and $E(X_{\alpha_2})$ we have, from the definition of N that this is equal to $[h'_1]_1 * [h'_2]_2 * [h_3]_{\alpha_3} * ... * [h_n]_{\alpha_n}$. Proceeding inductively we obtain $[h'_1]_1 * [h'_2]_2 * ... * [h'_n]_n$. Finally, since each υ'_s lies in $E(X_s) \cap E(X_{s+1})$ we may replace each ν'_s with τ_s so that $h'_s = (\nu'_s)^{-1}\nu'_{s+1} = \tau_{s-1}^{-1}\tau_s = g_s$ and obtain $[h'_1]_1 * [h'_2]_2 * ... * [h'_n]_n = [g_1]_1 * [g_2]_2 * ... * [g_n]_n$.

We have now demonstrated that the equivalence class of $[g_1]_1 * [g_2]_2 * ... * [g_n]_n$ in $*\delta_E(E(X_\alpha)/N$ is independent of how the E-loop $[\gamma]_E = [g_1]_E * [g_2]_E * ... * [g_n]_E$ is factored as above. We will show further that if ζ is any loop E-equivalent to γ (i.e. ζ is any other element in the equivalence class $[\gamma]_E$) then any factorization of ζ is in the same equivalence class as $[g_1]_1 * [g_2]_2 * ... * [g_n]_n$ in $*\delta_E(E(X_\alpha)/N$. Toward this end, suppose that $a \in X_a$ can be inserted into γ, i.e. that $\zeta = \{* = x_0, x_1, ...x_s, a, x_{s+1}, x_n = *\}$ is an expansion of $\gamma = \{* = x_0, x_1, ...x_n\}$. If $[\zeta]_E = [h_1]_E * [h_2]_E * ... * [h_s]_E * [h_a]_E * [h_{s+1}]_E * ... * [h_n]_E$ is a factorization of ζ then we wish to show that $[h_1]_1 * [h_2]_2 * ... * [h_s]_s * [h_a]_a * [h_{s+1}]_{s+1} * ... * [h_n]_n$ is in the same equivalence class as $[g_1]_1 * [g_2]_2 * ... * [g_n]_n$ in $*\delta_E(E(X_\alpha)/N$. Since (x_s, a) and $(a, x_{s+1}) \in E$ we have that $a \in E(X_s) \cap E(X_{s+1})$ and $x_s, x_{s+1} \in E(X_a)$. Using what we have already shown, we may assume that $\nu_s = \tau_s$ for all s with the added assumption that τ_s is an E-chain in $E(X_s) \cap E(X_a) \cap E(X_{s+1})$ and τ_{s+1} is an E-chain in $E(X_{s+1}) \cap E(X_a) \cap E(X_{s+2})$. This then implies that $h_t = g_t$ for all $t \neq a$ or $s+1$. We may also choose ν_a to lie in $E(X_s) \cap E(X_a) \cap E(X_{s+1})$. Then we have that the chain $g_{s+1} = \tau_s^{-1}\tau_{s+1}$ lies in both of the subspaces $E(X_{s+1})$ and $E(X_a)$, so that by the definition of N we have $[g_{s+1}]_{s+1} = [g_{s+1}]_a$ in $*\delta_E(E(X_\alpha)/N$. Further, $g_{s+1} = \tau_s^{-1}\tau_{s+1} \sim \tau_s^{-1}a\tau_{s+1}$ (by assumption on a)$\sim \tau_s^{-1}\tau_a\tau_a^{-1}\tau_{s+1}$ (by [chain inset lemma]). Since $\tau_s, \tau_a, \tau_{s+1}$ all lie in $E(X_a)$ we have $[g_{s+1}]_a = [\tau_s^{-1}\tau_a\tau_a^{-1}\tau_{s+1}]_a = [\tau_s^{-1}\tau_a]_a * [\tau_a^{-1}\tau_{s+1}]_a = [h_a]_a * [h_{s+1}]_a$ in $\delta_E(E(X_a))$. Since $\tau_a^{-1}\tau_{s+1}$ also lies in $E(X_{s+1})$ we have by the definition of N that $[h_{s+1}]_a = [h_{s+1}]_{s+1}$ in $*\delta_E(E(X_\alpha)/N$. All of this gives that $[g_{s+1}]_{s+1} = [h_a]_a * [h_{s+1}]_{s+1}$ in $*\delta_E(E(X_\alpha)/N$. Thus, if ζ is obtained from γ by an expansion or contraction, then any factorization of either is equivalent in $*\delta_E(E(X_\alpha)/N$.

To show that any factorization of ζ has a representative in $*\delta_E(E(X_\alpha)/N$ equivalent to that of γ we will need one other fact. Suppose $[\gamma]_\alpha \in *\delta_E(E(X_\alpha))$ and $[g_1]_E * [g_2]_E * ... * [g_n]_E$ is a factorization of γ. Then $[g_1]_1 * [g_2]_2 * ... * [g_n]_n$ is in the same equivalence class as $[\gamma]_\alpha$ in $*\delta_E(E(X_\alpha)/N$. γ_α in $E(X_\alpha)$ implies that $x_s \in E(X_\alpha)$ for all $0 \leq s \leq n$. We then form a factorization as above, with the additional assumption that each τ_s lies in $E(X_\alpha) \cap E(X_s) \cap E(X_{s+1})$. Then we have that $[\gamma]_\alpha = [g_1 g_2 ... g_n]_\alpha = [g_1]_\alpha * [g_2]_\alpha * ... * [g_n]_\alpha$ in $\delta_E(E(X_\alpha))$ and hence $[\gamma]_\alpha$ and $[g_1]_\alpha * [g_2]_\alpha * ... * [g_n]_\alpha$ are in the same equivalence class in $*\delta_E(E(X_\alpha))/N$. Also, by the definition of N, $[g_s]_\alpha = [g_s]_s$ in $*\delta_E(E(X_\alpha))/N$ and the result follows.

Now, suppose that $[g_1]_1 * [g_2]_2 * ... * [g_n]_n$ and $[h_1]_1 * [h_2]_2 * ... * [h_m]_m$ are two elements of $*\delta_E(E(X_\alpha))$ such that $\chi([g_1]_1 * [g_2]_2 * ... * [g_n]_n) = \chi([h_1]_1 * [h_2]_2 * ... * [h_m]_m)$ i.e. $[g_1 g_2 ... g_n]_E = [h_1 h_2 ... h_m]_E$ in $\delta_E(X)$. Then there exists E chains $\gamma_0, \gamma_1, ...\gamma_k$ such that γ_i is an expansion or contraction of γ_{i-1} and $g_1 g_2 ... g_n = \gamma_0$ whereas $\gamma_k = h_1 h_2 ... h_m$. Let f_i be a factorization of γ_i and let F_i be the corresponding element in $*\delta_E(E(X_\alpha)$. Then F_0 is equivalent to $[g_1]_1 * [g_2]_2 * ... * [g_n]_n$ (by what we have just shown), F_i is equivalent to F_{i-1} since γ_i is obtained from γ_{i-1} by an expansion or

contraction and F_k is equivalent to $[h_1]_1 * [h_2]_2 * ... * [h_m]_m$ again, by what we have just shown. Hence χ is injective and the proposition is proved. ■

We now consider the inverse limit. If N_E represents the normal subgroup defined by the previous proposition then for each E there is an isomorphism $\chi_E : *\delta_E(E(X_\alpha))/N_E \to \delta_E(X)$ we may define for $F \subset E$ homomorphisms $f_{EF} = \chi_E^{-1} \circ \phi_{EF} \circ \chi_F$. If $[g_1]_1 * [g_2]_2 * ... * [g_n]_n \in *\delta_F(F(X_\alpha))$ then

$$\phi_{EF} \circ \chi_F([g_1]_1 * [g_2]_2 * ... * [g_n]_n) = \phi_{EF}([g_1 g_2 ... g_n]_F) = [g_1 g_2 ... g_n]_E$$

However, each g_i is a loop in $\delta_F(F(X_i))$ and since $F \subset E$ each g_i is a loop in $\delta_E(E(X_i))$. Thus $[g_1]_E * [g_2]_E * ... * [g_n]_E$ is a factorization of $\phi_{EF}([g_1 g_2 ... g_n]_F)$ and hence $\chi_E^{-1}(\phi_{EF}([g_1 g_2 ... g_n]_F)) = [g_1]_1 * [g_2]_2 * ... * [g_n]_n$. In short, f_{EF} is the mapping obtained by the inclusion of each term $[g_i]_i$ in $\delta_F(F(X_i))$ into $\delta_E(E(X_i))$. f_{EF} are thus bonding maps for the groups $*\delta_E(E(X_\alpha))/N_E$. We let p_E be the projection map of this inverse system onto $*\delta_E(E(X_\alpha))/N_E$.

Corollary 60 *Let $\{X_\alpha\}$ be a collection of subsets of a uniform space X whose union is X. Let $*$ be a base point in X and suppose $* \in X_\alpha$ for all α. Suppose there exists a basis Ω for the uniformity on X such that for each entourage $E \in \Omega$ and each triple $X_\alpha, X_\beta, X_\gamma$ we have that $E(X_\alpha) \cap E(X_\beta) \cap E(X_\gamma)$ is E chain connected. Then for each E we define N_E to be the normal subgroup defined in proposition 2 for E. Then $\delta_1(X) = \varprojlim *\delta_E(E(X_\alpha)/N_E$.*

Proof. Since Ω forms a basis for the uniformity on X the collection of entourages forms a co-final subset of the index on δ_1. We may then define

$$I_E : \varprojlim *\delta_E(E(X_\alpha))/N_E \longrightarrow \delta_E(X) by \chi_E \circ p_E.$$

So that

$$\chi_E^{-1} \circ \phi_{EF} \circ I_F = \chi_E^{-1} \circ \phi_{EF} \circ \chi_F \circ p_F = f_{EF} \circ p_F = p_E$$

Using the fact that χ_E is an isomorphism we obtain that $\phi_{EF} \circ I_F = \chi_E \circ p_E = I_E$ and by the fundamental property of inverse limits we have a well defined homomorphism $I : \varprojlim *\delta_E(E(X_\alpha))/N_E \longrightarrow \delta_1(X)$. Similarly we define $J_E : \delta_1(X) \longrightarrow *\delta_E(E(X_\alpha))/N_E$ by $J_E = \psi_E \circ \chi_E^{-1}$ and by a proof similar to the one above we obtain a well defined homomorphism $J : \delta_1(X) \longrightarrow \varprojlim *\delta_E(E(X_\alpha))/N_E$. If $(x) \in \delta_1(X)$ then the Eth element in $(I \circ J)(x)$ is $(\chi_E \circ \chi_E^{-1})(x_E) = x_E$. Thus $I \circ J = id$ and similarly $J \circ I = id$ so that I is an isomorphism. ■

We wish to remove the E-neighborhoods from the previous statement, i.e. we want $\delta_1(X) = \varprojlim *\delta_E(X_\alpha)/H_E$, where H_E is now the normal subgroup of $*\delta_E(X_\alpha)$ generated by elements of the form $[\gamma]_E^\alpha([\gamma]_E^\beta)^{-1}$ for every E-chain γ which lies in the intersection $X_\alpha \cap X_\beta$. The bonding maps for this inverse limit will be denoted g_{EF} and are induced by the maps ϕ_{EF}. This statement may not be true on an individual E level, since the consideration of neighborhoods of X_α was necessary to insure that each loop of X had a factorization. In the event, however, that $E(X_\alpha) \cap E(X_\beta) = E(X_\alpha \cap X_\beta)$ for every E in a basis for the uniformity on X the inverse limits are

isomorphic. This will follow from the fact that for a suitably chosen entourage F, the F-equivalence classes of $F(X_\alpha)$ are E-equivalent to some E-equivalence class in X_α.

Theorem 61 *(Van Kampen Theorem for deck groups). Let $X, \{X_\alpha\}$ be as in the previous Corollary. If $E(X_\alpha) \cap E(X_\beta) = E(X_\alpha \cap X_\beta)$ for all α, β and $E \in \Omega$ then $\delta_1(X) = \varprojlim *\delta_E(X_\alpha)/H_E$.*

Proof. We will define maps

$$I : \varprojlim *\delta_E(X_\alpha)/H_E \longrightarrow \varprojlim *\delta_E(E(X_\alpha))/N_E$$

,

$$J : \varprojlim *\delta_E(E(X_\alpha))/N_E \longrightarrow \varprojlim *\delta_E(X_\alpha)/H_E$$

such that $IJ = JI = id$. Since $X_\alpha \subset E(X_\alpha)$ we have a (uniformly continuous) inclusion map from $X_\alpha \subset E(X_\alpha)$ which induces a homomorphism i_E^α from $\delta_E(X_\alpha) \to \delta_E(E(X_\alpha)) \hookrightarrow *\delta_E(E(X_\alpha))$. Hence (see [7] page 68) there exists a homomorphism

$$i_E : *\delta_E(X_\alpha) \to *\delta_E(E(X_\alpha))$$

If $[\gamma]_E^\alpha([\gamma]_E^\beta)^{-1} \in H_E$ then γ lies in $X_\alpha \cap X_\beta \subset E(X_\alpha) \cap E(X_\beta) = E(X_\alpha \cap X_\beta)$ and hence $i_E(H_E) \subset N_E$. Thus i_E induces a homomorphism from $*\delta_E(X_\alpha)/H_E$ to $*\delta_E(E(X_\alpha)/N_E$ which we deonte by $\tilde{\imath}_E$. If $F \subset E$ then $f_{EF} \circ \tilde{\imath}_F = \tilde{\imath}_E \circ g_{EF}$ since f_{EF}, g_{EF} are simply the maps obtained by renaming each F equivalence class as an E-equivalence class. If ψ_K represents the Kth projection mapping of $\varprojlim *\delta_E(X_\alpha)/H_E$ we may then define $I_K : \varprojlim *\delta_E(X_\alpha)/H_E \to \delta_K(K(X))$ by $I_K(\gamma) = \tilde{\imath}_K \circ \psi_K(\gamma)$. If $F \subset K$ then

$$I_K(\gamma) = \tilde{\imath}_K(\psi_K(\gamma)) = \tilde{\imath}_K(g_{KF}(\psi_F(\gamma)) = f_{KF}(\tilde{\imath}_F(\psi_F(\gamma))) = f_{KF}(I_F(\gamma))$$

and the $\tilde{\imath}_K$ satisfy the universal property of inverse limits defining the homomorphism I.

We must then define $J : \varprojlim *\delta_E(E(X_\alpha)/N_E \to \varprojlim *\delta_E(X_\alpha)/H_E$ such that $IJ = JI = id$. To define J we let E be an entourage of X and choose F so that $F^3 \subset E$. Then the elements in an F chain in the F neighborhood of X_α are each within F of some point of X. By the choice of F these points form an E-chain. More specifically, Let $[\gamma]_F$ be an F-loop equivalence class in $F(X_\alpha)$ where $\gamma = \{* = a_0, a_1, ...a_n = *\}$. Then for each $1 \leq i \leq n-1$ there exists an $x_i \in X$ such that $a_i \in B(x_i, F)$. Then, by the choice of F we have that $\eta = \{* = a_0, x_1, ...x_{n-1}, a_n = *\}$ is an E-chain. We then define $j_{FE}^\alpha : \delta_F(F(X_\alpha)) \to \delta_E(X_\alpha)$ by sending $[\gamma]_F$ to $[\eta]_E$. To see that this mapping is independent of the choice of x_i, suppose that $a_i \in B(y_i, F)$ for $1 \leq i \leq n-1$. Then each y_i is within F^3 of both x_{i-1} and x_i and so η is E-equivalent to the chain $\{* = a_0, y_1, x_1, y_2, x_2, ...y_{n-1}, x_{n-1}, a_n = *\}$ which is E-equivalent to $\{* = a_0, y_1, y_2, ...y_{n-1}, a_n = *\}$. This demonstrates that the mapping does not depend on the choice of x_i. Further, if $(a_j, a), (a, a_{j+1}) \in F$ so that $\{* = a_0, a_1, ...a_j, a, a_{j+1}, ...a_n = *\}$ is an F expansion of γ then, finding an x such that $a \in B(x, F)$ we can show that x is within F^3 of both x_j and x_{j+1} and thus $\{* =$

$a_0, x_1, ..., x_j, x, x_{j+1}, ...x_{n-1}, a_n = *\}$ is an E expansion of η. j^{α}_{FE} is then well defined. We must show that this mapping is independent of the choice of F. We first let F' be an entourage contained in F (so that in particular $(F')^3 \subset E$). We note that if $[\gamma]_{F'}$ is a loop in $\delta_{F'}(F'(X))$ then, since $f_{FF'}[\gamma]_{F'} = [\gamma]_F$ we know that every F'-chain in $[\gamma]_{F'}$ is an F-chain in $[\gamma]_F$. Further, choosing x to be within F' of a implies that x is within F of a. Since we have already shown that the mapping is independent of the choice of x or the representative of $[\gamma]_F$ we have that $j^{\alpha}_{F'E}(\psi_{F'}(\gamma)) = j^{\alpha}_{FE}(\psi_F(\gamma))$. We now consider any entourage K such that $K^3 \subset E$. We find $D \subset K \cap F$. By above $j^{\alpha}_{KE}([\gamma]_K) = j^{\alpha}_{DE}([\gamma]_D) = j^{\alpha}_{FE}([\gamma]_F)$.

The j^{α}_{FE} define a map from $*\delta_E(E(X_\alpha)) \to *\delta_E(X_\alpha)$. Further, if γ lies in $F(X_\alpha) \cap F(X_\beta) = F(X_\alpha \cap X_\beta)$ then η may be taken to lie in $X_\alpha \cap X_\beta$. This implies that the image of N_F is contained in H_E. We let j_{FE} represent the corresponding map from $*\delta_F(F(X_\alpha)) \to *\delta_E(X_\alpha)/H_E$. We now wish to define $\tilde{j}_K : \varprojlim *\delta_E(E(X_\alpha)/N_E \to *\delta_K(X_\alpha)/H_K$ by sending $(\gamma) \in \varprojlim *\delta_E(E(X_\alpha))/N_E$ to $j_{FK}(\psi_F(\gamma))$ for any F such that $F^3 \subset E$. We show that the collection $\{\tilde{j}_E\}$ satisfies the universal property of inverse limits. Suppose $R \subset K$ and D is an entourage such that $D^3 \subset R \cap K$. We must show that $g_{KR} \circ \tilde{j}_R = \tilde{j}_K$ or, $g_{KR} \circ j_{DR}(\psi_D(\gamma)) = j_{DK}(\psi_D(\gamma))$. If $\gamma = \{* = a_0, a_1, ...a_n = *\} \in \psi_D(\gamma)$ and $\eta = \{* = a_0, x_1, ...x_{n-1}, a_n = *\}$ is the corresponding chain in X then by the choice of D, η is both an R chain and an K chain. Hence the result follows since g_{KR} simply relabels each R equivalence class as a K equivalence class.

It remains to show that $IJ = JI = id$. We first note that in the mapping $j_{FE}([\gamma]_F) = [\eta]_E$ that $\gamma = \{* = a_0, a_1, ...a_n = *\}$ is E equivalent to $\eta = \{* = a_0, x_1, ...x_{n-1}, a_n = *\}$ in $E(X_\alpha)$. Hence we obtain $i_E j_{FE}([\gamma]_F) = [\gamma]_E = f_{EF}([\gamma]_F)$. Hence $IJ = id$. Further, if γ is a loop in $F(X)$ consisting of elements already in X then we can set $\eta = \gamma$. This gives us that $j_{FE} i_E([\gamma]_F) = [\gamma]_E = \phi_{EF}([\gamma]_F)$ and hence $JI = id$. ■

We conclude this section by noting that the bonding maps g_{EF} form an inverse system on the groups $*\delta_E(X_\alpha)$ and H_E separately. In fact $\varprojlim H_E$ is a normal subgroup of $\varprojlim *\delta_E(X_\alpha)$. Following the last paragraph of section III.7.1 of [3] we obtain the following corollary.

Corollary 62 *If $X, \{X_\alpha\}$ satisfy the requirements of the previous theorem, then there exists a homomorphism $\varphi : \varprojlim *\delta_E(X_\alpha)/\varprojlim H_E \to \varprojlim *\delta_E(X_\alpha)/H_E$.*

The homomorphism φ is not surjective in general.

6 Applications

We will show that the uniform space defined by the gluing uniformity on a finite connected CW complex is uniform coverable. As in the discussion preceding section 3, we let B^n be the unit ball in $\mathbb{R}^n$ and S^{n-1} the boundary of B^n. X_n is the n-skeleton of X. B^n is compact and thus has a unique uniformity with basis consisting of metric entourages by Theorem 1 in section II.4.1 of [3]. If $f_\alpha : S^{n-1}_\alpha \to X_{n-1}$ is a continuous map then it is uniformly continuous by Theorem 2 of the same section. Since S^{n-1}_α is compact in B^n we know that the glued uniformity on each n-skeleton is compatible with the quotient topology. The proof will be by induction using 56. Since the 0-skeleton X_0 of a connected CW complex is discrete however, it is not uniform coverable under the discrete uniformity and 56 does not apply. Thus we must prove that the 1-skeleton is uniform coverable separately.

We will show that there exists a basis of entourages whose balls are open and path connected, and the result will then follow by corollary 65 in [2]. We first consider the basis for each 1-cell B^1_α formed by the entourages $F_\alpha(1/n)$ for $n \in \mathbb{N}$, $n > 2$. Since X_0 is discrete we may take Δ as a basis (see example 2 in II.1.1 of [3]). We may then form a basis for the uniformity on X_1 consisting of entourages $\langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle$ where for each α, $F_\alpha(1/m_\alpha)^4 \subset F_\alpha(1/n_\alpha)$. $(\Delta, F(1/n_\alpha))$ is acceptable since $f_\alpha^{-1}(\Delta) = \{(0_\alpha, 0_\alpha), (1_\alpha, 1_\alpha)\}$ and, since $n > 2$, $F_\alpha(1/n_\alpha) \cap f_\alpha^{-1}(\Delta) = f_\alpha^{-1}(\Delta)$. The balls of this basis are path connected by 57, but they are not open. We must find a second, more suitable basis.

Lemma 63 *Let X_0 be a discrete space and $\{B^1_\alpha\}$ a collection of 1-cells with attachment maps $f_\alpha : S^0_\alpha \to X_0$. For each α we choose $K_\alpha(1/n_\alpha)$ such that $n_\alpha > 2$. Let $\langle K_\alpha(1/n_\alpha)\rangle$ be the set of all pairs $([a],[b])$ such that either*

1. *$[a] = [b]$*
2. *$([a],[b])$ is in $K_\alpha(1/n_\alpha)$ for some α.*
3. *There exists $x \in X_0$ such that $([a],[x])$ is in $K_\alpha(1/n_\alpha)$ for some α and $([x],[b])$ is in $K_\beta(1/n_\beta)$ for some β.*

Then the collection of all $\langle K_\alpha(1/n_\alpha)\rangle$ forms a basis for the uniformity on $X_1 = X_0 \hat{} \{B^1_\alpha\}$ whose balls are open and path connected.

Proof. Note that each $\langle K_\alpha(1/n_\alpha)\rangle$ is symmetric. To show that the collection of all $\langle K_\alpha(1/n_\alpha)\rangle$ forms a basis for the uniformity on X_1 it is sufficient to show that for each $\langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle$ in the basis for X_1 there exists $\langle K_\alpha(1/n_\alpha)\rangle, \langle K'_\alpha(1/n_\alpha)\rangle$ such that

$$\langle K_\alpha(1/n_\alpha)\rangle \subset \langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle \subset \langle K'_\alpha(1/n_\alpha)\rangle$$

The second inclusion implies that each $\langle K_\alpha(1/n_\alpha)\rangle$ is in the uniformity of X_1 and symmetry implies that it is an entourage. Then the first inclusion implies that the collection forms a basis for the uniformity. In fact, we will show that

$$\langle F_\alpha(1/m_\alpha)\rangle \subset \langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle \subset \langle F_\alpha(1/n_\alpha)\rangle$$

First, we show that $\langle F_\alpha(1/m_\alpha)\rangle \subset \langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle$. If a pair $([a],[b])$ meets condition 1 or 2 in the definition of $\langle F_\alpha(1/m_\alpha)\rangle$ then, since $F_\alpha(1/m_\alpha) \subset F_\alpha(1/n_\alpha)$ for each α and $\Delta \subset \Delta$ we have that the pair meets condition 1 in the definition of $\langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle$. Further, if $([a],[b])$ meets condition 3 of the definition of $\langle F_\alpha(1/m_\alpha)\rangle$ then, since $([x],[x])$ is in Δ we have that $([a],[b])$ meets condition 2 of the definition of $\langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle$. Thus $\langle F_\alpha(1/m_\alpha)\rangle \subset \langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle$.

Now, we claim that $\langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle \subset \langle F_\alpha(1/n_\alpha)\rangle$. If $([a],[b])$ meets condition 1 in the definition of $\langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle$ then either $([a],[b])$ is in Δ or in $F_\alpha(1/n_\alpha)$ for some α. If $x_1 \in X \cap [a]$ and $x_2 \in X \cap [b]$ such that $(x_1, x_2) \in \Delta$ then $x_1 = x_2$ and hence $[a] = [b]$. On the other hand if $([a],[b])$ is in $F_\alpha(1/n_\alpha)$ for some α then by definition $([a],[b])$ satisfies condition 2 of the definition of $\langle F_\alpha(1/n_\alpha)\rangle$. Now, suppose $([a],[b])$ meets condition 2 in the definition of $\langle \Delta, \Delta, F_\alpha(1/n_\alpha), F_\alpha(1/m_\alpha)\rangle$. Then there exists $x_1, x_2 \in X$ such that $([a],[x_1]), ([x_2],[b])$ is in $F_\alpha(1/m_\alpha)$ for some α or Δ and $(x_1, x_2) \in \Delta$. Notice that $(x_1, x_2) \in \Delta$ implies that $x_1 = x_2$. Thus if $([a],[x_1]), ([x_2],[b])$ are both in $F_\alpha(1/m_\alpha)$ for some α then since $F_\alpha(1/m_\alpha) \subset F_\alpha(1/n_\alpha)$, $([a],[b])$ would satisfy condition 3 of the definition of $\langle F_\alpha(1/n_\alpha)\rangle$. If $([a],[x_1])$ is in Δ while $([x_2],[b])$ is in $F_\alpha(1/m_\alpha)$ for some α then $[a] = [x_1] = [x_2]$ and hence $([a],[b])$ satisfies condition 2 in the definition of $\langle F_\alpha(1/n_\alpha)\rangle$. A similar argument applies if $([x_2],[b])$ is in Δ while $([a],[x_1])$ is in $F_\alpha(1/m_\alpha)$ for some α. Finally, if both $([a],[x_1]), ([x_2],[b])$ are in Δ then $[a] = [x_1] = [x_2] = [b]$ and $([a],[b])$ satisfies condition 1 in the definition of $\langle F_\alpha(1/n_\alpha)\rangle$. Thus the collection of all $\langle K_\alpha(1/n_\alpha)\rangle$ forms a basis for the uniformity on X_1.

$B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ is open in the quotient topology. First, $i_X^{-1}(B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ is open automatically since X_0 is discrete. Now, suppose that there exists $y \in B^1_\alpha$ such that $([a],[y]) \in \langle K_\alpha(1/n_\alpha)\rangle$. We will show that there exists a metric ball $B(y, \varepsilon)$ such that

$$i_{B_\alpha}(B(y,\varepsilon)) \subset B([a], \langle K_\alpha(1/n_\alpha)\rangle)$$

We must consider several cases.

Case 1) $([a],[y]) \in \langle K_\alpha(1/n_\alpha)\rangle$ by condition 1.

In this case $[a] = [y]$ and then $i_{B^1_\alpha}(B(y, 1/n_\alpha)) \subset B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ by virtue of condition 2 in the definition.

Case 2) $([a],[y]) \in \langle K_\alpha(1/n_\alpha)\rangle$ by condition 2.Then for some β, there exists $t_1 \in B^1_\beta \cap [a]$ and $t_2 \in B^1_\beta \cap [y]$ such that $d(t_1, t_2) < 1/n_\beta$. If $\beta = \alpha$, and y is not 0_β or 1_β then y is the unique element in $[y]$ and we must have $t_2 = y$. We then choose an ε ball around t_2 so that $B(y, \varepsilon) \subset B(t_1, 1/n_\beta)$. Then $i_{B_\alpha}(B(y,\varepsilon)) \subset B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ by virtue of condition 2. If $\beta = \alpha$, and y is 0_β or 1_β then $i_{B^1_\alpha}(B(y, 1/n_\alpha)) \subset B([a], \langle K_\alpha(1/n_\alpha)\rangle)$by virtue of condition 2 in the definition. If $\beta \neq \alpha$ then $f_\beta(t_2) = f_\alpha(y)$. Then if $y' \in B(y, 1/n_\alpha)$ we have $([a], [f_\beta(t_1)])$ in $K_\beta(1/n_\beta)$ and $([f_\alpha(y)], [y'])$ in $K_\alpha(1/n_\alpha)$, and hence $[y'] \in B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ by virtue of condition 3.

Case 3) $([a],[y]) \in \langle K_\alpha(1/n_\alpha)\rangle$ by condition 3. Let $x \in X_0$ such that $([a],[x])$ is in $K_\beta(1/n_\beta)$ for some β and $([x],[y])$ is in $K_\gamma(1/n_\gamma)$ for some γ. We know by assumption that $y \in B^1_\alpha(1/n_\alpha)$. If $y \neq 0_\alpha$ or 1_α then y is the unique element of $[y]$ and hence $\gamma = \alpha$. Further we have $f_\alpha(e) = x$ and $d(e, y) < 1/n_\alpha$ for one of the endpoints

$e = 0_\alpha$ or 1_α. We choose an ε small enough that $B(y,\varepsilon) \subset B(e, 1/n_\alpha)$ and then $i_{B^1_\alpha}(B(y,\varepsilon)) \subset B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ by condition 3 in the definition of $\langle K_\alpha(1/n_\alpha)\rangle$. On the other hand, if $y = 0_\alpha$, or 1_α then $([x],[y])$ in $K_\gamma(1/n_\gamma)$ implies that for some $y_\gamma \in B^1_\gamma$ we have $f_\gamma(y_\gamma) = f_\alpha(y) = x$. Hence $i_{B^1_\alpha}(B(y, 1/n_\alpha)) \subset B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ by virtue of condition 3 in the definition of $\langle K_\alpha(1/n_\alpha)\rangle$. We have thus exhausted the possibilities, and hence $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ is open in the quotient topology.

We finish the proof by showing that $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ is path connected. Notice that if $([a],[b])$ meets condition 1 then there is nothing to show. Suppose $([a],[b])$ meets condition 2. Let $y_1 \in B^1_\alpha \cap [a]$ and $y_2 \in B_\alpha \cap [b]$ such that $d(y_1, y_2) < 1/n_\alpha$. We let $p = ty_1 + (1-t)y_2$ be the straight line path in B_α. Then $d(y_1, p(t)) < 1/n_\alpha$ for all $t \in I$ and hence $i_{B^1_\alpha} \circ p$ is a path between $[a]$ and $[b]$ lying in $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$. Finally, suppose $([a],[b])$ meets condition 3. Let $x \in X_0$ such that $([a],[x])$ is in $K_\alpha(1/n_\alpha)$ while $([x],[b])$ is in $K_\beta(1/n_\beta)$. Let $y_\alpha \in B^1_\alpha \cap [a], e_\alpha \in B^1_\alpha \cap [x]$ such that $d(y_\alpha, e_\alpha) < 1/n_\alpha$ and $y_\beta \in B^1_\beta \cap [b], e_\beta \in B^1_\beta \cap [x]$ such that $d(y_\beta, e_\beta) < 1/n_\beta$. Letting p_1 be the straight line path from y_α to e_α and p_2 the straight line path from e_β to y_β we have that $d(y_\alpha, p_1(t)) < 1/n_\alpha$ for all $t \in I$ and $d(e_\beta, p_2(t)) < 1/n_\beta$ for all $t \in I$. Hence $i_{B^1_\alpha} \circ p_1$ is a path from $[a]$ to $[x]$ which lies in $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ by condition 2, while $i_{B^1_\beta} \circ p_2$ is a path from $[x]$ to $[b]$ which lies in $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ by condition 3. ■

Proposition 64 *If X is a connected finite dimensional CW complex then the gluing uniformity on X defines a uniform space which is uniform coverable.*

Proof. If $n = 0$ then $X = X_0$ which must be a single point since the space is assumed connected. If $n \geq 1$ then attaching $n+1$ cells cannot reduce the number of components. This is because each S^n is connected hence its image is connected hence its image is contained in some connected component of X_n. Thus the assumption that X is connected implies that X_n is connected. Further, the gluing uniformity on every connected 1-dimensional CW complex is uniform coverable by 63. The result then follows inductively by 51 (since S^{n-1} is a compact subset of B^n) and 56. ■

We now turn to the task of showing that for finite connected CW complexes, the fundamental group is equivalent to the deck group. We will prove the 1-dimensional case separately using Theorem 88 of [2] which states that for a connected uniformly locally path connected, uniformly semi-locally simply connected uniform space X, $\pi_1(X) = \delta_1(X)$. Let X_1 be a path connected 1-dimensional CW complex. We have already shown that such a space is uniformly locally path connected. Thus, to obtain the result it is sufficient to show that X_1 is uniformly semi-locally simply connected.

Lemma 65 *Every 1-dimensional CW complex X_1 is uniformly locally simply connected (hence uniformly semi-locally simply connected).*

Proof. We choose the basis element $\langle K_\alpha(1/n_\alpha)\rangle$, which is the entourage formed by choosing the metric entourage $K(1/n_\alpha)$ in each of the cells B^1_α. Let $[a] \in X_1$. We will show that any loop $L : I \to B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ is null-homotopic in $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$. Since we already know that $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ is path connected, it is sufficient to assume that the loop is based at $[a]$. We consider 2 cases.

Case 1) $X \cap [a]$ is non-empty. We let $x_0 \in X \cap [a]$. Notice from the definition of $\langle K_\alpha(1/n_\alpha)\rangle$ that $[x_0]$ is the only element of $i_{X_0}(X_0)$ in $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$. For each α we consider the set $U_\alpha = i_{B^1_\alpha}((0_\alpha, 1_\alpha))$. Each equivalence class of U_α consists of a single element from $(0_\alpha, 1_\alpha)$. Thus $i^{-1}_{B^1_\beta}(U_\alpha) = \emptyset$ if $\beta \neq \alpha$, and $(0_\alpha, 1_\alpha)$ if $\beta = \alpha$. Also, $i^{-1}_X(U_\alpha) = \emptyset$. Thus U_α is open in X_1 for each α. Hence $L^{-1}(\cup_\alpha U_\alpha)$ is open in I. $L^{-1}(\cup_\alpha U_\alpha)$ then consists of a (countable) collection of disjoint open intervals $\{J_m\}_{m=1}^\infty$. In fact, we may assume that for each m we have $L(J_m) \subset U_{\alpha(m)}$ for some $\alpha(m)$. Otherwise a standard argument will show that $L^{-1}(U_{\alpha(m)})$ and $L^{-1}(\cup_{\beta\neq\alpha(m)} U_\beta)$ are both open and closed in J_m which contradicts connectedness. Further, if $m_1 < m_2$ are the endpoints of J_m we know that $L(m_1), L(m_2) \notin U_{\alpha(m)}$ by the definition of J_m. We also know that $L(m_1), L(m_2) = [a]$. This is because $L(m_1) \in U_\beta$ ($L(m_2) \in U_\beta$) for some β implies that there exists an open interval $(m_1 - \varepsilon, m_1 + \varepsilon) \subset L^{-1}(U_\beta)$, $((m_2 - \varepsilon, m_2 + \varepsilon) \subset L^{-1}(U_\beta))$. The previous argument shows that $\beta = \alpha(m)$ but this then provides a contradiction. Finally, we note that if $t \in I$ such that $t \notin L^{-1}(\cup_\alpha U_\alpha)$ then $L(t) \in X_0 \cap B([a], \langle K_\alpha(1/n_\alpha)\rangle) = [x_0] = [a]$. All of this together implies that if $\bar{J}_m$ is the closure of J_m we have that $i^{-1}_{B^1_{\alpha(m)}}(L(\bar{J}_m))$ is a loop in $B^1_{\alpha(m)}$ based at either $0_{\alpha(m)}$ or $1_{\alpha(m)}$. Using the contractability of $B^1_{\alpha(m)}$we can find a homotopy $H_m : \bar{J}_m \times I \to B^1_{\alpha(m)}$ which takes $i^{-1}_{B^1_\alpha(m)} L(\bar{J}_m)$ to the trivial loop at either $0_{\alpha(m)}$ or $1_{\alpha(m)}$. Specifically, for the trivial loop at $0_{\alpha(m)}$ we can define $H_m(s,t) = 0_{\alpha(m)}$ for $m_1 \leq s \leq m_1 + (m_2 - m_1)t$ and $H_m(s,t) = L(s) - tL(s)$ for $m_1 + (m_2 - m_1)t \leq s \leq m_2$. We note that H_m is decreasing in t in this case and thus it's image lies in the metric ball $B(0_{\alpha(m)}, 1/n_{\alpha(m)})$. Thus, if we denote by H'_m the homotopy $i_{B^1_{\alpha(m)}} \circ H_m$ then H'_m is a homotopy which takes $L(\bar{J}_m)$ to the constant function at $[a]$ and the image of H'_m lies in $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$. We can obtain similar results for the trivial loop at $1_{\alpha(m)}$ in which case the homotopy H_m is increasing in t. β

We define the homotopy H to be H'_m on $\bar{J}_m \times I$ and constant on

$$L^{-1}((\cup_\alpha U_\alpha)^C) \times I = L^{-1}([a]) \times I$$

We have that H is continuous at any point $(s,t) \in J_m \times I$ by the continuity of H'_m. Now, suppose $(s,t) \in L^{-1}([a]) \times I$. We will prove that H is continuous at (s,t) directly. Let V be an open set of X_1 containing $[a]$ and find an entourage $\langle K_\alpha(1/k_\alpha)\rangle$ such that $1/k_\alpha < 1/n_\alpha$ for each α and $B([a], \langle K_\alpha(1/k_\alpha)\rangle) \subset V$. Since this ball is open, by 63 $L^{-1}(B([a], \langle K_\alpha(1/k_\alpha)\rangle)$ is open in I and we may find an ε such that $I_s = (s - \varepsilon, s + \varepsilon) \subset L^{-1}(B([a], \langle K_\alpha(1/n_\alpha)\rangle)$. Now, we claim that $H(I_s \times I) \subset B([a], \langle K_\alpha(1/k_\alpha)\rangle)$. If $s' \in I_s \cap L^{-1}([a])$ then H is defined to be constant on $\{s'\} \times I$ hence $H(\{s'\} \times I) = \{[a]\}$. On the other hand if $s' \in J_m$ for some m then $H(\{s'\} \times I) = H'_m(\{s'\} \times I) \subset B([a], \langle K_\alpha(1/k_\alpha)\rangle)$ since H_m is decreasing in t if 0_m is the endpoint or increasing in t if 1_m is the endpoint (H_m is defined by a contraction on $B^1_{\alpha(m)}$ to one endpoint). Thus $I_s \times I$ is an open set in $I \times I$ containing (s,t) and such that $H(I_s \times I) \subset B([a], \langle K_\alpha(1/k_\alpha)\rangle) \subset V$ and H is then a homotopy. Since $H(I \times \{1\}) = [a]$ we have shown that L is null homotopic in $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ and thus $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ is simply connected.

Case 2)$X \cap [a]$ is empty.

In this case $a \in B^1_\beta$ for some β is the unique element in $[a]$. If

$$\min\{d(a, 0_\beta), d(a, 1_\beta)\} \geq 1/n_\beta$$

then $B([a], \langle K_\alpha(1/n_\alpha)\rangle) = i_{B^1_\beta}(B(a, 1/n_\beta))$ and we can use the contractability of $B(a, 1/n_\beta)$ onto a to find a homotopy which takes L to $[a]$. Suppose $d(a, 0_\beta) < 1/n_\beta$. We note that in this case $B([a], \langle K_\alpha(1/n_\alpha)\rangle)$ is equal to $B([x_0], \langle K_\alpha(1/n_\alpha)\rangle) \cup i_{B^1_\beta}([0_\beta, a + 1/n_\beta])$ where $x_0 = f_\beta(0_\beta)$. Thus if there exists a first s_1 such that $L(s_1) = [x_0]$ and a last s_2 such that $L(s_2) = [x_0]$ then $L|_{[s_1,s_2]}$ is a loop based at $[x_0]$ and we can apply the previous case to obtain a homotopy which takes $L|_{[s_1,s_2]}$ to the constant loop at $[x_0]$. Thus it is possible to assume that L is a loop which lies in $i_{B^1_\beta}(B^1_\beta)$ and we can use the contractability of $[0_\beta, a+1/n_\beta]$ onto a to find a homotopy which takes L to $[a]$. A similar result applies if $d(a, 1_\beta) < 1/n_\beta$. ■

Corollary 66 *If X is a connected one dimensional CW complex then $\pi_1(X) \cong \delta_1(X)$.*

Proof. Since every connected one dimensional CW complex is connected, uniformly locally path connected by 63 and uniformly semi-locally simply connected by 65, the result then follows by theorem 88 in [2].

We conclude by considering two examples. Let S consist of a countable number of circles $\{C_n\}_{n=1}^\infty$all joined at a point. We will consider two distinct uniformites on this set. If we consider S to be formed by gluing the endpoints of a countable number of unit intervals to a single point $*$ then we can put the glued uniformity on S which induces the quotient topology. Then S becomes a 1-dimensional CW complex under the induced topology, which is the wedge of a countable number of circles. We will denote this space by $\{*\}\hat{}\{B^1_n\}$. On the other hand, if we allow the perimeter of the circles to shrink to 0 by making the nth circle a circle with perimeter $\frac{2\pi}{2^n}$ then we can create a distinct uniformity on S by assigning it the length metric uniformity. The distance between points under the length metric is the length of the shortest curve connecting them. The space S under the induced topology is called the Hawaiian Earring, which we will denote by H. The metric uniformity which creates the Hawaiian Earring is distinct from the glued uniformity on the countable wedge of circles. To see this we note that in the Hawaiian Earring, $B(*, \varepsilon)$ will contain an infinite number of circles under the length metric uniformity, whereas 63 demonstrates that in the countable wedge of circles, $B(*, \langle F_\alpha(1/n_\alpha)\rangle)$ contains only those points on each circle within a distance of $1/n_\alpha$ from the endpoints. In both cases, we show that it is possible to apply the Van Kampen Theorem for deck groups. In each case we will consider the subsets X_n to be the circles C_n. We note that $\cup_{n=1}^\infty C_n = S$.

Example 67 *(The Hawaiian Earring). If $E(\varepsilon)$ is a basis entourage for the Hawaiian Earring, then the $E(\varepsilon)$-neighborhood of any circle C_n consists of C_n together with points z on any circle such that the length of the shortest arc from z to $*$ is less than ε. This neighborhood is then path connected and hence $E(\varepsilon)$-chain connected. The $E(\varepsilon)$-neighborhood of any two distinct circles consists only of those points z on any circle such that the length of the shortest arc from z to $*$ is less than ε. This is, in particular, the $E(\varepsilon)$-neithborhood of $*$ and hence $E(\varepsilon)(C_n) \cap E(\varepsilon)(C_m) = E(\varepsilon)(*) = E(\varepsilon)(C_n \cap C_m)$. The intersection of the $E(\varepsilon)$ neighborhood of any three such circles*

$C_{n_1}, C_{n_2}, C_{n_3}$ is either $E(\varepsilon)(C_n)$ if $n_1 = n_2 = n_3 = n$ or $E(\varepsilon)(*)$ if any two are distinct. In either case, these neighborhoods are path connected and hence $E(\varepsilon)$-chain connected. Thus theorem 61 applies. The author has shown [10] that in the unit circle S^1, if $\varepsilon < \frac{2\pi}{3}$ then $\delta_{E(\varepsilon)}(S^1) = \mathbb{Z}$ and if $\varepsilon \geq \frac{2\pi}{3}$ then $\delta_{E(\varepsilon)}(S^1) = \langle e \rangle$ (the trivial group). A modification of that proof would establish that for any circle C_n with perimeter $\frac{2\pi}{2^n}$ if $\varepsilon < \frac{1}{3}\frac{2\pi}{2^n}$ then $\delta_{E(\varepsilon)}(C_n) = \mathbb{Z}$ whereas $\delta_{E(\varepsilon)}(S^1) = \langle e \rangle$ if $\varepsilon \geq \frac{1}{3}\frac{2\pi}{2^n}$. We let ε_n satisfy the inequality $\frac{1}{3}\frac{2\pi}{2^{n+1}} \leq \varepsilon_n < \frac{1}{3}\frac{2\pi}{2^n}$. Then for each n we have that $\delta_{E(\varepsilon_n)}(C_k) = \mathbb{Z}$ for $k \leq n$ and $\langle e \rangle$ for $k > n$. By the Van Kampen Theorem for Deck Groups, $\delta_1(H) \cong \varprojlim *\delta_{E(\varepsilon_n)}(C_n)/N_{E(\varepsilon_n)} = \varprojlim *\{\mathbb{Z}\}_n/N_{E(\varepsilon_n)}$ where $*\{\mathbb{Z}\}_n$ represents the free product of n copies of $\mathbb{Z}$. $N_{E(\varepsilon_n)}$ is the normal subgroup generated by terms $[\gamma]_{C_{n_1}}[\gamma^{-1}]_{C_{n_2}}$ for all $E(\varepsilon_n)$-loops which lie in $C_{k_1} \cap C_{k_2} = \{*\}$. Since the only $E(\varepsilon_n)$-loop in any such intersection is the trivial loop, we have that $*\{\mathbb{Z}\}_n/N_{E(\varepsilon_n)} \cong *\{\mathbb{Z}\}_n$. Further, for $m < n$, the bonding map $\phi_{E(\varepsilon_m)E(\varepsilon_n)}(*\{\mathbb{Z}\}_n) \to *\{\mathbb{Z}\}_m$ which sends every $E(\varepsilon_n)$-loop to its $E(\varepsilon_m)$-equivalence class will trivialize every loop which lies on the circles $C_{m+1}, C_{m+2}, ...C_n$. Thus, $\delta_1(H) = \varprojlim *\{\mathbb{Z}\}_n$ is exactly the inverse limit described in [9]. We note that $\delta_1(H)$ is thus not equivalent to the fundamental group of H. In [9] the fundamental group of the Hawaiian Earring is identified as a subgroup of this inverse limit.

■

Example 68 *We have already proved that $\delta_1(\{*\}\hat{}\{C_n\}) = \pi_1(\{*\}\hat{}\{C_n\})$. We now calculate this group. For $m = 4, 5, ...$ we consider the entourage $\langle F_n(1/m) \rangle$ where, for each n, we choose the entourage $F_n(1/m)$ in B^1_n. If C_k $(= i_{B^1_k}(B^1_k))$ is one of the circles in $\{*\}\hat{}\{B^1_n\}$ then the $\langle F_n(1/m) \rangle$-neighborhood of C_k would be the union of C_k with the inclusions of all points of B^1_n for $n \neq k$ which lie in the intervals $[0_n, 1/m)$ or $(1/m, 1_n]$. This neighborhood is path connected by [path ent] and hence $\langle F_n(1/m) \rangle$-chain connected. The intersection of the $\langle F_n(1/m) \rangle$-neighborhoods of any two distinct circles would consist of the equivalence classes of all intervals $[0_n, 1/m)$ or $(1/m, 1_n]$. This neighborhood is similarly $\langle F_n(1/m) \rangle$-chain connected. In particular we have*

$$\langle F_n(1/m) \rangle(C_n) \cap \langle F_n(1/m) \rangle(C_m) = \langle F_n(1/m) \rangle(*) = \langle F_n(1/m) \rangle(C_n \cap C_m)$$

Further, the intersection of the $\langle F_n(1/m) \rangle$ neighborhood of any three such circles $C_{n_1}, C_{n_2}, C_{n_3}$ is either $\langle F_n(1/m) \rangle(C_n)$ if $n_1 = n_2 = n_3 = n$ or $\langle F_n(1/m) \rangle(*)$ if any two are distinct. In either case, these neighborhoods are path connected and hence $E(\varepsilon)$-chain connected. Hence we may apply the Van Kampen Theorem for Deck groups. Since the intersection of any two circles is $*$, we have that $N_{\langle F_n(1/m) \rangle}$ is trivial and obtain that $\delta_1(\{*\}\hat{}\{B^1_n\}) \cong \varprojlim *\delta_{\langle F_n(1/m) \rangle}(C_n)$. However, for any m, $\delta_{\langle F_n(1/m) \rangle}(C_n) \cong \mathbb{Z}$ and $*\delta_{\langle F_n(1/m) \rangle}(C_n)$ is the free product of a countable number of copies of $\mathbb{Z}$. Further, the bonding maps are one-to-one and thus the deck group is equivalent to the countable free product of integers.

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