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Zero-Divisor Graphs, Commutative Rings of Quotients, and Boolean Algebras

John D. LaGrange
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by John D. LaGrange entitled "Zero-Divisor Graphs, Commutative Rings of Quotients, and Boolean Algebras." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

David F. Anderson, Major Professor

We have read this dissertation and recommend its acceptance:

Michael Langston, Shashikant Mulay, Pavlos Tzermias

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

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A Dissertation
Presented For the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

John D. LaGrange
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Abstract

The *zero-divisor graph* of a commutative ring is the graph whose vertices are the nonzero zero-divisors of the ring such that distinct vertices are adjacent if and only if their product is zero. We use this construction to study the interplay between ring-theoretic and graph-theoretic properties. Of particular interest are Boolean rings and commutative rings of quotients.

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Chapter 1

Introduction

During the last century, graph theory has been found to have applications in many areas. Physics, chemistry, computer science, and the behavioral sciences have all made use of graphs. More recently, graph theory has emerged as a model for studying the zero-divisor structure of a commutative ring. Specifically, the nonzero zero-divisors of a ring constitute vertices of a graph, where distinct vertices are adjacent if and only if their product is zero. The result is a simple connected graph, called the *zero-divisor graph*.

This dissertation utilizes the above construction to study the interplay between ring-theoretic and graph-theoretic properties. Given a zero-divisor graph, one can investigate what ring-theoretic properties can be associated to the given graph. In Chapter 2, graph-theoretic criteria are given which completely characterize zero-divisor graphs of Boolean rings. On the other hand, one can start with an arbitrary graph and investigate whether the graph is realizable as the zero-divisor graph of a ring. In Chapter 3, an algorithm is given to characterize graphs that are realizable as zero-divisor graphs of direct products of integral domains. Moreover, it is determined when a graph is realizable as the zero-divisor graph of a Boolean ring. This is accomplished by defining a partial order on the vertices of a graph and providing graph-theoretic conditions which make the partial ordering a Boolean algebra. The correspondence between Boolean algebras and Boolean rings is then used to show that any graph satisfying the given conditions is the zero-divisor graph of a Boolean ring. This correspondence is used in Chapter 6 to provide a graph-theoretic characterization of complete Boolean algebras.

Of great interest are particular extensions of total quotient rings known as *rings of quotients*. The elements of such extensions can be described as equivalence classes of homomorphisms (in the sense of modules) defined on ideals without nonzero annihilators. Some of the main results of Chapters 2, 4, and 6 determine when the zero-divisor graph of a ring is isomorphic to the zero-divisor graphs of its rings of quotients. A graph-theoretic condition is introduced in Chapter 2 which characterizes the zero-divisor graphs of *complete* rings of quotients of Boolean rings. In Chapter 4, a similar characterization is given for certain rings without nonzero nilpotents. These results are generalized to arbitrary commutative rings in Chapter 6, where sufficient conditions are given for determining when an isomorphism exists between the zero-divisor graph of any ring and that of any of its rings of quotients.

The results of Chapter 5 are purely ring-theoretic. They examine the behavior of rings of quotients with respect to iteration and direct products. Furthermore, the investigation in Chapter 4 yields set-theoretic results on Boolean algebras. Specifically, it is proved that the cardinality of the set of partitions of an infinite complete Boolean algebra does not exceed the cardinality of the Boolean algebra. Throughout, a significant amount of attention is given to examples which illustrate the implications of each result.

The results of this dissertation are numbered in the order that they appear in the chapter. Each chapter is an article with its own bibliography. Moreover, chapters are arranged in the order that they were submitted for publication. Incidentally, this arrangement is logical in the sense that results and definitions become increasingly more general.

Chapter 2

Complemented Zero-Divisor Graphs and Boolean Rings

Abstract. For a commutative ring R, the zero-divisor graph of R is the graph whose vertices are the nonzero zero-divisors of R such that the vertices x and y are adjacent if and only if $xy = 0$. In this paper, we classify the zero-divisor graphs of Boolean rings, as well as those of Boolean rings that are rationally complete. We also provide a complete list of those rings whose zero-divisor graphs have the property that every vertex is either an end or adjacent to an end.

2.1 Introduction

The idea of a zero-divisor graph was introduced by I. Beck in [3]. While he was mainly interested in colorings, we shall investigate the interplay between ring-theoretic properties and graph-theoretic properties. This approach begun in a paper by D. F. Anderson and P. S. Livingston [2], and has since continued to evolve (e.g., [2], [1], [5], [7], [13], [12], and [16]). For example, in [2, Theorem 2.3], it is shown that every zero-divisor graph is *connected* (i.e., there is a path between any two vertices) and that the *distance* between any two vertices is at most three (i.e., any two vertices can be joined by less than four edges). In particular, every vertex of a zero-divisor graph is adjacent to some other vertex if and only if the zero-divisor graph has at least two vertices.

Throughout, R will always be a commutative ring with $1 \neq 0$. Let $Z(R)$ denote the set of zero-divisors of R and $T(R) = R_{R\backslash Z(R)}$ its total quotient ring. As in [2], we define $\Gamma(R)$ to be the (undirected) graph with vertices $V(\Gamma(R)) = Z(R)\backslash\{0\}$, such that distinct $v_1, v_2 \in V(\Gamma(R))$ are adjacent if and only if $v_1v_2 = 0$. Note that $\Gamma(R)$ is the empty graph if and only if R is an integral domain. Moreover, a nonempty $\Gamma(R)$ is finite if and only if R is finite and not a field [2, Theorem 2.2].

We will call a ring *reduced* if $\mathrm{nil}(R) = (0)$. A ring R with $1 \neq 0$ is *Boolean* if $x^2 = x$ for all $x \in R$. It is well known that Boolean rings are commutative and reduced with characteristic 2. Moreover, $R\backslash Z(R) = \{1\}$, and thus $V(\Gamma(R)) = R\backslash\{0, 1\}$, whenever R is Boolean. A commutative ring R with $1 \neq 0$ is *von Neumann regular* if for each $x \in R$, there is a $y \in R$ such that $x = x^2y$ or, equivalently, R is reduced and zero-dimensional [6, Theorem 3.1]. Clearly a Boolean ring is von Neumann regular, but not conversely. For example, let $\{F_i\}_{i\in I}$ be a family of fields. Then $\prod_{i\in I} F_i$ is always von Neumann regular, but it is Boolean if and only if $F_i \cong \mathbb{Z}_2$ for all $i \in I$.

Let Γ be a graph and let $v \in V(\Gamma)$. As in [1], $w \in V(\Gamma)$ is called a *complement* of v if v is adjacent to w, and no vertex is adjacent to both v and w, i.e., the edge $v - w$ is not an edge of any triangle in Γ. In such a case, we write $v\perp w$. In ring-theoretic terms, this is the same as saying $v\perp w$ in $\Gamma(R)$ if and only if $0 \neq v, w \in R$ are distinct, $vw = 0$, and $\mathrm{ann}(v)\cap\mathrm{ann}(w) \subseteq \{0, v, w\}$. Moreover, we will follow the authors in [1] and say that Γ is *complemented* if every vertex has a complement, and is *uniquely complemented* if it is complemented and any two complements of a vertex are adjacent to the same vertices. From [1, Theorems 3.5 and 3.9], we know that $\Gamma(R)$ is uniquely complemented if and only if either R is nonreduced and $\Gamma(R)$ is a *star graph* (i.e., a graph with at least two vertices such that there exists a vertex which is adjacent to every other vertex, and these are the only adjacency relations), or R is reduced and $T(R)$ is von Neumann regular. While, in the reduced case, this hypothesis yields information about the total quotient ring of R, a slightly stronger assumption on $\Gamma(R)$ will reveal information about R (see Theorem 2.5). Moreover, a stronger assumption is necessary since the relation $\Gamma(R) \cong \Gamma(T(R))$ [1, Theorem 2.2] implies that the zero-divisor structure of a ring will not detect the von Neumann regular property. For example, let $R = \mathbb{Z} \times \mathbb{Z}$. Then R is not von Neumann regular, but $T(R) \cong \mathbb{Q} \times \mathbb{Q}$ *is* von Neumann regular and $\Gamma(R) \cong \Gamma(T(R))$.

Let $B(R) = \{e \in R : e^2 = e\}$, the set of idempotents of R. Then the relation "$\leq$" defined by $a \leq b$ if and only if $ab = a$ partially orders $B(R)$, and makes $B(R)$ a Boolean algebra with inf as multiplication in R, 1 as the largest element, 0 as the smallest element, and complementation defined by $a' = 1 - a$. One checks that $a \vee b = (a' \wedge b')' = a + b - ab$, where "$+$" is addition in R. For a reference on the Boolean algebra of idempotents, see [11].

An ideal D of a ring R is called *dense* if $r \in R$ with $rD = \{0\}$ implies $r = 0$. Let D_1 and D_2 be dense ideals of R and let $f_i \in \mathrm{Hom}_R(D_i, R)$ $(i = 1, 2)$. To define $Q(R)$, the *complete ring of quotients of* R, note that $f_1 + f_2$ is an R-module homomorphism on the dense ideal $D_1 \cap D_2$, and $f_1 \circ f_2$ is an R-module homomorphism on the dense ideal $f_2^{-1}(D_1) = \{r \in R : f_2(r) \in D_1\}$. Then $Q(R) = F/\sim$ is a commutative ring, where $F = \{f \in \mathrm{Hom}_R(D, R) : D \subseteq R \text{ is a dense ideal}\}$ and $\sim$ is the equivalence relation defined by $f_1 \sim f_2$ if and only if there exists a dense ideal $D \subseteq R$ such that $f_1(d) = f_2(d)$ for all $d \in D$; we will denote the equivalence class of f by $[f]$. For all $a/b \in T(R)$, the ideal bR of R is dense and $f_{a/b} \in \mathrm{Hom}_R(bR, R)$, where $f_{a/b}(br) = ar$. One checks that the mapping $a/b \mapsto [f_{a/b}]$ is a ring monomorphism, and that $[f_0]$ and $[f_1]$ are the additive and multiplicative identities of $Q(R)$, respectively. However, this mapping need not be onto. Moreover, unlike the case for $T(R)$, it may happen that $\Gamma(R) \ncong \Gamma(Q(R))$ (e.g., Example 2.12). For $S \subseteq T(R)$, let $[S]$ denote the image of S in $Q(R)$. A ring R is called *rationally complete* if $[R] = Q(R)$ (i.e., if $r \mapsto [f_r]$ is an isomorphism). Note that $Q(R)$ is von Neumann regular if and only if R is reduced [11, Proposition 2.4.1]. Thus every reduced rationally complete ring is von Neumann regular. Also, $Q(R)$ is Boolean if and only if R is Boolean [11, Lemma 2.4.4]. It is a gentle exercise to show that $Q(R) = \varinjlim(\mathrm{Hom}_R(D, R))$, where the direct limit is taken over the family of *all* dense ideals of R, with $D_1 \leq D_2$ if and only if $D_2 \subseteq D_1$, and $f \mapsto f|_{D_2}$ whenever $f \in \mathrm{Hom}_R(D_1, R)$ with $D_1 \leq D_2$ [4, 1.7]. This is the definition of $Q(R)$ used in [9]. A detailed exposition of $Q(R)$ can be found in [11].

In this paper, we continue the investigation in [1] of complemented zero-divisor graphs. We shall investigate the zero-divisor structure of Boolean rings, as well as ring-theoretic properties that arise when a ring has a specifically classified zero-divisor graph. In Section 2.2, we see that if R is Boolean, then the atoms of $B(R)$ are precisely the elements of $V(\Gamma(R))$ that are adjacent to an end (an *end* being a vertex that is adjacent to precisely one other vertex). Also, we show that every vertex of the zero-divisor graph of a Boolean ring (not isomorphic to $\mathbb{Z}_2$) has a unique complement. By excluding the rings $\mathbb{Z}_9$ and $\mathbb{Z}_3[X]/(X^2)$, we see that the converse is also true. In Section 2.3, we describe the elements that belong to the rational completion $Q(R)$ of a Boolean ring R, but not R, whenever R is *not* rationally complete. Also, we show that a reduced rationally complete ring has the property that every nonzero annihilator ideal I contains an element with complements in $\Gamma(R)$ that annihilate I. Moreover, this property is sufficient to conclude that a Boolean ring is rationally complete. However, there are rings whose zero-divisor structures do not detect rational completeness (see Example 2.10). In Section 2.4, we observe that a zero-divisor graph has a complete subgraph that contains every vertex adjacent to an end, and give both a graph-theoretic and a ring-theoretic proof. Also, we provide a complete list of those rings whose zero-divisor graphs have the property that every vertex is either an end, or adjacent to an end. In particular, unless $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, the zero-divisor graph $\Gamma(R)$ has a vertex of a cycle that is not adjacent to an end if and only if $\Gamma(R)$ contains a cycle.

2.2 The Zero-Divisor Graph of a Boolean Ring

The goal of this section is to characterize the zero-divisor graph of a Boolean ring. Recall that an *atom* in a Boolean algebra B is an element $0 \neq a \in B$ such that $0 \neq b \in B$ with $b \leq a$ implies $b = a$. Then in $B(R)$, the Boolean algebra of idempotents, $0 \neq a \in B(R)$ is an atom if and only if $b = a$ whenever $0 \neq b \in B(R)$ with $ba = b$. We shall call a vertex $v \in \Gamma$ an *end* if v is adjacent to exactly one vertex. Thus, in ring-theoretic terms, $v \in R$ is an end if and only if v is nonzero and $\mathrm{ann}(v) \setminus \{0, v\} = \{w\}$ for some $w \in R$. Note that if R is a Boolean ring, then $R = B(R)$ (as sets).

Lemma 2.1. *Let R be a Boolean ring. An element $1 \neq b \in B(R)$ is an atom if and only if $1 - b$ is the unique end adjacent to b in $\Gamma(R)$.*

Proof. Suppose that $1 \neq b$ is an atom in $B(R)$. If $1 - b \neq x \in R$ with $xb = 0$, then $1 - b - x \neq 0$ with $(1 - b - x)x = 0 = (1 - b - x)b$. Since R is reduced, $1 - b - x \notin \{0, x, b\}$, and hence x is not an end. So if $1 - b$ is an end, then it is the unique end that is adjacent to b. But if $x \neq 0$ with $x(1 - b) = 0$, then $x = bx$. Thus $x \leq b$, and hence $x = b$ since b is an atom. So $1 - b$ is an end.

Conversely, suppose that $1 - b$ is the unique end adjacent to b. If $0 \neq x \in B(R)$ with $x \leq b$, then $x = bx$, and hence $x(1 - b) = 0$. Thus $x \in \{0, b, 1 - b\}$, and hence $x = b$ since R is reduced. Thus b is an atom. □

Theorem 2.2. *Let $R \ncong \mathbb{Z}_2$ be a Boolean ring. Then the atoms of $B(R)$ are precisely the elements of $V(\Gamma(R))$ that are adjacent to an end.*

Proof. The atoms of $B(R)$ are adjacent to an end by Lemma 2.1. If x is an end and $b \neq 0$ with $bx = 0$, then $x(1 - x) = 0$ implies $b = 1 - x$ (R is reduced), and thus $x = 1 - b$. So $1 - b$ is the unique end which is adjacent to b, and hence b is an atom by Lemma 2.1. □

Lemma 2.3. *Let $R \ncong \mathbb{Z}_2$ be a Boolean ring. Then every element of $V(\Gamma(R))$ has a unique complement.*

Proof. Let $r \in R \setminus \{0, 1\}$. Note that $r(1 - r) = 0$. If $rt = 0 = (1 - r)t$, then $t = rt = 0$. Hence $r \perp (1 - r)$. Suppose that $r \perp t$. Then $t(1 - r - t) = 0 = r(1 - r - t)$ implies $1 - r - t = 0$; that is, $t = 1 - r$. □

Remark 2.4. *Suppose that R is a ring with nonzero zero-divisors such that every element of $V(\Gamma(R))$ has a unique complement. In particular, R is uniquely complemented. Clearly either $|V(\Gamma(R))| = 2$ or $\Gamma(R)$ is not a star graph. In the former case, R is isomorphic to one of the rings in the set $\{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2)\}$ [2, Theorem 3.2]. In the latter, $T(R)$ is von Neumann regular [1, Corollary 3.10]. Therefore, if $r \in T(R)$, say $r = r^2t$ for some $t \in R$, then rt is idempotent and $ann(r) = ann(rt)$. Also, if $e \in T(R)$ is idempotent with $ann(e) = ann(rt)$, then $1 - rt \in ann(e)$ and $1 - e \in ann(rt)$ implies $e = ert = rt$. Thus, for all $r \in T(R)$, there is a unique idempotent e of $T(R)$ with $ann(r) = ann(e)$ (c.f. the discussion prior to Theorem 4.1 in [1]). But every element of $V(\Gamma(T(R)))$ has a unique complement since $\Gamma(R) \cong \Gamma(T(R))$ [1, Theorem 2.2], and thus $ann(r) = ann(e)$ if and only if $r = e$. Hence every element of $T(R)$ is idempotent. Thus $T(R)$ is Boolean, and hence so is R $(= T(R))$.*

Although the previous argument is more compact, we shall provide an elementary proof, independent of the above remark. Note that we have omitted the rings $\mathbb{Z}_9$ and $\mathbb{Z}_3[X]/(X^2)$ below since their zero-divisor graphs are isomorphic to $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$, the zero-divisor graph of a Boolean ring.

Theorem 2.5. *A ring R is Boolean if and only if either $R \cong \mathbb{Z}_2$, or $\Gamma(R)$ is not the empty graph, $R \notin \{\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2)\}$, and $\Gamma(R)$ has the property that every vertex has a unique complement. In particular, if $|V(\Gamma(R))| \geq 3$, then R is Boolean if and only if every vertex of $\Gamma(R)$ has a unique complement.*

Proof. If R is Boolean, then the stated conditions follow from Lemma 2.3. To prove the converse, we first show that $R \setminus Z(R) = \{1\}$. Suppose not; say $1 \neq r \in R \setminus Z(R)$. Then $R \ncong \mathbb{Z}_2$, and thus $\Gamma(R)$ is not the empty graph by hypothesis. Also, there is a $g \in V(\Gamma(R))$ and $k \in V(\Gamma(R)) \setminus \{0, g\}$ with $g \perp k$. Note that $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ satisfies $R \setminus Z(R) = \{1\}$, and so we can assume that $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Case 1: Suppose that $rg = g$. Note that $rk \neq g$ since otherwise $rk = rg$ so that $r(k - g) = 0$; this cannot happen since $k \neq g$. Suppose that $rk \neq k$. So $rk \notin \{0, g, k\}$ and $rk \in Z(R)$ since $rkg = 0$. Then, by uniqueness of complements, there is a $t \in V(\Gamma(R)) \setminus \{0, g, rk\}$ with $tg = 0 = trk$. Also, $t \neq k$ since $g \perp k$. But $trk = 0$ implies $tk = 0$, a contradiction. Thus we may assume $rk = k$. Then $k(r - 1) = 0 =$

$g(r-1)$. Since $k \perp g$ and $r \neq 1$, either $r-1=k$ or $r-1=g$. If $r-1=k$, then $k^2=(r-1)k=0$. If $k+g=0$, then $k=-g$, and thus $|V(\Gamma(R))|=2$ since $\Gamma(R)$ is connected, $k \perp g$, and $\operatorname{ann}(k)=\operatorname{ann}(g)$. Then, as in Remark 2.4, R is isomorphic to one of three rings, two of which are excluded by hypothesis. Hence, in this case, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, a contradiction. Then we have $k+g \neq 0$, and thus $k+g \notin \{0,k,g\}$ since $k \neq 0 \neq g$. Thus $k(k+g)=0$ and uniqueness of complements implies there is a $t \in V(\Gamma(R)) \setminus \{0,k,k+g\}$ with $tk=0$ and $t(k+g)=0$. Also, $t \neq g$ since $g \perp k$. But then $tg=tg+tk=t(g+k)=0$ contradicts $g \perp k$. Thus $r-1 \neq k$, and a symmetric argument shows $r-1 \neq g$. Hence *case 1* cannot happen, and so we may assume
Case 2: Suppose that $rg \neq g$ for all $r \in R \setminus \big(Z(R) \cup \{1\}\big)$. Suppose that $rg \neq k$. Then $rg \notin \{0,g,k\}$ and $rgk=0$. Since complements are unique, there is a $t \in V(\Gamma(R)) \setminus \{0,rg,k\}$ with $trg=0=tk$. Also, $t \neq g$ since $g \perp k$. But $trg=0$ implies $tg=0$, contradicting $g \perp k$. So assume $rg=k$. As before, we can assume $|V(\Gamma(R))| \neq 2$. Then, since $\Gamma(R)$ is connected, there is a $t \in V(\Gamma(R)) \setminus \{0,g,k\}$ with either $tg=0$ or $tk=0$. If $tg=0$, then $tk=trg=0$, contradicting $g \perp k$. If $tk=0$, then $trg=tk=0$ implies that $tg=0$, contradicting $g \perp k$. Now all possibilities have been exhausted. Therefore $R \setminus Z(R)=\{1\}$.

It now follows that R is reduced since $r \neq 0$ with $r^n=0$ implies that $1+r$ is a unit, and hence $1+r \in R \setminus Z(R)$ with $1+r \neq 1$, a contradiction. Choose $g \in R \setminus \{0,1\}$; say $g \perp k$ for some $k \in V(\Gamma(R))$. Note that $g^2 \neq 0$ and $g^2k=0$. Also, $g^2 \neq k$ since otherwise $k^2=0$, a contradiction. Then if $g^2 \neq g$, uniqueness of complements implies that there is a $t \in V(\Gamma(R)) \setminus \{0,g^2,k\}$ with $tg^2=0=tk$. But $g^2 \neq 0$ and $k^2 \neq 0$ implies that $tg \notin \{g,k\}$. Also, $tgg=0=tgk$. Hence $g \perp k$ implies $tg=0$. But $\operatorname{nil}(R)=(0)$ implies that $t \neq g$, and this contradicts $g \perp k$. So $g^2=g$, and we have that R is Boolean.

The "in particular" statement in the theorem is clear. □

2.3 Rationally Complete Boolean Rings

This section contains two main results: Proposition 2.6 examines the nature of an element of $Q(R) \setminus [R]$ whenever R is a Boolean ring which is not rationally complete, and Theorem 2.9 characterizes the zero-divisor graphs of rationally complete Boolean rings. For a set $S \subseteq R$, let $\operatorname{ann}_R(S)=\{r \in R : rs=0 \text{ for all } s \in S\}$. When R is reduced,

$I + \mathrm{ann}_R(I)$ is dense for any ideal $I \subseteq R$. Note that if D is dense in R, then its image $[D]$ in $Q(R)$ is dense (as a set) in $Q(R)$. To see this, suppose that D' is a dense ideal of R and $f \in \mathrm{Hom}_R(D', R)$ such that $[f] \neq [f_0] = \mathbf{0}$. Then there is a $d' \in D'$ such that $f(d') \neq 0$. Therefore, since D is dense, there is a $d \in D$ such that $f(dd') = f(d')d \neq 0$. It follows that

$$[f][f_d][f_{d'}] = [f \circ f_d \circ f_{d'}] = [f_{f(dd')}] \neq \mathbf{0}.$$

Thus $[f][f_d] \neq \mathbf{0}$ and hence $[f][D] \neq \{\mathbf{0}\}$. Since $[f] \in Q(R)$ was chosen arbitrarily, $[D]$ is dense in $Q(R)$. Recall that a Boolean algebra is *complete* if every subset has an infimum. A Boolean ring R is rationally complete if and only if $B(R)$ (and hence $B([R])$) is complete [9, Theorem 12.3.4]. It is well known that every Boolean algebra B is a subalgebra of a complete Boolean algebra $D(B)$, where the infimum of a set in B (when it exists) is the same as its infimum in $D(B)$. Here, $D(B)$ is the "so called" *Dedekind-MacNeille* completion of B [11, c.f. Section 2.4].

Proposition 2.6. *Let R be a Boolean ring. Then R is not rationally complete if and only if there exists a nonempty family $S \subseteq B(R)$ such that $[f] = \inf[S] \in B(Q(R)) \setminus B([R])$, where $f \in Hom_R(D, R)$ with $D = \sum_{s\in S} R(1-s) + ann_R\big(\sum_{s\in S} R(1-s)\big)$ is defined by $f(r_1+r_2) = r_2$ for all $r_1 \in \sum_{s\in S} R(1-s)$ and $r_2 \in ann_R\big(\sum_{s\in S} R(1-s)\big)$. Moreover, this property characterizes every element of $Q(R) \setminus [R]$.*

Proof. Note that D is a dense ideal of R and that the function f is well-defined since $\big(\sum_{s\in S} R(1-s)\big) \cap \mathrm{ann}_R\big(\sum_{s\in S} R(1-s)\big) = (0)$. The given conditions are sufficient to conclude that R is not rationally complete since $B(Q(R)) \setminus B([R]) \neq \emptyset$ means $Q(R) \setminus [R] \neq \emptyset$. To prove that these conditions are necessary, suppose that R is not rationally complete. Then $B([R])$ is not complete, and hence there is an $S \subseteq B(R)$ with $[\overline{f}] = \inf[S] \in B(Q(R)) \setminus B([R])$ for some $\overline{f} \in F$ (indeed, $Q(R)$ is rationally complete [11, Proposition 2.3.5]). Let D and f be as in the statement of the proposition. Since $[\overline{f}] \leq [f_s]$ for all $s \in S$, we have $[\overline{f}][f_{1-s}] = [\overline{f}]([f_1] - [f_s]) = [f_0] = \mathbf{0}$ for all $s \in S$, and therefore $[\overline{f}]\big[\sum_{s\in S} R(1-s)\big] = \{\mathbf{0}\}$. But $[\overline{f}] \notin B([R])$ implies $[\overline{f}] \neq \mathbf{0}$; thus $\mathrm{ann}_R\big(\sum_{s\in S} R(1-s)\big) \neq (0)$ since D is a dense ideal of R. Let $0 \neq r \in \mathrm{ann}_R\big(\sum_{s\in S} R(1-s)\big)$. Then $[f_r] \leq [f_s]$ for all $s \in S$, and hence

$[f_r] \leq [\overline{f}]$. That is, $[\overline{f}][f_r] = [f_r]$. It follows that $([f] - [\overline{f}])[D] = \{\mathbf{0}\}$, and therefore $[f] - [\overline{f}] = \mathbf{0}$ since $[D]$ is dense, i.e., $[\overline{f}] = [f]$.

The "moreover" statement of the proposition follows since $B(Q(R)) \cong D(B([R]))$, and every element of $D(B([R]))$ is the infimum of some $[S] \subseteq B([R])$ [11, Proposition 2.4.5 and the corollary to Proposition 2.4.6]. □

In [4, Theorem 10.9], it is shown that $B(R)$ is complete whenever R is reduced and rationally complete. Alternatively, this fact can be established by noting that the "sufficiency" portion of the above proof only requires that R be reduced.

Since there is no longer any danger of losing precision, we shall, for the remainder of this section, identify R with its image in $Q(R)$. The following statement follows from the proof of Proposition 2.6.

Corollary 2.7. *Let R be a Boolean ring. If $S \subseteq B(R)$ has no infimum, then $ann_R(\{1 - s\}_{s \in S}) \neq (0)$.*

Let R be a von Neumann regular ring. If $x \in R$, say $x = x^2y$ for some $y \in R$, then $e_x = xy \in B(R)$. Clearly $\mathrm{ann}(x) = \mathrm{ann}(e_x)$ for all $x \in R$. Also, $(1 - e_x) \perp x$ since $(1 - e_x)x = 0$ and $tx = 0 = t(1 - e_x)$ implies $t = t(xy) = 0$. By [1, Theorem 3.5], $\Gamma(R)$ is uniquely complemented. Thus $\mathrm{ann}(x') = \mathrm{ann}(1 - e_x)$ for every complement x' of x.

If $S \subseteq V(\Gamma(R))$ is a family of vertices, we shall call v a *central vertex of S* if v is adjacent to s for all $s \in S$. Recall that a reduced rationally complete ring is von Neumann regular, and thus its zero-divisor graph is uniquely complemented.

Lemma 2.8. *Let R be a reduced rationally complete ring. If a nonempty set $S \subseteq V(\Gamma(R))$ has a central vertex, then there is a central vertex v of S that possesses a complement adjacent to every central vertex of S (and hence, since $\Gamma(R)$ is uniquely complemented, every complement of v is adjacent to every central vertex of S).*

Proof. Suppose that the stated conditions fail for some reduced rationally complete (hence von Neumann regular) ring R; that is, suppose that there is a $\emptyset \neq S \subseteq V(\Gamma(R))$ with central vertices such that, if v is any central vertex of S, then there exists a central vertex w of S with $(1 - e_v)w \neq 0$. Let $S' = \{1 - e_s \in B(R) : s \in S\}$, and let

$C = \{b \in B(R) \setminus \{0\} : be = b \text{ for all } e \in S'\}$. Note that $C \neq \emptyset$ since if v is any central vertex of S, then $e_v \in C$. Also, every element of C is a central vertex of S. Thus, to every $b \in C$ there corresponds a central vertex w of S such that $(1-b)w \neq 0$; hence $(1-b)e_w \neq 0$. Let $f = \inf S'$ (in $D(B(R))$). Note that $C \neq \emptyset$ forces $f \neq 0$ since $b \in C$ means $b \leq e$ for all $e \in S'$. So if $f \in B(R)$, then $f \in C$, and hence there is a central vertex w of S such that $fe_w \neq e_w$. Since $e_w \in C$, this contradicts $e_w \leq f$. Hence $f \notin B(R)$. Since the infimum of a set taken in $B(R)$ agrees with the infimum taken in $D(B(R))$, we have that $B(R)$ is not complete. Thus, by the comments that follow the proof of Proposition 2.6, R is not rationally complete. □

Note that the "reduced" hypothesis cannot be dropped from Lemma 2.8. For example, consider the ring $R = \mathbb{F}_4[X]/(X^2)$. Then R is rationally complete (see the first paragraph of the proof of Theorem 2.9), but $\Gamma(R)$ is the complete graph on three vertices. In particular, complements do not exist in $\Gamma(R)$.

Recall from Theorem 2.5 that every vertex v of the zero-divisor graph of a Boolean ring has a unique complement, namely, $1-v$. Thus half of the following theorem is immediate.

Theorem 2.9. *Let R be a Boolean ring. Then R is rationally complete if and only if whenever $\emptyset \neq S \subseteq V(\Gamma(R))$ is a family of vertices that has a central vertex, there exists a central vertex of S whose complement is adjacent to all of the central vertices of S.*

Put more formally, Theorem 2.9 says that a Boolean ring R is rationally complete if and only if whenever S is a family of vertices with $C = \{v \in V(\Gamma(R)) : v \text{ is adjacent to } s \text{ for all } s \in S\} \neq \emptyset$, there is a $v^* \in C$ such that the complement $1 - v^*$ of v^* is adjacent to every $v \in C$.

Proof. Note that the only dense ideal of a finite ring R is R itself (e.g., [9, Theorem 80]), and every R-module homomorphism $f \in \mathrm{Hom}_R(R, R)$ is determined by $f(1)$; that is, $f \sim f_{f(1)}$. So every finite ring is rationally complete. Therefore Theorem 2.9 holds for $\mathbb{Z}_2$ vacuously. Let us now assume that R is a Boolean ring which is not isomorphic to $\mathbb{Z}_2$; in particular, $V(\Gamma(R)) \neq \emptyset$.

Suppose that R is rationally complete. Since Boolean rings are reduced, the stated conditions hold by Lemma 2.8. Conversely, suppose that the stated conditions on $V(\Gamma(R))$ are satisfied. Let $\emptyset \neq S \subseteq R$ be any family of elements. In $B(R)$, it is clear that $\inf S = 0$ if $0 \in S$. Suppose that $0 \notin S$. If $S = \{1\}$, then $\inf S = 1$. If $S \neq \{1\}$ and contains 1, then we may remove 1 from S without changing $\inf S$. Thus we may assume $0, 1 \notin S$. If S has no infimum, then $C = \{v \in V(\Gamma(R)) : v$ is adjacent to $1 - s$ for all $s \in S\} \neq \emptyset$ by Corollary 2.7. In this case, by hypothesis, there is a $v^* \in C$ such that $1 - v^*$ is adjacent to v for all $v \in C$. Since $v^* \in C$, we have $v^*(1-s) = 0$ for all $s \in S$; that is, $v^* \leq s$ for all $s \in S$. Moreover, if $v \leq s$ for all $s \in S$, then $v \in C$ so that $v(1-v^*) = 0$; that is, $v \leq v^*$. But this shows that $\inf S = v^* \in B(R)$, a contradiction. Thus every $\emptyset \neq S \subseteq R$ has an infimum, and hence $B(R)$ is a complete Boolean algebra. Therefore R is rationally complete by [9, Theorem 12.3.4]. $\square$

We conclude this section with three examples: Example 2.10 shows that a von Neumann regular ring may satisfy the condition of Lemma 2.8 without being rationally complete. Moreover, in contrast to Boolean rings, rationally complete von Neumann regular rings cannot be characterized in terms of their zero-divisor graphs. Examples 2.11 and 2.12 illustrate the necessity and sufficiency, respectfully, of the condition stated in Theorem 2.9. Also, if R is the ring defined in Example 2.12, then $\Gamma(R) \ncong \Gamma(Q(R))$ by Theorem 2.9. Alternatively, suppose that R is a reduced total quotient ring which is not von Neumann regular (e.g., [6, Example 6]). Then $\Gamma(R)$ is *not* uniquely complemented. However, $Q(R)$ is von Neumann regular, and hence $\Gamma(Q(R))$ *is* uniquely complemented [1, Theorem 3.5]. Therefore, $\Gamma(R) \ncong \Gamma(Q(R))$.

Example 2.10. *Let Δ denote the space of real numbers endowed with the discrete topology, and let $C(\Delta)$ be the usual ring of real-valued continuous function on Δ (in this case, the ring of all real-valued functions on Δ). Let $F(\Delta) = \{f \in C(\Delta) : f(\Delta)$ is finite$\}$. Clearly $F(\Delta)$ is a von Neumann regular ring, but is not Boolean. By [4, 4.3], $Q(F(\Delta)) \cong C(\Delta)$. Moreover, $F(\Delta)$ is not rationally complete. To see this, let $D = \sum_{i \in \Delta} e_i F(\Delta)$, where $e_i : \Delta \to \mathbb{R}$ is defined by*

$$e_i(j) = \left\{ \begin{array}{ll} 1, & i = j \\ 0, & i \neq j \end{array} \right. .$$

Clearly D is dense in $F(\Delta)$. Note that every nonzero proper ideal of $F(\Delta)$ contained in D is of the form $\sum_{i \in I} e_i F(\Delta)$, where I is a *nonempty* proper *subset of* Δ*. In particular,* D *does not properly contain any dense ideal of* $F(\Delta)$*. Therefore, we will have that* $F(\Delta)$ *is not rationally complete if we can produce an element* $\varphi \in Hom_{F(\Delta)}(D, F(\Delta))$ *such that, for all* $f \in F(\Delta)$*, there exists a* $d \in D$ *with* $\varphi(d) \neq fd$*. But one can easily check that the homomorphism* $\varphi : D \to F(\Delta)$ *defined by*

$$\varphi(e_i)(j) = \left\{ \begin{array}{ll} i, & j = i \\ 0, & j \neq i \end{array} \right.$$

is such an element (let $d = e_\alpha$*, where* $\alpha \notin f(\Delta)$*). Finally, we will show that* $F(\Delta)$ *satisfies the condition in Lemma 2.8, and that* $\Gamma(F(\Delta)) \cong \Gamma(Q(F(\Delta)))$*.*

Suppose that $\emptyset \neq S \subseteq V(\Gamma(F(\Delta)))$ *has a central vertex. Then there is an* $i \in \Delta$ *such that* $f(i) = 0$ *for all* $f \in S$*. Let* $K = \{i \in \Delta : f(i) = 0 \text{ for all } f \in S\}$*. Clearly a function* $0 \neq g \in F(\Delta)$ *is a central vertex of* S *if and only if* g *vanishes on* $\Delta \setminus K$*. Define* $g : \Delta \to \mathbb{R}$ *by*

$$g(i) = \left\{ \begin{array}{ll} 0, & i \notin K \\ 1, & i \in K \end{array} \right. .$$

Then $g \in F(\Delta)$ *is a central vertex of* S*, and* $h \in F(\Delta)$ *is a complement of* g *if and only if*

$$h(i) = \left\{ \begin{array}{ll} r_i, & i \notin K \\ 0, & i \in K \end{array} \right. ,$$

where each $r_i \in \mathbb{R} \setminus \{0\}$*. Thus every complement of* g *is adjacent to every central vertex of* S*.*

Finally, $\Gamma(F(\Delta)) \cong \Gamma(Q(F(\Delta)))$ *by [1, Theorem 4.1] since* $B(F(\Delta)) = B(C(\Delta))$ *and* $|\{f \in F(\Delta) : f(i) = 0 \Leftrightarrow i \in K\}| = |\{f \in C(\Delta) : f(i) = 0 \Leftrightarrow i \in K\}|$ *for every* $K \subseteq \Delta$*.*

Example 2.11. *Let* R *be the Boolean ring* $\prod_{i \in I} \mathbb{Z}_2$ *for some nonempty indexing set* I*. Since* $\mathbb{Z}_2$ *is rationally complete, so is* R *[11, Proposition 2.3.8]. (Alternatively, it is easy to show that* $B(R)$ *is complete.) We will show that the conditions given in Theorem 2.9 are satisfied.*

Suppose that $S = \{(r_{ij})_{i \in I}\}_{j \in J} \subseteq V(\Gamma(R))$ *has a central vertex. Then there is an* $i \in I$ *such that* $r_{ij} = 0$ *for all* $j \in J$*. Let* $I^* = \{i \in I : r_{ij} = 0 \text{ for all } j \in J\}$*, and let* $v = (v_i)$*, where*

$$v_i = \left\{ \begin{array}{ll} 0, & i \notin I^* \\ 1, & i \in I^* \end{array} \right. .$$

If $w = (w_i)$ is any central vertex of S, then $w_i = 0$ for all $i \in I \setminus I^*$. Then since $1 - v_i = 0$ for all $i \in I^*$, we have $(\mathbf{1}_R - v)w = \mathbf{0}_R$ for every central vertex w of S; that is, v is a central vertex of S whose complement is adjacent to every central vertex of S.

Example 2.12. *Let $R = \{N \subseteq \mathbb{N} : |N| < \infty$ or $|\mathbb{N} \setminus N| < \infty\}$, where $\mathbb{N}$ is the set of natural numbers. It is well known that R is a Boolean ring with addition defined as "symmetric difference" and multiplication defined as "intersection." Moreover, $B(R)$ is not complete: Let $S = \{s_n\}_{n=1}^{\infty}$, where $s_n = \mathbb{N} \setminus \{2i\}_{i=1}^{n}$, and let $f = \{2i-1\}_{i=1}^{\infty}$. Then inf $S = f \notin B(R)$ (note that f and S are the same as in Proposition 2.6). Since $B(R)$ is not complete, R is not rationally complete [9, Theorem 12.3.4]. Let $S' = \left\{\{2i\}_{i=1}^{n}\right\}_{n=1}^{\infty}$. Clearly $S' \subseteq V(\Gamma(R))$, and $C = \left\{\{2i-1\}_{i \in I} : I \subseteq \mathbb{N}\text{ with } |I| < \infty\right\}$ is the set of central vertices of S. But for any $v \in C$, $\mathbf{1}_R - v$ (i.e., $\mathbb{N} \setminus v$) contains $2i-1$ for some $i \in \mathbb{N}$, and thus $(\mathbf{1}_R - v)\{2i-1\} \neq \mathbf{0}_R$. Since $\{2i-1\} \in C$, we have shown that R does not satisfy the conditions of Theorem 2.9.*

2.4 Zero-Divisor Graphs with Ends

The goal of this section is to answer the following question:

> *Which rings have the property that every element of $V(\Gamma(R))$ is either an end or is adjacent to an end?*

Of course, all star graphs have this property. As noted in [2, Remark 2.4], $\Gamma(R)$ is a star graph if and only if either $R \cong A$, where $A \in \{\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2), \mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_4[X]/(2X, X^2-2)\}$; $R \cong \mathbb{Z}_2 \times A$, where A is an integral domain; or there exists a $0 \neq x \in R$ such that $Z(R) = \mathrm{ann}(x)$, $\mathrm{nil}(R) = \{0, x\}$, and $R/\mathrm{nil}(R)$ is an infinite integral domain (e.g., $\mathbb{Z}[X]/(2X, X^2)$). We will see that only two other graphs that have this property are realizable as zero-divisor graphs (see Figure 2.1).

In [1, Theorem 4.1], it is shown that the zero-divisor graphs of two von Neumann regular rings R and S are isomorphic if and only if there is a Boolean algebra isomorphism $\varphi : B(R) \to B(S)$ such that $|[e]| = |[\varphi(e)]|$ for all $1 \neq e \in B(R)$, where $[e] = \{r \in R : eR = rR\}$. If one of the rings is Boolean, we can say even more:

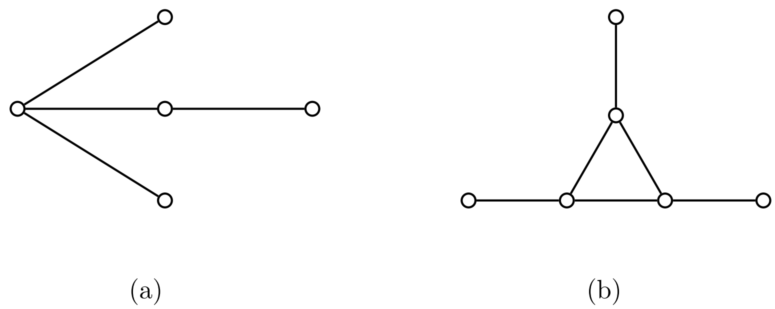

Figure 2.1: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Theorem 2.13. *Let R be a ring with nonzero zero-divisors, not isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$. If S is a Boolean ring such that $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$. In particular, if R and S are Boolean rings, then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$.*

Proof. By Theorem 2.5, we have that R is a Boolean ring. By [1, Theorem 4.1], there is a Boolean algebra isomorphism $\varphi : B(R) \to B(S)$. Since $B(R) = R$ and $B(S) = S$, the map $\varphi : R \to S$ is a bijection such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$. Note that $\varphi(1_R - a) = \varphi(a') = \varphi(a)' = 1_S - \varphi(a)$ for all $a \in R$. Since a Boolean ring has characteristic 2, we have

$$\begin{aligned}
\varphi(a+b) &= \varphi(a+b-2ab) \\
&= \varphi\big(a(1_R - b) \vee b(1_R - a)\big) \\
&= \varphi(a)\varphi(1_R - b) \vee \varphi(b)\varphi(1_R - a) \\
&= \varphi(a)\big(1_B - \varphi(b)\big) \vee \varphi(b)\big(1_B - \varphi(a)\big) \\
&= \varphi(a) + \varphi(b).
\end{aligned}$$

Therefore φ is an isomorphism of rings.

The "in particular" statement in the theorem is clear. ☐

Note that $\mathbb{Z}_2 \times \mathbb{R}$ and $\mathbb{Z}_2 \times \mathbb{C}$ are non-isomorphic von Neumann regular rings that have isomorphic zero-divisor graphs, where $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. For this, as well as examples with rings whose characteristic is finite, see [13].

For $a \in V(\Gamma(R))$, set $E(a) = \{x \in V(\Gamma(R)) \setminus \{a\} : \operatorname{ann}(x) \subseteq \{0, a, x\}\}$. Call an element of $E(a)$ an *end of* a. For $A \subseteq V(\Gamma(R))$, set $E(A) = \{x \in \Gamma(R) : x \in E(a) \text{ for some } a \in A\}$. Call an element of $E(A)$ an *end of* A. Call an element of $E(V(\Gamma(R)))$ an *end*. (This agrees with the definition that was given at the beginning of Section 2.2.) Thus, since zero-divisor graphs are connected, $x \in E(a)$ means a is the only vertex adjacent to x. It follows that if $0 \neq a, b \in Z(R)$ are distinct, then $E(a) \cap E(b) = \emptyset$. Also, $E(a) \subseteq \operatorname{ann}(ar)$ for all $r \in R$.

Suppose that $a, b \in V(\Gamma(R))$ are distinct with $E(a) \neq \emptyset \neq E(b)$; say $x \in E(a)$ and $y \in E(b)$. Then a is adjacent to b since otherwise the shortest path from x to y has at least four edges. This gives a graph-theoretic proof of the following lemma. A ring-theoretic proof is given below.

Lemma 2.14. *Let R be a ring and $\overline{V} = \{v \in V(\Gamma(R)) : E(v) \neq \emptyset\}$. Then the subgraph of $\Gamma(R)$ induced by $\overline{V}$ is complete.*

Proof. Suppose that $a, b \in \overline{V}$ are distinct. Then for all $x \in E(a)$ and $y \in E(b)$, the inclusion $E(a) \cup E(b) \subseteq \operatorname{ann}(ab)$ implies that

$$ab \in \operatorname{ann}(x) \cap \operatorname{ann}(y) \subseteq \{0, a, x\} \cap \{0, b, y\} = \{0\}.$$

Thus a and b are adjacent. □

We now provide the answer to the question posed at the beginning of this section. Observe that if R satisfies (1) or (3) below, then $\Gamma(R)$ is isomorphic to the graph in Figure 2.1(a) or Figure 2.1(b), respectively.

Theorem 2.15. *Let R be a ring with the property that every element of $V(\Gamma(R))$ is either an end or is adjacent to an end. Then exactly one of the following holds:*

(1) $R \cong A$, where $A \in \{\mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)\}$.

(2) $\Gamma(R)$ is a star graph.

(3) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Note that the hypothesis of Theorem 2.15 implies $|V(\Gamma(R))| \geq 2$. If $a, b \in V(\Gamma(R))$ are nonends with $a \perp b$, then $\{a, b\} = V(\Gamma(R)) \setminus E(V(\Gamma(R)))$; for, if c is a nonend distinct from a and b, then $ac = 0 = bc$

by Lemma 2.14, a contradiction. Then a and b are vertices that are not contained in any cycle and hence, since $E(a) \neq \emptyset \neq E(b)$ and $E(a) \cap E(b) = \emptyset$, the ring R is isomorphic to one of the rings in (1) by [16, (2.0) (ii)]. Thus a ring that satisfies the hypothesis of Theorem 2.15, but does not satisfy (1), has the property that $a \perp b$ implies either $a \in E(b)$ or $b \in E(a)$. Therefore, since the zero-divisor graph of any ring satisfying the hypothesis of Theorem 2.15 is complemented, Theorem 2.15 will be established upon proving

Theorem 2.16. *Let R be a ring such that $\Gamma(R)$ is complemented, and $a \perp b$ if and only if either $a \in E(b)$ or $b \in E(a)$. Then R satisfies one of (2) or (3) from Theorem 2.15.*

Proof. If a, b, and c are distinct vertices with $a \perp b$ and $a \perp c$, then $b, c \in E(a)$, and hence $\mathrm{ann}(b) \setminus \{b\} = \{0, a\} = \mathrm{ann}(c) \setminus \{c\}$. Therefore, $\Gamma(R)$ is uniquely complemented. By [1, Corollary 3.10], either $\Gamma(R)$ is a star graph or $T(R)$ is von Neumann regular. Suppose that $\Gamma(R)$ is not a star graph. Then $T(R)$ is von Neumann regular, and hence $\mathrm{nil}(R) = (0)$. In particular, $E(a) = \{x \in V(\Gamma(R)) : \mathrm{ann}(x) \subseteq \{0, a\}\}$. Let $a \in V(\Gamma(R))$, and let $b \in E(a)$. Then $a^2 b = 0$ implies $a^2 \in \{0, a\}$, and hence $a^2 = a$ since $\mathrm{nil}(R) = (0)$. So $V(\Gamma(R)) \setminus E(V(\Gamma(R)))$ is a set of idempotents. It now follows that $|V(\Gamma(R)) \setminus E(V(\Gamma(R)))| \leq 3$: Suppose that this claim is false, and let $a, b, c, d \in V(\Gamma(R))$ be distinct nonends. Lemma 2.14 implies that $a(c + d) = 0$; so $c + d \in Z(R)$. In fact, $c + d \in E(V(\Gamma(R)))$. If not, then $c + d \neq c$ (since $d \neq 0$) and Lemma 2.14 imply that $c^2 = c(c + d) = 0$, a contradiction. The same reasoning forces $c + d \neq 0$. But then $\{a, b\} \subseteq \mathrm{ann}(c + d)$ contradicts that $c + d$ is an end. So $\Gamma(R)$ has at most three nonends.

If $\Gamma(R)$ has precisely zero or one nonend, then clearly (2) holds. If $\Gamma(R)$ has two nonends, then (1) holds by [16, (2.0) (ii)], contradicting the condition that $a \perp b$ means either $a \in E(b)$ or $b \in E(a)$. Suppose that $|V(\Gamma(R)) \setminus E(V(\Gamma(R)))| = 3$. If a and b are distinct nonends, then $a + b \neq 1$ ($a + b \in Z(R)$ since it is annihilated by the third nonend) and, using Lemma 2.14 together with the observation that nonends are idempotent, we see that $(a + b)(1 - (a + b)) = 0$. Just as above, we have that $a + b$ is an end, and therefore $\{1 - (a + b), a, b\}$ is the set of nonends of $\Gamma(R)$. Let $c \in E(a)$. Then $c(1 - b) \in \mathrm{ann}(a + b)$, and thus $c(1 - b) = 1 - (a + b)$ since $a + b \in E(1 - (a + b))$ (note that $c(1 - b) \neq 0$ since $1 - b \in E(b)$). That is, $c(1 - b) = (1 - a)(1 - b)$, and hence $(c+a-1)(1-b)=0$. Thus $c+a-1 \in \{0,b\}$ since $1-b \in E(b)$. But $c+a-1=b$ implies that $bc+ba-b=b^2=b=-b$ ($b=-b$ since $(1-b)(b+b)=0$ implies $b+b \in \{0,b\}$). Then $bc=0$, a contradiction since $c \in E(a)$. So $c+a-1=0$, i.e., $c=1-a$. Symmetrically, $E(b)=\{1-b\}$ and $E(1-(a+b))=\{a+b\}$. Therefore, $\Gamma(R)=\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$. Thus $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by Theorem 2.13. $\square$

Corollary 2.17. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring with nonzero zero-divisors. Then $\Gamma(R)$ has a vertex of a cycle that is not adjacent to an end if and only if $\Gamma(R)$ contains a cycle.*

Proof. The necessity of the statement is trivial. Suppose that $\Gamma(R)$ contains a cycle. By contradiction, assume that every vertex of every cycle is adjacent to an end. By [16, (2.1) (i)], we have that every vertex of $\Gamma(R)$ that is not contained in a cycle must be an end. Then clearly R satisfies the hypothesis of Theorem 2.15. Therefore, since $\Gamma(R)$ contains a cycle, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, a contradiction. $\square$

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Chapter 3

On Realizing Zero-Divisor Graphs

Abstract. An algorithm is presented for constructing the zero-divisor graph of a direct product of integral domains. Moreover, graphs which are realizable as zero-divisor graphs of direct products of integral domains are classified, as well as those of Boolean rings. In particular, graphs which are realizable as zero-divisor graphs of finite reduced commutative rings are classified.

3.1 Introduction

Let R be a commutative ring (with $1 \neq 0$) and let $Z(R)$ be its set of zero-divisors. If R does not contain any nonzero nilpotents, then R will be called *reduced*. As usual, the set of positive integers will be denoted by $\mathbb{N}$, and $\mathbb{F}_q$ will be the finite field with q elements. We associate a (undirected) graph $\Gamma(R)$ to R whose vertices are the elements of $Z(R) \setminus \{0\}$, such that distinct vertices v_1 and v_2 are adjacent if and only if $v_1 v_2 = 0$. Thus, $\Gamma(R)$ is the empty graph if and only if R is an integral domain. The graph $\Gamma(R)$ is called the *zero-divisor graph of* R.

The notion of a zero-divisor graph was introduced by I. Beck in [3]. While he was mainly interested in colorings, we shall investigate the interplay between ring-theoretic and graph-theoretic properties. This approach begun in [2] and has since continued to evolve (cf. [1], [2], [1], [5], [6], [7], [11], [13], [12], [16], [14], and [9]).

A graph will be called *realizable* if it is isomorphic to $\Gamma(R)$ for some ring R. There are many results which imply that most graphs are not realizable. For example, it is known that zero-divisor graphs are *connected* with *diameter* at most three (i.e., any two vertices can be joined by three or less edges) [2, Theorem 2.3]. More generally, one uses ring-theoretic properties of a class of rings to reveal invariant graph-theoretic properties. On the other hand, there are algebraic properties of a ring that can be deduced when particular characteristics of $\Gamma(R)$ are known. For example, a reduced total quotient ring is zero-dimensional if and only if every vertex of its zero-divisor graph is incident with an edge that is *not* an edge of any triangle [1, Theorem 3.5]. However, it is easy to construct non-realizable graphs that are connected, have diameter at most three, and have the property that every vertex is incident with an edge which is not an edge of any triangle. For example, take any complete graph on n vertices where either $n = 2$ or $n \geq 4$, and assign an end to each vertex (an *end* being a vertex that is adjacent to precisely one other vertex) [11, Theorem 4.3]. On the other hand, this construction yields the zero-divisor graph of $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$ when $n = 3$.

If R and S are finite reduced rings which are not fields, then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$ [2, Theorem 4.1]. This result fails for infinite reduced rings. For example, in [1, Theorem 2.1] it is shown that if $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ are two families of integral domains (with $|I| \geq 2$), then $\Gamma(\prod_{i \in I} A_i) \cong \Gamma(\prod_{j \in J} B_j)$ if and only if there exists a bijection

$\varphi : I \to J$ such that $|A_i| = |B_{\varphi(i)}|$ for all $i \in I$. Of course, it is easy to find non-isomorphic integral domains of the same infinite cardinality. These findings are implicit in the results of this article.

In Section 3.2, we present an algorithm to construct realizable graphs. In particular, we classify graphs that are realizable as zero-divisor graphs of direct products of integral domains. It follows that the zero-divisor graph of any reduced ring is necessarily a subgraph of a member of a particular class of graphs (see Corollary 3.4). In [14], Redmond calculates every zero-divisor graph on n vertices, for all $n \leq 14$. Moreover, he gives an algorithm to find all finite reduced rings that have a zero-divisor graph on n vertices, for any $n \in \mathbb{N}$. These results are complemented by Corollary 3.4, which classifies graphs that are zero-divisor graphs of finite reduced rings. Furthermore, the results of Section 3.2 facilitate a lucid and timely construction of otherwise complicated zero-divisor graphs (see Example 3.5). In Section 3.3, graphs which are realizable as zero-divisor graphs of Boolean rings are characterized. This classification of graphs generalizes [1, Theorem 3.5] and [11, Theorem 2.5], which characterize zero-divisor graphs of commutative von Neumann regular rings and Boolean rings, respectively.

3.2 Direct Products of Integral Domains

In this section, we present an algorithm for constructing the zero-divisor graph of a direct product of integral domains. In [1, Theorem 2.2], it is shown that the zero-divisor graph of a ring is isomorphic to the zero-divisor graph of its total quotient ring. Suppose that R is a direct product of fields. Then [1, Proposition 4.5] implies that the zero-divisor graph induced by the set of idempotents of R is obtained by identifying vertices of $\Gamma(R)$ which share the same adjacency relations. These results motivate a less complicated representation of the zero-divisor graph of a direct product of integral domains.

Throughout, the letters κ, λ, and μ will denote (possibly infinite) cardinal numbers. If the set of vertices of a graph can be partitioned into a pair of nonempty disjoint sets A and B such that two vertices are adjacent if and only if one belongs to A and the other belongs to B, then the graph is called *complete bipartite*. We will refer to such graphs as (κ, λ)-*bipartite* when $|A| = \kappa$ and $|B| = \lambda$.

Let the labeled graph in Figure 3.1(a) denote the (κ, λ)-bipartite graph, and the labeled graph of Figure 3.1(b) denote the graph resulting from the obvious "gluing" of the (κ, λ)-bipartite graph with the (λ, μ)-bipartite graph (every bijective identification of the λ vertices will result in the same graph, up to isomorphism). One can denote a graph composed of an arbitrary set of "compatible" complete bipartite graphs in a similar fashion. For example, the labeled graph in Figure 3.3 represents a graph composed of twenty-five complete bipartite graphs.

Note that a graph may have many such representations. For example, the graph represented by Figure 3.1(a) with $\kappa = 2$ and $\lambda = 1$ is isomorphic to the graph represented by Figure 3.1(b) with $\kappa = \lambda = \mu = 1$. Given a graph Γ, define the *minimal representation of* Γ to be the labeled graph obtained by identifying vertices which share precisely the same adjacency relations, and then assigning the cardinality of the set of all such vertices to the corresponding representative vertex. For example, letting $\kappa = 2$ and $\lambda = 1$ in Figure 3.1(a) yields the minimal representation of the graph represented by Figure 3.1(b) with $\kappa = \lambda = \mu = 1$. In terms of zero-divisor graphs of reduced rings, one obtains a minimal representation by identifying vertices v_1 and v_2 if and only if $\mathrm{ann}(v_1) = \mathrm{ann}(v_2)$, and then assigning the cardinal $|\mathrm{ann}(v_1) \setminus \{0\}|$ to the corresponding representative vertex.

Recall that a ring R is *Boolean* if $x^2 = x$ for all $x \in R$. It is well-known that every Boolean ring has characteristic 2. Also, the set $B(R) = \{x \in R \mid x^2 = x\}$ of idempotents of a commutative ring R becomes a Boolean ring with multiplication defined the same as in R, and addition defined by the mapping $(a, b) \mapsto a + b - 2ab$. The

(a) (b)

Figure 3.1: Representations of complete bipartite graphs

following lemma is an immediate consequence of [11, Lemma 2.3], but is presented with an independent proof (also cf. [1, p. 232]).

Lemma 3.1. *Let $R \neq \mathbb{F}_2$ be a Boolean ring. Then the figure obtained by assigning the label* 1 *to each vertex of* $\Gamma(R)$ *is the minimal representation of* $\Gamma(R)$.

Proof. Suppose that v_1 and v_2 are vertices of $\Gamma(R)$ with $\mathrm{ann}(v_1) = \mathrm{ann}(v_2)$. Then $1 - v_2 \in \mathrm{ann}(v_2) = \mathrm{ann}(v_1)$ and $1 - v_1 \in \mathrm{ann}(v_1) = \mathrm{ann}(v_2)$. Thus $v_1(1 - v_2) = 0$ and $v_2(1 - v_1) = 0$. Hence $v_1 = v_1 v_2 = v_2$. $\square$

We now present the algorithm mentioned above for constructing the zero-divisor graph of a direct product of integral domains. Recall that the direct product $\prod_{i\in I} \mathbb{F}_2$ is isomorphic to the power set of I, endowed with addition as "symmetric difference" and multiplication as "intersection." This fact motivates (1) through (3). The last two steps are motivated by [1, Theorem 2.2, Proposition 4.5, and p. 233]. We define $[I]^\mu = \{J \subseteq I \mid |J| = \mu\}$.

1. Construct the complete graph on $\kappa \geq 2$ vertices. Denote the set of vertices of this graph by $A = \{a_i\}_{i\in I}$, where I is an indexing set with $|I| = \kappa$. If $\kappa = 2$, then go to (4). Otherwise, go to (2).

2. For each nonempty subset $A' \subseteq A$ such that $|A \setminus A'| \geq 2$, introduce a new vertex $v(A')$ such that $v(A')$ is adjacent to $a_i \in A$ if and only if $a_i \in A'$.

3. If A' and A'' are subsets of A as in (2), then declare $v(A')$ to be adjacent to $v(A'')$ if and only if $A' \cup A'' = A$.

   (*) The resulting graph is the zero-divisor graph of the Boolean ring $\prod_{i\in I} \mathbb{F}_2$. If κ is finite, this graph has $2^\kappa - 2$ vertices and

   $$\Sigma_{\mu=2}^{\kappa}(2^{\mu-1} - 1)\binom{\kappa}{\mu}$$

   edges.

4. Label a vertex a_i of A with a nonzero cardinal λ_i such that λ_i is finite if and only if $\lambda_i = p_i^{n_i} - 1$ for some prime number p_i and integer $n_i \in \mathbb{N}$. Do this for all $a_i \in A$. If $\kappa = 2$, then stop. Otherwise, go to (5).

5. For each subset A' of A as in (2), label the vertex $v(A')$ with $\Pi_{a_i \in A \setminus A'} \lambda_i$.

   (**) The resulting figure is the minimal representation of the zero-divisor graph of the ring $\prod_{i \in I} D_i$, where D_i is any integral domain of cardinality $\lambda_i + 1$. If κ is finite, this graph has

   $$\Sigma_{\mu=1}^{\kappa-1} \Sigma_{J \in [I]^\mu} \Pi_{j \in J} \lambda_j = \Pi_{i \in I}(\lambda_i + 1) - \Pi_{i \in I} \lambda_i - 1$$

   vertices and

   $$\Sigma_{\mu=2}^{\kappa} (2^{\mu-1} - 1) \Sigma_{J \in [I]^\mu} \Pi_{j \in J} \lambda_j$$

   edges.

Note that (**) implies (*) by letting $\lambda_i = 1$ for all $i \in I$. However, (*) will be verified implicitly in the proof of the following theorem. We will say that a graph is *representable* if either it is empty, or it is represented by a graph that can be constructed via the previous algorithm.

Theorem 3.2. *If a ring R is a direct product of integral domains, then $\Gamma(R)$ is representable. Moreover, a graph Γ is realizable as the zero-divisor graph of a direct product of integral domains if and only if Γ is representable.*

Proof. We first observe that it suffices to prove (**). Since $\Gamma(R)$ is empty if and only if R is an integral domain, we only need to consider nonempty graphs and direct products with at least two factors. Suppose that (**) is true. If R is a direct product of $\mu \geq 2$ integral domains, say $R = \prod_{i \in J} R_i$ (where J is an indexing set with $|J| = \mu$), then carry out (1) through (3) with $\kappa = \mu$ and $I = J$. Carry out (4) through

(5) by letting $\lambda_i = |R_i| - 1$ ($i \in I$). Then (**) implies that $\Gamma(R)$ is representable. This verifies the necessity portion of each statement in the theorem. The sufficiency portion of the last statement is an immediate consequence of (**).

To prove (**), let $R = \prod_{i \in I} D_i$, where I is an indexing set with $|I| = \kappa \geq 2$, and D_i is any integral domain of cardinality $\lambda_i + 1$ ($i \in I$). Let $B(R) \subseteq R$ be the set of elements $(r_i) \in R$ such that $r_i \in \{0, 1\}$ for all $i \in I$ ($B(R)$ is the set of idempotents of R). Let $A(R) \subseteq B(R)$ be the set of all elements with a 1 in precisely one coordinate and 0 elsewhere. Then the subgraph of $\Gamma(R)$ induced by $A(R)$ is complete on κ vertices.

Let $\emptyset \neq A' \subseteq A(R)$. If $A' = A(R)$, then there is no vertex of the graph induced by $B(R)$ that is adjacent to every element of A'. Suppose that $A' \subsetneq A(R)$. Let $v(A')$ be the element of $B(R)$ that has a 0 in the i-coordinate if and only if there exists an element of A' with a 1 in the i-coordinate. Then $v(A')$ is the unique element of $B(R)$ that satisfies the following property: The vertex $v(A')$ is adjacent to $a \in A(R)$ if and only if $a \in A'$. Since every zero-divisor of $B(R)$ annihilates some element of $A(R)$, it follows that every vertex of the subgraph induced by $B(R)$ is of the form $v(A')$ for some nonempty proper subset A' of $A(R)$. Moreover, $v(A') \in B(R) \setminus A(R)$ if and only if $v(A')$ has a 1 in at least two coordinates, i.e., if and only if $v(A')$ is not adjacent to at least two elements of $A(R)$, i.e., if and only if $|A(R) \setminus A'| \geq 2$.

Finally, two elements $v(A') = (r_i)$ and $v(A'') = (s_i)$ of $B(R)$ are adjacent in the subgraph induced by $B(R)$ if and only if $\{i \in I \mid r_i = 0 \text{ or } s_i = 0\} = I$, i.e., if and only if $\{a \in A(R) \mid a \text{ is adjacent to } (r_i)\} \cup \{a \in A(R) \mid a \text{ is adjacent to } (s_i)\} = A(R)$, i.e., if and only if $A' \cup A'' = A(R)$. Therefore, the subgraph induced by $B(R)$ is the graph that one obtains by carrying out steps (1) through (3) with $A = A(R)$.

Observe that $\Gamma(R)$ can be represented by the subgraph induced by $B(R)$; indeed, a vertex (r_i) of $\Gamma(R)$ is represented by the element of $B(R)$ that has a 0 in the i-coordinate if and only if $r_i = 0$. It is clear that the labeling of the elements of $A(R)$ is consistent with (4). In fact, if A' is any nonempty proper subset of $A(R)$, then $v(A')$ represents

$$\Pi_{a_i \in A(R) \setminus A'} |D_i \setminus \{0\}| = \Pi_{a_i \in A(R) \setminus A'} \lambda_i$$

vertices of $\Gamma(R)$, where a_i denotes the element of $A(R)$ whose i-coordinate is a 1. Thus $v(A')$ attains the desired label for every nonempty proper subset A' of $A(R)$. Therefore, $\Gamma(R)$ is represented by the figure obtained when (1) through (5) is carried out with $A = A(R)$ and the appropriate choice of λ_i ($i \in I$). Since the unlabeled version of this representation is the zero-divisor graph of the Boolean ring induced by $B(R)$, this representation is minimal by Lemma 3.1. Theorem 3.2 is now justified. To finish the proof of (**), it remains to verify the last statement.

If J is a proper nonempty subset of I, then the above argument shows that a vertex $v(\{a_i\}_{i\in I\setminus J})$ of the representative graph gets labeled with $\Pi_{j\in J}\lambda_j$; that is, for each proper nonempty subset $J \subset I$, there corresponds a vertex v_J representing $\Pi_{j\in J}\lambda_j$ vertices. Moreover, every vertex of $\Gamma(\prod_{i\in I} D_i)$ is represented by a unique vertex of the representative graph. Therefore, $\Gamma(\prod_{i\in I} D_i)$ has $\Sigma_{\mu=1}^{\kappa-1}\Sigma_{J\in[I]^\mu}\Pi_{j\in J}\lambda_j$ vertices. Furthermore, since $\prod_{i\in I} D_i$ has $\Pi_{i\in I}\lambda_i$ elements which are not zero-divisors, it has $\Pi_{i\in I}(\lambda_i + 1) - \Pi_{i\in I}\lambda_i - 1$ nonzero zero-divisors.

Finally, suppose that v is a vertex of the representative graph. Let $I_v = \{i \in I \mid a_i \text{ is not adjacent to } v\}$. It has been shown that the vertices $v(A')$ and $v(A'')$ ($A', A'' \subsetneq A$) are adjacent if and only if $A' \cup A'' = A$. Then taking complements shows that two distinct vertices v and w of the representative graph are adjacent if and only if $I_v \cap I_w = \emptyset$. Moreover, it is clear that an edge $v - w$ of the representative graph corresponds to $\Pi_{i\in I_v\cup I_w}\lambda_i$ edges of $\Gamma(\prod_{i\in I} D_i)$. But each subset J of I with $|J| \geq 2$ can be decomposed into nonempty (hence proper) subsets $H_1, H_2 \subseteq J$ such that $H_1 \cup H_2 = J$ and $H_1 \cap H_2 = \emptyset$. Moreover, such a decomposition is determined by the nonempty proper subset $H_1 \subsetneq J$, since then $H_2 = J \setminus H_1$. So if $|J| = \kappa$, then there are $2^\kappa - 2$ nonempty proper subsets of J, and thus half as many (that is, $2^{\kappa-1} - 1$) pairs of nonempty subsets H_1, H_2 of J such that $H_1 \cup H_2 = J$ and $H_1 \cap H_2 = \emptyset$. Since a set $H \subseteq I$ can be written as $H = I_v$ for some representative vertex v if and only if $\emptyset \neq H \subsetneq I$, this shows that every subset $J \subsetneq I$ with $|J| \geq 2$ corresponds to $(2^{\kappa-1} - 1)\Pi_{j\in J}\lambda_j$ edges of $\Gamma(\prod_{i\in I} D_i)$. Moreover, every edge of $\Gamma(\prod_{i\in I} D_i)$ is represented by a unique edge of the representative graph. Therefore, $\Gamma(\prod_{i\in I} D_i)$ has $\Sigma_{\mu=2}^{\kappa}(2^{\kappa-1} - 1)\Sigma_{J\in[I]^\mu}\Pi_{j\in J}\lambda_j$ edges. $\square$

Note that the converse to the first statement in the theorem is false. For example, $\Gamma(\mathbb{F}_3[X]/(X^2)) \cong \Gamma(\mathbb{F}_2 \times \mathbb{F}_2)$, but $\mathbb{F}_3[X]/(X^2)$ is not a product of integral domains (it has nilpotents). However, it is shown in [1, Theorem 5] that if a finite ring R is a direct product of integral domains and S is a ring which is not an integral domain, then $\Gamma(R) \cong \Gamma(S)$ implies that $R \cong S$ unless S is a local ring and either $R \cong \mathbb{F}_2 \times \mathbb{F}_2$ or $R \cong \mathbb{F}_2 \times \mathbb{F}_3$. In particular, the converse to the first statement in Theorem 3.2 is true whenever R is finite and not isomorphic to either of the rings $\mathbb{F}_2 \times \mathbb{F}_2$ and $\mathbb{F}_2 \times \mathbb{F}_3$.

Remark 3.3. *The above proof justifies the algorithm, from which the validity of Theorem 3.2 is attained. On the other hand, the following remarks outline a direct proof showing that a zero-divisor graph is the unlabeled minimal representation of the zero-divisor graph of a direct product of integral domains if and only if it is the graph induced by the power set of some set I (that is, the zero-divisor graph of $\prod_I \mathbb{F}_2$). Alternatively, this observation is an immediate consequence of [1, Proposition 4.5] (cf. the comments prior to Question 3.6 in Section 3.3). However, note that the labeling of the minimal representation of a zero-divisor graph is not arbitrary. Therefore, a graph need not be the zero-divisor graph of any ring, and yet its unlabeled minimal representation could be induced by the power set of some set (e.g., let $\kappa = 5$ and $\lambda = 1$ in Figure 3.1(a)).*

Let $R = \prod_{i \in I} D_i$, where $|I| \geq 2$ and each D_i is an integral domain. Define the set $\mathcal{P}^*(I) = \mathcal{P}(I) \setminus \{I, \emptyset\}$. For all $r \in R$, let $J(r) = \{i \in I \mid r(i) \neq 0\}$. Then $V(\Gamma(R)) = \{r \in R \mid J(r) \in \mathcal{P}^*\}$. Moreover, it is straightforward to check that the vertices r and s are adjacent in $\Gamma(R)$ if and only if $J(r) \cap J(s) = \emptyset$, and are represented by the same vertex in the minimal representation of $\Gamma(R)$ if and only if $J(r) = J(s)$. That is, the minimal representation of $\Gamma(R)$ is the graph Λ with vertex set $\mathcal{P}^*$, such that distinct vertices are adjacent if and only if their intersection is empty (i.e., $\Lambda = \Gamma(\prod_I \mathbb{F}_2)$).

Let R be a reduced ring. If $\mathcal{P}$ is the set of all prime ideals of R, then R can be regarded as a subring of $\prod_{P \in \mathcal{P}} R/P$ via the embedding $r \mapsto (r + P)$. If R is finite, then this mapping is onto (via the Chinese Remainder Theorem). Moreover, a nonempty zero-divisor graph $\Gamma(R)$ is finite if and only if R is a finite ring with nonzero zero-divisors [2, Theorem 2.2]. Therefore, we have the following descriptions of zero-divisor graphs of reduced rings.

Corollary 3.4. *The zero-divisor graph of a reduced ring is a subgraph of a representable graph. Moreover, a graph Γ is realizable as the zero-*

divisor graph of a finite reduced ring if and only if it is finite and representable.

We conclude this section with an example to illustrate the applicability of the above results. Note the virtual absence of any reference to algebraic properties of the given ring.

Example 3.5. *Let $R = \mathbb{F}_2 \times \mathbb{F}_{27} \times \mathbb{F}_{31} \times \mathbb{F}_{125}$. We shall construct the minimal representation of $\Gamma(R)$.*

1. *Since R is a product of four integral domains, construct the complete graph on a 4-element set A.*

2. *Choose a 2-element subset of A. Form a vertex that is adjacent to each element of that subset. Do this for each of the six 2-element subsets of A. Repeat this step for each of the four 1-element subsets of A.*

3. *Draw a line connecting any pair of nonadjacent vertices (not in A) which have the property that every vertex in A is adjacent to at least one of the elements in the pair.*

   Note that the resulting graph is the zero-divisor graph of the Boolean ring $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$. It has fourteen vertices and twenty-five edges (see Figure 3.2).

4. *Uniquely assign to each vertex in A an element from the set $\{1, 26, 30, 124\}$.*

5. *Choose a vertex not contained in A. Label this vertex with the product of the labels assigned to those vertices of A which are not adjacent to this vertex. Do this for each vertex not contained in A.*

   The resulting figure is the minimal representation of $\Gamma(R)$. It represents a graph with 112,529 vertices and 998,276 edges. (see Figure 3.3).

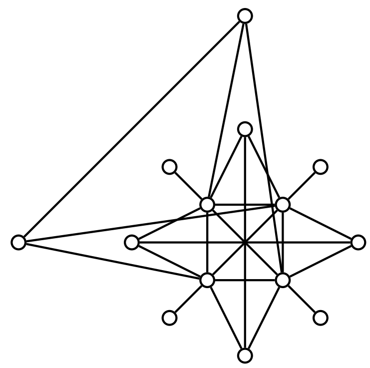

Figure 3.2: The zero-divisor graph of $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$

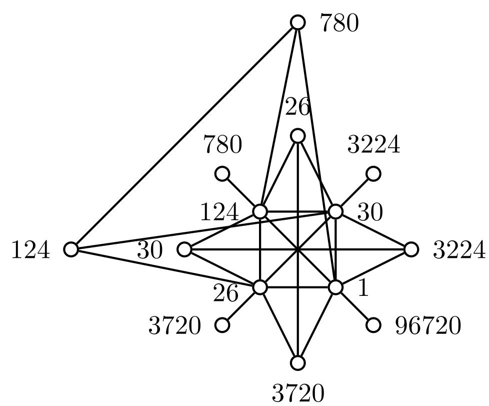

Figure 3.3: The minimal representation of $\Gamma(\mathbb{F}_2 \times \mathbb{F}_{27} \times \mathbb{F}_{31} \times \mathbb{F}_{125})$

3.3 Boolean Rings

The results in the previous section completely characterize graphs which are realizable as zero-divisor graphs of direct products of integral domains. In particular, the zero-divisor graphs of direct products of fields are characterized. In this section, the zero-divisor graphs of rings belonging to the "larger" class of commutative *von Neumann regular rings* (i.e., for all $r \in R$ there exists an $s \in R$ such that $r = r^2 s$) are examined.

Let Γ be an (undirected) graph. For any $\emptyset \neq V \subseteq V(\Gamma)$, let $C(V)$ denote the set of all vertices of Γ which are adjacent to every element of V. When $V = \{v_1, ..., v_n\}$ is finite, we shall write $C(V) = C(v_1, ..., v_n)$. Clearly distinct $v, w \in V(\Gamma)$ satisfy $v \in C(w)$ if and only if $w \in C(v)$. Also, if R is a ring and $\Gamma = \Gamma(R)$, then $C(V) = \text{ann}_R(V) \setminus (V \cup \{0\})$.

Given elements $v, w \in V(\Gamma)$, define $v \sim w$ if and only if v and w share the same adjacency relations in Γ. That is, $v \sim w$ if and only if $C(v) = C(w)$. Then the graph $\Gamma/\sim$, defined such that $[v]$ and $[w]$ are adjacent if and only if v and w are adjacent in Γ, is the unlabeled minimal representation of Γ.

Suppose that R is a commutative von Neumann regular ring. Recall that $B(R)$ is the Boolean ring induced by the idempotents of R. In [1, Proposition 4.5], it is shown that that mapping $V(\Gamma(B(R))) \rightarrow V(\Gamma(R)/\sim)$ defined by $e \mapsto [e]$ is a graph isomorphism. This result, Lemma 3.1, and the fact that Boolean rings are von Neumann regular imply that a graph Γ is the zero-divisor graph of a Boolean ring if and only if Γ is the unlabeled minimal representation of a von Neumann regular ring. Therefore, solutions to the problem of determining which graphs are realizable as von Neumann regular rings are given when the following two questions are answered:

Question 3.6. *Which graphs are realizable as Boolean rings?*

Question 3.7. *Given a Boolean ring R, when does a labeling of* $\Gamma(R)$ *induce the minimal representation of a von Neumann regular ring?*

Question 3.7 is examined in [1]. In this section, we provide a complete answer to Question 3.6

A partially ordered set B is called a *Boolean algebra* if it is a bounded distributive lattice such that all of its elements have complements, that is, a complemented distributive lattice. Given any $a, b \in B$, denote the supremum of a and b by $a \vee b$ and the infimum of a and b

by $a \wedge b$. Recall that b is a complement of a if and only if $a \vee b = 1$ and $a \wedge b = 0$ (by definition), where 1 and 0 denote the largest and smallest elements of B, respectively. By [9, Theorem I.6.4], a complemented lattice is a Boolean algebra if and only if all of its elements satisfy the condition

$$a \wedge b = 0 \text{ if and only if } a \leq b',$$

where b' is the complement of b (the antisymmetry of $\leq$ implies that b' is unique). Of course, using the associativity of $\wedge$, one checks that $a \wedge b = 0$ follows from $a \leq b'$ in any complemented lattice. The converse, however, is a consequence of distributivity. Note that a Boolean algebra becomes a Boolean ring by defining $ab = a \wedge b$ and $a + b = (a' \wedge b) \vee (a \wedge b')$. The Boolean algebra can then be recovered from the Boolean ring by declaring $a \leq b$ if and only if $ab = a$. In particular, to each Boolean ring there corresponds a Boolean algebra, and vice versa.

In what follows, a given element may be regarded as member of a ring, Boolean algebra, or vertex-set. We designate the characters r, s, t, and x to represent such elements. The class from which an operation or relation is considered shall be made explicit.

Let $\Gamma \neq \emptyset$ be a graph and $\varphi : V(\Gamma) \to V(\Gamma)$ a bijection. Define $\leq_\varphi$ on $V(\Gamma)$ by $r \leq_\varphi s$ if and only if $r \in C(\varphi(s))$. It is straightforward to check that $\leq_\varphi$ is a partial order on $V(\Gamma)$ if and only if φ satisfies the following properties:

(i) The containment $r \in C(\varphi(r))$ holds for all $r \in V(\Gamma)$.

(ii) If $r, s \in V(\Gamma)$ are distinct and $r \in C(\varphi(s))$, then $s \notin C(\varphi(r))$.

(iii) If $r, s, x \in V(\Gamma)$ with $r \in C(\varphi(s))$ and $s \in C(\varphi(x))$, then $r \in C(\varphi(x))$.

Thus, we will say that the bijection φ is *order-inducing* when (i)-(iii) are satisfied.

Theorem 3.8. *Let $\Gamma \neq \emptyset$ be a graph. Then $\Gamma = \Gamma(R)$ for some Boolean ring R if and only if there exists an order-inducing bijection $\varphi : V(\Gamma) \to V(\Gamma)$ which satisfies the following properties:*

(1) The map φ^2 is the identity on $V(\Gamma)$ (that is, φ can be defined by partitioning $V(\Gamma)$ into sets of order 2).

(2) For all $r, s \in V(\Gamma)$, either $C(r, s) = \emptyset$ or there exists an $x \in C(r, s)$ such that $C(r, s) \subseteq C(\varphi(x))$.

(3) If $r, s \in V(\Gamma)$, then $r \in C(s)$ if and only if $C(\varphi(r), \varphi(s)) = \emptyset$.

Proof. Suppose that $\Gamma = \Gamma(R)$ for some Boolean ring R. Define a map $\varphi : V(\Gamma(R)) \to V(\Gamma(R))$ by $\varphi(r) = 1 - r$. It is routine to show that φ is well-defined and bijective. Let $r, s \in V(\Gamma(R))$. In the Boolean algebra corresponding to R, $r \leq s$ if and only if $r \wedge s = r$. Since $\wedge$ is defined by multiplication in R, it follows that $r \leq s$ if and only if $r \in C(1 - s) = C(\varphi(s))$; that is, $r \leq s$ if and only if $r \leq_\varphi s$. Thus φ is order-inducing.

Clearly (1) holds. For (2), let $r, s \in V(\Gamma(R))$ such that $C(r, s) \neq \emptyset$. Let $x = 1 - (r + s - rs)$. If $t \in C(r, s)$, then $t \neq 0$ and $tx = t$. Hence $C(r, s) \neq \emptyset$ implies $x \neq 0$. But $xr = xs = 0$, and thus $x \in C(r, s)$. Finally, if $t \in C(r, s)$, then $t\varphi(x) = t(r + s + rs) = 0$, that is, $C(r, s) \subseteq C(\varphi(x))$.

It remains to prove (3). Let $r, s \in V(\Gamma(R))$. Then $rs(1 - r) = rs(1 - s) = 0$, that is, $rs\varphi(r) = rs\varphi(s) = 0$. Therefore, $C(\varphi(r), \varphi(s)) = \emptyset$ implies that $rs \notin V(\Gamma(R))$. Thus $rs = 0$ since $V(\Gamma(R)) = R \setminus \{0, 1\}$ whenever R is a Boolean ring. That is, $r \in C(s)$.

Conversely, suppose that $r \in C(s)$. Let $t \in R$ such that $t\varphi(r) = t\varphi(s) = 0$, i.e., $t(1 - r) = t(1 - s) = 0$. Then $t = tr = ts$. Hence $t = tr = (tr)r = (ts)r = 0$. It follows that $C(\varphi(r), \varphi(s)) = \emptyset$. This completes the "necessity" portion of the proof.

To prove the converse, let Γ be a graph with an order-inducing bijection $\varphi : V(\Gamma) \to V(\Gamma)$ which satisfies (1)-(3). Let 0 and 1 be any two distinct elements which do not belong to $V(\Gamma)$, and set $R = V(\Gamma) \cup \{0, 1\}$. Extend the map $\varphi : R \to R$ by letting $\varphi(1) = 0$ and $\varphi(0) = 1$. Define the relation $\leq$ on R by declaring $0 \leq 0 \leq r \leq 1 \leq 1$ for all $r \in V(\Gamma)$, and $r \leq s$ for all $r, s \in V(\Gamma)$ such that $r \leq_\varphi s$. It follows that $\leq$ is a partial order on R. To finish the proof, we shall utilize the following lemma, which is proved below.

Lemma 3.9. *The partial order $\leq$ makes R a Boolean algebra with complementation defined by $r' = \varphi(r)$ for all $r \in R$.*

By Lemma 3.9, R can be regarded as a Boolean ring with additive identity 0 and multiplicative identity 1, where multiplication in R is defined as $rs = r \wedge s$ for all $r, s \in R$. Let $r, s \in V(\Gamma)$. By definition,

$r \in C(s) = C(\varphi(\varphi(s)))$ if and only if $r \leq \varphi(s)$. But if $r \leq \varphi(s)$, then $r \wedge s \leq \varphi(s) \wedge s = s' \wedge s = 0$, i.e., $r \wedge s = 0$. Conversely, if $r \wedge s = 0$, then $\varphi(s) = \varphi(s) \vee (r \wedge s) = (\varphi(s) \vee r) \wedge (\varphi(s) \vee s) = (\varphi(s) \vee r) \wedge 1 = \varphi(s) \vee r$, that is, $r \wedge s = 0$ implies that $r \leq \varphi(s)$. Therefore, $r \in C(s)$ if and only if $r \wedge s = 0$, i.e., r is adjacent to s in Γ if and only if $rs = 0$ in the Boolean ring R. Since $V(\Gamma(R)) = R \setminus \{0, 1\} = V(\Gamma)$, it follows that $\Gamma = \Gamma(R)$. □

Proof of Lemma 3.9. To prove that every pair of elements in R has a supremum and infimum, let $r, s \in R$. If $1 \in \{r, s\}$, then $r \vee s = 1$. If $r = 1$, then $r \wedge s = s$, and if $s = 1$, then $r \wedge s = r$. Suppose that $1 \notin \{r, s\}$. If $0 \in \{r, s\}$, then $r \wedge s = 0$. If $r = 0$, then $r \vee s = s$; similarly, $r \vee s = r$ whenever $s = 0$. Therefore, assume that $\{r, s\} \subseteq R \setminus \{0, 1\}$, that is, $\{r, s, \varphi(r), \varphi(s)\} \subseteq V(\Gamma)$.

Suppose that the set $\{r, s\}$ has an upper bound $t \in R$ with $t \neq 1$. Then $0 \lneq r, s \leq t$ implies that $t \in V(\Gamma)$ and $\{r, s\} \subseteq C(\varphi(t))$, i.e., $\varphi(t) \in C(r, s)$. This shows that $r \vee s = 1$ whenever $C(r, s) = \emptyset$.

Suppose that $C(r, s) \neq \emptyset$. Then (2) implies that there exists an $x \in C(r, s)$ such that $C(r, s) \subseteq C(\varphi(x))$. In particular, $\{r, s\} \subseteq C(x) = C(\varphi(\varphi(x)))$, and thus $r, s \leq \varphi(x)$. Suppose that $t \in R$ with $r, s \leq t$. Then $t \neq 0$. If $t = 1$, then $\varphi(x) \leq t$. Suppose that $t \neq 1$. Then the containment $\{r, s\} \subseteq C(\varphi(t))$ follows since $r, s \leq t$, i.e., $\varphi(t) \in C(r, s)$, and therefore $\varphi(t) \in C(\varphi(x))$, that is, $\varphi(x) \in C(\varphi(t))$. Therefore $\varphi(x) \leq \varphi(\varphi(t)) = t$. Thus $r \vee s = \varphi(x)$, and every pair of elements in R has a supremum.

The relations $\varphi(r), \varphi(s) \leq \varphi(r) \vee \varphi(s)$ imply that $\varphi(\varphi(r) \vee \varphi(s)) \leq r, s$; this is clear when $\varphi(r) \vee \varphi(s) = 1$, and otherwise the former relations imply that $\{\varphi(r), \varphi(s)\} \subseteq C(\varphi(\varphi(r) \vee \varphi(s)))$, i.e., $\varphi(\varphi(r) \vee \varphi(s)) \in C(\varphi(r)) \cap C(\varphi(s))$. Thus $\varphi(\varphi(r) \vee \varphi(s)) \leq \varphi(\varphi(r)) = r$, and similarly $\varphi(\varphi(r) \vee \varphi(s)) \leq s$.

Suppose that $t \leq r, s$. Hence $t \neq 1$. If $t = 0$, then $t \leq \varphi(\varphi(r) \vee \varphi(s))$. Suppose that $t \neq 0$. Then $t \in V(\Gamma)$, and therefore $t \leq r, s$ implies that $t \in C(\varphi(r), \varphi(s))$. By (2), there exists an $x \in C(\varphi(r), \varphi(s))$ such that $C(\varphi(r), \varphi(s)) \subseteq C(\varphi(x))$. In particular, $t \in C(\varphi(x))$. That is, $t \leq \varphi(\varphi(x)) = \varphi(\varphi(r) \vee \varphi(s))$, where the last equality holds since the above argument shows that $\varphi(x) = \varphi(r) \vee \varphi(s)$. Thus $r \wedge s = \varphi(\varphi(r) \vee \varphi(s))$, and it follows that every pair of elements in R has an infimum.

Let $r \in R$. It is easy to check that $\varphi(r)$ is a complement of r when $r \in \{0, 1\}$. Suppose that $r \notin \{0, 1\}$, i.e., $r \in V(\Gamma)$. Note that

$r \in C(\varphi(r))$ since φ is order-inducing, and therefore $C(\varphi(r), r) = \emptyset$ by (3). Hence $r \vee \varphi(r) = 1$ by the above argument. Thus $r \wedge \varphi(r) = \varphi(\varphi(r) \vee \varphi(\varphi(r))) = \varphi(\varphi(r) \vee r) = \varphi(1) = 0$. Therefore, $\varphi(r)$ is a complement of r.

Let $r, s \in V(\Gamma)$. If $r \leq \varphi(s)$, then $r \wedge s \leq \varphi(s) \wedge s = 0$, that is, $r \wedge s = 0$. But if $r \wedge s = 0$, then $\varphi(r) \vee \varphi(s) = \varphi^2(\varphi(r) \vee \varphi(s)) = \varphi(r \wedge s) = 1$, and therefore $C(\varphi(r), \varphi(s)) = \emptyset$ by the above argument (indeed, $C(\varphi(s), \varphi(r)) \neq \emptyset$ implies that $\varphi(r) \vee \varphi(s) \in V(\Gamma)$). Thus $r \in C(s)$ by (3), i.e., $r \leq \varphi(s)$. Clearly the statement "$r \wedge s = 0$" is equivalent to "$r \leq \varphi(s)$" whenever $0 \in \{r, s\}$, and hence these statements are equivalent for all $r, s \in R$.

It has been verified that R is a lattice such that $\varphi(r)$ is a complement of r for all $r \in R$. Moreover, $r \wedge s = 0$ if and only if $r \leq \varphi(s)$. By [9, Theorem I.6.4], R is a Boolean algebra with complementation defined by $r' = \varphi(r)$ for all $r \in R$. □

Let Γ be a graph with order-inducing bijections $\varphi_1, \varphi_2 : V(\Gamma) \to V(\Gamma)$. It is known that if R and S are Boolean rings, then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$ [11, Theorem 4.1]. It follows that if φ_1 and φ_2 satisfy (1)-(3), then the induced Boolean rings (given by the proof of Lemma 3.9) are isomorphic. However, a stronger result is obtained from the following observation: If R is a Boolean ring and $r \in R$, then $1 - r$ is the unique element of R such that $r(1 - r) = 0$ and $\mathrm{ann}_R(r, 1 - r) = (0)$ (cf [11, Lemma 2.3]). Clearly $r(1 - r) = 0$. Let $t \in R$. Then $tr = t(1 - r) = 0$ implies that $t = tr = 0$. Moreover, if $tr = 0$, then $(1 - r - t) \in \mathrm{ann}_R(r, t)$. Therefore, either $t = 1 - r$ or $\mathrm{ann}_R(t, r) \neq (0)$. This proves the observation.

Suppose that φ is an order-inducing bijection on the vertices of a graph Γ satisfying (1)-(3). Let R be the Boolean ring induced by φ. By the above theorem, $C(V) = \mathrm{ann}_R(V) \setminus (V \cup \{0\})$ for all $\emptyset \neq V \subseteq V(\Gamma)$. Then the above observation together with (i) and (3) force $\varphi(r) = 1 - r$ for all $r \in R$. This gives a ring-theoretic proof of the following corollary. A graph-theoretic proof is given below.

Corollary 3.10. *Let $\Gamma \neq \emptyset$ be a graph with order-inducing bijections $\varphi_1, \varphi_2 : V(\Gamma) \to V(\Gamma)$ which satisfy (1)-(3). Then $\varphi_1 = \varphi_2$. In particular, the Boolean rings induced by φ_1 and φ_2 are equal (up to the choice of* 0 *and* 1*).*

Proof. Suppose that $r \in V(\Gamma)$ such that $\varphi_1(r) \neq \varphi_2(r)$. Recall that $C(r, \varphi_2(r)) = \emptyset$ by (i) and (3). That is, $C(\varphi_1^2(r), \varphi_1^2(\varphi_2(r))) = \emptyset$. Then (3) implies that $\varphi_1(r) \in C(\varphi_1(\varphi_2(r)))$. Thus (ii) implies that $\varphi_2(r) \notin C(\varphi_1(\varphi_1(r))) = C(r)$, contradicting (i). Therefore, $\varphi_1(r) = \varphi_2(r)$, and it follows that $\varphi_1 = \varphi_2$.

The "in particular" statement is clear. $\square$

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Chapter 4

The Cardinality of an Annihilator Class in a Von Neumann Regular Ring

Abstract. One defines an equivalence relation on a commutative ring R by declaring elements $r_1, r_2 \in R$ to be equivalent if and only if $\mathrm{ann}_R(r_1) = \mathrm{ann}_R(r_2)$. If $[r]_R$ denotes the equivalence class of an element $r \in R$, then it is known that $|[r]_R| = |[r/1]_{T(R)}|$, where $T(R)$ denotes the total quotient ring of R. In this paper, we investigate the extent to which a similar equality will hold when $T(R)$ is replaced by $Q(R)$, the complete ring of quotients of R. The results are applied to compare the zero-divisor graph of a reduced commutative ring to that of its complete ring of quotients.

4.1 Introduction

Let R be a commutative ring. One easily checks that an equivalence relation on R is given by declaring elements $r_1, r_2 \in R$ to be equivalent if and only if $\mathrm{ann}_R(r_1) = \mathrm{ann}_R(r_2)$. The cardinalities of such equivalence (annihilator) classes were considered in [13], where the authors were interested in ring-theoretic properties shared by von Neumann regular rings with identical zero-divisor structures. In [1], the authors show that every ring has the same zero-divisor structure as its total quotient ring. The proof of this result demonstrates that the cardinality of the annihilator class of an element does not change when the element is regarded as a member of its total quotient ring. We examine the degree to which this result can be generalized to a particular extension of a reduced total quotient ring.

Throughout, R will always be a commutative ring with $1 \neq 0$. Let $Z(R)$ denote the set of zero-divisors of R and $T(R) = R_{R \setminus Z(R)}$ its total quotient ring. A ring R will be called *reduced* if $\mathrm{nil}(R) = (0)$. A commutative ring R with $1 \neq 0$ is *von Neumann regular* if for each $x \in R$, there is a $y \in R$ such that $x = x^2 y$ or, equivalently, R is reduced with Krull dimension zero [6, Theorem 3.1].

A subset $D \subseteq R$ is *dense in* R if $\mathrm{ann}_R(D) = (0)$. Let D_1 and D_2 be dense ideals of R and let $\varphi_i \in \mathrm{Hom}_R(D_i, R)$ $(i = 1, 2)$. Note that $\varphi_1 + \varphi_2$ is an R-module homomorphism on the dense ideal $D_1 \cap D_2$, and $\varphi_1 \circ \varphi_2$ is an R-module homomorphism on the dense ideal $\varphi_2^{-1}(D_1) = \{r \in R \mid \varphi_2(r) \in D_1\}$. Then $Q(R) = F/\sim$ is a commutative ring, where $F = \{\varphi \in \mathrm{Hom}_R(D, R) \mid D \subseteq R \text{ is a dense ideal}\}$ and $\sim$ is the equivalence relation defined by $\varphi_1 \sim \varphi_2$ if and only if there exists a dense ideal $D \subseteq R$ such that $\varphi_1(d) = \varphi_2(d)$ for all $d \in D$ [11, Proposition 2.3.1]. In [11], J. Lambek calls $Q(R)$ the *complete ring of quotients of* R.

Let $\overline{\varphi} \in Q(R)$ denote the equivalence class containing φ. For all $a/b \in T(R)$, the ideal bR of R is dense and $\varphi_{a/b} \in \mathrm{Hom}_R(bR, R)$, where $\varphi_{a/b}(br) = ar$. One checks that the mapping $a/b \mapsto \overline{\varphi_{a/b}}$ is a ring monomorphism, and that $\overline{\varphi_0}$ and $\overline{\varphi_1}$ are the additive and multiplicative identities of $Q(R)$, respectively. In particular, the mapping $R \to Q(R)$ defined by $r \mapsto \overline{\varphi_r}$ is an embedding. However, these mappings need not be onto (see [11]). If the mapping $R \to Q(R)$ *is* onto (i.e., $r \mapsto \overline{\varphi_r}$ is an isomorphism), then R is called *rationally complete*. Note that $Q(R)$ is von Neumann regular if and only if R is reduced [11, Proposition 2.4.1]. Thus every reduced rationally complete ring is von Neumann regular.

A ring extension $R \subseteq S$ is called a *ring of quotients of* R if $f^{-1}R = \{r \in R \mid fr \in R\}$ is dense in S for all $f \in S$. In particular, $T(R)$ is a ring of quotients of R. If S is a ring of quotients of R, then there exists an extension of the mapping $R \to Q(R)$ which embeds S into $Q(R)$ [11, Proposition 2.3.6]. Therefore, every ring of quotients of R can be regarded as a subring of $Q(R)$. It follows that a dense set in R is dense in every ring of quotients of R. Also, R has a *unique* maximal (with respect to inclusion) ring of quotients, which is isomorphic to $Q(R)$ [11, Proposition 2.3.6]. In recognition of this observation, we shall abuse notation and denote the maximal ring of quotients of R by $Q(R)$. It is not hard to check that $Q(R) = Q(T(R))$ for any ring R. In fact, if $R \subseteq S \subseteq Q$, then Q is a ring of quotients of R if and only if Q is a ring of quotients of S and S is a ring of quotients of R (e.g., see the comments prior to Lemma 1.5 in [4]).

Let $B(R) = \{e \in R \mid e^2 = e\}$, the set of idempotents of R. Then the relation "$\leq$" defined by $a \leq b$ if and only if $ab = a$ partially orders $B(R)$, and makes $B(R)$ a Boolean algebra with inf as multiplication in R, the largest element as 1, the smallest element as 0, and complementation defined by $a' = 1 - a$. One checks that $a \vee b = (a' \wedge b')' = a + b - ab$, where "+" is addition in R. A set $E \subseteq B(R)$ is called a *set of orthogonal idempotents* if $e_1e_2 = 0$ for all distinct $e_1, e_2 \in E$. For a reference on the Boolean algebra of idempotents, see [11].

A Boolean algebra B is *complete* if $\inf E$ exists for every subset $E \subseteq B$. If B is a complete Boolean algebra, then $\sup E = \inf\{b \mid b \in B$ and $b \geq e$ for all $e \in E\}$. It is well known that every Boolean algebra B is a subalgebra of a complete Boolean algebra $D(B)$, where the infimum of a set in B (when it exists) is the same as its infimum in $D(B)$. Here, $D(B)$ is the "so called" *Dedekind-MacNeille* completion of B [11, c.f. Section 2.4]. Note that $D(B(R)) = B(Q(R))$ for every von Neumann regular ring R [4, Theorem 11.9]. In particular, $B(Q(R))$ is complete. Moreover, $B(R) = B(Q(R))$ whenever $B(R)$ is complete.

In this paper, we continue the investigations of [1] and [11]. We will denote the annihilator class of an element r in R by $[r]_R$, i.e., $[r]_R = \{s \in R \mid \mathrm{ann}_R(s) = \mathrm{ann}_R(r)\}$. As in [2], we define the *zero-divisor graph of* R, $\Gamma(R)$, to be the (undirected) graph with vertices $V(\Gamma(R)) = Z(R)\setminus\{0\}$, such that distinct $v_1, v_2 \in V(\Gamma(R))$ are adjacent if and only if $v_1v_2 = 0$. It is shown in [1, Theorem 2.2] that $\Gamma(R) \cong \Gamma(T(R))$ for any commutative ring R; the equality $|[r]_R| = |[r]_{T(R)}|$ for all $r \in R$ follows directly from the proof of this theorem (where we have identified R with its canonical image in $T(R)$). Both of these results fail when $T(R)$ is replaced by $Q(R)$ (e.g., Examples 4.10 and 4.11). In Section 4.2, we give necessary and sufficient conditions for the equality $|[r]_R| = |[r]_{Q(R)}|$ to hold, where R is a von Neumann regular ring such that $B(R)$ is complete and $2 \notin Z(R)$ (see Theorem 4.15). If either $B(R)$ is not complete or $2 \in Z(R)$, then the equality may or may not hold (see Examples 4.11, 4.17, and Corollary 4.16). This result is applied in Section 4.3 to give sufficient conditions for $\Gamma(R) \cong \Gamma(Q(R))$ to hold when R is a reduced ring. In particular, we provide a characterization of zero-divisor graphs which satisfy $\Gamma(R) \cong \Gamma(Q(R))$, where R is a reduced ring such that $|Z(R)| < \aleph_\omega$ and $2 \notin Z(R)$ (see Theorem 4.20).

4.2 The Cardinality of $[e]_{Q(R)}$

The investigation in this section involves a set-theoretic treatment of elements in a ring. The main theorems are numbered 4.4, 4.8, 4.15, and 4.16. The results numbered 4.1 through 4.8 develop useful relations within $Q(R)$, and ultimately provide an interpretation of elements in $Q(R)$ as subsets of a set. The results numbered 4.9 through 4.17 provide answers regarding the cardinalities of $[e]_R$ and $[e]_{Q(R)}$.

Throughout this section, R will always be a von Neumann regular ring unless stated otherwise. If $r \in R$, say $r = r^2s$, then $e_r = rs$ is the *unique* idempotent that satisfies $[r]_R = [e_r]_R$ (c.f. the discussion prior to Theorem 4.1 in [1], or Remark 2.4 of [11]). Moreover, $r = ue_r$ for some unit u of R [6, Corollary 3.3].

The following proposition shows that a nonzero element of a ring of quotients of R will map some idempotent of R into R nontrivially. Recall that $f^{-1}R$ is dense in S whenever f is a nonzero element of a ring of quotients S of R. In particular, there is an $r \in R$ such that $fr \in R \setminus \{0\}$.

Proposition 4.1. *Let R be a von Neumann regular ring. If $R \subseteq S$ is a ring of quotients of R, then for all $0 \neq f \in S$ there exists an $e \in B(R)$ such that $e \leq e_f$ and $0 \neq fe \in R$.*

Proof. Let $0 \neq f \in S$. Choose $r \in R$ such that $0 \neq fr \in R$. There is a unit u of R such that $r = ue_r$, and hence $fe_r = u^{-1}fr \in R \setminus \{0\}$. Let $e = e_f e_r$ (note that it makes sense to talk about e_f since $S \subseteq Q(R)$ and $Q(R)$ is von Neumann regular). Let $s \in Q(R)$ and $t \in R$ be elements such that $f = f^2 s$ and $r = r^2 t$. Then

$$e = e_f e_r = (fs)(rt) = (fr)(st) = e_{fr} \in R.$$

Moreover, $e \leq e_f$ and $fe = fe_r \in R \setminus \{0\}$. $\square$

For any set $A \subseteq R$, let $E_A = \{e_r \in B(R) \mid r \in A\}$. If $e \in B(R)$, then consider the set $\mathcal{R}_e(R) = \{\emptyset \neq A \subseteq R \mid e_{r_1} e_{r_2} = 0$ for all distinct $r_1, r_2 \in A$, and $\sup E_A = e\}$. Note that $\mathcal{R}_e(R) \neq \emptyset$ since $\{e\} \in \mathcal{R}_e(R)$. Also, if $\sup E_A = e$ and $0 \neq e' \in B(R)$ with $e' \leq e$, then there exists an $e'' \in E_A$ such that $e'e'' \neq 0$. Otherwise, $e'' \leq 1 - e'$ for all $e'' \in E_A$, and thus $e = \sup E_A \leq 1 - e'$. But this implies that $e'e = 0$, a contradiction. This fact is generalized in (1) of the following proposition.

Proposition 4.2. *Suppose that $E \subseteq B(R)$ is a set of orthogonal idempotents in a von Neumann regular ring R.*

(1) Let $e' \in B(R)$. Then $e' \sup E = 0$ if and only if $E \cup \{e'\}$ is a set of orthogonal idempotents. In particular, $r \sup E = 0$ if and only if $re' = 0$ for all $e' \in E$ $(r \in R)$.

(2) Suppose that E is finite; say $E = \{e_1, ..., e_n\}$. Then $\sup E = \sum_{j=1}^{n} e_j$.

(3) Let $e' \in B(R)$. If $f \in Q(R)$ such that $e' \leq e_f$, then $fe' \in [e']_{Q(R)}$.

(4) Let $e', e \in B(R)$ such that $e' \leq e$ and $2e' \in [e']_R$. Then $e' + e \in [e]_R$.

Proof. Note that $\sup E \in B(Q(R))$.

(1): If $e'e'' \neq 0$ for some $e'' \in E$, then $e'e'' \leq e'' \leq \sup E$ implies that $e'e'' \sup E = e'e'' \neq 0$; in particular, $e' \sup E \neq 0$. Conversely, suppose that $e' \sup E \neq 0$. Since $e' \sup E \leq \sup E$, the above comments show there exists an $e'' \in E$ such that $(e' \sup E)e'' \neq 0$; in particular, $e'e'' \neq 0$. Thus $E \cup \{e'\}$ is *not* a set of orthogonal idempotents.

The "in particular" statement holds since $[r]_R = [e_r]_R$ for all $r \in R$.

(2): It is easy to check that $e = \sum_{j=1}^{n} e_j \in B(R)$. Also, $e_j e = e_j$ for all $j \in \{1, ..., n\}$. Hence $\sup E \leq e$. But $e_j \leq \sup E$ for all $j \in \{1, ..., n\}$, and thus $e \sup E = e$; that is, $e \leq \sup E$. Therefore, $e = \sup E$.

(3): Clearly $\mathrm{ann}_{Q(R)}(e') \subseteq \mathrm{ann}_{Q(R)}(fe')$. Let $a \in \mathrm{ann}_{Q(R)}(fe')$. Then $ae' \in \mathrm{ann}_{Q(R)}(f) = \mathrm{ann}_{Q(R)}(e_f)$. Thus $0 = ae'e_f = ae'$; that is, $a \in \mathrm{ann}_{Q(R)}(e')$. Hence $\mathrm{ann}_{Q(R)}(e') = \mathrm{ann}_{Q(R)}(fe')$, i.e., $fe' \in [e']_{Q(R)}$.

(4): If $r \in \mathrm{ann}_R(e)$, then $re = 0$ and $re' = ree' = 0$. Hence $r \in \mathrm{ann}_R(e' + e)$, and therefore $\mathrm{ann}_R(e) \subseteq \mathrm{ann}_R(e' + e)$. To show the reverse inclusion, let $r \in \mathrm{ann}_R(e'+e)$. Note that $0 = re'(e'+e) = r(2e')$. Then $2e' \in [e']_R$ implies that $re' = 0$, and therefore $re = re' + re = r(e' + e) = 0$. Hence $\mathrm{ann}_R(e' + e) \subseteq \mathrm{ann}_R(e)$. □

In order to investigate cardinality, we shall translate the elements of an equivalence class $[e]_{Q(R)}$ into sets of elements of $\mathcal{R}_e(R)$. Such a correspondence is given in Theorem 2.4, and is motivated by the following example.

Example 4.3. *Let F be an infinite field and J an infinite indexing set. Let $F_j = F$ for all $j \in J$. Define $R = \{(r_j) \in \prod_{j \in J} F_j \mid \{r_j\}_{j \in J} \subseteq \{s_1, ..., s_n\}$ for some $\{s_1, ..., s_n\} \subseteq F$, for some $n \in \mathbb{N}\}$ (c.f. [11, Example 3.5]). Note that R is von Neumann regular. Let D be the dense ideal of R generated by the minimal nonzero idempotents of R (that is, the elements with a 1 in precisely one coordinate and 0 elsewhere). Then D is contained in $f^{-1}R$ for all $f \in \prod_{j \in J} F_j$. Thus $\prod_{j \in J} F_j$ is a ring of quotients of R. Moreover, $\prod_{j \in J} F_j$ is rationally complete [11, Proposition 2.3.8]. Therefore, $Q(R) = \prod_{j \in J} F_j$.*

Consider R from Example 4.3. Suppose that $F = \mathbb{Q}$, $J = \mathbb{N}$, and let e be the multiplicative identity of R (the largest element of $B(R)$). Note that there is a correspondence between $\mathcal{R}_e(R)$ and $[e]_{Q(R)}$, which is defined by taking the "sum" of the elements of a set in $\mathcal{R}_e(R)$. For example, the set

$$\{(1, 0, 0, ...), (0, 2, 0, ...), (0, 0, 3, ...), ...\} \in \mathcal{R}_e(R)$$

corresponds to the element $(1, 2, 3, ...) \in Q(R)$. This correspondence is generalized in the following theorem.

Theorem 4.4. *Let R be a von Neumann regular ring and suppose that $e \in B(R)$. The mapping $\sigma_e : \mathcal{R}_e(R) \to [e]_{Q(R)}$ defined by*

$$\sigma_e(A) = f \text{ if and only if } f \in [e]_{Q(R)} \text{ with } fe_r = r \text{ for all } r \in A$$

is a well-defined function. Moreover, $\sigma_e(A) \in R$ if and only if $\sigma_e(A) = \sigma_e(A')$ for some $A' \in \mathcal{R}_e(R)$ with $|A'| < \infty$.

Proof. Fix $e \in B(R)$. To show that σ_e is well-defined, we first show that every element of $\mathcal{R}_e(R)$ corresponds to some element in $[e]_{Q(R)}$. Let $A \in \mathcal{R}_e(R)$. Note that $D = (1-e, E_A)$ is a dense ideal of R: Any element $r \in R \setminus \{0\}$ that annihilates $1-e$ satisfies $re = r \neq 0$, and therefore does not annihilate all of E_A by Proposition 4.2 (1). Define $\varphi \in \mathrm{Hom}_R(D, R)$ by

$$\varphi\big(t(1-e) + \sum_{e_r \in E_A} t_r e_r\big) = \sum_{e_r \in E_A} t_r r.$$

(Indeed, φ is well-defined since multiplication by the appropriate idempotent will show that equal elements of D have equal "like terms," and clearly $t_r e_r = t'_r e_r$ implies that $t_r r = t'_r r$.) Then $\varphi(1-e) = 0$ and $\varphi(e_r) = r$ for all $r \in A$. Therefore, there exists an element $f \in Q(R)$ such that $f(1-e) = 0$ and $fe_r = r$ for all $r \in A$. It follows that $e_f \leq e$ (in $B(Q(R))$). To prove the reverse inequality, let $r \in A$. Then

$$r = fe_r = e_f f e_r = e_f r.$$

Thus $r(1-e_f) = 0$, which implies that $e_r \leq e_f$. Hence $e = \sup E_A \leq e_f$, and therefore $e = e_f$. This shows that $f \in [e]_{Q(R)}$, and therefore $\sigma_e(A) = f$. It remains to show that σ_e is single-valued. Suppose that A maps to both f and g. Then $(f-g)$ annihilates D. But D is dense in $Q(R)$, and thus $f - g = 0$, i.e., $f = g$. Therefore, σ_e is well-defined.

To see that the "moreover" statement is true, suppose that $\sigma_e(A) \in R$. Then $\sigma_e(A) = \sigma_e(A')$, where $A' = \{\sigma_e(A)\}$. Conversely, suppose that $A' \in \mathcal{R}_e(R)$ with $|A'| < \infty$; say $A' = \{r_1, ..., r_n\}$. Then

$$\sigma_e(A') = \sigma_e(A')e = \sigma_e(A')(\sum_{j=1}^{n} e_{r_j}) = \sum_{j=1}^{n}(\sigma_e(A')e_{r_j}) = \sum_{j=1}^{n} r_j \in R,$$

where the second equality follows from Proposition 4.2 (2) (c.f. the last paragraph prior to the statement of this theorem). $\square$

By the last part of the previous proof, we have

Corollary 4.5. *Let R be a von Neumann regular ring. If $A \in \mathcal{R}_e(R)$ is a finite set, then $\sigma_e(A) = \sum_{r \in A} r \in [e]_R$.*

Let $e \in B(R)$ and define the set

$$\mathcal{E}_e(R) = \{E_A \mid A \in \mathcal{R}_e(R)\}.$$

We shall write $f \prec E$ whenever $E \in \mathcal{E}_e(R)$ and $\sigma_e(A) = f$ for some $A \in \mathcal{R}_e(R)$ with $E_A = E$. By Proposition 4.2 (3), this is equivalent to declaring $f \prec E$ if and only if E is a set of orthogonal idempotents such that $\sup E = e_f = e$ and $fe' \in R$ for all $e' \in E$ (indeed, let $A = \{fe'\}_{e' \in E}$, c.f. the second paragraph in the proof of Theorem 4.8). In particular, if $r \in [e]_R$, then $r \prec E$ for all $E \in \mathcal{E}_e(R)$, i.e., $\{r \in [e]_R \mid r \prec E\} = [e]_R$ for all $E \in \mathcal{E}_e(R)$.

Corollary 4.6. *Let R be a von Neumann regular ring and suppose that $e \in B(R)$. If $E \in \mathcal{E}_e(R)$, then*

$$|\{A \in \mathcal{R}_e(R) \mid E_A = E\}| = |\{f \in [e]_{Q(R)} \mid f \prec E\}|.$$

Proof. The mapping $\{A \in \mathcal{R}_e(R) \mid E_A = E\} \to \{f \in [e]_{Q(R)} \mid f \prec E\}$ defined by $A \mapsto \sigma_e(A)$ is a well-defined surjection by Theorem 4.4 and the definition of $\prec$. It is injective since if $A_1, A_2 \in \{A \in \mathcal{R}_e(R) \mid E_A = E\}$ with $\sigma_e(A_1) = \sigma_e(A_2)$, then

$$A_1 = \{\sigma_e(A_1)e'\}_{e' \in E} = \{\sigma_e(A_2)e'\}_{e' \in E} = A_2.$$

Therefore,

$$|\{A \in \mathcal{R}_e(R) \mid E_A = E\}| = |\{f \in [e]_{Q(R)} \mid f \prec E\}|.$$

$\square$

Suppose that R is a reduced ring. Then the mapping $\mathrm{ann}_{Q(R)}(J) \mapsto \mathrm{ann}_R(J \cap R)$ $(J \subseteq Q(R))$ is a well-defined bijection of $\mathrm{Ann}(Q(R))$ onto $\mathrm{Ann}(R)$, where $\mathrm{Ann}(R) = \{\mathrm{ann}_R(J) \mid J \subseteq R\}$ [11, Proposition 2.4.3]; in particular, $[r]_R \subseteq [r]_{Q(R)}$ for all $r \in R$. Alternatively, suppose that R is a von Neumann regular ring. Then $[e]_R = \{r \in [e]_R \mid r \prec E\}$ for all $E \in \mathcal{E}_e(R)$. Since $\sigma_e(\mathcal{R}_e(R)) \subseteq [e]_{Q(R)}$, we have

Proposition 4.7. *Let R be a von Neumann regular ring and suppose that $e \in B(R)$. Then $[e]_R \subseteq \{f \in [e]_{Q(R)} \mid f \prec E\}$ for all $E \in \mathcal{E}_e(R)$. In particular, $[e]_R \subseteq [e]_{Q(R)}$.*

Of course, the "in particular" statement of the above proposition can be justified by the simpler argument that $r = ue$ for some unit u of R (and hence of $Q(R)$), for all $r \in [e]_R$. However, we will apply the first part of the proposition in the proof of Lemma 4.12.

Note that Theorem 4.4 implies that *some* of the elements of $[e]_{Q(R)}$ correspond to elements of $\mathcal{R}_e(R)$. The next theorem shows that *every* element in $[e]_{Q(R)}$ is of this type.

Theorem 4.8. *Let R be a von Neumann regular ring. Suppose that $e \in B(R)$. Then σ_e is surjective. In particular, $|[e]_{Q(R)}| \leq |\mathcal{R}_e(R)|$.*

Proof. Fix $e \in B(R)$. The result is trivial for the case $e = 0$. Suppose that $e \neq 0$. To show that σ_e is onto, choose any $f \in [e]_{Q(R)}$. Let $\mathcal{C} = \{\emptyset \neq E \subseteq B(R) \mid e'e'' = 0$ for all distinct $e', e'' \in E$, $e' \leq e$ for all $e' \in E$, and $fe' \in R$ for all $e' \in E\}$. Note that $\mathcal{C} \neq \emptyset$ since $\{0\} \in \mathcal{C}$. Let $\mathcal{C}$ be partially ordered by inclusion; then an application of Zorn's lemma shows that $\mathcal{C}$ has a maximal element, call it E. We will show that $\sup E = e$. If not, then consider $0 \neq e' = e - \sup E \in B(Q(R))$. Note that $fe' \in [e']_{Q(R)}$ by Proposition 4.2 (3). Hence Proposition 4.1 implies that there exists an $e'' \in B(R)$ such that $e'' \leq e'$ and $fe'' = fe'e'' \in R \setminus \{0\}$. Also, $e' \leq e$ implies $e'' \leq e$, and thus

$$e'' \sup E = e''(e - e') = e'' - e'' = 0.$$

But then $E \cup \{e''\} \in \mathcal{C}$ by Proposition 4.2 (1), contradicting the maximality of E. Therefore, $\sup E = e$.

Let $A = \{fe' \mid e' \in E\}$. Then Proposition 4.2 (3) implies $E_A = E$, and thus $A \in \mathcal{R}_e(R)$. Also, $e_{fe'} = e'$ implies that $fe_{fe'} = fe'$ for all $fe' \in A$. Hence $\sigma_e(A) = f$.

The "in particular" statement is clear. $\square$

We now turn our attention to the cardinality of $[e]_R$. The previous theorem allows one to derive information about the cardinality of $[e]_{Q(R)}$ from the set $\mathcal{R}_e(R)$. We will be able to relate the cardinalities of $[e]_{Q(R)}$ and $[e]_R$ if we can find a way to use the set $\mathcal{R}_e(R)$ to reveal information about $|[e]_R|$. The next three lemmas accomplish this by considering elements of the subset $\mathcal{E}_e(R)$ of $\mathcal{R}_e(R)$.

Lemma 4.9. *Let R be a von Neumann regular ring. Suppose that $E \subseteq B(R) \setminus \{0\}$ is a set of orthogonal idempotents with $\sup E = e$. Moreover, assume that $B(R)$ is complete and $2e' \in [e']_R$ for all $e' \in E$. Then $|[e]_R| \geq 2^{|E|}$.*

Proof. Define the mapping $\rho : \mathcal{P}(E) \to [e]_R$ by

$$\rho(E') = \sup E' + e,$$

where $\mathcal{P}(E)$ is the "power set" of E. Let $E' \subseteq E$. It is clear that $\operatorname{ann}_R(\sup E') \subseteq \operatorname{ann}_R(2 \sup E')$. Conversely, let $r \in \operatorname{ann}_R(2 \sup E')$. Then $2r \in \operatorname{ann}_R(\sup E')$, and hence $2re' = 0$ for all $e' \in E'$ by Proposition 4.2 (1). Thus $r \in \operatorname{ann}_R(2e') = \operatorname{ann}_R(e')$ for all $e' \in E'$, and therefore Proposition 4.2 (1) implies that $r \in \operatorname{ann}_R(\sup E')$. This shows that $\operatorname{ann}_R(\sup E') = \operatorname{ann}_R(2 \sup E')$, i.e., $2 \sup E' \in [\sup E']_R$. Hence ρ is well-defined by Proposition 4.2 (4). To show that ρ is injective, suppose that $E_1, E_2 \subseteq E$ with $E_1 \neq E_2$; say $0 \neq e' \in E_1 \setminus E_2$. Then

$$e' \sup E_1 = e' \neq 0 = e' \sup E_2,$$

where the last equality holds by Proposition 4.2 (1). It follows that $\sup E_1 \neq \sup E_2$. Thus $E_1 \neq E_2$ implies that $\rho(E_1) \neq \rho(E_2)$. Therefore, ρ is injective, and hence

$$|[e]_R| \geq |\mathcal{P}(E)| = 2^{|E|}.$$

$\square$

For the remainder of this section, it will be necessary to recall some facts from set theory. In what follows, we will assume the generalized continuum hypothesis. Given any cardinal $\mathfrak{m}$, let $\operatorname{cf}(\mathfrak{m})$ denote the cofinality of $\mathfrak{m}$. Note that $\operatorname{cf}(\mathfrak{m}) \leq \mathfrak{m}$, and $\operatorname{cf}(\mathfrak{m})$ is infinite whenever $\mathfrak{m}$ is infinite (e.g., see [7, Theorem 21.10]). An infinite cardinal $\mathfrak{m}$ is called *regular* if $\mathfrak{m} = \operatorname{cf}(\mathfrak{m})$. If $\mathfrak{m}$ is not regular, then it is called *singular*. Note that every successor cardinal is regular. Recall that $\mathfrak{m}^{\mathfrak{m}'}$ is defined to be the cardinal number $|A^B|$, where A and B are sets of cardinality $\mathfrak{m}$ and $\mathfrak{m}'$, respectively, and A^B is the set of all functions from B into A. If $\aleph_\alpha$ and $\aleph_\beta$ are infinite cardinals, then

$$\aleph_\alpha^{\aleph_\beta} = \left\{ \begin{array}{cc} \aleph_\alpha, & \aleph_\beta < \operatorname{cf}(\aleph_\alpha) \\ \aleph_{\alpha+1}, & \operatorname{cf}(\aleph_\alpha) \leq \aleph_\beta \leq \aleph_\alpha \\ \aleph_{\beta+1}, & \aleph_\alpha < \aleph_\beta \end{array} \right.$$

[7, Theorem 23.9]. Also, $m^{\aleph_\beta} = \aleph_{\beta+1}$ for every $2 \leq m < \infty$ [7, Theorem 22.13]. The notation $\sum_{i \in I} \mathfrak{m}_i$ is used to express the cardinality of the disjoint union $\coprod_{i\in I} A_i$, where $|A_i| = \mathfrak{m}_i$ for each $i \in I$. If I is an infinite indexing set with $\mathfrak{m}_i$ infinite for some $i \in I$, then $\sum_{i\in I} \mathfrak{m}_i = |I| \sup_{i\in I} \mathfrak{m}_i$. A detailed exposition of cardinal numbers can be found in chapter four of [7].

It is our goal to find conditions that ensure the equality $|[e]_{Q(R)}| = |[e]_R|$. We will see that it suffices to impose restrictions on the elements of the set $\mathcal{E}_e(R)$. The next two examples motivate such restrictions.

Example 4.10. *Let F be a field such that $|F| = \aleph_\omega$ and set $J = \mathbb{N}$. Suppose that R is the ring in Example 4.3. Choose an infinite subset I of $\mathbb{N}$, and let e be the idempotent with 1 in all coordinates $i \in I$ and 0 elsewhere. Then*

$$|[e]_R| = \aleph_\omega < \aleph_{\omega+1} = \aleph_\omega^{\aleph_0} = |[e]_{Q(R)}|,$$

where the second equality holds since cf$(\aleph_\omega) = \aleph_0$ [7, Theorem 22.11].

Example 4.11. *Let $K = \mathbb{Z}_2(X)$, and define the ring $R = \prod_{\mathbb{N}} \mathbb{Z}_2 + \bigoplus_{\mathbb{N}} K$. As in the Example 4.3, we have $Q(R) = \prod_{\mathbb{N}} K$. Choose an infinite subset I of $\mathbb{N}$, and let e be the idempotent with 1 in all coordinates $i \in I$ and 0 elsewhere. Then $|[e]_R| = \aleph_0 < \aleph_1 = |[e]_{Q(R)}|$.*

In Example 4.10, we found an element $e \in B(R)$ with an infinite set $E \in \mathcal{E}_e(R)$ such that $\mathrm{cf}(|[e']_R|) \leq |E| < |[e']_R|$ for some $e' \in E$ (namely, E was the set of minimal nonzero idempotents less than e, and e' could have been any element of E). In Example 4.11, we found an element $e \in B(R)$ with a set $E \in \mathcal{E}_e(R)$ such that $2e' \notin [e']_R$ for some $e' \in E$ (as before, E was the set of minimal nonzero idempotents less than e, and e' could have been any element of E). As a result, Lemma 4.9 fails for the element e. When R is a von Neumann regular ring such that $B(R)$ is complete, the desired equality will necessarily be obtained in the absence of such scenarios.

We shall say that an element $E \in \mathcal{E}_e(R)$ is *regular* if the relation $|E| < \sup\{|[e']_R| \mid e' \in E\}$ implies that either $\sup\{|[e']_R| \mid e' \in E\}$ is finite or $|E| < \mathrm{cf}(\sup\{|[e']_R| \mid e' \in E\})$. As a special case, $E \in \mathcal{E}_e(R)$ is regular if $|E| < \sup\{|[e']_R| \mid e' \in E\}$ implies that $\sup\{|[e']_R| \mid e' \in E\}$ is either finite or a regular cardinal. Clearly E is regular if it is finite.

Lemma 4.12. *Let R be a von Neumann regular ring, $e \in B(R)$, and $E \in \mathcal{E}_e(R)$. Assume that $B(R)$ is complete and $2e' \in [e']_R$ for all $e' \in E$. If $E \in \mathcal{E}_e(R)$ is regular, then $|[e]_R| = |\{f \in [e]_{Q(R)} \mid f \prec E\}|$.*

Proof. If E is finite, then $\{f \in [e]_{Q(R)} \mid f \prec E\} \subseteq [e]_R$ by Theorem 4.4. The reverse inclusion holds by Proposition 4.7, and hence the result follows. Suppose that E is infinite; say $|E| = \aleph_\alpha$ for some ordinal α. Let $\sup\{|[e']_R| \mid e' \in E\}| = \mathfrak{m}$. Define

$$F : \{A \in \mathcal{R}_e(R) \mid E_A = E\} \to \big(\cup\{[e']_R \mid e' \in E\}\big)^E$$

by the rule

$$F(A)(e) = r \text{ if and only if } e \in E \text{ with } e = e_r \text{ for some } r \in A.$$

Note that if $r_1, r_2 \in A$ with $r_1 \neq r_2$, then $e_{r_1}e_{r_2} = 0$. In particular, $r_1 \neq r_2$ implies that $e_{r_1} \neq e_{r_2}$, and therefore F is well-defined by definition. Also, F is injective since if $F(A_1) = F(A_2)$, then $r = F(A_1)(e_r) \in A_1$ for all $r \in A_2$, and similarly we have $A_1 \subseteq A_2$ so that $A_1 = A_2$. Hence

$$|\{A \in \mathcal{R}_e(R) \mid E_A = E\}| \leq |\big(\cup\{[e']_R \mid e' \in E\}\big)^E| = (\aleph_\alpha \mathfrak{m})^{\aleph_\alpha},$$

where the equality holds since the union is disjoint. Therefore,

$$|\{f \in [e]_{Q(R)} \mid f \prec E\}| \leq (\aleph_\alpha \mathfrak{m})^{\aleph_\alpha} = \left\{ \begin{array}{ll} \mathfrak{m}, & \mathfrak{m} > \aleph_\alpha \\ \aleph_{\alpha+1}, & \mathfrak{m} \leq \aleph_\alpha \end{array} \right.,$$

where the inequality follows by Corollary 4.6, and the equality follows since E is regular.

Let $e' \in E$ and $r \in [e']_R$. Then $e_r = e'$ and $e - e' \in B(R)$. Using Proposition 4.2 (2), it is easy to check that $\{r, e - e'\} \in \mathcal{R}_e(R)$, and thus Corollary 4.5 implies that $r + (e - e') \in [e]_R$. This shows that the mapping $[e']_R \to [e]_R$ given by $r \mapsto r + (e - e')$ is well-defined. Clearly it is also injective. Hence $|[e']_R| \leq |[e]_R|$ for all $e' \in E$, and therefore $\mathfrak{m} \leq |[e]_R|$. Also, $|E \setminus \{0\}| = \aleph_\alpha$, and thus

$$|[e]_R| \geq 2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

by Lemma 4.9. Therefore, we have $|[e]_R| = |\{f \in [e]_{Q(R)} \mid f \prec E\}|$ since Proposition 4.7 implies that the reverse inequality always holds. $\square$

Remark 4.13. *Although the following arguments generalize to arbitrary Boolean algebras, we shall assume that B is the Boolean algebra of idempotents of a commutative ring. Suppose that B is complete, and let $b \in B$. Then $B|_b = \{e \in B \mid e \leq b\}$ is a complete*

Boolean algebra, where the partial order on $B|_b$ *is inherited from* B. *Let* $s(b)$ *denote the least cardinal such that there is no set* $E \subseteq B|_b$ *of orthogonal idempotents with* $|E| = s(b)$. *Suppose that* B *is infinite. In [1, Corollary 2.7], it is shown that there exists a finite set of orthogonal idempotents* $\{b_1, ..., b_n\} \subseteq B$ *with* $\sup\{b_1, ..., b_n\} = 1$, *such that* $|B|_{b_i}| = \sum_{\mathfrak{m}<s(b_i)} |B|_{b_i}|^{\mathfrak{m}}$ *for each* $i = 1, ..., n$. *(In [1], this result is stated in the context of compact extremely disconnected topological spaces.) We will show that this implies* $|\mathcal{E}_e(R)| \leq |B(R)|_e|$ *whenever* e *is an element of a complete Boolean algebra* $B(R)$ *such that* $|B(R)|_e|$ *is infinite.*

Suppose that B *is complete and infinite. Let* $\mathcal{E} = \{E \subseteq B \mid e_1e_2 = 0$ *for all distinct* $e_1, e_2 \in E$ *and* $\sup E = 1\}$. *It suffices to show that* $|\mathcal{E}| \leq |B|$. *Note that the number of subsets of cardinality less than* $\mathfrak{n}$ *of a set* J *is at most* $\sum_{\mathfrak{m}<\mathfrak{n}} |J|^{\mathfrak{m}}$. *Using [1, Corollary 2.7], choose a set of orthogonal elements* $\{b_1, ..., b_n\} \subseteq B$ *such that* $\sup\{b_1, ..., b_n\} = 1$, *and*

$$|B|_{b_i}| = \sum_{\mathfrak{m}<\mathfrak{m}_i} |B|_{b_i}|^{\mathfrak{m}}$$

for each $i \in \{1, ..., n\}$, *where* $\mathfrak{m}_i$ *is the least cardinal such that there is no set* $E \subseteq B|_{b_i}$ *of orthogonal elements with* $|E| = \mathfrak{m}_i$. *By the choice of* $\mathfrak{m}_i$ *together with the fourth sentence of this paragraph, we have* $|\mathcal{E}_{b_i}| \leq \sum_{\mathfrak{m}<\mathfrak{m}_i} |B|_{b_i}|^{\mathfrak{m}} = |B|_{b_i}|$, *where* $\mathcal{E}_{b_i} = \{E \subseteq B|_{b_i} \mid e_1e_2 = 0$ *for all distinct* $e_1, e_2 \in E$, *and* $\sup E = b_i\}$. *Let* $\mathfrak{m}_j = \max_{1\leq i\leq n}\{\mathfrak{m}_i\}$. *Note that* $\mathfrak{m}_j$ *is infinite (and hence so is* $B|_{b_j}$*) since* B *is infinite (this is an application of König's Lemma, e.g., see [10, Exercise 25.12]). Let* $\mathcal{E}^* = \{E \in \mathcal{E} \mid e \leq b_i$ *for some* $i \in \{1, ..., n\}$ *for all* $e \in E\}$. *Noting that* $\{b_1, ..., b_n\} \subseteq B$ *is a set of orthogonal idempotents such that* $\sup\{b_1, ..., b_n\} = 1$, *we see that* $E \in \mathcal{E}^*$ *if and only if* $E = \cup_{i=1}^n E_i$ *for some* $E_i \in \mathcal{E}_{b_i}$ *(namely,* $E_i = B|_{b_i} \cap E$*). Then* $|\mathcal{E}^*| \leq \Pi_{i=1}^n |\mathcal{E}_{b_i}| \leq \max_{1\leq i\leq n}(|B|_{b_i}|)$, *where the second inequality follows since* $|\mathcal{E}_{b_i}| \leq |B|_{b_i}|$ *for each* $i \in \{1, ..., n\}$.

The mapping $\psi : \mathcal{E} \to \mathcal{E}^*$ *given by* $E \mapsto \{bb_i \mid b \in E$ *and* $i \in \{1, ..., n\}\}$ *is well-defined. Let* $E^* \in \mathcal{E}^*$*; say* $E^* = \cup_{i=1}^n E_i$, *(*$E_i \in \mathcal{E}_{b_i}$*). Note that every element belonging to a member of* $\psi^{-1}(\{E^*\})$ *is a* sum *(that is, supremum),* $b = bb_1 + \cdots + bb_n$, *with* $bb_i \in E_i \cup \{0\}$ *for each* $i \in 1, ..., n$. *Then an element* $b \in B$ *belongs to a member of* $\psi^{-1}(\{E^*\})$ *if and only if* $b = e_1 + \cdots + e_n$ *for some* $e_i \in E_i \cup \{0\}$. *This shows that* *the mapping*

$$(E_1 \cup \{0\}) \times \cdots \times (E_n \cup \{0\}) \to \cup\{E \mid E \in \psi^{-1}(\{E^*\})\}$$

given by the rule $(e_1, ..., e_n) \mapsto \sum_{i=1}^n e_i$ *is a well-defined surjection. Since the elements of* E^* *are orthogonal, this mapping is also injective. Hence,*

$$|\cup\{E \mid E \in \psi^{-1}(\{E^*\})\}| = \Pi_{i=1}^n |E_i \cup \{0\}| < \mathfrak{m}_j \le |B|_{b_j}|,$$

where the last inequality holds since a complete Boolean algebra is always strictly larger than any of its sets of orthogonal elements. (Indeed, if $E \subseteq B$ *is a set of nonzero orthogonal elements, then the mapping* $E' \mapsto \sup E'$ *defines an injection from the power set of* E *into* B*. In particular,* $|E| < |B|$ *for any set* $E \subseteq B$ *of orthogonal elements.) Also, if* $E' \in \psi^{-1}(\{E^*\})$*, then*

$$|E'| \le |\cup\{E \mid E \in \psi^{-1}(\{E^*\})\}| < \mathfrak{m}_j.$$

Since $|\cup\{E \mid E \in \psi^{-1}(\{E^*\})\}| < |B|_{b_j}|$*, it follows that the number of subsets of cardinality less than* $\mathfrak{m}_j$ *of* $\cup\{E \mid E \in \psi^{-1}(\{E^*\})\}$ *is at most* $\sum_{\mathfrak{m}<\mathfrak{m}_j} |B|_{b_j}|^{\mathfrak{m}}$*. But it has been shown that every member of* $\psi^{-1}(\{E^*\})$ *has cardinality strictly less than* $\mathfrak{m}_j$*, and thus*

$$|\psi^{-1}(\{E^*\})| \le \sum_{\mathfrak{m}<\mathfrak{m}_j} |B|_{b_j}|^{\mathfrak{m}} = |B|_{b_j}|.$$

Therefore,

$$\begin{aligned} |\mathcal{E}| &= |\cup_{E^* \in \mathcal{E}^*} \psi^{-1}(\{E^*\})| \\ &= \sum_{E^* \in \mathcal{E}^*} |\psi^{-1}(\{E^*\})| \\ &= |\mathcal{E}^*| \sup\{|\psi^{-1}(\{E^*\})| \mid E^* \in \mathcal{E}^*\} \\ &\le \big(\max_{1 \le i \le n}(|B|_{b_i}|)\big)|B|_{b_j}| \\ &= \max_{1 \le i \le n}(|B|_{b_i}|) \\ &\le |B|. \end{aligned}$$

If $2e' \in [e']_R$ for all $e' \le e$, then Proposition 4.2 (4) implies that the mapping $B(R)|_e \to [e]_R$ given by $e' \mapsto e' + e$ is well-defined. It is clearly injective, and thus the following lemma holds by the above remark.

Lemma 4.14. *Let R be a von Neumann regular ring. Suppose that $B(R)$ is complete and choose $e \in B(R)$. Assume that $2e' \in [e']_R$ for all $e' \le e$. If $|B(R)|_e|$ is infinite, then $|\mathcal{E}_e(R)| \le |[e]_R|$.*

Note that Lemma 4.14 can fail if $B(R)|_e$ is finite. For example, let $R = \prod_{i=1}^{5} \mathbb{Z}_3$ and let $e = (1,1,1,1,1)$. Then $\mathcal{E}_e(R) = 52$ (the fifth *Bell number*), but $|[e]_R| = 32$.

Given any element e of the complete Boolean algebra $B(R)$, we will say that a cardinal $\mathfrak{m}$ is *achieved by regular elements of* $\mathcal{E}_e(R)$ if there exists a set of regular elements $\{E_i\}_{i\in I} \subseteq \mathcal{E}_e(R)$ with $| \cup_{i\in I} \{f \in [e]_{Q(R)} \mid f \prec E_i\}| = \mathfrak{m}$. Let R be the ring in Example 4.10. Note that the regular elements of $\mathcal{E}_e(R)$ are precisely the finite elements. Letting $\{E_i\}_{i\in I}$ denote the family of all regular elements of $\mathcal{E}_e(R)$, we have $\cup_{i\in I}\{f \in [e]_{Q(R)} \mid f \prec E_i\} = [e]_R$, and hence $|[e]_{Q(R)}|$ is *not* achieved by regular elements.

We now state and prove the main theorem of this section.

Theorem 4.15. *Suppose that R is a von Neumann regular ring such that $B(R)$ is complete. Let $e \in B(R)$ be an element such that $2e' \in [e']_R$ for all $e' \le e$. Then $|[e]_{Q(R)}| = |[e]_R|$ if and only if $|[e]_{Q(R)}|$ is achieved by regular elements of $\mathcal{E}_e(R)$.*

Proof. The necessity is clear since $|[e]_{Q(R)}| = |[e]_R|$ implies that $|[e]_{Q(R)}|$ is achieved by the regular element $E = \{e\}$ (indeed, $[e]_R \subseteq [e]_{Q(R)}$ and $r \prec \{e\}$ for all $r \in [e]_R$).

To prove the converse, note that if $|E| < \infty$ for all $E \in \mathcal{E}_e(R)$, then $[e]_{Q(R)} = [e]_R$ by Theorems 4.4 and 4.8. In particular, $|[e]_{Q(R)}| = |[e]_R|$.

Suppose that $\mathcal{E}_e(R)$ contains an infinite element. Then $|[e]_R|$ is infinite by Lemma 4.9. Suppose that I is an indexing set such that $\{E_i\}_{i\in I} \subseteq \mathcal{E}_e(R)$ is a family of regular elements with $| \cup_{i\in I} \{f \in [e]_{Q(R)} \mid f \prec E_i\}| = |[e]_{Q(R)}|$. Then

$$\begin{aligned}
|[e]_{Q(R)}| &= | \cup_{i\in I} \{f \in [e]_{Q(R)} \mid f \prec E_i\}| \\
&\le |I| \sup_{i\in I} |\{f \in [e]_{Q(R)} \mid f \prec E_i\}| \\
&= |I||[e]_R| \\
&\le |\mathcal{E}_e(R)||[e]_R| \\
&= |[e]_R|,
\end{aligned}$$

where the second equality follows by Lemma 4.12 and the last equality follows by Lemma 4.14. Thus $|[e]_{Q(R)}| = |[e]_R|$ since Proposition 4.7 implies that the reverse inequality is always true. $\square$

It is known that $Q(\prod_{i\in I} R_i) = \prod_{i\in I} Q(R_i)$ for any family of rings $\{R_i\}_{i\in I}$ [11, Proposition 2.3.8]. It is easy to see that $|[(e_i)]_{\prod_{i\in I} R_i}| = \prod_{i\in I} |[e_i]_{R_i}|$ for any such product. Therefore, if $|[e_i]_{R_i}| = |[e_i]_{Q(R_i)}|$ for all $i \in I$, then $|[(e_i)]_{\prod_{i\in I} R_i}| = |[(e_i)]_{Q(\prod_{i\in I} R_i)}|$.

Note that a ring may have $|[e]_R| = |[e]_{Q(R)}|$ without satisfying the condition "$2e' \in [e]_R$ for all $e' \leq e$." For example, the equality is automatic whenever $R = Q(R)$. The following application of the previous theorem shows that a ring $R \neq Q(R)$ can have an idempotent e such that $2e \notin [e]_R$, and yet $|[e]_R| = |[e]_{Q(R)}|$. Moreover, this equality can hold even if $B(R)$ is not complete.

Recall that a ring R is *Boolean* if $x = x^2$ for all $x \in R$, i.e., $R = B(R)$ (as sets). In particular, a Boolean ring is von Neumann regular, and has characteristic 2. Moreover, a ring R is Boolean if and only if $Q(R)$ is Boolean [11, Lemma 2.4.4].

Corollary 4.16. *Suppose that I is an indexing set, $\{F_i\}_{i\in I}$ is a family of rationally complete rings, A is a von Neumann regular ring with $B(A)$ complete, $|A| < \aleph_\omega$, $2 \notin Z(A)$, and B is a Boolean ring. Let $\mathcal{S}$ be a nonempty subset of $\{A, B, \prod_{i\in I} F_i\}$. If $R \cong \prod_{S\in\mathcal{S}} S$, then $|[e]_R| = |[e]_{Q(R)}|$ for all $e \in B(R)$.*

Proof. We might as well assume that $R = \prod_{S\in\mathcal{S}} S$. By the above comments, it suffices to show that the result is true when $\mathcal{S}$ is a singleton set. Clearly it is true when $\mathcal{S} = \{\prod_{i\in I} F_i\}$ since $\prod_{i\in I} F_i = \prod_{i\in I} Q(F_i) = Q\big(\prod_{i\in I} F_i\big)$. To see that it holds for $\mathcal{S} = \{B\}$, recall that each equivalence class of a von Neumann regular ring is represented by a *unique* idempotent. Thus, since $Q(B)$ is Boolean, $|[e]_B| = 1 = |[e]_{Q(B)}|$ for all $e \in B$.

It remains to show that the result holds for $\mathcal{S} = \{A\}$. Since $\aleph_\omega$ is the smallest singular cardinal, every element of $\mathcal{E}_e(A)$ is regular. In particular, Theorem 4.8 implies that $|[e]_{Q(A)}|$ is achieved by regular elements of $\mathcal{E}_e(A)$ for all $e \in A$. Finally, 2 is a unit of A since it is not a zero-divisor, and thus $2e \in [e]_A$ for all $e \in B(A)$. Therefore, $|[e]_A| = |[e]_{Q(A)}|$ for all $e \in B(A)$ by Theorem 4.15. $\square$

It is easy to illustrate the convenience of the previous corollary. For example, let $F = \mathbb{Q}$ and $J = \mathbb{N}$ in Example 4.3. Then R is a von Neumann regular ring, $B(R)$ is complete, $|R| = \aleph_1 < \aleph_\omega$, and $2 \notin Z(R)$ (here, 2 is the element of R with the *integer* 2 in all coordinates). Therefore, $|[e]_R| = |[e]_{Q(R)}|$ for all $e \in B(R)$ by Corollary 4.16. Note that we were able to draw this conclusion without knowing anything about $Q(R)$.

Several of the previous results were proved under the assumption that "$B(R)$ is complete." We conclude this section with an example showing that this "completeness" statement must be included in all of those results. However, we only emphasize the necessity for Theorem 4.15.

Example 4.17. *Let $F_n = \mathbb{Q}$ for all $n \in \mathbb{N}$. Define $R \subseteq \prod_{n\in\mathbb{N}} F_n$ to be the ring such that $(r_n) \in R$ if and only if there exists $N \in \mathbb{N}$ such that $r_n = r_N$ for all $n \geq N$. As in Example 4.3, one shows that $Q(R) = \prod_{n\in\mathbb{N}} F_n$. For any $n \in \mathbb{N}$, let $e(n)$ be the element of $B(R)$ with 1 in the coordinate n and 0 elsewhere. Let $N \in \mathbb{N}$ and define $E = \{e(n) \in B(R) \mid n \geq N\}$. Then the idempotent $e = \sup E$ is clearly an element of $B(R)$ (e is the element with 1 in all coordinates $n \geq N$ and 0 elsewhere). Note that $B(R)$ is not complete since the set $\{e(2n+1)\}_{n=N}^{\infty} \subseteq B(R)$ has no supremum in $B(R)$. It is easy to see that $E \in \mathcal{E}_e(R)$ is regular and $|[e]_{Q(R)}|$ is achieved by E. However,*

$$|[e]_R| = \aleph_0 < \aleph_1 = |[e]_{Q(R)}|.$$

4.3 Zero-Divisor Graphs

The idea of a zero-divisor graph was introduced by I. Beck in [3]. While he was mainly interested in colorings, we shall investigate the interplay between ring-theoretic properties and graph-theoretic properties. This approach begun in a paper by D.F. Anderson and P.S. Livingston [2], and has since continued to evolve (e.g., [2], [1], [5], [7], [11], [13], [12], and [16]).

Let Γ be a graph and let $v \in V(\Gamma)$. As in [1], a vertex $w \in V(\Gamma)$ is called a *complement* of v if v is adjacent to w, and the edge $v - w$ is not an edge of any triangle in Γ. In ring-theoretic terms, this is the same as saying that w is a complement of v in $\Gamma(R)$ if and only if

$0 \neq v, w \in R$ are distinct, $vw = 0$, and $\text{ann}(v) \cap \text{ann}(w) \subseteq \{0, v, w\}$. As in [1], we will say that Γ is *complemented* if every vertex has a complement, and is *uniquely complemented* if it is complemented and any two complements of a vertex are adjacent to the same vertices. Note that $\Gamma(R)$ is uniquely complemented if and only if either R is nonreduced and $\Gamma(R)$ is a *star graph* (i.e., a graph with at least two vertices such that there exists a vertex which is adjacent to every other vertex, and these are the only adjacency relations), or R is reduced and $T(R)$ is von Neumann regular [1, Theorems 3.5 and 3.9]. Moreover, [1, Theorem 3.5] shows that a reduced ring is uniquely complemented if and only if it is complemented.

Suppose that R is a von Neumann regular ring. Let $x \in R$. Then there is a unit $u \in R$ such that $xu = e_x$, the unique idempotent satisfying $[x]_R = [e_x]_R$. Hence $1 - e_x$ is a complement of x since $(1 - e_x)x = 0$, and $tx = 0 = t(1 - e_x)$ implies $t = te_x = t(xu) = (tx)u = 0$. By [1, Theorem 3.5], $\Gamma(R)$ is uniquely complemented. Thus $\text{ann}(x') = \text{ann}(1 - e_x)$ for every complement x' of x.

In this section, we explore some consequences of the results given in Section 4.2. Theorem 4.19 gives sufficient conditions to conclude that a reduced ring R satisfies $\Gamma(R) \cong \Gamma(Q(R))$. In Theorem 4.20, we explain precisely when $\Gamma(R) \cong \Gamma(Q(R))$ for "small" reduced rings with $2 \notin Z(R)$. Finally, Theorem 4.21 shows that a Boolean ring R satisfies $\Gamma(R) \cong \Gamma(Q(R))$ if and only if $R \cong Q(R)$. Moreover, the zero-divisor graphs of such Boolean rings are completely characterized.

Let $S \subseteq V(\Gamma(R))$ be a family of vertices. As in [11], we shall call v a *central vertex of* S if v is adjacent to s for all $s \in S$. The following lemma is implicit in the proofs of Lemma 3.3 and Theorem 3.4 of [11].

Lemma 4.18. *Let R be a von Neumann regular ring. Then $B(R)$ is complete if and only if whenever $\emptyset \neq S \subseteq V(\Gamma(R))$ is a family of vertices that has a central vertex, there exists a central vertex v of S possessing a complement that is adjacent to all of the central vertices of S (and hence, since $\Gamma(R)$ is uniquely complemented, every complement of v is adjacent to every central vertex of S).*

Proof. To prove the necessity of the stated conditions, suppose that there is a $\emptyset \neq S \subseteq V(\Gamma(R))$ with central vertices such that, if v is any central vertex of S, then there exists a central vertex w of S with $(1 - e_v)w \neq 0$. Let $S' = \{1 - e_s \in B(R) \mid s \in S\}$, and let $C = \{b \in$

$B(R) \setminus \{0\} \mid be_s = 0$ for all $s \in S\}$. Note that $C \neq \emptyset$ since $e_v \in C$ whenever v is a central vertex of S. Moreover, every element of C is a central vertex of S. Therefore, to every $b \in C$ there corresponds a central vertex w of S such that $(1-b)w \neq 0$. In particular, $(1-b)e_w \neq 0$. Let $f = \inf S'$ (in $D(B(R))$). Note that $f \neq 0$ since $b \leq f$ whenever $b \in C$. Thus, if $f \in B(R)$, then $f \in C$ and hence there is a central vertex w of S such that $fe_w \neq e_w$. But $e_w \in C$, and hence $e_w \leq f$. This is a contradiction, and therefore $f \notin B(R)$. Since the infimum of a set taken in $B(R)$ agrees with the infimum taken in $D(B(R))$, we have that $B(R)$ is not complete.

Conversely, suppose that the stated conditions on $V(\Gamma(R))$ are satisfied. Let $\emptyset \neq S \subseteq B(R)$ be any family of elements. It is clear that $\inf S = 0$ if $0 \in S$. Suppose that $0 \notin S$. If $S = \{1\}$, then $\inf S = 1$. If $S \neq \{1\}$ and contains 1, then we may remove 1 from S without changing $\inf S$. Thus we may assume $0, 1 \notin S$.

Since R is reduced, $D = \mathrm{ann}_R(\{1-s\}_{s\in S})+(\{1-s\}_{s\in S})$ is dense in R, and hence in $Q(R)$. Let $\inf S = f \in B(Q(R))$. Then $f(\{1-s\}_{s\in S}) = (0)$. Suppose that S has no infimum in $B(R)$. Then $f \neq 0$ since $f \notin B(R)$. Evidently, $\mathrm{ann}_R(\{1-s\}_{s\in S}) \neq (0)$ since otherwise $fD = (0)$. That is, $C = \{v \in V(\Gamma(R)) \mid v$ is adjacent to $1-s$ for all $s \in S\} \neq \emptyset$. By hypothesis, there is a $v^* \in C$ whose complements are adjacent to every element of C. In particular, $v(1-e_{v^*}) = 0$ for all $v \in C$. Since $v^* \in C$, it follows that $e_{v^*}(1-s) = 0$ for all $s \in S$; that is, $e_{v^*} \leq s$ for all $s \in S$. Moreover, if $0 \neq v \in B(R)$ with $v \leq s$ for all $s \in S$, then $v \in C$ so that $v(1-e_{v^*}) = 0$; that is, $v \leq e_{v^*}$. But this shows that $f = \inf S = e_{v^*} \in B(R)$, a contradiction. Thus every $\emptyset \neq S \subseteq B(R)$ has an infimum, and hence $B(R)$ is a complete Boolean algebra. □

Let R be any ring. We shall say that $\Gamma(R)$ is *central vertex complete*, or *c.v.-complete*, if $\Gamma(R)$ satisfies the condition of Lemma 4.18. Thus Lemma 4.18 can be restated as follows:

Let R be a von Neumann regular ring. Then $B(R)$ is complete if and only if $\Gamma(R)$ is c.v.-complete.

As already noted, every ring R satisfies $\Gamma(R) \cong \Gamma(T(R))$ by [1, Theorem 2.2]. In [1, Theorem 4.1], it is shown that the zero-divisor graphs of two von Neumann regular rings R and S are isomorphic if and only if there is a Boolean algebra isomorphism $\varphi : B(R) \to B(S)$

such that $|[e]_R| = |[\varphi(e)]_S|$ for all $1 \neq e \in B(R)$. Therefore, Examples 4.10 and 4.11 illustrate that a von Neumann regular ring R may fail to satisfy the condition $\Gamma(R) \cong \Gamma(Q(R))$. (Indeed, if R is the ring in Example 4.10, then $|[e]_R| < \aleph_{\omega+1}$ for all $e \in B(R)$. If R is the ring in Example 4.11, then $|[e]_R| < \aleph_1$ for all $e \in B(R)$.)

Recall that a von Neumann regular ring R satisfies $B(R) = B(Q(R))$ if and only if $B(R)$ is complete [4, Theorem 11.9].

Theorem 4.19. *Let R be a reduced ring. Suppose that $\Gamma(R)$ is a complemented c.v.-complete graph. If $2e \in [e]_{T(R)}$ and $|[e]_{Q(T(R))}|$ is achieved by regular elements of $\mathcal{E}_e(T(R))$ for all $e \in B(T(R)) \setminus \{1\}$, then $\Gamma(R) \cong \Gamma(Q(R))$.*

Proof. Suppose that $\Gamma(R)$ is a complemented c.v.-complete graph. Note that it makes sense to speak of $\mathcal{E}_e(T(R))$ since $T(R)$ is von Neumann regular by [1, Theorem 3.5]. Also, $\Gamma(R) \cong \Gamma(T(R))$ implies that $B(T(R))$ is complete by Lemma 4.18 . Thus $B(T(R)) = B(Q(T(R)))$ by [4, Theorem 11.9]. Suppose that $2e \in [e]_{T(R)}$ and $|[e]_{Q(T(R))}|$ is achieved by regular elements of $\mathcal{E}_e(T(R))$ for all $1 \neq e \in B(T(R))$. Then Theorem 4.15 implies that $|[e]_{T(R)}| = |[e]_{Q(T(R))}|$ for all $1 \neq e \in B(T(R))$. Thus $\Gamma(T(R)) \cong \Gamma(Q(T(R))) = \Gamma(Q(R))$, where the isomorphism follows by [1, Theorem 4.1] and the equality follows since $Q(T(R)) = Q(R)$; hence $\Gamma(R) \cong \Gamma(Q(R))$ by [1, Theorem 2.2]. $\square$

To apply the previous result, one must have information regarding the zero-divisor graph of R, as well as information about the total quotient ring of R. However, information regarding $T(R)$ is unnecessary when R is "small."

Theorem 4.20. *Let R be a reduced ring. Suppose that $|V(\Gamma(R))| < \aleph_\omega$. If $2 \notin Z(R)$, then $\Gamma(R) \cong \Gamma(Q(R))$ if and only if $\Gamma(R)$ is a complemented c.v.-complete graph.*

Proof. Note that $|V(\Gamma(T(R)))| < \aleph_\omega$ since $\Gamma(R) \cong \Gamma(T(R))$. Also, 2 is a unit in $T(R)$ since $2 \notin Z(R)$ implies that $2 \notin Z(T(R))$. Finally,

$$|T(R)| \leq |Z(T(R))|^2 = (|V(\Gamma(T(R)))| + 1)^2 < \aleph_\omega$$

(the first inequality is an application of the first isomorphism theorem on the $T(R)$-module homomorphism $T(R) \to T(R)r$ defined by $s \mapsto sr$, where $0 \neq r \in Z(T(R))$).

If $\Gamma(R) \cong \Gamma(Q(R))$, then $\Gamma(R)$ is complemented since $Q(R)$ is von Neumann regular, and is c.v.-complete since $B(Q(R))$ is complete. Conversely, suppose that $\Gamma(R)$ is a complemented c.v.-complete graph. Then $\Gamma(R) \cong \Gamma(T(R))$ implies that $T(R)$ is von Neumann regular and $B(T(R))$ is complete. Therefore, $B(T(R)) = B(Q(T(R)))$. Moreover, $|[e]_{T(R)}| = |[e]_{Q(T(R))}|$ for all $e \in B(T(R))$ by Corollary 4.16. Thus

$$\Gamma(R) \cong \Gamma(T(R)) \cong \Gamma(Q(T(R))) = \Gamma(Q(R)),$$

where the second isomorphism follows from [1, Theorem 4.1]. □

Note that a Boolean ring R is rationally complete if and only if $B(R)$ is a complete Boolean algebra [9, Theorem 12.3.4]. The following theorem was proved in [11, Theorem 3.4 and Theorem 4.1]. However, a simpler proof is available with the aid of Lemma 4.18.

Theorem 4.21. *Let R be a Boolean ring. Then the following conditions are equivalent:*

(1) R is rationally complete.

(2) $\Gamma(R)$ is c.v.-complete.

(3) $\Gamma(R) \cong \Gamma(Q(R))$.

Proof. (1)⇔(2): Lemma 4.18 implies that $B(R)$ is complete if and only if $\Gamma(R)$ is c.v.-complete; that is, R is rationally complete if and only if $\Gamma(R)$ is c.v.-complete.

(3)⇒(2): Since $B(Q(R))$ is complete, (3) implies that $\Gamma(R)$ is c.v.-complete by Lemma 4.18.

(1)⇒(3): This is obvious. □

We end this section by observing that the zero-divisor graphs of rationally complete Boolean rings are completely characterized: It is known that a ring R is Boolean if and only if either $R \cong \mathbb{Z}_2$ or $R \notin \{\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2)\}$ and $\Gamma(R) \neq \emptyset$ has the property that every vertex has a unique complement [11, Theorem 2.5]. Taking this together with the previous theorem, we have the following corollary.

Corollary 4.22. *Suppose that R is a ring which is not isomorphic to either of the rings in the set $\{\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2)\}$. Then R is a rationally complete Boolean ring if and only if either $R \cong \mathbb{Z}_2$ or $\Gamma(R) \neq \emptyset$ is a c.v.-complete graph such that every vertex has a unique complement.*

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Chapter 5

Rationally $\aleph_\alpha$-Complete Commutative Rings of Quotients

Abstract. If $\{R_i\}_{i \in I}$ is a family of rings, then it is well-known that $Q(R_i) = Q(Q(R_i))$ and $Q\big(\prod_{i \in I} R_i\big) = \prod_{i \in I} Q(R_i)$, where $Q(R)$ denotes the maximal ring of quotients of R. This paper contains an investigation of how these results generalize to a particular class of rings of quotients of a commutative ring.

5.1 Introduction

Let R be a commutative ring with identity. Define the set of *zero-divisors* of R to be $Z(R) = \{r \in R \mid rs = 0 \text{ for some } 0 \neq s \in R\}$. The total quotient ring $T(R)$ of R is the ring $R_{R \setminus Z(R)}$. Multiplication by any element a/b of $T(R)$ defines an R-module homomorphism from the principal ideal bR into R. More generally, there are ring extensions of R consisting entirely of elements which can be viewed as R-module homomorphisms from *dense* ideals of R (a set $D \subseteq R$ is dense in R if $\mathrm{ann}_R(D) = \{0\}$) into R. When R is an integral domain, any such homomorphism can be defined by elements of $T(R)$. For example, if $\mathfrak{d} \subseteq R$ is a dense ideal and $f \in \mathrm{Hom}_R(\mathfrak{d}, R)$, then f can be identified with $f(d)/d \in T(R)$ for any $0 \neq d \in \mathfrak{d}$. Similarly, any such homomorphism whose domain contains an element of $R \setminus Z(R)$ can be identified with an element of $T(R)$. However, if $\mathfrak{d} \subseteq Z(R)$ is dense, then $\mathrm{Hom}_R(\mathfrak{d}, R)$ may have elements which cannot be defined as multiplication by a member of $T(R)$. In what follows, we define a class of ring extensions of $T(R)$ whose elements can be described as R-module homomorphisms on dense ideals having generating sets that meet certain cardinality restrictions. Motivated by two well-known results, we then inspect the way that passing to these extensions behaves with respect to iteration and direct products.

To begin the construction, let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be dense ideals of R, and suppose that $f_i \in \mathrm{Hom}_R(\mathfrak{d}_i, R)$ $(i = 1, 2)$. Then $\mathfrak{d}_1\mathfrak{d}_2$ is a dense ideal of R, and $\{f_1 + f_2, f_1 \circ f_2\} \subseteq \mathrm{Hom}_R(\mathfrak{d}_1\mathfrak{d}_2, R)$. Let $F = \cup_{\mathfrak{d}} \mathrm{Hom}_R(\mathfrak{d}, R)$, where the union is taken over all dense ideals of R. Then $Q(R) = F/{\sim}$ is a commutative ring, where $f_1 \sim f_2$ if and only if $f_1|_D = f_2|_D$ for some dense set D of R. In Section 2.3 of [11], $Q(R)$ is called the *complete ring of quotients of* R. One checks that R is embedded in $Q(R)$ by identifying any element $r \in R$ with the equivalence class containing the homomorphism defined via multiplication by r.

A ring extension $R \subseteq S$ is called a *ring of quotients of* R if $f^{-1}R = \{r \in R \mid fr \in R\}$ is dense in S for all $f \in S$. For example, $T(R)$ is a ring of quotients of R. Suppose that S is a ring of quotients of R. Then the correspondence given by identifying any element $f \in S$ with the equivalence class containing $(r \mapsto fr) \in \mathrm{Hom}_R(f^{-1}R, R)$ is an extension of the mapping $R \to Q(R)$ described above, and embeds S into $Q(R)$. Therefore, every ring of quotients of R can be regarded as a subring of $Q(R)$. It follows that any dense set in R is dense in every ring of quotients of R. Also, R has a *unique* maximal (with respect to inclusion) ring of quotients, which is isomorphic to $Q(R)$ [11, Proposition 2.3.6]. In this paper, isomorphic rings will not be distinguished. In particular, we shall denote the maximal ring of quotients of R by $Q(R)$.

Let $R \subseteq S \subseteq T$ be rings. Then T is a ring of quotients of R if and only if T is a ring of quotients of S and S is a ring of quotients of R [4, 1.4]. It follows that $Q(R) = Q(S)$ whenever $R \subseteq S$ is a ring of quotients. For more on rings of quotients, see [4] and [9].

Recall that a *cardinal number*, or *cardinal* for short, is any ordinal number that is not in bijective correspondence with any strictly smaller ordinal. The *cardinality* $|I|$ of a set I is the *unique* cardinal which is in bijective correspondence with I. We shall use the "aleph" notation to denote infinite cardinals. That is, any infinite cardinal will be denoted by $\aleph_\alpha$ for some ordinal α, where $\aleph_\alpha \leq \aleph_\beta$ if and only if $\alpha \leq \beta$. Note that ordinals are sets, and that the relation $\alpha < \beta$ is equivalent to $\alpha \in \beta$ for any ordinals α and β. Every infinite cardinal is a limit ordinal. A cardinal $\aleph_\alpha$ is a *limit cardinal* if α is a limit ordinal. If $\aleph_\alpha$ is a limit cardinal with $\alpha \neq 0$, then the set $\{\aleph_\theta\}_{\theta<\alpha}$ is cofinal in $\aleph_\alpha$. In particular, the *cofinality* of any infinite limit cardinal $\aleph_\alpha$ (i.e., the smallest cardinal that is cofinal in $\aleph_\alpha$) with $\alpha \neq 0$ is at most $|\alpha|$, that is, $\mathrm{cf}(\aleph_\alpha) \leq |\alpha|$. A cardinal $\aleph_\alpha$ is called *singular* if $\mathrm{cf}(\aleph_\alpha) < \aleph_\alpha$. Any cardinal which is *not* singular (i.e., any cardinal that is equal to its cofinality) is called *regular*. Every infinite singular cardinal is necessarily a limit cardinal. If $\{A_i\}_{i\in I}$ is a family of sets, then $|\cup_{i\in I} A_i| \leq |I| \sup\{|A_i| \mid i \in I\}$. In particular, if $|\cup_{i\in I} A_i|$ is infinite, then $|\cup_{i\in I} A_i| \leq \max\{|I|, \sup\{|A_i| \mid i \in I\}\}$. Also, if $|I| < \mathrm{cf}(\aleph_\alpha)$ and $|A_i| < \aleph_\alpha$ for all $i \in I$, then $|\cup_{i\in I} A_i| < \aleph_\alpha$; we shall refer to this fact as the *pigeonhole principle*. For a detailed exposition on infinite cardinals, see [7].

Let α be any ordinal. Given any infinite subsets D_1 and D_2 of a commutative ring, note that $|\{d_1 d_2 \mid d_1 \in D_1, d_2 \in D_2\}| \leq \max\{|D_1|, |D_2|\}$. It follows that the set $Q_\alpha(R) = \{f \in Q(R) \mid \text{there exists a } D \subseteq f^{-1}R$ such that $\mathrm{ann}_R(D) = \{0\}$ and $|D| < \aleph_\alpha\}$ is a commutative ring. Clearly $Q_\alpha(R)$ is a ring of quotients of R. Moreover, the inclusions

$$R \subseteq Q_\alpha(R) \subseteq Q_\beta(R) \subseteq Q(R)$$

hold for all $\alpha \leq \beta$. Also, there exists an ordinal α such that $Q_\beta(R) =$

$Q(R)$ whenever $\alpha \leq \beta$ (e.g., choose α so that $|R| < \aleph_\alpha$). We will say that R is *rationally $\aleph_\alpha$-complete* if $R = Q_\alpha(R)$. If R is rationally $\aleph_\alpha$-complete, then it is easy to see that R is rationally $\aleph_\beta$-complete for all $\beta \leq \alpha$. If R is rationally $\aleph_\alpha$-complete for all α (i.e., $R = Q(R)$), then we will say that R is *rationally complete*.

The ring $Q_0(R)$ was studied by T.G. Lucas in [12], where the main focus was on integral closure. For a ring of continuous functions R, the relationship between $Q_\alpha(R)$ and certain topological properties of the underlying space was examined by A.I. Singh and M.A. Swardson in [8]. In [5], A.W. Hager and J. Martinez study the ring $Q_\alpha(R)$ for a regular uncountable cardinal $\aleph_\alpha$. The results in [5] are mainly concerned with the case when R is an Archimedean f-ring. The investigation in this paper is motivated by, and has a significant role in the development of [4]. In [4], the author is interested in zero-divisor graphs of rings of quotients of a commutative ring.

The following two results are well-known: The ring $Q(R)$ is rationally $\aleph_\alpha$-complete for every ordinal α, i.e., $Q(Q(R)) = Q(R)$ [11, Proposition 2.3.5]. Also, $Q(\prod_{i\in I} R_i) = \prod_{i\in I} Q(R_i)$, where $\{R_i\}_{i\in I}$ is any family of rings [11, Proposition 2.3.8]. In this paper, we seek to generalize these two results by examining the rational $\aleph_\alpha$-completion of a ring $Q_\beta(R)$, where α and β are ordinal numbers. In particular, we provide set-theoretic bounds on the ring $Q_\alpha(Q_\beta(R))$, and give sufficient conditions to conclude that $Q_\alpha(Q_\beta(R)) = Q_\theta(R)$ for some ordinal θ (e.g., Theorem 5.1 and Corollary 5.10). Also, set-theoretic bounds are given on the ring $\prod_{i\in I} Q_\alpha(R_i)$, and sufficient conditions are provided to conclude that $Q_\alpha(\prod_{i\in I} R_i) = \prod_{i\in I} Q_\alpha(R_i)$ for a given family of commutative rings $\{R_i\}_{i\in I}$ (see Theorem 5.3). Examples are presented to illustrate the implications of each result.

5.2 The Rational $\aleph_\alpha$-Completion of $Q_\beta(R)$

The two main theorems of this paper are established in this section (Theorem 5.1 and Theorem 5.3). However, these results raise natural questions, which are studied in Section 5.3. We begin with a theorem which suggests that one may have $Q_\beta(Q_\beta(R)) \neq Q_\beta(R)$ for some ordinal β. Indeed, the existence of such a ring is proved by letting $\alpha = \beta = \omega$ in Example 5.8 (recall that ω is the smallest infinite ordinal, and $\aleph_\omega$ is the least infinite singular cardinal).

In [12, Lemma 5], it is shown that an element $f \in Q(R)$ is necessarily a member of $Q_0(R)$ whenever $f^{-1}Q_0(R)$ contains a dense finite set. The author accomplishes this by viewing $Q_0(R)$ as a subring of $T(R[X])$ [12, Lemma 4]. Since $Q(Q_0(R)) = Q(R)$, it follows that $Q_0(Q_0(R)) \subseteq Q_0(R)$. But $Q_0(R) \subseteq Q_0(Q_0(R))$ is always true, and therefore $Q_0(Q_0(R)) = Q_0(R)$. In [5, Proposition 2.2], it is shown that $Q_\alpha(Q_\alpha(R)) = Q_\alpha(R)$ whenever $\aleph_\alpha$ is a regular cardinal. These facts are generalized in the following theorem.

Theorem 5.1. *Let R be a commutative ring. Suppose that α and β are ordinal numbers, and let $\theta = \max\{\alpha, \beta\}$. Then $Q_\theta(R) \subseteq Q_\alpha\big(Q_\beta(R)\big) \subseteq Q_{\theta+1}(R)$. If $Q_\theta(R) \subsetneq Q_\alpha\big(Q_\beta(R)\big)$, then the following must hold:*

(1) β is a limit ordinal.

(2) $\alpha \leq \beta$.

(3) $\aleph_\alpha > cf(\aleph_\beta)$.

In particular, if $\aleph_\beta$ is a regular cardinal, then $Q_\theta(R) = Q_\alpha(Q_\beta(R))$.

Proof. If $\theta = \beta$, then $Q_\theta(R) = Q_\beta(R) \subseteq Q_\alpha\big(Q_\beta(R)\big)$. Suppose that $\theta = \alpha$, and let $f \in Q_\theta(R)$. In particular, $f \in Q(R) = Q\big(Q_\beta(R)\big)$. It is easy to see that $f^{-1}R \subseteq f^{-1}Q_\beta(R)$, and hence $f^{-1}Q_\beta(R)$ contains a dense set D such that $|D| < \aleph_\alpha$. Therefore, $f \in Q_\alpha\big(Q_\beta(R)\big)$. Thus $Q_\theta(R) \subseteq Q_\alpha\big(Q_\beta(R)\big)$.

Let $f \in Q_\alpha\big(Q_\beta(R)\big)$. There is a dense set $D \subseteq f^{-1}Q_\beta(R)$ such that $|D| < \aleph_\alpha$. Let $d \in D$. Then $d \in Q_\beta(R)$ implies that there exists a dense set $E_d \subseteq d^{-1}R$ such that $|E_d| < \aleph_\beta$. Also, $fd \in Q_\beta(R)$ and hence there exists a dense set $F_d \subseteq (fd)^{-1}R$ such that $|F_d| < \aleph_\beta$. Let $G_d = \{eg \mid e \in E_d \text{ and } g \in F_d\}$. Then G_d is dense, $G_d \subseteq (fd)^{-1}R$, and $|G_d| < \aleph_\beta$. Thus $|dG_d| < \aleph_\beta$ for all $d \in D$. Also, $dG_d \subseteq R$, and therefore $dG_d \subseteq f^{-1}R$. Let $H = \cup_{d\in D} dG_d$. Then $H \subseteq f^{-1}R$. Moreover, H is dense since if $rH = \{0\}$, then $rdG_d = \{0\}$ for all $d \in D$; but each G_d is dense, and therefore $rd = 0$ for all $d \in D$. That is, $r \in \text{ann}_R(D) = \{0\}$. It follows that $f \in Q_{\theta+1}(R)$ since

$$|H| \leq |D| \sup\{|dG_d| \mid d \in D\} \leq \aleph_\theta,$$

and therefore $Q_\alpha(Q_\beta(R)) \subseteq Q_{\theta+1}(R)$. This proves the first claim.

Note that if (2) fails, then $\sup\{|dG_d| \mid d \in D\} \leq \aleph_\beta < \aleph_\alpha$. Thus

$$|H| \leq |D| \sup\{|dG_d| \mid d \in D\} < \aleph_\alpha = \aleph_\theta.$$

If (3) fails to hold, then the inequalities $|dG_d| \leq |G_d| < \aleph_\beta$ and $|D| < \aleph_\alpha \leq \mathrm{cf}(\aleph_\beta)$ imply that $|H| < \aleph_\beta$ by the pigeonhole principle. Therefore, if (2) or (3) fail, then $|H| < \aleph_\theta$. If (2) and (3) hold, then $\mathrm{cf}(\aleph_\beta) < \aleph_\alpha \leq \aleph_\beta$. In particular, $\aleph_\beta$ is a singular cardinal, and thus (1) holds. So if (1) fails, then either (2) or (3) must fail, and hence $|H| < \aleph_\theta$. Therefore, if any of (1), (2), or (3) fails, then $f \in Q_\theta(R)$, and it follows that $Q_\theta(R) = Q_\alpha(Q_\beta(R))$.

The "in particular" statement holds since either (2) or (3) fails whenever $\aleph_\beta$ is a regular cardinal. $\square$

Corollary 5.2. *Let R be a commutative ring. Then $Q_\alpha(Q_\beta(R)) = Q_\beta(Q_\alpha(R)) = Q_\theta(R)$ for any ordinals $\alpha, \beta < \omega$, where $\theta = \max\{\alpha, \beta\}$. Moreover, $Q_0(Q_\beta(R)) = Q_\beta(Q_0(R)) = Q_\beta(R)$ for every ordinal β.*

Proof. Both statements are immediate consequences of Theorem 5.1. The first statement follows since $\aleph_\alpha$ and $\aleph_\beta$ are regular cardinals whenever $\alpha, \beta < \omega$. The second statement holds since the cofinality of any infinite cardinal is at least $\aleph_0$; that is, $\aleph_0 \leq \mathrm{cf}(\aleph_\beta)$ for every ordinal β. $\square$

We now turn our attention to rational $\aleph_\alpha$-completions of direct products. Note that Corollary 5.5 is a special case of a more general theorem of Utumi for noncommutative rings. In fact, the "$Q_\alpha\big(\prod_{j\in J} R_j\big) \subseteq \prod_{j\in J} Q_\alpha(R_j)$" portion in the proof of the following theorem is a close mimicry of the proof given in [11, Proposition 4.3.9].

Theorem 5.3. *Let $\{R_j\}_{j\in J}$ be a family of commutative rings. Then $Q_\alpha\big(\prod_{j\in J} R_j\big) \subseteq \prod_{j\in J} Q_\alpha(R_j) \subseteq Q_{\alpha+1}\big(\prod_{j\in J} R_j\big)$. If $Q_\alpha(\prod_{j\in J} R_j) \subsetneq \prod_{j\in J} Q_\alpha(R_j)$, then the following must hold:*

(1) α is a limit ordinal.

(2) $|J| \geq \mathrm{cf}(\aleph_\alpha)$.

Proof. Let α be an ordinal and suppose that $f \in Q_\alpha(\prod_{j\in J} R_j)$. There exists a dense set $D \subseteq f^{-1} \prod_{j\in J} R_j$ with $|D| < \aleph_\alpha$. For every $i \in J$, let $\iota_i : R_i \to \prod_{j\in J} R_j$ and $\pi_i : \prod_{j\in J} R_j \to R_i$ be the usual injection and projection maps. Let $i \in J$. If $r\pi_i(D) = \{0\}$ for some $r \in R_i$, then the element of $\prod_{j\in J} R_j$ with r in the i-coordinate and 0 elsewhere annihilates D. Since D is dense, this element must be (0); in particular, $r = 0$. Thus $\pi_i(D)$ is dense in R_i. Clearly $|\pi_i(D)| \le |D| < \aleph_\alpha$. Let $F \in \mathrm{Hom}_{\prod_{j\in J} R_j}(\langle D\rangle, \prod_{j\in J} R_j)$ be given by $F(d) = fd$, where $\langle D\rangle$ is the ideal of $\prod_{j\in J} R_j$ generated by D. Then $\pi_i \circ F \circ \iota_i \in \mathrm{Hom}_{R_i}(\pi_i(\langle D\rangle), R_i)$. Therefore, there exists an $r_i \in Q_\alpha(R_i)$ such that $r_i d_i = (\pi_i \circ F \circ \iota_i)(d_i)$ for all $d_i \in \pi_i(D)$. Consider the element $(r_j) \in \prod_{j\in J} Q_\alpha(R_j)$, where each r_j is defined as above. Let $e_j \in \prod_{j\in J} R_j$ be the element with 1 in the j-coordinate and 0 elsewhere. Then the map $\iota_j \circ \pi_j$ is given via multiplication by e_j, and $\pi_j(e_j) = 1$. If $d \in D$, then

$$\begin{aligned}
\big(r_j\big)d &= \big(r_j\pi_j(d)\big)\\
&= \big((\pi_j \circ F \circ \iota_j)(\pi_j(d))\big)\\
&= \big((\pi_j \circ F)(e_j d)\big)\\
&= \big(\pi_j(e_j)\pi_j(F(d))\big)\\
&= \big(1\pi_j(fd)\big)\\
&= fd.
\end{aligned}$$

Since D is dense, it follows that $f = (r_j) \in \prod_{j\in J} Q_\alpha(R_j)$. Therefore, $Q_\alpha(\prod_{j\in J} R_j) \subseteq \prod_{j\in J} Q_\alpha(R_j)$.

Suppose that $f \in \prod_{j\in J} Q_\alpha(R_j)$. Given any $j \in J$, let $D_j \subseteq \pi_j(f)^{-1}R_j$ be dense such that $|D_j| < \aleph_\alpha$. Also, let $\psi_j : |D_j| \to D_j$ be a bijection, and define $\Psi_j : \aleph_\alpha \to D_j \cup \{0\}$ by

$$\Psi_j(\beta) = \left\{ \begin{array}{cc} \psi_j(\beta), & \beta < |D_j| \\ 0, & |D_j| \le \beta < \aleph_\alpha \end{array} \right. .$$

For a fixed $\beta < \aleph_\alpha$, let $r_\beta = \big(\Psi_j(\beta)\big) \in \prod_{j\in J} R_j$. Define

$$D = \{r_\beta \mid \beta < \aleph_\alpha\}.$$

Note that the element $r_\beta \in D$ is the βth row of the $\aleph_\alpha \times |J|$ matrix $(m_{\beta j})$, where $m_{\beta j} = \Psi_j(\beta)$. As a set, the jth column of $(m_{\beta j})$ is $D_j \cup \{0\}$. Suppose that $r \in \prod_{j\in J} R_j$ with $rD = \{(0)\}$. If j is any element of J, then $\pi_j(r)\Psi_j(\beta) = 0$ for all $\beta < \aleph_\alpha$. But $\{\Psi_j(\beta) \mid \beta < \aleph_\alpha\} = D_j \cup \{0\}$ is dense, and thus $\pi_j(r) = 0$. Hence $r = \big(\pi_j(r)\big) = (0)$. Therefore, D is dense. Let $\beta < \aleph_\alpha$. Then $fr_\beta = \big(\pi_j(f)\Psi_j(\beta)\big) \in$

$\prod_{j\in J} R_j$, and it follows that $D \subseteq f^{-1}\prod_{j\in J} R_j$. Clearly $|D| \le \aleph_\alpha$. Hence $f \in Q_{\alpha+1}\big(\prod_{j\in J} R_j\big)$. Thus $\prod_{j\in J} Q_\alpha(R_j) \subseteq Q_{\alpha+1}\big(\prod_{j\in J} R_j\big)$.

If (1) fails, then $|D_j| \le \aleph_{\alpha-1}$ for all $j \in J$, and therefore, $r_\beta = (0)$ for all $\aleph_{\alpha-1} \le \beta < \aleph_\alpha$. It follows that $D' = \{r_\beta \mid \beta < \aleph_{\alpha-1}\}$ is dense. Also, $D' \subseteq D \subseteq f^{-1}\prod_{j\in J} R_j$. Clearly $|D'| < \aleph_\alpha$, and thus $f \in Q_\alpha\big(\prod_{j\in J} R_j\big)$.

If (2) fails, then $|\cup_{j\in J} D_j| < \aleph_\alpha$ by the pigeonhole principle. In particular, there exists a $\beta_0 < \alpha$ such that $|D_j| \le \aleph_{\beta_0}$ for all $j \in J$. As in the above case, the set $D' = \{r_\beta \mid \beta < \aleph_{\beta_0}\}$ is dense, $D' \subseteq f^{-1}\prod_{j\in J} R_j$, and $|D'| < \aleph_\alpha$. Hence $f \in Q_\alpha\big(\prod_{j\in J} R_j\big)$. Therefore, if either (1) or (2) fails to hold, then $Q_\alpha\big(\prod_{j\in J} R_j\big) = \prod_{j\in J} Q_\alpha(R_j)$. □

Corollary 5.4. *If J is a finite set, then $Q_\alpha\big(\prod_{j\in J} R_j\big) = \prod_{j\in J} Q_\alpha(R_j)$ for every ordinal α.*

Proof. The cofinality of any infinite cardinal is at least $\aleph_0$. Therefore, if J is finite, then Theorem 5.3 (2) fails for every ordinal α. □

As a second corollary to Theorem 5.3, we obtain the following well-known result.

Corollary 5.5. *Let $\{R_j\}_{j\in J}$ be a family of commutative rings. Then $Q\big(\prod_{j\in J} R_j\big) = \prod_{j\in J} Q(R_j)$.*

Proof. Let α be an ordinal such that $Q\big(\prod_{j\in J} R_j\big) = Q_\alpha\big(\prod_{j\in J} R_j\big)$ and $Q(R_j) = Q_\alpha(R_j)$ for all $j \in J$. Then Theorem 5.3 (1) fails for $\alpha + 1$, and thus

$$Q\big(\prod_{j\in J} R_j\big) = Q_{\alpha+1}\big(\prod_{j\in J} R_j\big) = \prod_{j\in J} Q_{\alpha+1}(R_j) = \prod_{j\in J} Q(R_j).$$

□

5.3 Applications and Examples

It is natural to question whether any of the inclusions in Theorem 5.1 or Theorem 5.3 can be replaced by "equality" or "proper inclusion." The following examples show that, in general, the answer is "no." However, this section contains results which strengthen Theorem 5.1 under some additional hypotheses (see Theorem 5.6, Corollary 5.9, and Corollary 5.10).

In [6], a ring R is said to satisfy (a.c.) (the *annihilator condition*) if for any finite set of elements $\emptyset \neq A \subseteq R$ there exists an element $r \in R$ such that $\mathrm{ann}_R(r) = \mathrm{ann}_R(A)$. This definition can be generalized by lifting the "finite" condition. We shall say that R satisfies (g.a.c.) (the *generalized annihilator condition*) if for every $\emptyset \neq A \subseteq R$ there exists an $r \in R$ such that $\mathrm{ann}_R(r) = \mathrm{ann}_R(A)$. Moreover, R will be called *reduced* if it has no nonzero nilpotents.

Theorem 5.1 provides a list of sufficient conditions to conclude that $Q_\alpha(Q_\beta(R)) = Q_\theta(R)$, where $\theta = \max\{\alpha, \beta\}$. In particular, the equality is guaranteed when $\theta = \alpha > \beta$. On the other hand, it may happen that $Q_\beta(R) \subsetneq Q_\alpha(Q_\beta(R)) \subsetneq Q_{\beta+1}(R)$ (see Example 5.13). The following result gives sufficient conditions for the equality $Q_\alpha(Q_\beta(R)) = Q_{\beta+1}(R)$ to hold.

Theorem 5.6. *Let R be a reduced commutative ring. Suppose that α and β are ordinals such that $Q_\beta(R) \subsetneq Q_\alpha(Q_\beta(R))$. If R satisfies (g.a.c.), then $Q_\alpha(Q_\beta(R)) = Q_{\beta+1}(R)$.*

Proof. Let $f \in Q_{\beta+1}(R)$. Since Theorem 5.1 implies that the inclusion $Q_\alpha(Q_\beta(R)) \subseteq Q_{\beta+1}(R)$ is always true, the desired result will hold if $f \in Q_\alpha(Q_\beta(R))$. If $f \in Q_\beta(R)$, then $f \in Q_\alpha(Q_\beta(R))$. Suppose that $f \in Q_{\beta+1}(R) \setminus Q_\beta(R)$. Then there exists a dense set $E \subseteq f^{-1}R$ such that $|E| = \aleph_\beta$. Let $\{E_\theta\}_{\theta<\mathrm{cf}(\aleph_\beta)}$ be a family of subsets of E such that $|E_\theta| < \aleph_\beta$ for all $\theta < \mathrm{cf}(\aleph_\beta)$, and $\cup_{\theta<\mathrm{cf}(\aleph_\beta)} E_\theta = E$. Let $\theta < \mathrm{cf}(\aleph_\beta)$. Since $E_\theta \subseteq R$ and R satisfies (g.a.c.), there exists a $d_\theta \in R$ such that $\mathrm{ann}_R(d_\theta) = \mathrm{ann}_R(E_\theta)$. Let $D = \{d_\theta\}_{\theta<\mathrm{cf}(\aleph_\beta)}$. Then D is dense in R since

$$\begin{aligned}\mathrm{ann}_R(D) &= \cap_{\theta<\mathrm{cf}(\aleph_\beta)} \mathrm{ann}_R(d_\theta)\\ &= \cap_{\theta<\mathrm{cf}(\aleph_\beta)} \mathrm{ann}_R(E_\theta)\\ &= \mathrm{ann}_R\big(\cup_{\theta<\mathrm{cf}(\aleph_\beta)} E_\theta\big)\\ &= \mathrm{ann}_R(E)\\ &= \{0\}.\end{aligned}$$

In particular, D is dense in $Q_\beta(R)$. Moreover, $|D| \leq \mathrm{cf}(\aleph_\beta) < \aleph_\alpha$, where the last inequality holds by Theorem 5.1. Thus, the desired result will follow if $D \subseteq f^{-1}Q_\beta(R)$. Since $D \subseteq R \subseteq Q_\beta(R)$, it remains to show that $fd_\theta \in Q_\beta(R)$ for all $\theta < \mathrm{cf}(\aleph_\beta)$. Fix some $\theta < \mathrm{cf}(\aleph_\beta)$. Since R satisfies (g.a.c.), there exists a $d \in R$ such that $\mathrm{ann}_R(d) = \mathrm{ann}_R\big(\mathrm{ann}_R(d_\theta)\big)$. Let $D' = E_\theta \cup \{d\}$. Then

$$\begin{aligned}\mathrm{ann}_R(D') &= \mathrm{ann}_R(E_\theta) \cap \mathrm{ann}_R(d)\\ &= \mathrm{ann}_R(E_\theta) \cap \mathrm{ann}_R\big(\mathrm{ann}_R(d_\theta)\big)\\ &= \mathrm{ann}_R(E_\theta) \cap \mathrm{ann}_R\big(\mathrm{ann}_R(E_\theta)\big)\\ &= \{0\},\end{aligned}$$

where the last equality holds since R is reduced. Thus $D' \subseteq R$ is dense. Also, $|D'| < \aleph_\beta$ since $|E_\theta| < \aleph_\beta$. Clearly $d_\theta \in \mathrm{ann}_R(d)$, and thus $(fd_\theta)d = f(d_\theta d) = f(0) = 0 \in R$. If $e \in E_\theta$, then $(fd_\theta)e = (fe)d_\theta \in R$, where the containment follows since $E_\theta \subseteq E \subseteq f^{-1}R$ and $d_\theta \in R$. Hence $D' \subseteq (fd_\theta)^{-1}R$. Therefore, $fd_\theta \in Q_\beta(R)$, and the proof is complete. □

Recall that the inclusions $R \subseteq S \subseteq Q(R)$ imply that S is a ring of quotients of R [4, 1.4]. In particular, a subset $D \subseteq R$ is dense in R if and only if it is dense in S. Clearly $f^{-1}R \subseteq f^{-1}S$ for all $f \in Q(R)$. Therefore, the inclusions $R \subseteq S \subseteq Q(R)$ imply that $Q_\alpha(R) \subseteq Q_\alpha(S)$ for every ordinal α. Also, note that $Q(K) = K$ for any field K since every dense set in K contains a unit (if $f \in Q(K)$ and $0 \neq u \in f^{-1}K$, then $f = (fu)u^{-1} \in K$). In particular, Corollary 5.5 implies that every direct product of fields is rationally complete.

A commutative ring R with $1 \neq 0$ is *von Neumann regular* if for each $x \in R$, there is a $y \in R$ such that $x = x^2y$ or, equivalently, R is reduced and zero-dimensional [6, Theorem 3.1]. It is well-known that von Neumann regular rings do not properly contain any finitely generated dense ideals. To prove this, suppose that R is von Neumann regular, and let $\mathfrak{d} \subseteq R$ be an ideal with generating set $\{r_1, ..., r_n\}$; say $r_i = r_i^2 s_i$ for some $s_i \in R$ $(i = 1, ..., n)$. It is easy to check that $(1 - r_1s_1)\cdots(1 - r_ns_n) \in \mathrm{ann}_R(\{r_1, ..., r_n\})$, and therefore $(1 - r_1s_1)\cdots(1 - r_ns_n) = 0$ whenever $\{r_1, ..., r_n\}$ is dense. In particular, if $\mathfrak{d}$ is dense, then $1 = f(r_1, ..., r_n) \in \mathfrak{d}$ for some $f(X_1, ..., X_n) \in R[X_1, ..., X_n]$. It follows that $Q_0(R) = R$ for every von Neumann regular ring R.

Let K be an infinite field, β an ordinal, I an indexing set with $|I| = \aleph_\beta$, and define $R_1(\beta, K) = R_1 \subseteq S_1 = \prod_I K$ to be the ring $R_1 = \{r \in S_1 \mid |\{r(i)\}_{i \in I}| < \infty\}$. Fix an $r \in R_1$, and suppose that

$s \in S_1$ is the element such that

$$s(i) = \begin{cases} r(i)^{-1}, & r(i) \neq 0 \\ 0, & r(i) = 0 \end{cases}.$$

The mapping $\{r(i)\}_{i \in I} \to \{s(i)\}_{i \in I}$ given by $r(i) \mapsto s(i)$ is easily checked to be a well-defined bijection. Thus $s \in R_1$ since $r \in R_1$ and $|\{r(i)\}_{i \in I}| = |\{s(i)\}_{i \in I}|$. Clearly $r = r^2 s$. Therefore, R_1 is von Neumann regular.

Let $D \subseteq R_1$ be the set consisting of all elements in R_1 with 1 in precisely one coordinate and 0 elsewhere. An element of R_1 having a nonzero j-coordinate does *not* annihilate the element of D that has a 1 in the j-coordinate. Therefore, D is dense. If $f \in S_1$ and $d \in D$, say $d(j) = 1$, then $\{(fd)(i)\}_{i \in I} \subseteq \{0, f(j)\}$. This shows that $D \subseteq f^{-1}R_1$ for all $f \in S_1$. Thus $R_1 \subseteq S_1 \subseteq Q(R_1)$, and therefore $S_1 \subseteq Q(R_1) \subseteq Q(S_1) = S_1$, where the equality holds since S_1 is a product of fields. That is, $Q(R_1) = S_1$. Clearly $|D| = \aleph_\beta$. Hence $Q_{\beta+1}(R_1) \subseteq Q(R_1) = S_1 \subseteq Q_{\beta+1}(R_1)$, that is, $Q_{\beta+1}(R_1) = Q(R_1)$.

Proposition 5.7. *Let α be an ordinal and suppose that R_1 is the ring defined above. Then $Q_\alpha(R_1) = \{f \in S_1 \mid |\{f(i)\}_{i \in I}| < \aleph_\alpha\}$. In particular, if $|K| = \aleph_\alpha$, then $R_1 = Q_0(R_1) \subsetneq Q_1(R_1) \subsetneq Q_2(R_1) \subsetneq \cdots \subsetneq Q_{\theta+1}(R_1) = Q(R_1)$, where $\theta = \min\{\alpha, \beta\}$.*

Proof. Let α be an ordinal and suppose that $f \in S_1$ with $|\{f(i)\}_{i \in I}| < \aleph_\alpha$. For each $j \in I$, let $e_j \in R_1$ be the element such that

$$e_j(i) = \begin{cases} 1, & f(i) = f(j) \\ 0, & f(i) \neq f(j) \end{cases}.$$

Note that $e_i(i) = 1$ for all $i \in I$. So if $r \in R_1$ with $r\{e_i\}_{i \in I} = \{(0)\}$, then $r(i) = r(i)e_i(i) = 0$ for all $i \in I$; that is, $r = (0)$. Hence $\{e_i\}_{i \in I}$ is dense. Also, it is easy to check that the mapping $\{f(i)\}_{i \in I} \to \{e_i\}_{i \in I}$ given by $f(i) \mapsto e_i$ is a well-defined bijection. Thus $|\{e_i\}_{i \in I}| = |\{f(i)\}_{i \in I}| < \aleph_\alpha$. But if $j \in I$, then $\{(fe_j)(i)\}_{i \in I} \subseteq \{0, f(j)\}$; a set of finite cardinality. Hence $\{e_i\}_{i \in I} \subseteq f^{-1}R_1$. Therefore, $f \in Q_\alpha(R_1)$, and it is proved that $\{f \in S_1 \mid |\{f(i)\}_{i \in I}| < \aleph_\alpha\} \subseteq Q_\alpha(R_1)$.

To verify the reverse inclusion, suppose that $f \in S_1$ such that $|\{f(i)\}_{i \in I}| \geq \aleph_\alpha$. Let $D \subseteq f^{-1}R_1$ be dense. Fix a $d \in D$, and suppose that $d' \in S_1$ is the element such that

$$d'(i) = \begin{cases} d(i)^{-1}, & d(i) \neq 0 \\ 0, & d(i) = 0 \end{cases}.$$

As before, $d' \in R_1$. Then $fd \in R_1$ implies that $fdd' = (fd)d' \in R_1$. Also, if $d(i) \neq 0$, then $(fdd')(i) = f(i)d(i)d(i)^{-1} = f(i)$. But for every $i \in I$ there exists a $d \in D$ such that $d(i) \neq 0$; otherwise, there exists an $i \in I$ such that $d(i) = 0$ for all $d \in D$, and hence the element of R_1 with a 1 in the i-coordinate and 0 elsewhere annihilates D, contradicting that D is dense. Thus $\{f(i)\}_{i \in I} \subseteq \cup_{d \in D}\{(fdd')(i)\}_{i \in I}$. Therefore, $\aleph_\alpha \leq |\{f(i)\}_{i \in I}| \leq |\cup_{d \in D} \{(fdd')(i)\}_{i \in I}|$, where the first inequality holds by hypothesis. But $fdd' \in R_1$ implies that $\{(fdd')(i)\}_{i \in I}$ is finite for all $d \in D$, and therefore $|D| \geq \aleph_\alpha$. Since the dense set $D \subseteq f^{-1}R_1$ was chosen arbitrarily, it follows that $f \notin Q_\alpha(R_1)$. Hence $Q_\alpha(R_1) \subseteq \{f \in S_1 \mid |\{f(i)\}_{i \in I}| < \aleph_\alpha\}$, and the proof of the first claim is complete.

To prove the "in particular" statement, suppose that $\theta = \alpha$. Since $\{f(i)\}_{i \in I} \subseteq K$, it follows that $|\{f(i)\}_{i \in I}| \leq \aleph_\alpha$ for all $f \in S_1$. Therefore, the above argument shows that $Q_{\alpha+1}(R_1) = Q_{\alpha+2}(R_1) = \cdots = Q_{\beta+1}(R_1) = Q(R_1)$. The proper inclusions $Q_0(R_1) \subsetneq \cdots \subsetneq Q_{\alpha+1}(R_1)$ follow from the above since for all $\kappa \leq \alpha$, there exists an $f \in S_1$ such that $|\{f(i)\}_{i \in I}| = \aleph_\kappa$. Finally, suppose that $\theta = \beta$; that is, $|K| \geq \aleph_\beta$. Then for all $\kappa \leq \beta$, there exists an $f \in S_1$ such that $|\{f(i)\}_{i \in I}| = \aleph_\kappa$. Thus $Q_\kappa(R_1) \subsetneq Q_{\kappa+1}(R_1)$ for all $\kappa \leq \beta$. It has already been observed that $Q_{\beta+1}(R_1) = Q(R_1)$. In either case, the equality $R_1 = Q_0(R_1)$ holds since R_1 is von Neumann regular. □

Example 5.8. *Let R_1 be the ring defined above, where I and K satisfy $|I| = |K| = \aleph_\beta$ for some ordinal β. If cf$(\aleph_\beta) < \aleph_\alpha \leq \aleph_\beta$, then $Q_\beta(R_1) \subsetneq Q_\alpha(Q_\beta(R_1)) = Q_{\beta+1}(R_1)$.*

Proof. Let $\emptyset \neq A \subseteq R_1$. Then $\mathrm{ann}_{R_1}(A) = \mathrm{ann}_{R_1}(r)$, where $r \in R_1$ is the element such that

$$r(i) = \begin{cases} 0, & a(i) = 0 \text{ for all } a \in A \\ 1, & \text{otherwise} \end{cases}.$$

Thus R_1 satisfies (g.a.c.). Therefore, it suffices to verify the proper inclusion $Q_\beta(R_1) \subsetneq Q_\alpha(Q_\beta(R_1))$ by Theorem 5.6.

Let $f \in S_1$ be any element which is bijective as a function from I onto K; in particular, $|\{f(i)\}_{i \in I}| = \aleph_\beta$. Then $f \notin Q_\beta(R_1)$ by Proposition 5.7. To show that $f \in Q_\alpha(Q_\beta(R_1))$, let $\psi : I \to \aleph_\beta$ be a bijection, and $\tau : \mathrm{cf}(\aleph_\beta) \to \aleph_\beta$ a mapping onto a cofinal set in $\aleph_\beta$. For each $\theta < \mathrm{cf}(\aleph_\beta)$, let $e_\theta \in R_1$ be the element such that

$$e_\theta(i) = \begin{cases} 1, & \psi(i) < \tau(\theta) \\ 0, & \psi(i) \geq \tau(\theta) \end{cases}.$$

Then $\{e_\theta\}_{\theta<\mathrm{cf}(\aleph_\beta)} \subseteq R_1 \subseteq Q_\beta(R_1)$. Clearly $|\{e_\theta\}_{\theta<\mathrm{cf}(\aleph_\beta)}| \leq \mathrm{cf}(\aleph_\beta) < \aleph_\alpha$. Suppose that $r \in Q_\beta(R_1)$ with $r\{e_\theta\}_{\theta<\mathrm{cf}(\aleph_\beta)} = \{(0)\}$. Let $i \in I$. Since $\aleph_\beta$ is a limit ordinal, there exists a $\theta < \mathrm{cf}(\aleph_\beta)$ with $\psi(i) < \tau(\theta)$. It follows that $r(i) = r(i)e_\theta(i) = 0$ for any such θ. Then $r = 0$ since $i \in I$ was chosen arbitrarily. Thus $\{e_\theta\}_{\theta<\mathrm{cf}(\aleph_\beta)}$ is dense. Fix some $\theta < \mathrm{cf}(\aleph_\beta)$. Then $\{(fe_\theta)(i)\}_{i\in I} = \{f(i) \mid \psi(i) < \tau(\theta)\} \cup \{0\}$; thus $|\{(fe_\theta)(i)\}_{i\in I}| \leq |\tau(\theta)| + 1 < \aleph_\beta$. Hence $fe_\theta \in Q_\beta(R_1)$ by Proposition 5.7, and it follows that $\{e_\theta\}_{\theta<\mathrm{cf}(\aleph_\beta)} \subseteq f^{-1}Q_\beta(R_1)$. Therefore, $f \in Q_\alpha(Q_\beta(R_1)) \setminus Q_\beta(R_1)$. The inclusion $Q_\beta(R_1) \subseteq Q_\alpha(Q_\beta(R_1))$ is always true, and thus $Q_\beta(R_1) \subsetneq Q_\alpha(Q_\beta(R_1))$. ☐

Suppose that R and S are rings which satisfy (g.a.c.). Define $\pi_R : R \times S \to R$ and $\pi_S : R \times S \to S$ to be the usual projection maps. Let $A \subseteq R \times S$, and choose elements $r \in R$ and $s \in S$ such that $\mathrm{ann}_R(r) = \mathrm{ann}_R(\pi_R(A))$ and $\mathrm{ann}_S(s) = \mathrm{ann}_S(\pi_S(A))$. Then it is straightforward to check that $\mathrm{ann}_{R\times S}\big((r,s)\big) = \mathrm{ann}_{R\times S}(A)$. Therefore, $R \times S$ satisfies (g.a.c.).

The proper inclusion $Q_\beta(R) \subsetneq Q_\alpha(Q_\beta(R))$ implies that $\mathrm{cf}(\aleph_\beta) < \aleph_\alpha \leq \aleph_\beta$ by Theorem 5.1. Therefore, the following corollary is stronger than Theorem 5.6.

Corollary 5.9. *Let R be a reduced ring, and suppose that cf$(\aleph_\beta) < \aleph_\alpha \leq \aleph_\beta$. If R satisfies (g.a.c.), then $Q_\alpha(Q_\beta(R)) = Q_{\beta+1}(R)$.*

Proof. Let R_1 be the ring defined in Example 5.8. As an application of Corollary 5.4, it follows that

$$\begin{aligned} Q_\beta(R_1 \times R) &= Q_\beta(R_1) \times Q_\beta(R) \\ &\subsetneq Q_\alpha\big(Q_\beta(R_1)\big) \times Q_\alpha\big(Q_\beta(R)\big) \\ &= Q_\alpha\big(Q_\beta(R_1) \times Q_\beta(R)\big) \\ &= Q_\alpha\big(Q_\beta(R_1 \times R)\big), \end{aligned}$$

where the proper inclusion holds by Example 5.8. If $Q_\alpha(Q_\beta(R)) \subsetneq Q_{\beta+1}(R)$, then a similar argument shows that $Q_\alpha(Q_\beta(R_1 \times R)) \subsetneq Q_{\beta+1}(R_1 \times R)$. But the proof of Example 5.8 shows that R_1 satisfies (g.a.c.). Also, R satisfies (g.a.c.) by hypothesis, and therefore $R_1 \times R$ satisfies (g.a.c.). Hence the proper containment $Q_\beta(R_1 \times R) \subsetneq Q_\alpha(Q_\beta(R_1 \times R))$ implies that $Q_\alpha(Q_\beta(R_1 \times R)) = Q_{\beta+1}(R_1 \times R)$ by Theorem 5.6, a contradiction. It follows that $Q_\alpha(Q_\beta(R))$ is *not* properly contained in $Q_{\beta+1}(R)$. But the inclusion $Q_\alpha(Q_\beta(R)) \subseteq Q_{\beta+1}(R)$ holds by Theorem 5.1, and therefore $Q_\alpha(Q_\beta(R)) = Q_{\beta+1}(R)$. $\square$

Note that the converse to the second assertion in Theorem 5.1 is false (see Example 5.12). However, the criteria given in (1)-(3) of Theorem 5.1 is equivalent to the inequalities $\mathrm{cf}(\aleph_\beta) < \aleph_\alpha \leq \aleph_\beta$, since conditions (2) and (3) imply that $\aleph_\beta$ is a limit cardinal. Therefore, Corollary 5.9 gives a partial converse to the second assertion in Theorem 5.1: If R is a reduced ring satisfying (g.a.c.) and $Q_\theta(R) \neq Q_{\theta+1}(R)$, then $Q_\alpha(Q_\beta(R)) = Q_{\theta+1}(R)$ if and only if $\mathrm{cf}(\aleph_\beta) < \aleph_\alpha \leq \aleph_\beta$. From this we obtain the following corollary, which exploits the rigidity of $Q_\alpha(Q_\beta(R))$ for reduced rings R satisfying (g.a.c.).

Corollary 5.10. *Let R be a reduced ring which satisfies (g.a.c.). Suppose that α and β are ordinals, and set $\theta = \max\{\alpha, \beta\}$. Then*

$$Q_\alpha(Q_\beta(R)) \in \{Q_\theta(R), Q_{\theta+1}(R)\}.$$

Proof. Either α and β fail some of the criteria given in (1)-(3) of Theorem 5.1, or $\mathrm{cf}(\aleph_\beta) < \aleph_\alpha \leq \aleph_\beta$. In particular, $\theta = \beta$ in the latter case. Thus $Q_\alpha(Q_\beta(R)) \in \{Q_\theta(R), Q_{\theta+1}(R)\}$ by Theorem 5.1 and Corollary 5.9. $\square$

The following example shows that the converse to Corollary 5.9 is false.

Example 5.11. *Let $R_1 \subseteq S_1$ be the rings defined above, where I and K satisfy $|I| = |K| = \aleph_\omega$. Define $R \subseteq R_1$ to be the ring $\{r \in R_1 \mid$ there exists a subset $I_r \subseteq I$ such that $|I \setminus I_r| < \aleph_\omega$ and $r(i) = r(i')$ for all $i, i' \in I_r\}$. Then $Q_\omega(R) \subsetneq Q_1(Q_\omega(R)) = Q_{\omega+1}(R)$, but R does* not *satisfy (g.a.c.).*

Proof. To prove that $Q_\omega(R) \subsetneq Q_1(Q_\omega(R)) = Q_{\omega+1}(R)$, it suffices to show that $Q_1(R) = Q_1(R_1)$; then it follows that

$$Q_\alpha(R) = Q_\alpha(Q_1(R)) = Q_\alpha(Q_1(R_1)) = Q_\alpha(R_1)$$

for every ordinal $\alpha \geq 1$, where the first and last equalities hold by Theorem 5.1 (since 1 is *not* a limit ordinal). In particular, Example 5.8 implies that $Q_\omega(R) = Q_\omega(R_1) \subsetneq Q_1(Q_\omega(R_1)) = Q_1(Q_\omega(R))$ and $Q_1(Q_\omega(R)) = Q_1(Q_\omega(R_1)) = Q_{\omega+1}(R_1) = Q_{\omega+1}(R)$.

Let $\{I_\alpha\}_{\alpha<\omega}$ be a family of subsets of I such that $|I_\alpha| < \aleph_\omega$ for all $\alpha < \omega$ and $\cup_{\alpha<\omega} I_\alpha = I$. For each $\alpha < \omega$, define $e_\alpha \in R_1$ by

$$e_\alpha(i) = \left\{ \begin{array}{ll} 1, & i \in I_\alpha \\ 0, & i \in I \setminus I_\alpha \end{array} \right. .$$

Note that each e_α is an element of R since the set $I_{e_\alpha} = I \setminus I_\alpha$ satisfies the conditions which define R. Suppose that $(0) \neq r \in R$, say $r(i) \neq 0$. Since $\cup_{\alpha<\omega} I_\alpha = I$, there exists an $\alpha < \omega$ such that $i \in I_\alpha$. Thus $r(i)e_\alpha(i) \neq 0$, and hence $r\{e_\alpha\}_{\alpha<\omega} \neq \{(0)\}$. Then $\{e_\alpha\}_{\alpha<\omega}$ is dense.

Suppose that $f \in R_1$. For each $\alpha < \omega$, let $I_{fe_\alpha} = I \setminus I_\alpha$. Then I_{fe_α} satisfies the conditions given in the definition of R. Thus $\{e_\alpha\}_{\alpha<\omega} \subseteq f^{-1}R$. This shows that R_1 is a ring of quotients of R. It follows that $Q_1(R) \subseteq Q_1(R_1)$. But clearly $|\{e_\alpha\}_{\alpha<\omega}| = \aleph_0 < \aleph_1$, and hence $R_1 \subseteq Q_1(R)$. Moreover, $R_1 \subseteq Q_1(R) \subseteq Q_1(R_1)$ implies that $Q_1(R)$ is a ring of quotients of R_1. Then $Q_1(R_1) \subseteq Q_1(Q_1(R)) = Q_1(R)$, where the equality follows from Theorem 5.1. Therefore, $Q_1(R) = Q_1(R_1)$. Hence $Q_\omega(R) \subsetneq Q_1(Q_\omega(R)) = Q_{\omega+1}(R)$.

It remains to show that R does not satisfy (g.a.c.). Let J_1 and J_2 be disjoint sets, each having cardinality $\aleph_\omega$. Then $|J_1 \cup J_2| = \aleph_\omega$. Let $\psi : I \to J_1 \cup J_2$ be a bijection. For each $j \in J_1$, let $e_j \in R$ be the element such that

$$e_j(i) = \left\{ \begin{array}{ll} 1, & \psi(i) = j \\ 0, & \psi(i) \neq j \end{array} \right. .$$

Suppose that $r \in R$ with $\mathrm{ann}_R(r) = \mathrm{ann}_R(\{e_j\}_{j\in J_1})$. Let $i \in I$. Then $r(i) = 0$ if and only if $e_j(i) = 0$ for all $j \in J_1$; otherwise the element of R with a 1 in the i-coordinate and 0 elsewhere annihilates either r or $\{e_j\}_{j\in J_1}$, but not both. Thus $\{i \in I \mid r(i) \neq 0\} = \psi^{-1}(J_1)$ and $\{i \in I \mid r(i) = 0\} = \psi^{-1}(J_2)$. Since $r \in R$, there exists a subset $I_r \subseteq I$ such that $|I \setminus I_r| < \aleph_\omega$ and $r(i) = r(i')$ for all $i, i' \in I_r$. Then $|\psi^{-1}(J_2)| = \aleph_\omega$ implies that $\psi^{-1}(J_2) \nsubseteq I \setminus I_r$; that is, $\psi^{-1}(J_2) \cap I_r \neq \emptyset$. It follows that $r(i) = 0$ for all $i \in I_r$. The same reasoning shows that $\psi^{-1}(J_1) \cap I_r \neq \emptyset$, and thus $r(i) \neq 0$ for some $i \in I_r$, a contradiction. Hence no such r exists. Therefore, R does not satisfy (g.a.c.). $\square$

The remainder of this section is devoted to showing that the inclusions implied by Theorem 5.1 and Theorem 5.3 cannot be improved without additional hypotheses. It has already been shown that there exists a ring R such that $Q_\beta(R) \subsetneq Q_\alpha(Q_\beta(R)) = Q_{\beta+1}(R)$ for some ordinals α and β (see Example 5.8). To proceed, we shall introduce another ring. Let I be an indexing set with $|I| = \aleph_\omega$, and define $R_2 \subseteq S_2 = \prod_{i \in I} \mathbb{Z}_2$ by $R_2 = \{r \in S_2 \mid \text{either } \{i \mid r(i) = 0\} \text{ is finite or } \{i \mid r(i) = 1\} \text{ is finite}\}$. As with R_1, one shows that $S_2 = Q(R_2)$.

Example 5.12. *Let R_2 be the ring defined above. Then*

$$R_2 = Q_\omega(R_2) = Q_1(Q_\omega(R_2)) \subsetneq Q_{\omega+1}(R_2).$$

Proof. Let $f \in Q_\omega(R_2)$, and suppose that $D \subseteq f^{-1}R_2$ is a dense set with $|D| < \aleph_\omega$. Note that $I = \cup_{d \in D}\{i \mid d(i) = 1\}$; otherwise, there exists an $i \in I$ such that $d(i) = 0$ for all $d \in D$. But then the element of R_2 with a 1 in the i-coordinate and 0 elsewhere annihilates D, contradicting that D is dense. Then D contains an element d' such that $\{i \mid d'(i) = 0\}$ is finite. If not, then $\{i \mid d(i) = 1\}$ is finite for all $d \in D$, and hence the equality $I = \cup_{d \in D}\{i \mid d(i) = 1\}$ forces $|D| = \aleph_\omega$, a contradiction. If $d' = \mathbf{1}$, then $f = fd' \in R_2$. Therefore, assume that $d' \neq \mathbf{1}$. Suppose that $\{i \mid d'(i) = 0\} = \{i_1, ..., i_n\}$. Since D is dense, there exists $d_1, ..., d_n \in D$ (not necessarily distinct) such that $d_k(i_k) = 1$ $(k = 1, ..., n)$. Then $\{d', d_1, ..., d_n\} \subseteq f^{-1}R_2$ is dense and finite. Thus $f \in Q_0(R_2) = R_2$, where the equality holds since R_2 is von Neumann regular. This shows that $Q_\omega(R_2) \subseteq R_2$. Hence $Q_\omega(R_2) = R_2$. Then

$$R_2 = Q_\omega(R_2) \subseteq Q_1(Q_\omega(R_2)) = Q_1(R_2) \subseteq Q_\omega(R_2) = R_2.$$

Therefore, $R_2 = Q_\omega(R_2) = Q_1(Q_\omega(R_2))$.

The proper inclusion is clear by the equalities $Q_1(Q_\omega(R_2)) = R_2$ and $Q_{\omega+1}(R_2) = Q(R_2) = S_2$ (indeed, the dense set consisting of elements with 1 in precisely one coordinate and 0 elsewhere has cardinality $\aleph_\omega < \aleph_{\omega+1}$). □

Example 5.13. *Let $R = R_1 \times R_2$, where R_1 is the ring defined above with $\beta = \omega$. Then $Q_\omega(R) \subsetneq Q_1(Q_\omega(R)) \subsetneq Q_{\omega+1}(R)$.*

Proof. Letting $R = R_2$, $\alpha = 1$, and $\beta = \omega$, the result follows from the first three sentences in the proof of Corollary 5.9. □

It has been shown that the inclusions implied by Theorem 5.1 cannot be improved without additional hypotheses. The following three examples demonstrate that the same is true for Theorem 5.3. The rings $R_1(\beta, K)$ and R_2 defined above are used to accomplish this task. Observe that some notational inconveniences are dealt with in Example 5.16 by changing the shorthand from R_1 to R_β.

Example 5.14. *Let R_2 be the ring defined above. Then*

$$Q_\omega(\prod_\omega R_2) = \prod_\omega Q_\omega(R_2) \subsetneq Q_{\omega+1}(\prod_\omega R_2).$$

Proof. Note that $\omega + 1$ is *not* a limit ordinal, and thus $Q_{\omega+1}(\prod_\omega R_2) = \prod_\omega Q_{\omega+1}(R_2)$ by Theorem 5.3. Therefore, the proper inclusion is an immediate consequence of Example 5.12. Moreover, Example 5.12 shows that $Q_\omega(R_2) = R_2$, and thus

$$\prod_\omega Q_\omega(R_2) = \prod_\omega R_2 \subseteq Q_\omega(\prod_\omega R_2) \subseteq \prod_\omega Q_\omega(R_2),$$

where the last inclusion holds by Theorem 5.3. Hence $Q_\omega(\prod_\omega R_2) = \prod_\omega Q_\omega(R_2)$. □

Example 5.15. *Let R_1 be the ring defined above, where I and K satisfy $|I| = |K| = \aleph_\omega$. Then $Q_\omega(\prod_\omega R_1) \subsetneq \prod_\omega Q_\omega(R_1) \subsetneq Q_{\omega+1}(\prod_\omega R_1)$.*

Proof. Note that $\omega + 1$ is *not* a limit ordinal. Therefore, $\prod_\omega Q_\omega(R_1) \subsetneq \prod_\omega Q_{\omega+1}(R_1) = Q_{\omega+1}(\prod_\omega R_1)$, where the proper inclusion holds by Example 5.8, and the equality holds by Theorem 5.3.

Let $\alpha < \omega$. By Proposition 5.7, there exists an $f_\alpha \in Q_{\alpha+1}(R_1) \setminus Q_\alpha(R_1)$. Let $f \in \prod_\omega Q_\omega(R_1)$ be the element such that $f(\alpha) = f_\alpha$ for all $\alpha < \omega$. Since $Q_\omega(\prod_\omega R_1) \subseteq \prod_\omega Q_\omega(R_1)$ holds by Theorem 5.3, it suffices to show that $f \notin Q_\omega(\prod_\omega R_1)$.

Suppose that $D \subseteq f^{-1} \prod_\omega R_1$ is dense. As in the proof of Theorem 5.3, $\pi_\alpha(D)$ is dense in R_1 for each $\alpha < \omega$. But $fD \subseteq \prod_\omega R_1$ implies that $f_\alpha \pi_\alpha(D) \subseteq R_1$ for each $\alpha < \omega$. Then $f_\alpha \notin Q_\alpha(R_1)$ implies that $|\pi_\alpha(D)| \geq \aleph_\alpha$ for all $\alpha < \omega$. Thus $|D| \geq \aleph_\alpha$ for all $\alpha < \omega$. Therefore, $|D| \geq \aleph_\omega$. This verifies that $f \notin Q_\omega(\prod_\omega R_1)$, and completes the proof. □

Example 5.16. *Let K be a field such that $|K| = \aleph_\omega$. For each $\beta < \omega$, let $R_1(\beta, K)$ be the ring defined above. For convenience, set $R_\beta = R_1(\beta, K)$. Then $Q_\omega(\prod_{\beta<\omega} R_\beta) \subsetneq \prod_{\beta<\omega} Q_\omega(R_\beta) = Q_{\omega+1}(\prod_{\beta<\omega} R_\beta)$.*

Proof. If $\beta < \omega$, then $Q_\beta(R_\beta) \subsetneq Q_{\beta+1}(R_\beta) = \cdots = Q(R_\beta)$ by Proposition 5.7. In particular, $Q_\omega(R_\beta) = Q_{\omega+1}(R_\beta)$ for all $\beta < \omega$. Note that $\omega + 1$ is *not* a limit ordinal. Hence $\prod_{\beta<\omega} Q_\omega(R_\beta) = \prod_{\beta<\omega} Q_{\omega+1}(R_\beta) = Q_{\omega+1}(\prod_{\beta<\omega} R_\beta)$, where the last equality follows by Theorem 5.3. The proof of the first proper inclusion in Example 5.15 extends to show that $Q_\omega(\prod_{\beta<\omega} R_\beta) \subsetneq \prod_{\beta<\omega} Q_\omega(R_\beta)$, where the element f is chosen such that $f_\beta \in Q_{\beta+1}(R_\beta) \setminus Q_\beta(R_\beta)$ for all $\beta < \omega$. $\square$

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Chapter 6

Invariants and Isomorphism Theorems for Zero-Divisor Graphs of Commutative Rings of Quotients

Abstract. Given a commutative ring R with $1 \neq 0$, the zero-divisor graph $\Gamma(R)$ of R is the graph whose vertices are the nonzero zero-divisors of R, such that distinct vertices are adjacent if and only if their product in R is 0. It is well-known that the zero-divisor graph of any ring is isomorphic to that of its total quotient ring. This result fails for more general rings of quotients. In this paper, conditions are given for determining whether the zero-divisor graph of a ring of quotients of R is isomorphic to that of R. Examples involving zero-divisor graphs of rationally $\aleph_0$-complete commutative rings are studied extensively. Moreover, several graph invariants are studied and applied in this investigation.

6.1 Introduction

Let R be a commutative ring with $1 \neq 0$, and let $Z(R)$ denote the set of zero-divisors of R. The *zero-divisor graph* $\Gamma(R)$ of R is the simple undirected graph with vertices $V(\Gamma(R)) = Z(R)\setminus\{0\}$, such that distinct vertices $v, w \in V(\Gamma(R))$ are adjacent if and only if $vw = 0$. The notion of a zero-divisor graph was first introduced by I. Beck in [3]. While he was mainly interested in colorings, we shall investigate the interplay between ring-theoretic and graph-theoretic properties. This approach begun in a paper by D. F. Anderson and P. S. Livingston [2], and has since continued to evolve.

Let Γ_1 and Γ_2 be simple undirected graphs. Then Γ_1 is *isomorphic* to Γ_2 if there exists an *isomorphism* $\varphi : V(\Gamma_1) \to V(\Gamma_2)$; that is, a bijection $\varphi : V(\Gamma_1) \to V(\Gamma_2)$ such that $v, w \in V(\Gamma_1)$ are adjacent if and only if $\varphi(v), \varphi(w) \in V(\Gamma_2)$ are adjacent. If Γ_1 is isomorphic to Γ_2, then we will write $\Gamma_1 \simeq \Gamma_2$.

In [1], it is shown that the zero-divisor graph of any ring is isomorphic to that of its total quotient ring. Related theorems on more general rings of quotients are given in [7] and [9]. While the latter investigations treat rings without nonzero nilpotents, this paper extends results to arbitrary commutative rings. However, rings without nonzero nilpotents shall be considered as well.

A ring R is called *reduced* if it does not have any nonzero nilpotents. We will say that R is *decomposable* if $R \cong R_1 \oplus R_2$ for some nonzero rings R_1 and R_2. If R is not decomposable, then R is *indecomposable*. A commutative ring R with $1 \neq 0$ is *von Neumann regular* if for each $r \in R$, there exists an $s \in R$ such that $r = r^2 s$ or, equivalently, R is reduced with Krull dimension zero [6, Theorem 3.1].

Given rings $R \subseteq S$ and a subset A of S, define $\mathrm{ann}_R(A) = \{r \in R \mid ra = 0 \text{ for all } a \in A\}$. If $A = \{a\}$, then we will write $\mathrm{ann}_R(A) = \mathrm{ann}_R(a)$. An equivalence relation on R is given by declaring elements $r, s \in R$ equivalent if and only if $\mathrm{ann}_R(r) = \mathrm{ann}_R(s)$. The equivalence class of an element $r \in R$ will be denoted by $[r]_R$; that is, $[r]_R = \{s \in R \mid \mathrm{ann}_R(r) = \mathrm{ann}_R(s)\}$. Suppose that R is von Neumann regular. If $r \in R$, say $r = r^2 s$, then $e_r = rs$ is the unique idempotent that satisfies $[r]_R = [e_r]_R$ (cf. [7, Remark 2.4] or the discussion prior to [1, Theorem 4.1]).

In [6], a ring R is said to satisfy (a.c.) (the *annihilator condition*) if, given any $r, s \in R$, there exists an $x \in R$ such that $\mathrm{ann}_R(r, s) = \mathrm{ann}_R(x)$. It follows (by induction) that if $A \subseteq R$ is any finite subset, then there exists an $r \in R$ such that $\mathrm{ann}_R(A) = \mathrm{ann}_R(r)$. We extend this definition, and say that a ring R satisfies $\aleph_\alpha$-(g.a.c.) (the $\aleph_\alpha$-*generalized annihilator condition*) if, given any subset $A \subseteq R$ with $|A| < \aleph_\alpha$, there exists an $r \in R$ such that $\mathrm{ann}_R(A) = \mathrm{ann}_R(r)$. We say that R satisfies (g.a.c.) if it satisfies $\aleph_\alpha$-(g.a.c.) for every ordinal α. Note that the definition in [6] coincides with our definition of $\aleph_0$-(g.a.c.).

A set $D \subseteq R$ is *dense in* R if $\mathrm{ann}_R(D) = \{0\}$. Let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be dense ideals of R, and suppose that $f_i \in \mathrm{Hom}_R(\mathfrak{d}_i, R)$ $(i = 1, 2)$. Then $\mathfrak{d}_1\mathfrak{d}_2$ is a dense ideal of R, and $\{f_1 + f_2, f_1 \circ f_2\} \subseteq \mathrm{Hom}_R(\mathfrak{d}_1\mathfrak{d}_2, R)$. Let $F = \cup_{\mathfrak{d}}\mathrm{Hom}_R(\mathfrak{d}, R)$, where the union is taken over all dense ideals of R. Then $Q(R) = F/\sim$ is a commutative ring, where $f_1 \sim f_2$ if and only if $f_1|_D = f_2|_D$ for some dense set D of R. One checks that R is embedded in $Q(R)$ by identifying any element $r \in R$ with the equivalence class containing the homomorphism $(s \mapsto rs) \in \mathrm{Hom}_R(R, R)$. In [11], J. Lambek calls $Q(R)$ the *complete ring of quotients of* R.

A ring extension $R \subseteq S$ is called a *ring of quotients of* R if $f^{-1}R = \{r \in R \mid fr \in R\}$ is dense in S for all $f \in S$. For example, the total quotient ring $T(R)$ of R is a ring of quotients of R. To see this, observe that sR is dense in $T(R)$ whenever $r/s \in T(R)$. Suppose that S is a ring of quotients of R. Then the correspondence given by identifying an element $f \in S$ with the equivalence class containing $(r \mapsto fr) \in \mathrm{Hom}_R(f^{-1}R, R)$ is an extension of the mapping $R \to Q(R)$ described above, and embeds S into $Q(R)$. Therefore, every ring of quotients of R can be regarded as a subring of $Q(R)$. It follows that a dense set in R is dense in every ring of quotients of R. Also, R has a *unique* maximal (with respect to inclusion) ring of quotients, which is isomorphic to $Q(R)$ [11, Proposition 2.3.6]. In this paper, isomorphic rings will not be distinguished. In particular, we shall denote the maximal ring of quotients of R by $Q(R)$. Note that a ring R is reduced if and only if $Q(R)$ is von Neumann regular [4, 1.11].

Let α be an ordinal. Given any subsets D_1 and D_2 of R such that $|D_i| < \aleph_\alpha$ $(i = 1, 2)$, it follows that $|\{d_1d_2 \mid d_1 \in D_1 \text{ and } d_2 \in D_2\}| < \aleph_\alpha$. Therefore, the set $Q_\alpha(R) = \{f \in Q(R) \mid \text{there exists a } D \subseteq f^{-1}R$ such that $\mathrm{ann}_R(D) = \{0\}$ and $|D| < \aleph_\alpha\}$ is a commutative ring. Clearly $Q_\alpha(R)$ is a ring of quotients of R. Also, there exists an ordinal β such that $Q_\alpha(R) = Q(R)$ for all $\alpha \geq \beta$. As in [10], we will say that R is *rationally $\aleph_\alpha$-complete* if $R = Q_\alpha(R)$. If R is rationally $\aleph_\alpha$-complete, then it is easy to see that R is rationally $\aleph_\beta$-complete for all $\beta \leq \alpha$. If R is rationally $\aleph_\alpha$-complete for all α (i.e., $R = Q(R)$), then we will say that R is *rationally complete*. In [12], T.G. Lucas calls $Q_0(R)$ the *ring of finite fractions of* R. In [5], A.W. Hager and J. Martinez refer to $Q_\alpha(R)$ as the *ring of* $\aleph_\alpha$*-quotients of* R. Examples and fundamental properties of rational $\aleph_\alpha$-completions of commutative rings are given in [10].

Let $R \subseteq S \subseteq T$ be rings. Then T is a ring of quotients of R if and only if T is a ring of quotients of S and S is a ring of quotients of R [4, 1.4]. It follows that $Q(S) = Q(R)$ whenever $R \subseteq S$ is a ring of quotients. Moreover, given any ordinal α, it is easy to check that $f^{-1}R \subseteq f^{-1}S$ for all $f \in Q_\alpha(R)$. Therefore, if S is a ring of quotients of R, then $Q_\alpha(R) \subseteq Q_\alpha(S)$ for every ordinal α.

The main focus of this paper is on the relationship between the zero-divisor graphs of R and $Q_\alpha(R)$ for a commutative ring R. In particular, criteria is sought for determining when these graphs are isomorphic. Using the fact that any ring of quotients of R can be embedded in $Q_\alpha(R)$ for some α, our results extend to *all* rings of quotients of R.

The ring-theoretic foundation for this study is established in a series of lemmas given in Section 6.2. Furthermore, these results motivate a ring-theoretic characterization of $\aleph_\alpha$-complete Boolean algebras (Theorem 6.4). In [8, Lemma 3.1], a graph-theoretic condition (see Theorem 6.7(4)) is presented for determining when the relation $\Gamma(R) \simeq \Gamma(Q(R))$ holds for a von Neumann regular ring R. However, this condition is meaningful only when certain graph-theoretic assumptions (known to be possessed by zero-divisor graphs of von Neumann regular rings) are met. In particular, this condition cannot be employed in the study of zero-divisor graphs of arbitrary rings. In Section 6.3, we expose the underlying mechanics of this condition. It turns out that $\aleph_\alpha$-(g.a.c.) is an appropriate generalizing criterion (Remark 6.12(1) and Theorem 6.13). In fact, if R is a von Neumann regular ring, then the key graph-theoretic condition of [8, Lemma 3.1] is possessed by $\Gamma(R)$ if and only if R satisfies (g.a.c.) (Theorem 6.7). In an effort to determine the relation $\Gamma(R) \simeq \Gamma(Q_\alpha(R))$ based on characteristics of $\Gamma(R)$, we investigate the graph-theoretic implications of the property $\aleph_\alpha$-(g.a.c.). Any ring that satisfies $\aleph_\alpha$-(g.a.c.) has a *weak central vertex* $\aleph_\alpha$*-complete* zero-divisor graph. If R is a decomposable ring, then R satisfies $\aleph_\alpha$-(g.a.c.) if and only if $\Gamma(R)$ is a weak central vertex $\aleph_\alpha$-complete graph (Theorem 6.18 and Corollary 6.19). On the other hand, if R is any reduced ring, then R satisfies $\aleph_\alpha$-(g.a.c.) if and only if $\Gamma(R)$ is a *central vertex* $\aleph_\alpha$*-complete* graph (Theorem 6.5 and Corollary 6.17). We conclude Section 6.3 with a lemma which provides sufficient conditions for the zero-divisor graphs of direct sums to be isomorphic. In Section 6.4, the results in Section 6.3 are applied to examples involving $\Gamma(Q_0(R))$, where R is a total quotient ring such that $R \subsetneq Q_0(R) \subsetneq Q(R)$. In particular, four of the five possible relations between $\Gamma(R)$, $\Gamma(Q_0(R))$, and $\Gamma(Q(R))$ are shown to exist (Theorem 6.22). Furthermore, examples are constructed to show that $\aleph_\alpha$-(g.a.c.) is not a necessary condition for the relation $\Gamma(R) \simeq \Gamma(Q_\alpha(R))$ to hold (Example 6.35 and Example 6.36).

6.2 Rings of Quotients and the Annihilator Conditions

In this section, we study the annihilator ideals of a ring of quotients. In particular, it is shown that the annihilator of an element in a ring of quotients of R is the annihilator of an element in R whenever R satisfies (g.a.c.) (Lemma 6.3). We conclude this section with a theorem which characterizes $\aleph_\alpha$-complete Boolean algebras (Theorem 6.4).

In [8], the inclusion $[r]_R \subseteq [r]_{Q(R)}$ is justified for a reduced ring by noting that the mapping $\mathrm{ann}_{Q(R)}(J) \mapsto \mathrm{ann}_R(J \cap R)$ $(J \subseteq Q(R))$ is a well-defined bijection of $\{\mathrm{ann}_{Q(R)}(J) \mid J \subseteq Q(R)\}$ onto $\{\mathrm{ann}_R(J) \mid J \subseteq R\}$ [11, Proposition 2.4.3]. Elementary proofs are given when R is von Neumann regular [8, Proposition 2.7]. The following lemma generalizes this observation with a simpler proof.

Lemma 6.1. *Let R be a commutative ring. Suppose that S is a ring of quotients of R, and let $f_1, f_2 \in S$. Then $ann_R(f_1) = ann_R(f_2)$ if and only if $ann_S(f_1) = ann_S(f_2)$.*

Proof. Clearly $\mathrm{ann}_S(f_1) = \mathrm{ann}_S(f_2)$ implies that $\mathrm{ann}_R(f_1) = \mathrm{ann}_R(f_2)$. Suppose that $\mathrm{ann}_R(f_1) = \mathrm{ann}_R(f_2)$, and let $g \in \mathrm{ann}_S(f_1)$. Then $g(g^{-1}R) \subseteq \mathrm{ann}_R(f_1) = \mathrm{ann}_R(f_2)$, and hence $f_2 g \in \mathrm{ann}_S(g^{-1}R) = \{0\}$. That is, $g \in \mathrm{ann}_S(f_2)$. A symmetric argument shows that $\mathrm{ann}_S(f_2) \subseteq \mathrm{ann}_S(f_1)$, and therefore the desired equality holds. $\square$

Lemma 6.2. *Let R be a commutative ring. Suppose that S is a ring of quotients of R, and let D be a dense set in R. If $f \in S$, then*

$$ann_R(f) = \cap_{d \in D} ann_R(fd) = ann_R\big(\cup_{d \in D}\{fd\}\big).$$

Proof. To prove the first equality, suppose that $r \in \cap_{d\in D}\mathrm{ann}_R(fd)$. Then $rfd = 0$ for all $d \in D$. That is, $rf \in \mathrm{ann}_S(D) = \{0\}$, where the equality holds since D is dense in every ring of quotients of R. Thus $r \in \mathrm{ann}_R(f)$. This shows that $\cap_{d\in D}\mathrm{ann}_R(fd) \subseteq \mathrm{ann}_R(f)$. The reverse inclusion is obvious, and therefore the equality holds.

The second equality is clear. □

Lemma 6.3. *Let R and S be commutative rings with $R \subseteq S \subseteq Q_\alpha(R)$. Suppose that R satisfies $\aleph_\alpha$-(g.a.c.). If $f \in S$, then there exists an $r \in R$ such that $[f]_S = [r]_S$.*

Proof. The inclusion $S \subseteq Q_\alpha(R)$ implies there exists a dense set $D \subseteq f^{-1}R$ such that $|D| < \aleph_\alpha$. Since R satisfies $\aleph_\alpha$-(g.a.c.), there exists an $r \in R$ such that $\mathrm{ann}_R\big(\cup_{d\in D}\{fd\}\big) = \mathrm{ann}_R(r)$. But $R \subseteq S \subseteq Q_\alpha(R)$ implies that S is a ring of quotients of R. Then by Lemma 6.2, it follows that $\mathrm{ann}_R(f) = \mathrm{ann}_R(r)$. Therefore, Lemma 6.1 implies that $\mathrm{ann}_S(f) = \mathrm{ann}_S(r)$, i.e., $[f]_S = [r]_S$. □

Given a commutative ring R, let $B(R) = \{r \in R \mid r^2 = r\}$; that is, let $B(R)$ denote the set of idempotents of R. Then the relation $\leq$, defined by $r \leq s$ if and only if $rs = r$, partially orders $B(R)$, and makes $B(R)$ a Boolean algebra with inf as multiplication in R, the largest element as 1, the smallest element as 0, and complementation defined by $r' = 1 - r$. A Boolean algebra B is called $\aleph_\alpha$-*complete* if $\inf A$ exists in B for all $A \subseteq B$ with $|A| \leq \aleph_\alpha$. If B is $\aleph_\alpha$-complete for every ordinal α, then B is called *complete*. By the de Morgan Laws, it follows that B is $\aleph_\alpha$-complete if and only if $\sup A$ exists in B for all $A \subseteq B$ with $|A| \leq \aleph_\alpha$ (e.g., see Section 20 in [13]).

Suppose that R is von Neumann regular. Let $A \subseteq B(R) \subseteq B(Q(R))$. It is known that $B(Q(R))$ is a complete Boolean algebra [4, Theorem 11.9]. Thus, $\inf A \in B(Q(R))$. If R satisfies (g.a.c.), then Lemma 6.3 implies that there exists an element $r \in R$ such that $[r]_{Q(R)} = [\inf A]_{Q(R)}$. But $\inf A$ is idempotent, and thus $\inf A = e_r \in R$. Hence $B(R)$ is complete whenever R satisfies (g.a.c.). The converse is also true. The following theorem generalizes these observations (without the hypothesis "$B(Q(R))$ is complete").

Theorem 6.4. *Let R be a von Neumann regular ring. Then $B(R)$ is $\aleph_\alpha$-complete if and only if R satisfies $\aleph_{\alpha+1}$-(g.a.c.).*

Proof. Suppose that $B(R)$ is $\aleph_\alpha$-complete. Let $A \subseteq R$ such that $|A| < \aleph_{\alpha+1}$. Since $|A| \leq \aleph_\alpha$, there exists an $e \in B(R)$ such that $e = \sup\{e_a \mid a \in A\}$. In particular, $e \geq e_a$ for all $a \in A$. That is, $e_a = ee_a$ for all $a \in A$.

Clearly $\mathrm{ann}_R(e) \subseteq \mathrm{ann}_R(e_a) = \mathrm{ann}_R(a)$ for all $a \in A$. Hence $\mathrm{ann}_R(e) \subseteq \mathrm{ann}_R(A)$. To show the reverse inclusion, suppose that $r \in \mathrm{ann}_R(A)$. Then $e_a(1-e_r) = e_a$ for all $a \in A$. That is, $e_a \leq 1 - e_r$ for all $a \in A$. Therefore, $e \leq 1 - e_r$, i.e., $e(1-e_r) = e$. Then $ee_r = 0$, and therefore $r \in \mathrm{ann}_R(e)$. Thus $\mathrm{ann}_R(e) = \mathrm{ann}_R(A)$, and it follows that R satisfies $\aleph_{\alpha+1}$-(g.a.c.).

Conversely, suppose that R satisfies $\aleph_{\alpha+1}$-(g.a.c.). Let $A \subseteq B(R)$ such that $|A| \leq \aleph_\alpha$. Since $|A| < \aleph_{\alpha+1}$, there exists an $r \in R$ such that $\mathrm{ann}_R(r) = \mathrm{ann}_R(\{1 - a \mid a \in A\})$. Hence $\mathrm{ann}_R(e_r) = \mathrm{ann}_R(\{1 - a \mid a \in A\})$. In particular, $(1-e_r)(1-a) = 0$ for all $a \in A$. It follows that $1 - e_r \leq a$ for all $a \in A$. Suppose that $b \in B(R)$ with $b \leq a$ for all $a \in A$. Then $b(1-a) = 0$ for all $a \in A$; that is, $b \in \mathrm{ann}_R(\{1 - a \mid a \in A\}) = \mathrm{ann}_R(e_r)$. Thus $b(1-e_r) = b$, i.e., $b \leq 1 - e_r$. Hence $\inf A = 1 - e_r \in B(R)$. Therefore, $B(R)$ is $\aleph_\alpha$-complete. ☐

Note that Theorem 6.4 gives ring-theoretic conditions which characterize $\aleph_\alpha$-complete Boolean algebras. Every Boolean algebra is of the form $B(R)$ for some *Boolean ring* R (that is, a ring R such that $r^2 = r$ for all $r \in R$), cf. [11, Proposition 1.1.3]. Therefore, a Boolean algebra $B(R)$ is $\aleph_\alpha$-complete if and only if R satisfies $\aleph_{\alpha+1}$-(g.a.c.). In Section 6.3, this ring-theoretic property will be translated into a graph-theoretic property (Theorem 6.5 and Theorem 6.18).

6.3 Invariants and Isomorphism Theorems

Let Γ be a graph, $V(\Gamma)$ the set of vertices of Γ, and $\emptyset \neq A \subseteq V(\Gamma)$. As in [8], a vertex $v \in V(\Gamma)$ will be called a *central vertex of A* if every element of A is adjacent to v. Let $C(A) \subseteq V(\Gamma)$ denote the set of all central vertices of A. If $A = \{a\}$, then we will write $C(A) = C(a)$.

Note that, if $\Gamma = \Gamma(R)$ for some ring R, then

$$C(A) = \mathrm{ann}_R(A) \setminus \big(A \cup \{0\}\big).$$

A graph Γ is said to be *central vertex $\aleph_\alpha$-complete*, or *c.v.-$\aleph_\alpha$-complete*, if for all $\emptyset \neq A \subseteq V(\Gamma)$ with $|A| < \aleph_\alpha$ and $C(A) \neq \emptyset$, there exists a $v \in V(\Gamma)$ such that $C(A) = C(v)$. If Γ is c.v.-$\aleph_\alpha$-complete for every ordinal α, then we will say that Γ is *c.v.-complete.* The following theorem translates the this definition into ring-theoretic terms.

Theorem 6.5. *Let R be a reduced ring. Then $\Gamma(R)$ is c.v.-$\aleph_\alpha$-complete if and only if R satisfies $\aleph_\alpha$-(g.a.c.).*

Proof. Observe that, since R is reduced, $C(A) = \mathrm{ann}_R(A) \setminus \{0\}$ for every $\emptyset \neq A \subseteq V(\Gamma(R))$. Therefore, the equality $C(A) = C(B)$ holds for nonempty sets $A, B \subseteq V(\Gamma(R))$ if and only if $\mathrm{ann}_R(A) = \mathrm{ann}_R(B)$.

Suppose that R satisfies $\aleph_\alpha$-(g.a.c.). Let $\emptyset \neq A \subseteq V(\Gamma(R))$ with $|A| < \aleph_\alpha$ and $C(A) \neq \emptyset$. Then $C(A) = C(r)$, where $r \in R$ is an element such that $\mathrm{ann}_R(A) = \mathrm{ann}_R(r)$. Hence $\Gamma(R)$ is a c.v.-$\aleph_\alpha$-complete.

Suppose that $\Gamma(R)$ is c.v.-$\aleph_\alpha$-complete. Let $\emptyset \neq A \subseteq R$ with $|A| < \aleph_\alpha$. If $\mathrm{ann}_R(A) = \{0\}$, then $\mathrm{ann}_R(A) = \mathrm{ann}_R(1)$. Suppose that $\mathrm{ann}_R(A) \neq \{0\}$. If $A = \{0\}$, then $\mathrm{ann}_R(A) = \mathrm{ann}_R(0)$. Suppose that $A \neq \{0\}$. Then $\mathrm{ann}_R(A) \neq \{0\}$ implies that $\emptyset \neq A \setminus \{0\} \subseteq V(\Gamma(R))$ and $C(A \setminus \{0\}) \neq \emptyset$. Therefore, $\mathrm{ann}_R(A) = \mathrm{ann}_R(A \setminus \{0\}) = \mathrm{ann}_R(r)$, where r is any element which satisfies $C(A \setminus \{0\}) = C(r)$. Thus R satisfies $\aleph_\alpha$-(g.a.c.). $\square$

Note that Theorem 6.5 can fail for rings with nonzero nilpotents. For example, the zero-divisor graph of $\mathbb{Z}_{25}$ is the complete graph on four vertices. In particular, $\Gamma(\mathbb{Z}_{25})$ is not c.v.-$\aleph_0$-complete. However, $\mathbb{Z}_{25}$ satisfies (g.a.c.) since the annihilator of any set in $\mathbb{Z}_{25}$ is either $\{0\} = \mathrm{ann}_{\mathbb{Z}_{25}}(1)$ or $Z(R) = \mathrm{ann}_{\mathbb{Z}_{25}}(5)$.

By Theorem 6.4 and Theorem 6.5, we have

Corollary 6.6. *Let R be a von Neumann regular ring. Then $B(R)$ is $\aleph_\alpha$-complete if and only if $\Gamma(R)$ is c.v.-$\aleph_{\alpha+1}$-complete.*

Let Γ be a graph, and suppose that $v \in V(\Gamma)$. As in [1], an element $w \in V(\Gamma)$ will be called a *complement of v* if w is adjacent to v, and no element of $V(\Gamma)$ is adjacent to both v and w. A graph Γ is *complemented* if every element of $V(\Gamma)$ has a complement. If Γ is a simple graph, then

v is a complement of w if and only if the edge $v - w$ is not an edge of any triangle in Γ. It is known that any reduced total quotient ring R is von Neumann regular if and only if $\Gamma(R)$ is complemented [1, Theorem 3.5].

Note that Corollary 6.6 is a generalization of [8, Lemma 3.1], which states the following: If R is a von Neumann regular ring, then $B(R)$ is a complete Boolean algebra if and only if whenever $\emptyset \neq A \subseteq V(\Gamma(R))$ is a family of vertices with $C(A) \neq \emptyset$, there exists a $v \in C(A)$ such that every complement of v is adjacent to every element of $C(A)$. In fact, the terminology *c.v.-complete* was first given in [8], where a zero-divisor graph was said to be c.v.-complete if it satisfied condition (4) of the following theorem.

Theorem 6.7. *The following statements are equivalent for a von Neumann regular ring R.*

(1) For all $\emptyset \neq A \subseteq R$, there exists a $v \in ann_R(A)$ such that $ann_R(A) = ann_R(1 - e_v)$.

(2) R satisfies (g.a.c.).

(3) $\Gamma(R)$ is c.v.-complete.

(4) If $\emptyset \neq A \subseteq V(\Gamma(R))$ is a family of vertices with $C(A) \neq \emptyset$, then there exists a $v \in C(A)$ such that every complement of v is adjacent to every element of $C(A)$.

Proof. Observe that (1) implies (2) by definition, (2) implies (3) by Theorem 6.5, and (3) implies (4) by Corollary 6.6 together with [8, Lemma 3.1]. It remains to show that (4) implies (1).

If $\text{ann}_R(A) = \{0\}$, then let $v = 0$. Suppose that $\text{ann}_R(A) \neq \{0\}$. If $A = \{0\}$, then let $v = 1$. If $A \neq \{0\}$, then we can regard A as a nonempty subset of $V(\Gamma(R))$ since $\text{ann}_R(A) = \text{ann}_R(A \setminus \{0\})$. Also, $\text{ann}_R(A) \neq \{0\}$ implies that $C(A) \neq \emptyset$, and therefore there exists a $v \in C(A)$ such that every complement of v is adjacent to every element of $C(A)$. But $\text{ann}_R(v) = \text{ann}_R(e_v)$ implies that v is adjacent to $1 - e_v \in B(R)$. Moreover, if $r \in R$ with $rv = 0 = r(1 - e_v)$, then $r = re_v = 0$. This shows that $1 - e_v$ is a complement of v, and thus $1 - e_v$ is adjacent to every element of $C(A)$. It follows that $\text{ann}_R(A) \subseteq \text{ann}_R(1 - e_v)$. But if $r \in \text{ann}_R(1 - e_v)$, then $r = re_v \in \text{ann}_R(A)$, where the containment holds since $v \in C(A)$ and $\mathrm{ann}_R(v) = \mathrm{ann}_R(e_v)$. Thus $\mathrm{ann}_R(1 - e_v) \subseteq \mathrm{ann}_R(A)$. Hence $\mathrm{ann}_R(A) = \mathrm{ann}_R(1 - e_v)$. $\square$

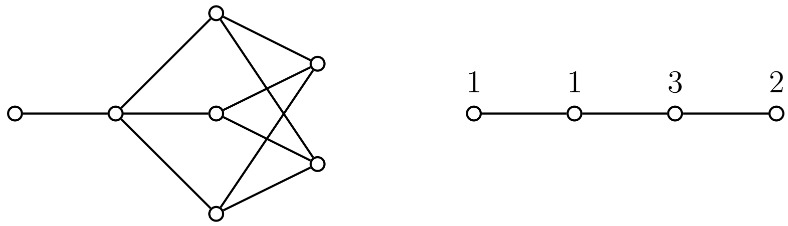
Suppose that R is a von Neumann regular ring such that $2 \notin Z(R)$ and $|R| < \aleph_\omega$. By [8, Theorem 3.3], $\Gamma(R) \simeq \Gamma(Q(R))$ if and only if $\Gamma(R)$ satisfies condition (4) of Theorem 6.7. Then Theorem 6.7 gives several efficient ways to determine whether the zero-divisor graph of a von Neumann regular ring R is isomorphic to that of $Q(R)$. For example, we have

Corollary 6.8. *Suppose that R is a von Neumann regular ring such that $2 \notin Z(R)$ and $|R| < \aleph_\omega$. Then $\Gamma(R) \simeq \Gamma(Q(R))$ if and only if R satisfies (g.a.c.).*

Given any $v \in V(\Gamma)$, define $V_v(\Gamma) = \{w \in V(\Gamma) \mid C(w) = C(v)\}$. If $\Gamma = \Gamma(R)$ for some ring R, then we will write $V_r(\Gamma(R)) = V_r(R)$. Note that the relation $\sim$ on $V(\Gamma)$ defined by $v \sim w$ if and only if $V_v(\Gamma) = V_w(\Gamma)$ is an equivalence relation. Let Γ^* be the graph with vertices $\{V_v(\Gamma) \mid v \in V(\Gamma)\}$, such that $V_v(\Gamma)$ and $V_w(\Gamma)$ are adjacent in Γ^* if and only if v and w are adjacent in Γ. The graph Γ^* was considered in [1], where it was shown that $\Gamma(R)^*$ is the zero-divisor graph of a Boolean ring whenever R is von Neumann regular [1, Proposition 4.5]. In [9], the *minimal representation* of a graph Γ was defined as the graph Γ^*, where the vertex $V_v(\Gamma)$ was labeled with the cardinal number $|V_v(\Gamma)|$.

If Γ is a simple graph, then every edge of Γ^* represents a complete bipartite graph (see Figure 6.1). In particular, any zero-divisor graph can be recovered from its minimal representation. Note that, if R is reduced, then $\Gamma(R)^*$ is the graph with vertices $\{[r]_R \mid r \in Z(R) \setminus \{0\}\}$, such that $[r]_R$ is adjacent to $[s]_R$ if and only if $rs = 0$. In fact, $[r]_R = V_r(R)$ for all $r \in Z(R) \setminus \{0\}$.

Clearly $\Gamma_1 \simeq \Gamma_2$ implies that $\Gamma_1^* \simeq \Gamma_2^*$. Although the converse is false, there are certain properties of Γ which are preserved by Γ^*. For example, if $n > 2$ is an integer, then the diameter of Γ is n if and only if the diameter of Γ^* is n (indeed, no two vertices of a minimal path in Γ having length $n > 2$ can belong to the same vertex in Γ^*). Also, a vertex w is a complement of v in Γ if and only if $V_w(\Gamma)$ is a complement of $V_v(\Gamma)$ in Γ^*. Furthermore, it is a routine exercise to show that a graph Γ is c.v.-$\aleph_\alpha$-complete if and only if Γ^* is c.v.-$\aleph_\alpha$-complete.



(a) Γ (b) Γ^*

Figure 6.1: A graph Γ and its minimal representation Γ^*

On the other hand, the following proposition gives necessary and sufficient conditions for $\Gamma_1 \simeq \Gamma_2$. Although it was not formally stated, the idea behind Proposition 6.9 was utilized in [1, Theorem 2.2], showing that $\Gamma(R) \simeq \Gamma(T(R))$ for any commutative ring R. Moreover, [1, Theorem 4.1] is a special case of this proposition.

Proposition 6.9. *Let Γ_1 and Γ_2 be simple undirected graphs. Then $\Gamma_1 \simeq \Gamma_2$ if and only if there exists an isomorphism $\varphi : V(\Gamma_1^*) \to V(\Gamma_2^*)$ such that $|v^*| = |\varphi(v^*)|$ for all $v^* \in V(\Gamma_1^*)$.*

Proof. If $\psi : V(\Gamma_1) \to V(\Gamma_2)$ is an isomorphism, then it is easy to check that the mapping $\varphi : V(\Gamma_1^*) \to V(\Gamma_2^*)$ given by $\varphi\big(V_v(\Gamma_1)\big) = V_{\psi(v)}(\Gamma_2)$ has the desired properties. Conversely, suppose that $\varphi : V(\Gamma_1^*) \to V(\Gamma_2^*)$ is an isomorphism such that $|v^*| = |\varphi(v^*)|$ for all $v^* \in V(\Gamma_1^*)$. For every $v^* \in V(\Gamma_1^*)$, let $\psi_{v^*} : v^* \to \varphi(v^*)$ be a bijection. Then one checks that the mapping $\psi : V(\Gamma_1) \to V(\Gamma_2)$, given by $\psi(v) = \psi_{v^*}(v)$ if and only if $v \in v^*$, is an isomorphism. $\square$

It is evident from Proposition 6.9 that the cardinality of $V_r(R)$ is valuable in determining whether two zero-divisor graphs are isomorphic. If the index of nilpotency of a ring-element $r \in R$ is 2, then the cardinality of $V_r(R)$ is necessarily equal to 1. This claim is made precise in the following theorem.

Theorem 6.10. *Let R be a commutative ring. Suppose that $0 \neq r \in R$ with $r^2 = 0$. Then $V_r(R) = \{r\}$.*

Proof. Suppose that $x \in V_r(R) \setminus \{r\}$. Then $\operatorname{ann}_R(x) \setminus \{x\} = \operatorname{ann}_R(r) \setminus \{r\}$. In particular, $xr \neq 0$. Thus $r(1+x) \neq r$. Since $r^2 = 0$, it follows that $r(1+x) \in \operatorname{ann}_R(r) \setminus \{r\} = \operatorname{ann}_R(x) \setminus \{x\}$. If $1+x$ is not a zero-divisor, then the equality $xr(1+x) = 0$ implies that $xr = 0$, a contradiction. Hence $1+x$ is a zero-divisor. In particular, x is not a nilpotent element. Thus $\operatorname{ann}_R(r) \setminus \{r\} = \operatorname{ann}_R(x)$.

Suppose that $rx \neq r$. Then $rx \in \operatorname{ann}_R(r) \setminus \{r\} = \operatorname{ann}_R(x)$ implies that $x^2 \in \operatorname{ann}_R(r)$. But $x^2 \neq r$ since x is not a nilpotent. Thus $x^2 \in \operatorname{ann}_R(r) \setminus \{r\} = \operatorname{ann}_R(x)$, contradicting that x is not a nilpotent. Therefore, it must be the case that $rx = r$.

If $1 - x = r$, then $1 = x + r = x + rx = x(1+r)$. This contradicts that x is a zero-divisor. Therefore, $1 - x \neq r$. Then $rx = r$ implies that $1 - x \in \operatorname{ann}_R(r) \setminus \{r\} = \operatorname{ann}_R(x)$. That is, $x^2 = x$. Hence $1 + r - x \in \operatorname{ann}_R(r)$ and $x(1+r-x) = r \neq 0$. Since $\operatorname{ann}_R(r) \setminus \{r\} = \operatorname{ann}_R(x)$, it follows that $1 + r - x = r$. But then $1 - x = 0$, and hence $x = 1$. This contradicts that x is a zero-divisor, and we have exhausted all possibilities. Therefore, no such element x exists. Thus $V_r(R) \setminus \{r\} = \emptyset$. Clearly $r \in V_r(R)$, and hence $V_r(R) = \{r\}$. $\square$

Corollary 6.11. *Let $R \subseteq S$ be commutative rings. If the mapping $\varphi : V(\Gamma(R)^*) \to V(\Gamma(S)^*)$ defined by $\varphi(V_r(R)) = V_r(S)$ is an isomorphism, then $\{r \in R \mid r^2 = 0\} = \{f \in S \mid f^2 = 0\}$.*

Proof. Suppose that $0 \neq f \in S$ with $f^2 = 0$. Since φ is surjective, there exists an $r \in R$ such that $V_r(S) = V_f(S)$. But Theorem 6.10 shows that $V_f(S) = \{f\}$, and it follows that $f = r \in R$. $\square$

For a von Neumann regular ring R, condition (4) of Theorem 6.7 is necessary and sufficient to conclude that the mapping $\varphi : V(\Gamma(R)^*) \to V(\Gamma(Q(R))^*)$ defined by $\varphi(V_r(R)) = V_r(Q(R))$ is an isomorphism ([1, Proposition 4.5], [4, Theorem 11.9], and [8, Lemma 3.1]). When trying to generalize this result to arbitrary rings, one is forced to seek other criteria. For example, any application of Theorem 6.7(4) is contingent upon the assumption that elements of $V(\Gamma(R))$ have complements. Remark 6.12(1) and Theorem 6.13 provide generalizations by considering condition (2) of Theorem 6.7.

Remark 6.12. *(1) Let $R \subseteq S$ be commutative rings. Suppose that the correspondence $\{[r]_R \mid 0 \neq r \in Z(R)\} \to \{[f]_S \mid 0 \neq f \in Z(S)\}$ given by $[r]_R \mapsto [r]_S$ is a bijection, and that $|[r]_R| = |[r]_S|$ for all $0 \neq$*

$r \in Z(R)$. Then a proof similar to that of the converse statement in Proposition 6.9 shows that $\Gamma(R) \simeq \Gamma(S)$ (this is precisely the method of proof used in [1, Theorem 2.2]). In particular, suppose that $R \subseteq S \subseteq Q_\alpha(R)$, and that R satisfies $\aleph_\alpha$-(g.a.c.). Then the correspondence $\{[r]_R \mid 0 \neq r \in Z(R)\} \to \{[f]_S \mid 0 \neq f \in Z(S)\}$ described above is a well-defined bijection by Lemma 6.1 and Lemma 6.3. Therefore, if $|[r]_R| = |[r]_S|$ for all $0 \neq r \in Z(R)$, then $\Gamma(R) \simeq \Gamma(S)$.

(2) Suppose that the mapping φ given in Corollary 6.11 is an isomorphism. Using Corollary 6.11, it is easy to see that $[f]_S \subseteq R$ for all $0 \neq f \in S$ with $f^2 = 0$. Also, $V_f(S) = [f]_S$ whenever $0 \neq f \in Z(S)$ with $f^2 \neq 0$. Therefore, if φ is an isomorphism and $|V_r(R)| = |V_r(S)|$ for all $0 \neq r \in Z(R)$, then the correspondence $\{[r]_R \mid 0 \neq r \in Z(R)\} \to \{[f]_S \mid 0 \neq f \in Z(S)\}$ described above induces an isomorphism from $V(\Gamma(R))$ onto $V(\Gamma(S))$. The converse is false (e.g., by the proof of [1, Theorem 2.2] and Corollary 6.11, the converse fails for the rings $R = \mathbb{Z}_4[X]$ and $S = T(R)$). In this sense, the isomorphisms induced by φ are stronger than the isomorphisms induced by the mapping $\{[r]_R \mid 0 \neq r \in Z(R)\} \to \{[f]_S \mid 0 \neq f \in Z(S)\}$ described above.

Theorem 6.13. *Let α be an ordinal, and suppose that R and S are commutative rings such that $R \subseteq S \subseteq Q_\alpha(R)$. Suppose that R satisfies $\aleph_\alpha$-(g.a.c.). Then the mapping $\varphi : V(\Gamma(R)^*) \to V(\Gamma(S)^*)$ defined by $\varphi(V_r(R)) = V_r(S)$ is an isomorphism if and only if $\{r \in R \mid r^2 = 0\} = \{f \in S \mid f^2 = 0\}$.*

Proof. If φ is an isomorphism, then the desired equality holds by Corollary 6.11. Conversely, suppose that $\{r \in R \mid r^2 = 0\} = \{f \in S \mid f^2 = 0\}$. To show that φ is well-defined, suppose that $r, x \in R$ with $V_r(R) = V_x(R)$. That is, $\mathrm{ann}_R(r) \setminus \{r\} = \mathrm{ann}_R(x) \setminus \{x\}$. Let $f \in \mathrm{ann}_S(r) \setminus \{r\}$. If $f \in R$, then $f \in \mathrm{ann}_R(x) \setminus \{x\} \subseteq \mathrm{ann}_S(x) \setminus \{x\}$. Therefore, assume that $f \in S \setminus R$.

By Lemma 6.3, there exists an element $t \in R$ such that $\mathrm{ann}_S(t) = \mathrm{ann}_S(f)$. If $t = r$, then $f^2 = 0$ since $fr = 0$ and $\mathrm{ann}_S(r) = \mathrm{ann}_S(f)$. But this contradicts that $f \in S \setminus R$ since $\{r \in R \mid r^2 = 0\} = \{f \in S \mid f^2 = 0\}$. Hence, $t \neq r$. Since $fr = 0$, it follows that $t \in \mathrm{ann}_R(r)$. Then $t \in \mathrm{ann}_R(r) \setminus \{r\} = \mathrm{ann}_R(x) \setminus \{x\}$. Thus $tx = 0$, and therefore $f \in \mathrm{ann}_S(x)$. Then the containments $f \in S \setminus R$ and $x \in R$ imply that $f \in \mathrm{ann}_S(x) \setminus \{x\}$. This shows that $\mathrm{ann}_S(r) \setminus \{r\} \subseteq \mathrm{ann}_S(x) \setminus \{x\}$. A symmetric argument proves the reverse inclusion, and thus

$\mathrm{ann}_S(r) \setminus \{r\} = \mathrm{ann}_S(x) \setminus \{x\}$. That is, $V_r(S) = V_x(S)$. Therefore, φ is well-defined.

Clearly the equality $\mathrm{ann}_S(r) \setminus \{r\} = \mathrm{ann}_S(x) \setminus \{x\}$ implies that $\mathrm{ann}_R(r) \setminus \{r\} = \mathrm{ann}_R(x) \setminus \{x\}$. Thus φ is injective. Also, it is straightforward to verify that φ preserves and reflects adjacency relations. It only remains to verify that φ is surjective.

Let $V_f(S) \in V(\Gamma(S)^*)$. By Lemma 6.3, there exists an element $t \in R$ such that $\mathrm{ann}_S(t) = \mathrm{ann}_S(f)$. If $f^2 = 0$, then $f \in R$. Thus $V_f(S)$ is the image of $V_f(R)$. Suppose that $f^2 \neq 0$. Then $t^2 \neq 0$. Therefore,

$$\mathrm{ann}_S(t) \setminus \{t\} = \mathrm{ann}_S(t) = \mathrm{ann}_S(f) = \mathrm{ann}_S(f) \setminus \{f\}.$$

Thus $V_f(S)$ is the image of $V_t(R)$. Hence φ is surjective. $\square$

Remark 6.14. *Note that the proof of the converse statement in Theorem 6.13 does not assume the fact that $V_r(R) = \{r\}$ for any $0 \neq r \in Z(R)$ with $r^2 = 0$. Of course, this fact is guaranteed by Theorem 6.10. Therefore, the mapping φ given in Theorem 6.13 can be shown to be a well-defined bijection by applying Lemma 6.1 and Lemma 6.3 to elements $V_r(R)$ with $r^2 \neq 0$, and then applying Theorem 6.10 to such elements with $r^2 = 0$.*

If R is reduced, then $Q(R)$ satisfies (g.a.c.) by Theorem 6.4 and [4, Theorem 11.9]. However, this observation does not generalize. For example, there exists a reduced ring R such that $Q_0(R)$ does not satisfy $\aleph_0$-(g.a.c.) (see Example 6.36). Moreover, there exists a ring R containing nonzero nilpotents such that $Q(R)$ does not satisfy $\aleph_0$-(g.a.c.) (see Example 6.35). In particular, the hypothesis $\aleph_\alpha$-(g.a.c.) is not a necessary condition for the conclusion of Theorem 6.13.

The following corollary is an immediate consequence of Proposition 6.9, Corollary 6.11, and Theorem 6.13.

Corollary 6.15. *Let α be an ordinal, and suppose that R and S are commutative rings such that $R \subseteq S \subseteq Q_\alpha(R)$. Suppose that R satisfies $\aleph_\alpha$-(g.a.c.) and $\{r \in R \mid r^2 = 0\} = \{f \in S \mid f^2 = 0\}$. If $|V_r(R)| = |V_r(S)|$ for all $r \in Z(R)$ with $r^2 \neq 0$, then $\Gamma(R) \simeq \Gamma(S)$.*

If R is reduced, then the hypotheses of Theorem 6.13 are reflected by $\Gamma(R)$. This is made evident in the following corollary.

Corollary 6.16. *Let α be an ordinal, and suppose that R and S are reduced commutative rings such that $R \subseteq S \subseteq Q_\alpha(R)$. If $\Gamma(R)$ is c.v.-$\aleph_\alpha$-complete, then the mapping $\varphi : V(\Gamma(R)^*) \to V(\Gamma(S)^*)$ defined by $\varphi([r]_R) = [r]_S$ is an isomorphism.*

Proof. Observe that $\{r \in R \mid r^2 = 0\} = \emptyset = \{f \in S \mid f^2 = 0\}$ since R and S are reduced. Moreover, $[r]_R = V_r(R)$ and $[r]_S = V_r(S)$ for all $0 \neq r \in Z(R)$. The result now follows from Theorem 6.5 and Theorem 6.13. □

Note that Example 6.36 shows that the converse to Corollary 6.16 is false. The following corollary is an immediate consequence of Corollary 6.16 and Proposition 6.9.

Corollary 6.17. *Let α be an ordinal, and suppose that R and S are reduced commutative rings such that $R \subseteq S \subseteq Q_\alpha(R)$. If $\Gamma(R)$ is c.v.-$\aleph_\alpha$-complete and $|[r]_R| = |[r]_S|$ for all $r \in Z(R) \setminus \{0\}$, then $\Gamma(R) \simeq \Gamma(S)$.*

Let Γ be a graph. We will say that Γ is *weakly central vertex $\aleph_\alpha$-complete*, or w.c.v.-$\aleph_\alpha$-complete, if for all $\emptyset \neq A \subseteq V(\Gamma)$ with $|A| < \aleph_\alpha$, either $C(A) = \emptyset$ or there exists a $v \in V(\Gamma)$ such that

$$C(v) \setminus A = C(A) \setminus \{v\}.$$

A graph Γ will be called w.c.v.-complete if it is w.c.v.-$\aleph_\alpha$-complete for every ordinal α. Note that every simple c.v.-$\aleph_\alpha$-complete graph is w.c.v.-$\aleph_\alpha$-complete. In particular, every c.v.-$\aleph_\alpha$-complete zero-divisor graph is w.c.v.-$\aleph_\alpha$-complete. The converse is false. For example, if Γ is a complete graph on at least three vertices, then Γ is w.c.v.-complete, but not c.v.-complete.

If $\Gamma = \Gamma(R)$ for some ring R, then Γ is w.c.v.-$\aleph_\alpha$-complete if and only if for all $\emptyset \neq A \subseteq R$ with $|A| < \aleph_\alpha$, there exists a $v \in R$ such that

$$\mathrm{ann}_R(v) \setminus \big(A \cup \{v\}\big) = \mathrm{ann}_R(A) \setminus \big(A \cup \{v\}\big).$$

Therefore, if R satisfies $\aleph_\alpha$-(g.a.c.), then $\Gamma(R)$ is w.c.v.-$\aleph_\alpha$-complete. The following theorem shows that the converse holds whenever R is decomposable.

Theorem 6.18. *Let α be an ordinal, and suppose that R is a decomposable commutative ring. Let R_1 and R_2 be nonzero rings such that $R \cong R_1 \oplus R_2$. Then the following statements are equivalent.*

(1) R_1 and R_2 satisfy $\aleph_\alpha$-(g.a.c.).

(2) R satisfies $\aleph_\alpha$-(g.a.c.).

(3) $\Gamma(R)$ is w.c.v.-$\aleph_\alpha$-complete.

In particular, if $\Gamma(R)$ is a w.c.v.-$\aleph_\alpha$-complete graph, then every direct summand of R has a w.c.v.-$\aleph_\alpha$-complete zero-divisor graph.

Proof. Without loss of generality, assume that $R = R_1 \oplus R_2$.

To prove (1) implies (2), suppose that R_1 and R_2 satisfy $\aleph_\alpha$-(g.a.c.). Let $\emptyset \neq A \subseteq R$ such that $|A| < \aleph_\alpha$. Note that $|\pi_i(A)| < \aleph_\alpha$, where π_i is the usual projection mapping ($i = 1, 2$). Let $r_i \in R_i$ be an element such that $\operatorname{ann}_{R_i}(r_i) = \operatorname{ann}_{R_i}(\pi_i(A))$. It is routine to check that $\operatorname{ann}_R((r_1, r_2)) = \operatorname{ann}_R(A)$. Thus R satisfies $\aleph_\alpha$-(g.a.c.).

Note that (2) implies (3) by the above comments. To show (3) implies (1), suppose that $\Gamma(R)$ is w.c.v.-$\aleph_\alpha$-complete. Let $\emptyset \neq A \subseteq R_1$ with $|A| < \aleph_\alpha$. We need to show that there exists an element $r \in R_1$ such that $\operatorname{ann}_{R_1}(r) = \operatorname{ann}_{R_1}(A)$. Then R_2 will satisfy $\aleph_\alpha$-(g.a.c.) by symmetry.

If $A = \{0\}$, then let $r = 0$. Suppose that $A \neq \{0\}$. Then $\operatorname{ann}_{R_1}(A) = \operatorname{ann}_{R_1}(A \setminus \{0\})$, and hence we can assume that $0 \notin A$.

If $\operatorname{ann}_{R_1}(A) = \{0\}$, then let $r = 1$. Suppose that $\operatorname{ann}_{R_1}(A) \neq \{0\}$. Then $A \times \{1\} \subseteq V(\Gamma(R_1 \oplus R_2))$. Also, $(x, 0) \in C(A \times \{1\})$ for all $0 \neq x \in \operatorname{ann}_{R_1}(A)$. Since $\Gamma(R)$ is w.c.v.-$\aleph_\alpha$-complete, there exists an element $(r_1, r_2) \in R$ such that

$$C((r_1, r_2)) \setminus (A \times \{1\}) = C(A \times \{1\}) \setminus \{(r_1, r_2)\}.$$

Suppose that $0 \neq x \in \operatorname{ann}_{R_1}(A)$. Then $(x, 0) \in \operatorname{ann}_R(A \times \{1\})$. If $(x, 0) = (r_1, r_2)$, then $0 \notin A$ implies $(0, 1) \in C((r_1, r_2)) \setminus (A \times \{1\})$. But clearly $(0, 1) \notin C(A \times \{1\})$, contradicting the choice of (r_1, r_2). Hence $(x, 0) \neq (r_1, r_2)$, and therefore $(x, 0) \in C(A \times \{1\}) \setminus \{(r_1, r_2)\}$. Thus $(x, 0) \in C((r_1, r_2))$. In particular, $x \in \operatorname{ann}_{R_1}(r_1)$. Since $0 \in \operatorname{ann}_{R_1}(r_1)$, this shows that $\operatorname{ann}_{R_1}(A) \subseteq \operatorname{ann}_{R_1}(r_1)$.

If $0 \neq x \in \operatorname{ann}_{R_1}(r_1)$, then $(x, 0) \in C((r_1, r_2)) \setminus (A \times \{1\})$. Thus $(x, 0) \in C(A \times \{1\})$. Hence $x \in \operatorname{ann}_{R_1}(A)$. Since $0 \in \operatorname{ann}_{R_1}(A)$, this verifies the inclusion $\operatorname{ann}_{R_1}(r_1) \subseteq \operatorname{ann}_{R_1}(A)$, and it follows that $\operatorname{ann}_{R_1}(r_1) = \operatorname{ann}_{R_1}(A)$. Therefore, R_1 satisfies $\aleph_\alpha$-(g.a.c.).

To prove the "in particular" statement, suppose that $\Gamma(R)$ is w.c.v.-$\aleph_\alpha$-complete. Then the result follows from the above argument since every ring satisfying $\aleph_\alpha$-(g.a.c.) has a w.c.v.-$\aleph_\alpha$-complete zero-divisor graph. $\square$

Note that "decomposable" cannot be omitted from the hypothesis in the previous theorem. Specifically, $\Gamma(R)$ may be w.c.v.-$\aleph_\alpha$-complete while R does not satisfy $\aleph_\alpha$-(g.a.c.). For example, let $R = \mathbb{Z}_4[X]/(X^2)$. Then $\Gamma(R)$ is w.c.v.-complete (see Figure 6.2). Moreover, $\mathrm{ann}_R(\{\overline{2}, \overline{2+X}\}) = \{\overline{0}, \overline{2X}\}$. Suppose that $\mathrm{ann}_R(f) = \{\overline{0}, \overline{2X}\}$ for some $f \in R$. Then $f \in \{\overline{0}, \overline{2X}\}$ since $f^2 = \overline{0}$ for all $f \in Z(R)$. But then $\overline{2} \in \mathrm{ann}_R(f)$, a contradiction. Therefore, no such f exists. Thus R does not satisfy $\aleph_0$-(g.a.c.). Incidently, we have proved that R is an indecomposable ring. Moreover, any ring having R as a direct summand does not have a w.c.v.-$\aleph_\alpha$-complete zero-divisor graph.

Corollary 6.19. *Let α be an ordinal, and suppose that R and S are commutative rings such that $R \subseteq S \subseteq Q_\alpha(R)$. Suppose that R is decomposable and $\{r \in R \mid r^2 = 0\} = \{f \in S \mid f^2 = 0\}$. If $\Gamma(R)$ is w.c.v.-$\aleph_\alpha$-complete and $|V_r(R)| = |V_r(S)|$ for all $r \in Z(R)$ with $r^2 \neq 0$, then $\Gamma(R) \simeq \Gamma(S)$.*

Proof. This result is a restatement of Corollary 6.15, where the $\aleph_\alpha$-(g.a.c.) hypothesis has been translated into its graph-theoretic counterpart. $\square$

In Section 6.4, there are several examples that are constructed by passing to direct sums. We conclude this section with a lemma which will be useful in such constructions.

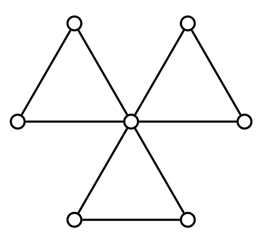


Figure 6.2: $\Gamma(\mathbb{Z}_4[X]/(X^2))$

Lemma 6.20. *Let $\varphi_1 : V(\Gamma(R_1)) \to V(\Gamma(R_1'))$ and $\varphi_2 : V(\Gamma(R_2)) \to V(\Gamma(R_2'))$ be isomorphisms. If $|R_i \setminus V(\Gamma(R_i))| = |R_i' \setminus V(\Gamma(R_i'))|$ for each $i \in \{1, 2\}$, then $\Gamma(R_1 \oplus R_2) \simeq \Gamma(R_1' \oplus R_2')$.*

Proof. Let $\psi_i : R_i \setminus V(\Gamma(R_i)) \to R_i' \setminus V(\Gamma(R_i'))$ be bijections with $\psi_i(0_{R_i}) = 0_{R_i'}$ $(i = 1, 2)$. Let $\Phi_i : R_i \to R_i'$ be defined by

$$\Phi_i(r) = \begin{cases} \varphi_i(r), & r \in V(\Gamma(R_i)) \\ \psi_i(r), & \text{otherwise} \end{cases}.$$

Finally, let $\Psi : R_1 \oplus R_2 \to R_1' \oplus R_2'$ be defined by the rule

$$\Psi(r_1, r_2) = (\Phi_1(r_1), \Phi_2(r_2)).$$

Then it is straightforward to show that

$$\Psi|_{V(\Gamma(R_1 \oplus R_2))} : V\big(\Gamma(R_1 \oplus R_2)\big) \to V\big(\Gamma(R_1' \oplus R_2')\big)$$

is an isomorphism. □

6.4 The Zero-Divisor Graph of $Q_0(R)$

Let R be a commutative ring. The zero-divisor graph of $Q(R)$ was studied in [7], where the relations $\Gamma(R) \simeq \Gamma(Q(R))$ and $\Gamma(R) \not\simeq \Gamma(Q(R))$ were shown to be realizable by von Neumann regular rings satisfying $R \subsetneq Q(R)$. With the results of Section 6.3, we are now equipped to identify relations between more general rings of quotients. In this section, we consider the zero-divisor graphs of R, $Q_0(R)$, and $Q(R)$. In particular, we examine the following hypotheses.

Relation 6.21. The following scenarios will be considered for a commutative ring R.

(1) $\Gamma(R) \not\simeq \Gamma(Q_0(R)) \not\simeq \Gamma(Q(R))$, and $\Gamma(R) \not\simeq \Gamma(Q(R))$.

(2) $\Gamma(R) \simeq \Gamma(Q_0(R)) \not\simeq \Gamma(Q(R))$.

(3) $\Gamma(R) \not\simeq \Gamma(Q_0(R)) \simeq \Gamma(Q(R))$.

(4) $\Gamma(R) \simeq \Gamma(Q_0(R)) \simeq \Gamma(Q(R))$.

(5) $\Gamma(R) \simeq \Gamma(Q(R)) \not\simeq \Gamma(Q_0(R))$.

Note that the existence of rings which satisfy (2), (3), or (4) of Relation 6.21 can be easily verified: Any ring R such that $R = Q_0(R)$ and $\Gamma(R) \not\simeq \Gamma(Q(R))$ will satisfy Relation (2) (e.g., [7, Example 3.7]). Any ring R such that $\Gamma(R) \not\simeq \Gamma(Q_0(R))$ and $Q_0(R) = Q(R)$ will satisfy (3) (see Example 6.30). Any rationally complete ring will satisfy (4) (e.g., any finite ring). Furthermore, if $T(R)$ is the total quotient ring of R, then $\Gamma(R) \simeq \Gamma(T(R))$ by [1, Theorem 2.2]. Therefore, it is easy to construct examples which satisfy $R \subsetneq T(R) = Q_0(R)$ and $\Gamma(R) \simeq \Gamma(Q_0(R))$ (e.g., let $R = \prod_{\mathbb{N}} \mathbb{Z}$). We shall avoid such trivialities, and consider total quotient rings which satisfy $R \subsetneq Q_0(R) \subsetneq Q(R)$.

If α is any ordinal, then $Q_0(Q_\alpha(R)) = Q_\alpha(R)$ by [10, Corollary 2.2]. Therefore, if $T(Q_\alpha(R))$ is the total quotient ring of $Q_\alpha(R)$, then $Q_\alpha(R) \subseteq T(Q_\alpha(R)) \subseteq Q_0(Q_\alpha(R)) = Q_\alpha(R)$. Thus $T(Q_\alpha(R)) = Q_\alpha(R)$. That is, $Q_\alpha(R)$ is a total quotient ring for every ordinal α.

The results of this section prove the following theorem.

Theorem 6.22. *Let $n \in \{1,2,3,4\}$. Then Relation 6.21(n) can be realized by a total quotient ring R such that $R \subsetneq Q_0(R) \subsetneq Q(R)$.*

The following examples involve versions of "$A+B$ rings" and "idealizations," as described in Sections 25 and 26 of [6]. All of the graph-isomorphisms of this section are "strong" in the sense of Remark 6.12(2). Let F be an infinite field. Set $D_1 = F[X,Y,Z]$ and $D_2 = F[\{XZ^n, YZ^n \mid n \geq 0\}]$, where X, Y, and Z are algebraically independent indeterminates. Throughout, $\mathcal{P}$ will be a set of prime ideals of D_1 containing infinitely many principal ideals. Let I be an indexing set for $\mathcal{P}$. Set $\mathcal{I} = I \times \mathbb{N}$. If $\alpha = (i,n) \in \mathcal{I}$, then set $P_\alpha = P_i$ and let K_α denote the quotient field of D_1/P_α.

Let $\Omega_k = \{f \in D_k \mid f \notin \cup_{\alpha\in\mathcal{I}} P_\alpha\}$ $(k = 1,2)$, and define $\varphi : (D_1)_{\Omega_1} \to \prod_{\alpha\in\mathcal{I}} K_\alpha$ to be the canonical homomorphism. Note that $\cap_{\alpha\in\mathcal{I}} P_\alpha = \{0\}$ since D_1 is a unique factorization domain and $\mathcal{P}$ contains infinitely many principal ideals. In particular, φ is an embedding. Let $R_1 = \varphi\big((D_1)_{\Omega_1}\big) + \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ and $R_2 = \varphi\big((D_2)_{\Omega_2}\big) + \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$. Then $R_2 \subseteq R_1 \subseteq \prod_{\alpha\in\mathcal{I}} K_\alpha$.

Suppose that $\varphi(f/g) + b \in R_k \setminus Z(R_k)$ $(f \in D_k,\ g \in \Omega_k,\ b \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha,\ k = 1,2)$. Then $(\varphi(f/g)+b)(\alpha) \neq \overline{0}$ for all $\alpha \in \mathcal{I}$. Since $b(\alpha) = \overline{0}$ for all but finitely many α, it follows that $\varphi(f/g)(\alpha) \neq \overline{0}$ for almost all α. Thus $f \notin P$ for all $P \in \mathcal{P}$. That is, $f \in \Omega_k$. Hence

$(\varphi(f/g)+b)^{-1} = \varphi(g/f)+b' \in R_k$, where

$$b'(\alpha) = \left\{ \begin{array}{ll} -\varphi(g/f)(\alpha) + \big((\varphi(f/g)+b)(\alpha)\big)^{-1}, & b(\alpha) \neq \overline{0} \\ \overline{0}, & \text{otherwise} \end{array} \right. .$$

Therefore, R_1 and R_2 are total quotient rings. In fact, it will be shown that $R_1 = Q_0(R_1)$ (Proposition 6.27).

Let J be a subset of $(D_k)_{\Omega_k}$ $(k = 1, 2)$. If an element of R_k with a nonzero α-coordinate annihilates $\varphi(J)$, then $J \subseteq (P_\alpha)_{\Omega_k}$. Conversely, if $J \subseteq (P_\alpha)_{\Omega_k}$, then the element of R_k having a $\overline{1}$ in the α-coordinate and $\overline{0}$ elsewhere annihilates $\varphi(J)$. Therefore, $\varphi(J)$ is dense in R_k if and only if $J \setminus P_{\Omega_k} \neq \emptyset$ for all $P \in \mathcal{P}$.

The dense set $E \subseteq R_2$ of elements having a $\overline{1}$ in precisely one coordinate and $\overline{0}$ elsewhere satisfies $E \subseteq r^{-1}R_2$ for all $r \in \prod_{\alpha\in\mathcal{I}} K_\alpha$. Thus $\prod_{\alpha\in\mathcal{I}} K_\alpha \subseteq Q(R_2)$. Being a direct product of fields, $\prod_{\alpha\in\mathcal{I}} K_\alpha$ is rationally complete. Hence $Q(R_2) = \prod_{\alpha\in\mathcal{I}} K_\alpha$. Similarly, $Q(R_1) = \prod_{\alpha\in\mathcal{I}} K_\alpha$.

The results of this section numbered 6.23 through 6.27 are derived from proofs found in [6] and [12]. The reader may wish to pass straight to Example 6.28. The following proposition shows that R_1 satisfies $\aleph_0$-(g.a.c.) whenever $\mathcal{P}$ consists entirely of principal ideals (cf. [6, Example 2]).

Proposition 6.23. *Let D be a subring of D_1, and suppose that $\mathcal{P} \subseteq \{fD_1 \mid f \in D\}$. Set $\Omega = \{f \in D \mid f \notin P$ for all $P \in \mathcal{P}\}$. Then $\varphi(D_\Omega) + \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ satisfies $\aleph_0$-(g.a.c.). In particular, the isomorphism $\Gamma\big(\varphi(D_\Omega) + \bigoplus_{\alpha\in\mathcal{I}} K_\alpha\big)^* \simeq \Gamma\big(Q_0(\varphi(D_\Omega) + \bigoplus_{\alpha\in\mathcal{I}} K_\alpha)\big)^*$ holds.*

Proof. Let $T = \varphi(D_\Omega) + \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$, and suppose that $t_1, t_2 \in T$; say $t_k = \varphi(f_k/g_k) + b_k$ $(f_k \in D,\ g_k \in \Omega,\ b_k \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha,\ k = 1, 2)$. Note that the set $\mathcal{I}' = \{\alpha \in \mathcal{I} \mid$ either $b_1(\alpha) \neq \overline{0}$ or $b_2(\alpha) \neq \overline{0}\}$ is finite. Let $\mathcal{I}'' = \{\alpha \in \mathcal{I}' \mid t_1(\alpha) = t_2(\alpha) = \overline{0}\}$. If $f_1/g_1 = f_2/g_2 = 0$, then let $f = 0$; otherwise, by hypothesis, there exists a (finite) set $J \subseteq D$ such that $\{P \in \mathcal{P} \mid \{f_1, f_2\} \subseteq P\} = \{pD_1 \mid p \in J\}$. If $J = \emptyset$, then let $f = 1$. Otherwise, let $f = \Pi_{p\in J} p \in D$. Define $b \in \prod_{\alpha\in\mathcal{I}} K_\alpha$ to be the element such that

$$b(\alpha) = \left\{ \begin{array}{ll} -\varphi(f)(\alpha), & \alpha \in \mathcal{I}'' \\ \overline{1} - \varphi(f)(\alpha), & \alpha \in \mathcal{I}' \setminus \mathcal{I}'' \\ \overline{0}, & \text{otherwise} \end{array} \right. .$$

Note that $b \in \bigoplus_{\alpha \in \mathcal{I}} K_\alpha$ since $\mathcal{I}'$ is finite. In particular, $\varphi(f) + b \in T$. If $\alpha \in \mathcal{I}''$, then $t_1(\alpha) = t_2(\alpha) = (\varphi(f) + b)(\alpha) = \overline{0}$. If $\alpha \in \mathcal{I}' \setminus \mathcal{I}''$, then $(\varphi(f) + b)(\alpha) = \overline{1}$, and either $t_1(\alpha) \neq \overline{0}$ or $t_2(\alpha) \neq \overline{0}$. Suppose that $\alpha \notin \mathcal{I}'$. Then $b_1(\alpha) = b_2(\alpha) = b(\alpha) = \overline{0}$. But clearly

$$\{P \in \mathcal{P} \mid \{f_1, f_2\} \subseteq P\} = \{pD_1 \mid p \in J\} = \{P \in \mathcal{P} \mid f \in P\}.$$

It follows that $\varphi(f)(\alpha) = \overline{0}$ if and only if $\varphi(f_1/g_1)(\alpha) = \varphi(f_2/g_2)(\alpha) = \overline{0}$ $(\alpha \in \mathcal{I})$. Therefore, $(\varphi(f) + b)(\alpha) = \overline{0}$ if and only if $t_1(\alpha) = t_2(\alpha) = \overline{0}$. Thus $\mathrm{ann}_T(t_1, t_2) = \mathrm{ann}_T(\varphi(f) + b)$, and it follows that T satisfies $\aleph_0$-(g.a.c.).

Clearly T is reduced, and hence the "in particular" statement follows from Theorem 6.13. □

Proposition 6.24. *Suppose that $\mathcal{P}$ is an infinite set of principal prime ideals of D_1 such that $ZD_1 \in \mathcal{P}$. Then R_2 does not satisfy $\aleph_0$-(g.a.c.).*

Proof. If R_2 satisfies $\aleph_0$-(g.a.c.), then there exists a $t \in R_2$ that satisfies $\mathrm{ann}_{R_2}(\varphi(XZ), \varphi(YZ)) = \mathrm{ann}_{R_2}(t)$. If $f/g \in (D_2)_{\Omega_2}$ and $P \in \mathcal{P}$, then $f/g \in P_{\Omega_2}$ if and only if $f \in P$. It follows that t can be chosen such that $t = \varphi(f) + b$ for some $f \in D_2$ and $b \in \bigoplus_{\alpha \in \mathcal{I}} K_\alpha$. Suppose that there exists a $P_\alpha \in \mathcal{P}$ such that either $\{XZ, YZ\} \subseteq P_\alpha$ or $f \in P_\alpha$, but not both. Say $\alpha = (i_0, n)$. Choose an $N \in \mathbb{N}$ such that $b(i_0, N) = \overline{0}$. Then the element of R_2 having a $\overline{1}$ in the (i_0, N)-coordinate and $\overline{0}$ elsewhere annihilates either $\{\varphi(XZ), \varphi(YZ)\}$ or t, but not both. This is a contradiction. Therefore, $\{XZ, YZ\} \subseteq P$ if and only if $f \in P$ $(P \in \mathcal{P})$. But $\{XZ, YZ\} \subseteq P \in \mathcal{P}$ if and only if $P = ZD_1$. Thus $f = uZ^n$ for some $0 \neq u \in F$ and $n \geq 1$. This contradicts the containment $f \in D_2$. Therefore, no such f exists. Hence R_2 does not satisfy $\aleph_0$-(g.a.c.). □

For any subset $J \subseteq (D_1)_{\Omega_1}$, let J^{-1} denote the set of elements in the quotient field of $(D_1)_{\Omega_1}$ that map J into $(D_1)_{\Omega_1}$ under multiplication. Note that the proofs of Lemma 6.25 and Proposition 6.27 are valid for *any* set $\mathcal{P}$ of prime ideals of D_1 which intersect in $\{0\}$.

Lemma 6.25. *Let $J \subseteq (D_1)_{\Omega_1}$ be a set such that $J \setminus P_{\Omega_1} \neq \emptyset$ for all $P \in \mathcal{P}$. Then $J^{-1} = (D_1)_{\Omega_1}$.*

Proof. Let $a/b \in J^{-1}$. We can assume that $a, b \in D_1$ such that a greatest common divisor of a and b is 1. Suppose that $a/b \notin (D_1)_{\Omega_1}$.

Then there exists a $P \in \mathcal{P}$ with $b \in P$. Let $r/q \in J \setminus P_{\Omega_1}$. In particular, $r \notin P$. But $(ar)/(bq) \in (D_1)_{\Omega_1} \subseteq (D_1)_P$ implies that $ars = bqt$ for some $s, t \in D_1$ with $s \notin P$. Since D_1 is a unique factorization domain and $\gcd(a, b) = 1$, it follows that b divides rs. This contradicts that $rs \notin P$. Therefore, $a/b \in (D_1)_{\Omega_1}$. This verifies that $J^{-1} \subseteq (D_1)_{\Omega_1}$. The reverse inclusion is obvious. $\square$

Lemma 6.26. *Suppose that $\mathcal{P}$ is an infinite set of principal prime ideals. Then $Q_0(R_1) = Q_0(R_2)$.*

Proof. There is no principal prime ideal containing both X and Y. Also, every element of D_1 maps the set $\{X, Y\}$ into D_2 under multiplication. Therefore, $\{\varphi(aX), \varphi(bY)\}$ is dense in R_2 for all $a, b \in \Omega_1$. Let $s \in R_1$; say $s = \varphi(f/g) + b$ for some $f \in D_1$, $g \in \Omega_1$, and $b \in \bigoplus_{\alpha \in \mathcal{I}} K_\alpha$. Then $\{\varphi(gX), \varphi(gY)\} \subseteq \varphi(f/g)^{-1}R_2$. Clearly $\{\varphi(gX), \varphi(gY)\} \subseteq b^{-1}R_2$ (indeed, $b \in R_2$), and thus $\{\varphi(gX), \varphi(gY)\} \subseteq s^{-1}R_2$. Hence $s \in Q_0(R_2)$. This shows that $R_1 \subseteq Q_0(R_2)$. Moreover, the inclusions $R_2 \subseteq R_1 \subseteq Q_0(R_2) \subseteq Q(R_2)$ imply that

$$Q_0(R_2) \subseteq Q_0(R_1) \subseteq Q_0(Q_0(R_2)) = Q_0(R_2),$$

where the equality holds by [10, Corollary 2.2]. Therefore, $Q_0(R_1) = Q_0(R_2)$. $\square$

The ring $Q_0(R)$ is calculated in [12, Theorem 11], where R is a ring constructed using the principle of idealization. The proof of the following proposition is a close mimicry of the one given for [12, Theorem 11(d)].

Proposition 6.27. *Let R_1 and R_2 be defined as above. Then $Q_0(R_1) = R_1$. If $\mathcal{P}$ consists entirely of principal ideals, then $Q_0(R_2) = R_1$.*

Proof. By Lemma 6.26, it suffices to show that $Q_0(R_1) = R_1$. Suppose that $s \in Q_0(R_1)$. There exists a finite set $J = \{j_1, ..., j_n\} \subseteq (D_1)_{\Omega_1} \setminus \{0\}$, and elements $b_k \in \bigoplus_{\alpha \in \mathcal{I}} K_\alpha$ $(k = 1, ..., n)$, such that the set

$$\{\varphi(j_1) + b_1, ..., \varphi(j_n) + b_n\}$$

is dense and contained in $s^{-1}R_1$. It follows that $\varphi(J)$ is dense. If not, then $J \subseteq (P_\beta)_{\Omega_1}$ for some $\beta = (i_0, m) \in \mathcal{I}$. But the set $\{\alpha \in \mathcal{I} \mid b_k(\alpha) \neq \overline{0}$ for some $k \in \{1, ..., n\}\}$ is finite. Hence there exists an integer N such that $b_k\big((i_0, N)\big) = \overline{0}$ for all $k \in \{1, ..., n\}$. Then the nonzero element of R_1 having a $\overline{1}$ in the (i_0, N)-coordinate and $\overline{0}$ elsewhere annihilates $\{\varphi(j_1) + b_1, ..., \varphi(j_n) + b_n\}$, a contradiction. Therefore, $J \setminus P_{\Omega_1} \neq \emptyset$ for all $P \in \mathcal{P}$. Thus $\varphi(J)$ is dense.

Clearly $sb_k \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha \subseteq R_1$ for each $k \in \{1, ..., n\}$. Hence $s\varphi(j_k) = s(\varphi(j_k) + b_k) - sb_k \in R_1$ for all $k \in \{1, ..., n\}$; say

$$s\varphi(j_k) = \varphi(f_k/g_k) + e_k$$

for some $f_k \in D_1$, $g_k \in \Omega_1$, and $e_k \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ $(k = 1, ..., n)$.

Consider the mapping $\psi : \sum_{k=1}^n \varphi\big(j_k(D_1)_{\Omega_1}\big) \to \varphi\big((D_1)_{\Omega_1}\big)$ defined by

$$\psi\Big(\sum_{k=1}^n \varphi(j_k r_k/q_k)\Big) = \sum_{k=1}^n \varphi\big((f_k/g_k)(r_k/q_k)\big),\ r_k \in D_1,\ q_k \in \Omega_1.$$

Note that ψ is well-defined since $\sum_{k=1}^n \varphi(j_k r_k/q_k) = (\overline{0})$ implies

$$\sum_{k=1}^n \varphi\big((f_k/g_k)(r_k/q_k)\big) + \sum_{k=1}^n e_k\varphi(r_k/q_k) = s\sum_{k=1}^n \varphi(j_k r_k/q_k) = (\overline{0}),$$

and thus

$$\sum_{k=1}^n \varphi\big((f_k/g_k)(r_k/q_k)\big) \in \varphi\big((D_1)_{\Omega_1}\big) \cap \bigoplus_{\alpha\in\mathcal{I}} K_\alpha = (\overline{0}).$$

Hence $\psi\big((\overline{0})\big) = (\overline{0})$. Then clearly

$$\psi \in \operatorname{Hom}_{\varphi\left((D_1)_{\Omega_1}\right)}\Big(\sum_{k=1}^n \varphi\big(j_k(D_1)_{\Omega_1}\big), \varphi\big((D_1)_{\Omega_1}\big)\Big).$$

Choose an element $j \in J$. Then $s_1 = \psi(\varphi(j))/\varphi(j)$ belongs to the quotient field of $\varphi\big((D_1)_{\Omega_1}\big)$, and

$$s_1\varphi(j_k) = \psi(\varphi(j_k)) = \varphi(f_k/g_k) \in \varphi\big((D_1)_{\Omega_1}\big)$$

for all $k \in \{1, ..., n\}$. Also, $J^{-1} = (D_1)_{\Omega_1}$ by Lemma 6.25, and it follows that $s_1 \in \varphi\big((D_1)_{\Omega_1}\big)$.

Consider the mapping $\rho : \sum_{k=1}^{n} \varphi\big(j_k(D_1)_{\Omega_1}\big) \to \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ defined by

$$\rho\big(\sum_{k=1}^{n} \varphi(j_k r_k/q_k)\big) = \sum_{k=1}^{n} e_k\varphi(r_k/q_k),\ r_k \in D_1,\ q_k \in \Omega_1.$$

Note that ρ is well-defined since the above computations show that the equality $\sum_{k=1}^{n} e_k\varphi(r_k/q_k) = (\overline{0})$ holds whenever $\sum_{k=1}^{n} \varphi(j_k r_k/q_k) = (\overline{0})$. Hence

$$\rho \in \operatorname{Hom}_{\varphi\left((D_1)_{\Omega_1}\right)}\big(\sum_{k=1}^{n} \varphi\big(j_k(D_1)_{\Omega_1}\big), \bigoplus_{\alpha\in\mathcal{I}} K_\alpha\big).$$

For each $\alpha \in \mathcal{I}$, choose an element $t_\alpha \in \varphi\big(J \setminus (P_\alpha)_{\Omega_1}\big)$. Let $j_k \in J$. Then

$$t_\alpha(\alpha)\big(\rho(\varphi(j_k))(\alpha)\big) = \big(\varphi(j_k)(\alpha)\big)\big(\rho(t_\alpha)(\alpha)\big).$$

This shows that $\rho(\varphi(j_k)) = s_2\varphi(j_k)$ for all $j_k \in J$, where $s_2 \in \prod_{\alpha\in\mathcal{I}} K_\alpha$ is the element such that $s_2(\alpha) = t_\alpha(\alpha)^{-1}\big(\rho(t_\alpha)(\alpha)\big)$ for all $\alpha \in \mathcal{I}$. That is,

$$s_2\varphi(j_k) = e_k \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$$

for each $k \in \{1, ..., n\}$.

Since $J \setminus P_{\Omega_1} \neq \emptyset$ for all $P \in \mathcal{P}$, it follows that

$$\{\alpha \in \mathcal{I} \mid s_2(\alpha) \neq \overline{0}\} = \cup_{k=1}^{n}\{\alpha \in \mathcal{I} \mid \big(s_2\varphi(j_k)\big)(\alpha) \neq \overline{0}\}.$$

But $s_2\varphi(j_k) \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ for all $k \in \{1, ..., n\}$, and therefore $\{\alpha \in \mathcal{I} \mid s_2(\alpha) \neq \overline{0}\}$ is a finite union of finite sets. Thus $\{\alpha \in \mathcal{I} \mid s_2(\alpha) \neq \overline{0}\}$ is finite. Hence $s_2 \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$.

By the above arguments, it follows that s and $s_1 + s_2$ are elements of $\prod_{\alpha\in\mathcal{I}} K_\alpha = Q(R_1)$ which agree on the dense set $\varphi(J)$. Thus $s = s_1 + s_2$. But the above arguments also show that $s_1 + s_2 \in \varphi\big((D_1)_{\Omega_1}\big) + \bigoplus_{\alpha\in\mathcal{I}} K_\alpha = R_1$. Hence $s \in R_1$, and it follows that $Q_0(R_1) \subseteq R_1$. The reverse inclusion is clear, and therefore $Q_0(R_1) = R_1$. ☐

Example 6.28. *Suppose that $\mathcal{P}$ is the set of all principal prime ideals of D_1. Then R_2 is a total quotient ring which satisfies $R_2 \subsetneq Q_0(R_2) \subsetneq Q(R_2)$ and Relation 6.21(1).*

Proof. The discussion prior to Proposition 6.23 shows that R_2 is a total quotient ring. The proper inclusions will follow immediately upon establishing the validity of Relation 6.21(1). Note that $Q_0(R_2) = R_1$ by

Proposition 6.27. That $\Gamma(R_2) \not\cong \Gamma(Q_0(R_2))$ follows from Theorem 6.5, Proposition 6.23, and Proposition 6.24. Also, [1, Theorem 3.5] shows that a reduced total quotient ring is von Neumann regular if and only if its zero-divisor graph is complemented. In particular, $\Gamma(Q(R_2))$ is complemented. On the other hand, R_2 is a total quotient ring which is *not* von Neumann regular (e.g., the prime ideal $\varphi(\{0\}) + \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ is not maximal), and hence $\Gamma(R_2)$ is not complemented. Thus $\Gamma(R_2) \not\cong \Gamma(Q(R_2))$. Similarly, $\Gamma(Q_0(R_2)) \not\cong \Gamma(Q(R_2))$. □

Example 6.29. *Let $\mathcal{P}$ be the family of principal prime ideals belonging to the set $\{fD_1 \mid f \in D_2\}$. Then R_2 is a total quotient ring which satisfies $R_2 \subsetneq Q_0(R_2) \subsetneq Q(R_2)$ and Relation 6.21(2).*

Proof. The discussion prior to Proposition 6.23 shows that R_2 is a total quotient ring. The containment $R_2 \subsetneq Q_0(R_2)$ holds since Proposition 6.27 shows that $\varphi(Z) \in Q_0(R_2) \setminus R_2$. That $\Gamma(Q_0(R_2)) \not\cong \Gamma(Q(R_2))$ follows as in Example 6.28. This also verifies that $Q_0(R_2) \subsetneq Q(R_2)$. Note that $\Gamma(R_2)$ is c.v.-$\aleph_0$-complete by Theorem 6.5 and Proposition 6.23. By Corollary 6.17, it only remains to show that $|[r]_{R_2}| = |[r]_{Q_0(R_2)}|$ for all $r \in Z(R_2) \setminus \{0\}$.

Let $r \in Z(R_2) \setminus \{0\}$. Observe that $|F| \leq |[r]_{R_2}|$ since $\varphi(u)r \in [r]_{R_2}$ for all $u \in F$. Also, the inequality $|[r]_{R_2}| \leq |[r]_{Q_0(R_2)}|$ follows from Lemma 6.1. Furthermore, $\mathcal{P}$ consists entirely of principal ideals, and hence $|\mathcal{I}| \leq |D_1| = |F|$. Since $Q_0(R_2) = R_1$, it follows that $|Q_0(R_2)| = |F|$. Therefore,

$$|F| \leq |[r]_{R_2}| \leq |[r]_{Q_0(R_2)}| \leq |Q_0(R_2)| = |F|.$$

Thus $|[r]_{R_2}| = |[r]_{Q_0(R_2)}|$ for all $r \in Z(R_2) \setminus \{0\}$. □

Let R be a commutative ring and M an (unitary) R-module. The *idealization* $R(+)M$ *of* M is the commutative ring (with unity) $R \times M$, where addition is defined componentwise and multiplication is defined by the rule $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. Note that $(1, 0)$ is the multiplicative identity in $R(+)M$.

Define $S_1 = (D_1)_{\Omega_1}(+)\bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ and $S_2 = (D_2)_{\Omega_2}(+)\bigoplus_{\alpha\in\mathcal{I}} K_\alpha$. Making the appropriate modifications to Proposition 6.27 will show that $S_1 = Q_0(S_1)$. Alternatively, this is an immediate consequence of Lemma 6.25 taken together with [12, Theorem 11(f)]. If $\mathcal{P}$ consists entirely of principal ideals, then $Q_0(S_2) = S_1$. To see this, note that the set $\{\big(X,(\overline{0})\big),\big(Y,(\overline{0})\big)\}$ is dense and contained in $s^{-1}S_2$ for all $s \in S_1$. Therefore,

$$S_1 \subseteq Q_0(S_2) \subseteq Q_0(S_1) = S_1.$$

Hence $Q_0(S_2) = S_1$.

If $(r,m) \in S_2$, then

$$\operatorname{ann}_{S_2}\big((r,m)\big) = \big\{(s,n) \in S_2 \mid rs = 0 \text{ and } \{rn, sm\} \subseteq \{(\overline{0})\}\big\},$$

where the inclusion $\{rn, sm\} \subseteq \{(\overline{0})\}$ holds since $rs = 0$ forces either $r = 0$ or $s = 0$. Then it is straightforward to check that the non-zero-divisors of S_2 are precisely those elements of the form $(f/g, a)$, where $f, g \in \Omega_2$. Any such element is a unit in S_2 with $(f/g,a)^{-1} = (g/f, b)$, where $b(\alpha) = -(g/f)^2 a(\alpha)$ for all $\alpha \in \mathcal{I}$. Thus S_2 is a total quotient ring.

Example 6.30. *Let $\mathcal{P}$ be the set of all principal prime ideals of D_1. Then S_2 is a total quotient ring which satisfies $\Gamma(S_2) \not\cong \Gamma(Q_0(S_2)) = \Gamma(Q(S_2))$.*

Proof. The above comments show that S_2 is a total quotient ring. Let $D \subseteq S_2$. Suppose that there exists a $P \in \mathcal{P}$ such that $f \in P$ for all $(f,a) \in D$. Then $(0,b) \in \operatorname{ann}_{S_2}(D)$, where b is any element which satisfies $b(\alpha) = \overline{0}$ for all $\alpha \in \mathcal{I}$ with $P_\alpha \neq P$. Conversely, if no such P exists, then for all $(\overline{0}) \neq b \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ there exists an element $(f,a) \in D$ such that $fb \neq (\overline{0})$. It follows that a set $D \subseteq S_2$ is dense if and only if it has the property that, for all $P \in \mathcal{P}$, there exists elements $f \in D_2 \setminus P$ and $a \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ such that $(f,a) \in D$. But any element of D_2 is contained in only finitely many members of $\mathcal{P}$. Therefore, $D \subseteq S_2$ is dense if and only if it contains a finite set $\{(f_i,a_i)\}_{i=1}^n$ such that, for all $P \in \mathcal{P}$, there exists a $j \in \{1,...,n\}$ with $f_j \notin P$. In particular, every dense set in S_2 contains a finite dense set. Thus $Q_0(S_2) = Q(S_2)$, and hence $\Gamma(Q_0(S_2)) = \Gamma(Q(S_2))$.

Note that $\Gamma(Q_0(S_2))$ is w.c.v.-$\aleph_0$-complete. To see this, suppose that $\{(f,a),(g,b)\} \subseteq Q_0(S_2)$. If either $f \neq 0$ or $g \neq 0$, then let h be a greatest common divisor of f and g in D_1. If $f = g = 0$, then let $h = 0$. Suppose that $c \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ is the element defined by

$$c(\alpha) = \left\{ \begin{array}{cc} \overline{0}, & a(\alpha) = b(\alpha) = \overline{0} \\ \overline{1}, & \text{otherwise} \end{array} \right. .$$

Since $\mathcal{P}$ is a set of principal ideals, it follows that $\{f, g\} \subseteq P$ if and only if $h \in P$ ($P \in \mathcal{P}$). Using this fact, it is straightforward to check that

$$\operatorname{ann}_{Q_0(S_2)}\big((h, c)\big) = \operatorname{ann}_{Q_0(S_2)}\big((f, a), (g, b)\big).$$

It follows that $Q_0(S_2)$ satisfies $\aleph_0$-(g.a.c.). Hence $\Gamma(Q_0(S_2))$ is w.c.v.-$\aleph_0$-complete by the comments prior to Theorem 6.18.

It remains to show that $\Gamma(S_2)$ is *not* w.c.v.-$\aleph_0$-complete. Consider the set $A = \{(XZ, (\overline{0})), (YZ, (\overline{0}))\} \subseteq V(\Gamma(S_2))$. Note that

$$\operatorname{ann}_{S_2}(A) = \{(0, a) \in S_2 \mid a(\alpha) = \overline{0} \text{ whenever } P_\alpha \neq ZD_1\}.$$

Therefore, if

$$C(A) \setminus \{(f, b)\} = C((f, b)) \setminus A$$

for some $(f, b) \in S_2$, then

$$\{P \in \mathcal{P} \mid f \in P\} = \{ZD_1\}.$$

But then $f = uZ^n$ for some $u \in F$ and $n \geq 1$. This contradicts that $f \in D_2$, and hence no such element exists. Thus $\Gamma(S_2)$ is not w.c.v.-$\aleph_0$-complete. $\square$

Let R be a von Neumann regular ring. Then R does not properly contain any finitely generated dense ideals. To see this, let $\{r_1, ..., r_n\} \subseteq R$ be dense. For each $i \in \{1, ..., n\}$, there exists an $s_i \in R$ such that $r_i = r_i^2 s_i$. Then

$$(1 - r_1 s_1) \cdots (1 - r_n s_n) \in \operatorname{ann}_R(r_1, ..., r_n) = \{0\}.$$

Thus $1 = f(r_1, ..., r_n) \in r_1 R + ... + r_n R$ for some $f(X_1, ..., X_n) \in R[X_1, ..., X_n]$. It follows that $Q_0(R) = R$ whenever R is von Neumann regular.

Let α be an ordinal. Then $Q_\alpha(R \oplus S) = Q_\alpha(R) \oplus Q_\alpha(S)$ for any rings R and S [10, Corollary 2.4]. This property will be used freely in the following examples.

Example 6.31. *Suppose that $\mathcal{P}$ is the set of all principal prime ideals of D_1. Let R be any von Neumann regular ring such that $R \neq Q(R)$, the isomorphism $\Gamma(R) \simeq \Gamma(Q(R))$ holds, and $|R \setminus V(\Gamma(R))| = |Q(R) \setminus V(\Gamma(Q(R))|$. Define $W = S_2 \oplus R$. Then W is a total quotient ring which satisfies $W \subsetneq Q_0(W) \subsetneq Q(W)$ and Relation 6.21(3).*

Proof. Note that there exists a ring R possessing the properties given in the hypothesis (e.g., [7, Example 3.5]). Being the direct sum of total quotient rings, W is a total quotient ring. Also, the above comments show that $Q_0(W) = Q_0(S_2) \oplus Q_0(R) = Q_0(S_2) \oplus R \subsetneq Q(S_2) \oplus Q(R) = Q(W)$. The proper inclusion $W \subsetneq Q_0(W)$ will follow upon establishing Relation 6.21(3).

The isomorphism $\Gamma(Q_0(S_2) \oplus R) \simeq \Gamma(Q_0(S_2) \oplus Q(R))$ follows from Lemma 6.20. Also, Example 6.30 shows that $Q_0(S_2) = Q(S_2)$. Thus

$$\begin{aligned}
\Gamma(Q_0(W)) &= \Gamma(Q_0(S_2) \oplus Q_0(R)) \\
&= \Gamma(Q_0(S_2) \oplus R) \\
&\simeq \Gamma(Q_0(S_2) \oplus Q(R)) \\
&= \Gamma(Q(S_2) \oplus Q(R))) \\
&= \Gamma(Q(W)).
\end{aligned}$$

Note that $B(Q(R))$ is a complete Boolean algebra by [4, Theorem 11.9]. Thus $Q(R)$ satisfies (g.a.c.) by Theorem 6.4. Since $\Gamma(Q_0(R)) = \Gamma(R) \simeq \Gamma(Q(R))$, Theorem 6.7 implies that $Q_0(R)$ satisfies (g.a.c.). In particular, $Q_0(R)$ satisfies $\aleph_0$-(g.a.c.). The proof of Example 6.30 shows that $Q_0(S_2)$ satisfies $\aleph_0$-(g.a.c.). Therefore, $\Gamma(Q_0(W))$ is w.c.v.-$\aleph_0$-complete by Theorem 6.18. However, the proof of Example 6.30 also shows that $\Gamma(S_2)$ is not w.c.v.-$\aleph_0$-complete. Hence, Theorem 6.18 implies that $\Gamma(W)$ is not w.c.v.-$\aleph_0$-complete. Thus $\Gamma(W) \not\simeq \Gamma(Q_0(W))$. □

Example 6.32. *Suppose that $\mathcal{P}$ is the family of principal prime ideals belonging to the set $\{fD_1 \mid f \in D_2\}$. Then S_2 is a total quotient ring which satisfies $\Gamma(S_2) \simeq \Gamma(Q_0(S_2)) = \Gamma(Q(S_2))$.*

Proof. The comments prior to Example 6.30 show that S_2 is a total quotient ring. The equality $\Gamma(Q_0(S_2)) = \Gamma(Q(S_2))$ holds as in Example 6.30. Let $\{(f_1/g_1, b_1), (f_2/g_2, b_2)\} \subseteq S_2$ ($f_k \in D_2$, $g_k \in \Omega_2$, $b_k \in \bigoplus_{\alpha \in \mathcal{I}} K_\alpha$, $k = 1, 2$). If $f_1/g_1 = f_2/g_2 = 0$, then let $h = 0$. If either $f_1/g_1 \neq 0$ or $f_2/g_2 \neq 0$, then there exists a (finite) set $J \subseteq D_2$ such that $\{P \in \mathcal{P} \mid \{f_1, f_2\} \subseteq P\} = \{pD_1 \mid p \in J\}$. If $J = \emptyset$, then let $h = 1$. If $J \neq \emptyset$, then let $h = \Pi_{p \in J} p \in D_2$. Clearly $\{f_1, f_2\} \subseteq P$ if and only if $h \in P$ ($P \in \mathcal{P}$). Thus S_2 satisfies $\aleph_0$-(g.a.c.) by the same argument used for the ring $Q_0(S_2)$ in Example 6.30. Also, $\{t \in S_2 \mid t^2 = 0\} =$

$\{(0,a) \mid a \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha\} = \{f \in Q_0(S_2) \mid f^2 = 0\}$. An argument similar to the one given in Example 6.29 shows that

$$|F| \leq |V_t(S_2)| \leq |V_t(Q_0(S_2))| \leq |Q_0(S_2)| = |F|$$

for all $t = (f/g, a) \in Z(S_2)$ with $f/g \neq 0$. Therefore, Corollary 6.15 implies that $\Gamma(S_2) \simeq \Gamma(Q_0(S_2))$. $\square$

Example 6.33. *Suppose that $\mathcal{P}$ is the family of principal prime ideals belonging to the set $\{fD_1 \mid f \in D_2\}$. Let R be any von Neumann regular ring such that $R \neq Q(R)$, the isomorphism $\Gamma(R) \simeq \Gamma(Q(R))$ holds, and $|R \setminus V(\Gamma(R))| = |Q(R) \setminus V(\Gamma(Q(R))|$. Define $W = S_2 \oplus R$. Then W is a total quotient ring which satisfies $W \subsetneq Q_0(W) \subsetneq Q(W)$ and Relation 6.21(4).*

Proof. There exists a ring R possessing the properties given in the hypothesis (e.g., [7, Example 3.5]). Being the direct sum of total quotient rings, W is a total quotient ring. Observe that $(Z, (\overline{0})) \in Q_0(S_2) \setminus S_2$, and hence $W \subsetneq Q_0(S_2) \oplus Q_0(R) = Q_0(W)$. The inclusion $Q_0(W) \subsetneq Q(W)$ holds as in Example 6.31. It remains to verify Relation 6.21(4).

Observe that $S_k \setminus V(\Gamma(S_k)) = \{(f/g, a) \in S_k \mid f, g \in \Omega_k\} \cup \{(0, (\overline{0}))\}$ for each $k \in \{0, 1\}$ (cf. the comments prior to Example 6.30). But $F \subseteq \Omega_k \subseteq D_1$ and $|F| = |D_1|$. Hence $|\Omega_1| = |\Omega_2|$. It is now easy to check that $|S_1 \setminus V(\Gamma(S_1))| = |S_2 \setminus V(\Gamma(S_2))|$. That is, $|Q_0(S_2) \setminus V(\Gamma(Q_0(S_2)))| = |S_2 \setminus V(\Gamma(S_2))|$. By Lemma 6.20 and Example 6.32, it follows that $\Gamma(S_2 \oplus R) \simeq \Gamma(Q_0(S_2) \oplus R)$. Thus

$$\Gamma(W) \simeq \Gamma(Q_0(S_2) \oplus R) = \Gamma(Q_0(W)),$$

where the equality holds since $Q_0(R) = R$. Finally, note that the isomorphism $\Gamma(Q_0(W)) \simeq \Gamma(Q(W))$ holds as in Example 6.31. $\square$

It has been shown that (1), (2), (3), and (4) of Relation 6.21 can be met, in fact, by total quotient rings R which satisfy $R \subsetneq Q_0(R) \subsetneq Q(R)$. However, we do not know the answer to the following question.

Question 6.34. *Does there exist a ring R which satisfies Relation 6.21(5)?*

The remaining two examples show that an $\aleph_\alpha$-rationally complete ring may have a zero-divisor graph whose vertices do not satisfy any of the completeness criteria introduced in this paper. Using the fact that finite rings are rationally complete (indeed, finite rings do not properly contain any dense ideals), the comments prior to Corollary 6.19 show that it is easy to construct a rationally complete ring whose zero-divisor graph is not w.c.v.-$\aleph_0$-complete. A less trivial example is provided in Example 6.35. Every reduced rationally complete ring has a c.v.-complete zero-divisor graph (cf. the comments prior to Corollary 6.15). However, Example 6.36 shows that a reduced $\aleph_\alpha$-rationally complete ring need not have a w.c.v.-$\aleph_\alpha$-complete zero-divisor graph. In particular, the zero-divisor graph of such a ring need not be c.v.-$\aleph_\alpha$-complete. Since a graph Γ is c.v.-$\aleph_\alpha$-complete if and only if Γ^* is c.v.-$\aleph_\alpha$-complete, the converse to Corollary 6.16 is false. Moreover, Example 6.35 shows that the conclusion of Corollary 6.19 can hold without the w.c.v.-$\aleph_\alpha$-complete hypothesis.

Example 6.35. *Let $\mathcal{P}'$ be the set of all principal prime ideals of D_1, and let $\mathcal{P} = \mathcal{P}' \cup \{YD_1 + ZD_1\}$. Then $S_1 = Q(S_1)$, but $\Gamma(S_1)$ is not w.c.v.-$\aleph_0$-complete. In particular, $Q(S_1)$ does not satisfy $\aleph_0$-(g.a.c.).*

Proof. The equality $S_1 = Q_0(S_1)$ holds by Lemma 6.25 together with [12, Theorem 11(f)], and $Q_0(S_1) = Q(S_1)$ holds as in Example 6.30.

Note that XD_1 is the only principal prime ideal containing the set $\{XY, XZ\}$. Therefore, if $f \in D_1$ and $a \in \bigoplus_{\alpha\in\mathcal{I}} K_\alpha$ such that

$$\mathrm{ann}_{S_1}\big((f,a)\big) = \mathrm{ann}_{S_1}\big((XY,(\overline{0})),(XZ,(\overline{0}))\big),$$

then $f = uX^n$ for some $u \in F$ and $n \geq 1$. But then $f \notin YD_1 + ZD_1$, a contradiction. Thus no such f exists. This proves the "in particular" statement. Since D_1 is an integral domain, it immediately follows that $\Gamma(S_1)$ is not w.c.v.-$\aleph_0$-complete. ☐

Example 6.36. *Let $\mathcal{P}'$ be the set of all principal prime ideals of D_1, and let $\mathcal{P} = \mathcal{P}' \cup \{YD_1 + ZD_1\}$. Then $R_1 = Q_0(R_1)$, but $\Gamma(R_1)$ is not w.c.v.-$\aleph_0$-complete. In particular, $Q_0(R_1)$ does not satisfy $\aleph_0$-(g.a.c.).*

Proof. Note that $R_1 = Q_0(R_1)$ by Proposition 6.27. Replacing S_1, (f,a), $(XY,(\overline{0}))$, and $(XZ,(\overline{0}))$ by R_1, $\varphi(f)+a$, $\varphi(XY)$, and $\varphi(XZ)$, respectively, the desired results follow from the proof of Example 6.35. ☐

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Vita

John David LaGrange was born in Danville, Indiana on March 9, 1977. He graduated from Danville Community High School in 1995. He received a Bachelor of Science degree in Psychology with a minor in Mathematics in 1999 from the University of Southern Indiana. He received a Master of Science degree in Mathematics from Western Kentucky University in 2001. He began the Ph.D. program in Mathematics at the University of Tennessee in January of 2004, and received the degree in May of 2008 under the direction of David F. Anderson.