8-1995

Study of the Equilibria of Parabolic Differential Equations with Interfaces Intersecting the Boundary

Michal Kowalczyk
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Michal Kowalczyk entitled "Study of the Equilibria of Parabolic Differential Equations with Interfaces Intersecting the Boundary." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Nicholas Alikakos, Major Professor

We have read this dissertation and recommend its acceptance:

Don Hinton, A. Freire, H. Simpson, S. Georghiou

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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C W Minkel

Associate Vice Chancellor

and Dean of the Graduate School

STUDY OF THE EQUILIBRIA OF PARABOLIC DIFFERENTIAL EQUATIONS WITH INTERFACES INTERSECTING THE BOUNDARY

A Thesis

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Michał Kowalczyk

August 1995

ACKNOWLEDGEMENT. This work is the result of arduous deliberation much of which were sustained as a result of the support of many people.I would like to thank all of those people who played such a role in my graduate studies at the University of Tennessee.

My deepest gratitude goes out to my advisor Professor Nicholas Alikakos for his optimism and patience, never-ending support and encouragment, both professionally and personally, and his continued trust in me.

I also wish to thank Professor Gorgio Fusco for suggesting the problem in Chapter 1 and many enlightening discussion during our meetings.

Much gratitude goes to Professor Xinfu Chen for helping to shape my point of view within the area of PDEs.

I would also like to thank Professor Peter Bates for the discussions we had during my stay at Brigham Young University and Professor Samuel Jordan for his continued support at UT.

I extend my gratitude to my teacher Professor Wojciech Stasiak who encouraged me to study analysis.

Much thanks goes out to the members of my doctoral committee who were both willing and able to serve: professors A.Freire, D.Hinton, H.Simpson, and S.Georghiou.

I also wish to thank the Department of Mathematics at UT for giving me a chance.

ABSTRACT. Existence of steady state solutions for the Allen-Cahn and Cahn-Hilliard equations in two dimensional domains is discussed. We are in particular interested in establishing existence of single layered equilibria with the property that their transition layer intersects the boundary. In the case of the Allen-Cahn equation we consider bone-like domains and seek solutions intersecting the flat part of the boundary. We establish conditions for the domain which ensure existence of such equilibria. Their stability is also analyzed. For the Cahn-Hilliard equations we show that there exist equilibria near every point of a local maximum of the curvature of the boundary.

Contents

Introduction 5

1 The existence and stability of equilibria for the Allen-Cahn equation
- 1.1 Preliminaries 13
- 1.2 The existence and stability of equilibria-proofs of the main results 20
- 2.1 The eigenvalue problem-proof of Lemma B 44
- 3.1 The quasi-invariant manifold-proof of Lemma A 55

2 The existence of equilibria for the Cahn-Hilliard equation
- 1.1 Preliminaries 66
- 2.1 Minimizers in the circular class 70
- 2.2 Minimizers in the BV class 81

References 94

Vita 98

INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the following two problems:

$$\text{(AC)} \qquad \begin{cases} \epsilon^2 \Delta u - F'(u) = u_t & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\text{(CH)} \qquad \begin{cases} \Delta\left(-\epsilon^2 \Delta u + F'(u)\right) = u_t & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where F is a double well potential (see figure 1) , $F > 0, F(-1) = F(1) = 0$ (typical choice is $F(u) = \frac{1}{8}(u^2-1)^2), \frac{\partial}{\partial n}$ exterior normal derivative and $\epsilon \ll 1$ a small parameter.In the sequel we assume that, unless stated otherwise, $N = 2$.

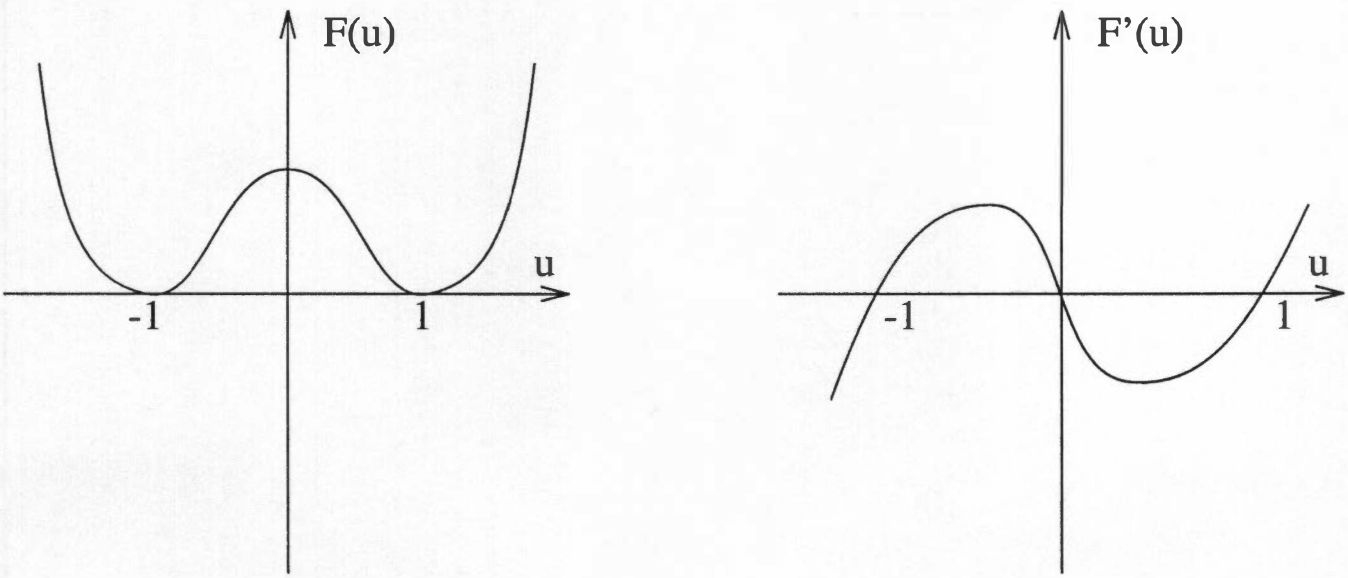

FIGURE 1. *F* AND ITS DERIVATIVE

The first equation was introduced in [AC] as a model for the motion of the antiphase boundary that separates two phases of crystalline solid.Here u represents the long range order parameter, the function F repersents the free energy per unit volume and its two wells correspond to different stable material phases.The equilibrium order parameters are $u = 1$ and $u = -1$ and the antiphase boundary Γ is simply the interface between the two regions, one with u approximately equal to 1 and the other with u approximately equal to -1.We refer to (AC) as the Allen-Cahn equation.

The so called Cahn-Hilliard equation (CH) serves as a model for phase separation and coarsening phenomena in binary alloys at a temperature at which only two different concentration phases can coexist.The evolution of the concentration divides into two stages.During the first relatively short stage called phase separation the alloy becomes inhomogeneous, a fine-grained mixture of the two phases corresponding to stable concentration configuration.

In terms of (CH) the solution u is approximately equal to 1 in a subregion Ω^+ of Ω and is approximately equal to -1 in another subregion Ω^- of Ω.These two subregions are separated by a thin interface Γ.

The first stage is followed by a very slow coarsening process during which the originally fine grained structure becomes less fine while the average concentration remains constant.

At the level of (CH) this phenomena corresponds to such a behaviour of u in which the interface Γ moves and eventually tends to a surface minimizing the area subject to a volume constraint. More details on the physical background for (CH) can be found in [Cahn-H,Cahn].

From the mathematical point of view the formation of spatial patterns and coexistence of two different phases in both models are due to the presence of the bistable term $F'(u)$.We will explain this point by a simple formal analysis.Consider for example the Allen-Cahn equation and set $\epsilon = 0$ so that (AC) becomes an ODE:

$$u_t = F'(u).$$

By an elementary analysis we see that as t increases we have $u(x,t) \to \pm 1$ depending on whether $u(x,0) > 0$ or $u(x,0) < 0$.Therefore we expect that after a short time there will be regions where $u \approx \pm 1$ separeted by diffused (with the width $O(\epsilon)$) interfaces.It turns out that a similiar in principle effect is observed for the solutions of the Cahn-Hilliard equation, however even formal analysis is considerably more difficult in this case.We refer to [F3] for more details.

It is perhaps beyond the scope of this work to give a full description of the dynamics generated by (AC) and (CH).We mention only that the presence of the moving spacial patterns allows us to reduce the very complex dynamics to much simpler approximate system consisting of rules for the evolution of patterns.For example it is known for the Allen-Cahn equation that the interfaces move according to the law (the Mean Curvature Motion)

$$\text{(MCM)} \qquad \mathcal{V} = -\epsilon^2\kappa,$$

where $\mathcal{V}$ denotes the velocity in the direction normal to Γ and κ is its mean curvature.Rigorous analysis of (MCM) and its connection to (AC) can be found for example in [Ch1,deM-S2,I,E-S-S,R-S-K2,So].

In the Cahn-Hilliard case Pego [Pe] by applying the method of matched asymptotic expansions derived the following law of motion for the interface Γ

$$\begin{aligned} &\Delta\mu = 0 \qquad x \in \Omega \subset \Gamma, \\ &\mu = \epsilon\alpha_1\kappa \qquad x \in \Gamma, \\ &\frac{\partial\mu}{\partial n} = 0 \qquad x \in \partial\Omega, \\ \text{(HS)} \qquad &\mathcal{V} = \alpha_2\left[\frac{\partial\mu}{\partial n}\right]_\Gamma, \end{aligned}$$

where μ is an auxillary function, α_1, α_2 are constants, κ is the curvature of Γ, $\mathcal{V}$ is the velocity in the direction normal to Γ and $[\frac{\partial\mu}{\partial n}]_\Gamma$ is the jump of the normal derivative of μ across Γ.We usually refer to (HS) as the Helle-Shaw problem.Existence of the solutions to (HS) and their relation to (CH) has beeen established in [Ch2,A-B-C].

Notice that in both cases the interfaces move with the speed which is of the algebraic order in terms of ϵ ($O(\epsilon^2)$ for (MCM) and $O(\epsilon)$ for (HS)). Without futher analyzing (MCM) and (HS) we observe that the interfaces with zero curvature are the equilibria for (MCM) and that the interfaces with constant curvature are the equilibria for (HS).It is natural to ask: do the equilibria of the geometric problem always correspond to the equilibria of the parabolic problem?

To explain this point we first describe a very important feature of both (AC) and (CH), namely the presence of the so called metastable states.Consider the case when $\Omega = [-1, 1]$.In one space dimension we of course do not have the curvature effect and the previous laws of motion do not apply.The formal analysis which was done by Neu [Neu] suggests that the layered solutions for the one dimensional (AC) move towards the equilibrium states without changing its structure (number of layers) for the lenght of time of the order $O(e^{\frac{1}{\epsilon}})$.In this case the speed with which the interfaces move is of the order $O(e^{-\frac{1}{\epsilon}})$.Those superslowly evolving solutions are called the metastable states.We refer to [F-H,C-P1,2,F,Bro-K2,A-B-F2] for the rigorous results and the detailed description of the dynamics in the context of the one dimensional (AC).Similiarly the presence of the metastable states was established for the one dimensional (CH) by Alikakos, Fusco and Bates [A-B-F1] (see also [B-X] and [Gr]) and the multidemensional (CH) by Alikakos and Fusco [A-F2,3] (see also [A-Bro-F]).In particular for the two dimensional (CH) they showed that if initially the transition layer is very close to a circle (equilibrium for (HS)) then it moves superslowly towards the boundary of the domain while retaining its shape.

This is not however always the case that all the equilibria for the geometric problem correspond to the metastable states for the PDE.It was shown for example in [K-S] that providing that there exists an isolated equilibrium of (MCM) (segment of straight line partitioning a non-convex domain) then there exist an equilibrium of (AC) whose transition layer is close to this equilibrium of (MCM).These remarks suggest that more detailed information about the domain Ω may be needed in order to describe the relation between equilibria of (MCM) and (HS) and equilibria of (AC) and (CH).

The main purpose of this work is to analyze equilibrium states of (AC) and (CH) whose spatial structure is characterized by the presence of a single interface which intersects the boundary of Ω.For (CH) those equilibria are the limits as $t \to \infty$ of the solutions of the evolution problem and they correspond to the isolated equilibria of the geometric problem.For (AC) we chose the domain Ω in such a way that there are continuum of non-isolated equilibria of the Mean Curvature Motion.In this case the equilibria of (AC) lie on the invariant manifold of extremely slowly evolving (metastable) solutions of the evolution problem.

For both models discussed here the steady state-state solutions we are interested in correspond to the final stages of the evolution described by (AC) and (CH).We remark here that the time scales involved make this phase of the process difficult to observe in either actual physical experiments or computer simulations.In fact at a first glance the metastable states described above may look like equilibria since for a very long time no significant change in the location of the interfaces will be observed.

The spatial structure of the steady state solutions we are after allows us to describe them in terms of the location of the interface.Here we define the interface Γ_ϵ as the zero level set of the equilibrium solution u^ϵ, $\Gamma_\epsilon = \{u^\epsilon = 0\}$.In order to develop some intuition about the geometric properties of Γ_ϵ we recall that equilibria of both (AC) and (CH) are critical points of the free energy functional:

$$\mathcal{J}_\epsilon[u] = \int_\Omega \epsilon^2 |\nabla u|^2 + F(u),$$

where for (CH) we additionally impose mass constraint $\int_\Omega u = m, m \in (-|\Omega|, |\Omega|)$.Let's consider the constrained problem.Since $F(-1) = F(1) = 0$ therefore a minimizer u^ϵ should stay close to ± 1.However since the average concentration is conserved $\int_\Omega u = const.$ we can not have $u^\epsilon \equiv 1$ or $u^\epsilon \equiv 1$ throughout Ω. Therefore there must be a transition layer between the regions where $u^\epsilon \approx \pm 1$.It is intuitevely clear that the gradient term in the expression for $\mathcal{J}_\epsilon$ is proportional to the length of the interface.This suggests that the partition of Ω given by Γ_ϵ should be optimal in the sense that the length of the transition layer should be minimal.The rigorous result proven in [M,S] reads as follows:

Theorem (Modica, Sternberg). *If u^ϵ is a sequence of minimizers of $\mathcal{J}_\epsilon$ satisfying area con-*

straint $\int_\Omega u = m$ then:

(1)

$$\mathbf{u}^\epsilon \to \pm 1 \qquad a.e.\ in\ \Omega,$$

(2)

$$E_\epsilon = \{\mathbf{u}^\epsilon \le 0\} \to E,$$

where $|\partial E| \le |\partial F|$ for all sets $F \subset \Omega$ such that $|F \cap \Omega| = \frac{|\Omega| - m}{2}$.

In Chapter 2 of this work we shall analyze sets minimizing the perimeter subject to the area constraint for two dimensional domains.We apply the idea of Kohn and Sternberg [K-S] which allows us to reduce the problem of minimizing $\mathcal{J}_\epsilon$ to the geometric problem of minimizing the lenght subject to the area constraint.We show existence of the steady state solutions $\mathbf{u}^\epsilon$ to (CH) and characterize the location of the set $\mathbf{u}^\epsilon = 0$ in terms of the geometry of the domain.More precisely we prove that near each local maximum of the curvature of $\partial\Omega$ there is an equilibrium of (CH).We refer the reader to [R-S-K1, S-Z] where this approach was used for a different model and to work of Gurtin and Matano [Gu-M] where existence and location of equilibria was established providing certain symmetry of the domain.We also mention the work of Ni and Takagi [N-T] who obtain similiar in the spirit results in yet another context.

The applicability of the variational methods for the equilibrium problem is limited to those cases when the corresponding geometric problem is nondegenerate, namely there exists a unique set which locally minimizes the perimeter.For dealing with the class of problems with degenerate geometry we adopt a method developed in [FH,CP,F,ABF1,2] for the one dimensional (AC) and (CH).This techinique is based on the following key ingredients:

(1) Construction of the approximate solution by utilizing the method of formal asymptotic expansion (inner expansion);

(2) Refinement of the approximate solution by solving the so called v equation and reduction of the infinite dimensional problem to a finite dimensional one (formula for the speed);

(3) Analysis of the linearized operator and the associated eigenvalue problem (the gap).

At this point we briefly describe the main idea behind the step (1) for the one dimensional (AC) with $F(u) = \frac{1}{8}(u^2-1)^2$.More details on the other steps the reader will find in Chapter 1 of the present work.

Consider the equation:

$$-\epsilon^2 u_{xx} + \frac{1}{2}u(u^2-1) = 0, \qquad -1 < x < 1,$$

(AC1)
$$u_x(-1) = u_x(1) = 0.$$

Stretching variables $\eta = \frac{x}{\epsilon}$ we get:

$$-u_{\eta\eta} + \frac{1}{2}u(u^2-1) = 0, \qquad -\frac{1}{\epsilon} < \eta < \frac{1}{\epsilon},$$
$$u_\eta(-\frac{1}{\epsilon}) = u_\eta(\frac{1}{\epsilon}) = 0.$$

Letting $\epsilon \to 0$ we obtain:

$$-u_{\eta\eta} + \frac{1}{2}u(u^2-1) = 0, \qquad -\infty < \eta < \infty,$$
$$u_\eta(-\infty) = u_\eta(\infty) = 0.$$

A unique solution to this problem U satisfying $U(-\infty) = -1$ and $U(\infty) = 1$ can be determined explicitly because of the special form of the nonlinear term.We have $U = \tanh(\frac{\eta}{2})$.In phase plane U is represented by the heteroclinic orbit joining $(-1,0)$ with $(1,0)$. Notice that if we set

$$u^\xi = U\left(\frac{x-\xi}{\epsilon}\right)$$

then u^ξ satisfies (AC1) but does not satisfy the boundary conditions.However u^ξ is a very good approximation of the true solution since we have:

$$u^\xi_x(-1) = O(e^{-\frac{1}{\epsilon}|1+\xi|}) \qquad u^\xi_x(1) = O(e^{-\frac{1}{\epsilon}|1-\xi|})$$

The idea is to consider a one parameter family of approximate solutions $\mathcal{M}^A = \{u^\xi \mid \xi \in (-1+l, 1-l), l > 0\}$ and look for the true solution to (AC1) near $\mathcal{M}^A$.It turns out that we can reduce the problem of finding equilibria of the PDE to the problem of finding zeroes of the function c^{ξ} of a single variable ξ. In terms of the evolution problem c^{ξ} corresponds to the speed of the evoving interface.It can be shown that for (AC1) the unique zero of c^{ξ} is approximately located in the middle of the interval $(-1, 1)$.Chapter 1 of the present work contains a generalization of this result to two dimensional bone-like domains. In Section 1.2 of this chapter we give asymptotic formula for c^{ξ} as well as for its the derivative with respect to ξ.These formulas allow us to describe the location of the equilibria corresponding to the zeroes of the speed (Theorem 1.13) and their analyze their stability (Theorem 1.15).

CHAPTER I

THE EXISTENCE AND STABILITY OF EQUILIBRIA FOR THE ALLEN-CAHN EQUATION

1.1 Preliminaries

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a C^3 function satisfying:

(F1) $F \geq 0$ and F has exactly two zeros $F(\pm 1) = 0$;

(F2) F' has exactly 3 zeros $F'(\pm 1) = F'(0) = 0$;

(F3) $F''(\pm 1) = \beta^2 > 0, F''(0) < 0$.

(F4) There exist constants $w_0 > 0$ and $F_0 > 0$ such that for all $u \in [-1, 1]$ and w_1, w_2 such that $\bar{w} \colon = \max\{|w_1|, |w_2|\} < w_0$ we have:

$$|F'(u + w_1) - F''(u)w_1 - F'(u + w_2) + F''(u)w_2| \leq F_0 \bar{w} |w_1 - w_2|$$

Function F is called a double well potential.Observe that the assumptions (F1)-(F4) are satisfied if for example $F(u) = \frac{1}{8}(u^2 - 1)^2$.The reader may notice that the last assumption is redundant since as it can easily be proved (F4) follows from the fact that F is a C^3 function.For the reasons explained in Section 3.1 it is however convenient to state it here in this form.In the sequel we shall denote $F' = f$.

Under the hypothesis (F1)-(F3) there exists a unique solution U of;

$$(1.1) \qquad \begin{cases} -U'' + f(U) = 0, & U = U(x), \qquad x \in \mathbb{R}, \\ U(0) = 0, & \lim_{x \to \pm\infty} U(x) = \pm 1. \end{cases}$$

In the phase plane U is the heteroclinic orbit connecting $(-1, 0)$ to $(1, 0)$ (see figure 2).Notice that U is strictly increasing.

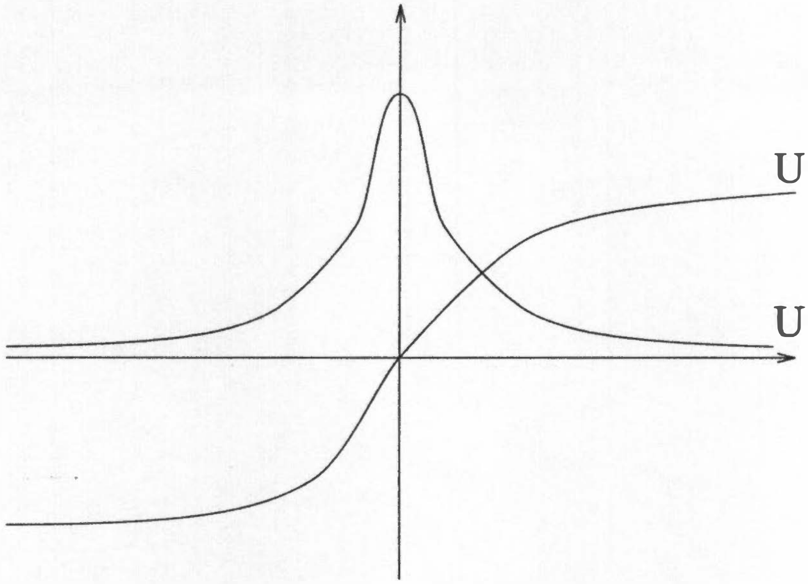

FIGURE 2. U AND ITS DERIVATIVE

The following asymptotic formulas will be used frequently:

$$\pm 1 \mp U(x) = \beta' e^{-\beta|x|} + o(e^{-\beta|x|}), \quad \text{as } x \to \pm\infty,$$

$$U'(x) = \beta\beta' e^{-\beta|x|} + o(e^{-\beta|x|}), \quad \text{as } x \to \pm\infty,$$

$$U''(x) = \mp\beta^2\beta' e^{-\beta|x|} + o(e^{-\beta|x|}), \quad \text{as } x \to \pm\infty, \tag{1.2}$$

where β' is a positive constant depending only on F.

Let Ω be a bounded simply connected region in $\mathbb{R}^2$ satisfying the following assumptions:

(D1) $\Omega = [0,1] \times (0,b) \cup \Omega_R \cup \Omega_L$, where $b > 0$ is a fixed number and Ω_R, Ω_L are bounded subregions of Ω such that:

$$\partial\Omega_L \cap [0,1] \times (0,b) = \{0\} \times (0,b),$$

$$\partial\Omega_R \cap [0,1] \times (0,b) = \{1\} \times (0,b).$$

(D2) There exists $a > 0$ such that for all $a' \in (-a, 1+a)$ each vertical line $x_1 = a'$ in the (x_1, x_2)-plane intersects $\partial\Omega$ at exactly 2 points.

(D3) $\partial\Omega$ is of class C^2 except at the points $C_1 = (1,0), C_2 = (1,b), C_3 = (0,b), C_4 = (0,0)$.We will call those points the corners of Ω.

(D4) From the assumptions (D1), (D2) it follows that near each corner $\partial\Omega$ can be represented as a graph of a function. More precisely there exists $r > 0$ such that $\partial\Omega \cap B_r(C_i) = \{(x_1,x_2) \mid x_2 = \phi_i(x_1)\}$ where $B_r(C_i)$ denotes a ball with radius r centered at C_i.We assume that there exist numbers $\kappa_i, \lambda_i > \frac{3}{2}, i = 1,\ldots,4, |\kappa_i| + |\kappa_{i+1}| > 0, i = 1,2$ such that:

$$\phi_1'(x_1) = \kappa_1(x_1-1)^{\lambda_1-1} + O((x_1-1)^{\lambda_1}), \qquad \text{as } x_1 \to 1^+,$$

$$\phi_2'(x_1) = -\kappa_2(x_1-1)^{\lambda_2-1} + O((x_1-1)^{\lambda_2}), \qquad \text{as } x_1 \to 1^+,$$

$$\phi_3'(-x_1) = \kappa_3 x_1^{\lambda_3-1} + O(x_1^{\lambda_3}), \qquad \text{as } x_1 \to 0^+,$$

$$\phi_4'(-x_1) = -\kappa_4 x_1^{\lambda_4-1} + O(x_1^{\lambda_4}), \qquad \text{as } x_1 \to 0^+,$$

with similiar formulas holding for ϕ_i''.

We call κ_i the curvature of the ith corner (or simply the curvature of the corner).The condition (D4) will sometimes be referred to as the nondegeneracy condition.We point out that the numbers κ_i can be seen as a natural generalization of the usual curvature (see fig.3).

We consider the following semilinear elliptic problem:

$$\begin{aligned} -\epsilon^2\Delta u + f(u) &= 0, \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

Here $\epsilon << 1$ and $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative. In the sequel we will use notation $-\epsilon^2\Delta u + f(u) = \mathcal{L}^\epsilon(u)$.

The main goal of this chapter is to establish existence of the layered solution u to (1.3) with the property that the layer is located in the rectangular part of Ω and $u \approx \pm 1$ on $\Omega_L \cup \Omega_R$. We see easily that the rescaled heteroclinic $U(\frac{x_1-\xi}{\epsilon})$ solves for any ξ the equation (1.3) but does not satisfy the boundary conditions. We first construct a one parameter family of approximate

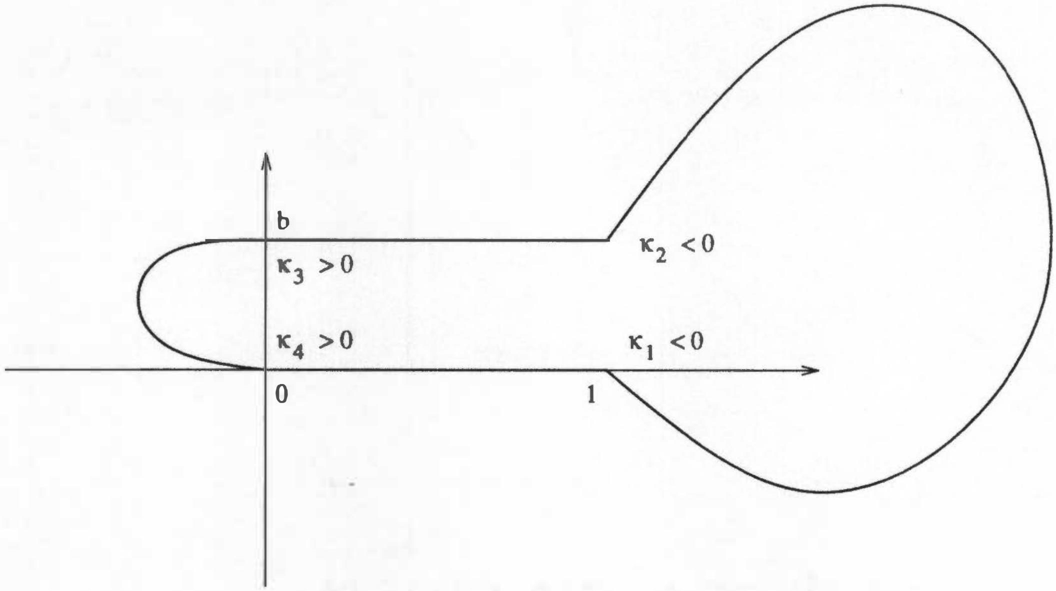

FIGURE 3. A TYPICAL BONE-LIKE DOMAIN

solutions (ansatz) to (1.3) which resembles the heteroclinic and also satisfies the boundary conditions. We clearly need to adjust $U(\frac{x_1-\xi}{\epsilon})$ only near the curved part of $\partial\Omega$. To accomplish this we take a monnotone function $\omega_1 \in C_0^\infty(\mathbb{R})$ such that:

$$\omega_1(s) = \begin{cases} 0 & \text{for } s \geq 1, \\ 0 \leq \omega_1 < 1 & \text{for } 0 < s < 1, \\ 1 & \text{for } s \leq 0. \end{cases}$$

It is easy to see that there exists a $C_0^\infty(\mathbb{R}^2)$ function ω_2^ϵ such that $\omega_2^\epsilon(\partial\Omega) = 1$, $\dfrac{\partial \omega_2^\epsilon}{\partial n} = 0$ on $\partial\Omega$ and $\operatorname{supp}(\omega_2^\epsilon) = \{x \mid \operatorname{dist}(x, \partial\Omega) \leq \epsilon\}$. We set:

$$\omega^\epsilon(x) = \left[\omega_1\left(\frac{x_1}{\epsilon}\right) + \omega_1\left(\frac{1-x_1}{\epsilon}\right)\right]\omega_2^\epsilon(x)$$

Let $l, l > 0$ be a fixed small number.Unless stated otherwise we always assume that $\xi \in (l, 1-l)$.

Let α^ξ be a smooth function such that:

$$\alpha^\xi(s) = \begin{cases} U(\frac{-\xi}{\epsilon}) & \text{for } s < \frac{1}{4}, \\ U(\frac{-\xi}{\epsilon}) \leq \alpha^\xi \leq U(\frac{1-\xi}{\epsilon}) & \text{for } \frac{1}{4} \leq s < \frac{3}{4}, \\ U(\frac{1-\xi}{\epsilon}) & \text{for } s \geq \frac{3}{4}. \end{cases}$$

We set

$$w^\xi(x) = \omega^\epsilon(x)\left[\alpha^\xi(x_1) - U\left(\frac{x_1-\xi}{\epsilon}\right)\right].$$

For each $\xi \in (l, 1-l)$ we define:

$$u^{\xi}(x) = U\left(\frac{x_1 - \xi}{\epsilon}\right) + w^{\xi}(x).$$

It follows from the definition of ω^{ϵ} that $\operatorname{supp}\omega^{\epsilon} \cap (\epsilon, 1-\epsilon) \times (0, b) = \emptyset, 0 \leq \omega^{\epsilon} \leq 1$ and $\dfrac{\partial \omega^{\epsilon}}{\partial n} = 0$ on $\partial\Omega$ hence $\dfrac{\partial u^{\xi}}{\partial n} = 0$ on $\partial\Omega$. It is also easy to see from the asymptotic formulas (1.2) that:

$$\left|u^{\xi}(x) - U\left(\frac{x_1 - \xi}{\epsilon}\right)\right| \begin{cases} = 0 & \text{if } x \in \Omega \setminus \operatorname{supp}\omega^{\epsilon} \\ \leq |w^{\xi}(x)| \leq C\left(e^{-\beta\frac{\xi}{\epsilon}} + e^{-\beta\frac{1-\xi}{\epsilon}}\right) & \text{if } x \in \operatorname{supp}\omega^{\epsilon}, \end{cases}$$

$$(1.4) \qquad |f(u^{\xi}) - f(U)| \begin{cases} = 0 & \text{if } x \in \Omega \setminus \operatorname{supp}\omega^{\epsilon}, \\ \leq C|w^{\xi}| \leq C\left(e^{-\beta\frac{\xi}{\epsilon}} + e^{-\beta\frac{1-\xi}{\epsilon}}\right) & \text{if } x \in \operatorname{supp}\omega^{\epsilon}. \end{cases}$$

Here and below we adopt the rule that C stands for a positive constant independent on ϵ and its value may change from line to line.

It turns out that the term $e^{-\beta\frac{\xi}{\epsilon}} + e^{-\beta\frac{1-\xi}{\epsilon}}$ and its square appear frequently in this chapter and therefore for simplification we shall denote:

$$\delta^{\xi} = e^{-\beta\frac{\xi}{\epsilon}} + e^{-\beta\frac{1-\xi}{\epsilon}},$$

$$\gamma^{\xi} = (\delta^{\xi})^2.$$

The set

$$\mathcal{M}^A = \{u^{\xi} \mid \xi \in (l, 1-l)\}$$

will be called the approximate invariant manifold. Elementary calculations show that $\mathcal{M}^A$ is very close to the set $\{u \mid \mathcal{L}^{\epsilon}(u) = 0\}$. Indeed by (1.1), (1.2) and the definition of u^{ξ} we have:

$$(1.5) \qquad \left|\mathcal{L}^{\epsilon}(u^{\xi}(x))\right| = \left|-\epsilon^2 \Delta w^{\xi} + f(U) - f(u^{\xi})\right| \leq C\delta^{\xi}.$$

Essential in our work is the analysis of the operator $\mathcal{L}^{\epsilon}$ linearized about u^{ξ}:

$$-\epsilon^2 \Delta u + f'(u^{\xi})u = \Phi \qquad \text{in } \Omega,$$

$$(1.6) \qquad \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega,$$

together with the associated eigenvalue problem

$$
\begin{aligned}
-\epsilon^2 \Delta V + f'(u^\xi)V = \lambda V \qquad & \text{in } \Omega, \\
\frac{\partial V}{\partial n} = 0 \qquad & \text{on } \partial\Omega.
\end{aligned}
\tag{1.7}
$$

In the sequel we shall denote $L^{\epsilon,\xi}u = -\epsilon^2\Delta u + f'(u^\xi)u$.

If we assume for example that $\Phi \in L^2(\Omega)$ and $\int_\Omega \Phi V_1 := \langle \Phi, V_1 \rangle = 0$, where V_1 is the eigenfunction of (1.7) corresponding to $\lambda = 0$ then it follows that there exists a unique weak solution to (1.6) $u \in W^{1,2}(\Omega)$.Moreover we have (the first fundamental inequality):

$$
\|\nabla u\|_{L^2(\Omega)} \leq C\epsilon^{-1}\big(\|\Phi\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big).
\tag{1.8}
$$

Notice however that it is not entirely obvious that this weak solution satisfies $u \in W^{2,2}(\Omega)$.The reason is that we have only assumed that $\partial\Omega$ is piecewise C^2 while usually $W^{2,2}(\Omega)$ regularity of the solutions of the linear elliptic PDE is obtained under stronger assumption $\partial\Omega \in C^2$.Smoothness of the domain allows us to weaken the conditions on the coefficients of the elliptic operator, typically we need to assume for example that the coefficients of the second derivatives are in $W^{2,q}(\Omega), q >$ dimension of the space.It is therefore natural to expect that by requiring sufficient smoothness of the coefficients one can establish $W^{2,2}$ regularity of the solutions under weaker assumptions on the regularity of the domain.Consider the following problem

$$
\begin{aligned}
-\Delta u + q(x)u = \Phi(x) \qquad & \text{in } \Omega, \\
\frac{\partial u}{\partial n} + \sigma(x)u = \psi(x) \qquad & \text{on } \partial\Omega,
\end{aligned}
\tag{1.9}
$$

where for our purposes it sufficies to assume $\Phi \in L^2(\Omega), q \in C^0(\Omega), \sigma, \psi \in C^1(\Omega)$.We also assume that Ω satisfies the conditions (D1)-(D4).We claim that if u is a weak $W^{1,2}(\Omega)$ solution to (1.9) then $u \in W^{2,2}(\Omega)$ and moreover

$$
\|u\|_{W^{2,2}(\Omega)} \leq C(\|\Phi\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} + \|\psi\|_{W^{1,2}(\partial\Omega)}),
\tag{1.10}
$$

where constant C depends only on Ω, q, σ.The proof of the claim is based on the well known idea of flattening the boundary by using an appropriate local diffeomorphism; we omitt the details.We shall refer to (1.10) as the second fundamental inequality.Finally notice that from the Sobolev embedding $W^{2,2}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega), \alpha < 1$ and the second fundamental inequality it follows that the solutions to (1.9) are Hölder continuous as well.

Before we proceed further we would like to quote two usefull interpolation inequalities (see [L-U] pp.48–49 (2.19) and (2.25)).Let $u \in W^{1,2}(\Omega)$ and $\varepsilon, 0 < \varepsilon < 1, m > 1$ be given.There exists a constant c_ε depending on ε, m and Ω only such that

$$\|u\|_{L^m(\partial\Omega)} \leq \varepsilon\|\nabla u\|_{L^2(\Omega)} + c_\varepsilon\|u\|_{L^2(\Omega)}. \tag{1.11}$$

Assuming additionally that $m > 2$ we also have:

$$\|u\|^2_{L^{\frac{2m}{m-2}}(\Omega)} \leq \varepsilon\|\nabla u\|^2_{L^2(\Omega)} + c_\varepsilon\|u\|^2_{L^2(\Omega)}. \tag{1.12}$$

We remark that these estimates play a crucial role in showing (1.10).

We conclude this section with some observations regarding the eigenvalue problem (1.7). From the general theory of the selfadjoint operators we know that the spectrum of $L^{\varepsilon,\xi}$ is discrete and real with ∞ as the only possible limit point.By (λ_i^ξ, V_i^ξ) we shall denote the ith eigenvalue and eigenfunction of $L^{\varepsilon,\xi}$, respectively. The Krein-Rutman theorem implies that the principal eigenvalue λ_1^ξ is simple and the principal eigenfunction V_1^ξ is positive.

We shall present now the main idea in investigating V_1^ξ. Observe that differentiating (1.1) with respect to x we obtain:

$$-U''' + f'(U)U' = 0,$$

hence 0 is an eigenvalue of the operator in (1.1) linearized about U and U' is the corresponding eigenfunction. Since from the estimates (1.4) $\mathbf{u}^\xi$ can be viewed as a small perturbation of $U(\frac{x_1-\xi}{\varepsilon})$ therefore it is not unreasonable to expect that (1.6) has a small eigenvalue and that the corresponding eigenfunction resembles $\mathbf{u}^\xi_\xi$. In Section 3.1 we show that in fact $V_1^\xi \approx q^\varepsilon \mathbf{u}^\xi_\xi$

where $q^\epsilon = -\|u_\xi^\xi\|_{L^2(\Omega)}^{-1}$. By using (1.2) we can show easily that:

$$q^\epsilon = -\frac{\epsilon^{\frac{1}{2}} b^{\frac{1}{2}}}{\|U'\|_{L^2(\mathbb{R})}} + O(e^{-\frac{C}{\epsilon}}). \tag{1.13}$$

The key points of our analysis in the next sections are:

(1) The existence of the solutions to the reduced equation and the improved v estimate (Lemma **A**).

(2) The analysis of the associated eigenvalue problem including the estimates on the gap in the spectrum and the on the rate of decay of the principal eigenfunction (Lemma B).

(3) Derivation of the explicit asymptotic formula for the speed c^ξ.

This chapter is organized as follows.In Section 1.2 we state Lemma A and Lemma B and evaluate the speed.We also give there proofs of the main existence and stability results (Theorems 1.13, 1.15).Subsequent sections 2.1 and 3.1 are devoted to the proofs of Lemmas B and A respectively.

1.2 The Existence and Stability of the Equilibria

The purpose of this section is to show existence of the solutions of (1.3) lying close to the manifold $\mathcal{M}^A$ and describe their stability. In order to execute this plan we need several technical results whose proofs will be given in the following sections. We first outline the method applied in showing the existence so that the role played by Lemma A and Lemma B will become clear.

The main idea is to construct a new one parameter family of aproximate solutions to (1.3) by solving the following problem:

$$\begin{aligned} -\epsilon^2\Delta(u^\xi + v^\xi) + f(u^\xi + v^\xi) &= c^\xi u_\xi^\xi \qquad \text{in } \Omega, \\ \langle V_1^\xi, v^\xi\rangle &= 0, \\ \frac{\partial v^\xi}{\partial n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.14}$$

where V_1^ξ is the principal eigenfunction of the linear eigenvalue problem (1.7) and an uknown function v^ξ and number c^ξ are to be determined for each $\xi \in (l, 1-l)$. Throughout this chapter the symbol $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $L^2(\Omega)$.The second condition in (1.14) will be referred to as the orthogonality condition. To find a solution to (1.3) it sufficies to find a solution to the reduced equation:

$$c^\xi = 0, \qquad \xi \in (l, 1-l). \tag{1.15}$$

From this point of view our method is nothing else but a variation of the classical Liapunov-Schmidt reduction. We notice that investigating c^ξ provides not only a precise information about the location of equilibria but also gives as a very good approximation of a portion of the true invariant manifold of the Allen-Cahn equation. In fact it can be shown that the set:

$$\mathcal{M}^Q = \{\mathbf{u}^\xi + v^\xi \mid \xi \in (l, 1-l),\ v^\xi \text{ solves } (1.14)\}$$

called a quasi-invariant manifold, both contains equilibria of the Allen-Cahn equation and is tangential to the true invariant manifold at equilibria. We shall not pursue this here and refer the reader to [F-H,CP,F,A-B-F2] in the context of the Allen-Cahn equation and to [A-B-F1,A-F12] in the context of the Cahn-Hilliard equation for further details.

Existence of solutions to (1.14) is established in the following

Lemma A. *Given $l > 0$ there exists $\epsilon_0 > 0$ such that for each $\epsilon < \epsilon_0$, $\xi \in (l, 1-l)$ and $\mathbf{u}^\xi \in \mathcal{M}^A$ there exists a unique pair $(v^\xi, c^\xi) \in W^{2,2}(\Omega) \times \mathbb{R}$ solving (1.14).*

Moreover we have estimates:

$$|c^\xi| \leq C\epsilon^{-\frac{15}{2}}\gamma^\xi, \tag{1.16 a}$$

$$\|v^\xi\|_{W^{2,2}(\Omega)} \leq C\epsilon^{-4}\delta^\xi, \tag{1.16 b}$$

$$\|v^\xi(x)\|_{W^{1,2}(R_{5\tilde{l}})} \leq C\delta^\xi e^{-\frac{Q}{\epsilon}}, \tag{1.16 c}$$

where $\tilde{l} = \frac{l}{11}$ and $R_{5\tilde{l}} = \{(x_1, x_2) \in \Omega \mid 5\tilde{l} < x_1 < 1 - 5\tilde{l}\}$. In addition (v^ξ, c^ξ) are both differentiable with respect to ξ and:

$$|c^\xi_\xi| \leq C\epsilon^{-17}\gamma^\xi, \tag{1.17 a}$$

$$\|v^\xi_\xi\|_{W^{2,2}(\Omega)} \leq C\epsilon^{-9}\delta^\xi. \tag{1.17 b}$$

Remark 1.1. The estimate (1.16 c) is of special importance for establishing existence of solutions to (1.15) and this fact was first pointed out in [F-H] and then later in [A-B-F2] where it was referred to as the improved v estimate.

We prove Lemma A in Section 3. Using the above lemma we can derive a formula for c^{ξ}.We will do this now in order to justify our interest in the eigenvalue problem (1.7). The nonlinear elliptic equation in (1.14) can be recast as:

$$L^{\epsilon,\xi}v^{\xi} = c^{\xi}u^{\xi}_{\xi} - L^{\epsilon,\xi}w^{\xi} + \mathcal{N}(v^{\xi}) + \mathcal{M}(w^{\xi}),$$

where

$$L^{\epsilon,\xi} = -\epsilon^2\Delta + f'(u^{\xi})$$

$$\mathcal{N}(v^{\xi}) = -f(u^{\xi}+v^{\xi}) + f(u^{\xi}) + f'(u^{\xi})v^{\xi},$$

$$\mathcal{M}(w^{\xi}) = f(U) - f(U+w^{\xi}) + f'(U+w^{\xi})w^{\xi}.$$

From the orthogonality condition $\langle V_1^{\xi}, v^{\xi}\rangle = 0$ and the Fredholm Alternative we obtain after elementary computations:

$$c^{\xi} = -\frac{\langle -L^{\epsilon,\xi}w^{\xi} + \mathcal{N}(v^{\xi}) + \mathcal{M}(w^{\xi}), V_1^{\xi}\rangle}{\langle V_1^{\xi}, u^{\xi}_{\xi}\rangle}. \tag{1.18}$$

It turns out that the principal part in the asymptotic expansion of c^{ξ} comes from the boundary integral in the expression:

$$\langle L^{\epsilon,\xi}w^{\xi}, V_1^{\xi}\rangle = \langle w^{\xi}, \lambda_1^{\xi}V_1^{\xi}\rangle - \epsilon^2\int_{\partial\Omega} V_1^{\xi}\frac{\partial w^{\xi}}{\partial n}\,dS. \tag{1.19}$$

From the definition of w^{ξ} we see that

$$\frac{\partial w^{\xi}}{\partial n} = -\frac{\partial}{\partial n}U\left(\frac{x_1-\xi}{\epsilon}\right),$$

therefore denoting:

$$\int_{\partial\Omega} V_1^{\xi}\frac{\partial}{\partial n}U\left(\frac{x_1-\xi}{\epsilon}\right)dS = I^{\xi}, \tag{1.20}$$

the boundary integral in (1.19) becomes

$$\int_{\partial\Omega} V_1^\xi \frac{\partial w^\xi}{\partial n}\, dS = -I^\xi .$$

The intuitive argument suggests that $c^\xi = \epsilon^2 q^\epsilon I^\xi +$ higher order terms. However we need to have a very refined information about V_1^ξ in order to establish this statement rigorously. In this direction we have:

Lemma B. *With the notation as in Section 1.1, for l and ϵ given as in Lemma A and $x = (x_1, x_2) \in \Omega$ we have:*

(i)

(1.21 a) $$|\lambda_1^\xi| \le C\epsilon^{-5}\gamma^\xi ,$$

(1.21 b) $$\lambda_2^\xi > C\epsilon^2 ,$$

where λ_i^ξ denotes the ith eigenvalue of (1.7).In addition

(1.22 a) $$\|V_1^\xi - q^\epsilon u_\xi^\xi\|_{L^2(\Omega)} \le C\epsilon^{-2}\delta^\xi ,$$

(1.22 b) $$\|V_1^\xi - q^\epsilon u_\xi^\xi\|_{W^{2,2}(\Omega)} \le C\epsilon^{-4}\delta^\xi ,$$

where V_1^ξ is the principal eigenfunction of (1.7) and $q^\epsilon = -\|u_\xi^\xi\|_{L^2(\Omega)}^{-1}$.

(ii) *For the rate of decay of V_1^ξ we have:*

(1.23) $$V_1^\xi(x) \le C\epsilon^{-\frac{5}{2}} e^{-\beta\frac{|x_1-\xi|}{\epsilon}} .$$

(iii) *Similiar to (1.21)-(1.23) estimates hold for the derivatives of λ_1^ξ, V_1^ξ with respect to ξ, namely,*

(1.24 a) $$|\lambda_{1,\xi}^\xi| \le C\epsilon^{-6}\gamma^\xi ,$$

(1.24 b) $$\|V_{1,\xi}^\xi - q^\epsilon u_{\xi\xi}^\xi\|_{L^2(\Omega)} \le C\epsilon^{-5}\delta^\xi ,$$

(1.24 c) $$\|V_{1,\xi}^\xi - q^\epsilon u_{\xi\xi}^\xi\|_{W^{2,2}(\Omega)} \le C\epsilon^{-7}\delta^\xi ,$$

(1.24 d) $$|V_{1,\xi}^\xi(x)| \le C\epsilon^{-\frac{7}{2}} e^{-\beta\frac{|x_1-\xi|}{\epsilon}} .$$

The proof of the above lemma will be given in Section 2.1.

Our goal is to find the asymptotic form of the integral I^ξ defined in (1.20).This calculation is somewhat involved and we need several intermediate steps in order to complete it.First we recall the following classical result.

Lemma 1.1. *The fundamental solution $G(x,y)$, $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$ of the operator $-\epsilon^2\Delta + \beta^2$ is given by:*

$$G(x,y) = \frac{1}{2\pi\epsilon^2} K_0\left(\frac{\beta}{\epsilon}|x-y|\right), \tag{1.25}$$

where K_0 is the modified Bessel function of order 0.Moreover K_0 satisfies:

$$K_0(r) = \int_0^\infty e^{-r\cosh t}\,dt, \qquad r > 0,$$
$$K_0(r) = -\ln\frac{r}{2} - \mu + O(r), \qquad \text{where } \mu \text{ is the Euler constant,} \qquad r \ll 1,$$
$$K_0(r) = \sqrt{\frac{\pi}{2r}}e^{-r}\left[1 + O\left(\frac{1}{r}\right)\right], \qquad r \gg 1.$$

Notice that all we need to know in order to compute I^ξ are the boundary values of V_1^ξ.The next lemma shows that they can be determined from certain integral equation.

Lemma 1.2. *For every $x \in \partial\Omega, \xi \in (l, 1-l)$ we have:*

$$V_1^\xi(x) + \mathcal{K}^\epsilon V_1^\xi(x) = \psi^\xi(x), \tag{1.26}$$

where

$$\mathcal{K}^\epsilon V_1^\xi(x) = 2\epsilon^2 \int_{\partial\Omega} V_1^\xi(y)\frac{\partial G}{\partial n_y}(x,y)\,dS_y, \tag{1.27}$$

$G(x,y)$ is the fundamental solution defined in (1.25) and

$$\psi^\xi(x) = 2\int_\Omega G(x,y)h^\xi(y)\,dy, \tag{1.28 a}$$

$$h^\xi(y) = \left(\beta^2 - f'(u^\xi) - \lambda_1^\xi\right)V_1^\xi(y) \tag{1.28 b}$$

Proof of Lemma 1.2. We fix a point $x \in \Omega$ and set $\mathcal{S}(x,\delta) = \{y \in \overline{\Omega} \mid |x-y| = \delta\}$.Let $\Omega_\delta = \Omega \setminus B_\delta(x)$ and $\partial\Omega'_\delta = \partial\Omega_\delta \setminus \mathcal{S}(x,\delta)$.We recast the equation (1.7) in the form:

$$-\epsilon^2 \Delta V_1^\xi + \beta^2 V_1^\xi = h^\xi, \tag{1.29}$$

where $\beta^2 = f'(\pm 1)$.Multiplying (1.29) by $G(x,y), y \in \Omega_\delta$ and integrating over Ω_δ with respect to y we obtain:

$$\begin{aligned} \int_{\Omega_\delta} G(x,y)h^\xi(y)\,dy &= \epsilon^2 \int_{\partial\Omega'_\delta} V_1^\xi(y)\frac{\partial G}{\partial n_y}(x,y)\,dS_y \\ &+ \epsilon^2 \int_{\mathcal{S}(x,\delta)} V_1^\xi(y)\frac{\partial G}{\partial n_y}(x,y)\,dS_y - \epsilon^2 \int_{\mathcal{S}(x,\delta)} \frac{\partial V_1^\xi}{\partial n_y}(y)G(x,y)\,dS_y, \end{aligned} \tag{1.30}$$

where we have made use of $\dfrac{\partial V_1^\xi}{\partial n} = 0$ on $\partial\Omega$.Consider the last integral in the above expression.Taking δ sufficiently small we get from Lemma 1.1:

$$G(x,y) = -\frac{1}{2\pi\epsilon^2}\ln\left(\frac{\beta}{2\epsilon}r\right) - \mu + O\left(\frac{r}{\epsilon}\right),$$

where $r = |x-y|$.Therefore, the Schwarz inequality and the estimate (1.11) with $m = 2$ we implies

$$\left|\epsilon^2 \int_{\mathcal{S}(x,\delta)} \frac{\partial V_1^\xi}{\partial n_y}(y)G(x,y)\,dS_y\right| \le C\|V_1^\xi\|_{W^{2,2}(\Omega)}\delta^{\frac{1}{2}}\ln\left(\frac{\beta\delta}{2\epsilon}\right) \to 0, \qquad \text{as } \delta \to 0. \tag{1.31}$$

Observe that on $\mathcal{S}(x,\delta)$ we have

$$\frac{\partial G}{\partial n_y}(x,y) = \frac{1}{2\pi\epsilon^2\delta} + \frac{\beta}{\epsilon}O(1), \qquad \text{as } \delta \to 0,$$

hence

$$\epsilon^2 \int_{\mathcal{S}(x,\delta)} V_1^\xi(y)\frac{\partial G}{\partial n_y}(x,y)\,dS_y = \frac{1}{2\pi\delta}\int_{\mathcal{S}(x,\delta)} V_1^\xi(y)\,dS_y + O(\delta), \qquad \text{as } \delta \to 0.$$

Since

$$\lim_{\delta\to 0}\frac{1}{2\pi\delta}\int_{\mathcal{S}(x,\delta)} V_1^\xi(y)\,dS_y = \frac{1}{2}V_1^\xi(x), \tag{1.32}$$

therefore (1.26) follows now by letting $\delta \to 0$ in (1.30) and utilizing (1.31), (1.32).

□

Corollary 1.3. *The formula (1.26) remains valid if V_1^ξ is replaced by $V_{1,\xi}^\xi$ with h^ξ, ψ^ξ replaced by h_ξ^ξ, ψ_ξ^ξ respectilvely.*

It can be shown that the operator $\mathcal{K}^\epsilon$ is completely continuous on $C^0(\partial\Omega)$ into $C^0(\partial\Omega)$ and that $\|\mathcal{K}^\epsilon\| \le C\epsilon^\alpha$ providing that $\partial\Omega \in C^{1,\alpha}$.Thus the operator $I + \mathcal{K}^\epsilon$ is invertible and in principle it should be possible to determine V_1^ξ by solving the equation (1.26).We shall not however pursue it here since by utilizing the equation itself we can obtain sufficient for our purposes information about V_1^ξ. We will analyze the right hand side of (1.26) first.It turns out that we only need to know $\psi^\xi(x)$ for $x \in \gamma^M$ where for $\xi \in (l, 1-l)$ and $M > 0$ we set:

$$\gamma^M = \{x \in \partial\Omega \mid -\beta^{-1}M\epsilon|\ln\epsilon| < x_1 < \xi - \frac{l}{2} \text{ or } \xi + \frac{l}{2} < x_1 < 1 + \beta^{-1}M\epsilon|\ln\epsilon|\}.$$

The reader should keep in mind that γ^M depends also on ϵ. In the sequel we shall denote:

$$d^\xi(x) = \operatorname{dist}(x, \Gamma^\xi) \qquad \text{where } \Gamma^\xi = \{x \in \Omega \mid x_1 = \xi\}.$$

Lemma 1.4. *If $\psi^\xi(x)$ is the function defined in (1.28) then the following formula is satisfied for all sufficiently large M:*

$$\psi^\xi(x) = \tilde{q}^\epsilon e^{-\frac{\beta}{\epsilon}d^\xi(x)}(1 + o(1)), \qquad x \in \gamma^M, \qquad \text{as } \epsilon \to 0, \tag{1.33}$$

where:

$$\tilde{q}^\epsilon = -\epsilon^{-1}q^\epsilon \int_{-\infty}^{\infty} [\beta^2 - f'(U(t))]\, U'(t) e^{-t\frac{\xi - x_1}{|\xi - x_1|}}\, dt. \tag{1.34}$$

In particular there exists a positive constant c such that $c^{-1}\epsilon^{-\frac{1}{2}} \le \tilde{q}^\epsilon \le c\epsilon^{-\frac{1}{2}}$.

Proof of Lemma 1.4. Set:

$$D^\xi = \{y \in \Omega \mid d^\xi(y) < \frac{l}{4}\}.$$

For fixed $x \in \gamma^M$ we write:

$$\psi^\xi(x) = \int_{\Omega \setminus D^\xi} G(x, y) h^\xi(y)\, dy + \int_{D^\xi} G(x, y) h^\xi(y)\, dy =: \psi_1^\xi(x) + \psi_2^\xi(x).$$

We first estimate ψ_1^ξ. Observe that on $\Omega \setminus D^\xi$ we have by Lemma B:

$$|h^\xi(y)| \le C\epsilon^{-\frac{5}{2}} e^{-\beta\frac{|y_1-\xi|}{\epsilon}}(e^{-\beta\frac{|y_1-\xi|}{\epsilon}} + \delta^\xi). \tag{1.35}$$

Let $\rho > 0$ be a fixed number. From Lemma 1.1 we know that there exist constants c_1, c_2 depending on ρ such that:

$$\begin{aligned} &|K_0(r)| \le c_1 |\ln \frac{r}{2}| \qquad \text{for } r \le \rho, \\ &|K_0(r)| \le c_2 r^{-\frac{1}{2}} e^{-r} \qquad \text{for } r > \rho, \end{aligned} \tag{1.36}$$

where K_0 is the modified Bessel function of order 0. We now further decompose:

$$\psi_1^\xi(x) = \int_{(\Omega\setminus D^\xi)\cap B_{\rho\epsilon}(x)} G(x,y)h^\xi(y)\,dy + \int_{\Omega\setminus(D^\xi\cup B_{\rho\epsilon}(x))} G(x,y)h^\xi(y)\,dy =: \psi_{11}^\xi(x) + \psi_{12}^\xi(x).$$

Since for $y \in (\Omega \setminus D^\xi) \cap B_{\rho\epsilon}(x)$ we have by (1.35):

$$|h^\xi(y)| \le C\epsilon^{-\frac{5}{2}} e^{-\beta\frac{|x_1-\xi|}{\epsilon}}(e^{-\beta\frac{|x_1-\xi|}{\epsilon}} + \delta^\xi),$$

therefore via (1.36) we get:

$$\begin{aligned} |\psi_{11}^\xi(x)| &\le C\epsilon^{-\frac{9}{2}} e^{-\beta\frac{|x_1-\xi|}{\epsilon}}(e^{-\beta\frac{|x_1-\xi|}{\epsilon}} + \delta^\xi)\int_{B_{\rho\epsilon}(x)} \left|\ln\left(\frac{\beta|x-y|}{2\epsilon}\right)\right| dy \\ &\le C\epsilon^{-\frac{7}{2}} e^{-\beta\frac{|x_1-\xi|}{\epsilon}}(e^{-\beta\frac{|x_1-\xi|}{\epsilon}} + \delta^\xi). \end{aligned}$$

For $x \in \gamma^M$ we have:

$$\begin{aligned} &e^{-\beta\frac{|x_1-\xi|}{\epsilon}} \le e^{-\frac{\beta}{\epsilon}(\frac{1}{2})}, \\ &e^{-\beta\frac{|x_1-\xi|}{\epsilon}}\delta^\xi \le e^{-\beta\frac{|x_1-\xi|}{\epsilon} - \frac{\beta}{\epsilon}l} \le e^{-\frac{\beta}{\epsilon}d^\xi(x)}e^{-\frac{\beta}{\epsilon}(\frac{1}{2})}. \end{aligned} \tag{1.37}$$

Since on γ^M

$$|d^\xi(x) - |x_1 - \xi|| = O((\epsilon|\ln\epsilon|)^\lambda), \qquad \text{where } \lambda = \min_{i=1,\dots,4}\{\lambda_i\},$$

therefore it follows:

$$|\psi_{11}^\xi(x)| \le Ce^{-\frac{\beta}{\epsilon}d^\xi(x)}e^{-\frac{C}{\epsilon}}. \tag{1.38}$$

To estimate $\psi_{12}^{\xi}(x)$ we observe that for $y \in \Omega \setminus D^{\xi}$ we have $|x-y| + |y_1 - \xi| - |x_1 - \xi| \geq 0$.It follows from (1.36) that:

$$|\psi_{12}^{\xi}(x)| \leq C\epsilon^{-4} e^{-\beta \frac{|x_1-\xi|}{\epsilon}} \int_{\Omega\setminus(D^{\xi}\cup B_{\rho_0}(x))} r^{-\frac{1}{2}} (e^{-\beta\frac{|y_1-\xi|}{\epsilon}} + \delta^{\xi})\, dy$$

$$\leq C\epsilon^{-4} e^{-\beta\frac{|x_1-\xi|}{\epsilon}} e^{-\frac{\beta}{\epsilon}(\frac{l}{4})} \leq C\epsilon^{-4} e^{-\frac{\beta}{\epsilon} d^{\xi}(x)} e^{-\frac{\rho}{\epsilon}}, \tag{1.39}$$

where we have made use of $|y_1 - \xi| \geq \frac{l}{4}$ in $\Omega \setminus D^{\xi}$ and (1.37).

It remains to evaluate $\psi_2^{\xi}(x)$.Let $z^x = (z_1^x, z_2^x) \in \overline{\Omega}$ be a unique point such that $d^{\xi}(x) = |x - z^x|$.Observe that either $z^x = (\xi, x_2)$ or $z^x = (\xi, 0)$ or $z^x = (\xi, b)$ depending on Ω and the location of x.From Lemma 1.1 it follows that for $x \in \gamma^M$, $y \in D^{\xi}$ we have:

$$G(x,y) = \frac{\epsilon^{-\frac{3}{2}}}{2\sqrt{2\pi\beta}} |x-y|^{-\frac{1}{2}} e^{-\frac{\beta}{\epsilon}|x-y|} \left[1 + O\left(\frac{\epsilon}{|x-y|}\right)\right] \qquad \text{as } \epsilon \to 0.$$

Since $|x-y| > \frac{l}{4}$, $y \in D^{\xi}$ therefore it sufficies to compute:

$$\tilde{\psi}^{\xi}(x) = \frac{\epsilon^{-\frac{3}{2}}}{\sqrt{2\pi\beta}} \int_{D^{\xi}} |x-y|^{-\frac{1}{2}} e^{-\frac{\beta}{\epsilon}|x-y|} h^{\xi}(y)\, dy.$$

From Lemma B (see (1.22 b)) and the definition of u^{ξ} it follows that for $y \in D^{\xi}$ we have:

$$h^{\xi}(y) = q^{\epsilon}\left\{\beta^2 - f'\left[U\left(\frac{y_1-\xi}{\epsilon}\right)\right] + \lambda_1^{\xi}\right\} U_{\xi}\left(\frac{y_1-\xi}{\epsilon}\right)(1+o(1)). \tag{1.40}$$

Taking into account only the principal term in the above expression, stretching variables $y - z^x = \epsilon\eta$ in the formula for $\tilde{\psi}^{\xi}$ and utilizing (1.40) we obtain:

$$\tilde{\psi}^{\xi}(x) = \frac{\epsilon^{\frac{1}{2}} e^{-\frac{\beta}{\epsilon}d^{\xi}(x)}}{\sqrt{2\pi\beta}} \int_{D_{\epsilon}^{\xi}} |x - z^x - \epsilon\eta|^{-\frac{1}{2}} H^{\epsilon}(\eta)\, d\eta, \tag{1.41}$$

where

$$H^{\epsilon}(\eta) = -\epsilon^{-1} q^{\epsilon}\left\{\beta^2 - f'(U(\eta_1)) + \lambda_1^{\xi}\right\} U'(\eta_1) \exp\left\{-\frac{\eta[2(z^x - x) + \epsilon\eta]}{\beta(|x - z^x - \epsilon\eta| + |x - z^x|)}\right\} \tag{1.42}$$

and

$$D_{\epsilon}^{\xi} = \left(-\frac{l}{4\epsilon}, \frac{l}{4\epsilon}\right) \times \left(-\frac{z_2^x}{\epsilon}, \frac{b - z_2^x}{\epsilon}\right)$$

It follows that $H^\epsilon(\eta)$ is integrable in D^ξ_ϵ. Observe also that we can break $H^\epsilon(\eta)$ into $H^\epsilon(\eta) = H^\xi_1(\eta)H^\xi_2(\eta)$, where

$$H^\xi_2 = -\epsilon^{-1}q^\epsilon\left\{\beta^2 - f'(U(\eta_1)) + \lambda^\xi_1\right\}U'(\eta_1)\exp\left\{-\frac{\eta_1[2(\xi - x_1) + \epsilon\eta_1]}{\beta(|x - z^x - \epsilon\eta| + |x - z^x|)}\right\}$$
$$H^\xi_2 = \exp\left\{-\frac{\eta_2[2(z^x_2 - x_2) + \epsilon\eta_2]}{\beta(|x - z^x - \epsilon\eta| + |x - z^x|)}\right\}.$$

Notice that if $x \in \gamma^M$ then $|z^x_2 - x_2| \le C(M\beta^{-1}\epsilon|\ln\epsilon|)^\lambda$ with $\lambda > \frac{3}{2}$ and therefore by integrating with respect to η_2 we obtain that asymptotically:

$$\int_{-\frac{z^x_2}{\epsilon}}^{\frac{b - z^x_2}{\epsilon}} H^\xi_2(\eta)\,d\eta_2 = \epsilon^{-\frac{1}{2}}\sqrt{\frac{\beta}{2}}|x - z^x|^{\frac{1}{2}}\left\{\int_0^{\frac{(z^x_2)^2}{2\epsilon\beta|x - z^x|}} + \int_0^{\frac{(b - z^x_2)^2}{2\epsilon\beta|x - z^x|}}\right\} s^{-\frac{1}{2}}e^{-s}\,ds + o(1)$$

$$= \epsilon^{-\frac{1}{2}}\sqrt{\frac{\beta}{2}}|x - z^x|^{\frac{1}{2}}\Gamma(\frac{1}{2}) + o(1), \tag{1.43}$$

where we have made use of $\min\{|z^x_2|, |b - z^x_2|\} < C(\epsilon|\ln\epsilon|)^\lambda$. Since by Lemma B, $|\lambda^\xi_1| \le C\epsilon^{-5}\gamma^\xi$ therefore combining (1.41), (1.42) and (1.43) yields:

$$\tilde{\psi}^\xi(x) = -\epsilon^{-1}\frac{q^\epsilon}{2}e^{-\frac{\beta}{\epsilon}d^\epsilon(x)}\left\{\int_{-\infty}^{\infty}\left[\beta^2 - f'(U(t))\right]U'(t)e^{-t\frac{\xi - x_1}{|\xi - x_1|}}\,dt\right\}(1 + o(1)). \tag{1.44}$$

Thus by (1.38), (1.39) and (1.44) the lemma follows.

□

Corollary 1.5. *Under the assumptions of the previous lemma the following formula holds for $x \in \gamma^M$ for all sufficiently large M:*

$$\psi^\xi_\xi(x) = Q^\epsilon e^{-\frac{\beta}{\epsilon}d^\epsilon(x)}(1 + o(1)), \tag{1.45}$$

where

$$Q^\epsilon = \epsilon^{-2}q^\epsilon\beta\int_{-\infty}^{\infty}\left\{\left[\beta^2 - f'(U(t))\right]U''(t) - f''(U(t))\left(U'(t)\right)^2\right\}e^{-t\frac{\xi - x_1}{|\xi - x_1|}}\,dt.$$

In particular there exists a constant $c > 0$ such that $c^{-1}\epsilon^{-\frac{3}{2}} \le |Q^\epsilon| \le c\epsilon^{-\frac{3}{2}}$ and $Q^\epsilon < 0$ if $x_1 < \xi$ and $Q^\epsilon > 0$ if $x_1 > \xi$.

The proof of the above corollary is similiar to the proof of Lemma 1.4 and we omitt the details.

Next we shall estimate the second term in the equation (1.26), namely $\mathcal{K}^\epsilon V_1^\xi(x)$ for $x \in \tilde{\gamma}^M$ where

$$\tilde{\gamma}^M = \{x \in \partial\Omega \mid -M\beta^{-1}\epsilon|\ln\epsilon| < x_1 < 1 + M\beta^{-1}\epsilon|\ln\epsilon|\}$$

(note that $\gamma^M \subset \tilde{\gamma}^M$).We first need a technical lemma.

Lemma 1.6. *Let $\theta > 0$ be such that $\lambda = \min_{i=1,\dots,4}\lambda_i > \dfrac{3}{2} + \theta > \dfrac{3}{2}$ and $M > 2 + \theta$. Then there exists τ, $\tau > \min\{\frac{\theta}{2}, \frac{1}{2}\}$ such that*

$$\left|\epsilon^2 \int_{\partial\Omega} e^{-\beta\frac{|y_1-\xi|}{\epsilon}} \frac{\partial G}{\partial n_y}(x,y)\,dS_y\right| \le C\epsilon^\tau e^{-\beta\frac{|x_1-\xi|}{\epsilon}}, \tag{1.46}$$

for all $x \in \tilde{\gamma}^M$.

Proof of Lemma 1.6. Let ρ, c_3, c_4 be chosen so that the following inequalities are satisfied

$$|K_0'(r)| \le c_3 r^{-1} \qquad \text{for } r \le \rho,$$
$$|K_0'(r)| \le c_4 r^{-\frac{1}{2}} e^{-r} \qquad \text{for } r > \rho. \tag{1.47}$$

For definitness we assume that $x \in \tilde{\gamma}^M$ satisfies $\xi \le x_1$ and $x_2 < b$ (x lies near the corner C_1).Set $z = (\xi, 0)$.We break the integral in (1.46) into:

$$\epsilon^2 \int_{\partial\Omega} e^{-\beta\frac{|y_1-\xi|}{\epsilon}} \frac{\partial G}{\partial n_y}(x,y)\,dS_y = \epsilon^2 \int_{\partial\Omega\cap B_{\rho\epsilon}(x)} e^{-\beta\frac{|y_1-\xi|}{\epsilon}} \frac{\partial G}{\partial n_y}(x,y)\,dS_y$$
$$+\,\epsilon^2 \int_{\partial\Omega\setminus B_{\rho\epsilon}(x)} e^{-\beta\frac{|y_1-\xi|}{\epsilon}} \frac{\partial G}{\partial n_y}(x,y)\,dS_y =: I_1 + I_2.$$

By using the first of the estimates in (1.47) we obtain:

$$|I_1| \le C\left\{\int_{r<\rho\epsilon} r^{-2}|(y-x)n_y|\,dS_y\right\} e^{-\beta\frac{|x_1-\xi|}{\epsilon}}.$$

Notice that if $\partial\Omega \in C^{1,\alpha}$ then $|(y-x)n_y| \le Cr^{1+\alpha}$, where $r = |x-y|$, hence:

$$|I_1| \le C\epsilon^{\lambda-1} e^{-\beta\frac{|x_1-\xi|}{\epsilon}}. \tag{1.48}$$

For estimating I_2 we use the second of the formulas in (1.47).

$$|I_2| \le C\epsilon^{-\frac{1}{2}} \int_{\partial\Omega\setminus B_{\rho\epsilon}(x)} r^{-\frac{3}{2}} e^{-\frac{\beta}{\epsilon}(r+|y_1-\xi|)}|(y-x)n_y|\,dS_y.$$

Set

$$\gamma_1 = \{y \in \partial\Omega \mid \xi - 2M\beta^{-1}\epsilon|\ln\epsilon| \le y_1 \le 1 + 2M\beta^{-1}\epsilon|\ln\epsilon|, y_2 < b\}.$$

We have

$$|I_2| \le C\epsilon^{-\frac{1}{2}} \int_{\gamma_1} r^{-\frac{3}{2}} e^{-\frac{\beta}{\epsilon}(r+|y_1-\xi|)} |(y-x)n_y|\, dS_y$$

$$(1.49) \qquad + C\epsilon^{-\frac{1}{2}} \int_{\partial\Omega\setminus\gamma_1} r^{-\frac{3}{2}} e^{-\frac{\beta}{\epsilon}(r+|y_1-\xi|)} |(y-x)n_y|\, dS_y := I_{21} + I_{22},$$

Observe that for $y \in \partial\Omega \setminus \gamma_1$ we have $r + |y_1 - \xi| \ge |x_1 - \xi| + M\beta^{-1}\epsilon|\ln\epsilon|$, hence

$$(1.50) \qquad I_{22} = C \le C\epsilon^{-\frac{1}{2}+M}(M\epsilon|\ln\epsilon|)^{-\frac{1}{2}} e^{-\beta\frac{|x_1-\xi|}{\epsilon}} \le C\epsilon^{M-2} e^{-\beta\frac{|x_1-\xi|}{\epsilon}}.$$

To estimate the first integral in (1.49) we introduce parametrization of γ_1 as follows:

$$\gamma_1 = \{y \in \partial\Omega \mid \xi - 2M\beta^{-1}\epsilon|\ln\epsilon| \le y_1 \le 1 + 2M\beta^{-1}\epsilon|\ln\epsilon|, y_2 = \phi_1(y_1)\}.$$

For a given $x \in \tilde{\gamma}^M$ we consider function $g_1 : \gamma_1 \to \mathbb{R}$ defined by:

$$g_1(y) = |x - y| + |y_1 - \xi| - |x_1 - \xi|.$$

Notice that $g_1(y) \ge 0$. From the Mean Value Theorem it also follows that there exists $\vartheta, 0 \le \vartheta \le 1$ such that for $y_1^* = \vartheta y_1 + (1-\vartheta)x_1$ we have

$$|(y-x)n_y| = |(x_1 - y_1)\phi_1'(y_1) - (\phi_1(x_1) - \phi_1(y_1))| = |x_1 - y_1|\,(|\phi_1'(y_1) - \phi_1'(y_1^*)|)\,.$$

By using the assumption (D4) we get $|\phi_1'(y_1)| + |\phi_1(y^*)| \le C(\epsilon|\ln\epsilon|)^{\lambda-1}$ hence

(1.51)

$$I_{12} \le Ce^{-\beta\frac{|x_1-\xi|}{\epsilon}} \int_{\gamma_1} e^{-\frac{\beta}{\epsilon}g_1(y)} |x_1 - y_1|^{-\frac{1}{2}} (|\phi_1'(y_1)| + |\phi_1'(y_1^*)|)\, dy_1 \le Ce^{-\beta\frac{|x_1-\xi|}{\epsilon}}(\epsilon|\ln\epsilon|)^{\lambda-1}.$$

By combining (1.48), (1.50) and (1.51) we obtain (1.46).

□

Remark 1.2. Note that from (1.49), (1.50) it follows that if a function $V(x)$ satisfies:

$$V(x)\begin{cases} \leq Ce^{-\beta\frac{|x_1-\xi|}{\epsilon}} & \text{for } x \in \tilde{\gamma}^{2M}, \\ \leq C\epsilon^{-p}e^{-\beta\frac{|x_1-\xi|}{\epsilon}} & \text{for } x \in \partial\Omega \setminus \tilde{\gamma}^{2M}, \end{cases}$$

where $M > 0, p > 0$ are such that $M - 2 - p > \theta > 0$ then there exists $\tau > 0$ depending on θ such that:

$$\left|\epsilon^2 \int_{\partial\Omega} V(x)\frac{\partial G}{\partial n_y}(x,y)\,dS_y\right| \leq C\epsilon^{\tau}e^{-\beta\frac{|x_1-\xi|}{\epsilon}},$$

for $x \in \tilde{\gamma}^{M}$.This observation will be used in the proof of the following corollary.

Corollary 1.7. *For all sufficiently large M the following estimate holds for V_1^{ξ}, the principal eigenfunction of (1.7):*

$$|V_1^{\xi}(x)| \leq C\epsilon^{-\frac{1}{2}}e^{-\beta\frac{|x_1-\xi|}{\epsilon}}, \tag{1.52}$$

where $x \in \tilde{\gamma}^{2M}$.

Proof of Corollary 1.7. Notice that by Lemma B (see (1.22 b)) the estimate (1.52) is satisfied for $x \in \partial\Omega$ such that $\xi - \frac{l}{2} < x_1 < \xi + \frac{l}{2}\}$.It sufficies to prove (1.52) for $x \in \gamma^{2M}$.Let for given M and λ a positive number τ be chosen as in Lemma 1.6 and let k be a positive integer such that $k\tau - \frac{5}{2} \geq -\frac{1}{2}$.From Lemma B (1.23) we have:

$$|V_1^{\xi}(y)| \leq C\epsilon^{-\frac{5}{2}}e^{-\beta\frac{|y_1-\xi|}{\epsilon}}, \qquad y \in \partial\Omega,$$

hence from Lemma 1.6 it follows:

$$\left|\epsilon^2 \int_{\partial\Omega} V_1^{\xi}(y)\frac{\partial G}{\partial n_y}(x,y)\,dS_y\right| \leq C\epsilon^{-\frac{5}{2}+\tau}e^{-\beta\frac{|x_1-\xi|}{\epsilon}},$$

for $x \in \gamma^{(k+1)M}$.Utilizing (1.26) we get for $x \in \gamma^{(k+1)M}$:

$$|V_1^{\xi}(x)| \leq |\psi^{\xi}(x)| + \left|\epsilon^2 \int_{\partial\Omega} V_1^{\xi}(y)\frac{\partial G}{\partial n_y}(x,y)\,dS_y\right| \leq C(\epsilon^{-\frac{1}{2}}e^{-\beta\frac{|x_1-\xi|}{\epsilon}} + \epsilon^{-\frac{5}{2}+\tau}e^{-\beta\frac{|x_1-\xi|}{\epsilon}}), \tag{1.53}$$

where we have made use of Lemma 1.4 and the fact that for $x \in \gamma^{(k+1)M}$

$$\big||x_1 - \xi| - d^{\xi}(x)\big| \leq C(\epsilon|\ln\epsilon|)^{\lambda}.$$

Combining (1.53) and Lemma B (1.23) yields:

$$V_1^\xi(x)\begin{cases} \leq C\epsilon^{-\frac{5}{2}+\tau}e^{-\beta\frac{|x_1-\xi|}{\epsilon}}, & x\in\tilde{\gamma}^{(k+1)M},\\ \leq C\epsilon^{-\frac{5}{2}}e^{-\beta\frac{|x_1-\xi|}{\epsilon}}, & x\in\partial\Omega\setminus\tilde{\gamma}^{(k+1)M}.\end{cases}$$

From Remark 1.2 by utilizing the last estimates and repeating the argument leading to (1.53) we obtain:

$$|V_1^\xi(x)|\leq C\epsilon^{-\frac{5}{2}+2\tau}e^{-\beta\frac{|x_1-\xi|}{\epsilon}},$$

for $x\in\gamma^{kM}$.We can continue the above procedure of improving the upper bound on V_1^ξ as long as $-\frac{5}{2}+m\tau<-\frac{1}{2}$ and thus (1.52) follows.

□

Combining Lemma 1.6 and Corollary 1.7 we immediately obtain:

Corollary 1.8. *Providing that $M>2$ there exists $\tau>0$ depending on $M,\lambda=\min_{i=1,\dots,4}\lambda_i$ such that*

$$\left|\epsilon^2\int_{\partial\Omega}V_1^\xi(y)\frac{\partial G}{\partial n_y}(x,y)\,dS_y\right|\leq C\epsilon^{-\frac{1}{2}+\tau}e^{-\beta\frac{|x_1-\xi|}{\epsilon}}, \tag{1.54}$$

for $x\in\tilde{\gamma}^M$.

By using Corollary 1.5 and Lemma 1.6 we can easily prove results analogous to Corollary 1.7 and Corollary 1.8 with V_1^ξ replaced by $V_{1,\xi}^\xi$.We summarize the corresponding estimates in the next corollary:

Corollary 1.9. *For all sufficiently $M>2$ there exists $\tau>0$ depending on M,λ such that for $x\in\tilde{\gamma}^M$:*

$$|V_{1,\xi}^\xi(x)|\leq C\epsilon^{-\frac{3}{2}}e^{-\beta\frac{|x_1-\xi|}{\epsilon}} \tag{1.55}$$

and consequently

$$\left|\epsilon^2\int_{\partial\Omega}V_{1,\xi}^\xi(y)\frac{\partial G}{\partial n_y}(x,y)\,dS_y\right|\leq C\epsilon^{-\frac{1}{2}+\tau}e^{-\beta\frac{|x_1-\xi|}{\epsilon}}. \tag{1.56}$$

We are in the position now to derive an asymptotic formula for the integral I^ξ defined in (1.20).

Lemma 1.10. *The following asymptotic formula holds true:*

$$
\begin{aligned}
I^\xi = \tilde{q}^\epsilon \epsilon^{-1}\beta\beta' \Big[\sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i) \left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} & e^{-2\beta(\frac{1-\xi}{\epsilon})} \\
- \sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i) \left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} & e^{-2\beta(\frac{\xi}{\epsilon})} \Big] (1+o(1)) + R^\xi,
\end{aligned}
\tag{1.57}
$$

where

$$
|R^\xi| \le C\Big(\epsilon^{-\frac{7}{2}+M} + \epsilon^{-\frac{3}{2}+\tau} \sum_{i=1}^{4} |\kappa_i| \epsilon^{\lambda_i}\Big)\gamma^\xi,
$$

$\tilde{q}^\epsilon$ is as in Lemma 1.4, $M, \tau > 0$ are as in Lemma 1.6 and such that $M - 2 > \tau + \sum \lambda_i$, numbers κ_i, λ_i are defined in (D4) and Γ stands for the standard gamma function.

Proof of Lemma 1.10. Let M, τ be chosen as in Lemma 1.6.Observe that by taking M sufficiently large we can always achieve $M - 2 > \tau + \sum \lambda_i$. We decompose I^ξ to:

$$
I^\xi = \int_{\partial\Omega\setminus\gamma^M} V_1^\xi \frac{\partial}{\partial n} U\left(\frac{x_1 - \xi}{\epsilon}\right) dS + \int_{\gamma^M} V_1^\xi \frac{\partial}{\partial n} U\left(\frac{x_1-\xi}{\epsilon}\right) dS =: I_1^\xi + I_2^\xi.
$$

The idea is to estimate I_1^ξ and evaluate I_2^ξ.Utilizing (1.23) and the asymptotic formula (1.2) we get:

$$
|I_1^\xi| \le C\epsilon^{-\frac{7}{2}} \int_{\partial\Omega\setminus\gamma^M} e^{-2\beta\frac{|x_1-\xi|}{\epsilon}} dS.
$$

Since $x \in \partial\Omega \setminus \gamma^M$ therefore $|x_1 - \xi| \ge M\beta^{-1}\epsilon|\ln\epsilon| + \min\{\xi, 1-\xi\}$, hence

$$
|I_1^\xi| \le C\epsilon^{-\frac{7}{2}}\left(e^{-2\frac{\beta}{\epsilon}(\xi + \frac{M}{\beta}\epsilon|\ln\epsilon|)} + e^{-2\frac{\beta}{\epsilon}[(1-\xi)+\frac{M}{\beta}\epsilon|\ln\epsilon|]}\right) \le C\epsilon^{-\frac{7}{2}+M}\gamma^\xi. \tag{1.58}
$$

We shall now evaluate I_2^ξ.By employing (1.26) we can recast the expression for I_2^ξ in the form:

$$
I_2^\xi = -\int_{\gamma^M} K^\epsilon V_1^\xi \frac{\partial}{\partial n} U\left(\frac{x_1-\xi}{\epsilon}\right) dS + \int_{\gamma^M} \psi^\xi \frac{\partial}{\partial n} U\left(\frac{x_1-\xi}{\epsilon}\right) dS =: I_{21}^\xi + I_{22}^\xi.
$$

From Lemmas 1.4, 1.6 it follows that it is convenient to evaluate first

$$
\tilde{I}^\xi :\, = \int_{\gamma^M} e^{-\frac{\beta}{\epsilon}d^\xi(x)} \frac{\partial}{\partial n} U\left(\frac{x_1-\xi}{\epsilon}\right) dS.
$$

Let $C_i = (C_{i1}, C_{i2})$ denote the ith corner of Ω and let

$$
\gamma_i^M = \{x \in \gamma^M \mid |x_1 - C_{i1}| < M\beta^{-1}\epsilon|\ln\epsilon| \text{ and } x \in \operatorname{supp}\Big(\frac{\partial U}{\partial n}\Big)\}.
$$

In each γ_i^M the local coordinates are given as in (D4), namely if $x = (x_1, x_2) \in \gamma_i^M$ then $x_2 = \phi_i(x_1)$.Observe that for $x \in \gamma_i^M$ we have

$$d^\xi(x) + |x_1 - \xi| = 2|x_1 - \xi| + O((\epsilon|\ln\epsilon|)^{\lambda_i}), \tag{1.59}$$

where we have made use of the assumption (D4).We also have $\frac{\partial U}{\partial n}(\frac{x_1-\xi}{\epsilon}) = \epsilon^{-1}U'(\frac{x_1-\xi}{\epsilon})\tilde{\phi}_i(x_1)$, where

$$\tilde{\phi}_i(x_1) = \begin{cases} \phi_i'(x_1) & \text{if } i = 1,4; \\ -\phi_i'(x_1) & \text{if } i = 2,3. \end{cases} \tag{1.60}$$

From the assumption (D4) we further conclude:

$$\phi_i'(x_1) = \begin{cases} \kappa_i(|x_1 - C_{i1}|)^{\lambda_i - 1} + O(|x_1 - C_{i1}|^{\lambda_i}), i = 1,3; \\ -\kappa_i(|x_1 - C_{i1}|)^{\lambda_i - 1} + O(|x_1 - C_{i1}|^{\lambda_i}), i = 2,4. \end{cases} \tag{1.61}$$

Thus combining (1.59),(1.60),(1.61) and the well known asymptotic formula for the Laplace integrals (see [E] p. 36) we obtain:

$$\begin{aligned} \int_{\gamma_i^M} e^{-\frac{\beta}{\epsilon}d^\xi(x)}\frac{\partial}{\partial n}U\left(\frac{x_1-\xi}{\epsilon}\right) dS &= \epsilon^{-1}\int_{\gamma_i^M} e^{-\frac{\beta}{\epsilon}d^\xi(x)}U'\left(\frac{x_1-\xi}{\epsilon}\right)\tilde{\phi}_i(x_1)[1 + (\phi_i')^2]^{\frac{1}{2}}\, dy_1 \\ &= \begin{cases} \epsilon^{-1}\beta\beta'\kappa_i\Gamma(\lambda_i)(\frac{\epsilon}{2\beta})e^{-2\frac{\beta}{\epsilon}(1-\xi)}(1 + o(1)), & i = 1,2; \\ -\epsilon^{-1}\beta\beta'\kappa_i\Gamma(\lambda_i)(\frac{\epsilon}{2\beta})e^{-2\frac{\beta}{\epsilon}\xi}(1 + o(1)), & i = 3,4. \end{cases} \end{aligned}$$

This and Lemma 1.4 yields:

$$I_{22}^\xi = \tilde{q}^\epsilon\epsilon^{-1}\beta\beta'\left[\sum_{i=1}^{2}\kappa_i\Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta(\frac{1-\xi}{\epsilon})} - \sum_{i=3}^{4}\kappa_i\Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta(\frac{\xi}{\epsilon})}\right](1+o(1)). \tag{1.62}$$

The integral $|I_{21}^\xi|$ can be treated similiarly hence:

$$|I_{21}^\xi| \le C\epsilon^{-\frac{3}{2}+\tau}\left(\sum_{i=1}^{4}|\kappa_i|\epsilon^{\lambda_i}\right)\gamma^\xi. \tag{1.63}$$

Combining (1.59), (1.62) and (1.63) yields (1.57).

□

By employing Corollary 1.5 and Corollary 1.9 we obtain:

Corollary 1.11. *The following asymptotic formula holds true:*

$$
(1.64)\quad
\begin{aligned}
I_\xi^\xi &= \int_{\partial\Omega} V_{1,\xi}^\xi \frac{\partial}{\partial n} U\left(\frac{x_1-\xi}{\epsilon}\right) dS + \int_{\partial\Omega} V_1^\xi \frac{\partial}{\partial n} U_\xi \left(\frac{x_1-\xi}{\epsilon}\right) dS = \\
&\tilde{Q}^\epsilon \Bigg[\sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta(\frac{1-\xi}{\epsilon})} + \sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta(\frac{\xi}{\epsilon})}\Bigg](1+o(1)) + \tilde{R}^\xi,
\end{aligned}
$$

where $\tilde{Q}^\epsilon$ is a positive constant depending only on F such that there exists $c > 0, c^{-1}\epsilon^{-\frac{5}{2}} \leq \tilde{Q}^\epsilon \leq c\epsilon^{-\frac{5}{2}}$ and

$$
|\tilde{R}^\xi| \leq C\left(\epsilon^{-\frac{9}{2}+M} + \epsilon^{-\frac{5}{2}+\tau} \sum_{i=1}^{4} |\kappa_i| \epsilon^{\lambda_i}\right)\gamma^\xi
$$

for $M, \tau > 0$ as in Lemma 1.6 and such that $M - 2 > \tau + \sum \lambda_i$.

The proof is essentially a repetition of the argument in the proof of Lemma 1.10 and we omitt it.

In the sequel we shall denote:

$$
(1.65)\quad
\begin{aligned}
b_0^\xi &= \tilde{q}^\epsilon q^\epsilon \epsilon\beta\beta' \Bigg[\sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta(\frac{1-\xi}{\epsilon})} - \sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta(\frac{\xi}{\epsilon})}\Bigg] \\
b_1^\xi &= \tilde{Q}^\epsilon q^\epsilon \epsilon^2 \Bigg[\sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta(\frac{1-\xi}{\epsilon})} + \sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta(\frac{\xi}{\epsilon})}\Bigg],
\end{aligned}
$$

where $q^\epsilon, \tilde{q}^\epsilon, Q^\epsilon$ are defined in Lemma B, (1.4) and Corollary 1.11 respectively. We can now establish the asymptotic formula for the speed c^ξ.

Lemma 1.12. *The following formula holds:*

$$
(1.66)\qquad c^\xi = b_0^\xi(1+o(1)) + r_0^\xi,
$$

where

$$
|r_0^\xi| \leq C\epsilon^{\frac{5}{2}}\left(\epsilon^{-\frac{3}{2}+\tau} \sum_{i=1}^{4} |\kappa_i| \epsilon^{\lambda_i} + \epsilon^{-\frac{5}{2}+M} + e^{-\frac{C}{\epsilon}}\right)\gamma^\xi
$$

and $M, \tau > 0$ are as in Lemma 1.10.

Remark 1.3. In verifying (1.66) the improved v estimate plays an essential role.

Proof of Lemma 1.12. By using (1.18), (1.19) we get

$$c^\xi = \epsilon^2 I^\xi \langle V_1^\xi, u_\xi^\xi \rangle^{-1} + R_1^\xi,$$

where

$$R_1^\xi = \frac{\langle \lambda_1^\xi w^\xi - \mathcal{N}(v^\xi) - \mathcal{M}(w^\xi), V_1^\xi \rangle}{\langle V_1^\xi, u_\xi^\xi \rangle}.$$

Since from Lemma B we have:

$$\langle V_1^\xi, u_\xi^\xi \rangle = \frac{1}{q^\epsilon} + e^{-\frac{C}{\epsilon}},$$

therefore by Lemma 1.10 and (1.65) the proof will be complete if we can show:

$$|R_1^\xi| \le C\gamma^\xi e^{-\frac{C}{\epsilon}}. \tag{1.67}$$

Let $R_1^\xi = (R_{11}^\xi + R_{12}^\xi + R_{13}^\xi)\langle V_1^\xi, u_\xi^\xi \rangle^{-1}$ where $R_{11}^\xi = \langle \lambda_1^\xi w^\xi, V_1^\xi \rangle$, $R_{12}^\xi = -\langle \mathcal{N}(v^\xi), V_1^\xi \rangle$, $R_{13}^\xi = -\langle \mathcal{M}(w^\xi), V_1^\xi \rangle$. By Lemma B ((1.21 a),(1.23)) and the definition of w^ξ we easily get

$$|R_{11}^\xi| \le C\gamma^\xi e^{-\frac{C}{\epsilon}}. \tag{1.68}$$

For estimating R_{12}^ξ we shall use the improved v estimate from Lemma A.We have

$$\langle \mathcal{N}(v^\xi), V_1^\xi \rangle = \int_{R_{5\bar{\xi}}} \mathcal{N}(v^\xi) V_1^\xi + \int_{\Omega \setminus R_{5\bar{\xi}}} \mathcal{N}(v^\xi) V_1^\xi,$$

where $R_{5\bar{\xi}}$ is defined in Lemma A.Since $|\mathcal{N}(v^\xi)| < C|v^\xi|^2$ therefore from the improved v estimate and (1.12) with $m = 4$ we get

$$\left| \int_{R_{5\bar{\xi}}} \mathcal{N}(v^\xi) V_1^\xi \right| \le \|V_1^\xi\|_{L^2(\Omega)} \|v^\xi\|^2_{L^4(R_{5\bar{\xi}})} \le C\gamma^\xi e^{-\frac{C}{\epsilon}}.$$

Similiar argument utilizing Lemma A and Lemma B (1.23) yields

$$\left| \int_{\Omega \setminus R_{5\bar{\xi}}} \mathcal{N}(v^\xi) V_1^\xi \right| \le C\epsilon^{-8} \|V_1^\xi\|_{C^0(\Omega \setminus R_{\bar{\xi}})} \gamma^\xi \le C\gamma^\xi e^{-\frac{C}{\epsilon}}.$$

From the last two estimates it follows

$$|R_{12}^\xi| \le C\gamma^\xi e^{-\frac{C}{\epsilon}}. \tag{1.69}$$

The estimate for R_{13}^{ξ} is obtained by a similiar argument.Observe that $|\mathcal{M}(w^{\xi})| \le C|w^{\xi}|^2$ and $\operatorname{supp} w^{\xi} \cap R_{5\tilde{l}} = \emptyset$ hence

$$|R_{13}^{\xi}| \le C\gamma^{\xi} e^{-\frac{C}{\epsilon}}.$$

Combining this with (1.68), (1.69) yields (1.67).The proof of the lemma is complete.

□

Now we shall prove the main existence result in this chapter.

Theorem 1.13.

(1) *Suppose that there exists* $\epsilon_0 > 0$ *such that for all* ϵ, $0 < \epsilon < \epsilon_0$ *we have*

$$\left[\sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i}\right]\left[\sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i}\right] > 0. \tag{1.70}$$

Then there exists $\epsilon_1 > 0$ *such that for all* ϵ, $0 < \epsilon < \epsilon_1$ *there exists* $\hat{\xi} \in (\xi, 1-\xi)$ *such that:*

$$c^{\hat{\xi}} = 0. \tag{1.71}$$

Moreover

$$|\hat{\xi} - \frac{1}{2}| \le C\epsilon|\ln \epsilon| \tag{1.72}$$

(2) *If there exists* $\epsilon_0 > 0$ *such that for all* ϵ, $0 < \epsilon < \epsilon_0$ *we have*

$$\left[\sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i}\right]\left[\sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i}\right] < 0, \tag{1.73}$$

then there exists $\epsilon_1 > 0$ *such that for all* ϵ, $0 < \epsilon < \epsilon_1$ *we have either* $c^{\xi} > 0$ *or* $c^{\xi} < 0$ *for all* $\xi \in (l, 1-l)$.

*Proof of Theorem 1.13. Proof of part (1).*Observe that (1.70) implies that both terms in the brackets are of the same sign.For definitness we assume that for all sufficiently small ϵ:

$$\sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} > 0 \quad \text{and} \quad \sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} > 0.$$

We can choose constant $K > 0$ so that for all sufficiently small ϵ

$$0 < \sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i - K\beta} - \sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i + K\beta},$$
$$0 > \sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i + K\beta} - \sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i - K\beta}.$$

This implies:

$$b_0^{\xi} > 0, \qquad \text{for all } \xi \in (\frac{1}{2} + K\epsilon|\ln\epsilon|, 1 - l),$$
$$b_0^{\xi} < 0 \qquad \text{for all } \xi \in (l, \frac{1}{2} - K\epsilon|\ln\epsilon|).$$

From Lemma 1.12 it follows (taking K bigger if necessary)

$$c^{\xi} < 0 \qquad \text{for all } \xi \in (\frac{1}{2} + K\epsilon|\ln\epsilon|, 1 - l),$$
$$c^{\xi} < 0 \qquad \text{for all } \xi \in (l, \frac{1}{2} - K\epsilon|\ln\epsilon|),$$

for all sufficiently small ϵ. The proof of part (1) is now complete.

*Proof of part (2).*For the proof of the second assertion of the theorem we notice that from (1.73) it follows that the terms in brackets are of opposite signs.As in the proof of part (1) we can assume, for definitness, that:

$$\sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} > 0 \qquad \text{and} \ \sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} < 0.$$

It follows that for all $\xi \in (l, 1-l)$ we have $b_0^{\xi} > 0$, hence by the formula (1.66) in Lemma 1.12 we obtain

$$0 < c^{\xi} \qquad \text{for all } \xi \in (l, 1 - l).$$

This ends the proof of the theorem.

□

From Theorem 1.13 we immediatelly obtain:

Corollary 1.14. *Under the assumptions of part (1) of Theorem 1.13 there exists* $\hat{\xi} \in (l, 1-l)$ *and a function* $\hat{u} = u^{\hat{\xi}} + v^{\hat{\xi}}$ *such that* $\hat{u}$ *is a solution to the steady state Allen-Cahn equation (eq. (1.14)).*

The last result in this chapter deals with stability of the equilibria whose existence we have just proved.

Theorem 1.15. *Suppose that the assumptions of part (1) of Theorem 1.13 are satisfied and let* $\hat{\xi} \in (l, 1-l)$ *be such that* $c^{\hat{\xi}} = 0$.*Let further* $\hat{u} = u^{\hat{\xi}} + v^{\hat{\xi}}$ *be an equilibrium solution to (1.14) and* $\hat{\lambda}_1$ *denote the principal eigenvalue of:*

$$
\begin{aligned}
-\epsilon^2 \Delta \hat{V} + f'(\hat{u})\hat{V} &= \hat{\lambda}\hat{V}, &&\text{in } \Omega, \\
\frac{\partial \hat{V}}{\partial n} &= 0, &&\text{on } \partial\Omega.
\end{aligned}
\tag{1.74}
$$

Then the following statements hold true.

(1) *If* $c_{\xi}^{\hat{\xi}}$ *denotes the derivative of* c^{ξ} *with respect to* ξ *taken at* $\xi = \hat{\xi}$ *then*

$$
c_{\xi}^{\hat{\xi}} = b_1^{\hat{\xi}}(1 + o(1)) + r_1^{\hat{\xi}},
\tag{1.75}
$$

where

$$
|r_1^{\hat{\xi}}| \le C\epsilon^{\frac{5}{2}}\left(\epsilon^{-\frac{5}{2}+\tau}\sum_{i=1}^{4}|\kappa_i|\epsilon^{\lambda_i} + \epsilon^{-\frac{9}{2}+M} + e^{-\frac{\sigma}{\epsilon}}\right)\gamma^{\hat{\xi}},
$$

$b_1^{\hat{\xi}}$ *is defined in (1.65) and constants* $M, \tau > 0$ *are as in Lemma 1.10.*

(2) *If*

$$
\sum_{i=1}^{2}\kappa_i\Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} > 0 \quad \text{and} \quad \sum_{i=3}^{4}\kappa_i\Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} > 0,
\tag{1.76}
$$

for all sufficiently small ϵ *then* $c_{\xi}^{\hat{\xi}} < 0$.*If on the other hand*

$$
\sum_{i=1}^{2}\kappa_i\Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} < 0 \quad \text{and} \quad \sum_{i=3}^{4}\kappa_i\Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} < 0,
\tag{1.77}
$$

then $c_{\xi}^{\hat{\xi}} > 0$.

(3) *The following asymptotic formula holds*

$$\hat{\lambda}_1 = c_\xi^{\hat{\xi}} + \gamma^{\hat{\xi}} e^{-\frac{\varrho}{\epsilon}}. \tag{1.78}$$

In particular $\hat{u}$ is unstable when (1.76) holds and is stable when (1.77) holds.

Remark 1.4. It is well known from the work of Casten and Holland [Ca-H] and Matano [Ma] that equilibria of (1.14) are unstable if the domain Ω is convex.Theorem 1.15 shows that for non-convex domains both stability and instability may occur.

Proof of Theorem 1.15. Proof of part (1). Differentiating the expression (1.18) with respect to ξ yields

$$c_\xi^\xi = -\left[\frac{\partial}{\partial \xi}(\langle \mathcal{N}(v^\xi) + \mathcal{M}(w^\xi) - \lambda_1^\xi w^\xi, V_1^\xi\rangle - \epsilon^2 I^\xi)\right]\langle V_1^\xi, u_\xi^\xi\rangle^{-1} + c^\xi\langle V_1^\xi, u_\xi^\xi\rangle \frac{\partial}{\partial \xi}\langle V_1^\xi, u_\xi^\xi\rangle,$$

hence at $\xi = \hat{\xi}$ we have

$$c_\xi^{\hat{\xi}} = \epsilon^2 I_\xi^{\hat{\xi}} \langle V_1^\xi, u_\xi^\xi\rangle^{-1}\Big|_{\xi=\hat{\xi}} - \left.\frac{\frac{\partial}{\partial \xi}\langle \mathcal{N}(v^\xi) + \mathcal{M}(w^\xi) - \lambda_1^\xi w^\xi, V_1^\xi\rangle}{\langle V_1^\xi, u_\xi^\xi\rangle}\right|_{\xi=\hat{\xi}}.$$

By applying Corollary 1.11 we have

$$\epsilon^2 I_\xi^{\hat{\xi}} \langle V_1^{\hat{\xi}}, u_\xi^{\hat{\xi}}\rangle = b_1^{\hat{\xi}}(1 + o(1)) + \tilde{R}_1^{\hat{\xi}},$$

where

$$|\tilde{R}_1^{\hat{\xi}}| \leq C\epsilon^{\frac{3}{2}}\left(\epsilon^{-\frac{5}{2}+\tau}\sum_{i=1}^{4}|\kappa_i|\epsilon^{\lambda_i} + \epsilon^{-\frac{5}{2}+M}\right).$$

It sufficies to show that for $\xi = \hat{\xi}$

$$\left|\frac{\partial}{\partial \xi}\langle \mathcal{N}(v^\xi) + \mathcal{M}(w^\xi) - \lambda_1^\xi w^\xi, V_1^\xi\rangle\right| \leq C\gamma^\xi e^{-\frac{\varrho}{\epsilon}}.$$

But this last estimate follows from Lemma A and Lemma B by the argument similiar to the one in Lemma 1.12.We again make use of the improved v estimate since

$$\left|\frac{\partial}{\partial \xi}\mathcal{N}(v^\xi)\right| \leq C(|v^\xi||v_\xi^\xi| + |v^\xi|^2).$$

*Proof of part (2).*The argument we applied in the proof of the part (2) of Theorem 1.13 can be used here with only minor modifications.We omitt the details.

*Proof of part (3).*For the proof of the third assertion of the theorem we observe that since at the equilibrium $c^{\hat{\xi}} = 0$ therefore after differentiating (1.14) with respect to ξ we obtain for $\xi = \hat{\xi}$:

$$-\epsilon^2 \Delta \hat{u}_\xi + f'(\hat{u})\hat{u}_\xi = c_\xi^{\hat{\xi}} u_\xi^{\hat{\xi}} \qquad \text{in } \Omega,$$
$$\frac{\partial \hat{u}_\xi}{\partial n} = 0 \qquad \text{on } \partial\Omega.$$

From the definition of $\hat{u}$ we have:

$$-\epsilon^2 \Delta \hat{u}_\xi + f'(\hat{u})\hat{u}_\xi = c_\xi^{\hat{\xi}} \hat{u}_\xi - c_\xi^{\hat{\xi}} v_\xi^{\hat{\xi}}. \tag{1.79}$$

Setting:

$$\frac{\hat{u}_\xi}{\|\hat{u}_\xi\|_{L^2(\Omega)}} = \hat{V}, \qquad \frac{v_\xi^{\hat{\xi}} c_\xi^{\hat{\xi}}}{\|\hat{u}_\xi\|_{L^2(\Omega)}} = \hat{r},$$

we can recast (1.79) as:

$$-\epsilon^2 \Delta \hat{V} + f'(\hat{u})\hat{V} = c_\xi^{\hat{\xi}} \hat{V} + \hat{r}.$$

Notice that $c_\xi^{\hat{\xi}}$ is almost the principal eigenvalue of $\mathcal{L}^\epsilon$ linearized about $\hat{u}$.Indeed we have

$$\|\hat{r}\|_{L^2(\Omega)} \leq C\gamma^\xi e^{-\frac{C}{\epsilon}}.$$

The following lemma is needed to complete the proof:

Lemma 1.16 (The Abstract Perturbation Result). *Let A be selfadjoint operator on $\mathcal{H}$, a Hilbert space, I a compact interval in $\mathbb{R}$, $\{\psi_1, \ldots, \psi_n\}$ linearly independent normalized elements in $D(A)$.Assume that the following conditions hold true:*

(i)

$$A\psi_j = \lambda_j \psi_j + r_j, \qquad \|r_j\| \leq \epsilon',$$
$$\lambda_j \in I, \qquad j = 1, \ldots, N.$$

(ii) *There is a number $a > 0$ such that I is a-isolated in the spectrum of A:*

$$(\sigma(A) \setminus I) \cap (I + (-a, a)) = \emptyset.$$

Then

$$\vec{d}(\mathcal{E}, \mathcal{F}) = \sup_{\substack{\phi \in E \\ \|\phi\|=1}} d(\phi, \mathcal{F}) \leq \frac{N^{\frac{1}{2}}\epsilon'}{a(\lambda^{\min})},$$

where $\mathcal{E} = span\{\psi_1, \ldots, \psi_N\}$, $\mathcal{F}$ is the closed subspace of $\mathcal{H}$ associated to $\sigma(A) \cap I$ and $\lambda^{\min}$ is the smallest eigenvalue of the matrix $(\langle \psi_i, \psi_j \rangle)$.

The proof of this lemma can be found in [H-S] (see also [A-F1,2] for more details on applications). Let $A = -\epsilon^2 \Delta + f'(\hat{u})$.We first show that $\hat{\lambda}_2 \geq C\epsilon^2$.By the variational characterization of eigenvalues we have:

$$\hat{\lambda}_2 = \sup_{\phi} \inf_{v \perp \phi} \frac{\int_\Omega \epsilon^2 |\nabla v|^2 + f'(\hat{u})v^2}{\int_\Omega v^2} \geq \inf_{v \perp V_1^\xi} \frac{\int_\Omega \epsilon^2 |\nabla v|^2 + f'(\hat{u})v^2}{\int_\Omega v^2}$$

$$\geq \inf_{v \perp V_1^\xi} \frac{\int_\Omega \epsilon^2 |\nabla v|^2 + f'(u^\xi)v^2}{\int_\Omega v^2} - \sup_{v \perp V_1^\xi} \frac{\int_\Omega \left| f'(\hat{u}) - f'(u^\xi) \right| v^2}{\int_\Omega v^2}$$

$$\geq \lambda_2^{\hat{\xi}} - C\delta^{\hat{\xi}} \geq C\epsilon^2, \tag{1.80}$$

where the last inequality follows from Lemma B. Let a constant $K > 0$ be chosen such that:

$$\|\hat{r}\|_{L^2(\Omega)} (\gamma^{\hat{\xi}} e^{\frac{-K}{\epsilon}})^{-1} = o(1) \qquad \text{as } \epsilon \to 0. \tag{1.81}$$

Let $I = (c_{\hat{\xi}}^{\hat{\xi}} - \gamma^{\hat{\xi}} e^{\frac{-K}{\epsilon}}, c_{\hat{\xi}}^{\hat{\xi}} + \gamma^{\hat{\xi}} e^{\frac{-K}{\epsilon}})$ and set $a = \gamma^{\hat{\xi}} e^{\frac{-K}{\epsilon}}$.Suppose that $\hat{\lambda}_1 \notin (c_{\hat{\xi}}^{\hat{\xi}} - 2\gamma^{\hat{\xi}} e^{\frac{-K}{\epsilon}}, c_{\hat{\xi}}^{\hat{\xi}} + 2\gamma^{\hat{\xi}} e^{\frac{-K}{\epsilon}})$.This,Lemma 1.16 and (1.80) implies that the subspace $\mathcal{F}$ associated to $\sigma(A) \cap I$ contains only 0 vector hence $\vec{d}(\mathcal{E}, \mathcal{F}) = 1$.On the other hand by (1.81) we have

$$\vec{d}(\mathcal{E}, \mathcal{F}) \leq \|\hat{r}\|_{L^2(\Omega)} (\gamma^{\hat{\xi}} e^{\frac{-K}{\epsilon}})^{-1} = o(1),$$

a contradiction.The proof of the theorem is complete.

□

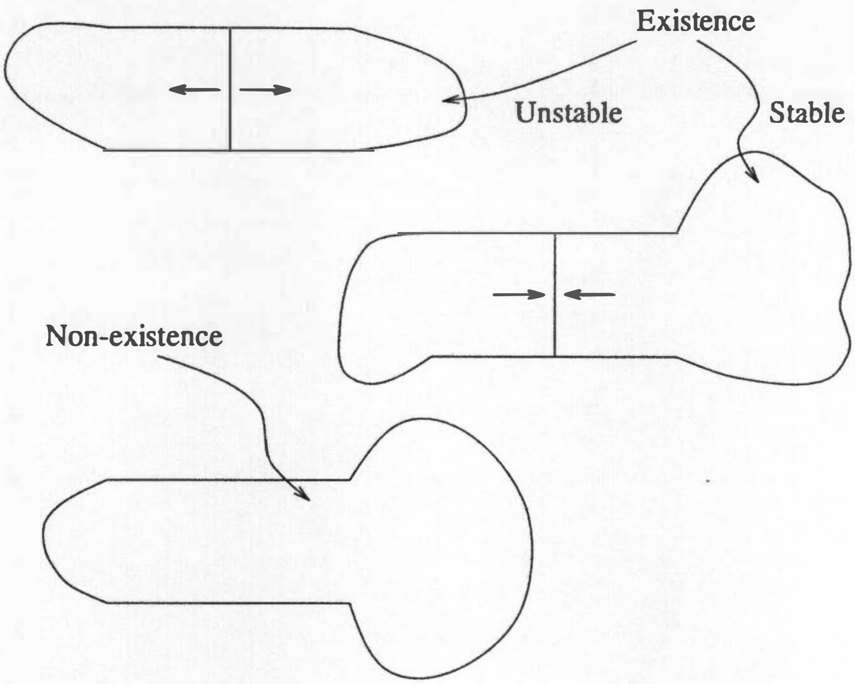

FIGURE 4. EXISTENCE AND STABILITY OF EQUILIBRIA IN DIFFERENT DOMAINS

Figure 4 illustrates some typical examples of domains to which the results of this chapter apply.

2.1 The Eigenvalue Problem-proof of Lemma B.

Before we give the proof of Lemma B we will consider an example in which the spectrum of the operator $L^{\epsilon,\xi}$ can be calculated explicitly.The importance of the result of these calculations will become apparent in the proof of Lemma 2.2 below.

The Sola-Morales Example. Let $R_\epsilon = (\epsilon, 1-\epsilon)\times(0,b)$.We claim that the estimates (1.22) hold true for $\Omega = R_\epsilon$.

Proof of the claim. We follow here [Ste]. Since $\operatorname{supp} w^\xi \subset \Omega \setminus R_\epsilon$ therefore (1.7) becomes:

$$-\epsilon^2\Delta V + f'\left[U\left(\frac{x_1-\xi}{\epsilon}\right)\right]V = \mu V \qquad \text{in } R_\epsilon,$$

$$\frac{\partial}{\partial n}V = 0 \qquad \text{on } \partial R_\epsilon.$$

The potential $f'(U)$ depends only on x_1 and we can solve this eigenvalue problem explicitly by separation of variables.Setting $V = w_1(x_1)w_2(x_2)$ we obtain:

$$-\epsilon^2 w_1'' + f'\left[U\left(\frac{x_1-\xi}{\epsilon}\right)\right] w_1 = \sigma w_1, \qquad 0 < x_1 < 1, \qquad w_1'(0) = w_1'(1) = 0,$$

$$-w_2'' = \nu w_2, \qquad 0 < x_2 < b, \qquad w_2'(0) = w_2'(b) = 0.$$

Simple calculations show that:

$$\mu_{mn} = \sigma_m + \epsilon^2 \nu_n = \sigma_m + \epsilon^2 n^2 \frac{\pi^2}{b^2} \qquad m, n = 0, 1, \ldots$$

It is well known that (see [F-H,C-P,DeM-S2,A-B-F1,2]):

$$\sigma_0 = O(\epsilon^{-5}\gamma^\xi),$$

$$\sigma_1 > C.$$

It follows that:

$$\mu_{00} = O(\epsilon^{-5}\gamma^\xi),$$

$$\mu_{01} = \epsilon^2\frac{\pi^2}{b^2} + O(\epsilon^{-\frac{5}{2}}\gamma^\xi).$$

□

For convenience we present the proof of Lemma B in several steps.

Proof of part (i) of Lemma B. For the proof of (1.21) we need two lemmas.

Lemma 2.1. *Given $l > 0$ there exist $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ and $\xi \in (l, 1-l)$ we have:*

$$\text{(2.1)} \quad \lambda_1^\xi \leq -C\epsilon^{-1}\left[\sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta\left(\frac{1-\xi}{\epsilon}\right)} + \sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta\left(\frac{\xi}{\epsilon}\right)}\right](1 + o(1))$$

Proof of Lemma 2.1. From the variational characterization of eigenvalues by choosing $U'(\frac{x_1-\xi}{\epsilon})$ as a test function we obtain:

$$\text{(2.2)} \qquad \lambda_1^\xi \leq \frac{\int_\Omega \epsilon^2 \left|\nabla U'(\frac{x_1-\xi}{\epsilon})\right|^2 + f'(u^\xi)\left|U(\frac{x_1-\xi}{\epsilon})\right|^2}{\int_\Omega \left|U'(\frac{x_1-\xi}{\epsilon})\right|^2}.$$

We can estimate the numerator in (2.2) with the help of Green's formula:

$$\begin{aligned}
&\int_\Omega \epsilon^2 \left|\nabla U'\left(\frac{x_1-\xi}{\epsilon}\right)\right|^2 + f'(u^\xi)\left|U'\left(\frac{x_1-\xi}{\epsilon}\right)\right|^2 \\
&\le \int_\Omega \left[-\epsilon^2 \Delta U'\left(\frac{x_1-\xi}{\epsilon}\right) + f'(U)U'\left(\frac{x_1-\xi}{\epsilon}\right)\right] U'\left(\frac{x_1-\xi}{\epsilon}\right) \\
&+ \int_\Omega |f'(u^\xi) - f'(U)| \left[U'\left(\frac{x_1-\xi}{\epsilon}\right)\right]^2 \\
&+ \epsilon^2 \int_{\partial\Omega} U'\left(\frac{x_1-\xi}{\epsilon}\right) \frac{\partial}{\partial n} U'\left(\frac{x_1-\xi}{\epsilon}\right) dS =: I_1 + I_2 + I_3.
\end{aligned}$$

By (1.1) $I_1 = 0$. From the asymptotic formulas for U', U'' and the definition of u^ξ it follows that I_2 can be estimated by $\gamma^\xi e^{-\frac{C}{\epsilon}}$.It sufficies now to estimate the boundary integral I_3. Since from the asymptotic formulas (1.2) only small neighborhoods of the corners matter standard computations show:

$$I_3 \le -C\left[\sum_{i=1}^{2} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta\left(\frac{1-\xi}{\epsilon}\right)} + \sum_{i=3}^{4} \kappa_i \Gamma(\lambda_i)\left(\frac{\epsilon}{2\beta}\right)^{\lambda_i} e^{-2\beta\left(\frac{\xi}{\epsilon}\right)}\right](1+o(1)). \tag{2.3}$$

On the other hand the denominator in (2.2) satisfies

$$\int_\Omega \left|U'\left(\frac{x_1-\xi}{\epsilon}\right)\right|^2 = O(\epsilon). \tag{2.4}$$

From (2.2), (2.3) and (2.4) the lemma follows.

□

Now we will establish the rest of (1.21) in Lemma B.

Lemma 2.2. *Under the hypothesis of Lemma 2.1 we have:*

$$\lambda_1^\xi \ge -C\epsilon^{-5}\gamma^\xi, \tag{2.5}$$

$$\lambda_2^\xi \ge C\epsilon^2. \tag{2.6}$$

Proof of Lemma 2.2. We decompose Ω to $\Omega = R_\epsilon \cup \Omega_L^\epsilon \cup \Omega_R^\epsilon$ where

$$R_\epsilon = (\epsilon, 1-\epsilon) \times (0, b),$$

$$\Omega_L^\epsilon = \{x \in \Omega \mid x_1 \le \epsilon\},$$

$$\Omega_R^\epsilon = \{x \in \Omega \mid x_1 \ge 1-\epsilon\}.$$

By $\mu_i^{\xi,L}, \mu_i^{\xi,R}, \mu_i^{\xi}$ we will denote the eigenvalues of (1.7) restricted to $\Omega_L^{\xi}, \Omega_R^{\xi}, R_{\epsilon}$ respectively. We rearrange the set $\{\mu_i^{\xi,L}\} \cup \{\mu_i^{\xi,R}\} \cup \{\mu_i^{\xi}\}$ in the order of increasing magnitude (counting multiplicities) and denote the resulting sequence by μ_i^*. From the Hilbert-Courant comparison principle we have:

$$\lambda_i^{\xi} \geq \mu_i^* \qquad \text{for } i = 1, 2, \ldots \tag{2.7}$$

Since for all sufficiently small ϵ and $\xi \in (l, 1-l)$ we have

$$f'(u^{\xi}) \geq \frac{1}{2}\beta^2,$$

therefore a simple argument using the Rayleigh quotient for (1.7) restricted to Ω_L^{ξ}, Ω_R^{ξ} implies:

$$\mu_1^{\xi,L} \geq \frac{1}{2}\beta^2, \qquad \mu_1^{\xi,R} \geq \frac{1}{2}\beta^2.$$

This and Lemma 2.1 shows that without loss of generality we can assume that

$$\mu_1^* = \mu_1^{\xi}, \qquad \mu_2^* = \mu_2^{\xi}.$$

From the Sola-Morales example we know that:

$$\mu_1^{\xi} \geq -C\epsilon^{-5}\gamma^{\xi}, \qquad \mu_2^{\xi} \geq C\epsilon^2.$$

The above estimates and (2.7) yield (2.5), (2.6).

□

Combining the previous two lemmas gives (1.21).

Proof of the inequalities (1.22) in Lemma B.

Lemma 2.3. *Set as in Section 1.1* $q^{\epsilon} = -\|u_{\xi}^{\xi}\|_{L^2(\Omega)}^{-1}$ *and let*

$$H^{\xi} = q^{\epsilon}u_{\xi}^{\xi}, \qquad P^{\xi} = V_1^{\xi} - H^{\xi}.$$

Under the hypothesis of the previous lemmas we have:

$$\|P^{\xi}\|_{L^2(\Omega)} \leq C\epsilon^{-2}\delta^{\xi}. \tag{2.8}$$

Proof of Lemma 2.3. We follow the proof of Proposition B in [ABF2]. We first decompose P^ξ to:

$$P^\xi = \alpha V_1^\xi + r^\xi, \qquad \text{where } \langle r^\xi, V_1^\xi \rangle = 0. \tag{2.9}$$

Applying $L^{\epsilon,\xi}$ on both sides of (2.9), multiplying by r^ξ and using the orthogonality condition for r^ξ, V_1^ξ we obtain:

$$\langle -L^{\epsilon,\xi} H^\xi, r^\xi \rangle = \langle L^{\epsilon,\xi} r^\xi, r^\xi \rangle. \tag{2.10}$$

Since

$$u_\xi^\xi = -\epsilon^{-1} U' \left(\frac{x_1 - \xi}{\epsilon} \right) + w_\xi^\xi(x) = -(1 - \omega^\epsilon(x))\, \epsilon^{-1} U' \left(\frac{x_1 - \xi}{\epsilon} \right) + \omega^\epsilon(x) \alpha_\xi^\xi(x_1),$$

therefore we get:

$$L^{\epsilon,\xi} u_\xi^\xi = \epsilon^{-1} \left(f'(u^\xi) - f'(U) \right) (1 - \omega^\epsilon) U' + f'(u^\xi) \omega^\epsilon \alpha_\xi^\xi - 2\epsilon \nabla U' \nabla \omega^\epsilon - \epsilon U' \Delta \omega^\epsilon - \epsilon^2 \Delta(\omega^\epsilon \alpha_\xi^\xi).$$

From the definiton of ω^ϵ we know that $|\operatorname{supp} \omega^\epsilon| = O(\epsilon)$ hence:

$$\|f'(u^\xi) \omega^\epsilon \alpha_\xi^\xi\|_{L^2(\Omega)} \leq \|f'(u^\xi) \alpha_\xi^\xi\|_{L^2(\operatorname{supp} \omega^\epsilon)} \leq C \epsilon^{-\frac{1}{2}} \delta^\xi.$$

In a similiar manner we can estimate each term in the expression for $L^{\epsilon,\xi} u_\xi^\xi$ and using the formula (1.13) it follows:

$$\|L^{\epsilon,\xi} H^\xi\|_{L^2(\Omega)} \leq C q^\epsilon \epsilon^{-\frac{1}{2}} \delta^\xi \leq C \delta^\xi. \tag{2.12}$$

On the other hand from Lemma 2.2 and the variational characterization of eigenvalues we obtain:

$$\langle L^{\epsilon,\xi} r^\xi, r^\xi \rangle \geq C \epsilon^2 \|r^\xi\|_{L^2(\Omega)}^2. \tag{2.13}$$

Using the Schwartz inequality on the right hand side of (2.10) and combining (2.12) and (2.13) yields:

$$\|r^\xi\|_{L^2(\Omega)} \leq C \epsilon^{-2} \delta^\xi. \tag{2.14}$$

Writing $-H^\xi = (\alpha - 1)V_1^\xi + r^\xi$ and taking L^2 norm on both sides we conclude via (2.14):

$$1 + C\epsilon^{-2}\delta^\xi \geq |1 - \alpha| \geq 1 - C\epsilon^{-2}\delta^\xi,$$

hence

$$|\alpha| \leq C\epsilon^{-2}\delta^\xi. \tag{2.15}$$

This together with (2.14) completes the proof of the lemma.

□

To finish the proof of (1.22) we need the following:

Corollary 2.4. *With the assumptions as in Lemma 2.3 we have:*

$$\langle V_1^\xi, H^\xi \rangle = 1 + O(\epsilon^{-2}\delta^\xi),$$
$$\|P^\xi\|_{W^{2,2}(\Omega)} \leq C\epsilon^{-4}\delta^\xi. \tag{2.16}$$

In particular for each $l', 0 < l' < l$ there exists a constant K such that $|x_1 - \xi| \leq l'$ implies:

$$V_1^\xi(x) = q^\epsilon u_\xi^\xi(x)\big(1 + O(e^{-\frac{K}{\epsilon}})\big). \tag{2.17}$$

Proof. The first formula in (2.16) follows immediately from (2.15).For the other estimate we observe that P^ξ satisfies:

$$L^{\epsilon,\xi}P^\xi = \lambda_1^\xi V_1^\xi - L^{\epsilon,\xi}H^\xi \qquad \text{in } \Omega,$$
$$\frac{\partial P^\xi}{\partial n} = 0 \qquad \text{on } \partial\Omega.$$

Using Lemma 2.3 and (2.12) we get:

$$\|\lambda_1^\xi V_1^\xi - L^{\epsilon,\xi}H^\xi\|_{L^2(\Omega)} \leq C\delta^\xi \tag{2.18}$$

To estimate P^ξ we employ the second fundamental inequality (1.10) applied to the operator $L^{\epsilon,\xi}$ and estimates (2.8), (2.18):

$$\|P^\xi\|_{W^{2,2}(\Omega)} \leq C\epsilon^{-2}(\|L^{\epsilon,\xi}P^\xi\|_{L^2(\Omega)} + \|P^\xi\|_{L^2(\Omega)}) \leq C\epsilon^{-4}\delta^\xi.$$

The second assertion of the lemma follows from (2.16) and the Sobolev embedding $W^{2,2}(\Omega) \hookrightarrow C^0(\Omega)$.

$\square$

The proof of the part (i) of the Lemma B is complete.

Proof of part (ii) of Lemma B.

Fix $l', 0 < l' < l$ and set:

$$R^\xi = \{x = (x_1, x_2) \in \Omega \mid |x_1 - \xi| \leq l'\},$$

$$D_L^\xi = \{x = (x_1, x_2) \in \Omega \mid x_1 \leq \xi - l'\},$$

$$D_R^\xi = \{x = (x_1, x_2) \in \Omega \mid x_1 \geq \xi + l'\}.$$

From the previous lemma and the asymptotic formulas for U' we can easily show that the estimate (1.23) holds for $x \in R^\xi$.Our goal is to extend it to the rest of Ω. We shall use the idea of Agmon [Ag1,2] and Hellfer-Sjöstrand [H-S] and estimate the expression $V_1^\xi e^{\beta \frac{|x_1-\xi|}{\epsilon}}$. Let $\beta^- = \beta - \epsilon$.Straightforward calculations give:

$$\epsilon^2 \int_{D_L^\xi \cup D_R^\xi} \left|\nabla \left(V_1^\xi e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}\right)\right|^2 + \int_{D_L^\xi \cup D_R^\xi} \left(V_1^\xi e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}\right)^2 \left(f'(u^\xi) - \lambda_1^\xi - (\beta^-)^2\right)$$
$$= \epsilon^2 \int_{\partial(D_L^\xi \cup D_R^\xi)} e^{2\beta^- \frac{|x_1-\xi|}{\epsilon}} V_1^\xi \frac{\partial}{\partial n} V_1^\xi \, dS.$$

(2.19)

Since V_1^ξ satisfies the Neumann boundary conditions it follows from (2.17):

$$\epsilon^2 \int_{\partial(D_L^\xi \cup D_R^\xi)} e^{2\beta^- \frac{|x_1-\xi|}{\epsilon}} V_1^\xi \frac{\partial}{\partial n} V_1^\xi \, dS = \epsilon^2 \int_{\partial R^\xi} e^{2\beta^- \frac{|x_1-\xi|}{\epsilon}} V_1^\xi \frac{\partial}{\partial n} V_1^\xi \, dS$$
$$= \epsilon^2 \int_{\partial R^\xi} \left[P^\xi \frac{\partial}{\partial n} P^\xi + q^\epsilon u_\xi^\xi \frac{\partial}{\partial n} P^\xi + P^\xi \frac{\partial}{\partial n}(q^\epsilon u_\xi^\xi) + (q^\epsilon u_\xi^\xi) \frac{\partial}{\partial n}(q^\epsilon u_\xi^\xi)\right] e^{2\beta^- \frac{|x_1-\xi|}{\epsilon}} \, dS : = \sum_{i=1}^4 I_i,$$

where P^ξ is defined in Lemma 2.3.Utilizing the interpolation inequality (1.11) with $m = 2$ and the Corollary (2.4) we get

$$|I_1| \leq e^{2\beta^- \frac{l'}{\epsilon}} \|P^\xi\|_{L^2(\partial R^\xi)} \|\frac{\partial}{\partial n} P^\xi\|_{L^2(\partial R^\xi)} \leq C e^{2\beta^- \frac{l'}{\epsilon}} \|P^\xi\|^2_{W^{2,2}(\Omega)} \leq C.$$

The other integrals I_i can be treated similiarly so that we obtain $|I_i| \leq C, i = 1, \dots, 4$, hence from $0 < f'(u^\xi) - \lambda_1^\xi - (\beta^-)^2 = O(\epsilon)$ for $|x_1 - \xi| > l'$ it follows

$$\|V_1^\xi e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}\|_{L^2(D_L^\xi \cup D_R^\xi)} \leq C\epsilon^{-\frac{1}{2}},$$

$$\|\nabla \left(V_1^\xi e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}\right)\|_{L^2(D_L^\xi \cup D_R^\xi)} \leq C\epsilon^{-1}. \tag{2.20}$$

Since

$$\Delta \left(V_1^\xi e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}\right) = \epsilon^{-2} \left(\lambda_1^\xi - f'(u^\xi)\right) V_1^\xi e^{\beta^- \frac{|x_1-\xi|}{\epsilon}} + 2\nabla V_1^\xi \nabla e^{\beta^- \frac{|x_1-\xi|}{\epsilon}} + V_1^\xi \Delta e^{\beta^- \frac{|x_1-\xi|}{\epsilon}},$$

in Ω,

$$\frac{\partial}{\partial n} V_1^\xi e^{\beta^- \frac{|x_1-\xi|}{\epsilon}} - V_1^\xi e^{\beta^- \frac{|x_1-\xi|}{\epsilon}} \frac{\partial}{\partial n}\left(\beta^- \frac{|x_1-\xi|}{\epsilon}\right) = 0, \qquad \text{on } \partial\Omega,$$

therefore another application of the second fundamental inequality and the Sobolev embedding implies the required estimates.

□

Proof of part (iii) of Lemma B. We shall denote:

$$L_\xi^{\epsilon,\xi} = \left[-\epsilon^2 \Delta + f'(u^\xi)\right] \frac{\partial}{\partial \xi} + f''(u^\xi) u_\xi^\xi.$$

We first observe that from classical perturbation theory (λ_i^ξ, V_i^ξ) are differentiable functions of the parameter ξ (see [K] chp.VII, sec.2).This follows from the differentiability properties of $f'(u^\xi)$ as a function of ξ.

Proof of the estimate for $\lambda_{1,\xi}^\xi$. Differentiating both sides of the equation (1.7) with respect to ξ we obtain the following problem for $(\lambda_{1,\xi}^\xi, V_{1,\xi}^\xi)$:

$$L_\xi^{\epsilon,\xi} V_1^\xi = \lambda_{1,\xi}^\xi V_1^\xi + \lambda_1^\xi V_{1,\xi}^\xi \qquad \text{in } \Omega,$$

$$\frac{\partial V_{1,\xi}^\xi}{\partial n} = 0 \qquad \text{on } \partial\Omega. \tag{2.21}$$

Notice that $\langle V_1^\xi, V_{1,\xi}^\xi\rangle = 0 = \langle H_\xi^\xi, H^\xi\rangle$ where $H^\xi = q^\epsilon u_\xi^\xi$. Multiplying both sides of (2.21) by V_1^ξ and integrating over Ω we obtain:

$$\langle L_\xi^{\epsilon,\xi} V_1^\xi, V_1^\xi\rangle = \langle L^{\epsilon,\xi} V_{1,\xi}^\xi + f''(u^\xi) u_\xi^\xi V_1^\xi, V_1^\xi\rangle$$
$$= \langle \lambda_{1,\xi}^\xi V_1^\xi + \lambda_1^\xi V_{1,\xi}^\xi, V_1^\xi\rangle = \lambda_{1,\xi}^\xi.$$

Orthogonality of V_1^ξ and $V_{1,\xi}^\xi$ implies that $\langle L^{\epsilon,\xi} V_{1,\xi}^\xi, V_1^\xi\rangle = 0$, hence:

$$\lambda_{1,\xi}^\xi = \langle f''(u^\xi) u_\xi^\xi V_1^\xi, V_1^\xi\rangle. \tag{2.22}$$

For $R_\epsilon = (\epsilon, 1-\epsilon) \times (0, b)$ we set:

$$\langle u, v\rangle_\epsilon = \int_{R_\epsilon} uv.$$

From (2.22) and (1.23) it follows:

$$|\lambda_{1,\xi}^\xi| \le \int_{\Omega\setminus R_\epsilon} f''(u^\xi) u_\xi^\xi \left(V_1^\xi\right)^2 + \langle f''(u^\xi) u_\xi^\xi V_1^\xi, V_1^\xi\rangle_\epsilon$$
$$\le C\epsilon^{-5}\gamma^\xi\delta^\xi + \langle f''(u^\xi) u_\xi^\xi V_1^\xi, V_1^\xi\rangle_\epsilon.$$

On R_ϵ we have $u^\xi(x) = U\left(\frac{x_1-\xi}{\epsilon}\right)$.It is convenient to denote $U^\xi = U\left(\frac{x_1-\xi}{\epsilon}\right)$.Since U^ξ satisfies $-\epsilon^2 \Delta U^\xi + f(U^\xi) = 0$ therefore after differentiating this last equation twice with respect to ξ we obtain:

$$L_\xi^{\epsilon,\xi} U_\xi^\xi = L^{\epsilon,\xi} U_{\xi\xi}^\xi + f''(U^\xi)(U_\xi^\xi)^2 = 0 \qquad \text{in } R_\epsilon.$$

This and the decomposition $P^\xi = V_1^\xi - H^\xi$ as in Lemma 2.3 yields:

$$\langle f''(U^\xi) U_\xi^\xi P^\xi, V_1^\xi\rangle_\epsilon = \langle f''(U^\xi) U_\xi^\xi V_1^\xi, V_1^\xi\rangle_\epsilon + \langle f''(U^\xi) H^\xi U_\xi^\xi, V_1^\xi\rangle_\epsilon$$
$$= \langle f''(U^\xi) U_\xi^\xi, (P^\xi)^2\rangle_\epsilon + q^\epsilon \langle f''(U^\xi)(U_\xi^\xi)^2, P^\xi + V_1^\xi\rangle_\epsilon$$
$$= \langle f''(U^\xi) U_\xi^\xi, (P^\xi)^2\rangle_\epsilon - q^\epsilon \langle L^{\epsilon,\xi} U_{\xi\xi}^\xi, P^\xi + V_1^\xi\rangle_\epsilon. \tag{2.23}$$

From the estimate (2.8) we have:

$$\langle f''(U^\xi) U_\xi^\xi, (P^\xi)^2\rangle_\epsilon \le C\epsilon^{-\frac{9}{2}}\gamma^\xi. \tag{2.24}$$

Integration by parts yields the estimate for the second term in (2.23):

$$\begin{aligned}\left|q^\epsilon (L^{\epsilon,\xi}U^\xi_\xi, P^\xi + V_1^\xi\rangle_\epsilon\right| &= \left|q^\epsilon\langle U^\xi_{\xi\xi}, L^{\epsilon,\xi}(P^\xi + V_1^\xi)\rangle_\epsilon\right| \\ &+ \left|q^\epsilon\epsilon^2 \int_{\partial R_\epsilon} [U^\xi_{\xi\xi}\frac{\partial}{\partial n}(V_1^\xi + P^\xi) - (V_1^\xi + P^\xi)\frac{\partial}{\partial n}U^\xi_{\xi\xi}]\, dS\right| \\ &\le \left|q^\epsilon\langle U^\xi_{\xi\xi}, 2\lambda_1^\xi V_1^\xi\rangle_\epsilon\right| + C\epsilon^{-5}\gamma^\xi \le C\epsilon^{-6}\gamma^\xi,\end{aligned}$$

where we have made use of $V_1^\xi + P^\xi = H^\xi + 2P^\xi$ and (1.11) The last inequality combined with (2.24) gives (1.24 a).

Proof of (1.24 b). From the decomposition $P^\xi = V_1^\xi - H^\xi$ as we have:

$$P^\xi_\xi = V^\xi_{1,\xi} - H^\xi_\xi = V^\xi_{1,\xi} - \frac{\partial}{\partial\xi}(q^\epsilon u^\xi_\xi),$$

hence

$$\begin{aligned} L^{\epsilon,\xi}P^\xi_\xi &= L^{\epsilon,\xi}(V^\xi_{1,\xi} - q^\epsilon u^\xi_{\xi\xi}) - q^\epsilon_\xi L^{\epsilon,\xi}u^\xi_\xi \\ &= -\frac{\partial}{\partial\xi}(q^\epsilon L^{\epsilon,\xi}u^\xi_\xi) - f''(u^\xi)u^\xi_\xi P^\xi + \lambda_1^\xi V^\xi_{1,\xi} + \lambda^\xi_{1,\xi}V_1^\xi. \end{aligned} \tag{2.25}$$

Let $P^\xi_\xi = \alpha^\xi V_1^\xi + Q^\xi$ with $\langle V_1^\xi, Q^\xi\rangle = 0$. We first estimate α^ξ:

$$\begin{aligned}\langle P^\xi_\xi, V_1^\xi\rangle = \alpha^\xi &= -\langle H^\xi_\xi, V_1^\xi\rangle \\ &= -\langle H^\xi_\xi, V_1^\xi - H^\xi\rangle = -\langle H^\xi_\xi, P^\xi\rangle\end{aligned}$$

From the formula for q^ϵ we get:

$$\begin{aligned}|q^\epsilon_\xi| &= \left|-\langle u^\xi_\xi, u^\xi_{\xi\xi}\rangle(q^\epsilon)^2\right| \le (q^\epsilon)^2\left|\langle u^\xi_\xi, u^\xi_{\xi\xi}\rangle_\epsilon\right| + C\epsilon^{-2}\gamma^\xi \\ &\le C\epsilon^{-2}\gamma^\xi.\end{aligned} \tag{2.26}$$

where the last inequality follows from $\int_{-\infty}^{\infty} U'U'' = 0$ and the asymptotic formulas (1.2). Consequently:

$$|\alpha^\xi| \le \left(|q^\epsilon|\|u^\xi_{\xi\xi}\|_{L^2(\Omega)} + |q^\epsilon_\xi|\|u^\xi_\xi\|_{L^2(\Omega)}\right)\|P^\xi\|_{L^2(\Omega)} \le C\epsilon^{-1}\|P^\xi\|_{L^2(\Omega)}. \tag{2.27}$$

From the representation formula for P_ξ^ξ we obtain:

$$\langle L^{\epsilon,\xi} P_\xi^\xi, Q^\xi \rangle = \langle L^{\epsilon,\xi} Q^\xi, Q^\xi \rangle.$$

Orthogonality of V_1^ξ and Q^ξ yields as in Lemma 2.3:

$$\langle L^{\epsilon,\xi} Q^\xi, Q^\xi \rangle \geq C\epsilon^2 \|Q^\xi\|^2_{L^2(\Omega)}. \tag{2.28}$$

On the other hand from (2.25):

$$\langle L^{\epsilon,\xi} P_\xi^\xi, Q^\xi \rangle = -\langle \frac{\partial}{\partial \xi}\left(q^\epsilon L^{\epsilon,\xi} u_\xi^\xi\right), Q^\xi \rangle - \langle f''(u^\xi) u_\xi^\xi P^\xi, Q^\xi \rangle + \langle \dot\lambda_1^\xi V_{1,\xi}^\xi, Q^\xi \rangle. \tag{2.29}$$

Differentiating $L^{\epsilon,\xi} u_\xi^\xi = L^{\epsilon,\xi} w_\xi^\xi + \left(f'(u^\xi) - f'(U^\xi)\right) U_\xi^\xi$ with respect to ξ and using (2.26) we obtain:

$$\left| \langle \frac{\partial}{\partial \xi}\left(q^\epsilon L^{\epsilon,\xi} u_\xi^\xi\right), Q^\xi \rangle \right| \leq C\epsilon^{-2}\delta^\xi \|Q^\xi\|_{L^2(\Omega)}. \tag{2.30}$$

For the second term in (2.29) we get:

$$\left| \langle f''(u^\xi) u_\xi^\xi P^\xi, Q^\xi \rangle \right| \leq C\epsilon^{-1} \|P^\xi\|_{L^2(\Omega)} \|Q^\xi\|_{L^2(\Omega)}. \tag{2.31}$$

Finally for the last term in (2.29) we have:

$$\begin{aligned} \left| \langle \dot\lambda_1^\xi V_{1,\xi}^\xi, Q^\xi \rangle \right| &= |\lambda_1^\xi| \left| \langle P_\xi^\xi + H_\xi^\xi, Q^\xi \rangle \right| \\ &= |\lambda_1^\xi| \left| \langle Q^\xi, Q^\xi \rangle + \langle H_\xi^\xi, Q^\xi \rangle \right| \leq |\lambda_1^\xi| \|Q^\xi\|^2_{L^2(\Omega)} + C\epsilon^{-6}\gamma^\xi \|Q^\xi\|_{L^2(\Omega)}. \end{aligned} \tag{2.32}$$

Combining (2.27)-(2.32) yields:

$$\|Q^\xi\|_{L^2(\Omega)} \leq C\epsilon^{-5}\delta^\xi. \tag{2.33}$$

This and (2.26) completes the proof of (1.24 b). Argument analogous to the one in Lemma 2.3 implies the inequality (1.24 c) as well.We omitt the details.

□

Proof of (1.28 d). The expression $V_{1,\xi}^{\xi} e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}$ can be treated quite similiarly as the expression $V_1^{\xi} e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}$ in the proof of part (ii) of Lemma B.Notice that in this case the identity (2.19) requires slight modification since we have an extra term $(\lambda_{1,\xi}^{\xi} - f''(u^{\xi})u_{\xi}^{\xi})V_1^{\xi}$.In fact:

$$\epsilon^2 \int_{D_L^{\xi}\cup D_R^{\xi}} \left|\nabla \left(V_{1,\xi}^{\xi} e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}\right)\right|^2 + \int_{D_L^{\xi}\cup D_R^{\xi}} \left(V_{1,\xi}^{\xi} e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}\right)^2 \left(f'(u^{\xi}) - \lambda_1^{\xi} - (\beta^-)^2\right)$$
$$= \epsilon^2 \int_{\partial(D_L^{\xi}\cup D_R^{\xi})} e^{2\beta^- \frac{|x_1-\xi|}{\epsilon}} V_{1,\xi}^{\xi} \frac{\partial}{\partial n} V_{1,\xi}^{\xi}\, dS + \int_{(D_L^{\xi}\cup D_R^{\xi})} \left(\lambda_{1,\xi}^{\xi} - f''(u^{\xi})u_{\xi}^{\xi}\right) \left(V_1^{\xi} e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}\right)^2.$$

However the extra term in the above expression as well as the boundary integral can be estimated by using (2.20) and the inequality (1.24 c):

$$\left|\epsilon^2 \int_{\partial(D_L^{\xi}\cup D_R^{\xi})} e^{2\beta^- \frac{|x_1-\xi|}{\epsilon}} V_{1,\xi}^{\xi} \frac{\partial}{\partial n} V_{1,\xi}^{\xi}\, dS + \int_{(D_L^{\xi}\cup D_R^{\xi})} \left(\lambda_{1,\xi}^{\xi} - f''(u^{\xi})u_{\xi}^{\xi}\right) \left(V_1^{\xi} e^{\beta^- \frac{|x_1-\xi|}{\epsilon}}\right)^2\right|$$
$$\leq C\epsilon^{-2}.$$

From this point we can follow the proof of part (ii) of Lemma B to complete the proof of (1.24).

□

3.1 The Quasi Invariant Manifold-proof of Lemma A.

Recall from Section 1.2 that we are after a pair $(v^{\xi}, c^{\xi}) \in W^{2,2}(\Omega) \times \mathbb{R}$ such that:

$$L^{\epsilon,\xi} v^{\xi} = c^{\xi} u_{\xi}^{\xi} - L^{\epsilon,\xi} w^{\xi} + \mathcal{N}(v^{\xi}) + \mathcal{M}(w^{\xi}) \qquad \text{in } \Omega,$$
$$\langle V_1^{\xi}, v^{\xi}\rangle = 0,$$
$$\frac{\partial}{\partial n} v^{\xi} = 0 \qquad \text{on } \partial\Omega, \tag{3.1}$$

where

$$\mathcal{N}(v^{\xi}) = -f(u^{\xi} + v^{\xi}) + f(u^{\xi}) + f'(u^{\xi})v^{\xi},$$
$$\mathcal{M}(w^{\xi}) = f(U) - f(U + w^{\xi}) + f'(U + w^{\xi})w^{\xi}. \tag{3.2}$$

Set for $w \in W^{2,2}(\Omega)$:

$$c^{\xi}(w) = -\frac{\langle -L^{\epsilon,\xi} w^{\xi} + \mathcal{N}(w) + \mathcal{M}(w^{\xi}), V_1^{\xi}\rangle}{\langle V_1^{\xi}, u_{\xi}^{\xi}\rangle},$$
$$\Phi^{\xi}(w) = c^{\xi}(w) u_{\xi}^{\xi} - L^{\epsilon,\xi} w^{\xi} + \mathcal{N}(w) + \mathcal{M}(w^{\xi}). \tag{3.3}$$

Notice that $\langle \Phi^\xi(w), V_1^\xi \rangle = 0$. Let $\mathcal{K}^\xi, \xi \in (l, 1-l)$ be a one parameter family of maps $\mathcal{K}^\xi : W^{2,2}(\Omega) \to W^{2,2}(\Omega)$ defined by

$$\mathcal{K}^\xi(w) = v \qquad \text{if and only if } v \in W^{2,2}(\Omega) \qquad \text{and } L^{\epsilon,\xi} v = \Phi^\xi(w).$$

It is clear that v^ξ is a solution to (3.1) if and only if $\mathcal{K}^\xi(v^\xi) = v^\xi$ and therefore the idea is to prove Lemma A by applying the Banach Contraction Mapping theorem.We first need a technical lemma:

Lemma 3.1. *Let $\xi \in (l, 1-l)$ and $w \in W^{2,2}(\Omega)$.We have the following estimates:*

$$\|L^{\epsilon,\xi} w^\xi\|_{L^2(\Omega)} \leq C\delta^\xi,$$

$$\|\mathcal{M}(w^\xi)\|_{L^2(\Omega)} \leq C\gamma^\xi,$$

$$\|\mathcal{N}(w)\|_{L^2(\Omega)} \leq C\|w\|^2_{W^{2,2}(\Omega)},$$

$$|c^\xi(w)| \leq C\epsilon^{\frac{1}{2}}\|w\|^2_{W^{2,2}(\Omega)} + C\epsilon^{-4}\gamma^\xi. \tag{3.4}$$

Proof of Lemma 3.1. The first two estimates in (3.4) follow from (1.4) and (1.5) and the definition of w^ξ and $\mathcal{M}$.For the third estimate we notice that $\mathcal{N}(w)$ is a "quadratic" function of w, hence we have:

$$\|\mathcal{N}(w)\|_{L^2(\Omega)} \leq \sup\left(|f''|^2\right) \|w\|^2_{C^0(\Omega)} \leq C\|w\|^2_{W^{2,2}(\Omega)}.$$

The last estimate follows from

$$\left|\langle L^{\epsilon,\xi} w^\xi, V_1^\xi \rangle\right| \leq C\epsilon^{-\frac{5}{2}}\gamma^\xi,$$

the first estimate in (3.4), (1.23) and $\operatorname{supp} w^\xi \cap (\epsilon, 1-\epsilon) \times (0, b) = \emptyset$.

□

The following lemma is a simple consequence of the theory of elliptic PDE and the Fredholm Alternative (see also Section 1.1 of this chapter).

Lemma 3.2. *For each $w \in W^{2,2}(\Omega)$ and $\xi \in (l, 1-l)$ there exists a unique pair $(c^{\xi}(w), v)$ such that c^{ξ} is determined from the formula (3.3) and $L^{\epsilon,\xi}v = \Phi^{\xi}(w)$. In particular the map $\mathcal{K}^{\xi}$ is well defined as a map from $W^{2,2}(\Omega)$ to the subspace of $W^{2,2}(\Omega)$ consisting of functions orthogonal (in L^2 inner product) to V_1^{ξ}.*

Set:

$$B(w,\rho) = \{u \in W^{2,2}(\Omega) \mid \|u - w\|_{W^{2,2}(\Omega)} < \rho\},$$
$$B(\rho) = B(0,\rho),$$
$$\rho^{\xi} = \epsilon^{-4}\delta^{\xi}.$$

As a first step in showing the existence of a fixed point for $\mathcal{K}^{\xi}$ we establish:

Lemma 3.3. *There exists $\epsilon_0 > 0$ such that for each $\epsilon < \epsilon_0$ we have:*

$$\mathcal{K}^{\xi}\left(B(\rho^{\xi})\right) \subset B(\rho^{\xi})$$

for all $\xi \in (l, 1-l)$.

Proof of Lemma 3.3. Let $w \in B(\rho^{\xi})$ be fixed and let $v = \mathcal{K}^{\xi}(w)$.From Lemma 3.2 we have $L^{\epsilon,\xi}v = \Phi^{\xi}(w)$.It sufficies to show $v \in B(\rho^{\xi})$.Since:

$$\langle L^{\epsilon,\xi}v, v\rangle = \langle \Phi^{\xi}(w), v\rangle,$$

therefore from the variational characterization of eigenvalues,orthogonality condition for v and Lemma B (1.21 b) we get:

$$\langle L^{\epsilon,\xi}v, v\rangle \geq \lambda_2^{\xi}\|v\|^2_{L^2(\Omega)}. \tag{3.5}$$

From Lemma 3.1 we obtain:

$$\begin{aligned} \|\Phi^{\xi}(w)\|_{L^2(\Omega)} &\leq |c^{\xi}(w)|\|u^{\xi}_{\xi}\|_{L^2(\Omega)} + \|\mathcal{N}(w)\|_{L^2(\Omega)} + \|\mathcal{M}(w^{\xi})\|_{L^2(\Omega)} + \|L^{\epsilon,\xi}w^{\xi}\|_{L^2(\Omega)} \\ &\leq \left(C\epsilon^{\frac{1}{2}}\|w\|^2_{W^{2,2}(\Omega)} + C\epsilon^{-5}\gamma^{\xi} + \|w^{\xi}\|_{L^2(\Omega)} + C\delta^{\xi}\right) \leq C\delta^{\xi}, \end{aligned} \tag{3.6}$$

hence

$$\left|\langle \Phi^\xi(w), v\rangle\right| \leq C\delta^\xi \|v\|_{L^2(\Omega)}.$$

Consequently:

$$\|v\|_{L^2(\Omega)} \leq C(\lambda_2^\xi)^{-1}\delta^\xi \tag{3.7}$$

From (3.5), (3.6) we also get by the second fundamental inequality (1.10):

(3.8)

$$\|v\|_{W^{2,2}(\Omega)} \leq C\epsilon^{-2}\left(\|L^{\epsilon,\xi}v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right) \leq C\epsilon^{-2}\left(\|\Phi^\xi(w)\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right) \leq C\epsilon^{-4}\delta^\xi,$$

hence the lemma follows.

□

Remark 3.1. By the assumption (F4) if w_1, w_2 are sufficiently small then $|\mathcal{N}(w_1) - \mathcal{N}(w_2)| \leq F_0|w_1 - w_2|\big(|w_1| + |w_2|\big)$.This inequality will be used in the next lemma.

Lemma 3.4. *For each sufficiently small ϵ the map $\mathcal{K}^\xi$ restricted to $B(\rho^\xi)$ is a uniform contraction map, namely for each $w_1, w_2 \in B(\rho^\xi)$:*

$$\|\mathcal{K}^\xi(w_1) - \mathcal{K}^\xi(w_2)\|_{W^{2,2}(\Omega)} \leq \theta\|w_1 - w_2\|_{W^{2,2}(\Omega)} \tag{3.9}$$

where $0 < \theta < 1$ can be chosen uniformly for $\xi \in (l, 1-l)$.

Proof of Lemma 3.4. Let ξ be fixed and for given w_1, w_2 let $\mathcal{K}^\xi(w_i) = v_i, i = 1,2$.From the definiton of $\mathcal{K}^\xi$ we have:

$$c^\xi(w_1) - c^\xi(w_2) = -\frac{\langle \mathcal{N}(w_1) - \mathcal{N}(w_2), V_1^\xi\rangle}{\langle V_1^\xi, u_\xi^\xi\rangle},$$

$$L^{\epsilon,\xi}(v_1 - v_2) = \Phi^\xi(w_1) - \Phi^\xi(w_2) = \left(c^\xi(w_1) - c^\xi(w_2)\right) + \mathcal{N}(w_1) - \mathcal{N}(w_2). \tag{3.10}$$

Set $\delta w = w_1 - w_2, \delta v = v_1 - v_2$.From Remark 3.1 we get:

$$\|\mathcal{N}(w_1) - \mathcal{N}(w_2)\|_{L^2(\Omega)} \leq C\|\delta w\|_{L^2(\Omega)}\left(\|w_1\|_{L^2(\Omega)} + \|w_2\|_{L^2(\Omega)}\right),$$

$$|c^\xi(w_1) - c^\xi(w_2)| \leq C\epsilon^{\frac{1}{2}}\|\delta w\|_{L^2(\Omega)}\left(\|w_1\|_{L^2(\Omega)} + \|w_2\|_{L^2(\Omega)}\right)\|V_1^\xi\|_{C^0(\Omega)},$$

hence,

$$\|\Phi^{\xi}(w_1) - \Phi^{\xi}(w_2)\|_{L^2(\Omega)} \leq C\|\delta w\|_{L^2(\Omega)} \left(\|w_1\|_{L^2(\Omega)} + \|w_2\|_{L^2(\Omega)}\right).$$

Multiplying both sides of the second equation in (3.10) by δv and integrating by parts over Ω we get after arguing as in the previous lemma:

$$\|\delta v\|_{L^2(\Omega)} \leq C(\lambda_2^{\xi})^{-1}\|\delta w\|_{L^2(\Omega)} \left(\|w_1\|_{L^2(\Omega)} + \|w_2\|_{L^2(\Omega)}\right),$$

hence

$$\begin{aligned}\|\delta v\|_{W^{2,2}(\Omega)} &\leq C\epsilon^{-2} \left(\|\Phi^{\xi}(w_1) - \Phi^{\xi}(w_2)\|_{L^2(\Omega)} + \|\delta v\|_{L^2(\Omega)}\right) \\ &\leq C\epsilon^{-4}\|\delta w\|_{L^2(\Omega)} \left(\|w_1\|_{L^2(\Omega)} + \|w_2\|_{L^2(\Omega)}\right) \leq C\epsilon^{-4}\rho^{\xi}\|\delta w\|_{L^2(\Omega)}.\end{aligned}$$

and the lemma follows.Notice that since $l > 0$ is independent on ϵ therefore θ in the assertion of the lemma can be chosen uniformly for all ξ.

□

The proof of Lemma A will be given in two parts.First we shall establish all but the assertion (1.16 c) of this lemma.Proof of the improved v estimate will be presented separetely since it requires different techniques and is more involved.

Lemma 3.5 (first part of Lemma A). *For each $\xi \in (l, 1-l)$ and all ϵ sufficiently small there exists a unique pair (v^{ξ}, c^{ξ}) such that:*

$$\mathcal{K}^{\xi}(v^{\xi}) = v^{\xi},$$

$$c^{\xi} = c^{\xi}(v^{\xi}). \tag{3.11}$$

Moreover we have estimates:

$$\|v^{\xi}\|_{W^{2,2}(\Omega)} \leq C\epsilon^{-4}\delta^{\xi}, \tag{3.12 a}$$

$$|c^{\xi}| \leq C\epsilon^{-\frac{15}{2}}\gamma^{\xi}. \tag{3.12 b}$$

In addition (v^ξ, c^ξ) *are differentiable with respect to* ξ *and we have:*

$$\|v^\xi_\xi\|_{W^{2,2}(\Omega)} \le C\epsilon^{-9}\delta^\xi, \tag{3.13 a}$$

$$|c^\xi_\xi| \le C\epsilon^{-17}\gamma^\xi. \tag{3.13 b}$$

Proof of Lemma 3.5. The existence and uniqueness part of the lemma is a consequence of the Banach Contraction Mapping Theorem and Lemma 3.4.(3.12) follows from Lemma 3.3 (estimate (3.12 a)) and Lemma 3.1 (estimate on (3.12 b)).Since the constant θ in the statement of Lemma 3.4 can be chosen uniformly with respect to $\xi \in (l, 1-l)$ therefore it follows from the results in [H,He] that v^ξ is continuous with respect to ξ hence c^ξ is continuous as well.In order to show differentiability it sufficies to show that $\mathcal{K}^\xi(w)$ is Frechet differentiable with respect to ξ (see [He]).Fix $\xi \in (l, 1-l)$, $w \in B(\rho^\xi)$ and $v = \mathcal{K}^\xi(w)$.Set $\Delta^{\xi,h}u(\xi) = u(\xi + h) - u(\xi)$. We first consider $\Delta^{\xi,h}\Phi^\xi(w)$.From Lemma B we obtain:

$$\left|\frac{\partial}{\partial\xi}\langle u^\xi_\xi, V^\xi_1\rangle\right| \le C\epsilon^{-\frac{11}{2}}\delta^\xi.$$

Straightforward calculations show:

$$\|\Delta^{\xi,h}\mathcal{N}(w)\|_{L^2(\Omega)} \le C\epsilon^{-1}\|w\|^2_{W^{2,2}(\Omega)}h,$$

$$\|\Delta^{\xi,h}L^{\epsilon,\xi}w^\xi\|_{L^2(\Omega)} \le C\epsilon^{-1}\delta^\xi,$$

$$\|\Delta^{\xi,h}\mathcal{M}(w^\xi)\|_{L^2(\Omega)} \le C\epsilon^{-1}\gamma^\xi.$$

Consequently:

$$\|\Delta^{\xi,h}\Phi^\xi(w)\|_{L^2(\Omega)} \le C\epsilon^{-1}\delta^\xi h. \tag{3.14}$$

For $\Delta^{\xi,h}\mathcal{K}^\xi(w) = \Delta^{\xi,h}v$ we have:

$$L^{\epsilon,\xi}\Delta^{\xi,h}v = -\Delta^{\xi,h}f'(u^\xi)(\Delta^{\xi,h}v - v) + \Delta^{\xi,h}\Phi^\xi(w). \tag{3.15}$$

From (3.7), (3.14) we get:

$$\|L^{\epsilon,\xi}\Delta^{\xi,h}v\|_{L^2(\Omega)} \le C\left(\epsilon^{-3}\delta^\xi\delta^{\xi+h}h + \epsilon^{-1}\delta^\xi h\right) \le C\epsilon^{-1}\delta^\xi h.$$

Using the technique from the previous section, namely decomposing $\Delta^{\xi,h}v = a^{\xi,h}V_1^{\xi} + r^{\xi,h}$, where $\langle V_1^{\xi}, r^{\xi,h}\rangle = 0$ and estimating the remainder and the coefficient as in Lemma 2.3 we get:

$$|a^{\xi,h}| \le \left|\langle V_1^{\xi}, \Delta^{\xi,h}v\rangle\right| \le \left|\langle V_1^{\xi+h} - V_1^{\xi}, \mathcal{K}^{\xi+h}(w)\rangle\right| \le C\epsilon^{-\frac{7}{3}}\rho^{\xi}h,$$

$$\|\Delta^{\xi,h}v\|_{L^2(\Omega)} \le C\epsilon^{-\frac{7}{3}}\rho^{\xi}h,$$

hence

$$\|\Delta^{\xi,h}v\|_{W^{2,2}(\Omega)} \le C\epsilon^{-\frac{9}{3}}\rho^{\xi}h \le C\epsilon^{-9}\delta^{\xi}h. \tag{3.16}$$

From the Rellich-Kondratchov Theorem we conclude that for each sequence $\{h_n\}$, $h_n \to 0$ there exists a subsequence, which we again denote for convenience by $\{h_n\}$ and a function $\tilde{v} \in W^{2,2}(\Omega)$ such that:

$$\frac{1}{h_n}\Delta^{\xi,h_n}v \to \tilde{v} \qquad \text{strongly in } L^2(\Omega) \text{ and weakly in } W^{2,2}(\Omega).$$

Now we need to show that the limiting function $\tilde{v}$ does not depend on the choice of a sequence $\{h_n\}$.To this end assume that there are sequences $\{h_n^i\}, i = 1, 2$ such that:

$$\frac{1}{h_n^i}\Delta^{\xi,h_n^i}v \to \tilde{v}^i \qquad \text{strongly in } L^2(\Omega) \text{ and weakly in } W^{2,2}(\Omega) \text{ for } i = 1, 2.$$

After passing to the limit (as $n \to \infty$) in the equation (3.15) for $i = 1, 2$ we get:

$$L^{\epsilon,\xi}(\tilde{v}^1 - \tilde{v}^2) = 0 \qquad \text{in } \Omega,$$

$$\frac{\partial}{\partial n}(\tilde{v}^1 - \tilde{v}^2) = 0 \qquad \text{on } \partial\Omega.$$

This implies that there exists α such that $\tilde{v}^1 = \tilde{v}^2 + \alpha V_1^{\xi}$.Since for each $\xi \in (l, 1-l)$ we have $\langle v, V_1^{\xi}\rangle = 0$ therefore:

$$\langle V_1^{\xi}, \frac{\Delta^{\xi,h_n^i}v}{h_n^i}\rangle = -\langle \frac{V_1^{\xi+h_n^i} - V_1^{\xi}}{h_n^i}, \Delta^{\xi,h_n^i}v - v\rangle \qquad i = 1, 2.$$

Taking the limit on both sides we get:

$$\langle V_1^{\xi}, \tilde{v}^i\rangle = -\langle V_{1,\xi}^{\xi}, v\rangle,$$

hence

$$\langle V_1^\xi, \tilde{v}^1 \rangle - \alpha = \langle V_1^\xi, \tilde{v}^2 \rangle = -\langle V_{1,\xi}^\xi, v \rangle = \langle V_1^\xi, \tilde{v}^1 \rangle.$$

Consequently $\alpha = 0$ and $\tilde{v}^1 = \tilde{v}^2 : = \tilde{v}$. By using (3.15) with the help of (1.10) we obtain the strong convergence of $\frac{1}{h_n}\Delta^{\xi,h_n} v$ in $W^{2,2}(\Omega)$.It remains to show now that $\tilde{v} = \frac{d}{d\xi}\mathcal{K}^\xi(w)$ is continuous with respect to ξ.The proof of this standard and we omitt it.The estimates (3.13) follow now from (3.16) and Lemma 3.1.

□

Proof of the improved v-estimate. Before we begin the proof we point out that in some sense $W^{2,2}(\Omega)$ estimate in Lemma A is already optimal.This is due to the fact that $\|L^{\epsilon,\xi} u_\xi^\xi\|_{L^2(\Omega)} = O(\epsilon^{-1}\delta^\xi)$ and therefore we can not hope on improving (3.12) globally on the whole Ω.On the other hand on the rectangular part of Ω we have $-\epsilon^2 \Delta u^\xi + f(u^\xi) = 0$ and thus the ansatz $U(\frac{x_1-\xi}{\epsilon})$ is there a better approximation of the true solution then on the rest of Ω.This is why we can improve the estimate on v^ξ only locally, near the layer. First we need a classical result:

Lemma 3.6. *Let Σ be an infinite strip in $\mathbb{R}^2$, $\Sigma = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_2 < b\}$.Consider the following equation:*

$$\begin{aligned} -\epsilon^2 \Delta u + \beta^2 u = \psi \qquad & \text{in } \Sigma, \\ \frac{\partial u}{\partial n} = 0 \qquad & \text{on } \partial\Sigma. \end{aligned}$$

Then for each $\psi \in L^2(\Sigma)$ the above equation has a unique solution $u \in L^2(\Sigma)$ which can be represented by:

$$u(x) = \epsilon^{-2} \int_\Sigma \psi(y) G(x, y)\, dy,$$

where for $x = (x_1, x_2), y = (y_1, y_2)$ we have:

$$G(x,y) = b^{-1} \sum_{n=0}^{\infty} \sigma_n^{-1} e^{-\sigma_n |x_1 - y_1|} \cos(\frac{\pi n}{b} x_2) \cos(\frac{\pi n}{b} y_2), \qquad 0 < y_2 < x_2 < b,$$
$$\sigma_n = \sqrt{\beta^2 \epsilon^{-2} + \left(\frac{\pi n}{b}\right)^2}.$$

Proof can be found in [D-N]. We set:

$$\Omega(a_1, a_2) = \{x = (x_1, x_2) \in \Omega \mid a_1 < x_1 < a_2\},$$

$$\Gamma(a) = \{x = (x_1, x_2) \in \Omega \mid x_1 = a\},$$

$$\tilde{\partial} D = \partial D \cap \partial \Omega \qquad \text{for any } D \subset \mathbb{R}^2,$$

Set $\tilde{l} = \frac{l}{11}$. The next lemma gives us estimate on v^ξ on the "boundary" of the rectangular part of Ω.

Lemma 3.7. *For each $\xi \in (l, 1-l)$ and ϵ sufficiently small we have:*

$$|v^\xi(x)| \leq C\epsilon^{-4}\delta^\xi e^{-\frac{2\beta}{\epsilon}\tilde{l}} \tag{3.17}$$

for $x \in \Omega(3\tilde{l}, 7\tilde{l}) \cup \Omega(1-7\tilde{l}, 1-3\tilde{l})$.

Proof of Lemma 3.7. Fix $x \in \Omega(3\tilde{l}, 7\tilde{l})$.Recall that on $(\tilde{l}, 1-\tilde{l}) \times (0, b)$ we have $w^\xi = 0$ hence from (3.3):

$$\Phi^\xi(v^\xi) = c^\xi(v^\xi)u^\xi_\xi + \mathcal{N}(v^\xi) \qquad \text{in } \Omega(\tilde{l}, 9\tilde{l}).$$

We recast the equation for v^ξ:

$$-\epsilon^2\Delta v^\xi + \beta^2 v^\xi = \Phi^\xi(v^\xi) + \left((\beta)^2 - f'(u^\xi)\right)v^\xi. \tag{3.18}$$

Multiplying (3.18) by $G(x,y)$ and integrating by parts with respect to y over $\Omega(\tilde{l}, 9\tilde{l})$ we obtain:

$$\begin{aligned} v^\xi(x) = &\int_{\Omega(\tilde{l},9\tilde{l})} \left[\Phi^\xi(v^\xi) + (\beta^2 - f'(u^\xi))\,v^\xi\right](y)G(x,y)\,dy \\ &- \epsilon^2 \int_{\partial\Omega(\tilde{l},9\tilde{l})} G(x,y)\frac{\partial}{\partial n}v^\xi(y)\,dS_y + \epsilon^2 \int_{\partial\Omega(\tilde{l},9\tilde{l})} v^\xi(y)\frac{\partial}{\partial n}G(x,y)\,dS_y \\ =:& \; I_1 - \epsilon^2 I_2 + \epsilon^2 I_3. \end{aligned} \tag{3.19}$$

From Lemmas 3.1 and 3.3 we can easily estimate:

$$|I_1| \leq C\|\Phi^\xi(v^\xi) + (\beta^2 - f'(u^\xi))\,v^\xi\|_{C^0(\Omega(\tilde{l},9\tilde{l}))} \leq C\epsilon^{-4}\delta^\xi e^{-2\frac{\beta}{\epsilon}\tilde{l}}. \tag{3.20}$$

Note that the first inequality above follows from the fact that the series $\int_{\Omega(\tilde{l},9\tilde{l})} |G(x,y)|\,dx$ is absolutely and uniformly (in ϵ) convergent since $\int_0^\infty e^{-\sigma t}\,dt = \frac{1}{\sigma}$. From

$$\frac{\partial v^\xi}{\partial n} = 0, \quad \frac{\partial}{\partial n_y} G(x,y) = 0 \qquad \text{on } \tilde{\partial}\Omega(\tilde{l}, 9\tilde{l})$$

it follows for $x \in \Omega(3\tilde{l}, 7\tilde{l})$ by making use of (1.11) again that

$$|I_i| \leq C\epsilon^{-4}\delta^\xi e^{-\frac{2\rho\tilde{l}}{\epsilon}} \qquad \text{for } i = 2, 3.$$

This completes the proof of for $x \in \Omega(3\tilde{l}, 7\tilde{l})$, the argument for the other part is analogous and we omitt it.

□

Recall that

$$R_{5\tilde{l}} = (5\tilde{l}, 1 - 5\tilde{l}) \times (0, b).$$

Let $(\tilde{V}_i^\xi, \tilde{\lambda}_i^\xi)$ denote respectively the ith eigenfunction and eigenvalue of the problem (1.7) with the set Ω replaced by $R_{5\tilde{l}}$.It is not very difficult to show (see Lemma B and the Sola-Morales example) that:

$$\|\tilde{V}_1^\xi - V_1^\xi\|_{L^2(R_{5\tilde{l}})} \leq C\epsilon^{-2}\delta^\xi e^{\frac{5\rho}{\epsilon}\tilde{l}}. \tag{3.21}$$

This fact will be used in the next lemma in which we extend (3.17) to the set $R_{5\tilde{l}}$.

Lemma 3.8. *Under the assumptions of Lemma 3.7 we have:*

$$\|v^\xi(x)\|_{W^{1,2}(R_{5\tilde{l}})} \leq C\epsilon^{-12}\delta^\xi e^{-\frac{3\rho}{\epsilon}\tilde{l}}.$$

Proof of Lemma 3.8. From $\langle V_1^\xi, v^\xi\rangle = 0$ we obtain via Lemma B and Lemma 3.5:

$$\left|\langle V_1^\xi, v^\xi\rangle_{L^2(R_{5\tilde{l}})}\right| = \left|-\langle V_1^\xi, v^\xi\rangle_{L^2(\Omega\setminus R_{5\tilde{l}})}\right| \leq C\epsilon^{-6}\gamma^\xi e^{\frac{5\rho\tilde{l}}{\epsilon}}. \tag{3.22}$$

This and (3.21) yields:

$$\left|\langle \tilde{V}_1^\xi, v^\xi\rangle_{L^2(R_{5\tilde{l}})}\right| = \left|\langle V_1^\xi, v^\xi\rangle_{L^2(R_{5\tilde{l}})}\right| + \left|\langle \tilde{V}_1^\xi - V_1^\xi, v^\xi\rangle_{L^2(R_{5\tilde{l}})}\right| \leq C\epsilon^{-6}\gamma^\xi e^{\frac{5\rho\tilde{l}}{\epsilon}}. \tag{3.23}$$

Let $\bar{v}^{\xi}$ be the restriction of v^{ξ} to $R_{5\tilde{l}}$.We can decompose:

$$\bar{v}^{\xi} = \tilde{v}^{\xi} + \langle \tilde{V}_1^{\xi}, \bar{v}^{\xi} \rangle_{L^2(R_{5\tilde{l}})} \tilde{V}_1^{\xi}. \tag{3.24}$$

It is easy to see that $\dfrac{\partial \bar{v}^{\xi}}{\partial n} = 0$ on $\tilde{\partial} R_{5\tilde{l}}$ and

$$\|\bar{v}^{\xi}(x)\|_{W^{2,2}(R_{5\tilde{l}})} \leq \|v^{\xi}\|_{W^{2,2}(\Omega)} + C\epsilon^{-\frac{17}{2}} \gamma^{\xi} e^{\frac{5\beta}{\epsilon}\tilde{l}} \leq C\epsilon^{-4}\delta^{\xi}. \tag{3.25}$$

Multiplying the equation $L^{\epsilon,\xi}\tilde{v}^{\xi} = \Phi^{\xi}(\bar{v}^{\xi}) - \tilde{\lambda}_1^{\xi}\langle \tilde{V}_1^{\xi}, \bar{v}^{\xi}\rangle_{L^2(R_{5\tilde{l}})}\tilde{V}_1^{\xi}$ by $\tilde{v}^{\xi}$ and integrating by parts over $R_{5\tilde{l}}$ we obtain via (3.26) and (1.11)

$$\begin{aligned}
\int_{R_{5\tilde{l}}} \epsilon^2|\nabla \tilde{v}^{\xi}| + f'(u^{\xi})(\tilde{v}^{\xi})^2 &\leq |\langle \Phi^{\xi}(\bar{v}^{\xi}) - \tilde{\lambda}_1^{\xi}\langle \tilde{V}_1^{\xi}, \bar{v}^{\xi}\rangle_{L^2(R_{5\tilde{l}})}\tilde{V}_1^{\xi}, \tilde{v}^{\xi}\rangle| + \epsilon^2 \left| \int_{\partial R_{5\tilde{l}}} \tilde{v}^{\xi} \frac{\partial}{\partial n}\bar{v}^{\xi}\, dS \right| \\
&\leq C\epsilon^{-8}\gamma^{\xi}\|\tilde{v}^{\xi}\|_{L^2(R_{5\tilde{l}})} + \epsilon^2\|\tilde{v}^{\xi}\|_{C^0(\Omega(4\tilde{l},6\tilde{l}))}\|\bar{v}^{\xi}(x)\|_{W^{2,2}(R_{5\tilde{l}})} \\
&\leq C\epsilon^{-12}\gamma^{\xi}\delta^{\xi} + C\epsilon^{-2}\delta^{\xi}\|\tilde{v}^{\xi}\|_{C^0(\Omega(4\tilde{l},6\tilde{l}))}.
\end{aligned} \tag{3.26}$$

From Lemma (3.7) we obtain

$$\begin{aligned}
\|\tilde{v}^{\xi}\|_{C^0(\Omega(4\tilde{l},6\tilde{l}))} &\leq \|\bar{v}^{\xi}\|_{C^0(\Omega(4\tilde{l},6\tilde{l}))} + \epsilon^{-1}|\langle \tilde{V}_1^{\xi}, \bar{v}^{\xi}\rangle_{L^2(R_{5\tilde{l}})}| \\
&\leq C(\epsilon^{-4}\delta^{\xi}e^{-\frac{3\beta}{\epsilon}\tilde{l}} + \epsilon^{-7}\gamma^{\xi}e^{\frac{5\beta}{\epsilon}\tilde{l}}) \leq C\epsilon^{-4}\delta^{\xi}e^{-\frac{3\beta}{\epsilon}\tilde{l}},
\end{aligned}$$

hence by (3.26) we get from $\langle \tilde{v}^{\xi}, \tilde{V}_1^{\xi}\rangle = 0$

$$\|\tilde{v}^{\xi}\|^2_{L^2(R_{5\tilde{l}})} \leq C(\epsilon^{-14}\gamma^{\xi}\delta^{\xi} + \epsilon^{-8}\gamma^{\xi}e^{-\frac{3\beta}{\epsilon}\tilde{l}})$$

and therefore

$$\|\tilde{v}^{\xi}\|_{W^{1,2}(R_{5\tilde{l}})} \leq C\delta^{\xi}e^{-\frac{c}{\epsilon}}.$$

Combining this with (3.23) and (3.24) we get

$$\|v^{\xi}\|_{W^{1,2}(R_{5\tilde{l}})} \leq C\delta^{\xi}e^{-\frac{c}{\epsilon}}.$$

The proof of the improved v estimate is complete.

□

CHAPTER II

THE EXISTENCE OF EQUILIBRIA FOR THE CAHN-HILLIARD EQUATION

1.1 Preliminaries

In the present chapter we will be interested in establishing existence of certain equilibrium states of the nonlinear Cahn-Hiliard equation

$$\text{(CH)}\qquad \begin{aligned} u_t &= \Delta(-\epsilon^2\Delta u + F'(u)), x \in \Omega, \\ \frac{\partial}{\partial n}u &= \frac{\partial}{\partial n}(-\epsilon^2\Delta u + F'(u)) = 0, x \in \partial\Omega. \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ a smooth, bounded domain, $\dfrac{\partial}{\partial n}$ the exterior normal derivative, $0 < \epsilon \ll 1$ small parameter. Here $F \in C^2(\Omega)$ is a double well potential: $F \geq 0$,$F(-1) = F(1) = 0$.In addition we assume that there exist positive constants p, u_0 such that $C^{-1}|u|^p \leq F(u) \leq C|u|^p$ for $u > u_0$.

It is convienent to rescale the free energy functional $\mathcal{J}_\epsilon = \int_\Omega \epsilon^2|\nabla u|^2 + F(u)$ associated to (CH) in the form

$$(1.1)\qquad \mathcal{G}_\epsilon[u,m] = \begin{cases} \int_\Omega \epsilon|\nabla u|^2 + \epsilon^{-1}F(u) & \text{if} \quad \int_\Omega u = m, \\ +\infty & \text{otherwise}. \end{cases}$$

We introduce now a class of functions of bounded variation which turns out to be a natural function space for our problem. For $f \in L^1(\Omega)$ we define:

$$\int_\Omega |Df| = \sup\{\int_\Omega f \operatorname{div} w \mid w = (w_1, w_2) \in C_0^1(\Omega;\mathbb{R}^2)\}.$$

If $\int_\Omega |Df| < \infty$ then we say that f has bounded variation.By $BV(\Omega)$ we denote the Banach space of all such functions equipped with the norm:

$$||f||_{BV} = ||f||_{L^1} + \int_\Omega |Df|.$$

For any measurable set $E \subset \mathbb{R}^2$ by ϕ_E we denote the characteristic function of E.If $E \subset \Omega$ and $||\phi_E||_{BV} < \infty$ then we say that E has finite perimeter and we denote:

$$\mathrm{Per}_\Omega(E) = \int_\Omega |D\phi_E|.$$

Let γ be a plane curve.We call γ a $C^{1,\alpha}$ curve if locally it can be represented as a graph of $C^{1,\alpha}$ function. It is known for such curves that if $\mathcal{L}(\gamma)$ denotes the lenght of γ and if E_γ is the set cut off from Ω by γ then

$$\mathrm{Per}_\Omega(E_\gamma) = \mathcal{L}(\gamma).$$

In fact similiar statement holds if we only assume that γ is a rectifiable curve.

Of the special importance is the fact that the Rellich-Kondratchov theorem holds in $BV(\Omega)$, namely bounded BV sets are precompact in L^1. More details about the space of functions of bounded variation the reader can find in [G1].

For any given $A \in (0, |\Omega|)$ we define:

$$\mathcal{D}_A = \{E \subset \Omega \mid |E| = A \quad \text{and} \quad \int_\Omega |D\phi_E| < \infty\}.$$

Definition 1.1. Set $E \subset \Omega$, $E \in \mathcal{D}_A$ is called a local minimizer of the perimeter in the class $\mathcal{D}_A$ if there exists $\rho > 0$ such that for each set $F \subset \Omega$, $F \in \mathcal{D}_A$ satisfying:

$$\int_\Omega |\phi_E - \phi_F| < \rho,$$

we have:

$$\int_\Omega |D\phi_E| \leq \int_\Omega |D\phi_F|, \tag{1.2}$$

We say that E is an isolated local minimizer of the perimeter in the class $\mathcal{D}_A$ if the equality in (1.2) implies $E = F$ a.e. in Ω

It turns out that to the functional $\mathcal{G}_\epsilon$ we can relate the following functional:

$$\mathcal{G}_0[u,m] = \begin{cases} 2c_0 \int_\Omega |Du| & \text{if } u \in BV(\Omega) \text{ and } \int_\Omega u = m \text{ and } F(u(x)) = 0 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise} \end{cases} \tag{1.3}$$

where $c_0 = \displaystyle\int_{-1}^{1} F^{1/2}(s)\, ds$. The relation between the global minimizers of $\mathcal{G}_\epsilon$ and $\mathcal{G}_0$ was first pointed out in the work of Modica [M] (see also [S]).Later Kohn and Sternberg [K-S] observed that the methods used previously for the global minimizers extend to local minimizers as well and that, moreover, certain existence result can be established.We will define now local minimizers for both $\mathcal{G}_\epsilon$ and $\mathcal{G}_0$.

Definition 1.2. We call u^ϵ an L^1-local minimizer of $\mathcal{G}_\epsilon$ if for some $\rho > 0$, $\mathcal{G}_\epsilon[u^\epsilon, m] \leq \mathcal{G}_\epsilon[v, m]$ whenever $0 < \|v - u^\epsilon\|_{L^1(\Omega)} \leq \rho$. We also call u^0 an isolated L^1-local minimizer of $\mathcal{G}_0$ if for some $\rho > 0$, $\mathcal{G}_0[u^0, m] < \mathcal{G}_0[v, m]$ whenever $0 < \|v - u^0\|_{L^1(\Omega)} < \rho$.

Using the Γ-convergence method it can be shown [K-S] that the following holds.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary and suppose that u^0 is an isolated L^1-local minimizer of $\mathcal{G}_0$.Then there exists $\epsilon_0 > 0$ and a family $\{u^\epsilon\}_{\epsilon<\epsilon_0}$ such that:*

(1) *u^ϵ is an L^1-local minimizer of $\mathcal{G}_\epsilon$;*

(2) *$\|u^\epsilon - u^0\|_{L^1(\Omega)} \to 0$ as $\epsilon \to 0$.*

The main goal of this chapter is to give a geometric characterization of isolated L^1-local minimizers of $\mathcal{G}_0$ and by doing so showing the existence of certain equilibrium states for (CH). Observe that $W(u^0(x)) = 0$ a.e. in Ω is equivalent to $u^0 = \pm 1$ a.e in Ω.Denote $E = \{u^0 = 1\}$.From the Coarea Formula we have

$$\mathcal{G}_0[u^0, m] = 4c_0 \int_\Omega |D\phi_E|,$$

with $|E| = \dfrac{m + |\Omega|}{2}$. Thus the problem of minimizing $\mathcal{G}_0[\cdot, m]$ is equivalent to the problem of minimizing $\mathrm{Per}_\Omega(\cdot)$ in the class $\mathcal{D}_A$ with $A = \dfrac{m + |\Omega|}{2}$. This is the the latter problem we shall concentrate on here.

In order to characterize minimizers of the perimeter in $BV(\Omega)$ class in terms of the geometric properties of the domain we consider a family C_A of circular arcs lying in Ω whith the endpoints lying on $\partial\Omega$ and enclosing area A.We give a precise definition of the circular class C_A in the next section.We also prove there that it is possible to introduce in C_A a differentiable function $L(s, \sigma)$ which to each circular arc $w \in C_A$ joining points $z(s), z(\sigma) \in \partial\Omega$ assigns its lenght.It turns out (Lemma 2.3) that the local minima of $L(s, \sigma)$ correspond to the shortest (locally) smooth curves with their ends lying on $\partial\Omega$ and enclosing area A.Now the main result of this chapter can be stated:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded region with $\partial\Omega \in C^4$.*

(i) *If $z(\bar{s})$ is the point of a local (strict) maximum of the curvature of $\partial\Omega$ then for all sufficiently small $A > 0$ there exist points s_A, σ_A such that the function $L(s_A, \sigma_A)$ defined on $\mathcal{C}_A$ attains its local (strict) minimum at $s = s_A, \sigma = \sigma_A$.Moreover $s_A \to \bar{s}, \sigma_A \to \bar{s}$ as $A \to 0$.*

(ii) *Let $A, A \in (0, |\Omega|)$ be given and s_A, σ_A be the points of a strict local minimum of $L(s, \sigma)$ on $\mathcal{C}_A$.Let $w \in \mathcal{C}_A$ be the circular arc with endpoints $z(s_A), z(\sigma_A)$ and E be the set with area A cut off from Ω by w.Then E is an isolated local minimizer of the perimeter in the class $\mathcal{D}_A$.*

Figure 5 below illustrates the situation.

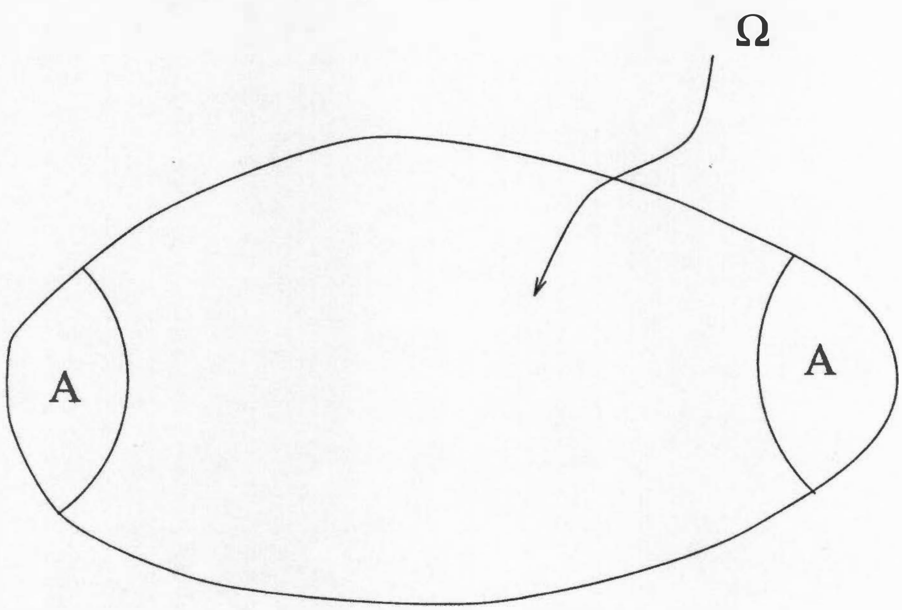

FIGURE 5. LOCATION OF THE LOCAL MINIMIZERS IN THE CLASS $\mathcal{C}_A$

From Theorems 1.1 and 1.2 existence of equilibria to (CH) whose transition layers are close to circular arcs intersecting the boundary orthogonally follows immediately.Notice that if A is sufficiently small (which in terms of the mass constraint corresponds to $m \approx 0$ or $m \approx |\Omega|$) then the Four Vertex Theorem guarantees that we have at least two such equilibria.It is natural to conjecture that in fact the number of equilibria of the type described above is equal to the number of critical points of the curvature of $\partial\Omega$.However the method used in this chapter allows us only to establish the correspondence between local minima of the free energy functional and local maxima of the curvature.

The rest of this chapter is organized as follows: in Section 2.1 we investigate the family C_A and the function L.Part (i) of the above theorem is contained in Corollary 2.6.In Section 2.2 we consider class $\mathcal{D}_A$, in particular part (ii) of Theorem 1.2 is contained in Lemma 2.12 of this section.

We are greatly indebted to Professor Xinfu Chen for suggesting the argument in Section 2.1.

2.1 Minimizers in the Circular Class

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded,simply connected domain with C^4 boundary.We will assume that $\partial\Omega$ is oriented counterclockwise and equipped with the arc lenght parametrization $z(s)$.In order to motivate our considerations we first give the following:

Example 2.1. Let $A \in (0, |\Omega|)$ be a given number.Find the shortest smooth curve with its ends lying on $\partial\Omega$ and dividing Ω on two parts one of which has area A.

Let $w(x), x \in [0,1]$ denote the parametrization of the curve we are after and let $w(0) = z(s), w(1) = z(\sigma)$ be the endpoints of w lying on $\partial\Omega$.Set:

$$\mathcal{L}[w] = \int_0^1 |w'|\,dx,$$

$$\mathcal{A}[w] = \frac{1}{2}\left\{\int_0^1 (w_1w_2' - w_1'w_2)\,dx + \int_\sigma^s (z_1z_2' - z_1'z_2)\,ds'\right\}. \tag{2.1}$$

We seek a solution of the problem of minimizing $\mathcal{L}[w]$ subject to the constraint $\mathcal{A}[w] = A$.It is well known that minimizers are critical points of:

$$\mathcal{F}_\lambda[w]:\,=\mathcal{L}[w] - 2\lambda\mathcal{A}[w]. \tag{2.2}$$

From classical calculus of variations any critical point of (2.2) satisfies:

$$\begin{cases} \frac{d}{dx}\left(\frac{w_1'}{|w'|} - \lambda w_2\right) = \lambda w_2', \\ \frac{d}{dx}\left(\frac{w_2'}{|w'|} + \lambda w_2\right) = -\lambda w_2'. \end{cases} \tag{2.3}$$

where λ is the Lagrange multiplier and w, z satisfy the transversality conditions:

$$\langle w'(0), z'(s)\rangle = 0,$$

$$\langle w'(1), z'(\sigma)\rangle = 0. \tag{2.4}$$

From (2.3) we conclude that unless $\lambda = 0$ there exist numbers a_1, a_2, r such that:

$$(\lambda w_1 + a_1)^2 + (\lambda w_2 + a_2)^2 = r^2.$$

If $\lambda = 0$ then there exist constants a, b such that $aw_1 + b = w_2$ and in any case the solution to (2.3) must be a curve of constant curvature.Thus any critical point of $\mathcal{F}_\lambda$ must be a curve of constant curvature which meets $\partial\Omega$ orthogonally. One could try to solve (2.3) directly and then by using classical calculus of variations methods find all critical points of $\mathcal{L}$ in the contrained class and then describe their properties (perhaps depending on $\partial\Omega$).We shall not pursue this approach here and instead we shall reduce the problem of minimizing the functional $\mathcal{L}$ to the problem of minimizing appropriately constructed function of two variables.

In the sequel any curve of constant curvature will be reffered to as a circular arc or simply a circle.

Before we proceede further we remark that all calculations in this sections are greatly simplified if we treat Ω and various curves appearing later as subsets of the complex plane.For example $z(s)$, the arc lenght parametrization of $\partial\Omega$ is to be understood as $z(s) = \text{Re}z(s) + i\text{Im}z(s)$.The formula for the area in (2.2) takes form

$$\mathcal{A}[w] = \frac{1}{2}\text{Im}\left\{\int_0^1 \overline{w}\, dw + \int_\sigma^s \overline{z}\, dz\right\}.$$

We first need the following lemma.

Lemma 2.1. *Let A be a given positive real number.For any two points $z(s), z(\sigma) \in \partial\Omega$ there exists a circular arc w joining them, whose arc lenght parametrization can be written:*

$$w(t, s, \sigma) = z(s) + z'(s)e^{i\psi}\int_0^t e^{i\kappa\tau}\, d\tau. \tag{2.5}$$

Moreover κ the curvature of w, ψ the contact angle between w and $\partial\Omega$ at $z(s)$ and L the lenght of w are all smooth functions of (s,σ) determined from the equations:

$$\left\{\begin{array}{l} w(L,s,\sigma)=z(\sigma), \\ \frac{1}{2}Im\left\{\int_0^L \overline{w}\,dw+\int_\sigma^s \overline{z}\,dz\right\}=A. \end{array}\right. \tag{2.6}$$

Proof of Lemma 2.1. Let $\beta(s,\sigma)$ be the angle between the tangent $z'(s)$ and the chord $z(\sigma)-z(s)$ measured in the counterclockwise direction from $z'(s)$ to the chord.If we denote $R=|z(\sigma)-z(s)|$ then we can write:

$$z(\sigma)-z(s)=Rz'(s)e^{i\beta}. \tag{2.7}$$

Let $\psi\in[0,2\pi)$ and $\theta=2(\beta-\psi)$.If we define:

$$\begin{aligned} \kappa &= \frac{2\sin\left(\frac{\theta}{2}\right)}{R}, \\ L &= \frac{\theta}{\kappa}=\frac{\frac{\theta}{2}}{\sin\left(\frac{\theta}{2}\right)}R. \end{aligned} \tag{2.8}$$

then an elementary geometric argument (or direct calculations) shows that the circular arc whose representation is given by (2.5) satisfies the first equation in (2.6).In other words w is the circular arc joining $z(s)$ and $z(\sigma)$ with the contact angle at $z(s)$ equal to ψ.To complete the proof we need to show that ψ can be chosen so that the second condition in (2.6) (the area constraint) is satisfied. Because of the relation (2.8) it sufficies to find a smooth function $\theta=\theta(s,\sigma)$ such that (2.6) holds.From the definition of w:

$$\int_0^L \overline{w}\,dw=\overline{z(s)}z'(s)e^{i\psi}\int_0^L e^{i\kappa\tau}\,d\tau+\int_0^L\int_0^t e^{i\kappa(t-\tau)}\,d\tau dt,$$

hence after standard calculations:

$$A=\frac{1}{2}\mathrm{Im}\left\{\overline{z(s)}(z(\sigma)-z(s))+\int_\sigma^s \overline{z}\,dz+R^2H(\theta)\right\}, \tag{2.9}$$

where

$$H(\theta)=\frac{1-\cos\theta}{4\sin^2\frac{\theta}{2}}+i\frac{\theta-\sin\theta}{4\sin^2\frac{\theta}{2}}=:\frac{1}{2}+iF(\theta), \tag{2.10}$$

thus

$$R^2F(\theta) = 2A - \mathrm{Im}\left\{\overline{z(s)}(z(\sigma) - z(s)) + \int_\sigma^s \bar{z}\,dz\right\}. \tag{2.11}$$

Since

$$F'(\theta) = \frac{\sin\frac{\theta}{2} - \frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^3\frac{\theta}{2}} > 0, \qquad \theta \in (-2\pi, 2\pi)$$

and $F(\pm 2\pi) = \pm\infty$ therefore F^{-1} is smooth.This immediately implies that (2.11) can be solved for θ and that θ is a smooth function of (s, σ).The proof is complete.

□

For any $A \in (0, |\Omega|)$ a family of circular arcs $\mathcal{C}_A$ defined below will be called the circular class:

$$\mathcal{C}_A = \{w(\cdot, s, \sigma) \mid z(s), z(\sigma) \in \partial\Omega,\ w(t, s, \sigma) \in \Omega \text{ for } t \in (0, L) \text{ and } w \text{ satisfies (2.6)}\}.$$

From Lemma 2.1 we conclude:

Corollary 2.2. *The set:*

$$\mathcal{S}_A = \{(s, \sigma) \mid w(\cdot, s, \sigma) \in \mathcal{C}_A\}$$

is open.

From what we have said above it is clear that we would like to reduce the problem of investigating geometric properties of minima of (2.2) to the problem of investigating minima of the function $L(s, \sigma)$ in the set $\mathcal{S}_A$.The next lemma shows that this reduction is legitimate.

Lemma 2.3. *Let (s_0, σ_0) be a local minimum of $L(s, \sigma)$ for $(s, \sigma) \in \mathcal{S}_A$.There exists $\rho > 0$ such that for any $(s, \sigma) \in B_\rho(s_0, \sigma_0)$ (ball with radius ρ and center at (s_0, σ_0)) and any $C^{1,\alpha}, \alpha > 0$ curve γ lying in Ω, joining points $z(s), z(\sigma)$ and satisfying the area constraint $\mathcal{A}[\gamma] = A$ we have:*

(i)

$$\mathcal{L}[\gamma] \geq L(s_0, \sigma_0).$$

Moreover if L attains its strict local minimum at (s_0, σ_0) then the inequality above is strict unless $\gamma = w(\cdot, s_0, \sigma_0)$.

(ii) *Circular arc $w(\cdot, s_0, \sigma_0)$ satisfies the orthogonality conditions (2.4).*

Proof of Lemma 2.3. Let ρ be chosen so that for any $(s, \sigma) \in B_\rho(s_0, \sigma_0)$ there exists a circular arc $w(\cdot, s, \sigma) \in \mathcal{C}_A$.Let γ be a given curve joining $z(s), z(\sigma)$ and $w = w(\cdot, s, \sigma) \in \mathcal{C}_A$.If the curvature of w is zero then Lemma 2.3 follows immediately.Assume that $\kappa \neq 0$ and extend w outside Ω to the full circle (with radius κ^{-1}).We also extend γ outside Ω by the same circular arc that we used to extend w to the full circle.Denote the resulting closed curves by $\tilde{w}, \tilde{\gamma}$ respectively.Since $w \in \mathcal{C}_A$ and γ satisfies the area constraint we conclude that the areas enclosed by the circle $\tilde{w}$ and curve $\tilde{\gamma}$ are equal.Isoperimetric property of circle implies therefore that the lenght of $\tilde{\gamma}$ can not be smaller than the lenght of $\tilde{w}$.The first assertion of the lemma follows now from the fact that $\tilde{\gamma}$ and $\tilde{w}$ coincide outside Ω.Since from (i) $w(\cdot, s_0, \sigma_0)$ is a critical point of $\mathcal{L}$ subject to the constraint $\mathcal{A}[w] = A$ therefore the second assertion of the lemma follows.

□

In order to find the location of local minima of $L(s, \sigma)$ we shall further restrict the family of "competing" curves.Set:

$$\mathcal{C}_A^\perp = \left\{ w(\cdot, s, \sigma) \mid \psi = \frac{\pi}{2} \right\}.$$

From the necessary conditions for the minima in (2.3) and the transversality conditions (2.4) we see that $\mathcal{C}_A^\perp$ contains minimizers of $\mathcal{L}$.

We now define the contact angle ϕ between $\partial\Omega$ and $w(\cdot, s, \sigma)$ at $z(\sigma)$ and the angle α between $z'(\sigma)$ and the chord $z(s) - z(\sigma)$ as:

$$e^{i\phi} = -w_t(L, s, \sigma)\overline{z'(\sigma)},$$

$$e^{i\alpha} = (z(s) - z(\sigma))\overline{z'(\sigma)}. \tag{2.12}$$

A simple geometric argument shows that:

$$\theta = 2(\phi - \alpha),$$

$$\alpha + \beta = \phi + \psi. \tag{2.13}$$

where θ is defined in Lemma 2.1. The next lemma gives the parametrization of $\mathcal{C}_A^{\perp}$.

Lemma 2.4. *Let $(s_0, \sigma_0) \in \mathcal{S}_A$ be such that $\psi(s_0, \sigma_0) = \frac{\pi}{2}$.Providing that $\sin\phi(s_0, \sigma_0) \neq 0$ we have:*

(i) *There exists an open interval (s_1, s_2) containing s_0 and a smooth function $\sigma : (s_1, s_2) \to [0, |\partial\Omega|)$ such that:*

$$\sigma(s_0) = \sigma_0,$$

$$\psi(s, \sigma(s)) = \frac{\pi}{2}. \tag{2.14}$$

(ii) *The following formulae hold:*

$$\sigma' = \frac{1}{2F'(\theta)\sin\phi}, \tag{2.15 a}$$

$$R(\kappa' + \nu) = 2\cos\frac{\theta}{2} = 2\sin\beta, \tag{2.15 b}$$

$$L' = -\sigma' \cos\phi. \tag{2.15 c}$$

where $R = |z(s) - z(\sigma)|$, F is defined in Lemma 2.1 and $\nu = \nu(s)$ denotes the curvature of $\partial\Omega$ at $z(s)$.

Proof of Lemma 2.4. For the first assertion of the lemma it sufficies to show that if w, θ, κ and L are defined as in (2.5) and (2.8) of Lemma 2.1 with $\psi = \frac{\pi}{2}$ then for each s sufficiently close to s_0 there exists σ such that the area constraint (2.6) is satisfied.Let $A(s, \sigma)$ denote the number of the right hand side of the expression (2.6).The proof will be completed if we show that:

$$\frac{\partial A}{\partial \sigma}(s_0, \sigma_0) \neq 0. \tag{2.16}$$

For verifying (2.16) it is convenient to compute $\frac{\partial \kappa}{\partial \sigma} = \kappa_\sigma$ first. We claim that:

$$\kappa_\sigma = -2\frac{\sin\phi}{R^2}. \tag{2.17}$$

To show (2.17) observe that from (2.8) with $\psi = \frac{\pi}{2}$ we have:

$$\kappa = 2\frac{\sin\beta}{R}.$$

From the definition of β we obtain:

$$z'(\sigma) = z'(s)R_\sigma e^{i\beta} + iz'(s)\beta_\sigma Re^{i\beta}.$$

Multiplying both sides by $\overline{z'(s)e^{-i\beta}}$ yields:

$$-e^{-i\alpha} = R_\sigma + i\beta_\sigma R,$$

hence

$$\begin{aligned} R_\sigma &= -\cos\alpha = -\cos\left(\phi - \frac{\theta}{2}\right), \\ \beta_\sigma &= R^{-1}\sin\alpha = R^{-1}\sin\left(\phi - \frac{\theta}{2}\right). \end{aligned} \tag{2.18}$$

Now (2.17) follows after direct calculations. We shall verify (2.16). Denoting for simplification $w(t) = w(t, s, \sigma)$ we get after differentiating (2.6):

$$\begin{aligned} \frac{\partial A}{\partial \sigma} &= \frac{1}{2}\mathrm{Im}\left\{\overline{w(L)}w_t(L)L_\sigma + \int_0^L (\overline{w}_\sigma w_t + \overline{w}w_{t\sigma})\,dt - \overline{z(\sigma)}z'(\sigma)\right\} \\ &= \frac{1}{2}\mathrm{Im}\left\{\int_0^L (\overline{w}_\sigma w_t - \overline{w}_t w_\sigma)\,dt\right\} \\ &= \mathrm{Im}\left\{\int_0^L \overline{w}_\sigma w_t\,dt\right\} = \mathrm{Im}\left(-i\kappa_\sigma H(\theta)R^2\right) \\ &= -\frac{1}{2}\kappa_\sigma R^2, \end{aligned}$$

hence $\frac{\partial}{\partial \sigma}A(s_0, \sigma_0) = \sin\phi(s_0, \sigma_0) \neq 0$.

Now we shall show the second assertion of the lemma. We begin with the proof of the formula (2.15 a). From $\kappa' = \kappa_s + \kappa_\sigma \sigma'$ we get:

$$\sigma' = -\frac{\kappa_s}{\kappa_\sigma}. \tag{2.19}$$

We first compute

$$
(2.20) \qquad R_s = \frac{\partial}{\partial s}|z(\sigma) - z(s)| = \frac{\mathrm{Re}\left[(z(\sigma) - z(s))\overline{z'(s)}\right]}{R} = \cos\beta.
$$

Differentiating (2.11) with respect to s we get:

$$
R\theta_s = \frac{-\sin\beta - 2R_s F(\theta)}{F'(\theta)},
$$

hence by using the definition of κ and $F(\theta)$ we obtain:

$$
\kappa_s = \frac{1}{R^2 F'(\theta)}.
$$

and the expression for σ' follows.To show the second formula in (2.15) we evaluate:

$$
\begin{aligned}
0 &= \frac{d}{ds}A(s, \sigma(s)) = \frac{d}{ds}\frac{1}{2}\mathrm{Im}\left\{\int_0^L \overline{w}\,dw + \int_\sigma^s \overline{z}\,dz\right\} \\
&= \frac{1}{2}\mathrm{Im}\left\{\overline{w}w_t L' + \int_0^L (\overline{w}_s w_t + \overline{w}w_{ts})\,dt + \overline{z(s)}z'(s) - \overline{z(\sigma)}z'(\sigma)\sigma'\right\} \\
(2.21) \qquad &= \mathrm{Im}\left\{\int_0^L \overline{w}_s w_t\,dt\right\}.
\end{aligned}
$$

Straightforward calculations using the definition of w and $z''(s) = i\nu z'(s)$ yield (2.15 b). For the proof of the expression for L' in (2.15 c) we consider:

$$
\mathrm{Re}\left[\frac{d}{ds}z(\sigma)\overline{w_t(L)}\right] = \sigma'\mathrm{Re}\left[z'(\sigma)\overline{w_t(L)}\right] = -\sigma'\cos\phi.
$$

From the definition of w we have on the other hand

$$
\begin{aligned}
\mathrm{Re}\left[\frac{d}{ds}z(\sigma)\overline{w_t(L)}\right] &= \mathrm{Re}\left[\frac{d}{ds}w(L)\overline{w_t(L)}\right] \\
&= L' + \mathrm{Re}\left[z'(s)\overline{w_t(L)}\right] + \mathrm{Re}\left\{\left[iz''(s)\int_0^L e^{i\kappa\tau}\,d\tau\right]\overline{w_t(L)}\right\} \\
&\quad + \kappa'\mathrm{Re}\left\{\left[i^2 z'(s)\int_0^L e^{i\kappa\tau}\,d\tau\right]\overline{w_t(L)}\right\} \\
&= \frac{\sin\theta}{R} - (\kappa + \nu)\sin\frac{\theta}{2} + L' = L'.
\end{aligned}
$$

The last equality above follows from (2.15 b). The proof of the lemma is complete.

$\square$

In order to relate minimizers of L with some geometric properties of the domain we need to "localize" the minimization problem by considering circular arcs enclosing a very small area.The lemma below provides the asymptotic formulae for L the circular arc shrinks to a point.

Lemma 2.5.

(i) *There exists a positive constant A_Ω such that for all $A < A_\Omega$ the function $\sigma(s)$ described in the previous lemma is defined on the whole interval $[0, |\partial\Omega|)$.*

(ii) *For each fixed $s \in [0, |\partial\Omega|)$ we have the following approximate formulae:*

$$
\begin{aligned}
L'(s,\sigma(s)) &= -\frac{1}{6}\nu'(s)(\sigma-s)^2 + O\left((\sigma-s)^3\right), \\
L''(s,\sigma(s)) &= -\frac{1}{6}\nu''(s)(\sigma-s)^2 + O\left((\sigma-s)^3\right), \qquad \text{as } \sigma(s) \to s.
\end{aligned}
\tag{2.22}
$$

Consequently,

$$
\begin{aligned}
L'(s,\sigma(s)) &= -\frac{4A}{3\pi}\nu'(s) + O(A^{\frac{3}{2}}), \\
L''(s,\sigma(s)) &= -\frac{4A}{3\pi}\nu''(s) + O(A^{\frac{3}{2}}), \qquad \text{as } A \to 0.
\end{aligned}
\tag{2.23}
$$

Proof of Lemma 2.5. The first assertion of the lemma is easy to show by using the fact that $\partial\Omega$ is compact.We omitt the details. We shall show now the second part of the lemma. Combining the first and the last formula in (2.15) yields:

$$
L' = \frac{\cos\phi}{2F'(\theta)\sin\phi}
$$

Using $\psi = \frac{\pi}{2}$ and the definitions of α, β we get:

$$
L' = \frac{\operatorname{Im}\left\{(z(\sigma)-z(s))^2\,\overline{z'(s)z'(\sigma)}\right\}}{2F'(\theta)\operatorname{Re}\left\{(z(\sigma)-z(s))^2\,\overline{z'(s)z'(\sigma)}\right\}}
\tag{2.24}
$$

From $z''(s) = i\nu(s)z'(s), z'''(s) = \left(i\nu'(s) - \nu^2\right)z'(s)$ and $\theta \to \pi$ as $\sigma \to s$ we obtain for each term appearing in (2.24):

$$
\begin{aligned}
z(\sigma)-z(s) &= \left[(\sigma-s)+\frac{i\nu}{2}(\sigma-s)^2+\frac{i\nu'-\nu^2}{6}(\sigma-s)^3+O\left((\sigma-s)^4\right)\right]z'(s),\\
z'(\sigma) &= \left[1+i\nu(\sigma-s)+\frac{i\nu'-\nu^2}{2}(\sigma-s)^2+O\left((\sigma-s)^3\right)\right]z'(s), \qquad \text{as } \sigma\to s,\\
F'(\theta) &= 1+O\left((\theta-\pi)^2\right),\\
\phi &= \frac{\pi}{2}+o(1), \qquad \text{as } \theta\to\pi.
\end{aligned}
$$

Combining the last two formulae with the expressions for σ', θ we get:

$$
\begin{aligned}
\sigma' &= 1+O\left((\sigma-s)^2\right),\\
F'(\theta) &= 1+O\left((\sigma-s)^2\right), \qquad \text{as } \sigma\to s.
\end{aligned}
$$

The first approximate formula in (2.22) now follows by substituting the above expressions in (2.24).The second fromula in (2.22) is obtained by differentiating the first one with respect to s and using $\sigma' = 1+O\left((\sigma-s)^2\right)$.To get (2.23) we observe that:

$$
\begin{aligned}
R^2 = |z(\sigma)-z(s)|^2 &= (\sigma-s)^2+O\left((\sigma-s)^3\right),\\
\operatorname{Im}\left\{\overline{z(s)}\,(z(\sigma)-z(s))+\int_\sigma^s \bar{z}\,dz\right\} &= O\left((\sigma-s)^3\right),\\
F(\theta) &= \frac{\pi}{4}+O\left((\sigma-s)\right), \qquad \text{as } \sigma\to s,
\end{aligned}
$$

hence by (2.9) we obtain:

$$
A=\frac{\pi}{8}(\sigma-s)^2+O\left((\sigma-s)^3\right),
$$

and therefore

$$
(\sigma-s)=\left(\frac{8A}{\pi}\right)^{\frac{1}{2}}+O(A).
$$

Substituting the last expression into (2.22) ends the proof.

□

As an immediate consequence of (2.23) we obtain

Corollary 2.6. *Let $\partial\Omega \in C^4$ and $z(\tilde{s})$ be the point of a strict local maximum of the curvature of $\partial\Omega$ i.e. $\nu(\tilde{s}) = 0, \nu''(\tilde{s}) < 0$.For each sufficiently small A there exists a point s_A such that the function $L = L(s, \sigma(s))$ has a strict local minimum at s_A and $s_A \to \tilde{s}$ as $A \to 0$. Analogous statements hold if either $z(\tilde{s})$ is a local minimum or a saddle point of the curvature of $\partial\Omega$.*

Corollary 2.7. *Suppose that there exists an open interval $(s_1, s_2) \in [0, |\partial\Omega|)$ and a positive number A^* such that for each $s \in (s_1, s_2)$ and $A \in (0, A^*)$ the function $L = L(s, \sigma(s))$ has a critical point at s.Then the curvature $\nu(s) = const.$ on the interval (s_1, s_2).*

Proof of Corollary 2.7. Since the asymptotic formulas (2.22) and the hypothesis of the corollary imply that

$$0 = L' = -\frac{4A}{3\pi}\nu'(s) + O(A^{\frac{3}{2}})$$

holds for all $s \in (s_1, s_2)$ and $A < A^*$ therefore $\nu'(s) = 0$ hence the corollary follows.

□

The next corollary shows that it may happen, even if $\nu' \neq 0$ that minima of L are not strict for some values of A.

Corollary 2.8. *Let γ be a given C^4 curve and $z(s)$ be its arc lenght parametrization.Fix $A > 0, L_0 > 0, \kappa_0 > 0$ and a point $z(s_0) \in \gamma$.There exists a curve $\tilde{\gamma}$ whose parametrization with arc lenght $y(s)$ satisfies:*

$$y(s) = w(L, s),$$

where

$$w(t, s) = z(s) + z'(s)e^{i} \int_0^t e^{i\kappa\tau}\, d\tau,$$

the functions L, κ are to be determined from the system of equations

$$\text{(2.25)} \qquad \begin{cases} R(\kappa' + \nu) - 2\cos\frac{\theta}{2} = 0, \\ L' = 0, \qquad L(s_0) = L_0, \kappa(s_0) = \kappa_0 \end{cases}$$

and $\theta = \kappa L$, $R = \frac{2}{\kappa}\sin\frac{\theta}{2}$.Moreover for any $z(s), z(\sigma) \in \gamma$ we have:

$$0 = \frac{1}{2} Im\left\{ \int_0^{L(s)} \overline{w}\, dw + \int_s^{\sigma} \overline{y}\, dy + \int_{L(\sigma)}^{0} \overline{w}\, dw + \int_{\sigma}^{s} \overline{z}\, dz \right\}.$$

Proof of Corollary 2.8. From the second equation in (2.25) we get $L(s) = L_0$.Substituting this in the first equation and using that $K(\theta) = 2R^{-1}\cos\frac{\theta}{2} - \nu$ is Lipschitz and $|K(\theta)| \leq C(|\theta| + 1)$ we can establish existence of κ satisfying the first equation.The second assertion follows now from Lemma 2.4.

□

2.2 Minimizers in the BV Class

Recall that in Section 1.1 of the present chapter we defined for $A \in (0, |\Omega|)$:

$$\mathcal{D}_A = \{E \subset \Omega \mid |E| = A \quad \text{and} \int_\Omega |D\phi_E| < \infty\} \tag{2.28}$$

Notice that if a circular arc $c \in \mathcal{C}_A$ and $E_c \in \mathcal{D}_A$ is the set cut off from Ω by c then $\mathcal{L}(c) = \mathrm{Per}_\Omega(E_c)$.Our goal is to prove that strict local minimizers of $\mathcal{L}$ in $\mathcal{C}_A$ (which as we know from the previous section correspond to strict local minima of the function L on $\mathcal{S}_A$) are also isolated local minimizers of $\mathrm{Per}_\Omega(\cdot)$ in $\mathcal{D}_A$. One of the difficulties in executing this plan is the fact that the sets "competing" with a local minimizer in the circular class have to satisfy the area constraint.Therefore we need to know how the perimeter of the set is affected when we change its area.The key point for our consideration is the following perturbation result which is due to Giusti [G2].We state the lemma only for two dimensional domains although it remains true in any dimension.

Lemma 2.9. *Let E be a given set of finite perimeter, D be an open domain such that $\int_D |D\phi_E| > 0$ and $K > 0$ be a positive constant.There exist positive constants ρ_0, v_0, Q_0 depending on $E, D \cap E$, and K such that for every set F of finite perimeter which satisfies:*

$$\begin{aligned} &\int_D |D\phi_F| < K \int_D |D\phi_E|, \\ &\int_D |\phi_F - \phi_E| < \rho_0. \end{aligned} \tag{2.29}$$

and for every $v \in \mathbb{R}$, $|v| < v_0$ there exists set F^v coinciding with F outside D with the following properties:

$$|F^v| = |F| + v, \tag{2.30}$$

$$\left| \int_D |D\phi_{F^v}| - \int_D |D\phi_F| \right| \le Q_0 |v|, \tag{2.31}$$

$$\int_D |\phi_{F^v} - \phi_F| \le Q_0 |v| \int_D |D\phi_E|. \tag{2.32}$$

Proof of Lemma 2.9. For completeness we reproduce the proof from [G2] adopted for simplification to two dimensional case. From the definition of the perimeter it follows that there exists a function $w \in C_0^1(D)$, $|w| \le 1$ such that:

$$\int_D \phi_E \operatorname{div} w \ge \frac{1}{2} \int_D |D\phi_E|.$$

Choose ρ_0 satisfying:

$$\rho_0 \|Dw\|_{C^0(D)} \le \frac{1}{4} \int_D |D\phi_E|.$$

For any set F satisfying (2.29) we have:

$$\int_D \phi_F \operatorname{div} w = \int_D \phi_E \operatorname{div} w + \int_D (\phi_F - \phi_E) \operatorname{div} w,$$

hence from the choice of ρ_0 we get:

$$\int_D |D\phi_F| \ge \frac{1}{4} \int_D |D\phi_E|.$$

For $t \in \mathbb{R}$ we define a one parameter family of functions $g_t \colon D \to \mathbb{R}^2$ by: $g_t(x) = x + tw(x)$.It is clear that g_t is a difeomorphism into D providing that t is sufficiently small.Let $F_t = g_t(F)$.We have:

$$|F_t| = \int_F |\det Dg_t|,$$

$$\int_D |D\phi_{F_t}| = \int_D |\det Dg_t| |(Dg_t)^{-1} D\phi_F|. \tag{2.33}$$

Notice that the last formula is standard if ϕ_{F_t}, ϕ_F are replaced by smooth functions.Its generalized version can be found in [G1]. Direct calculations show that:

$$\det Dg_t = 1 + t \operatorname{div} w + t^2 P(Dw),$$
$$(Dg_t)^{-1} = I - tM_1(Dw) + t^2 M_2(Dw), \tag{2.34}$$

where P is a second degree polynomial and M_i's are matrices whose entries are bounded functions of Dw.Using (2.34) in (2.33) we obtain

$$|F_t| = |F| + t\int_D \phi_F \operatorname{div} w + t^2 \int_D \phi_F P(Dw) =: |F| + \lambda(t).$$

Since

$$\lambda'(t) = \int_D \phi_F \operatorname{div} w + 2t \int_D \phi_F P(Dw) > \frac{1}{4} - 2|t||D|\|P(Dw)\|_{C^0(D)},$$

therefore there exists $t_0 > 0$ depending only on D and w such that $\lambda'(t) > 0$ for $|t| < t_0$.This implies that there exists $v_0 > 0$ depending again only on D and w such that for each v, $|v| < v_0$ there exists t^v such that $\lambda(t^v) = v$.Moreover:

$$|t^v| \leq |\lambda^{-1}(v)| \leq Q_0|v|, \tag{2.35}$$

for some $Q_0 = Q_0(D, E \cap D)$.Let $F^v = F_{t^v}$.It is clear thar F^v satisfies (2.30). Combining (2.33) and (2.34) yields:

$$\int_D |D\phi_{F^v}| \leq \int_D |D\phi_F| + t^v \|M_1(Dw)\|_{C^0(D)} \int_D |D\phi_F| + (t^v)^2 H, \tag{2.36}$$

where H is a bounded constant depending only on D, $D \cap E$ and K.Taking t_0 smaller if necessary and using (2.35) we conclude (2.31) from (2.36).It remains now to prove the last assertion of the lemma.We use the fact that C^1 functions are dense in $BV(D)$.Let $f \in C^1(D)$ and $f_t = f \circ g_t^{-1}$.We have:

$$f_t(x) - f(x) = f_t(x) - f_t(g_t(x)) = -\int_0^1 \langle tw, Df_t(x + tsw)\rangle\, ds,$$

hence

$$\int_D |f_t - f| \le |t| \left| \int_D \int_0^1 \langle w, Df(x + tsw)\rangle \, ds \right| \le Q_1 |t| \int_D |Df|, \tag{2.37}$$

where Q_1 depends only on $D, E \cap D$ and K.Taking the approximating sequence $f_n \to \phi_F$ in $BV(D)$, using $\phi_F \circ g_t^{-1} = \phi_{F_t}$ and passing to the limit in (2.37) we obtain (2.32).The proof is complete.

□

For $E \in \mathcal{D}_A$ we define:

$$\mathcal{S}(E, \delta) := \{F \in \mathcal{D}_A \mid \int_\Omega |\phi_F - \phi_E| = \delta\}.$$

We consider now the problem of minimizing $\mathrm{Per}_\Omega(\cdot)$ in the set $\mathcal{S}(E, \delta)$.

Lemma 2.10. *Let $A \in (0, |\Omega|)$ and $E \in \mathcal{D}_A$ be fixed.There exist a positive constant δ_0 depending on E and A only, such that the minimum of $\mathrm{Per}_\Omega(\cdot)$ in $\mathcal{S}(E, \delta)$ is for each $\delta, 0 < \delta < \delta_0$ attained for some F_δ in $\mathcal{S}(E, \delta)$.Moreover the boundary of F_δ consists of $C^{1,\alpha}, \alpha > 0$ curves.*

Proof of Lemma 2.10. The argument in the proof is motivated by the similiar result in [G2]. First we need to chose δ_0.Let r_0 be the largest positive number such that for each $r < r_0$ there exist $x \in E, y \in \Omega \setminus E$ and two balls with centers at x, y and radii r such that $B_r(x) \subset E$ and $B_r(y) \subset \Omega \setminus E$.Let $\delta_0 = 2\pi r_0^2$.

Now we show existence of a set F_δ minimizing $\mathrm{Per}_\Omega(\cdot)$ in $\mathcal{S}(E, \delta)$ m.Fix $\delta, 0 < \delta < \delta_0$ and let $\{F_n\} \subset \mathcal{S}(E, \delta)$ be a minimizing sequence for $\mathrm{Per}_\Omega(\cdot)$.Since the set $G = ((E \setminus B_r(x)) \cup B_r(y)$ with $r = \sqrt{\frac{\delta}{2\pi}}$ satisfies:

$$\int_\Omega |D\phi_G| \le \int_\Omega |D\phi_E| + 4\pi r,$$

therefore for the minimizing sequence we have:

$$\int_\Omega \phi_{F_n} + \int_\Omega |D\phi_{F_n}| \le const.$$

This implies that there exists $\phi \in L^1(\Omega)$ such that $\phi_{F_n} \to \phi$ in $L^1(\Omega)$ (after passing to a subsequence if necessary).Since $\phi_{F_n} = 1$ or 0 therefore ϕ must be a characteristic function of some set $F_\delta, \phi = \phi_{F_\delta}$.From the lower semicontinuity of $\mathrm{Per}_\Omega(\cdot)$ we obtain:

$$\int_\Omega |D\phi_{F_\delta}| \le \liminf_{n\to\infty} \int_\Omega |D\phi_{F_n}| \le const.$$

hence $F_\delta \in \mathcal{S}(E,\delta)$. To show the second assertion of the lemma we shall employ the perturbation result from Lemma 2.9.Notice that from $F_\delta \in \mathcal{S}(E,\delta)$ we have:

$$|(\Omega \setminus F_\delta) \cap E| = \frac{1}{2}\delta = |F_\delta \cap (\Omega \setminus E)|.$$

Let $x \in \partial F_\delta \cap \Omega$ be fixed and chose $\rho_0 > 0$ small such that there exist two open sets D^+, D^- satisfying:

$$D^- \subset E, \qquad D^+ \subset (\Omega \setminus E),$$

$$\int_{D^-} |D\phi_{F_\delta}| > 0, \qquad \int_{D^+} |D\phi_{F_\delta}| > 0,$$

$$\overline{B_{\rho_0}}(x) \cap (D^- \cup D^+) = \emptyset, \tag{2.38}$$

For $\rho < \rho_0$ we define function:

$$\psi(F_\delta, \rho) = \int_{\overline{B_\rho(x)}} |D\phi_{F_\delta}| - \inf\left\{ \int_{\overline{B_\rho(x)}} |D\phi_G| \mid F_\delta \setminus B_\rho(x) = G \setminus B_\rho(x) \right\}.$$

In order to show that ∂F_δ is a $C^{1,\alpha}$ curve near x it sufficies to show that for $\rho < \rho_0$:

$$\psi(F_\delta, \rho) \le C\rho^2 \tag{2.39}$$

(see [G1,2,T-M] for details).Let M minimize $\int_\Omega |D\phi_G|$ among all G such that $G = F_\delta$ outside $B_\rho(x)$.We clearly have:

$$\int_\Omega |\phi_M - \phi_{F_\delta}| \le \pi\rho^2. \tag{2.40}$$

Let numbers v^+, v^- be defined by:

$$\int_{B_\rho(x)} \phi_{M\cap E} - \int_{B_\rho(x)} \phi_{F_\delta \cap E} = -v^-,$$

$$\int_{B_\rho(x)} \phi_{M\cap(\Omega\setminus E)} - \int_{B_\rho(x)} \phi_{F_\delta\cap(\Omega\setminus E)} = -v^+. \tag{2.41}$$

From the definition of M and (2.40) we conclude that $|v^+|, |v^-| \leq \pi\rho^2$ We shall apply Lemma 2.9 with $E = F_\delta, F = F_\delta, D = D^-, D = D^+$.Let $\rho_0^+, v_0^+, Q_0^+, \rho_0^-, v_0^-, Q_0^-$ be constants depending on F_δ and D^+, D^- respectively as in Lemma 2.9.Taking ρ_0 smaller if necessary we can guarantee that $\pi\rho_0^2 \leq \min(\rho_0^+, \rho_0^-)$ and $|v^+| \leq v_0^+, |v^-| \leq v_0^-$ so that we can utilize Lemma 2.9 on D^+, D^-.Therefore there exist sets F_δ^+, F_δ^- satisfying the assertions of the lemma on D^+, D^- respectively.In particular:

$$
\begin{aligned}
&|F_\delta^+| = |F_\delta \cap D^+| + v^+, \\
\text{(2.42)} \qquad &|F_\delta^-| = |F_\delta \cap D^-| + v^-.
\end{aligned}
$$

Define set M_δ as:

$$
M_\delta = \begin{cases} M & \text{in } \overline{B_\rho}(x), \\ F_\delta^+ & \text{in } D^+, \\ F_\delta^- & \text{in } D^-, \\ F_\delta & \text{otherwise.} \end{cases}
$$

We claim that $M_\delta \in \mathcal{S}(E,\delta)$.From (2.41), (2.42) we have:

$$
\begin{aligned}
\int_\Omega \phi_{M_\delta} - \int_\Omega \phi_{F_\delta} &= \int_{D^+\cup D^-\cup B_\rho(x)} (\phi_{M_\delta} - \phi_{F_\delta}) \\
&= |F_\delta^+| - |F_\delta \cap D^+| + |F_\delta^-| - |F_\delta \cap D^-| \\
&+ \int_{B_\rho(x)} (\phi_{M\cap E} + \phi_{M\cap(\Omega\setminus E)} - \phi_{F_\delta\cap E} - \phi_{F_\delta\cap(\Omega\setminus E)}) \\
\text{(2.43)} \qquad &= 0.
\end{aligned}
$$

(2.41) and the definition of M_δ also yields:

$$
\begin{aligned}
\int_\Omega |\phi_{M_\delta} - \phi_E| - \int_\Omega |\phi_{F_\delta} - \phi_E| &= \int_{D^+} (\phi_{M_\delta} - \phi_{F_\delta}) + \int_{B_\rho(x)} (\phi_{M_\delta\cap(\Omega\setminus E)} - \phi_{F_\delta\cap(\Omega\setminus E)}) \\
&+ \int_{D^-} (\phi_{\Omega\setminus M_\delta} - \phi_{\Omega\setminus F_\delta}) + \int_{B_\rho(x)} (\phi_{E\cap(\Omega\setminus M_\delta)} - \phi_{E\cap(\Omega\setminus F_\delta)}) \\
&= v^+ - v^+ + \int_{D^-} (-\phi_{M_\delta} + \phi_{F_\delta}) + \int_{B_\rho(x)} (-\phi_{M_\delta\cap E} + \phi_{F_\delta\cap E}) \\
\text{(2.44)} \qquad &= -v^- + v^- = 0.
\end{aligned}
$$

From (2.43) and (2.44) the claim follows.For each $\rho < \rho_0$ the function $\psi(F_\delta, \rho)$ satisfies:

$$
\begin{aligned}
\psi(F_\delta, \rho) &= \int_\Omega |D\phi_{F_\delta}| - \int_\Omega |D\phi_M| \leq \int_\Omega |D\phi_{M_\delta}| - \int_\Omega |D\phi_M| \\
&= \int_{D^+} |D\phi_{F_\delta^+}| + \int_{D^-} |D\phi_{F_\delta^-}| - \int_{D^+} |D\phi_{F_\delta}| - \int_{D^-} |D\phi_{F_\delta}|.
\end{aligned}
$$

From Lemma 2.9 we have:

$$\left|\int_{D^+} |D\phi_{F_\delta^+}| - \int_{D^+} |D\phi_{F_\delta}|\right| \le Q_0^+|v^+|,$$
$$\left|\int_{D^-} |D\phi_{F_\delta^-}| - \int_{D^-} |D\phi_{F_\delta}|\right| \le Q_0^-|v^-|.$$

From the choice of v^+, v^- the estimate (2.39) follows.This ends the proof.

□

Remark 2.2.

(1) Lemma 2.10 easily generalizes to higher space dimensions.

(2) Similiar to Lemma 2.10 results can be found for example in [M,S].

It is intuitionally clear that if a set E is connected then the global minimizer of the perimeter in $\mathcal{S}(E,\delta), F_\delta$ should also be connected, at least for sufficiently small δ.For our purposes the following lemma is sufficient.

Lemma 2.11. *Let $A \in (0, |\Omega|)$ be a given number and let $E \in \mathcal{D}_A$ be a connected, proper subset of Ω.Assume that $\partial E = \gamma$ is a $C^{1,\alpha}$ curve with endpoints $\gamma^0, \gamma^1 \in \partial\Omega, \gamma_0 \neq \gamma_1$.Let $\{E_n\}$ be a sequence of open subsets of Ω whose boundaries consist of $C^{1,\alpha}$ curves satisfying:*

$$\lim_{n\to\infty} \int_\Omega |\phi_{E_n} - \phi_E| = 0,$$
$$\lim_{n\to\infty} \int_\Omega |D\phi_{E_n}| = \int_\Omega |D\phi_E|. \tag{2.45}$$

Then there exist a sequence $\{G_n\}$ of open, connected, proper subsets of Ω such that $\partial G_n = \gamma_n$ is for each sufficiently large n a $C^{1,\alpha}$ curve in Ω.Moreover:

(i)

$$\lim_{n\to\infty} \sup_{x\in\gamma_n} \operatorname{dist}(x,\gamma) = 0.$$

In addition if γ_n^0, γ_n^1 denote the endpoints of γ_n then we can choose the orientation on γ_n such that $\gamma_n^i \to \gamma^i, i = 1, 2$ as $n \to \infty$.

(ii) *(2.45) holds with G_n in place of E_n and moreover:*

$$\int_\Omega |D\phi_{G_n}| \le \int_\Omega |D\phi_{E_n}|, \tag{2.46}$$

with equality holding only if $\partial E_n \cap \Omega$ is a single $C^{1,\alpha}$ curve.

Proof of Lemma 2.11. Let $E_n = \bigcup_j E_{n_j}$ be for each E_n its decomposition to open, disjoint, connected components.The idea of the proof is to find γ_n's among the components of ∂E_{n_j}.First observe that (2.45) implies that:

$$\int_\Omega |D\phi_{E_n}| \le const. \tag{2.47}$$

We claim that there exists a constant $d > 0$ depending only on E such that for each sufficiently large n there exists $\tilde{E}_n$, a component of E_n, satisfying:

$$|\tilde{E}_n| > d. \tag{2.48}$$

We show the claim by contradiction.Let

$$b_n := \sup_j \int_\Omega \phi_{E_{n_j}}$$

and assume that $\liminf_n b_n = 0$.Let $x \in E$ be a fixed point and ρ be such that $\overline{B_\rho}(x) \subset E$.From (2.45) there exists a positive integer N_ρ such that for all $n > N_\rho$ we have:

$$\int_{B_\rho(x)} \phi_{E_n} \ge \frac{1}{2}\pi\rho^2.$$

Let $\{b_{n_k}\}$ be a subsequence of $\{b_n\}$ such that:

$$b_{n_k} \le \frac{\pi\rho^2}{2^{k+1}}.$$

By the definition of N_ρ and b_{n_k} there must exists at least 2^k disjoint subsets of each set E_{n_k}, which we denote by $A_{ki}, i = 1, \ldots, 2^k$ satisfying:

$$\frac{\pi\rho^2}{2^{k+1}} \ge \int_{B_\rho(x)} \phi_{A_{ki}} = |B_\rho(x) \cap A_{ki}| \ge \frac{\pi\rho^2}{2^{k+2}}, \qquad i = 1, \ldots, 2^k.$$

The choice of the sets A_{ki} can be accomplished by selecting first all those disjoint connected components of $E_{n_k} \cap B_\rho(x)$ whose measure falls between $\frac{\pi\rho^2}{2^{k+1}}$ and $\frac{\pi\rho^2}{2^{k+3}}$.Next we take those among remaining connected components whose measure is between $\frac{\pi\rho^2}{2^{k+2}}$ and $\frac{\pi\rho^2}{2^{k+3}}$ and group them in pairs.We can continue this procedure untill the sets A_{ki} are constructed.From the Isoperimetric Inequality we have (with constant c_2 independent on ρ):

$$\int_{B_\rho(x)} |D\phi_{A_{ki}}| \geq c_2 \frac{\rho}{2^{\frac{k}{2}}} \qquad i = 1, \ldots, 2^k,$$

hence

$$\int_\Omega |D\phi_{E_{n_k}}| \geq \sum_{i=1}^{2^k} \int_{B_\rho(x)} |D\phi_{A_{ki}}| \geq c_2 \pi \rho 2^{\frac{k}{2}},$$

therefore taking k large enough we get a contradiction with (2.47).The proof of the claim is complete.

Now we shall choose sets G_n.Notice that it is natural to take as G_n this component of E_n which has the largest measure.This however in general will not gaurantee that ∂G_n is a single curve.That is why we need an intermediate step.Let for each n the set F_n be the largest in the sense of measure among those connected components of E_n which satisfy (2.48).Smoothness of ∂E_n implies that $\partial E_n \cap \Omega = \bigcup_j \gamma_{nj}$ where each γ_{nj} is a $C^{1,\alpha}$ curve.Each such curve divides Ω into two disjoint subregions one of which includes F_n.We denote this subregion by $G_{\gamma_{nj}}$ so that $F_n \subset G_{\gamma_{nj}}$.Finally we define γ_n to be the longest among γ_{nj}'s and we set $G_n = G_{\gamma_n}$.

We will first establish (i).We claim that there exists a sequence of points $x_n \in \gamma_n$ such that:

$$\lim_{n \to \infty} \text{dist}(x_n, \gamma) = 0. \tag{2.49}$$

We show (2.49) by contradiction.For $\epsilon > 0$ we set:

$$\mathcal{N}_\epsilon = \{x \in \Omega \mid \text{dist}(x, \gamma) < \epsilon\}.$$

If (2.49) does not hold then there exists ϵ such that for each sufficiently large n we have $\gamma_n \cap \mathcal{N}_\epsilon = \emptyset$.Consequently:

$$\int_\Omega |D\phi_{E_n}| \geq \int_{\mathcal{N}_\epsilon} |D\phi_{E_n}| + |\gamma_n|.$$

Passing to the limit in the above expression we obtain by using the first assumption in (2.45) and the lower semicontinuity of the perimeter:

$$\begin{aligned}\int_\Omega |D\phi_E| = \lim_{n\to\infty}\int_\Omega |D\phi_{E_n}| &\geq \liminf_{n\to\infty}\int_{\mathcal{N}_\epsilon} |D\phi_{E_n}| + \liminf_{n\to\infty}|\gamma_n| \\ &\geq \int_{\mathcal{N}_\epsilon} |D\phi_E| + \liminf_{n\to\infty}|\gamma_n|\end{aligned}$$

hence $\liminf_{n\to\infty}|\gamma_n| = 0$.But this contradicts the choice of γ_n since neither $|F_n| \to 0$ nor $|F_n| \to |\Omega|$.This ends the proof of the claim.

To finish the proof of the first part of (i) it sufficies to show that for each $\epsilon > 0$ there exists n_ϵ such that $\gamma_\epsilon \subset \mathcal{N}_\epsilon$ for $n > n_\epsilon$.We again establish this by contradiction.Let ϵ be such that that there exists a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ and a seqeunce of points $y_k \in \gamma_{n_k}$ such that $\{y_k\}\cap\mathcal{N}_\epsilon = \emptyset$ for all $k \geq 1$.Consider the set $\mathcal{N}_{\frac{\epsilon}{2}}$.From the previous claim we conclude that there exists a sequence of points $\{x_k\} \in \gamma_{n_k}\cap\mathcal{N}_{\frac{\epsilon}{2}}$ converging to γ as $k\to\infty$.We have:

$$\int_\Omega |D\phi_E| \geq \liminf_{k\to\infty}\int_{\mathcal{N}_{\frac{\epsilon}{2}}} |D\phi_{E_k}| + \liminf_{k\to\infty}\int_{\Omega\setminus\mathcal{N}_{\frac{\epsilon}{2}}} |D\phi_{E_k}| \geq \int_{\mathcal{N}_{\frac{\epsilon}{2}}} |D\phi_E| + \frac{\epsilon}{2},$$

a contradiction with $\epsilon > 0$.

We shall establish the second assertion in (i).By γ_n^0,γ_n^1 we will denote the endpoints of γ_n.If $\mathrm{dist}(\gamma_n^0,\gamma_n^1) \to 0$ as $n\to\infty$ then for each sufficiently small $\epsilon > 0$ we have either $G_n \subset \mathcal{N}_\epsilon$ or $\Omega\setminus\mathcal{N}_\epsilon \subset G_n\setminus\mathcal{N}_\epsilon$.But this would imply that either $|G_n| \to 0$ or $|G_n| \to |\Omega|$ which is impossible.Thus we have $\lim_{n\to\infty}\mathrm{dist}(\gamma_n^0,\gamma_n^1) > 0$.Since $\gamma_n \to \gamma$ by (i) therefore $\gamma_n^i \to \gamma^i, i = 1,2$ (after possibly changing the orientation of γ_n).The whole assertion (i) is now established.

From (i) it follows that the first equality in (2.45) holds with G_n in place of E_n.From the lower semicontinuity of the perimeter we also get:

$$\begin{aligned}\int_\Omega |D\phi_E| = \limsup_{n\to\infty}\int_\Omega |D\phi_{E_n}| &\geq \limsup_{n\to\infty}\int_\Omega |D\phi_{G_n}| \\ &\geq \liminf_{n\to\infty}\int_\Omega |D\phi_{G_n}| \geq \int_\Omega |D\phi_E|.\end{aligned}$$

The last part of (ii) follows directly from the construction.

□

Now we are in the position to prove the main result of this section.

Lemma 2.12. *Let (s_0, σ_0) be a strict local minimum of L in $\mathcal{S}_A$ and $w_0 = w(\cdot, s_0, \sigma_0)$ be the corresponding circular arc.Then the set E cut off from Ω by w_0 is an isolated local minimizer of the perimeter (in the sense of the Definition 2.1) in $\mathcal{D}_A$.*

Proof of Lemma 2.12. We prove the lemma by contradiction.For each sufficiently small δ by F_δ we denote as in Lemma 2.10 the minimizer of the perimeter in $\mathcal{S}(E,\delta)$.If E is not an isolated local minimizer of the perimeter then there exists a sequence $\{\delta_n\}, \delta_n \to 0$ as $n \to \infty$ such that the sets $F_n := F_{\delta_n}$ satisfy:

$$\int_\Omega |D\phi_{F_n}| \leq \int_\Omega |D\phi_E|. \tag{2.50}$$

Since $F_n \in \mathcal{S}(E, \delta_n)$ therefore the lower semicontinuity of the perimeter implies:

$$\int_\Omega |D\phi_E| \geq \limsup_{n\to\infty} \int_\Omega |D\phi_{F_n}| \geq \liminf_{n\to\infty} \int_\Omega |D\phi_{F_n}| \geq \int_\Omega |D\phi_E|,$$

hence the family $\{F_n\}$ satisfies the assumptions of Lemma 2.11.Let $\{G_n\}$ denote a family of sets satisfying assertions (i), (ii) of the same lemma and set $\gamma_n = \partial G_n$.

In general $G_n \notin \mathcal{D}_A$ but we will show now that we can modify G_n for each sufficiently large n in such a way that the new set $\tilde{G}_n$ satisfies $\tilde{G}_n \in \mathcal{D}_A$.Let:

$$F_n^- = G_n \setminus F_n,$$

$$F_n^+ = F_n \setminus G_n,$$

$$|F_n \setminus G_n| - |G_n \setminus F_n| = v_n.$$

Notice that $A = |F_n| = |G_n| + v_n$.From Lemma 2.11 we know that $v_n \to 0$ as $n \to \infty$.Let ϵ be chosen such that $(s_0 - \epsilon, s_0 + \epsilon) \times (\sigma_0 - \epsilon, \sigma_0 + \epsilon) \subset \mathcal{S}_A$ and let $\mathcal{N}_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, w_0) < \epsilon\}$.We shall apply Lemma 2.9 with G_n in place of F and $D = \mathcal{N}_\epsilon$.Let v_0, ρ_0, Q_0 be constants defined in Lemma 2.9 and therefore depending only on $E,\mathcal{N}_\epsilon$.By choosing n sufficiently large we can guarantee that:

$$\gamma_n \subset \mathcal{N}_\epsilon \qquad \text{by Lemma 2.11,}$$

$$v_n < v_0, \qquad \int_\Omega |\phi_E - \phi_{G_n}| < \rho_0 \qquad \text{by Lemma 2.9.}$$

Therefore Lemma 2.9 applies for the set G_n in place of F hence there exists a set $\tilde{G}_n$ such that:

$$|\tilde{G}_n| = |G_n| + v_n = A,$$

$$\left| \int_\Omega |D\phi_{G_n}| - \int_\Omega |D\phi_{\tilde{G}_n}| \right| \leq Q_0 |v_n|. \tag{2.51}$$

Moreover from the proof of Lemma 2.9 it follows that γ_n and $\tilde{\gamma}_n = \partial \tilde{G}_n$ are diffeomorphic and $\tilde{\gamma}_n \subset \mathcal{N}_\epsilon$.From the Isoperimetric Inequality there exists a constant $C(\Omega)$ such that:

$$\int_\Omega |D\phi_{G_n \setminus F_n}| \geq C(\Omega)|G_n \setminus F_n|^{\frac{1}{2}},$$

$$\int_\Omega |D\phi_{F_n \setminus G_n}| \geq C(\Omega)|F_n \setminus G_n|^{\frac{1}{2}}. \tag{2.52}$$

From (2.51), (2.52) it follows that for all sufficiently large n we have:

$$\begin{aligned}
\int_\Omega |D\phi_{F_n}| - \int_\Omega |D\phi_{\tilde{G}_n}| &= \int_\Omega |D\phi_{G_n}| - \int_\Omega |D\phi_{\tilde{G}_n}| + \int_\Omega |D\phi_{G_n \setminus F_n}| + \int_\Omega |D\phi_{F_n \setminus G_n}| \\
&\geq -Q_0|v_n| + C(\Omega)|G_n \setminus F_n|^{\frac{1}{2}} + C(\Omega)|F_n \setminus G_n|^{\frac{1}{2}} \\
&= -Q_0 \big| -|G_n \setminus F_n| + |F_n \setminus G_n| \big| + C(\Omega)|G_n \setminus F_n|^{\frac{1}{2}} + C(\Omega)|F_n \setminus G_n|^{\frac{1}{2}} \\
&\geq -Q_0|G_n \setminus F_n| + C(\Omega)|G_n \setminus F_n|^{\frac{1}{2}} - Q_0|F_n \setminus G_n| + C(\Omega)|F_n \setminus G_n|^{\frac{1}{2}} \\
&\geq 0.
\end{aligned} \tag{2.53}$$

We first show that the last inequality is strict.For if we have equality in (2.53) then $|G_n \setminus F_n| = 0 = |F_n \setminus G_n|$ hence we must have $F_n = G_n \in \mathcal{D}_A$ for all sufficiently large n which implies that $\partial F_n = \gamma_n$ is a $C^{1,\alpha}$ embedded curve.By Lemma 2.3:

$$\int_\Omega |D\phi_{F_n}| \geq L(s_0, \sigma_0)$$

and this inequality is strict unless $F_n = E$.But this contradicts the fact that $F_n \in \mathcal{S}(E, \delta_n)$ with $\delta_n > 0$.We conclude therefore that the inequality in (2.53) is strict.This, however leads to a contradiction as well.Since $\tilde{G}_n \in \mathcal{D}_A$ and $\tilde{\gamma}_n \subset \mathcal{N}_\epsilon$ is a $C^{1,\alpha}$ curve Lemma 2.3 implies:

$$L(s_0, \sigma_0) = \int_\Omega |D\phi_E| \leq \int_\Omega |D\phi_{\tilde{G}_n}|,$$

while from (2.50) and (2.53) we get:

$$\int_\Omega |D\phi_{\tilde{G}_n}| < \int_\Omega |D\phi_{F_n}| \leq \int_\Omega |D\phi_E|,$$

a contradiction.The proof of the lemma is complete.

□

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