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Properties of Ideal-Based Zero-Divisor Graphs of Commutative Rings

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To the Graduate Council:

I am submitting herewith a dissertation written by Jesse Gerald Smith Jr. entitled "Properties of Ideal-Based Zero-Divisor Graphs of Commutative Rings." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Properties of Ideal-Based Zero-Divisor Graphs of Commutative Rings

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Lovingly dedicated to my parents Jesse Gerald Smith Sr. and Barbara Faye Smith.

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This I recall to my mind, therefore have I hope.

It is of the LORD's mercies that we are not consumed, because his compassions fail not. They are new every morning: great is thy faithfulness.

Lamentations 3:21-23 (King James Version)

Abstract

Let R be a commutative ring with nonzero identity and I an ideal of R. The focus of this research is on a generalization of the zero-divisor graph called the ideal-based zero-divisor graph for commutative rings with nonzero identity. We consider such a graph to be nontrivial when it is nonempty and distinct from the zero-divisor graph of R. We begin by classifying all rings which have nontrivial ideal-based zero-divisor graph complete on fewer than 5 vertices. We also classify when such graphs are complete on a prime number of vertices. In addition we classify all rings which admit nontrivial planar ideal-based zero-divisor graph. The ideas of complemented and uniquely complemented are considered for such graphs, and we classify when they are uniquely complemented. The relationship between graph isomorphisms of the ideal-based zero divisor graph with respect to I and graph isomorphisms of the zero-divisor graph of R/I [R mod I] is also considered. In the later chapters, we consider properties of ideal-based zero-divisor graphs when the corresponding factor rings are Boolean or reduced. We conclude by giving all nontrivial ideal based zero-divisor graphs on less than 8 vertices, a few miscellaneous results, and some questions for future research.

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Chapter 1

Preliminaries

Let us begin by stating our set theory notation. Given a set X, we denote the cardinality of X by $|X|$. We write $A \subseteq B$ if A is a subset of B. We consider A to be a proper subset of B if $A \subseteq B$ and $A \neq B$. We will write $A \subsetneq B$ to denote that A is a proper subset of B. The notation $A \setminus B$ means the set-theoretic difference of B from A. For example, $\mathbb{R} \setminus \mathbb{Q}$ is the set of irrational real numbers.

Throughout this paper, by a *graph* we mean a simple graph. A *simple graph* is an undirected graph without multiple edges or loops. A graph G is a pair of sets V and E, of vertices and edges respectively, where E consists of sets $\{a, b\}$ and $a, b \in V$. Graphs are often visualized by drawing the vertices as dots and the edges by lines connecting the dots.

Example 1.1. Let G be defined by the vertex set $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{a, c\}, \{d, e\}\}$. We can visualize the graph as in Figure 1.1.

In graph theory, it is of no interest what the vertices are named or how the edges are drawn. This is formalized by the concept of a graph isomorphism.

Definition 1.2. Let the graph G be defined by vertex set V and edge set E. Let H be the graph defined by vertex set V' and edge set E'. A *graph isomorphism* from G to H is a bijection $\phi : V \to V'$ such that $\{\phi(a), \phi(b)\} \in E'$ if and only if $\{a, b\} \in E$.

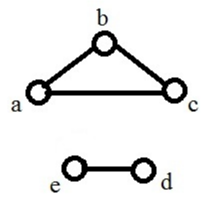


Figure 1.1: Graph defined as in Example 1.1

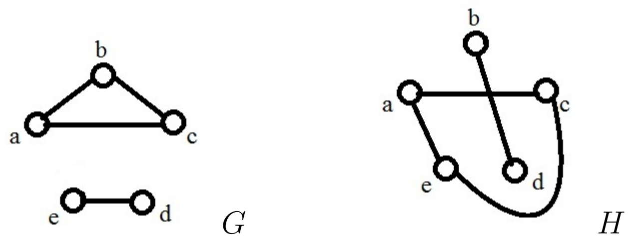


Figure 1.2: Isomorphic Graphs

In other words, a graph isomorphism is a bijection between the vertex sets which preserves edges.

The following example gives an isomorphism between two graphs.

Example 1.3. The two graphs G and H in Figure 1.2 are isomorphic by the following isomorphism: $a \to a, b \to c, c \to e, d \to d, e \to b$.

Before proceeding, we define some basic descriptive terminology for graphs. We say that two edges are *incident* if they share a common vertex, and we say that two vertices are *adjacent* if there is an edge between them. A *path* in graph is a sequence of incident edges or adjacent vertices. For example, $a - b - c$ is a path in Figure 1.1. One can say that the latter is a path from vertex a to vertex c. The *length* of a path is the number of edges traversed during the path. We define the *distance* between two distinct vertices x and y, denoted $d(x, y)$, to be the length of a shortest path from x to y provided a path exists, and we define $d(x, y) = \infty$ otherwise. Moreover, we set $d(x, x) = 0$. For example, the graph in Figure 1.1 has $d(a, c) = 1$ and $d(a, d) = \infty$. The *diameter* of a graph G, denoted $diam(G)$, is defined as $\sup\{d(x,y) \mid x$ and y are distinct vertices of $G\}$ provided G has at least two vertices and 0 otherwise. A *cycle* in a graph is a path that begins and ends at the same vertex. For example, $a - b - c - a$ is a cycle in the graph in Figure 1.1. The *girth* of a graph G, denoted $gr(G)$, is defined to the length of a shortest cycle in G provided a cycle exists and ∞ otherwise. The graph in Figure 1.1 has girth 3. A graph G is *connected* provided there is a path between any two distinct vertices. We will consider a graph on one vertex to be connected. The graph in Figure 1.1 is not connected. At times, we will let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G.

A graph H is a *subgraph* of G, denoted $H \subseteq G$, if the $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *subgraph generated by a subset of vertices* W is the subgraph consisting of those vertices together with all edges of the original graph between the vertices of the subset W; this is denoted $\langle W \rangle$. We say that H is an *induced subgraph* of G, if $\langle V(H) \rangle = H$. A *complete graph* on n vertices is a graph consisting of n vertices where every pair of distinct vertices is adjacent; we denote such a graph by K^n. A *complete bipartite graph* G is a graph for which there exists disjoint non-empty subsets A, B of vertices such that two vertices of G are adjacent if and only if one vertex is in A and and the other vertex is B (we sometimes write this as $G = A \cup B$); we denote such a graph by $K^{m,n}$, where $m = |A|$ and $n = |B|$. A well-known complete bipartite graph is $K^{3,3}$; it is often referred to as the utility graph. The utility graph and K^5 will be important subgraphs to look for when considering the concept of planarity (which will be defined later in this introduction). The graphs $K^{1,n}$ are often called *star graphs* because of their resemblence to a star shape. Another special graph that will arise is $\overline{K}^{1,3}$. In general, we let $\overline{K}^{m,3}$ be the graph defined by joining $G = K^{m,3} (= A \cup B$, where $|A| = m$ and $|B| = 3)$ to the star graph $H = K^{1,m}$ by identifying the center of H with a point of B [3, p 2]. In particular, $\overline{K}^{1,3}$ is the graph in Figure 1.3.

Let R be a commutative ring with nonzero identity. We call $x \in R$ a *zero-divisor* of R if there exists $0 \neq r \in R$ such that $rx = 0$. In the 1980's, Beck used the idea of zero-divisors to produce a simple graph given a ring R [12]. This was called the

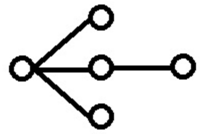


Figure 1.3: $\overline{K}^{1,3}$

zero-divisor graph of R. Beck was interested in colorings of these graphs. Given a graph G, a *coloring* is an assignment of "colors" to vertices in such a way that no two adjacent vertices have the same assigned "color." When considering colorings, the most common questions deal with a graph's chromatic number. A graph's *chromatic number* is the fewest number of colors required to "color" the graph. In the late 1990s, David F. Anderson and Philip S. Livingston modified Beck's definition to be the following [7, 23]:

Definition 1.4. Let R be a commutative ring with nonzero identity. Let $Z(R)$ be the zero-divisors of R and set $Z(R)^* = Z(R) \setminus \{0\}$. Then the *zero-divisor graph* of R, denoted $\Gamma(R)$, is the graph on the vertex set $Z(R)^*$, where two distinct vertices x and y are adjacent if and only if $xy = 0$.

While Beck was primarily interested in colorings of a ring's associated graph, Anderson and Livingston shifted the focus to the interplay between ring-theoretic properties and graph-theoretic properties. That is, if the ring R has certain properties, does this then induce certain graph-theoretic properties on its associated graph (and vice-versa). Anderson and Livingston's modified definition soon became the accepted "modern" definition of the zero-divisor graph of a ring R. This definition can be extended to non-commutative rings rather easily in several different ways (i.e., we could consider the left-sided zero-divisor graph of a given ring R) [27, 28, 29]. Beck's definition of the zero-divisor graph differed from the current definition in that it was a graph on the vertex set R. With Beck's definition, every graph contained a star subgraph where 0 was adjacent to every other vertex.

Many authors have considered properties of the zero-divisor graph. The following theorem encapsulates the basic properties of zero-divisor graphs. For a survey of research done on the zero-divisor graph, we recommend the recent article by Anderson et al. [3].

Theorem 1.5. *(Basic Properties of the Zero-divisor Graph) Let R be a commutative ring with nonzero identity and let $\Gamma(R)$ be the zero-divisor graph of R.*

1. *$\Gamma(R)$ is connected [23, Theorem 8], [7, Theorem 2.3].*
2. *$\Gamma(R)$ is empty if and only if R is an integral domain.*
3. *$\Gamma(R)$ has finitely many vertices if and only if R is finite or an integral domain [7, Theorem 2.2].*
4. *$diam(\Gamma(R)) \in \{0, 1, 2, 3\}$ [7, Theorem 2.3].*
5. *$gr(\Gamma(R)) \in \{3, 4, \infty\}$.*

Property 5 in Theorem 1.5 has an interesting history as described in [3]. The proof was first shown only for Artinian rings in [7]. The result was proven for general commutative rings independently by several authors [11, 17, 25, 37].

The primary focus of this dissertation is a generalization of the zero-divisor graph called the *ideal-based zero-divisor graph*. In 2001, S. P. Redmond gave the following definition ([27] and [30]).

Definition 1.6. Let R be a commutative ring with nonzero identity and I an ideal of R. Define $\Gamma_I(R)$ to be the graph on vertices $\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. This is called the *ideal-based zero-divisor graph* of R with respect to the ideal I.

This turns out to be a very natural generalization of the zero-divisor graph. This can be seen in the following theorem. Property (7) of the following will be of great use throughout this work.

Theorem 1.7. *[30] Let R be a commutative ring with nonzero identity and I an ideal of R.*

1. *If $I = \{0\}$, then $\Gamma_I(R) = \Gamma(R)$.*
2. *$\Gamma_I(R) = \emptyset$ if and only if I is a prime ideal of R.*
3. *$\Gamma_I(R)$ is connected.*
4. *$diam(\Gamma_I(R)) \in \{0, 1, 2, 3\}$.*
5. *$gr(\Gamma_I(R)) \in \{3, 4, \infty\}$.*
6. *$\Gamma_I(R)$ contains $|I|$ disjoint subgraphs each isomorphic to $\Gamma(R/I)$.*
7. *$|V(\Gamma_I(R))| = |I||V(\Gamma(R/I))|$.*

Proof. Properties (1) and (2) are [30, Proposition 2.2]. Properties (3) and (4) are [30, Theorem 2.4]. Property (5) is [30, Theorem 5.5]. Finally, Property (6) is [30, Corollary 2.7]. Property (7) is an easy consequence of (6). □

Properties of the ideal-based zero-divisor graph have been studied by various authors. In both [27, 30], Redmond notes a strong connection between $\Gamma_I(R)$ and $\Gamma(R/I)$. Redmond describes a three step construction method for $\Gamma_I(R)$ based on $\Gamma(R/I)$. The method is described below. Notice that the key factors in the construction method are $\Gamma(R/I)$, $|I|$, and the concept of connected columns.

Redmond's Three Step Construction for Building $\Gamma_I(R)$

1. Let $\{a_\lambda\}_{\lambda\in\Lambda}$ be a set of coset representatives of $V(\Gamma(R/I))$. For each $i \in I$, define a graph G_i with vertices $\{a_\lambda + i\}_{\lambda\in\Lambda}$, where $a_\lambda + i$ is adjacent to $a_\beta + i$ if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(R/I)$. To generate $\Gamma_I(R)$, define G to have vertices $\cup_{i\in I} G_i$. To draw the graph, draw each G_i (each are isomorphic) in successive rows including the edges contained in each G_i.

2. For distinct $\lambda, \beta \in \Lambda$ and for each $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\beta + j$ if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(R/I)$. (At this stage, we are connecting the rows together (avoiding connecting along a column)).

3. For each $\lambda \in \Lambda$ and distinct $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\lambda + j$ if and only if $a_\lambda^2 \in I$. In this step, we are connecting the columns as necessary (i.e., if and only if $a_\lambda^2 \in I$). Such columns are called *connected columns*.

Example 1.8. Example (A): Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = 0 \times 0 \times \mathbb{Z}_2$. Then $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Whence $\Gamma(R/I)$ is as in the first row of Step 1 of (A) from Figure 1.4. Since $|I| = 2$, we draw 2 rows of this graph in Step 1. Notice that for R/I, no zero-divisor has the property that its square is zero. Therefore there are no connected columns in Step 3.

Example (B): Let $S = \mathbb{Z}_9 \times \mathbb{Z}_2$ and $J = 0 \times \mathbb{Z}_2$. Then $S/J \cong \mathbb{Z}_9$. Notice that $\Gamma(R/I) \cong \Gamma(S/J)$ (both are line graphs on two vertices, i.e., K^2) and $|I| = |J| = 2$; hence Steps 1 and 2 are the same as in part (A). However, here $Z(S/J)^* = \{3, 6\}$ and $3^2 = 0$ and $6^2 = 0$. Thus both "columns" are connected. This requires connections to be made in Step 3.

This example shows that $\Gamma(R/I) \cong \Gamma(S/J)$ and $|I| = |J|$ does not imply $\Gamma_I(R) \cong \Gamma_J(S)$.

Redmond studied the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$, and this study was continued by other authors. An example of this continued investigation is [9]. Notice that in [9], most results are considered in light of two separate cases: whether or not I is a radical ideal of R. When I is not a radical ideal, then $\Gamma_I(R)$ will have connected columns. On the other hand, when I is a radical ideal, then $\Gamma_I(R)$ will not have connected columns. Hence considering the properties of $\Gamma_I(R)$ under each case (I radical or non-radical) separately is of great aide.

A graph G is *planar* if it can drawn in a plane such that no edges cross. In this paper, one of our goals is to classify when an ideal-based zero-divisor graph is planar. We consider $\Gamma_I(R)$ to be *nontrivial* if I is a nonzero, proper, non-prime ideal of R.

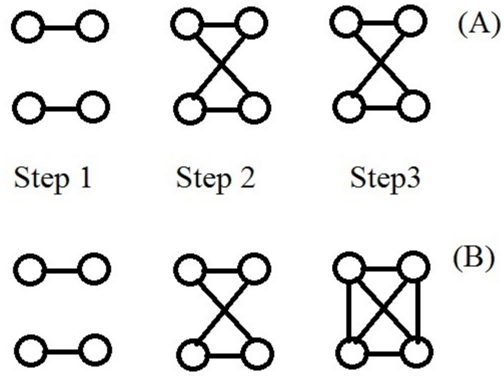


Figure 1.4: Graph defined as in Example 1.8

The latter requirements forces $\Gamma_I(R)$ to be distinct from $\Gamma(R)$ and to be nonempty. In order to achieve this goal, we will use the celebrated Kurtowski's Theorem from Graph Theory [14, Theorem 6.13]. To state the theorem, we need to define a graph subdivision.

Definition 1.9. Let G and H be graphs. Then H is a *subdivision* of G if H can be derived from G by applying the following operations:

1. Adding a vertex on an edge, that is, replacing $v - w$ (vertices v, w are adjacent) by $v - a - w$, where a is a new vertex.
2. Replacing a vertex adjacent to only two vertices by only an edge (undoing item 1).

Theorem 1.10. *(Kuratowski's Theorem) A graph G is planar if and only if it does not contain a subgraph which is a subdivision of K^5 or $K^{3,3}$.*

Example 1.11. Consider the graphs G and H as in Figure 1.5. Notice that G is a subdivision of H. Moreover, $H = K^{3,3}$, and therefore G has a subgraph (the graph itself) that is a subdivision of $K^{3,3}$. Hence G is not planar by Kuratowski's Theorem.

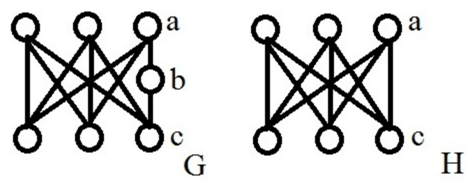


Figure 1.5: Graphs G and H are subdivisions of each other. (Example 1.11)

In Chapter 2, we begin by classifying which finite commutative rings have zero-divisor graphs isomorphic to K^n for small values of n. Using this information and Kuratowski's Theorem, we will determine, in Chapter 3, all rings up to isomorphism with non-trivial planar ideal-based zero-divisor graph. This will be accomplished by first finding what restraints planarity forces on $|I|$ and the girth of $\Gamma(R/I)$.

Before proceeding, we recall some definitions and facts from Abstract Algebra. Let R be a commutative ring with nonzero identity. We say that $e \in R$ is *idempotent* if $e^2 = e$, and we denote the set of all idempotent elements by $Idem(R)$. A ring R is called a *Boolean ring* if $Idem(R) = R$. This is a special case of what is called a it von Neumann regular ring. A ring R is a *von Neumann regular ring* if for every $x \in R$, there exists a $y \in R$ such that $x = xyx$. A good reference for properties of von Neumann regular rings can be found in [4]. Let X be an indeterminate; then $R[X]$ is the polynomial ring with coefficients in R and indeterminate X. If I is an ideal of $R[X]$, we will often write $R[X]/I = R[x]$, where $x = X + I$ (the image of the indeterminate X). We define a ring R to be local if it has a unique maximal ideal. We let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{F}_q$ denote the natural numbers, the integers, the integers modulo n, and the field of q elements, respectively. We will denote the ideal of nilpotent elements of a ring R by $nil(R)$. We say that ring R is *reduced* if $nil(R) = 0$. Given an ideal I of R, we define $\sqrt{I} = \{r \in R \mid r^k \in I \text{ for some } k \in \mathbb{N}\}$. Notice that R/I is reduced if and only if $I = \sqrt{I}$. We say that an ideal I is a *radical* ideal if $I = \sqrt{I}$. For all other undefined algebra concepts, we direct the reader to [19].

We will use a special case of the following result throughout this paper. If R is an Artinian commutative ring, then R is isomorphic to a finite direct product of local

Artinian rings [10, Theorem 8.7]. In particular, if R is a finite commutative ring, then R is isomorphic to a finite direct product of finite local rings. We will also use that local rings have only the trivial idempotents (i.e., $\{0, 1\}$). To see this, note that if R has a non-trivial idempotent e then $R \cong Re \times R(1-e)$, where Re and $R(1-e)$ are both non-zero. It is clear then that R has at least two maximal ideals.

Another standard result we will use from Abstract Algebra is that the ideals of the ring $\prod_{\lambda \in \Lambda} R_\lambda$ (where each R_λ is a ring and Λ is finite) are of the form $\prod_{\lambda \in \Lambda} I_\lambda$ where each I_λ is an ideal of R_λ for all $\lambda \in \Lambda$.

It has been seen in [6] that a ring being von Neumann regular is closely related to its zero-divisor graph being complemented (this will be defined at the beginning of Chapter 4). A generalization of the latter relationship to $\Gamma_I(R)$ is considered in Chapter 4. In Chapter 5, we consider how isomorphisms on the level of the ideal-based zero-divisor graph relate to isomorphisms on the level of the factor graph (i.e., $\Gamma(R/I)$). We consider properties induced on $\Gamma_I(R)$ when either R or R/I is Boolean in Chapter 6. Inspired by research on isomorphisms of ideal-based zero-divisor graphs, we turn to the consideration of the number of vertices and edges of $\Gamma(R)$ when R is reduced. In Chapter 7, a computer program is given to determine if assuming two reduced rings have zero-divisor graphs with the same number of vertices and same number of edges forces the two rings to be isomorphic. The answer turns out to be negative, but the example is not as small as one might expect. In the penultimate chapter, we determine all possible graphs for $\Gamma_I(R)$ on a small number of vertices. Finally, we close with a few miscellaneous results and some questions for future research.

Chapter 2

When $\Gamma_I(R)$ is Complete on up to Five Vertices

Our first goal is to determine when $\Gamma_I(R)$ is planar. In order to do this, we will need to consider local rings of order 4, 8, and 16. We will also need to consider when $\Gamma_I(R)$ is complete on up to 5 vertices. In researching the previous question, we found that we can classify up to isomorphism all commutative rings with $\Gamma_I(R)$ isomorphic to K^p, where p is an odd, non-Mersenne prime. This chapter consists of the work on classifying when $\Gamma_I(R)$ is complete.

2.1 When $|Z(R)| = 2$

We begin this section with a well-known result.

Lemma 2.1. *Let R be a commutative local ring with nonzero identity and $|R| = 4$. Then*

$$R \cong \mathbb{Z}_4,\ \mathbb{Z}_2[X]/(X^2),\ \text{or}\ \mathbb{F}_4.$$

Proof. Since $|R| = 4$, $char(R) \in \{2, 4\}$. If $char(R) = 4$, then it is evident that $R \cong \mathbb{Z}_4$. Otherwise, assume $char(R) = 2$. Let M be the unique maximal ideal of R. Then we must have $|M| = 1$ or $|M| = 2$. If $|M| = 1$, then R must be a field with four elements. Since finite fields are determined up to isomorphism by their cardinality, $R \cong \mathbb{F}_4$. If $|M| = 2$, then $M = \{0, x\}$. Since R is local, R has no nontrivial idempotents, and thus $x^2 = 0$. Consider the evaluation homomorphism $\phi : \mathbb{Z}_2[X] \to R$ given by sending X to x. Notice that $X^2 \in \ker(\phi)$ since $x^2 = 0$. It is clear that ϕ is onto; so $\mathbb{Z}_2[X]/ker(\phi) \cong R$ by the First Isomorphism Theorem. But $|\mathbb{Z}_2[X]/(X^2)| = 4 = |R|$; thus $ker(\phi) = (X^2)$, and therefore $R \cong \mathbb{Z}_2[X]/(X^2)$. □

The following result is also a well-known. But for completeness, we again provide a detailed proof.

Proposition 2.2. *Let R be a commutative ring with nonzero identity. Then $|Z(R)| = 2$ if and only if $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Moreover, $|V(\Gamma(R))| = 1$ if and only if $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$.*

Proof. Since $|Z(R)| = 2$, there exists a $0 \neq x \in Z(R)$. Consider the map $\phi : R \to Z(R)$ by $\phi(r) = xr$. It is evident that this map is an onto (module) homomorphism with $ker(\phi) = Z(R)$; so it follows from the First Isomorphism Theorem that $|R| = |Z(R)|^2 = 4$. Writing R as product of local rings, we then have either that R is local or isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Now $\mathbb{Z}_2 \times \mathbb{Z}_2$ has 3 zero-divisors and thus R must be a local ring of order 4. The desired result then follows from Lemma 2.1. The "moreover statement" follows from the fact that $|V(\Gamma(R))| = |Z(R)^*|$. □

The following result may be deduced using Theorem 1.7 and Redmond's Construction Method for $\Gamma_I(R)$ from $\Gamma(R/I)$. We will prove it as a corollary to the preceeding result.

Corollary 2.3. *Let R be a commutative ring with nonzero identity and I an ideal of R. If $\Gamma(R/I) \cong K^1$, then $\Gamma_I(R) \cong K^{|I|}$.*

Proof. By Proposition 2.2, $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Thus $V(\Gamma(R/I)) = \{a + I\}$, where $a^2 \in I$. Then $V(\Gamma_I(R)) = \{a+i\}_{i \in I}$. Notice that $(a+i)(a+j) \in I$ for all $i, j \in I$. Moreover $|V(\Gamma_I(R))| = |I||V(\Gamma(R/I))| = |I| \cdot 1 = |I|$. Thus $\Gamma_I(R) \cong K^{|I|}$. □

We note that the content of the preceding lemma appears in [17] and [18] as a lemma used for proving a result about cycles in zero-divisor graphs.

2.2 When $\Gamma_I(R) \cong K^2$

We wish to classify all rings R and nonzero ideals I for which $\Gamma_I(R)$ is isomorphic to K^2. Recall from Property (7) of Theorem 1.7 that $|V(\Gamma_I(R))| = |I||V(\Gamma(R/I))|$. So in this case, $2 = |I||V(\Gamma(R/I))|$. Since we are interested in the case when I is nonzero, the latter forces $|I| = 2$ and $|V(\Gamma(R/I)| = 1$. Under these hypotheses, $|Z(R/I)| = 2$ and thus $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by Proposition 2.2. We begin by classifying all such rings R up to isomorphism. Throughout this section, keep in mind that R/I must be isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$.

This question has previously been answered (in a variety of forms) and published in several papers [2, 9, 7, 12]; however the details of the classification are not given due to the length of the calculation. Since this situation has arisen as a special case of many results, we prove the result in somewhat excruciating detail below.

Recall that $|R/I| = 4$ and $|I| = 2$; therefore $|R| = |R/I||I| = 8$. Hence $char(R) \in \{2, 4, 8\}$. Since R is finite, it is a product of finite local rings. Because $|R| = 8$, we must have that R is isomorphic to a product of either 1, 2, or 3 local rings. If R is isomorphic to a product of 3 local rings, we have that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$; this will give $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ which is a contradiction of our choices for R/I. Thus either R is a finite local ring or a product of two finite local rings.

Assume that $R \cong R_1 \times R_2$, where R_1, R_2 are finite local rings. Since $|R| = 8$, these local rings have cardinality 2 and 4. Without loss of generality, say $|R_1| = 4$ and $|R_2| = 2$. Then $R_2 \cong \mathbb{Z}_2$. Notice that I cannot be of the form $I_1 \times 0$, where I_1 is an ideal of R_1 of order 2, as then we would have $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (which as before can not occur). Thus I must be of the form $0 \times \mathbb{Z}_2$, and hence $R/I \cong R_1$. Using that R/I is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, R must be isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2$ with $I = 0 \times \mathbb{Z}_2$.

Otherwise, R must be a finite local ring. If $char(R) = 8$, then $R \cong \mathbb{Z}_8$. In this case, we can see that $I = (4)$ satisfies our requirements. Otherwise, $char(R) \in \{2, 4\}$. Let M be the maximal ideal of R. Since R/I is not a field (it has a nonzero zero-divisor), we must have that $I \subsetneq M \subsetneq R$. Since $|I| = 2$ and $|M| \mid 8$, we must have $|M| = 4$.

Case: char(R)=4

Since $char(R) = 4$, we have $2 \in Z(R)^*$, and thus $2 \in M$. Since $|M| = 4$, there exists an $a \in M \setminus \{0, 2\}$. Then notice that $a + 2 \in M$. If $a + 2 \in \{0, 2, a\}$, we get a contradiction in each case (recall that $-2 = 2$ since the ring has characteristic 4).

Thus $M = \{0, 2, a, a + 2\}$. We can see that $2a \neq a$ and $2a \neq 2 + a$ (if either $2a = a$ or $2a = 2 + a$, we have in the first case that $a = 0$, and in the second case that $a = 2$; both are contradictions to the choice of a). Thus either $2a = 0$ or $2a = 2$, since $2a \in M$.

We now show that the $2a = 2$ case cannot hold. Assume to the contrary, that is, $2a = 2$. Note that we must have $a^2 \in M$. Since in a local ring there are no nontrivial idempotents, we must have $a^2 = 0, 2,$ or $a + 2$.

If either $a^2 = 0$ or $a^2 = 2$, then $2a = 2 \Rightarrow 0 = 2a^2 = 2a = 2$, which is a contradiction as $char(R) = 4$.

Assume $2a = 2$ and $a^2 = a + 2$. Then $2a = 2 \Rightarrow 2a^2 = 2a \Rightarrow 2a^2 - 2a = 0$. Thus $2a(a - 1) = 0$ and $2a = 2 \neq 0$; therefore $a - 1 \in Z(R) \subseteq M$. But $a \in M$ and $a - 1 \in M$ implies $-1 \in M$, which is a contradiction.

In all the preceding cases, we derived a contradiction. Thus we must have $2a = 0$. As before, R being local eliminates the case that $a^2 = a$. Thus we must have $2a = 0$ and $a^2 \in \{0, 2, a + 2\}$.

If $a^2 = a + 2$ and $2a = 0$, then multiplying both sides by a yields $a^3 = a^2 + 2a = a^2 + 0 = a^2$, and hence $a^4 = a^3$. Thus $a^2 = a^4 = (a^2)^2$. Again, using that local rings only have trivial idempotents, we have $a^2 \in \{0, 1\}$. Since M is a maximal (hence proper) ideal of R, we have $a^2 \neq 1$. Hence $a^2 = 0$. Whence $0 = a^2 = a + 2 \Rightarrow a = 2$, which is a contradiction.

Thus we have reduced to two cases when R is local with maximal ideal $M = \{0, 2, a, a+2\}$ and $char(R) = 4$:

1. $2a = 0$ and $a^2 = 0$, or

2. $2a = 0$ and $a^2 = 2$.

Consider the evaluation homomorphism $\phi : \mathbb{Z}_4[X] \to R$ given by $X \mapsto a$. Notice that $\phi(\mathbb{Z}_4[X])$ has cardinality at least 5. Since the cardinality of the image of ϕ must divide 8, it follows that the cardinality of the image must be 8; thus the map is onto. We now must compute $ker(\phi)$ in the above two cases.

Case 1: $2a = 0$ and $a^2 = 0$.

Notice that $X^2, 2X \in \ker(\phi)$. Moreover, $\mathbb{Z}_4[X]/(X^2, 2X)$ consists of precisely 8 elements. Thus $ker(\phi) = (X^2, 2X)$. Therefore by the First Isomorphism Theorem, $R \cong \mathbb{Z}_4[X]/(X^2, 2X)$.

Case 2: $2a = 0$ and $a^2 = 2$.

We must have $2X, X^2 - 2 \in ker(\phi)$. Again, $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ has 8 elements; whence $\ker(\phi) = (2X, X^2 - 2)$. Therefore by the First Isomorphism Theorem, $R \cong \mathbb{Z}_4[X]/(2X, X^2 - 2)$.

Case: char(R)=2

Recall that M has 4 elements. We consider the situation under two sub-cases: either M contains a nonzero element whose square is nonzero OR $x^2 = 0$ for every $x \in M$.

Assume there exists an $a \in M$ such that $a^2 \neq 0$. Then since R is local, we also have $a^2 \neq a$. Thus $\{0, a, a^2\} \subsetneq M$. We claim the fourth element of M must be $a + a^2$. Certainly the latter must be an element of M; we must establish that it is not in the set $\{0, a, a^2\}$. It is evident that $a + a^2 \notin \{a, a^2\}$ because in both cases we get a contradiction with the facts $a \neq 0$ and $a^2 \neq 0$. If $a + a^2 = 0$, then $a^2 = -a = a$ (because $char(R) = 2$), which is a contradiction as R has no nontrivial idempotents. Thus $M = \{0, a, a^2, a + a^2\}$. We know that $a^3 \in \{0, a, a^2, a + a^2\}$.

If $a^3 = a$, then $(a^2)^2 = a^4 = a^3a = aa = a^2$; whence $a^2 \in Idem(R) \setminus \{0, 1\}$, which is a contradiction as R is local. If $a^3 = a^2$, then $a^4 = a^3a = a^2a = a^3 = a^2$, which yields the same contradiction as before. If $a^3 = a + a^2$, then $a^4 = (a + a^2)a = a^2 + a^3 = a^2 + (a + a^2) = 2a^2 + a = 0 + a = a$ (using that $char(R) = 2$). Thus $a^4 = a$ which implies that $a^6 = a^3$. Therefore $a^3 \in Idem(R) \setminus \{0, 1\}$, again a contradiction. Thus $a^3 = 0$.

Thus in the case that M contains a nonzero element a such that $a^2 \neq 0$, it follows that $M = \{0, a, a^2, a + a^2\}$, where $a^3 = 0$.

Now assume that the square of every element of M is 0. Then we must have $M = \{0, a, b, a + b\}$. We claim that $ab = 0$. If $ab = a$, then $ab - a = 0 \Rightarrow a(b - 1) = 0 \Rightarrow b - 1 \in Z(R) \subseteq M$ (since $a \neq 0$); whence $1 \in M$, which is a contradiction. We get a similar contradiction if $ab = b$. If $ab = a + b$, then $0 = a^2b = a(a + b) = a^2 + ab = ab$; thus $b = 0$, which is a contradiction.

Thus in the case that M consists of elements whose square is zero, we must have that $M = \{0, a, b, a + b\}$, where $ab = 0$ (that is, $M^2 = 0$).

Therefore if $char(R) = 2$, we have the following two cases:

1. $M = \{0, a, a^2, a + a^2\}$, where $a^3 = 0$, or
2. $M = \{0, a, b, a + b\}$, where $a^2 = b^2 = ab = 0$ (that is, $M^2 = 0$).

If $M = \{0, a, a^2, a + a^2\}$, where $a^3 = 0$, consider the evaluation homomorphism $\phi : \mathbb{Z}_2[X] \to R$ given by $X \mapsto a$. Then $im(\phi)$ consists of at least the 5 elements $\{0, 1, a, a^2, a + a^2\}$; whence this map must be onto as R has 8 elements. Since $a^3 = 0$, we have $X^3 \in ker(\phi)$. Notice that $\mathbb{Z}_2[X]/(X^3)$ consists of 8 elements, whence $ker(\phi) = (X^3)$. Therefore $R \cong \mathbb{Z}_2[X]/(X^3)$ by the First Isomorphism Theorem.

If $M = \{0, a, b, a + b\}$, where $ab = 0$ (that is, $M^2 = 0$), then consider the evaluation homomorphism $\phi : \mathbb{Z}_2[X, Y] \to R$ given by $X \mapsto a$ and $Y \mapsto b$. As before, this map must be onto. Since $M^2 = 0$, we see that $(X^2, XY, Y^2) \subseteq ker(\phi)$. As before, $\mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$ consists of 8 elements; thus $ker(\phi) = (X^2, XY, Y^2)$. Therefore, $R \cong \mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$ by the First Isomorphism Theorem.

Notice that in the local cases, choosing any ideal consisting of two elements will give that R/I is a local ring (not a field) consisting of four elements. Thus by Proposition 2.1, R/I will be isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. In conclusion, we have the following result. Recall our following convention: if I is an ideal of $R[X]$, we will often write $R[X]/I = R[x]$, where $x = X + I$ (the image of the indeterminate X).

Proposition 2.4. *Let R be a commutative ring with nonzero identity and I a nonzero ideal of R. Then $\Gamma_I(R) \cong K^2$ if and only if R is isomorphic to one of the* **7** *rings with corresponding ideal I from Table 2.1.*

Table 2.1: When $\Gamma_I(R) \cong K^2$ and $I \neq 0$.

Ring	Ideal (s)
$\mathbb{Z}_8$	$(4) = \{0, 4\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$0 \times \mathbb{Z}_2$
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2$	$0 \times \mathbb{Z}_2$
$\mathbb{Z}_4[X]/(X^2, 2X)$	(x), (2), or $(x+2)$
$\mathbb{Z}_4[X]/(2X, X^2 - 2)$	(2)
$\mathbb{Z}_2[X]/(X^3)$	(x^2)
$\mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$	(x), (y), or $(x+y)$

During our search for when $\Gamma_I(R) \cong K^2$, we showed the following result. This lemma will be useful for the work on when $\Gamma_I(R)$ is planar.

Lemma 2.5. *Let R be a commutative local ring of order 8 with nonzero identity and a maximal ideal consisting of 4 elements. Then R is isomorphic to one of the following* **five** *rings: $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$, $\mathbb{Z}_4[X]/(X^2, 2X)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$.*

2.3 When $\Gamma_I(R) \cong K^3$

We now investigate when a nontrivial $\Gamma_I(R)$ is the complete graph on 3 vertices. Using $|V(\Gamma_I(R))| = |I||V(\Gamma(R/I))|$ and $|I| \geq 2$, we must have $|V(\Gamma(R/I))| = 1$ and

$|I| = 3$. By Proposition 2.2, $|V(\Gamma(R/I))| = 1$ implies that $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Thus $|R/I| = |R|/|I|$ gives $|R| = 12$. We express R as a product of finite local rings. Using that finite local rings have cardinality a power of a prime, the only possibilities (up to isomorphism) for the factorization are as follows:

1. $R_1 \times R_2 \times R_3$, where $|R_1| = |R_2| = 2$ and $|R_3| = 3$,
2. $R_1 \times R_2$, where $|R_1| = 3$ and $|R_2| = 4$.

In the first possible factorization, $|R_1| = |R_2| = 2$ and $|R_3| = 3 \Rightarrow R_1 \cong R_2 \cong \mathbb{Z}_2$ and $R_3 \cong \mathbb{Z}_3$. Thus $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. The only ideal of this ring consisting of three elements is $0 \times 0 \times \mathbb{Z}_3$; thus $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. But this fails to meet the hypothesis that $|Z(R/I)^*| = 1$.

In the second factorization, $|R_1| = 3 \Rightarrow R_1 \cong \mathbb{Z}_3$. Then $R \cong \mathbb{Z}_3 \times R_2$, where $|R_2| = 4$. Since the only ideal of $\mathbb{Z}_3 \times R_2$ consisting of three elements is $\mathbb{Z}_3 \times 0$, it follows that $R/I \cong R_2$. Using that R/I must be isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, we have $R_2 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$.

Therefore we have the following result.

Proposition 2.6. *Let R be a commutative ring with nonzero identity and I a nonzero ideal of R. Then $\Gamma_I(R) \cong K^3$ if and only if R is isomorphic to one of the* **2** *rings with respective ideal I from Table 2.2.*

Table 2.2: When $\Gamma_I(R) \cong K^3$ and $I \neq 0$

Ring	Ideal
$\mathbb{Z}_3 \times \mathbb{Z}_4$	$\mathbb{Z}_3 \times 0$
$\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$	$\mathbb{Z}_3 \times 0$

2.4 When $\Gamma_I(R) \cong K^4$

We now consider when a nontrivial $\Gamma_I(R)$ is the complete graph on 4 vertices (K^4). When $4 = |V(\Gamma_I(R))| = |I||V(\Gamma(R/I))|$ and $|I| \geq 2$, we have two possibilities:

1. $|I| = 4$ and $|V(\Gamma(R/I))| = 1$, or
2. $|I| = 2$ and $|V(\Gamma(R/I))| = 2$.

Case 1. $|I| = 4$ **and** $|V(\Gamma(R/I))| = 1$ Recall that $|V(\Gamma(R/I))| = 1$ implies that R/I is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by Proposition 2.2 . So $|I| = 4$ and $|R/I| = 4$, and therefore $|R| = 16$. Again, we will proceed by writing R as a product of finite local rings. Using that $|S| = 2 \Rightarrow S \cong \mathbb{Z}_2$, we see that R must be isomorphic to one of the following products, where each R_i is a finite local ring.

1. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,
2. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times R_1$, where $|R_1| = 4$,
3. $\mathbb{Z}_2 \times R_1$, where $|R_1| = 8$,
4. $R_1 \times R_2$, where $|R_1| = |R_2| = 4$, or
5. R_1, where $|R_1| = 16$.

Factorization 1)

It is evident (since $|I| = 4$) that I will be a product of 2 $\mathbb{Z}_2$'s and 2 zero's. Thus $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ which (as several times before) violates the hypothesis that $1 = |V(\Gamma(R/I))| = |Z(R/I)^*|$.

Factorization 2)

Since $|I| = 4$, I must be of one of the following types.

$$I = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \times 0, \\ 0 \times 0 \times R_1, \text{ or} \\ 0 \times \mathbb{Z}_2 \times I_1 \text{ or } \mathbb{Z}_2 \times 0 \times I_1. \end{cases}$$

In the above, I_1 is an ideal of R_1 with $|I_1| = 2$. Then in each respective case, we have that

$$R/I \cong \begin{cases} R_1, \\ \mathbb{Z}_2 \times \mathbb{Z}_2, \text{ or} \\ \mathbb{Z}_2 \times 0 \times \mathbb{Z}_2 \text{ or } 0 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2. \end{cases}$$

As before, we can not have $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$; whence we must have that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times R_1$, $I = \mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$, and $R/I \cong R_1$. Since $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, it follows that the only possibilities are as follows:

1. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, $I = \mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$, or

2. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, $I = \mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$.

Factorization 3)

We have $R \cong \mathbb{Z}_2 \times R_1$, where R_1 is a local ring with 8 elements. Then since $|I| = 4$, we must have that (a) $I = \mathbb{Z}_2 \times I_1$, where $|I_1| = 2$ OR (b) $I = 0 \times I_2$, where $|I_2| = 4$.

One can see that case (b) will not occur, since as before, $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ under such circumstances. In the case of (a), $R/I \cong 0 \times R_1/I_1$. Thus we must have $R_1/I_1 \cong \mathbb{Z}_4$ or $Z_2[X]/(X^2)$. Since R_1 is a local ring of order 8, it must be isomorphic to one of the 5 rings from Lemma 2.5. Notice that each of these rings contain a non-maximal ideal I_1 of order 2. Hence each R_1/I_I will be isomorphic to either $\mathbb{Z}_4$ or $Z_2[X]/(X^2)$ by Lemma 2.1. Thus in the case of the third possible factorization, R must be isomorphic to one of the following rings with corresponding ideal I:

1. $\mathbb{Z}_2 \times \mathbb{Z}_8$, $I = \mathbb{Z}_2 \times (4)$,

2. $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(X^2, 2X)$, $I = \mathbb{Z}_2 \times I_1$, $I_1 = \{0, x\}$, $\{0, 2\}$, or $\{0, x+2\}$,

3. $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$, $I = \mathbb{Z}_2 \times \{0, 2\}$,

4. $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$, $I = \mathbb{Z}_2 \times \{0, x^2\}$, or

5. $\mathbb{Z}_2 \times \mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$, $I = \mathbb{Z}_2 \times I_1$, $I_1 = \{0, x\}$, $\{0, y\}$, or $\{0, x+y\}$.

Factorization 4)

In this case, R is isomorphic to a product of two local rings each of order 4. By Lemma 2.1, R is isomorphic to one of the following rings:

$$R \cong \begin{cases} \mathbb{Z}_4 \times \mathbb{Z}_4, \\ \mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2), \\ \mathbb{Z}_4 \times \mathbb{F}_4, \\ \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2), \\ \mathbb{Z}_2[X]/(X^2) \times \mathbb{F}_4, \text{ or} \\ \mathbb{F}_4 \times \mathbb{F}_4. \end{cases}$$

It now suffices to find all ideals of order 4 in the preceding rings such that $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Notice that $\mathbb{F}_4 \times \mathbb{F}_4$ only has ideals $0 \times \mathbb{F}_4$ and $\mathbb{F}_4 \times 0$ of order 4; but in both cases $R/I \cong \mathbb{F}_4$ which does not meet our hypothesis. Using that ideals of $R \times S$ are of the form $I \times J$, where I is an ideal of R and J is an ideal of S, we can rather easily consider all ideals of order 4 from the remaining possible rings. By inspection, we get that R must be isomorphic to one of the following rings with corresponding ideal I:

$$R \cong \begin{cases} \mathbb{Z}_4 \times \mathbb{Z}_4, I = \mathbb{Z}_4 \times 0 \text{ or } 0 \times \mathbb{Z}_4, \\ \mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2), I = \mathbb{Z}_4 \times 0 \text{ or } 0 \times \mathbb{Z}_2[X]/(X^2), \\ \mathbb{Z}_4 \times \mathbb{F}_4, I = 0 \times \mathbb{F}_4 \\ \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2), I = \mathbb{Z}_2[X]/(X^2) \times 0 \text{ or } 0 \times \mathbb{Z}_2[X]/(X^2), \text{ or} \\ \mathbb{Z}_2[X]/(X^2) \times \mathbb{F}_4, I = 0 \times \mathbb{F}_4. \end{cases}$$

Factorization 5)

We have now arrived at the final possible factorization: where R is a finite local ring with 16 elements. For this case, we turn to a pair of papers by Corbas and Williams ([15] and [16]). In the paper "Planar zero-divisor graphs" ([13]), the authors use Corbas' and Williams' paper to conclude there are 21 commutative local rings with identity of order 16, up to isomorphism. Since the rings we are interested in must have a non-maximal ideal with 4 elements (because R/I is not a field since it has 2 zero-divisors), we must have $|M| = 8$. Below, we give each ring and their corresponding maximal ideal. We use the facts that $Z(R) = M$ and $|M| \in \{1, 2, 4, 8\}$ to find each maximal ideal. The calculations are fairly easy when the indeterminates are zero-divisors, and hence in the maximal ideal. Recall our following convention: if I is an ideal of $R[X]$, we will often write $R[X]/I = R[x]$, where $x = X + I$ (the image of the indeterminate X).

1. $\mathbb{F}_{16}$, $M = 0$.

2. $\mathbb{F}_4[X]/(X^2)$, $M = \{0, x, ax, bx\}$ (adding any other element to this ideal yields $1 \in M$), where $\mathbb{F}_4 = \{0, a, b, c\}$.

3. $\mathbb{Z}_2[X]/(X^4)$, $M = \{0, x, x^2, x^3, x^2 + x, x^3 + x, x^3 + x^2, x^3 + x^2 + x\}$.

4. $\mathbb{Z}_2[X, Y]/(X^3, XY, Y^2)$, $M = \{0, x, y, x + y, x^2, x^2 + x, x^2 + y, x^2 + x + y\}$.

5. $\mathbb{Z}_2[X, Y]/(X^2 - Y^2, XY)$, $M = \{0, x, y, x + y, x^2, x^2 + x, x^2 + y, x^2 + x + y\}$.

6. $\mathbb{Z}_2[X, Y]/(X^2, Y^2)$, $M = \{0, x, y, x + y, xy, x + xy, y + xy, x + y + xy\}$.

7. $\mathbb{Z}_2[X, Y, Z]/(X, Y, Z)^2$. Here $(X, Y, Z)^2 = (X^2, Y^2, Z^2, XY, YZ, XZ)$ and $M = \{0, x, y, z, x + y, x + z, y + z, x + y + z\}$.

8. $\mathbb{Z}_4[X]/(X^2 + X + 1)$. Here $x^2 = -x - 1 = 3x + 3$.

   Notice that in this case $x \notin M$. As otherwise, $x^2 = 3x + 3$, but $2 \in Z(R) \Rightarrow x + 2$ and $3x + 3 \in M$. Whence $1 = (x + 2) + (3x + 3) \in M$, which is impossible.

   Since 2 is a zero-divisor, we see that $\{0, 2, 2x, 2 + 2x\} \subseteq M$.

   Notice in this ring that x is a unit with inverse $x^2 = 3x + 3$ since $xx^2 = x(3x + 3) = 3x^2 + 3x = 3(3 + 3x) + 3x = 1 + x + 3x = 1$. This, along with the fact that 3 is a unit in this ring, gives $3, x, 3+3x, 3x$ are all units of the ring. Using this, we can see that adjoining any additional elements to the set $\{0, 2, 2x, 2+2x\}$ will produce a unit in the ideal, whence $M = \{0, 2, 2x, 2x+2\}$.

9. $\mathbb{Z}_4[X]/(2X, X^3 - 2)$, $M = \{0, 2, x, x+2, x^2, x^2+2, x^2+x, x^2+x+2\}$.

10. $\mathbb{Z}_4[X]/(X^2 - 2)$. Here $x^2 = 2$, whence $2x(x) = 2x^2 = 0$. Thus $M = \{0, x, 2, x+2, 2x, 3x, 2x+2, 3x+2\}$.

11. $\mathbb{Z}_4[X]/(X^2 - 2X - 2)$. Here $x^2 = 2x + 2$, and thus $2x^2 = 0 \Rightarrow (2x)(x) = 0$. So $M = \{0, 2, x, 2x, x+2, 3x, 2x+2, 3x+2\}$.

12. $\mathbb{Z}_4[X,Y]/(X^2 - 2, XY, Y^2, 2X)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$.

13. $\mathbb{Z}_4[X,Y]/(X^2 - 2, XY, Y^2 - 2, 2X)$.

    In this case, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$ as before. Notice that the multiplication structure is different for this ring.

14. $\mathbb{Z}_4[X,Y]/(X^2, XY - 2, Y^2)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$.

15. $\mathbb{Z}_4[X]/(2X, X^3)$, $M = \{0, x, 2, x^2, x+2, x^2+2, x^2+x, x^2+x+2\}$.

16. $\mathbb{Z}_4[X]/(X^2)$, $M = \{0, 2, x, x+2, 2x, 2x+2, 3x, 3x+2\}$.

17. $\mathbb{Z}_4[X]/(X^2 - 2X)$, $M = \{0, 2, x, 2x, x+2, 2x+2, 3x, 3x+2\}$.

18. $\mathbb{Z}_4[X,Y]/(2, X, Y)^2$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$.

19. $\mathbb{Z}_8[X]/(2X, X^2)$, $M = \{0, 2, 4, 6, x, x+2, x+4, x+6\}$.

20. $\mathbb{Z}_8[X]/(2X, X^2 - 4)$ This ring has the "same" maximal ideal as the preceding case, $M = \{0, 2, 4, 6, x, 2+x, 4+x, 6+x\}$. Notice that the multiplication structure is different for this ring.

21. $\mathbb{Z}_{16}$, $M = (2) = \{0, 2, 4, 6, 8, 10, 12, 14\}$.

In the preceding 21 cases, we see that only cases 1, 2, and 8 are local rings with maximal ideals of cardinality not equal to 8. Thus there are 18 local rings with 16 elements and maximal ideal consisting of 8 elements. Notice that in each of these cases, it suffices simply to find an ideal consisting of 4 elements from M. This is the case since $|R/I| = 4$ and R/I will be a local ring which is not a field. We have seen that the local rings of order 4 which are not fields are precisely those with $|Z(R)^*| = 1$ (Lemma 2.1).

We now proceed to investigate the ideals of four elements from the 18 aforementioned rings.

In the ring $\mathbb{Z}_4[X]/(2X, X^3 - 2)$, we notice that for any element $a \in M \setminus \{0, 2\}$, we have $(a) = M$. Moreover, $(2) = \{0, 2\}$. From this, we see that the ring has no ideal I such that $|I| = 4$. The rest of the rings have at least one ideal consisting of four elements. We list each ring below and *all* corresponding ideals of order 4. To find all ideals of order 4 for each ring, we found all possible subsets of the maximal ideal consisting of zero and three other elements. In each case, we had $\binom{7}{5} = 35$ possible subsets. For each, we found those which were ideals of R.

1. $\mathbb{Z}_2[X]/(X^4)$, $M = \{0, x, x^2, x^3, x + x^2, x + x^3, x^2 + x^3, x + x^2 + x^3\}$;

   $I = \{0, x^2, x^3, x^2 + x^3\}$.

2. $\mathbb{Z}_2[X, Y]/(X^3, XY, Y^2)$, $M = \{0, x, y, x + y, x^2, x + x^2, y + x^2, x + y + x^2\}$;

   $I = \{0, x, x^2, x + x^2\}$, $\{0, y, x^2, y + x^2\}$, or $\{0, x + y, x^2, x + y + x^2\}$

3. $\mathbb{Z}_2[X, Y]/(X^2 - Y^2, XY)$, $M = \{0, x, y, x^2, x + y, y + x^2, x + y + x^2, x + x^2\}$;

   $I = \{0, x, x^2, x + x^2\}$, $\{0, y, x^2, y + x^2\}$, or $\{0, x^2, x + y, x + y + x^2\}$.

4. $\mathbb{Z}_2[X, Y]/(X^2, Y^2)$, $M = \{0, x, y, x + y, xy, x + xy, y + xy, x + y + xy\}$;

   $I = \{0, x, xy, x + xy\}$, $\{0, y, xy, y + xy\}$, or $\{0, x + y, xy, x + y + xy\}$.

5. $\mathbb{Z}_2[X, Y, Z]/(X, Y, Z)^2$, $M = \{0, x, y, z, x + y, x + z, y + z, x + y + z\}$;

$I = \{0, z, x+y, x+y+z\}$, $\{0, x, y+z, x+y+z\}$, $\{0, x, z, x+z\}$, $\{0, y, z, y+z\}$, $\{0, x, y, x+y\}$, $\{0, x+y, x+z, y+z\}$, or $\{0, y, x+z, x+y+z\}$.

6. $\mathbb{Z}_4[X]/(X^2-2)$, $M = \{0, 2, x, 2+x, 2x, 3x, 2x+2, 3x+2\}$;

   $I = \{0, 2, 2x, 2+2x\}$.

7. $\mathbb{Z}_4[X]/(X^2-2X-2)$, $M = \{0, 2, x, 2x, x+2, 3x, 2x+2, 3x+2\}$;

   $I = \{0, 2, 2x, 2+2x\}$ (notice that $(x) = M$, from this one can see this is the only ideal with four elements).

8. $\mathbb{Z}_4[X,Y]/(X^2-2, XY, Y^2, 2X)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$;

   $I = \{0, 2, x, 2+x\}$, $\{0, 2, y, y+2\}$, or $\{0, 2, x+y, x+y+2\}$.

9. $\mathbb{Z}_4[X,Y]/(X^2-2, XY, Y^2-2, 2X)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$;

   $I = \{0, 2, x, 2+x\}$, $\{0, 2, y, y+2\}$, or $\{0, 2, x+y, x+y+2\}$.

10. $\mathbb{Z}_4[X,Y]/(X^2, XY-2, Y^2)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$;

    $I = \{0, 2, x, x+2\}$, $\{0, 2, y, y+2\}$, or $\{0, 2, x+y, x+y+2\}$ (notice that $(x, y) = M$).

11. $\mathbb{Z}_4[X]/(2X, X^3)$, $M = \{0, 2, x, x^2, x+2, x^2+2, x+x^2, x+x^2+2\}$;

    $I = \{0, 2, x^2, x^2+2\}$, $\{0, x, x^2, x+x^2\}$, or $\{0, x^2, x+2, x+x^2+2\}$ (notice that $(x, 2) = M$).

12. $\mathbb{Z}_4[X]/(X^2) = \{0, 2, x, x+2, 2x, 2x+2, 3x, 3x+2\}$;

    $I = \{0, 2, 2x, 2x+2\}$, $\{0, 2x, 3x+2, x+2\}$, or $\{0, x+2, 2x, 3x+2\}$.

13. $\mathbb{Z}_4[X]/(X^2-2X)$, $M = \{0, x, 2, 2x, x+2, 2x+2, 3x, 3x+2\}$;

    $I = \{0, x, 2x, 3x\}$, $\{0, 2, 2x, 2x+2\}$, or $\{0, 2x, x+2, 3x+2\}$.

14. $\mathbb{Z}_4[X,Y]/(2, X, Y)^2$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$;

$I = \{0, 2, x, x+2\}$, $\{0, 2, y, y+2\}$, $\{0, 2, x+y, x+y+2\}$, $\{0, x, y, x+y\}$, $\{0, x, y+2, x+y+2\}$, $\{0, y, x+2, x+y+2\}$, or $\{0, x+y, x+2, y+2\}$.

15. $\mathbb{Z}_8[X]/(2X, X^2)$, $M = \{0, 2, 4, 6, x, x+2, x+4, x+6\}$;

    $I = \{0, 2, 4, 6\}$, $\{0, 4, x, x+4\}$, or $\{0, 4, x+2, x+6\}$.

16. $\mathbb{Z}_8[X]/(2X, X^2 - 4)$, $M = \{0, 2, 4, 6, x, x+2, x+4, x+6\}$;

    $I = \{0, 2, 4, 6\}$, $\{0, 4, x, x+4\}$, or $\{0, 4, x+2, x+6\}$.

17. $\mathbb{Z}_{16}$, $M = (2) = \{0, 2, 4, 6, 8, 10, 12, 14\}$;

    $I = \{0, 4, 8, 12\}$.

Thus there are 17 possible rings up to isomorphism which contain an ideal I of order 4 such that $\Gamma_I(R) \cong K^4$.

This concludes the first case ($|I| = 4$ and $|V(\Gamma(R/I))| = 1$). We now proceed to the second case for $\Gamma_I(R) \cong K^4$.

Case 2. $|I| = 2$ **and** $|V(\Gamma(R/I))| = 2$.

We begin by noticing that R/I has the zero-divisor graph K^2 and both vertices must be nilpotent (in order for both to be connected columns). Thus we must have that R/I is not reduced. From [5, Example 2.1(a)], we have that $\Gamma(R/I) \cong K^2 \Leftrightarrow R/I \cong \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_3[X]/(X^2)$. Notice that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is reduced; thus $\Gamma_I(R)$ will not be a complete graph if $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. So we have that $R/I \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$. Thus $|R| = 18$ since $|\mathbb{Z}_9| = |\mathbb{Z}_3[X]/(X^2)| = 9$ and $|R| = |I||R/I|$.

Using that R will be isomorphic to a product of finite local rings and that finite local rings must have cardinality a power of a prime, we have that one of the following two cases must hold:

1. $R \cong \mathbb{Z}_2 \times R_1$, where $|R_1| = 9$ and R_1 is local, or

2. $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Notice that the second case can not happen as then $|I| = 2 \Rightarrow R/I \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. But $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong K^{2,2} \ncong K^2$. In the first case, the only ideal consisting of two elements is $\mathbb{Z}_2 \times 0$; whence $R/I \cong R_1$. Therefore, $R_1 \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$. This concludes the investigation of when $\Gamma_I(R) \cong K^4$. The latter work gives the following proposition.

Proposition 2.7. *Let R be a commutative ring with nonzero identity and I a* **nonzero** *ideal of R. Then $\Gamma_I(R) \cong K^4$ if and only if R is isomorphic to one of the* **31** *rings with corresponding ideal I from Table 2.3.*

Table 2.3: When $\Gamma_I(R) \cong K^4$ and $I \neq 0$

Ring	Ideal(s)
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\mathbb{Z}_2 \times (4)$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(X^2, 2X)$	$\mathbb{Z}_2 \times I_1$, $I_1 = (x)$, (2), or $(x+2)$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$	$\mathbb{Z}_2 \times (2)$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$	$\mathbb{Z}_2 \times (x^2)$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$	$\mathbb{Z}_2 \times I_1$, $I_1 = (x)$ or $I_1 = (y)$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$0 \times \mathbb{Z}_4$ or $\mathbb{Z}_4 \times 0$
$\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$	$0 \times \mathbb{Z}_2[X]/(X^2)$ or $\mathbb{Z}_4 \times 0$
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$	$0 \times \mathbb{Z}_2[X]/(X^2)$ or $\mathbb{Z}_2[X]/(X^2) \times 0$
$\mathbb{Z}_4 \times \mathbb{F}_4$	$0 \times \mathbb{F}_4$
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{F}_4$	$0 \times \mathbb{F}_4$
$\mathbb{Z}_2[X]/(X^4)$	$\{0, x^2, x^3, x^2 + x^3\}$
$\mathbb{Z}_2[X,Y]/(X^3, XY, Y^2)$	$\{0, x, x^2, x + x^2\}$, $\{0, y, x^2, y + x^2\}$, or $\{0, x + y, x^2, x + y + x^2\}$
$\mathbb{Z}_2[X,Y]/(X^2 - Y^2, XY)$	$\{0, x, x^2, x + x^2\}$, $\{0, y, x^2, y + x^2\}$, or $\{0, x^2, x + y, x + y + x^2\}$
$\mathbb{Z}_2[X,Y]/(X^2, Y^2)$	$\{0, x, xy, x + xy\}$, $\{0, y, xy, y + xy\}$, or $\{0, x + y, xy, x + y + xy\}$
$\mathbb{Z}_2[X,Y,Z]/(X,Y,Z)^2$	$I = \{0, z, x + y, x + y + z\}, \{0, x, y + z, x + y + z\}$, $\{0, x, z, x + z\}$, $\{0, y, z, y + z\}$, $\{0, x, y, x + y\}$, $\{0, x + y, x + z, y + z\}$, or $\{0, y, x + z, x + y + z\}$
$\mathbb{Z}_4[X]/(X^2 - 2)$	$\{0, 2, 2x, 2 + 2x\}$
$\mathbb{Z}_4[X]/(X^2 - 2X - 2)$	$\{0, 2, 2x, 2 + 2x\}$
$\mathbb{Z}_4[X,Y]/(X^2 - 2, XY, Y^2, 2X)$	$\{0, 2, x, 2 + x\}$, $\{0, 2, y, y + 2\}$, or $\{0, 2, x + y, x + y + 2\}$
$\mathbb{Z}_4[X,Y]/(X^2 - 2, XY, Y^2 - 2, 2X)$	$\{0, 2, x, 2 + x\}$, $\{0, 2, y, y + 2\}$, or $\{0, 2, x + y, x + y + 2\}$
$\mathbb{Z}_4[X,Y]/(X^2, XY - 2, Y^2)$	$\{0, 2, x, x + 2\}$, $\{0, 2, y, y + 2\}$, or $\{0, 2, x + y, x + y + 2\}$
$\mathbb{Z}_4[X]/(2X, X^3)$	$\{0, 2, x^2, x^2 + 2\}$, $\{0, x, x^2, x + x^2\}$, or $\{0, x^2, x + 2, x + x^2 + 2\}$
$\mathbb{Z}_4[X]/(X^2)$	$\{0, 2, 2x, 2x + 2\}$, $\{0, 2x, 3x + 2, x + 2\}$, or $\{0, x + 2, 2x, 3x + 2\}$
$\mathbb{Z}_4[X]/(X^2 - 2X)$	$\{0, x, 2x, 3x\}$, $\{0, 2, 2x, 2x + 2\}$, or $\{0, 2x, x + 2, 3x + 2\}$
$\mathbb{Z}_4[X,Y]/(2,X,Y)^2$	$\{0, 2, x, x + 2\}$, $\{0, 2, y, y + 2\}$, $\{0, 2, x + y, x + y + 2\}$, $\{0, x, y, x + y\}$, $\{0, x, y + 2, x + y + 2\}$, $\{0, y, x + 2, x + y + 2\}$, or $\{0, x + y, x + 2, y + 2\}$
$\mathbb{Z}_8[X]/(2X, X^2)$	$\{0, 2, 4, 6\}$, $\{0, 4, x, x + 4\}$, or $\{0, 4, x + 2, x + 6\}$
$\mathbb{Z}_8[X]/(2X, X^2 - 4)$	$\{0, 2, 4, 6\}$, $\{0, 4, x, x + 4\}$, or $\{0, 4, x + 2, x + 6\}$
$\mathbb{Z}_{16}$	$(4) = \{0, 4, 8, 16\}$
$\mathbb{Z}_2 \times \mathbb{Z}_9$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$	$\mathbb{Z}_2 \times 0$

2.5 Summary and K^p

We summarize the work in the preceding three sections as follows:

Proposition 2.8. *Let R be a commutative ring with nonzero identity and I a* **nonzero** *ideal of R. Then $\Gamma_I(R)$ is a complete graph on fewer than five vertices if and only if R is isomorphic to one of the* **38** *rings with corresponding ideal as found in Table 2.1, Table 2.2, or Table 2.3.*

One might ask where $\mathbb{Z}_{18}$ appears in these tables when $|R| = 18$. This ring is listed; note that $\mathbb{Z}_{18} \cong \mathbb{Z}_2 \times \mathbb{Z}_9$. To see the preceding, we have $9^2 = 9$ in $\mathbb{Z}_{18}$; whence $\mathbb{Z}_{18} \cong 9\mathbb{Z}_{18} \times (1-9)\mathbb{Z}_{18} \cong \mathbb{Z}_2 \times \mathbb{Z}_9$.

The next natural question is when do we have a nontrivial $\Gamma_I(R)$ which is complete on 5 vertices. In researching the latter, we quickly saw a pattern and produced a more general result.

Proposition 2.9. *Let R be a commutative ring with nonzero identity and p an odd prime. Then $\Gamma_I(R) \cong K^p$ for I a nonzero ideal of R if and only if R is isomorphic to one of the following rings with corresponding ideal I: $\mathbb{Z}_p \times \mathbb{Z}_4$ or $\mathbb{Z}_p \times \mathbb{Z}_2[X]/(X^2)$, where $I = \mathbb{Z}_p \times 0$.*

Proof. The reverse implication is evident. Assume that $\Gamma_I(R) \cong K^p$. Since $|I| \geq 2$, $p = |V(\Gamma(R))| = |I||V(\Gamma(R/I))|$, and p is prime, it follows that $|I| = p$ and $|V(\Gamma(R/I))| = 1$. By Proposition 2.2, $|V(\Gamma(R/I))| = 1 \Rightarrow R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Thus $|R| = 4|I| = 4p$.

We now use that R is isomorphic to a finite product of finite local rings. Since p is an odd prime, we have that the only possible factorizations are as follows (where R_1 is a local ring with four elements):

$$R \cong \begin{cases} \mathbb{Z}_p \times R_1, \\ \mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2. \end{cases}$$

Notice that in the above we are using that a finite local ring must have cardinality a power of prime (since p is an odd prime, neither $2p$ nor $4p$ is a power of prime).

It is evident that the only ideal of the second factorization consisting of p elements is $\mathbb{Z}_p \times 0 \times 0$, but then $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ which has $|V(\Gamma(R/I))| = 2$. Thus the second factorization can not occur. Whence we must have that $R \cong \mathbb{Z}_p \times R_1$, where R_1 is a local ring with 4 elements. Again we must have $I = \mathbb{Z}_p \times 0$; whence $R/I \cong R_1$. Thus $R_1 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. The desired result then follows. $\square$

Recall that a prime is called a *Mersenne prime* if and only if it is of the form $2^k - 1$, where $k \in \mathbb{N}$. By considering odd primes that are not Mersenne primes, we may remove the nonzero ideal hypothesis in the previous proposition.

Corollary 2.10. *Let R be a commutative ring with nonzero identity and I an ideal of R (possibly the zero ideal). Let p be an odd prime number that is* **not** *a Mersenne prime. Then $\Gamma_I(R) \cong K^p$ if and only if R is isomorphic to one of the following rings with corresponding ideal I: $\mathbb{Z}_p \times \mathbb{Z}_4$ or $\mathbb{Z}_p \times \mathbb{Z}_2[X]/(X^2)$, where $I = \mathbb{Z}_p \times 0$.*

Proof. In [7, Theorem 2.10 and Example 2.11(a)], the authors showed that there is a ring R such that $\Gamma(R)$ is complete on p vertices if and only if $p = q^n - 1$ for some prime q. Now $q^n - 1$ is prime implies that $q = 2$ or $n = 1$. If $n = 1$, then $q - 1$ is prime if and only if $q = 3$. In this case, we must have $p = 2$ which does not meet the hypothesis. Hence $q = 2$; so it follows that the only odd primes p for which there is a ring R that has $\Gamma(R)$ complete on p vertices are Mersenne primes. Thus for all primes p that are not Mersenne, in order for $\Gamma_I(R)$ to be complete on p vertices, we must have $I \neq 0$. The result then follows from Proposition 2.9. $\square$

Chapter 3

When $\Gamma_I(R)$ is Planar

Now that the classification work is out of the way, we will continue with the investigation of when ideal-based zero-divisor graphs are planar. Recall that we consider $\Gamma_I(R)$ to be a *nontrivial* ideal-based zero divisor graph provided I is a nonzero, proper, non-prime ideal of R. The latter requirements force $\Gamma_I(R)$ to be distinct from $\Gamma(R)$ and to be nonempty.

Recall that a graph G is *planar* if it can be drawn in a plane so that no two edges cross. Research on classifying all finite commutative rings with nonzero identity having nonempty planar zero-divisor graph has been done in [1, 5, 13, 33, 34, 36]. Work has also be done regarding when infinite commutative rings have planar zero-divisor graphs [35].

This chapter will utilize the well-known Kuratowski's Theorem from graph-theory. The statement of this theorem is included in the Introduction (Theorem 1.10). Example 1.11 shows an application of Kuratowski's Theorem.

3.1 Restraints on $|I|$ and $gr(\Gamma(R/I))$

We begin by investigating what restraints planarity forces on the graphs of $\Gamma(R/I)$ and $\Gamma_I(R)$.

Proposition 3.1. *Let R be a commutative ring with nonzero identity and I an ideal of R. If $\Gamma_I(R)$ is planar, then $|I| \leq 2$ or $|V(\Gamma(R/I))| \leq 1$.*

Proof. (By Contrapositive) Assume $|I| \geq 3$ and $|V(\Gamma(R/I))| \geq 2$. Then there are distinct adjacent vertices $x + I, y + I$ in $\Gamma(R/I)$. Since $|I| \geq 3$, there are distinct elements $0, i, j$ of I. Note that the subgraph of $\Gamma_I(R)$ generated by $\{x, y, x+i, y+i, x+j, y+j\} = \{x, x+i, x+j\} \cup \{y, y+i, y+j\}$ contains a subgraph isomorphic to $K^{3,3}$. Thus $\Gamma_I(R)$ is nonplanar by Kuratowski's Theorem. □

Remark 3.2. In the proof of Proposition 3.1, we considered the subgraph of $\Gamma_I(R)$ generated by $\{x, y, x+i, y+i, x+j, y+j\}$; call this subgraph G. We noted that G contained a subgraph isomorphic to $K^{3,3}$. It is possible that G contains more edges than $K^{3,3}$ (this would be the case when either $x^2 \in I$ or $y^2 \in I$).

Proposition 3.3. *Let R be a commutative ring with nonzero identity and I an ideal of R. If $|V(\Gamma(R/I))| = 1$, then $\Gamma_I(R)$ is planar if and only if $1 \leq |I| \leq 4$.*

Proof. By Proposition 2.2, $|\Gamma(R/I)| = 1 \Leftrightarrow |Z(R/I)^*| = 1 \Leftrightarrow R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. In both cases, I is not a radical ideal. Thus by Redmond's construction method of $\Gamma_I(R)$, $\Gamma_I(R) = K^{|I|}$. The result then follows since $K^{|I|}$ is planar if and only if $1 \leq |I| \leq 4$ (by Kuratowski's Theorem). □

It now suffices to consider the case when $|I| = 2$ and $\Gamma(R/I)$ has at least two distinct vertices. We will approach the problem by considering the different possibilities for $gr(\Gamma(R/I))$. The *girth* of a graph G, denoted $gr(G)$, is defined to the length of a shortest cycle in G provided a cycle exists and ∞ otherwise. Recall that $gr(\Gamma(R/I)) \in \{3, 4, \infty\}$ [11, 17, 25, 37].

Proposition 3.4. *Let R be a commutative ring with nonzero identity and I an ideal of R. If $|I| = 2$ and $gr(\Gamma(R/I)) = 4$, then $\Gamma_I(R)$ is nonplanar. Moreover, if I is nonzero and $gr(\Gamma(R/I)) = 4$, then $\Gamma_I(R)$ is nonplanar.*

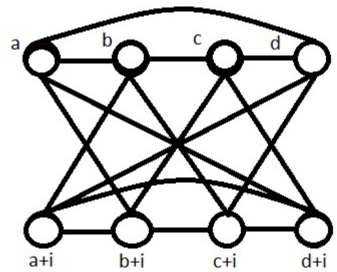


Figure 3.1: Subgraph when $gr(\Gamma(R/I)) = 4$

Proof. Since $gr(\Gamma(R/I)) = 4$, there exists vertices $a+I, b+I, c+I, d+I$ of $\Gamma(R/I)$ that form a 4-cycle. Moreover, since $|I| = 2$, there exists $0 \neq i \in I$. Thus by Redmond's construction of $\Gamma_I(R)$ based on $\Gamma(R/I)$, we see that $\Gamma_I(R)$ will have a subgraph as in Figure 3.1.

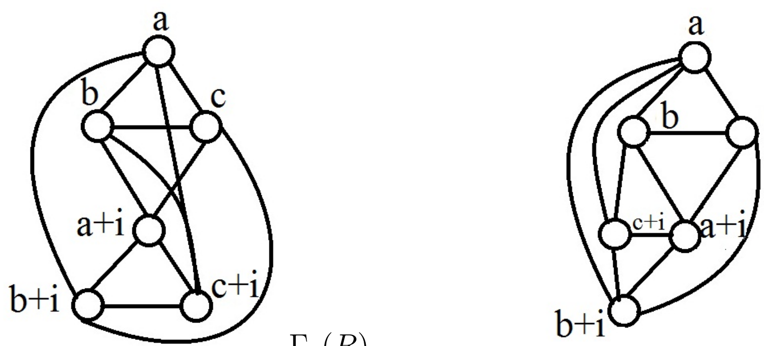
Notice that the vertex sets $A = \{a, a+i, c\}$ and $B = \{b, b+i, d+i\}$ induce a subgraph isomorphic to $K^{3,3}$. Whence by Kuratowski's Theorem, $\Gamma_I(R)$ is nonplanar. The "moreover statement" follows by combining Proposition 3.1 and this result. □

The following is [30, Theorem 7.2]:

Let I be a proper, nonzero ideal of a ring R that is not a prime ideal. Then $\Gamma_I(R)$ is planar if and only if $\omega(\Gamma(R/I)) \leq 2$ (i.e., $\Gamma(R/I)$ has no cycles) and either (a) $|I| = 2$ or (b) $\Gamma(R/I)$ consists of a single vertex and $|I| \leq 4$.

Here $\omega(G)$ is the clique number of a graph G. A *clique* of a graph G is a subgraph of G such that $G \cong K^n$ for some $n \in \mathbb{N}$. If a graph has no cliques, we set the *clique number* of G to be zero; otherwise we set the clique number to the $\sup\{n \mid K^n$ is isomorphic to a subgraph of $G\}$. Notice that the clique number of a graph can be infinity.

If $\Gamma(R/I)$ consists solely of a four-cycle (as the subgraph in our previous proof), then $\omega(\Gamma(R/I)) = 2$. So Redmond's Theorem 7.2 would imply that the induced subgraph from the preceding proof would be planar. However, we exhibited that this was not the case. For a concrete counterexample, consider $R = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ and $I = 0 \times 0 \times \mathbb{Z}_2$ (See Figure 3.1 for an isomorphic copy of $\Gamma_I(R)$) . We note that in



$\Gamma_I(R)$ Re-drawn $\Gamma_I(R)$

Figure 3.2: Subgraph when $gr(\Gamma(R/I)) = 3$ and I radical

Redmond's statement of the theorem, Redmond wrote "$\omega(\Gamma(R/I)) \leq 2$ (i.e., $\Gamma(R/I)$ has no cycles)". Although this statement is invalid, the theorem holds if we replace the clique number hypothesis with "$\Gamma(R/I)$ has no cycles" (i.e., $gr(\Gamma(R/I) = \infty)$.

We continue our investigation of the problem by now considering the girth 3 case.

Proposition 3.5. *Let R be a commutative ring with nonzero identity and I an ideal of R. If $|I| = 2$, $I = \sqrt{I}$, and $\Gamma(R/I) = K^3$, then $\Gamma_I(R)$ is planar.*

Proof. Under these assumptions, we have that $I = \{0, i\}$ and $\Gamma(R/I)$ is a 3-cycle on vertices $a + I$, $b + I$, and $c + I$. Since $I = \sqrt{I}$, $\Gamma_I(R)$ will be isomorphic to the graph in Figure 3.2. We can then see by inspection that this graph can be re-drawn so that edges do not cross. Thus $\Gamma_I(R)$ is planar. $\square$

It turns out the preceding result is rendered mute. In Example 2.1 of [5, pp. 2,3], it was shown that $\Gamma(R) \cong K^3$ if and only R is isomorphic to one of the following four rings: $\mathbb{F}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2 + X + 1), \mathbb{Z}_4[X]/(2, X)^2$, or $\mathbb{Z}_2[X, Y]/(X, Y)^2$. Thus in the preceding proposition, $\Gamma(R/I) \cong K^3$ if and only if R/I is isomorphic to one of the four previously mentioned rings. Since these rings are non-reduced, it follows that R/I is non-reduced. Since R/I is non-reduced if and only if I is not a radical ideal of R, it follows that the hypothesis of the preceding proposition is vacuous.

This observation lends light to Redmond's argument in [30, Theorem 7.2] in the following manner. He argues in his proof that if $\Gamma(R/I) \cong K^3$, then one can verify

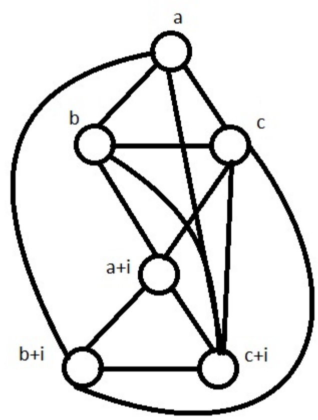


Figure 3.3: Subgraph of $\Gamma_I(R)$ when $gr(R/I) = 3$ and $|I| \geq 2$

that $\Gamma_I(R)$ is nonplanar by exhibiting a subgraph of $\Gamma_I(R)$ isomorphic to $K^{3,3}$. This is the case if one takes into consideration that I must be a non-radical ideal of R, and hence $\Gamma_I(R)$ has a connected column.

In light of this observation, we come to the following proposition.

Proposition 3.6. *Let R be a commutative ring with nonzero identity and I an ideal of R. If $gr(\Gamma(R/I)) = 3$ and $|I| = 2$, then $\Gamma_I(R)$ is nonplanar. Moreover, if $gr(\Gamma(R/I)) = 3$ and I is nonzero, then $\Gamma_I(R)$ is nonplanar.*

Proof. First assume that $\Gamma(R/I) \cong K^3$. By the preceding observations, it follows that I is not a radical ideal. Since $\Gamma(R/I) \cong K^3$, we have that $\Gamma(R/I)$ consists solely of a three-cycle, say $a + I - b + I - c + I - a + I$. Since $I \neq \sqrt{I}$, at least one of the elements a, b, c is in $\sqrt{I}$. Without loss of generality, assume that $c^2 \in I$. Then using Redmond's Construction Method, $\Gamma_I(R)$ will have a subgraph as in Figure 3.3.

The vertex sets $\{a, c, a+i\}$ and $\{b, b+i, c+i\}$ induce a subgraph of the preceding graph isomorphic to $K^{3,3}$. Thus $\Gamma_I(R)$ is nonplanar by Kuratowski's Theorem.

If $\Gamma(R/I) \ncong K^3$, then since $gr(\Gamma(R/I)) = 3$, we have that $\Gamma(R/I)$ does not consist solely of a three-cycle. Thus it follows that $\Gamma(R/I)$ would have a subgraph as in Figure 3.4 (A). Therefore, again using Redmond's construction method, $\Gamma_I(R)$ would have the subgraph as in Figure 3.4 (B).

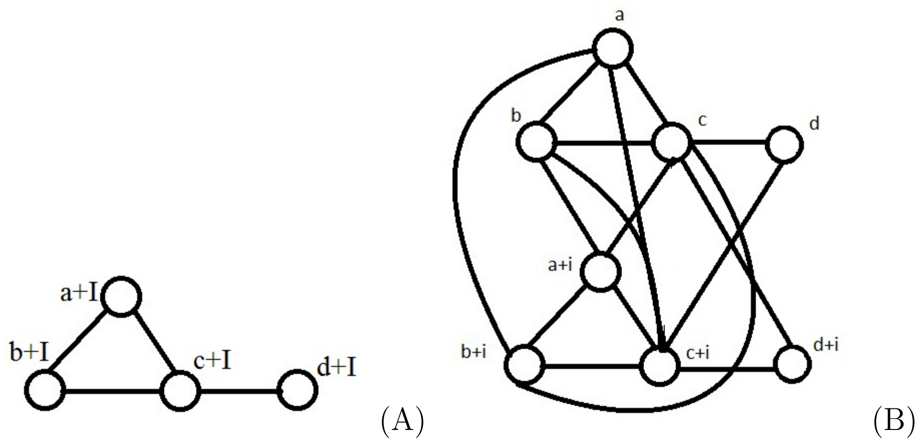


Figure 3.4: Subgraphs when $gr(\Gamma(R/I)) = 3$, $|I| = 2$, and $\Gamma_I(R) \ncong K^3$

Taking a subdivision of this graph by replacing $c - d - c + i$ with $c - c + i$, we get a subdivision of $\Gamma_I(R)$ which contains a subgraph isomorphic to the graph in Figure 3.3. Thus $\Gamma_I(R)$ is nonplanar (since we have already shown that the graph in Figure 3.3 was nonplanar). The "moreover statement" follows from Proposition 3.1 and this result. □

It now remains only to investigate the case when $gr(\Gamma(R/I)) = \infty$ (i.e., $\Gamma(R/I)$ has no cycles) and $|I| = 2$. A natural question is whether or not I being a radical ideal of R will affect the planarity of $\Gamma_I(R)$; in this case, it turns out that it does not.

Proposition 3.7. *Let R be a* **finite** *commutative ring with nonzero identity and I an ideal of R. If $|I| = 2$ and $gr(\Gamma(R/I)) = \infty$, then $\Gamma_I(R)$ is planar.*

Proof. If I is a prime ideal of R or $I = R$, then both $\Gamma(R/I)$ and $\Gamma_I(R)$ are empty, and hence planar. Assume that I is a proper, non-prime ideal of R.

Now $\Gamma(R/I)$ is nonempty since I is a proper, non-prime ideal of R. We handled the case when $V(\Gamma(R/I))$ is a singleton in Proposition 3.3; so we may assume that $|V(\Gamma(R/I))| \geq 2$. It then follows from [8, Theorems 2.4 and 2.5] that $\Gamma(R/I)$ is isomorphic to either $\overline{K}^{1,3}$ or $K^{1,n}$ for some $n \geq 1$.

We begin with the case that $\Gamma(R/I)$ is a star graph ($\Gamma(R/I) \cong K^{1,n}$), say with center c, ends a_k, and $I = \{0, i\}$. Using Redmond's construction method and the fact

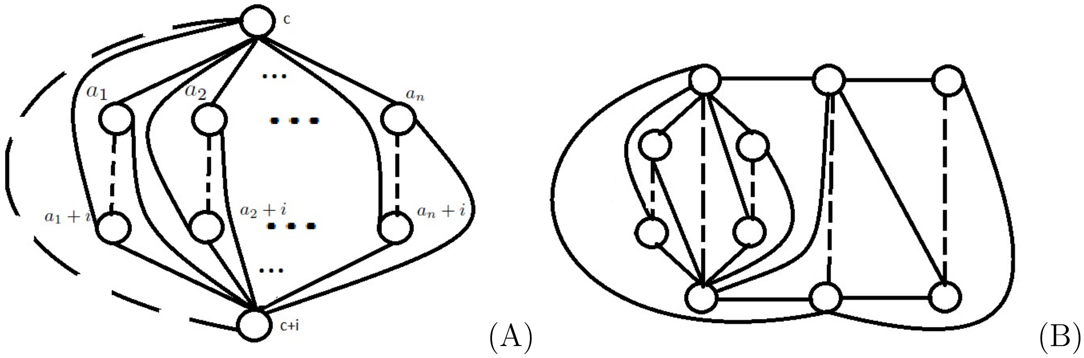


Figure 3.5: Graphs for Proposition 3.7

that $|I| = 2$, we can draw $\Gamma_I(R)$ as in Figure 3.5 (A). The dotted or hash-mark lines indicate lines that occur if and only if the vertex is in a connected column (recall connected columns exist if and only if I is a non-radical ideal). As drawn, we see that $\Gamma_I(R)$ is planar. It is important to note that we are using the finite hypothesis here. In order for the drawing of Figure 3.5 (A) to be make *sense*, there needs to be some constraints on the cardinality of vertices of $\Gamma(R/I)$.

If $\Gamma(R/I) \cong \overline{K}^{1,3}$, then one can see (regardless of whether or not $I = \sqrt{I}$) that $\Gamma_I(R)$ is planar. Using dotted or hash-mark lines as before, we can draw $\Gamma_I(R)$ as in Figure 3.5 (B).

Thus in all cases, $\Gamma_I(R)$ is planar as desired. □

We hypothesize that that the preceding result holds provided n is either finite or $\aleph_0$.

Combining all these results, we get a theorem which turns out to be only a slight modification of Redmond's Theorem 7.2. As previously mentioned, Redmond's statement of the theorem seems incorrect due to using the hypothesis $\omega(\Gamma(R/I)) \leq 2$ instead of $gr(\Gamma(R/I)) = \infty$. Moreover, it seems that in Redmond's proof a key observation (that appears to go unmentioned) was that $\Gamma(R/I) \cong K^3$ implies that R/I is non-reduced. Combining the previous propositions (and noting that Proposition 3.5 can not happen) yields the following theorem.

Theorem 3.8. *Let I be a nonzero, proper, non-prime ideal of a* **finite** *commutative ring R with nonzero identity. Then $\Gamma_I(R)$ is planar if and only if $gr(\Gamma(R/I)) = \infty$ and either (a) $|I| = 2$ or (b) $|V(\Gamma(R/I))| = 1$ and $|I| \in \{2, 3, 4\}$.*

Notice that the only place we required the finite hypothesis in the preceding was when $|I| = 2$ and $gr(R/I) = \infty$. In the next section, we will classify all commutative rings with nonzero identity satisfying the preceding theorem. During this classification, the finite hypothesis will again only be required for the case when $|I| = 2$ and $gr(\Gamma(R/I)) = \infty$. So this result and the classification (to follow) can be generalized simply by handling the case when R is infinite, $|I| = 2$, and $gr(\Gamma(R/I)) = \infty$.

3.2 Classifying Commutative Rings with nontrivial Planar $\Gamma_I(R)$

We now proceed to use the the previous theorem to classify the finite commutative rings R with nonzero identity such that R admits a nontrivial planar $\Gamma_I(R)$.

Proposition 3.9. *Let I be a nonzero, proper, non-prime,* **radical** *ideal of a* **finite** *ring commutative R with nonzero identity. Then $gr(\Gamma(R/I)) = \infty$ and $|I| = 2$ if and only if R is isomorphic to a ring with corresponding ideal from Table 3.1, where K is a finite field.*

Table 3.1: Rings for Proposition 3.9

Ring	Ideal
$\mathbb{Z}_4 \times K$	$(2) \times 0$
$\mathbb{Z}_2[X]/(X^2) \times K$	$(x) \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times K$	$\mathbb{Z}_2 \times 0 \times 0$, $0 \times \mathbb{Z}_2 \times 0$, or $0 \times 0 \times K$ (when $K = \mathbb{Z}_2$)

Proof. In the following argument, we use, without direct reference, that an ideal of $\prod_{i=1}^n R_i$ is of the form $\prod_{i=1}^n I_i$, where I_i is an ideal of R_i. In the case of an ideal with only two elements, it must be of the form $\prod_{i=1}^{n} I_i$, where $I_i = 0$ for all i except a fixed $k \in \{1, ..., n\}$ and $|I_k| = 2$.

Since I is a radical ideal, we have that R/I is reduced. By [8, Theorem 2.4], we have that $T(R/I) \cong \mathbb{Z}_2 \times K$, where K is a field. Since R is finite, and hence also R/I, we have $R/I \cong T(R/I)$; whence $R/I \cong \mathbb{Z}_2 \times K$. Since R is a finite commutative ring, we have that $R \cong \prod_{i=1}^{n} R_i$, where each R_i is a finite local ring. If $n \geq 4$, then R/I will be isomorphic to a product of at least 3 nonzero local rings. But this is a contradiction as R/I is a product of 2 local rings. Thus we must have that $n \leq 3$.

If $n = 1$, then R is local. Thus R/I is also local; so R/I can be expressed as a product of only one local ring. Thus $n \neq 1$ as R/I is a product of two local rings.

If $n = 2$, then $R \cong R_1 \times R_2$, where R_1, R_2 are local. Hence either $I = I_1 \times 0$ or $I = 0 \times I_2$, where $|I_1| = |I_2| = 2$. Thus $R/I \cong R_1/I_1 \times R_2$ or $R/I \cong R_1 \times R_2/I_2$. In either case, we have that R_i is a local ring with ideal I_i such that $|I_i| = 2$ and R_i/I_i is a field. Thus I_i is a maximal ideal of R_i. Notice that $Z(R_i)$ is nonzero, since otherwise R_i would be a finite integral domain, and hence a field (which contradicts the existence of a proper ideal with 2 elements). Since $0 \subsetneq Z(R_i) \subseteq I_i$ and $|I_i| = 2$, it follows that $Z(R_i) = I_i$. Therefore $|Z(R_i)| = 2$. Hence either $|Z(R_1)| = 2$ or $|Z(R_2)| = 2$. Thus by Proposition 2.2, either R_1 or R_2 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. The only constraint on the remaining factor (the one which is neither $\mathbb{Z}_4$ nor $\mathbb{Z}_2[X]/(X^2)$) is that it must a field or $\mathbb{Z}_2$. Since $\mathbb{Z}_2$ is a field, the preceding requirement reduces to simply being a field.

If $n = 3$, then since R/I is a product of 2 local rings, I is of the form one of the three rings times 2 zero ideals. Thus we must have that $R \cong \mathbb{Z}_2 \times K \times \mathbb{Z}_2$, where $I = \mathbb{Z}_2 \times 0 \times 0$, $0 \times 0 \times \mathbb{Z}_2$, or $0 \times K \times 0$ (in the case that $K = \mathbb{Z}_2$).

Thus in conclusion, R is isomorphic to one of the rings in Table 3.1 with corresponding ideal I. The converse is evident. $\square$

Notice that in each of the cases from Table 3.1, we have that $R/I \cong \mathbb{Z}_2 \times K$; whence we have that $\Gamma(R/I)$ is a star graph. So the graph of $\Gamma_I(R)$ in these cases corresponds to graph (A) in Figure 3.5, where the dotted edges are not present. We now proceed to the case that I is not a radical ideal. Before we proceed, we prove a couple of lemmas. These lemmas may seem unmotivated, but they will be used in the classification of the non-radical ideal case.

Lemma 3.10. *Let $R \cong R_1 \times \mathbb{Z}_2$, where R_1 is a local ring with 8 elements and I is an ideal of R with 2 elements. Then R/I is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2-2)$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$ if and only if R is isomorphic to one of the* **5** *rings with corresponding ideal I as in Table 3.2.*

Table 3.2: Rings for Lemma 3.10

Ring	Ideal
$\mathbb{Z}_8 \times \mathbb{Z}_2$	$0 \times \mathbb{Z}_2$ or $(4) \times 0$
$\mathbb{Z}_2[X]/(X^3) \times \mathbb{Z}_2$	$0 \times \mathbb{Z}_2$ or $(x^2) \times 0$
$\mathbb{Z}_4[X]/(2X, X^2-2) \times \mathbb{Z}_2$	$0 \times \mathbb{Z}_2$ or $(2) \times 0$
$\mathbb{Z}_4[X]/(X^2, 2X) \times \mathbb{Z}_2$	$(x) \times 0$, $(2) \times 0$, or $(x+2) \times 0$
$\mathbb{Z}_2[X,Y]/(X^2, XY, Y^2) \times \mathbb{Z}_2$	$(x) \times 0$, $(y) \times 0$, or $(x+y) \times 0$

Proof. Notice that $I = I_1 \times 0$ or $0 \times \mathbb{Z}_2$, where I_1 is an ideal of R_1 consisting of 2 elements.

We proceed to prove the forward implication. Assume that R/I is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2-2)$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.

If $I = 0 \times \mathbb{Z}_2$, then $R/I \cong R_1$. Thus using that R_1 is local, we must have $R_1 \cong \mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2-2)$ (these are the only rings for R/I, in our forward hypothesis, that are local). Hence $R \cong \mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_2[X]/(X^3) \times \mathbb{Z}_2$, or $\mathbb{Z}_4[X]/(2X, X^2-2) \times \mathbb{Z}_2$, where $I = 0 \times \mathbb{Z}_2$.

If $I = I_1 \times 0$, then $R/I \cong R_1/I_1 \times \mathbb{Z}_2$. Since R_1 is local with 8 elements, R_1/I_1 is a local ring with 4 elements. By Lemma 2.1, R_1/I_1 is isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, or $\mathbb{F}_4$. But since R/I must be isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2-2)$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, it follows that $R_1/I_1 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$.

Thus it suffices to find all local rings R_1 with 8 elements that contain a non-maximal ideal I, with 2 elements such that R_1/I_1 is isomorphic either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. In Section 2.2, we showed that there are 7 rings R of order 8 which contain an ideal of I of order 2 such that $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. These rings are listed in Table 2.1, along with the possible choices of the ideal. Notice that 2 of these rings are not local. Thus it follows that R_1 must be isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_4[X]/(X^2, 2X)$, $\mathbb{Z}_4[X]/(2X, X^2-2)$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$ (with ideal I_1 as chosen in Table 2.1).

The converse is evident. $\square$

Lemma 3.11. *Let R be a local ring of order 16 and I be an ideal of R consisting of 2 elements. Then R/I is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2-2)$ if and only if R is isomorphic to one of the* **8** *rings with corresponding ideal from Table 3.3.*

Table 3.3: Rings for Lemma 3.11

Ring	Ideal
$\mathbb{Z}_2[X,Y]/(X^3, XY, Y^2)$	(y) or $(y+x^2)$
$\mathbb{Z}_4[X]/(2X, X^3-2)$	(2)
$\mathbb{Z}_4[X]/(X^2-2)$	$(2x)$
$\mathbb{Z}_4[X]/(X^2-2X-2)$	$(2x)$
$\mathbb{Z}_4[X,Y]/(X^2-2, XY, Y^2, 2X)$	(y) or $(y+2)$
$\mathbb{Z}_4[X]/(2X, X^3)$	(2) or (x^2+2)
$\mathbb{Z}_8[X]/(2X, X^2)$	(x) or $(x+4)$
$\mathbb{Z}_{16}$	(8)

Proof. "$\Rightarrow$" Given a local ring R of order 16 and I an ideal with 2 elements, then R/I will be a local ring of order 8. All three possibilities for R/I have a maximal ideal consisting of 4 elements. Thus R is a local ring of order 16 with a maximal ideal consisting of 8 elements. By considering all local rings of order 16, as in Chapter 2, we see that that such a ring R must be isomorphic to one of the following 18 rings (we also include each ring's maximal ideal below):

1. $\mathbb{Z}_2[X]/(X^4)$, $M = \{0, x, x^2, x^3, x^2+x, x^3+x, x^3+x^2, x^3+x^2+x\}$.
2. $\mathbb{Z}_2[X,Y]/(X^3, XY, Y^2)$, $M = \{0, x, y, x+y, x^2, x^2+x, x^2+y, x^2+x+y\}$.
3. $\mathbb{Z}_2[X,Y]/(X^2-Y^2, XY)$, $M = \{0, x, y, x+y, x^2, x^2+x, x^2+y, x^2+x+y\}$.
4. $\mathbb{Z}_2[X,Y]/(X^2, Y^2)$, $M = \{0, x, y, x+y, xy, x+xy, y+xy, x+y+xy\}$.
5. $\mathbb{Z}_2[X,Y,Z]/(X,Y,Z)^2$, $M = \{0, x, y, z, x+y, x+z, y+z, x+y+z\}$.
6. $\mathbb{Z}_4[X]/(2X, X^3-2)$, $M = \{0, 2, x, x+2, x^2, x^2+2, x^2+x, x^2+x+2\}$.
7. $\mathbb{Z}_4[X]/(X^2-2)$, $M = \{0, x, 2, x+2, 2x, 3x, 2x+2, 3x+2\}$.
8. $\mathbb{Z}_4[X]/(X^2-2X-2)$, $M = \{0, 2, x, 2x, x+2, 3x, 2x+2, 3x+2\}$.
9. $\mathbb{Z}_4[X,Y]/(X^2-2, XY, Y^2, 2X)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$.
10. $\mathbb{Z}_4[X,Y]/(X^2-2, XY, Y^2-2, 2X)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$.
11. $\mathbb{Z}_4[X,Y]/(X^2, XY-2, Y^2)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$.
12. $\mathbb{Z}_4[X]/(2X, X^3)$, $M = \{0, x, 2, x^2, x+2, x^2+2, x^2+x, x^2+x+2\}$.
13. $\mathbb{Z}_4[X]/(X^2)$, $M = \{0, 2, x, x+2, 2x, 2x+2, 3x, 3x+2\}$.
14. $\mathbb{Z}_4[X]/(X^2-2X)$, $M = \{0, 2, x, 2x, x+2, 2x+2, 3x, 3x+2\}$.
15. $\mathbb{Z}_4[X,Y]/(2,X,Y)^2$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$.
16. $\mathbb{Z}_8[X]/(2X, X^2)$, $M = \{0, 2, 4, 6, x, x+2, x+4, x+6\}$.
17. $\mathbb{Z}_8[X]/(2X, X^2-4)$, $M = \{0, 2, 4, 6, x, 2+x, 4+x, 6+x\}$.
18. $\mathbb{Z}_{16}$, $M = (2) = \{0, 2, 4, 6, 8, 10, 12, 14\}$.

For any of the above rings, if we choose an ideal I with two elements (provided one exists), then R/I will be a local ring of order 8 with a maximal ideal consisting of 4 elements. Moreover, since R/I is a local ring of order 8 with a maximal ideal consisting of 4 elements, it must be isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$, $\mathbb{Z}_4[X]/(X^2, 2X)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ by Lemma 2.5. Then to ensure that R/I is isomorphic to one of the three rings in the hypothesis, it suffices to ensure that it is not isomorphic to $\mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$ or $\mathbb{Z}_4[X]/(X^2, 2X)$. Notice that of the five possible rings, these two are the only ones such that $M^2 = \{\alpha^2 \mid \alpha \in M\} = 0$ (where M is the maximal ideal of the ring). Thus it suffices to do the following calculations for each of the 17 local rings of order 16 in question:

1. Find all the ideals of R with $|I| = 2$;

2. For each such ideal I, square each element of M and ensure that not all of the squares lie in I.

The second calculation above ensures that the maximal ideal of R/I (namely M/I) does not have the property that $(M/I)^2 = \{\overline{\alpha}^2 \mid \overline{\alpha} \in M/I\} = \{0 + I\}$. The rings for which this holds will then have the property that R/I must be isomorphic to one of the three rings in question.

In order to find all the ideals consisting of two elements, we note that such an ideal must be principal of the form $(\alpha) = R\alpha$, where $\alpha^2 = 0$ (local rings only have trivial idempotents). In each calculation, we compute M^2 (as defined above) and, when necessary, compute $\Omega = \{\alpha \in M \mid \alpha^2 = 0 \text{ and } \alpha \neq 0\}$ (these are potential generators of ideals of order 2). We then list all ideals with two elements. Some elements will not generate an ideal of two elements; this will be the case when $\{0, \alpha\}$ (where $\alpha \in \Omega$) is not an ideal of R. Those ideals with $M^2 \not\subseteq I$ will produce R/I of the appropriate form. The calculations are as follows.

1. $\mathbb{Z}_2[X]/(X^4)$, $M = \{0, x, x^2, x^3, x^2+x, x^3+x, x^3+x^2, x^3+x^2+x\}$, $M^2 = \{0, x^2\}$, $\Omega = \{x^2, x^3, x^2 + x\}$. Notice that $x(x^2) = x^3 \notin \{0, x^2\}$ and $x(x^2 + x) = x^3 + x^2 \notin \{0, x^2 + x\}$, whence neither x^2 nor $x^2 + x$ generate an ideal consisting of two elements. Although $(x^2) = \{0, x^2\}$, $M^2 \subseteq (x^2)$ which does not meet our requirements. So this ring does **not** meet our criteria.

2. $\mathbb{Z}_2[X,Y]/(X^3,XY,Y^2)$, $M=\{0,x,y,x+y,x^2,x^2+x,x^2+x,x^2+x+y\}$, $M^2=\{0,x^2\}$, $\Omega=\{y,x^2,x^2+y\}$. The only ideals of order 2 are (x^2), (y), (x^2+y). Moreover (y), (x^2+y) meet our requirements. So this ring does meet our criteria.

3. $\mathbb{Z}_2[X,Y]/(X^2-Y^2,XY)$, $M=\{0,x,y,x^2,x+y,y+x^2,x+y+x^2,x+x^2\}$, $M^2=\{0,x^2=y^2\}$, $\Omega=\{x^2,x+y\}$. Notice that $\{0,x+y\}$ is not an ideal as $(x+y)x=x^2\notin\{0,x+y\}$. So the only possible ideal of two elements is M^2; so this ring does **not** meet our criteria.

4. $\mathbb{Z}_2[X,Y]/(X^2,Y^2)$, $M=\{0,x,y,x+y,xy,x+xy,y+xy,x+y+xy\}$, $M^2=0$. For any ideal I, $I\subseteq M^2$; so this ring can **not** meet our criteria. There is no need to compute the ideals of cardinality two in this case.

5. $\mathbb{Z}_2[X,Y,Z]/(X,Y,Z)^2$, $M=\{0,x,y,z,x+y,x+z,y+z,x+y+z\}$. Here $M^2=0$ as before; so this ring does **not** meet our criteria.

6. $\mathbb{Z}_4[X]/(2X,X^3-2)$, $M=\{0,2,x,x+2,x^2,x^2+2,x^2+x,x^2+x+2\}$, $M^2=\{0,x^2\}$, $\Omega=\{2,x+2,x^2,x^2+2,x^2+x\}$. Notice (x^2) won't work even if $(x^2)=\{0,x^2\}$ since then it would contain M^2. Notice that $x(x^2+2)=x^3\notin\{0,x^2+2\}$ and $x(x^2+x)=x^2+2\notin\{0,x^2+x\}$; whence neither $\{0,x^2+2\}$ nor $\{0,x^2+x\}$ are ideals. Moreover, $\{0,x+2\}$ is not an ideal since $x(x+2)=x^2\notin\{0,x+2\}$. The only remaining possibility is $(2)=\{0,2\}$, which does meet our criteria.

7. $\mathbb{Z}_4[X]/(X^2-2)$, $M=\{0,x,2,x+2,2x,3x,2x+2,3x+2\}$, $M^2=\{0,2\}$, $\Omega=\{2,2x,2x+2\}$. Notice that $x(2)=2x\notin\{0,2\}$ and $x(2x+2)=2x\notin\{0,2x+2\}$; whence neither $\{0,2\}$ nor $\{0,2x+2\}$ are ideals of the ring. However, $(2x)=\{0,2x\}$ and $M^2\not\subseteq(2x)$. Thus this ring meets our criteria.

8. $\mathbb{Z}_4[X]/(X^2-2X-2)$, $M=\{0,2,x,2x,x+2,3x,2x+2,3x+2\}$, $M^2=\{0,2x+2\}$, $\Omega=\{2,2x,2x+2\}$. Notice that $M\subseteq(2x+2)$; so regardless of the cardinality of $(2x+2)$, it would not satisfy the criteria. Since $x(2)=2x\notin\{0,2\}$, $\{0,2\}$ is

not an ideal of the ring. However, we have that $(2x) = \{0, 2x\}$ does satisfy the criteria.

9. $\mathbb{Z}_4[X,Y]/(X^2-2, XY, Y^2, 2X)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$, $M^2 = \{0, 2\}$, $\Omega = \{2, y, y+2\}$. Notice that each of these generate an ideal with two elements. However, $M^2 \subseteq (2)$; so the ideals (y) and $(y+2)$ meet our criteria.

10. $\mathbb{Z}_4[X,Y]/(X^2-2, XY, Y^2-2, 2X)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$, $M^2 = \{0, 2\}$, $\Omega = \{2, x+y, x+y+2\}$. The ideal (2) contains M^2; so it does not meet the criteria. Moreover, $x(x+y) = 2 \notin \{0, x+y\}$ and $x(x+y+2) = 2 \notin \{0, x+y+2\}$; whence neither of the other two elements of Ω generate an ideal consisting of two elements. Thus this ring does **not** meet our criteria.

11. $\mathbb{Z}_4[X,Y]/(X^2, XY-2, Y^2)$, $M = \{0, 2, x, y, x+y, x+2, y+2, x+y+2\}$. Here $M^2 = 0$; so regardless of the choice of I, this ring does **not** meet our criteria.

12. $\mathbb{Z}_4[X]/(2X, X^3)$, $M = \{0, x, 2, x^2, x+2, x^2+2, x^2+x, x^2+x+2\}$, $M^2 = \{0, x^2\}$, $\Omega = \{2, x^2, x^2+2\}$. Notice even if $\{0, x^2\}$ is an ideal of the ring, it would not meet our criteria as it would contain M^2. Notice that both $\{0, 2\}$ and $\{0, X^2+2\}$ are ideals and meet our criteria.

13. $\mathbb{Z}_4[X]/(X^2)$, $M = \{0, 2, x, x+2, 2x, 2x+2, 3x, 3x+2\}$, $M^2 = 0$. So, again, regardless of the choice of ideal, the criteria will not hold. Thus this ring does **not** meet our criteria.

14. $\mathbb{Z}_4[X]/(X^2-2X)$, $M = \{0, x, 2, 2x, x+2, 2x+2, 3x, 3x+2\}$, $M^2 = \{0, 2x\}$, $\Omega = \{2, 2x, 2x+2\}$. Notice that $x(2) = 2x \notin \{0, 2\}$ and $x(2x+2) = 2x \notin \{0, 2x+2\}$; hence $2, 2x+2$ do not generate ideals consisting of two elements. Even though $2x$ does generate an ideal of two elements, we have $M^2 \subseteq (2x)$. Thus this ring does **not** meet our criteria.

15. $\mathbb{Z}_4[X,Y]/(2,X,Y)^2$, $M = \{0, x, y, 2, x+y, x+2, y+2, x+y+2\}$, $M^2 = 0$. Therefore this ring will **not** meet our criteria regardless of the choice of ideal.

16. $\mathbb{Z}_8[X]/(2X, X^2)$, $M = \{0, 2, 4, 6, x, x+2, x+4, x+6\}$, $M^2 = \{0, 4\}$, $\Omega = \{4, x, x+4\}$. Regardless of whether or not $\{0, 4\}$ is an ideal, it will not meet our criteria. However, notice that x and $x+4$ do generate ideals of two elements meeting the criteria.

17. $\mathbb{Z}_8[X]/(2X, X^2-4)$, $M = \{0, 2, 4, 6, x, x+2, x+4, x+6\}$, $M^2 = \{0, 4\}$, $\Omega = \{4, x+2, x+6\}$. Notice that $x(x+2) = 4 \notin \{0, x+2\}$ and $2(x+6) = 12 = 4 \notin \{0, x+6\}$; whence neither $x+2$ nor $x+6$ generate ideals of two elements. Moreover, $M^2 \subseteq \{0, 4\} \subseteq (4)$; whence (4) would not meet our criteria. Thus this ring does **not** meet our criteria.

18. $\mathbb{Z}_{16}$, $M = (2) = \{0, 2, 4, 6, 8, 10, 12, 14\}$, $M^2 = \{0, 4\}$. Here the only ideal of the ring consisting of two elements is $\{0, 8\}$. This ideal does indeed satisfy our criteria.

The forward direction then follows.

"$\Leftarrow$" This direction is evident from the observations made during the proof of the forward direction. □

Proposition 3.12. *Let R be a* **finite** *commutative ring with nonzero identity and let I be a nonzero, proper,* **non-radical** *ideal of R. Then $gr(\Gamma(R/I)) = \infty$ and $|I| = 2$ if and only if R is isomorphic to one of the* **27** *rings from Table 3.4 (with appropriately chosen ideal I).*

Proof. For the reader's ease, each ring appears in Table 3.4 in the order it is adressed in this proof.

Note that I non-radical implies that I is non-prime. Thus $\Gamma_I(R)$ is nonempty.

Recall that I is not a radical ideal of R if and only if R/I is not reduced. So we begin our search by considering which non-reduced rings have corresponding zero-divisor graph with infinite girth. By [8, Theorem 2.5], we have that a non-reduced

Table 3.4: Non-radical case: $gr(\Gamma(R/I)) = \infty$ and $|I| = 2$

Ring	Ideal
$\mathbb{Z}_9 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	$Z_2 \times 0 \times 0$ or $0 \times \mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	$Z_2 \times 0 \times 0$ or $0 \times \mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\mathbb{Z}_2 \times 0$ or $0 \times \mathbb{Z}_4$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2)$	$0 \times (x)$, $0 \times (2)$, or $0 \times (x+2)$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2-2)$	$\mathbb{Z}_2 \times 0$ or $0 \times (2)$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$	$\mathbb{Z}_2 \times 0$ or $0 \times (x^2)$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$	$\mathbb{Z}_2 \times 0$, $0 \times (x)$, $0 \times (y)$, or $0 \times (x+y)$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(2) \times 0$ or $0 \times (2)$
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$	$(x) \times 0$ or $(x) \times 0$
$\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$	$(2) \times 0$ or $0 \times (x)$
$\mathbb{Z}_2[X,Y]/(X^3, XY, Y^2)$	(y) or $(y+x^2)$
$\mathbb{Z}_4[X]/(2X, X^3-2)$	(2)
$\mathbb{Z}_4[X]/(X^2-2)$	$(2x)$
$\mathbb{Z}_4[X]/(X^2-2X-2)$	$(2x)$
$\mathbb{Z}_4[X,Y]/(X^2-2, XY, Y^2, 2X)$	(y) or $(y+2)$
$\mathbb{Z}_4[X]/(2X, X^3)$	(2) or (x^2+2)
$\mathbb{Z}_8[X]/(2X, X^2)$	(x) or $(x+4)$
$\mathbb{Z}_{16}$	(8)
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_8$	(4)
$\mathbb{Z}_4[X]/(X^2, 2X)$	(x), (2), or $(x+2)$
$\mathbb{Z}_4[X]/(2X, X^2-2)$	(2)
$\mathbb{Z}_2[X]/(X^3)$	(x^2)
$\mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$	(x), (y), or $(x+y)$

ring A has $gr(\Gamma(A)) = \infty$ if and only if $A \cong B$ or $A \cong B \times \mathbb{Z}_2$, where $B \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, or $\Gamma(A)$ is a star graph. In the proof of the proceeding result, the authors show that $\Gamma(A)$ is complemented when $nil(R) \neq 0$ and $gr(\Gamma(R)) = \infty$. They then split the situation into two cases: when the graph is uniquely complemented or not. The uniquely complemented case is when $\Gamma(A)$ is a star graph. But using [6, Theorem 3.9], we have that $\Gamma(A)$ uniquely complemented with $nil(R)$ nonzero implies that either $\Gamma(A)$ is a star graph on at most two edges or an infinite star graph. However, since we are considering finite rings, $\Gamma(A)$ must be a star graph on at most 2 edges. Again from [6, Remark 3.12 (a)], we have that if $\Gamma(A)$ is a star graph on at most two edges, then A is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2), \mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_4[X]/(2X, X^2-2)$. However, among the preceding rings, only $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$ are reduced. Thus R/I must be isomorphic to one of the following rings: $\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2), \mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_4[X]/(2X, X^2-2)$, $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$. Using that $|I||R/I| = |R|$ and $|I| = 2$, we have that $|R| \in \{8, 16, 18\}$.

If $|R| = 18$, then $R/I \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$. Since $18 = 3 \cdot 3 \cdot 2$, using that R can be written as a product of finite local rings and that a local ring has cardinality a power of a prime, we have that $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ or $R_1 \times \mathbb{Z}_2$, where R_1 is a local ring of cardinality 9. The only ideal consisting of 2 elements in $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ is $0 \times 0 \times \mathbb{Z}_2$, but in this case, we have $R/I \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ which is not isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$. Thus $R \cong R_1 \times \mathbb{Z}_2$, where R_1 is local and $|R_1| = 9$. In this ring, the only ideal of order 2 is $0 \times \mathbb{Z}_2$, whence $R/I \cong R_1$. Thus $R \cong \mathbb{Z}_9 \times \mathbb{Z}_2$ or $\mathbb{Z}_3[X]/(X^2) \times \mathbb{Z}_2$, where $I = 0 \times \mathbb{Z}_2$. Hence when $|R| = 18$, there are only **2** possible rings up to isomorphism.

If $|R| = 16$, then R/I is isomorphic to one of the following rings:

$$\mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_4[X]/(2X, X^2-2), \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2). \tag{3.1}$$

Again, by writing R as a product of finite local rings, R must be isomorphic to one of the following types, where R_i are local rings:

$$R \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \\ R_1 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \text{ where } |R_1| = 4 \\ R_2 \times \mathbb{Z}_2, \text{ where } |R_2| = 8 \\ R_1 \times R_2, \text{ where } |R_1| = |R_2| = 4 \\ R, \text{ where } |R| = 16 \end{cases} \tag{3.2}$$

Notice that the first case can not occur as then $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ which is not isomorphic to any of four rings from 3.1. In the second factorization, we must have that $I = I_1 \times 0 \times 0$, $0 \times \mathbb{Z}_2 \times 0$, or $0 \times 0 \times \mathbb{Z}_2$, where I_1 is an ideal of R_1 of order 2. Then either $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (when $I = I_1 \times 0 \times 0$) or $R_1 \times \mathbb{Z}_2$ (when $I = 0 \times \mathbb{Z}_2 \times 0$, or $0 \times 0 \times \mathbb{Z}_2$). Thus I is of the form of the second two possibilities and $R/I \cong R_1 \times \mathbb{Z}_2$, where R_1 is a local ring of order 4. It is then evident that $R_1 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ (by Lemma 2.1 and the fact that R is not reduced). Thus in the second factorization of 3.2, we must have that $R \cong \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where I is either $0 \times \mathbb{Z}_2 \times 0$ or $0 \times 0 \times \mathbb{Z}_2$. Thus among the first and second factorizations in 3.2, we have only **2** possible rings.

The third factorization in 3.2 requires a bit more work. We have done the considerations of the third possible factorization in Lemma 3.10; in the third factorization we get **8** possible rings.

Finally by Lemma 2.1, we have that in the fourth case of 3.2, R_i is isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, or $\mathbb{F}_4$. Thus $R_1 \times R_2$ is isomorphic to one of the following:

1. $\mathbb{Z}_4 \times \mathbb{Z}_4$,

2. $\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$,

3. $\mathbb{Z}_4 \times \mathbb{F}_4$,

4. $\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$,

5. $\mathbb{Z}_2[X]/(X^2) \times \mathbb{F}_4$, or

6. $\mathbb{F}_4 \times \mathbb{F}_4$.

Notice that (6) above has no ideals of order 2. Possibility (3) above has only one ideal of order 2 namely $(2) \times 0$; but then $R/I \cong \mathbb{Z}_2 \times \mathbb{F}_4$ (which is not isomorphic to a ring in 3.1) A similar argument rules out possibility (5). The remaining possiblities each contain ideals of order 2 with R/I isomorphic to one of the rings in 3.1 (notice all ideals of order 2 from the remaining rings meet the hypothesis). This case yeilds **3** rings.

All that remains in the case when $|R| = 16$ and R is a local ring (that is the last factorization in 3.2). We must have that R contains an ideal consisting of two elements such that R/I is isomorphic to one of the five ring from 3.1. In Chapter 2, we noted that there are 21 local rings of order 16; so the task at hands seems a bit unruly. We considered this case separately in Lemma 3.11. So in the local case when $|R| = 16$, we have that R is isomorphic to one of the **8** rings from Lemma 3.11.

If $|R| = 8$, then R/I is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Then again by writing R as a product of finite local rings, R must be isomorphic to one of the following types, where R_i are local rings:

$$R \cong \begin{cases} R_1, \text{ where } |R_1| = 8 \\ R_1 \times \mathbb{Z}_2, \text{ where } |R_1| = 4 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \end{cases}$$

The last factorization does not occur as $|I| = 2$ implies that $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In the second case, we must have that I is of the form $0 \times \mathbb{Z}_2$ (as otherwise $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$). Hence in the second possible factorization, $R_1 \cong R/I$ which must be isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ (by Lemma 2.1 and the fact that R is not reduced). Thus in the second factorization, R is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2$ with $I = 0 \times \mathbb{Z}_2$. The only remaining case is when R is a local ring of order 8. Since R must have an ideal with two elements which is not prime, it follows that R is a local ring with 8 elements and a maximal ideal consisting of 4 elements. Such rings are classified in Lemma 2.5. Thus R is isomorphic to one of the following **5** rings: $\mathbb{Z}_8$, $\mathbb{Z}_4[X]/(X^2, 2X)$, $\mathbb{Z}_4[X]/(2X, X^2 - 2)$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$. Possible choices for the ideal I are also given in Section 2.1. $\square$

Proposition 3.13. *Let I be an ideal of a commutative ring R with nonzero identity. Then $|V(\Gamma(R/I))| = 1$ and $|I| \in \{2, 3, 4\}$ if and only if R is isomorphic to one the* **40** *rings with corresponding ideal as found in Table 2.1, Table 2.2, or Table 2.3.*

Proof. We know that $|V(\Gamma(R/I))| = 1$ if and only if $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. The desired result then follows from Proposition 2.8. □

By combining Propositions 3.9, 3.12, and 3.13, we have classified up to isomorphism all such **finite** commutative rings with nonzero identity such that $\Gamma_I(R)$ is planar. Notice in the reduced case, we get three infinite classes of rings, and in the non-reduced case we get **41** different finite rings as listed in Table 3.5.

The restriction of finite graphs only prevented us from considering the following situation:

$$\Gamma(R/I) \text{ is an infinite star graph and } I \neq \sqrt{I}. \tag{3.3}$$

Other than the preceding case, we have found up to isomorphism all rings with $\Gamma_I(R)$ planar with I nonzero. Currently, we leave the classification of 3.3 **open**, but plan to return to it in later research.

3.3 Graphs of Finite Planar Non-trivial $\Gamma_I(R)$ with $I \neq \sqrt{I}$

We can now draw all finite planar graphs corresponding to non-empty $\Gamma_I(R)$ with I a non-radical, nonzero ideal of a ring R. One way is to tread through our 39 rings and their ideals, graphing each possibility. However using some observations made in Proposition 3.12, we can make short work of determining these graphs.

Proposition 3.14. *Let R be a commutative ring with nonzero identity and I a non-radical, nonzero ideal of R. Then $\Gamma_I(R)$ is planar if and only if $\Gamma_I(R)$ is isomorphic to one of the 5 graphs in Figure 3.6.*

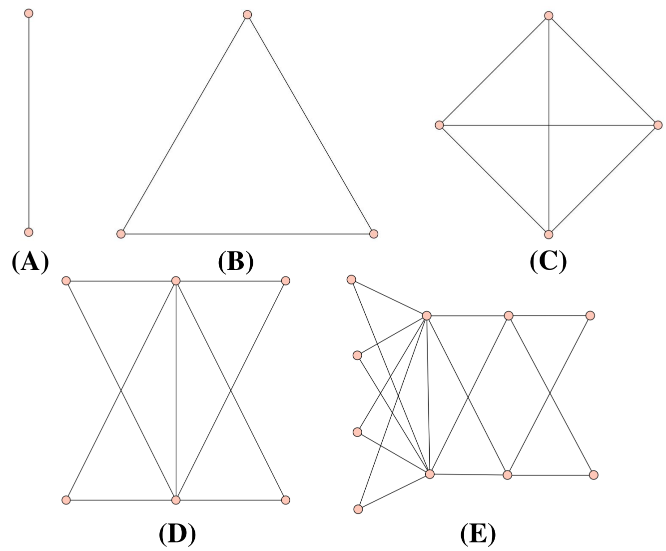


Figure 3.6: The 5 finite planar graphs with I non-radical and nonzero.

Proof. The converse is evident.

For the forward direction, by Theorem 3.8 we have that $gr(\Gamma(R/I)) = \infty$ and either (a) $|I| = 2$ or (b) $|V(\Gamma(R/I))| = 1$ and $|I| \in \{2, 3, 4\}$. In case (b), $\Gamma_I(R)$ is isomorphic to K^2, K^3, or K^4. Assume case (a) holds. Recall that in Proposition 3.12, we noted that under these conditions that $R/I \cong B$, $\mathbb{Z}_2 \times B$ (where $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$), or $\Gamma(R/I)$ is a star graph on at most two vertices (i.e., $K^{1,1}$ or $K^{1,2}$, here we are using the finite hypothesis). Now using Redmond's construction of $\Gamma_I(R)$ from $\Gamma(R/I)$, we can deduce the possible graphs for $\Gamma_I(R)$.

If $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, then $|V(\Gamma(R/I))| = 1$. Hence $|I| = 2$ gives that $\Gamma_I(R) \cong K^2$.

If $R/I \cong \mathbb{Z}_2 \times B$ (where B is as before), then R/I is $\overline{K}^{1,3}$. Notice that the vertex of degree 3 is the only element whose square is zero, thus $|I| = 2$ implies that $\Gamma_I(R)$ is isomorphic to (E) in Figure 3.6.

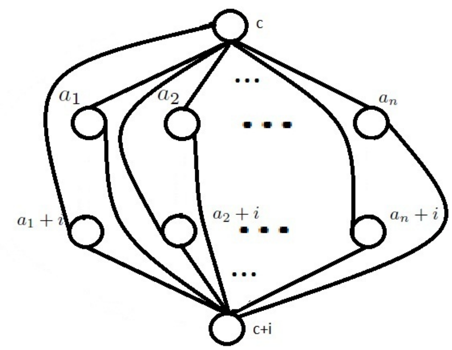


Figure 3.7: Finite planar graphs with I radical and nonzero.

If $\Gamma(R/I) \cong K^{1,1}$, then $R/I \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$ [6, pp. 2-3], and whence each vertex of the graph has the property that its square is zero. So with $|I| = 2$, we get $\Gamma_I(R) \cong K^4$.

If $\Gamma(R/I) \cong K^{1,2}$ and $I \neq \sqrt{I}$, then $R/I \cong \mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2-2)$ [6, pp. 2-3]. In each of the latter cases, the only vertex whose square is zero is the center vertex. Thus with $|I| = 2$, we have that $\Gamma_I(R)$ will be isomorphic to (D) of Figure 3.6. □

We now consider when $\Gamma_I(R)$ is a finite, planar graph and I is a radical ideal. In this case, we can see by Proposition 3.9 that $\Gamma(R/I)$ will be a star graph and $|I| = 2$. It then follows that $\Gamma_I(R)$ will be isomorphic to the graph in Figure 3.7.

Table 3.5: When $\Gamma_I(R)$ is planar for nonzero, non-radical ideal I

Ring	Ideal(s) when $gr(\Gamma(R/I)) = \infty$ and $\lvert I\rvert = 2$	Ideal(s) when $\Gamma_I(R) = K^{\lvert I\rvert}$ and $\lvert I\rvert \in \{3,4\}$
$\mathbb{Z}_3 \times \mathbb{Z}_4$		$\mathbb{Z}_3 \times 0$
$\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$		$\mathbb{Z}_3 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_9$	$\mathbb{Z}_2 \times 0$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$	$\mathbb{Z}_2 \times 0$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times 0 \times 0$ or $0 \times \mathbb{Z}_2 \times 0$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	$\mathbb{Z}_2 \times 0 \times 0$ or $0 \times \mathbb{Z}_2 \times 0$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\mathbb{Z}_2 \times 0$ or $0 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times (4)$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2)$	$0 \times (x)$, $0 \times (2)$, or $0 \times (x+2)$	$\mathbb{Z}_2 \times (x)$, $\mathbb{Z}_2 \times (2)$, or $\mathbb{Z}_2 \times (x+2)$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2-2)$	$\mathbb{Z}_2 \times 0$ or $0 \times (2)$	$\mathbb{Z}_2 \times (2)$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$	$\mathbb{Z}_2 \times 0$ or $0 \times (x^2)$	$\mathbb{Z}_2 \times (x^2)$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$	$\mathbb{Z}_2 \times 0$, $0 \times (x)$, $0 \times (y)$, or $0 \times (x+y)$	$\mathbb{Z}_2 \times (x)$ or $\mathbb{Z}_2 \times (y)$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(2) \times 0$ or $0 \times (2)$	$\mathbb{Z}_4 \times 0$ or $0 \times \mathbb{Z}_4$
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$	$(x) \times 0$ or $(x) \times 0$	$\mathbb{Z}_2[X]/(X^2) \times 0$ or $0 \times \mathbb{Z}_2[X]/(X^2)$
$\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$	$(2) \times 0$ or $0 \times (x)$	$\mathbb{Z}_4 \times 0$ or $0 \times \mathbb{Z}_2[X]/(X^2)$
$\mathbb{Z}_2[X,Y]/(X^3, XY, Y^2)$	(y) or $(y+x^2)$	$\{0, x, x^2, x+x^2\}$, $\{0, y, x^2, y+x^2\}$, or $\{0, x+y, x^2, x+y+x^2\}$
$\mathbb{Z}_4[X]/(2X, X^3-2)$	(2)	
$\mathbb{Z}_4[X]/(X^2-2)$	$(2x)$	$\{0, 2, 2x, 2+2x\}$
$\mathbb{Z}_4[X]/(X^2-2X-2)$	$(2x)$	$\{0, 2, 2x, 2+2x\}$
$\mathbb{Z}_4[X,Y]/(X^2-2, XY, Y^2, 2X)$	(y) or $(y+2)$	$\{0, 2, x, 2+x\}$, $\{0, 2, y, y+2\}$, or $\{0, 2, x+y, x+y+2\}$
$\mathbb{Z}_4[X]/(2X, X^3)$	(2) or (x^2+2)	$\{0, 2, x^2, x^2+2\}$, $\{0, x, x^2, x+x^2\}$, or $\{0, x^2, x+2, x+x^2+2\}$
$\mathbb{Z}_8[X]/(2X, X^2)$	(x) or $(x+4)$	$\{0, 2, 4, 6\}$, $\{0, 4, x, x+4\}$, or $\{0, 4, x+2, x+6\}$
$\mathbb{Z}_{16}$	(8)	(4)
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times 0$	
$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	$\mathbb{Z}_2 \times 0$	
$\mathbb{Z}_8$	(4)	
$\mathbb{Z}_4[X]/(X^2, 2X)$	(x), (2), or $(x+2)$	
$\mathbb{Z}_4[X]/(2X, X^2-2)$	(2)	
$\mathbb{Z}_2[X]/(X^3)$	(x^2)	
$\mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$	(x), (y), or $(x+y)$	
$\mathbb{Z}_4 \times \mathbb{F}_4$		$0 \times \mathbb{F}_4$
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{F}_4$		$0 \times \mathbb{F}_4$
$\mathbb{Z}_2[X]/(X^4)$		$\{0, x^2, x^3, x^2+x^3\}$
$\mathbb{Z}_2[X,Y]/(X^2-Y^2, XY)$		$\{0, x, x^2, x+x^2\}$, $\{0, y, x^2, y+x^2\}$, or $\{0, x^2, x+y, x+y+x^2\}$
$\mathbb{Z}_2[X,Y]/(X^2, Y^2)$		$\{0, x, xy, x+xy\}$, $\{0, y, xy, y+xy\}$, or $\{0, x+y, xy, x+y+xy\}$
$\mathbb{Z}_2[X,Y,Z]/(X,Y,Z)^2$		$I = \{0, z, x+y, x+y+z\}, \{0, x, y+z, x+y+z\}$, $\{0, x, z, x+z\}$, $\{0, y, z, y+z\}$, $\{0, x, y, x+y\}$, $\{0, x+y, x+z, y+z\}$, or $\{0, y, x+z, x+y+z\}$
$\mathbb{Z}_4[X,Y]/(X^2-2, XY, Y^2-2, 2X)$		$\{0, 2, x, 2+x\}$, $\{0, 2, y, y+2\}$, or $\{0, 2, x+y, x+y+2\}$
$\mathbb{Z}_4[X,Y]/(X^2, XY-2, Y^2)$		$\{0, 2, x, x+2\}$, $\{0, 2, y, y+2\}$, or $\{0, 2, x+y, x+y+2\}$
$\mathbb{Z}_4[X]/(X^2)$		$\{0, 2, 2x, 2x+2\}$, $\{0, 2x, 3x+2, x+2\}$, or $\{0, x+2, 2x, 3x+2\}$
$\mathbb{Z}_4[X]/(X^2-2X)$		$\{0, x, 2x, 3x\}$, $\{0, 2, 2x, 2x+2\}$, or $\{0, 2x, x+2, 3x+2\}$
$\mathbb{Z}_4[X,Y]/(2,X,Y)^2$		$\{0, 2, x, x+2\}$, $\{0, 2, y, y+2\}$, $\{0, 2, x+y, x+y+2\}$, $\{0, x, y, x+y\}$, $\{0, x, y+2, x+y+2\}$, $\{0, y, x+2, x+y+2\}$, or $\{0, x+y, x+2, y+2\}$
$\mathbb{Z}_8[X]/(2X, X^2-4)$		$\{0, 2, 4, 6\}$, $\{0, 4, x, x+4\}$, or $\{0, 4, x+2, x+6\}$

Chapter 4

When $\Gamma_I(R)$ is Complemented

In this chapter, we investigate the concepts of complemented and uniquely complemented graphs as considered in [27], [22], and [6]. In the following, there is some overlap with results from [27]. The primary overlap is in Lemma 4.6.

Recall that a ring R is von Neumann regular if for every $x \in R$, there exists $y \in R$ such that $x = xyx$. In [6], the authors find a connection between a ring being von Neumann regular and a graph property called complemented. Given vertices a and b of a graph G, we define a relation $a \leq b$ if and only a and b are not adjacent and each vertex of G adjacent to b is also adjacent to a. We define $a \sim b$ if $a \leq b$ and $b \leq a$. Here we have that $a \sim b$ if and only if a and b are not adjacent, yet they are adjacent to exactly the same vertices of G. Given two vertices a and b of G, we say that the vertices are *orthogonal*, denoted $a \perp b$, if a and b are adjacent and there is no vertex adjacent to both a and b. Notice that $a \perp b$ if and only if a and b are adjacent and the edge $a - b$ is not part of triangle (a 3-cycle) in G. A graph G is called *complemented* if given any vertex a of G, there exists a vertex b of G such that $a \perp b$. A graph G is *uniquely complemented* if it is complemented and $a \perp b$ and $a \perp c$ imply that $a \sim c$. The preceding relations and definitions are from [6] and [22]. In [6, Theorem 3.5], the authors show for a reduced ring that $\Gamma(R)$ is uniquely complemented if and only if

$\Gamma(R)$ is complemented, if and only if $T(R)$ is von Neumann regular. It is the goal of this chapter to extend this result to $\Gamma_I(R)$.

Proposition 4.1. *Let R be a commutative ring with nonzero identity and I a nonzero, proper ideal of R. If I is a non-radical ideal of R and $|V(\Gamma(R/I))| \geq 2$, then $\Gamma_I(R)$ is not complemented.*

Proof. Since $I \neq \sqrt{I}$, there exists an $r \in R \setminus I$ such that $r^2 \in I$. Then $r \in V(\Gamma_I(R))$. We claim that r has no complement in $\Gamma_I(R)$. Let s be any vertex of $\Gamma_I(R)$ adjacent to r; so $rs \in I$. Notice that $r \neq s$ as they are distinict adjacent vertices of $\Gamma_I(R)$. Then there are two possibilities: (1) there exists an $i \in I$ such that $s = r + i$ or (2) $s \neq r + i$ for all $i \in I$.

Case (1): Assume there exists an $i \in I$ such that $s = r + i$. Then $r + I = s + I$ in R/I. Since $|V(\Gamma(R/I))| \geq 2$, there exists a vertex $t + I$ adjacent to $r + I = s + I$ in $\Gamma(R/I)$. Notice that $t, r, s = r + i$ are all distinct vertices of $\Gamma_I(R)$ that are mutually adjacent. Thus the edge $r - s$ is part of a triangle in $\Gamma_I(R)$; so s is not a complement of r in $\Gamma_I(R)$.

Case (2): Assume $s \neq r + i$ for all $i \in I$. Since I is non-zero, choose $0 \neq i \in I$. Then the vertices $s, r, r+i$ are distinct mutually adjacent vertices of $\Gamma_I(R)$. Thus the edge $r - s$ is part of a triangle in $\Gamma_I(R)$; so, as before, s is not a complement of r in $\Gamma_I(R)$.

Thus we have shown that no vertex adjacent to r is a complement of r; so $\Gamma_I(R)$ is not complemented. $\square$

Theorem 4.2. *Let R be a commutative ring with nonzero identity and I a non-radical, nonzero, proper ideal of R. Then the following statements are equivalent.*

1. *$\Gamma_I(R)$ is uniquely complemented.*

2. *$\Gamma_I(R)$ is complemented.*

3. *$\Gamma_I(R) \cong K^2$.*

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are clear. It suffices to prove (2) $\Rightarrow$ (3).

Assume that $\Gamma_I(R)$ is complemented. Then by the contrapositive of the previous proposition, it follows that $|V(\Gamma(R/I))| \leq 1$. Since I is not prime (as it is non-radical), it follows that $|V(\Gamma(R/I))| = 1$. Thus $\Gamma_I(R) \cong K^{|I|}$ by Corollary 2.3. Since the only complete graph which is complemented is K^2, it follows that $|I| = 2$ and $\Gamma_I(R) \cong K^2$. $\square$

Notice that if $|V(\Gamma(R/I))| = 1$, then $I \neq \sqrt{I}$. Moreover, in this case, $\Gamma_I(R)$ is complemented if and only if $|I| = 2$ by the preceding theorem. Thus it remains to investigate the case when $|V(\Gamma(R/I))| \geq 2$.

Theorem 4.3. *Let R be a commutative ring with nonzero identity and I a nonzero, proper ideal of R. Then $\Gamma_I(R)$ is complemented and $|V(\Gamma(R/I))| \geq 2$ if and only if $\Gamma(R/I)$ is complemented and $I = \sqrt{I}$.*

Proof. "$\Rightarrow$" Assume that $\Gamma_I(R)$ is complemented and $|V(\Gamma(R/I))| \geq 2$. Since $\Gamma_I(R)$ is complemented and $|V(\Gamma(R/I))| \geq 2$, it follows from Proposition 4.1 that $I = \sqrt{I}$. So it remains to show that $\Gamma(R/I)$ is complemented. Let $r+I$ be a vertex of $\Gamma(R/I)$. Then r is a vertex of $\Gamma_I(R)$. By assumption, $\Gamma_I(R)$ is complemented; so there exists a vertex s of $\Gamma_I(R)$ such that $r \perp s$. We first show that $r+I \neq s+I$. Assume to the contrary, then $r - s = i \in I$. Thus $r(r-s) = ri \in I$. Since $r \perp s$, then $rs \in I$. Hence $r^2 = ri + rs \in I$, and thus $r \in I$ since $I = \sqrt{I}$. This is a contradiction since $r + I \neq I$, and hence $r \notin I$. Thus $r + I \neq s + I$. Since $r \perp s$ in $\Gamma_I(R)$ and $r + I \neq s + I$, it follows that $r + I$ is adjacent to $s + I$ in $\Gamma(R/I)$. It now remains only to show there is no other vertex in $\Gamma(R/I)$ adjacent to both of these. Assume to the contrary; then there exists a vertex $t + I$ adjacent to both $r + I$ and $s + I$ (hence $t + I, r + I$, and $s + I$ are distinct elements of R/I). Then notice that r, t, s are distinct mutually adjacent vertices of $\Gamma_I(R)$. But this is a contradiction as $r \perp s$ in $\Gamma_I(R)$. Therefore $r + I \perp s + I$. Since $r + I \in V(\Gamma(R/I))$ was chosen arbitrarily, it follows that $\Gamma(R/I)$ is complemented.

"$\Leftarrow$" Assume that $\Gamma(R/I)$ is complemented and $I = \sqrt{I}$. Since $\Gamma(R/I)$ is complemented, it follows that $|V(\Gamma(R/I)| \geq 2$. Let $r \in V(\Gamma_I(R))$; then $r + I \in V(\Gamma(R/I))$. Since $\Gamma(R/I)$ is complemented, there exists a vertex $s + I$ in $\Gamma(R/I)$ such that $r + I \perp s + I$. Since these are vertices in $\Gamma(R/I)$, it follows that neither is zero in R/I; hence $r, s \notin I$, but $rs \in I$. Thus r and s are adjacent vertices in $\Gamma_I(R)$. We claim that $r \perp s$ in $\Gamma_I(R)$. Assume to the contrary, then there exists a $t \in R \setminus I$, such that r, s, and t are distinct and mutually adjacent in $\Gamma_I(R)$. Using that $I = \sqrt{I}$, a similar argument to that in the forward implication shows that $r+I, s+I$, and $t+I$ are distinct vertices of $\Gamma(R/I)$. It then follows that $r+I, s+I$, and $t+I$ are distinct, mutually adjacent vertices of $\Gamma(R/I)$; but this is a contradiction as $r + I \perp s + I$. Therefore $r \perp s$ in $\Gamma_I(R)$. Since $r \in \Gamma_I(R)$ was chosen arbitrarily, it follows that $\Gamma_I(R)$ is complemented. □

Corollary 4.4. *Let R be a commutative ring with nonzero identity and I a nonzero ideal of R that is proper and not prime. Then $\Gamma_I(R)$ is complemented if and only if exactly one of the following statements holds.*

1. *$R/I \cong \mathbb{Z}_4$ or $R/I \cong \mathbb{Z}_2[X]/(X^2)$, and $|I| = 2$.*
2. *$\Gamma(R/I)$ is complemented and I is a radical ideal of R.*

Proof. In case (1), R/I is nonreduced and hence I is non-radical. Therefore (1) and (2) can not happen simultaneously.

"$\Rightarrow$" If I is a radical ideal of R, then we must have that $|V(\Gamma(R/I))| \geq 2$. Then (2) holds by Theorem 4.3. If I is not a radical ideal of R, then (1) holds by Theorem 4.2 and the proof of Propostion 2.4.

"$\Leftarrow$" In case (1), $\Gamma_I(R) = K^2$ and is therefore complemented. In case (2), $\Gamma_I(R)$ is complemented by Theorem 4.3.

□

Using the fact that R/I is reduced if and only if $I = \sqrt{I}$, we can extend the previous theorem to the following corollary using [6, Theorem 3.5]. In the following, note that if I is a prime ideal of the ring, then all of the graphs in question are empty. We will consider the empty graph to be vacuously complemented and uniquely complemented.

Corollary 4.5. *Let I an ideal of a commutative ring R. If I is a proper radical ideal of R, then the following statements are equivalent.*

1. *$\Gamma_I(R)$ is complemented.*

2. *$\Gamma(R/I)$ is complemented.*

3. *$\Gamma(R/I)$ is uniquely complemented.*

4. *$T(R/I)$ is von Neumann regular.*

We proceed to consider when $\Gamma_I(R)$ is uniquely complemented. Based on the preceding results, we are led to conjecture that when I is a radical ideal, we will have $\Gamma_I(R)$ uniquely complemented if and only $\Gamma_I(R)$ is complemented. The following results are similar to those found in [27, pp. 55-56]. It was after working on the following lemmas that the results from [27, pp. 55-56] were noticed by this author.

Lemma 4.6. *Let R be a commutative ring with nonzero identity and I a nonzero ideal of R. If I is a proper radical ideal, then $x \perp y$ in $\Gamma_I(R)$ if and only if $x + I \perp y + I$ in $\Gamma(R/I)$.*

Proof. "$\Rightarrow$" First notice that $I = \sqrt{I}$ and $xy \in I$ implies that $x + I \neq y + I$. As otherwise, $y = x + i$ for some $i \in I$. Then $x^2 = x(x + i) - xi = xy - xi \in I$. But $x \in V(\Gamma_I(R))$ implies that $x \notin I$. Hence $x \in \sqrt{I}$ and $x \notin I$, but this is a contradiction as $I = \sqrt{I}$.

Also, $(x + I)(y + I) = 0 + I$, so that $x + I$ and $y + I$ are adjacent vertices of $\Gamma(R/I)$. Assume to the contrary, that there exists $z + I \in V(\Gamma(R/I))$ such that $x+I-y+I-z+I-x+I$ is a triangle in $\Gamma(R/I)$. But then $x-y-z-x$ is a triangle in $\Gamma_I(R)$, which is a contradiction as $x \perp y$ in $\Gamma_I(R)$. Therefore, $x + I \perp y + I$ in $\Gamma(R/I)$ as desired.

"$\Leftarrow$" Assume that $x+I \perp y+I$ in $\Gamma(R/I)$. Then $xy \in I$; whence x and y are adjacent in $\Gamma_I(R)$. Assume that $x \not\perp y$. Then there exists a vertex c adjacent to both x and y in $\Gamma_I(R)$. We claim that then $c+I$ is distinct from $x+I$ and $y+I$ and each of these three is adjacent. To see that $c+I$ is distinct from $x+I$ and $y+I$, assume to the contrary. Without loss of generality, assume $c+I = x+I$. Then $c = x+i$ for some $i \in I$. Then $cx \in I$ implies that $x^2 \in I$, which is a contradiction as $I = \sqrt{I}$ and $x+I$ is nonzero. Since $x+I$, $y+I$, and $c+I$ are distinct and xy, yc, and xc are all in I, it follows that $x+I$, $y+I$, and $c+I$ is a three-cycle in $\Gamma(R/I)$. But this is a contradiction as $x+I \perp y+I$ in $\Gamma(R/I)$. $\square$

Lemma 4.7. *Let R be a commutative ring with nonzero identity and I a nonzero proper radical ideal of R. If $\Gamma(R/I)$ is uniquely complemented, $x \perp y$ and $x \perp z$ in $\Gamma_I(R)$, and $\alpha \in R \setminus I$, then*

$$\alpha y \in I \text{ if and only if } \alpha z \in I.$$

Proof. The statement is symmetric in terms of y and z; so it suffices to show that $\alpha y \in I \Rightarrow \alpha z \in I$.

By Lemma 4.6, $x+I \perp y+I$ and $x+I \perp z+I$ in $\Gamma(R/I)$. Since $\Gamma(R/I)$ is uniquely complemented, it follows that $\operatorname{ann}_{R/I}(y+I) = \operatorname{ann}_{R/I}(z+I)$ (here we also using the fact $\operatorname{ann}_{R/I}(y+I)\setminus\{y+I\} = \operatorname{ann}_{R/I}(y+I)$ and $\operatorname{ann}_{R/I}(x+I)\setminus\{x+I\} = \operatorname{ann}_{R/I}(x+I)$ since $I = \sqrt{I}$).

Assume $\alpha y \in I$. Then $\alpha+I \in \operatorname{ann}_{R/I}(y+I) = \operatorname{ann}_{R/I}(z+I)$. Hence $(\alpha+I)(z+I) = 0+I$, and therefore $\alpha z \in I$ as desired. $\square$

Theorem 4.8. *Let R be a commutative ring with nonzero identity and I a radical ideal of R. Then $\Gamma_I(R)$ is complemented if and only if $\Gamma_I(R)$ is uniquely complemented.*

Proof. If $I = (0)$, then the result follows from [6, Theorem 3.5]. If $\Gamma_I(R)$ is the empty graph, the statement holds vacuously.

Assume then that $I \neq (0)$ and that $\Gamma_I(R)$ is not the empty graph.

The reverse implication is by definition.

Assume $\Gamma_I(R)$ is complemented. Then $\Gamma_I(R)$ has at least two elements, and thus $V(\Gamma(R/I))$ must be non-empty. Since I is a radical ideal, it follows that $|V(\Gamma(R/I))| \neq 1$ (since there are only two rings up to isomorphism with exactly 2 zero-divisors, and they are both non-reduced rings). Thus $|V(\Gamma(R/I))| \geq 2$, and hence $\Gamma(R/I)$ is complemented by Theorem 4.3. Moreover, $\Gamma(R/I)$ is uniquely complemented by Corollary 4.5. The desired result then follows from Lemma 4.7. $\square$

Theorem 4.9. *Let R be a commutative ring with nonzero identity and I a non-prime, proper,* **radical** *ideal of R. Then the following statements are equivalent.*

1. *$\Gamma_I(R)$ is complemented.*

2. *$\Gamma_I(R)$ is uniquely complemented.*

3. *$\Gamma(R/I)$ is complemented.*

4. *$\Gamma(R/I)$ is uniquely complemented.*

5. *$T(R/I)$ is von Neumann regular.*

Moreover, if I is a non-prime, proper ideal of R then $\Gamma_I(R)$ is complemented if and only if $\Gamma_I(R)$ is uniquely complemented.

Proof. Assume that I is a nonzero, proper, non-prime, radical ideal of R. The equivalences follow from Corollary 4.5 and Theorem 4.8. For the "moreover statement," recall that if I is not a radical ideal, then $\Gamma_I(R)$ is complemented if and only if $\Gamma_I(R) \cong K^2$ by Theorem 4.2. However, K^2 is uniquely complemented. Thus, regardless of whether or not I is a radical ideal of R, we have $\Gamma_I(R)$ is uniquely complemented if and only if $\Gamma_I(R)$ is complemented.

If $I = 0$ and radical, then $\Gamma_I(R) = \Gamma(R/I) = \Gamma(R)$ and the result holds by [6, Theorem 3.5]. $\square$

Remarks: The hypothesis that I is proper is necessary in the preceding result in order for the statement $T(R/I)$ to make sense. Thus we can modify the hypothesis to yield the following result. Moreover, if I is prime, then all the graphs in question are empty and R/I is a field, so that all of the conditions hold.

Chapter 5

Isomorphisms of $\Gamma_I(R)$

We begin by considering the following result from [24, Theorem 2.2]. It was stated as follows:

Let I be a finite ideal of R and J be a finite ideal of S such that $I = \sqrt{I}$ and $J = \sqrt{J}$. Then the following hold:

(a) If $|I| = |J|$ and $\Gamma(R/I) \cong \Gamma(S/J)$, then $\Gamma_I(R) \cong \Gamma_J(S)$.

(b) If $\Gamma_I(R) \cong \Gamma_J(S)$, then $\Gamma(R/I) \cong \Gamma(S/J)$.

Remark 5.1. We first note that (b) of the preceding result does not hold. Consider the following example. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $I = 0$. Let $S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $J = \mathbb{Z}_2 \times 0 \times 0$. Then $\Gamma_I(R)$ and $\Gamma_J(S)$ are both 4-cycles, and hence isomorphic. In both cases, I and J are finite radical ideals of their respective rings. However, $\Gamma(R/I)$ is a 4-cycle and $\Gamma(S/J)$ is a line graph on 2 vertices; thus $\Gamma(R/I) \ncong \Gamma(S/J)$. This example also provides a counterexample to [9, Theorem 5.3] as both I and J are also non-maximal ideals.

Figure 5.1: $\Gamma(S/J)$, where $S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $J = \mathbb{Z}_2 \times 0 \times 0$

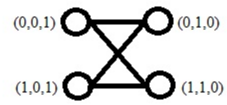


Figure 5.2: $\Gamma_J(S)$, where $S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $J = \mathbb{Z}_2 \times 0 \times 0$

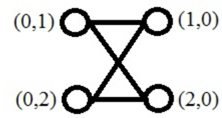


Figure 5.3: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$

The proof of [24, Theorem 2.2(b)] seems to have two shortcomings. The first is in the line: "Now if $a, b \in V(K')$, then $a + J \neq b + J$; otherwise, $a^2 \in J = \sqrt{J}$, and hence $a \in J$, which is a contradiction." It appears that the authors are using the assumption (which does not necessarily hold) that a and b are adjacent in K'. It is a common argument in proofs regarding ideal-based zero-divisor graphs, that if a, b are adjacent vertices of $\Gamma_J(S)$ and $J = \sqrt{J}$, then $a + J \neq b + J$. However, we do not have here that the two vertices in the argument are necessarily adjacent. By considering the example in the above remark, one can see that two different coset representatives in $\Gamma(R/I)$ may map to equivalent coset representatives in $\Gamma(S/J)$.

The second shortcoming of the attempted proof of [24, Theorem 2.2(b)] can be seen in the following example. The example shows that the the restriction of a graph isomorphism between $\Gamma_I(R)$ and $\Gamma_J(S)$ to a set of coset representatives for $V(\Gamma(R/I))$ may not map to a set of distinct coset representatives of $V(\Gamma(S/J))$.

Example 5.2. Let $R = S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$ and $I = J = \mathbb{Z}_2 \times 0 \times 0$, where $\mathbb{F}_4 = \{0, 1, a, b\}$ is the field with 4 elements. For a graph of $\Gamma(R/I)$, see Figure 5.4. For a graph of $\Gamma_I(R)$, see Figure 5.5. We may choose a complete set of coset representatives for $\Gamma(R/I)$ to be $K = \{(0, 0, 1), (0, 1, 0), (0, 0, a), (0, 0, b)\}$. Then consider the graph isomorphism given by Table 5.1

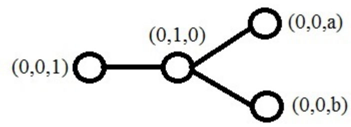


Figure 5.4: $\Gamma(R/I)$, where $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$ and $I = \mathbb{Z}_2 \times 0 \times 0$

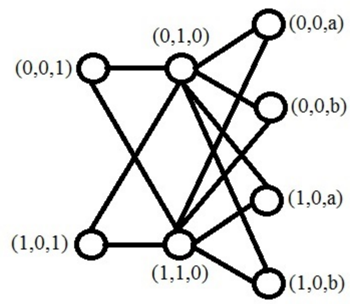


Figure 5.5: $\Gamma_I(R)$, where $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$ and $I = \mathbb{Z}_2 \times 0 \times 0$

We then have that $K' = \phi(K) = \{(1,0,1),(0,1,0),(0,0,a),(1,0,a)\}$. But K' is not a set of **distinct** coset representatives for $\Gamma(S/J)$ as $(0,0,a)+J = (1,0,a)+J$ since $(1,0,0) \in J$.

We believe that the proposition in question will hold if we assume beforehand that $|I| = |J|$. That is, changing [24, Theorem 2.2(b)] to be "If $|I| = |J|$ and $\Gamma_I(R) \cong \Gamma_J(S)$, then $\Gamma(R/I) \cong \Gamma(S/J)$."

Table 5.1: A Graph Isomorphism

$x \in \Gamma_I(R)$	$\phi(x)$
(0,0,1)	(1,0,1)
(0,1,0)	(0,1,0)
(0,0,a)	(0,0,a)
(0,0,b)	(1,0,a)
(1,0,1)	(0,0,1)
(1,1,0)	(1,1,0)
(1,0,a)	(1,0,b)
(1,0,b)	(0,0,b)

Because of the shortcomings in the proof of [24, Theorem 2.2(b)] and reservations of this author regarding the statement "Part (a) is an easy consequence of Theorem 2.1" [24], we seek to first give a direct proof of part (a) of [24, Theorem 2.2]. We should note that the following results should be fairly intuitive from Redmond's three step construction method for $\Gamma_I(R)$. In this proof, notice the subtle use of the radical ideal hypothesis. It would not be hard for one to construct an incorrect proof overlooking the requirement that the ideals must be radical.

Theorem 5.3. *Let R and S be commutative rings with nonzero identity and I and J radical ideals of R and S, respectively. If $\Gamma(R/I) \cong \Gamma(S/J)$ and $|I| = |J|$, then $\Gamma_I(R) \cong \Gamma_J(S)$.*

Proof. Since $\Gamma(R/I) \cong \Gamma(S/J)$, there exists a graph isomorphism $\phi : \Gamma(R/I) \to \Gamma(S/J)$. Let $K = \{a_\lambda\}_{\lambda\in\Lambda}$ be a complete set of distinct coset representatives of $V(\Gamma(R/I))$. Consider $\phi(K) = \{\phi(a_\lambda)\}_{\lambda\in\Lambda}$; this will be a complete set of distinct coset representatives of $V(\Gamma(S/J))$ as $\phi : V(\Gamma(R/I)) \to V(\Gamma(S/J))$ is a bijection. For ease of notation, set $\phi(a_\lambda) = b_\lambda$ and $\phi(K) = \{b_\lambda\}_{\lambda\in\Lambda}$. Since $|I| = |J|$, there exists a bijection $f : I \to J$. Consider the correspondence $\psi : V(\Gamma_I(R)) \to V(\Gamma_J(S))$ given by $\psi(a_\lambda + i) = \phi(a_\lambda) + f(i) = b_\lambda + f(i)$. This correspondence is a well-defined function by [30, Corollary 2.7]; the fact that ψ is onto follows from [30, Corollary 2.7] and that both ϕ and f are onto. Assume that $b_{\lambda_1} + f(i_1) = \phi(a_{\lambda_1} + i_1) = \phi(a_{\lambda_2} + i_2) = b_{\lambda_2} + f(i_2)$. Then $b_{\lambda_1} - b_{\lambda_2} \in J$, and hence $b_{\lambda_1} + J = b_{\lambda_2} + J$. But $\phi(K)$ is a set of distinct coset representatives of $V(\Gamma(S/J))$, and therefore $\lambda_1 = \lambda_2$. It is then evident that $i_1 = i_2$ as f is injective. Therefore ψ is also injective.

We now show that ψ preserves edges. Let r and s be adjacent in $V(\Gamma_I(R))$. Then since I is a radical ideal, $r + I \neq s + I$ and $r + I$ is adjacent to $s + I$ [30, Theorem 2.5]. Since $r + I \neq s + I$, there exist distinct $\lambda_1, \lambda_2 \in \Lambda$ and $i, j \in I$ such that $r = a_{\lambda_1} + i$ and $s = a_{\lambda_1} + j$. Since ϕ is a graph isomorphism, $\phi(r + I) = \phi(a_{\lambda_1} + I) = b_{\lambda_1} + J$ is adjacent to $\phi(s + I) = \phi(a_{\lambda_2} + I) = b_{\lambda_2} + J$. In other words, $b_{\lambda_1} b_{\lambda_2} \in J$. Therefore, $\psi(r) = b_{\lambda_1} + f(i)$ is adjacent to $\psi(s) = b_{\lambda_2} + f(j)$ in $\Gamma_J(S)$. The proof of the reverse direction for edge preservation is similar. Thus, $\psi : \Gamma_I(R) \to \Gamma_J(S)$ is a graph isomorphism. $\square$

Notice the subtle use of the radical ideal hypothesis in the preceding proof. If we leave out the radical ideal hypothesis, we know the result does not hold (e.g., Example 1.8). For the non-radical case, the proof fails when we try to prove that edges are preserved. In particular, if we had an edge created by a connected column, we would not be guaranteed that a corresponding edge exists in the second graph.

We return our focus to finding a converse of this result; we begin by proving a weaker result. Instead of assuming that R/I and S/J are reduced, let us assume that they are Boolean rings. We then quickly get that the desired implication holds.

Lemma 5.4. *Let R and S be finite commutative rings with nonzero identity and I and J ideals of R and S respectively. If $|I| = |J|$ and $\Gamma_I(R) \cong \Gamma_J(S)$, then $|V(\Gamma(R/I))| = |V(\Gamma(S/J))|$.*

Proof. This follows from the fact that $|V(\Gamma_I(R))| = |I||V(\Gamma(R/I))|$ (Theorem 1.7). $\square$

Lemma 5.5. *Let R and S be finite Boolean rings. Then $R \cong S$ if and only if $|Z(R)| = |Z(S)|$.*

Proof. "$\Leftarrow$" It is well known that for finite Boolean rings R and S, we have $R \cong \prod_{i=1}^{m} \mathbb{Z}_2$ and $S \cong \prod_{i=1}^{n} \mathbb{Z}_2$, for $m, n \in \mathbb{Z}^+$. It suffices to show that $m = n$. Assume to the contrary, that is $m \neq n$. Without loss of generality, $m < n$. Then R can be viewed as a subring of S in the natural way, namely $R \cong R' = \prod_{i=1}^{m} \mathbb{Z}_2 \times \prod_{j=m+1}^{n} 0 \subseteq S$. Let $x = (1, 1, 1, ..., 1, 0)$. Then $x \in Z(S) \setminus Z(R')$ since $m < n$. Hence $|Z(R)| = |Z(R')| < |Z(S)|$. But this is a contradiction of the hypothesis, and therefore we must have that $m = n$, and hence $R \cong S$. The "$\Rightarrow$" direction is trivial. $\square$

Alternative Proof

Proof. Since R and S are finite Boolean rings, they are isomorphic to a product of $\mathbb{Z}_2$'s. Thus $|R| = 2^m$ and $|S| = 2^n$. Notice then that $R \cong S$ if and only if $m = n$. Thus $R \cong S$ if and only if $|R| = |S|$. Notice that if $1 \neq x \in R$, then $1 - x \neq 0$ and $x(1 - x) = 0$. Thus $x \in Z(R)$. Since $1 \neq x \in R$ was arbitrary, it follows that $R = Z(R) \cup \{1\}$. Hence $|R| = |S|$ if and only if $|Z(R)| = |Z(S)|$. □

Notice that the argument in the *Alternative Proof* gives the following result.

Lemma 5.6. *Let R be a Boolean ring. Then $R = Z(R) \cup \{1\}$.*

Proposition 5.7. *Let R and S be finite commutative rings with nonzero identities and ideals I and J, respectively. Moreover, assume that R/I and S/J are Boolean and $|I| = |J|$. Then $\Gamma_I(R) \cong \Gamma_J(S)$ implies that $\Gamma(R/I) \cong \Gamma(S/J)$.*

Proof. By the Lemma 5.4, $|V(\Gamma(R/I))| = |V(\Gamma(S/J))|$. Hence $|Z(R/I)| = |V(\Gamma(R/I))| + 1 = |V(\Gamma(S/J))| + 1 = |Z(S/J)|$. Thus $R/I \cong S/J$ by Lemma 5.5, and hence $\Gamma(R/I) \cong \Gamma(S/J)$. □

The preceding arguments gave rise to the following conjecture and its proof. Here we find that if we assume that R and S are finite Boolean rings, then $\Gamma_I(R) \cong \Gamma_J(S) \Rightarrow \Gamma(R/I) \cong \Gamma(S/J)$ and $|I| = |J|$.

Proposition 5.8. *Let R and S be finite Boolean rings with I and J proper non-prime ideals of R and S, respectively. Then $\Gamma_I(R) \cong \Gamma_J(S)$ if and only if $R \cong S$ and $|I| = |J|$.*

Proof. Assume that $\Gamma_I(R) \cong \Gamma_J(S)$. Then $|I||V(\Gamma(R/I))| = |J||V(\Gamma(S/J))|$ (by Theorem 1.7(7)). Since R and S are isomorphic to a direct product of $\mathbb{Z}_2$'s, we have $|R| = 2^m$ and $|S| = 2^n$ for some positive integers m, n.

If either $m = 1$ or $n = 1$, then the corresponding ring(s) will be isomorphic to $\mathbb{Z}_2$, and hence will not contain a proper non-prime ideal; the only proper ideal of $\mathbb{Z}_2$

is $\{0\}$, which is maximal, and hence prime. Thus $m \geq 2$ and $n \geq 2$, and therefore $m-2 \geq 0$ and $n-2 \geq 0$.

Moreover, we know that I will be isomorphic to a product of $\mathbb{Z}_2$'s and $\{0\}$'s, whence $|I| = 2^i$ for some $0 \leq i < m$. Similarly, $|J| = 2^j$ for some $0 \leq j < n$. Note that if $i = m-1$, then R/I will be an integral domain, and hence I a prime ideal. But this is contrary to the hypothesis. Hence $0 \leq i \leq m-2$; similarly, $0 \leq j \leq m-2$. Since the number of zero-divisors of a Boolean ring is one less than the cardinality of the ring, we have $|Z(R/I)| = |R/I| - 1 = 2^{m-i} - 1$ and $|Z(S/J)| = 2^{n-j} - 1$. Hence $|V(\Gamma(R/I))| = 2^{m-i} - 2$ and $|V(\Gamma(S/J))| = 2^{n-j} - 2$ with $0 \leq i \leq m-2$ and $0 \leq j \leq n-2$ (using that $|Z^*(R)| = |Z(R)| - 1$). Therefore, $|I||V(\Gamma(R/I))| = |J||V(\Gamma(S/J))|$ implies that $2^m - 2^{i+1} = 2^n - 2^{j+1}$.

We claim that $m = n$ and $i = j$. It is evident that $m = n \Leftrightarrow i = j$; hence it suffices to show that $m \neq n$ and $i \neq j$ (along with $0 \leq i \leq m-2$ and $0 \leq j \leq n-2$) implies that $2^m - 2^{i+1} \neq 2^n - 2^{j+1}$.

Assume that $m \neq n$, $i \neq j$, $0 \leq i \leq m - 2\text{m}$ and $0 \leq j \leq n-2$. Without loss of generality, we may assume that $m < n$. First notice that for all $0 \leq j \leq n-2$, we have that $2^n - 2^{j+1} \geq 2^n - 2^{n-1}$ (since $f(x) = 2^n - 2^x$ is decreasing). Now $2^n - 2^{n-1} = 2^n(1-2^{-1}) = 2^n(2^{-1}) = 2^{n-1}$; thus $2^n - 2^{j+1} \geq 2^{n-1}$ for all $0 \leq j \leq n-2$. Since $m < n$, we have $2^m < 2^n$, and hence $2^m \leq 2^{n-1}$. Thus $m < n$ implies that $2^m - 2 < 2^m \leq 2^{n-1}$. Hence

$$2^n - 2^{j+1} \geq 2^{n-1} > 2^m - 2 \text{ for all } 0 \leq j \leq m-2.$$

But $2^m - 2 \geq 2^m - 2^{i+1}$ for all $0 \leq i \leq n-2$. Thus:

$$2^n - 2^{j+1} > 2^m - 2 \geq 2^m - 2^{i+1} \text{ for } 0 \leq i \leq m-2 \text{ and } 0 \leq j \leq n-2.$$

Hence $2^n - 2^{j+1} \neq 2^m - 2^{i+1}$ for all $0 \leq i \leq m-2$ and $0 \leq j \leq n-2$, as desired.

Thus we must have $m = n$ and $i = j$, whence $|R| = |S|$ and $|I| = |J|$. Since R and S are finite Boolean rings, $|R| = |S| \Rightarrow R \cong S$. Therefore, $R \cong S$ and $|I| = |J|$, as desired.

For the converse, note that Boolean rings are reduced, and thus I and J are radical ideals. The result then follows by Theorem 5.3. $\square$

Remark 5.9. If we simply assume that at least one of the ideals in the preceding proposition is non-prime, it follows that the ideal-based zero-divisor graph relative to the non-prime ideal will be non-empty. Thus in the forward implication, the other graph is non-empty; therefore we also have the remaining ideal is non-prime. In the reverse implication, $Spec(R) = Max(R)$ and $Spec(S) = Max(S)$. Thus the prime ideals are those that are maximal. By viewing R and S as a product of $\mathbb{Z}_2$'s, it is evident that an ideal I is prime if and only if maximal, if and only if $|I| = |R|/2$. Thus the conditions $|I| = |J|$ and $R \cong S$ ensure that if at least one of the two ideals is prime, then so is the other.

Therefore, although the theorem is true if we only assume that one of the ideals is non-prime, we are not losing generality by assuming both are non-prime.

If both of the ideals are prime, then the theorem does not hold. Consider $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $I = 0 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $S = \mathbb{Z}_2 \times \mathbb{Z}_2$, $J = 0 \times \mathbb{Z}_2$. Then $\Gamma_I(R)$ and $\Gamma_J(S)$ are empty, hence isomorphic; however, $R \ncong S$ and $|I| \neq |J|$.

Remark 5.10. The converse of Proposition 5.8 does not hold for infinite Boolean rings. Consider $R = S = \prod_{i=1}^{\infty} \mathbb{Z}_2$, where $I = 0 \times 0 \times \prod_{i=3}^{\infty} \mathbb{Z}_2$ and $J = 0 \times 0 \times 0 \times \prod_{i=4}^{\infty} \mathbb{Z}_2$. Then $S/J \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, whence $gr(\Gamma(S/J)) = 3 = gr(\Gamma_J(S))$. However $\Gamma_I(R) \cong K^{\aleph_0,\aleph_0}$; to see this, consider the vertex sets $V = \{(a_i)_{i\in\mathbb{N}} \in R \mid a_1 = 0, a_2 = 1\}$ and $W = \{(a_i)_{i\in\mathbb{N}} \in R \mid a_1 = 1, a_2 = 0\}$. Thus we have that $gr(\Gamma_I(R)) = 4$ and $gr(\Gamma_J(S)) = 3$. Therefore $\Gamma_I(R) \ncong \Gamma_J(S)$; however $R \cong S$ and $|I| = |J|$.

Recall that one of our goals is to determine when the following implication holds: $\Gamma_I(R) \cong \Gamma_J(S) \Rightarrow \Gamma(R/I) \cong \Gamma(S/J)$.

We have seen that even in the finite radical case that the above implication does not hold. We then assumed that we need the ideals to have the same cardinality. That is, we hoped to prove the following implication (at least in the reduced case):

$\Gamma_I(R) \cong \Gamma_J(S)$ and $|I| = |J| \Rightarrow \Gamma(R/I) \cong \Gamma(S/J)$.

However, the following example dashes the hopes of this holding in the case the ideals are infinite.

Example 5.11. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$, $S = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}$, $I = 0 \times 0 \times \mathbb{Z}$, and $J = 0 \times 0 \times \mathbb{Z}$. Then $|I| = |J|$. Notice that $\Gamma(R/I) \cong K^2$ and $\Gamma(S/J)$ is a 4-cycle; hence $\Gamma(R/I) \ncong \Gamma(S/J)$. However, $\Gamma_I(R) \cong \Gamma_J(S)$ since both graphs are a $K^{\aleph_0,\aleph_0}$.

To see the that $\Gamma_J(S) = K^{\aleph_0,\aleph_0}$, consider the sets $V = \{(0,1,k) \mid k \in \mathbb{Z}\} \cup \{(0,2,k) \mid k \in \mathbb{Z}\}$ and $W = \{(1,0,k) \mid k \in \mathbb{Z}\} \cup \{(2,0,k) \mid k \in \mathbb{Z}\}$. Notice that no vertex of V is adjacent to any other vertex of V. The same is true of W. However, every vertex of V is adjacent to every vertex of W (and vice-versa). Thus $\Gamma_J(S) = K^{\aleph_0,\aleph_0}$.

Similarly, $\Gamma_I(R) = K^{\aleph_0,\aleph_0}$.

In this example, we have commutative rings R and S with radical ideals I and J, respectively, such that $\Gamma_I(R) \cong \Gamma_J(S)$ and $|I| = |J|$, but $\Gamma(R/I) \ncong \Gamma(S/J)$.

Chapter 6

When R or R/I is Boolean

When looking at the research on $\Gamma(R)$ when R is Boolean, one of the graph-theoretic concepts considered is that of an end. An *end* in a graph G is a vertex v that is adjacent to exactly one other vertex in G. We begin by considering for non-zero ideals I, when $\Gamma_I(R)$ has ends.

Lemma 6.1. *Let R be a commutative ring with nonzero identity and I a nonzero ideal of R. If $|V(\Gamma(R/I))| \geq 2$, then $\Gamma_I(R)$ has no ends.*

Proof. Let $x \in V(\Gamma_I(R))$. Since $|V(\Gamma_I(R))| = |I||V(\Gamma(R/I))| \geq 2$ and $\Gamma_I(R)$ is connected (Theorem 1.7(3)), there exists $y \in V(\Gamma_I(R))$ adjacent to x. Then either $y = x + i$ for some $0 \neq i \in I$ or $y \neq x + i$ for all $i \in I$.

If $y = x + i$ for some $0 \neq i \in I$, then $x + I = y + I$ in R/I. Since $|V(\Gamma(R/I))| \geq 2$ and $\Gamma(R/I)$ is connected (Theorem 1.5(1)), there exists a $z+I$ adjacent to $x+I = y+I$ in $\Gamma(R/I)$. It then follows that y, z are adjacent to x in $\Gamma_I(R)$.

If $y \neq x + i$ for all $i \in I$, then choose $0 \neq i \in I$ (here we use that I is non-zero). Then $y, y + i$ are both adjacent to x in $\Gamma_I(R)$.

In both cases, we have shown that x was not an end in $\Gamma_I(R)$. Since $x \in V(\Gamma_I(R))$ was arbitrary, the desired result follows. □

Lemma 6.2. *Let R be a commutative ring with nonzero identity and I an ideal of R. If $|V(\Gamma(R/I))| = 1$, then $\Gamma_I(R)$ has an end if and only if $|I| = 2$.*

Proof. Recall that $|V(\Gamma(R/I))| = 1$ implies that $\Gamma_I(R) = K^{|I|}$ (this follows from [9, Theorem 4.7]). The desired result then follows since K^n has ends if and only if $n = 2$. $\square$

Proposition 6.3. *Let R be a commutative ring with nonzero identity and I a nonzero ideal of R. Then $\Gamma_I(R)$ has ends if and only if $|I| = 2$ and $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$.*

Proof. If I is a prime ideal, then $V(\Gamma(R/I)) = \emptyset$, and hence $\Gamma_I(R) = \emptyset$. Thus $\Gamma_I(R)$ has no ends.

Assume then that I is not a prime ideal of R. Thus $|V(\Gamma(R/I))| \geq 1$. If $|V(\Gamma(R/I))| \geq 2$, it follows from Lemma 6.1 that $\Gamma_I(R)$ has no ends. Hence $|V(\Gamma(R/I))| = 1$. So by Lemma 6.2, it follows that $\Gamma_I(R)$ has ends if and only if $|I| = 2$ and $|V(\Gamma(R/I))| = 1$. Recall that $|V(\Gamma(R/I))| = 1$ if and only if $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by Proposition 2.2. Thus the desired result holds. $\square$

In the preceding proposition, notice that a non-trivial $\Gamma_I(R)$ has ends if and only if it is isomorphic to K^2. By Proposition 2.4, we have the following result.

Proposition 6.4. *Let R be a commutative ring with nonzero identity and I a nonzero ideal of R.Then $\Gamma_I(R)$ has ends if and only if R is isomorphic to one of the seven rings with the appropriately chosen ideal I from Table 2.1.*

Notice that for nonzero ideals I, the graph $\Gamma_I(R)$ rarely has ends (this follows because of the many connections we get in $\Gamma_I(R)$ when I is non-zero).

Recall from the section on complemented and uniquely complemented, that we had the following lemma. (Lemma 4.6) Let R be a commutative ring with nonzero identity and I a nonzero ideal of R. If I is a radical ideal, the $x \perp y$ in $\Gamma_I(R)$ if and only if $x + I \perp y + I$ in $\Gamma(R/I)$.

Proposition 6.5. *Let R be a commutative ring with nonzero identity and I an ideal of R such that R/I is a Boolean ring. If $R/I \ncong \mathbb{Z}_2$, then every element of $V(\Gamma_I(R))$ has $|I|$ complements.*

Proof. By Lemma 2.3 from [20], each element of $V(\Gamma(R/I))$ has a unique complement in $\Gamma(R/I)$. Since R/I is a Boolean ring (hence reduced), I is a radical ideal of R. Hence the result follows from Lemma 4.6. $\square$

Corollary 6.6. *Let R be a Boolean ring and I an ideal of R with $\Gamma_I(R) \neq \emptyset$ (i.e., I is not a prime ideal). If each vertex of $\Gamma_I(R)$ has a unique complement, then I is the zero ideal and $R/I \ncong \mathbb{Z}_2$.*

Proof. If $R/I \cong \mathbb{Z}_2$, then $\Gamma_I(R) = \emptyset$. Therefore $R/I \ncong \mathbb{Z}_2$. Thus by Proposition 6.5, each vertex of $\Gamma_I(R)$ has $|I|$ complements. Whence each element has a unique complement if and only if $|I| = 1$, if and only if $I = 0$. $\square$

Let v be a vertex of a graph G, and let $N(v) = \{a \in V(G) \mid a \text{ is a adjacent to } v\}$. Recall from the introduction, that if $A \subseteq V(G)$, then $< A >$ is the subgraph of G generated (or induced) by A.

Definition 6.7. Let R be a commutative ring and I an ideal of R. We say that $v \in \Gamma_I(R)$ is a **core** of $\Gamma_I(R)$ if $< \{v\} \cup N(v) > = K^{1,|I|}$.

Note that if v is a core of $\Gamma_I(R)$, then it is the center of a star subgraph. The idea behind a core is to be a generalization of the concept of an end for the ideal-based zero-divisor graph. Recall that when I is nonzero, $\Gamma_I(R)$ rarely has ends. Thus we created the concept of a core in hopes of generalizing the idea of an end when $\Gamma_I(R)$ is nontrivial. Notice that when $I = 0$, the definition of a vertex v being a core yields that $< \{v\} \cup N(v) >= K^{1,1}$. Therefore $|N(v)| = 1$, and hence v is an end. The following lemmas show that when I is a finite radical ideal, the concept of core is an appropriate generalization of end.

Lemma 6.8. *Let R be a commutative ring with nonzero identity and I a radical ideal of R. Assume that $v + I$ is adjacent to exactly one vertex in $\Gamma(R/I)$, say $r + I$. Then a vertex s is a adjacent to v in $\Gamma_I(R)$ if and only if $s = r + i$ for some $i \in I$.*

Proof. It is evident that the vertices of the form $r+i$ are adjacent to v. Assume that some other vertex w is adjacent to v, where w is not of the form $r+i$ for some $i \in I$. Then $r+I \neq w+I$. But then since $v+I$ is adjacent to only $r+I$ in $\Gamma(R/I)$, we must have that $w+I = v+I$. Therefore $w-v \in I$ and $wv \in I$; but then $w^2, v^2 \in I$, which is a contradiction as I is a radical ideal of R. □

Lemma 6.9. *Let R be a commutative ring R with nonzero idenity and I a finite radical ideal of R. Then v is a core of $\Gamma_I(R)$ if and only if $v+I$ is an end in $\Gamma(R/I)$.*

Proof. We prove the forward direction by contrapositive. Assume that $v+I$ is not an end in $\Gamma(R/I)$. Then there exists distinct vertices $r+I, s+I$ both adjacent to $v+I$ in $\Gamma(R/I)$. Then for each $i \in I$, $r+i, s+i$ will be distinct vertices both adjacent to v in $\Gamma_I(R)$. Since $r+I \neq s+I$, no $r+i = s+j$, where $i, j \in I$. Also, no $r+i, s+i$ equals v, as otherwise $r+I, s+I, v+I$ would not be distinct vertices of $\Gamma(R/I)$. Hence v will be adjacent to at least $2|I|$ distinct elements in $\Gamma_I(R)$. Therefore $\{v\} \cup N(v)$ will have at least $2|I|+1$ elements. Thus $< \{v\} \cup N(v) > \neq K^{1,|I|}$ as $|V(K^{1,|I|})| = |I|+1 < 2|I|+1$ (this is true for all $|I| \geq 1$ since $|I|$ is finite).

For the reverse implication, assume that $v+I$ is an end in $\Gamma(R/I)$. Then $v+I$ is adjacent to exactly one vertex in $\Gamma(R/I)$, say $r+I$. Using that $I = \sqrt{I}$, by Lemma 6.8, a vertex s is adjacent to v in $\Gamma_I(R)$ if and only if $s = r+i$ for some $s \in I$. Thus $N(v) = \{r+i \mid i \in I\}$, whence $|N(v)| = |I|$. Since $I = \sqrt{I}$, we have that $r+i$ is not adjacent to $r+j$ for $i, j \in I$. Thus $< \{v\} \cup N(v) >= K^{1,|N(v)|} = K^{1,|I|}$; so v is a core of $\Gamma_I(R)$. □

Example 6.10. Notice that the "⇒" implication in Lemma 6.9 did not require the radical ideal hypothesis. Thus a core always comes from an end when $|I| < \infty$. The reverse direction utilizes the radical ideal hypothesis. The following example shows that the radical ideal hypothesis is necessary.

Let $R = \mathbb{Z}_3[X]/(X^2) \times \mathbb{Z}_2$ and $I = 0 \times \mathbb{Z}_2$. Then $\Gamma_I(R) = K^4$. Notice that the ends of $\Gamma(R/I)$ do not become cores in $\Gamma_I(R)$ (in fact, there are no cores). Notice that in this example, $|I| = 2$. In fact, if $I \neq \sqrt{I}$, then any end of $\Gamma(R/I)$ whose representative is in $\sqrt{I}$ will be a core if and only if $|I| = 1$. This follows since the subgraph in question will always have a 3-cycle if $|I| \geq 2$.

Remark 6.11. In Lemma 6.9, the finite hypothesis is necessary. Consider $R = \mathbb{Z}_6 \times \mathbb{Z}$ and $I = 0 \times \mathbb{Z}$. Then the vertex $(3, 0)$ will be a core of $\Gamma_I(R)$. Here $N((3, 0)) = \{(2, k) \mid k \in \mathbb{Z}\} \cup \{(4, k) \mid k \in \mathbb{Z}\}$, where no vertex in $N((3, 0))$ is adjacent to any other vertex in that set. Thus $< \{(3, 0)\} \cup N((3, 0)) >= K^{1,|N((3,0))|}$, where $|N((0, 3))| = |I| = |\mathbb{Z}|$; thus $(3, 0)$ is a core of $\Gamma_I(R)$, but $(3, 0) + I$ is not an end of $\Gamma(R/I)$. Perhaps more interestingly, every vertex of $\Gamma_I(R)$ is a core. This example not only shows that that the finite hypothesis is required in Lemma 6.9, but also that the concept of core (at least with its current definition) is most likely not very useful in the infinite setting.

We can construct a Boolean algebra on the set of idempotents of a ring R in the following way ([20]). The relation "$\leq$" defined on the set of idempotents of R given by $a \leq b$ if and only $ab = a$ partially orders $B(R) = Idem(R)$. This relation makes $B(R)$ into a Boolean algebra with inf defined as multiplication in R, 1 as the largest element, 0 as the smallest element, and complementation given by $a' = 1 - a$. Also $a \vee b = (a' \wedge b')' = a + b - ab$. An *atom* in a Boolean algebra B is an element $0 \neq a \in B$ such that $0 \neq b \in B$ with $b \leq a$ implies $b = a$. Thus given a ring R, we can define a Boolean algebra on the set of idempotents; this will be called the Boolean algebra of idempotents and will be denoted $B(R)$. For more on the definition of Boolean algebra and properties of $B(R)$ see [21]. Recall that in $B(R)$, $0 \neq a \in B(R)$ is an atom if and only if $b = a$ whenever $0 \neq b \in B(R)$ with $ba = b$.

Proposition 6.12. *Let R be a commutative ring with nonzero identity and I a finite ideal of R such that R/I be a Boolean ring with $R/I \ncong \mathbb{Z}_2$. Then $v + I$ is an atom of $B(R/I)$ if and only if v is adjacent to a core in $\Gamma_I(R)$.*

Proof. Since R/I is Boolean, R/I is reduced, and hence I is a radical ideal of R.

Let $v + I$ be an atom of $B(R/I)$. Then by [20, Theorem 2.2], $v + I$ is an atom of $B(R/I)$ if and only if $v + I$ is adjacent to an end, say $w + I$, in $\Gamma(R/I)$. By Lemma

6.9, $w+I$ is an end in $\Gamma(R/I)$ if and only w is a core of $\Gamma_I(R)$. Thus $v+I$ is an atom in $B(R/I)$ if and only if v is adjacent to a core in $\Gamma_I(R)$. □

Lemma 6.13. *Let R be a commutative ring with nonzero identity and I a finite ideal of R such that $R/I \ncong \mathbb{Z}_2$ is a Boolean ring. Then v is adjacent to a core in $\Gamma_I(R)$ if and only if v is adjacent to exactly $|I|$ cores of $\Gamma_I(R)$.*

Proof. The reverse implication is clear. To prove the "⇒" implication, let v be adjacent to a core w of $\Gamma_I(R)$. Then v is adjacent to $w+i$ for all $i \in I$ and $N(w) = N(w+i)$ for all $i \in I$. Thus $< \{w\} \cup N(w) >=< \{w+i\} \cup N(w+i) >$ for all $i \in I$. Therefore $w+i$ is a core of $\Gamma_I(R)$ adjacent to v for all $i \in I$. Assume to the contrary that there exists a $x \in V(\Gamma_I(R))$ such that x is a core of $\Gamma_I(R)$ adjacent to v, but $x \neq w+i$ for any $i \in I$. Then $x+I \neq w+I$. Furthermore, since I is a radical ideal of R (this is since R/I is Boolean and hence reduced), $v+I, x+I$ are distinct elements of R/I.

Since x, w are cores of $\Gamma_I(R)$, the vertices $x+I$ and $w+I$ are ends of $\Gamma(R/I)$ by Lemma 6.9. Since v is adjacent to a core, $v+I$ is an atom of $B(R/I)$ by Proposition 6.12. Thus the atom $v+I$ is adjacent to two distinct ends of $\Gamma(R/I)$ (namely $w+I, x+I$). But this contradicts [20, Lemma 1.2]. Thus the cores adjacent to v are precisely those of the form $w+i$, where $i \in I$. Hence v is adjacent to exactly $|I|$ cores. □

Theorem 6.14. *Let R be a commutative ring with nonzero identity and I a finite ideal of R such that R/I be a Boolean ring with $R/I \ncong \mathbb{Z}_2$. Then the following statements are equivalent.*

1. *$v+I$ is an atom of $B(R/I)$.*
2. *$v+I$ is adjacent to an end in $\Gamma(R/I)$.*
3. *v is adjacent to a core in $\Gamma_I(R)$.*
4. *v is adjacent to exactly $|I|$ cores in $\Gamma_I(R)$.*

Proof. The theorem follows from [20, Theorem 2.2], Proposition 6.12, and Lemma 6.13. □

Chapter 7

The Number of Vertices and Edges of $\Gamma(R)$ when R is Reduced

We begin by proving a lemma about prime numbers.

Lemma 7.1. *Let p, q, x, y be prime numbers and $r, s, a, b \in \mathbb{N}$. Assume that the following conditions hold.*

1. $p^r q^s = x^a y^b$.
2. $p^r + q^s = x^a + y^b$.

Then there exists a permutation π such that $\pi((x, y)) = (p, q)$ and $\pi((a, b)) = (r, s)$ (as ordered pairs).

Proof. We proceed in two cases.

Case 1: First assume that $p = q$. Then $p^r q^s = p^{r+s} = x^a y^b$. Thus $x = y = p$; so $p^{r+s} = p^{a+b}$. Therefore $r + s = a + b$.

Assume that a, b, r, s are all distinct. Without loss of generality, assume that a is the smallest among a, b, r, s. Then dividing the equation $p^a + p^b = p^r + p^s$ by p^a gives $1 + p^{b-a} = p^{r-a} + p^{s-a}$, where all the exponents are natural numbers. Then p divides the right-hand side; so $p \mid 1 + p^{b-a}$. But this is impossible as then $p \mid 1$.

Hence the values a, b, r, s are not distinct. If $a = r$, $a = s$, $b = r$, or $b = s$, the result follows from the equation $r + s = a + b$. If $a = b$ and $r = s$, then the desired result follows from $a + b = r + s$. Without loss of generality, the only remaining case is when $r = s$ and $a \neq b$. Then $2p^r = p^a + p^b$. If $r < a, r < b$, then $2 = p^{a-r} + p^{b-r}$, which implies that $a - r = 0 = b - r$. Thus $a = r, b = r$, which is a contradiction to the current hypothesis. Otherwise, either a is the smallest or b is the smallest (among the values in question). Both cases are analogous. If a is smaller than r, s, and b, it then follows that $2p^{r-a} = 1 + p^{b-a}$. This implies that $p \mid 1$, hence a contradiction as p is prime. Thus the desired result follows in the case that $p = q$.

Case 2: Now assume that $p \neq q$. The desired result follows from properties of Unique Factorization in $\mathbb{Z}$. (Notice that Property (2) was not required in Case (2).) $\square$

Let $R \cong K_1 \times K_2$ for finite fields K_1, K_2. Then from [5, Theorem 3.11], we have that

1. $|E(\Gamma(R))| = \frac{1}{2}\left[(2|K_1| - 1)(2|K_2| - 1) - 2|K_1||K_2| + 1\right]$, and
2. $|V(\Gamma(R))| = (|K_1| - 1) + (|K_2| - 1)$.

The right-hand side of the first equation simplifies to $|K_1||K_2| - (|K_1| + |K_2|) + 1$.

Proposition 7.2. *Let R and S be finite reduced commutative rings with nonzero identity whose zero divisor graphs are complete bipartite. Then the following statements are equivalent.*

1. $|V(\Gamma(R))| = |V(\Gamma(S))|$ *and* $|E(\Gamma(R))| = |E(\Gamma(S))|$.
2. $\Gamma(R) \cong \Gamma(S)$.
3. $R \cong S$.

Proof. Notice that (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are clear. Thus it suffices to show (1) $\Rightarrow$ (3).

Assume (1) holds, that is, $|V(\Gamma(R))| = |V(\Gamma(S))|$ and $|E(\Gamma(R))| = |E(\Gamma(S))|$. Since the graphs are complete bipartite, we have by [18, Theorem 1.14] that $R \cong K_1 \times K_2$ and $S \cong F_1 \times F_2$, where K_i, F_i are finite fields.

Without loss of generality, let $R = K_1 \times K_2$ and $S = F_1 \times F_2$, where K_i, F_i are finite fields. Then we have the following two equalities.

1. $|K_1||K_2| - (|K_1| + |K_2|) + 1 = |F_1||F_2| - (|F_1| + |F_2|) + 1$.

2. $(|K_1| - 1) + (|K_2| - 1) = (|F_1| - 1) + (|F_2| - 1)$.

Equality 2 above implies that $|K_1| + |K_2| = |F_1| + |F_2|$. Combining the previous equality with Equality 1 implies that $|K_1||K_2| = |F_1||F_2|$. Thus the following two conditions hold.

1. $|K_1||K_2| = |F_1||F_2|$.

2. $|K_1| + |K_2| = |F_1| + |F_2|$.

Now, using that the cardinality of a finite field is a power of prime, it follows from Lemma 7.1 that either $|F_1| = |K_2|$ and $|F_2| = |K_2|$, or $|F_1| = |K_2|$ and $|F_2| = |K_1|$. Moreover, since finite fields are isomorphic if and only if they have the same cardinality, it follows that $R \cong K_1 \times K_2 \cong F_1 \times F_2 \cong S$. Thus (1) $\Rightarrow$ (3). $\square$

We note that in Propostion 7.2, (2) if and only if (3) is a special case of [5, Theorem 4.1].

Thus we have that the finite complete bipartite graphs arising as zero-divisor graphs (from reduced rings) can be classified by their number of edges and number of vertices. This holds for finite complete bipartite graphs in general. The following argument shows this:

Consider the set of complete bipartite graphs $K^{m,n}$ with a fixed number of vertices v. Then $v = m + n$. The number of edges for such a graph will be $e = mn$. Using that $v = m + n$, we have $m = v - n$, and therefore $e = (v - n)n = vn - n^2$. Consider this as a function of n. Then notice that this is a parabola increasing on the interval

$(-\infty, v/2)$ and decreasing on the interval $(v/2, \infty)$ (if we extend the domain to all real numbers).

Let us assume we have two graphs K^{n_1,m_1} and K^{n_2,m_2}, both on v vertices. Then from the fact that $|V(K^{m,n})| = m + n$, $n_1 = n_2$ implies that $m_1 = m_2$, and therefore the two graphs are isomorphic. Assume $n_1 \neq n_2$ Using that $e(n)$ is a parabola, we have that $e(n_1) = e(n_2)$ if and only if $|n_1 - v/2| = |n_2 - v/2|$. Without losss of generality, assume that $n_1 < v/2$. Then $|n_1 - v/2| = v/2 - n_1$ and $|n_2 - v/2| = n_2 - v/2$. Thus $n_1 + n_2 = v = n_1 + m_1 \Rightarrow n_2 = m_1$. It then follows that $m_2 = n_1$. Hence $K^{n_1,m_1} \cong K^{m_2,n_2} \cong K^{n_2,m_2}$.

Hence, if two complete bipartite graphs have the same number of edges and the same number of vertices, they must be isomorphic. This gives another proof of the first proposition of this section.

The above argument does not show (in the complete bipartite case) that solely the number of vertices of a zero-divisor graph for a reduced ring determines the graph or ring (neither does the number of edges). Consider the following examples.

Example 7.3. This example shows that if the number of vertices of two zero-divisor graphs for reduced rings are the same, they need not have the same number of edges.

Let $R = \mathbb{Z}_2 \times \mathbb{F}_4$ and $S = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $|V(\Gamma(R))| = 4 = |V(\Gamma(S))|$, but $|E(\Gamma(R))| = 3$ and $|E(\Gamma(S))| = 4$.

Example 7.4. This example shows that if the number of edges of two zero-divisor graphs for reduced rings are the same, they need not have the same number of vertices.

Let $R = \mathbb{Z}_2 \times \mathbb{Z}_7$ and $S = \mathbb{Z}_3 \times \mathbb{F}_4$. Then $|E(\Gamma(R))| = 6 = |E(\Gamma(S))|$, but $|V(\Gamma(R))| = 7$ and $|V(\Gamma(S))| = 5$.

The preceding examples show that the number of vertices (or the number of edges) alone does not determine when two reduced rings have isomorphic zero-divisor graphs.

We conjectured that if two reduced rings have zero-divisor graphs with the same number of edges and the same number of vertices, then their graphs would have to be isomorphic. We have seen that this holds if both rings are isomorphic to a product of two fields. In order to check the conjecture for the products of more than two fields, we wrote a Matlab program to compute the number of vertices and edges for a given product of fields. We use the equations from [5] for the number of vertices and edges based on the cardinality of the fields. Using that the cardinality of a finite field is a power of a prime (and that for every power of a prime there is a corresponding field with that cardinality), we check the equations for various powers of primes up to a fixed bound.

The Matlab program has two parts. One is called 'powerofprimes.' This function takes a postive integer n and returns an array in ascending order, consisting of all powers of primes from 2 up to the postive integer n. The Matlab code is as follows:

```
function list = powerofprimes(n)
plist=primes(n); %Matlab function giving primes from 2 to n
list=[];
for i=1:length(plist)
    a=plist(i);
    p=plist(i);
    while a<= n
        list=[list,a];
        a=a*p;
    end
end
list=sort(list);
end
```

The main part of the program is called 'productfields.' It takes 3 inputs:'number', 'upperbound', and 'vBound'. The variable 'number' is the number of fields in the product, 'upperbound' is the highest cardinality of any field we consider (this inequality is not strict), and 'vBound' is the largest number of vertices we wish to consider (this inequality is strict). For example, 'productfields(2,8,1000)' will check all products of two fields where the field cardinalities range over the values 2,3,4,5,7, and 8, and only rings whose zero-divisor graphs have less than 1000 edges will make the list. The output is a matrix, where each row represents a different possible product of fields. The first 'number' of columns will list the powers of primes used in the calculation. The second to last column will contain the number of vertices, while the last column gives the number of edges. The Matlab code is as follows:

```
function Results=productfields(number,upperbound,vBound)
primes=powerofprimes(upperbound);
temp=[];
for i=1:number %Allow repeated fields
    temp=[temp,primes];
end
primes=temp;
field=combntns(primes,number); %Get All Possible Choices (Order Matters (i))
tempsize=size(field);
for i=1: tempsize(1) %Sort Row Entries
    field(i,:)=sort(field(i,:));
end
field=union(field,field,'rows'); % Remove duplicate rows (Fixing (i))
RC=size(field); %Number of Rows and Columns of Field Matrix
Results=[];

for i=1:RC(1)
    A=1; B=1; C=1; D=1;
    for j=1:RC(2)
        A=A*(2*field(i,j)-1);
        B=B*field(i,j);
```

```
            D=D*(field(i,j)-1);
        end
        V=B-D-1;
        B=2*B;
        E=(1/2)*(A-B+1);
        if V< vBound
                Results=[Results; field(i,:), V, E];
        end
    end
end
```

We begin by checking if a product of 2 finite fields could yield a zero-divisor graph with the same number of edges and vertices as a product of 3 finite fields. The answer was yes. We found such an example after 56,543 data points within the tables given by 'productfields(2,1000,2147483647)' and 'productfields(3,1000,2147483647)'. The number of edges and vertices in question being $V = 64$ and $E = 240$.

Example 7.5. Let $R = \mathbb{Z}_5 \times \mathbb{Z}_{61}$ and $S = \mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{F}_8$. Then using the equations for the number of edges and vertices, we see that both of these two rings have zero-divisor graphs with the same number of vertices and same number of edges (64 and 240, respectively). It is evident that the two graphs are not isomorphic (one is complete bipartite and the other is not). Another way of seeing that $\Gamma(R) \ncong \Gamma(S)$ is that $gr(\Gamma(R)) = 4$, while $gr(\Gamma(S)) = 3$.

The next obvious question would be: is the number of vertices being 64 minimal? That is, does there exist a counterexample to the conjecture where the number of vertices is less than 64. The answer is no. We notice that for a product of 7 fields, the smallest number of possible vertices will correspond to the graph of $(\mathbb{Z}_2)^7$, which will have $2^7 - 2 = 126 > 64$ vertices. So it suffices to check up to products of 6 fields, where the number of vertices is less than 65.

Thus we need only check powers of primes less than 65 (because if any factor exceeds 64, then so will the number of vertices). Thus it suffices to check all possible products of fields (2 up to 6 factors) with cardinalities among the set $A = \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37, 41, 43, 47, 49, 53, 59, 61, 64\}$.

Hence in Matlab, it will suffice to compare the vertices and edges entries of the following tables:

1. productfields(2,65,65),
2. productfields(3,65,65),
3. productfields(4,65,65),
4. productfields(5,65,65), and
5. productfields(6,65,65).

Because of the number of iterations required to get all possible field combinations, the execution of the last three function calls fail because of insufficient memory. In order to avoid this problem, we notice that for 4 and 5 factors, we can reduce the set A to just considering powers of primes up to 11.

If we have 4 factors, then the smallest number of vertices where 11 appears in a factor will be of the form $(\mathbb{Z}_2)^3 \times \mathbb{Z}_{11}$. But this will have $2^3 \cdot 11 - 10 - 1 = 77$ vertices. (Notice that if any field has more than 11 elements, then the total number of vertices must exceed 77.)

Also, if we have 5 factors, then the smallest number of vertices where 11 appears in a factor will be of the form $(\mathbb{Z}_2)^4 \times \mathbb{Z}_{11}$. But this will have $2^4 \cdot 11 - 10 - 1 = 165$ vertices. (Notice that if any field has more than 11 elements, then the total number of vertices must exceed 165.)

Similarly, for 6 factors, if a field F_1 in the factorization has more than 3 elements, then $|V(\prod_{i=1}^{6} F_i)| \geq |V(\mathbb{Z}_3 \times (\mathbb{Z}_2)^5)| = 3 \cdot 2^5 - 2 - 1 = 93 > 64$. So in the case of 6 fields, we can drop the list A down to $\{2\}$.

Thus it suffices to check productfields(4,11,65), productfields(5,11,65), and productfields(6,3,65). Hence we need only compare the number of vertices and edges given from the following tables in Matlab:

1. productfields(2,65,65),

2. productfields(3,65,65),

3. productfields(4,11,65),

4. productfields(5,11,65),

5. productfields(6,3,65) (notice there is only one to check here: $(\mathbb{Z}_2)^6$).

We compared the preceding tables using the Matlab command "intersect(...)." In each comparison, there was no overlap. Thus it follows that 64 is a minimal counterexample.

To summarize: Let $\Gamma(R)$ and $\Gamma(S)$ be the zero-divisor graphs of finite reduced rings R and S, where $|V(\Gamma(R))|, |V(\Gamma(S))| < 64$. Then $\Gamma(R) \cong \Gamma(S)$ if and only if $|V(\Gamma(R))| = |V(\Gamma(S))|$ and $|E(\Gamma(R))| = |E(\Gamma(S))|$.

We would provide the output for all 5 commands above, but each table consists of 256, 52, 14, 3, and 1 row(s), respectively.

We note that a program from [26] influenced how this author handled enumerating the different possible products of fields.

Chapter 8

$\Gamma_I(R)$ of Small Finite Commutative Rings

Inspired by work in [31] and [32], here we extend some of our work from Chapters 1 and 2 to classify the graphs of all nontrivial $\Gamma_I(R)$ on fewer than 7 vertices. We proceed by the means of the equation: $|V(\Gamma_I(R)| = |I||\Gamma_I(R)|$ (Theorem 1.7 (7)). When $|I| = 1$ (if and only if $I = 0$), we would be considering the case of zero-divisor graphs on n vertices. This has already been investigated. So we will consider only the cases when $|I| \geq 2$.

In the case of $n = 2, 3, 5, 7$, we must have that $|I| = n$ and $|V(\Gamma(R/I))| = 1$. This has already been considered in Chapter 2 (Proposition 2.9).

So it suffices to consider the cases when $n = 1, 4, 6$. Notice the case $n = 1$ does not occur when $I \neq 0$. Hence we may consider only the cases of $n = 4$ and $n = 6$.

If $n = 4$, then $|I| = 2$ or $|I| = 4$. We will consider each case separately.

Assume $n = 4$ and $|I| = 2$. Then we must have $|V(\Gamma(R/I))| = 2$. Using that $\Gamma(R/I)$ must be connected, we have $\Gamma(R/I) \cong K^2$. By [5, Example 2.1], $R/I \cong \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_3[X]/(X^2)$. From this observation, we can quickly see that $\Gamma_I(R)$ is isomorphic to either a 4-cycle or K^4, by taking note of connected columns and using Redmond's construction method for $\Gamma_I(R)$ from $\Gamma(R/I)$.

Note that both $\mathbb{Z}_9$ and $\mathbb{Z}_3[X]/(X^2)$ give rise to K^4. It has already been discussed for which rings non-trivial $\Gamma_I(R)$ yield K^4 in Chapter 2. Thus it suffices to discuss when $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We must have $|R| = 8$. Since $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is not local, it follows that R is not local. So by writing R as a product of finite local rings, it follows that either (1) $R \cong \mathbb{Z}_2 \times R_1$ (where R_1 is a local ring of order 4) or (2) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. By Lemma 2.1, it follows that R is isomorphic to one of the following four rings: $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_2 \times \mathbb{F}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. By examination, we see that R/I will be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ provided we pick R to be one of the following rings with corresponding ideal I:

1. $\mathbb{Z}_2 \times \mathbb{Z}_4$, $I = 0 \times (2)$,

2. $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, $I = 0 \times (x)$,

3. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $I = 0 \times 0 \times \mathbb{Z}_2$, $0 \times \mathbb{Z}_2 \times 0$, or $\mathbb{Z}_2 \times 0 \times 0$.

Notice that that the preceding three rings are not in rings for K^4 in Table 2.3. We also note that the above argument is similar to that of [9, Example 4.14(c)].

Now when $|I| = 4$, it must be that $|V(\Gamma(R/I))| = 1$. From which it will follow that $\Gamma_I(R) \cong K^4$. This case was already consider in Chapter 2.

Combining the previous observations, yields the following:

Proposition 8.1. *Let R be a commutative ring with nonzero identity and I a proper, non-prime, non-zero ideal of R. Then $\Gamma_I(R)$ is a graph on 4 vertices if and only if exactly one of the following two statements hold.*

1. *$\Gamma_I(R)$ is a 4-cycle; in which case, R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with appropriately chosen ideal I.*

2. *$\Gamma_I(R) = K^4$; in which case, R is isomorphic to one of the 29 rings from Table 2.3 with appropriately chosen ideal I.*

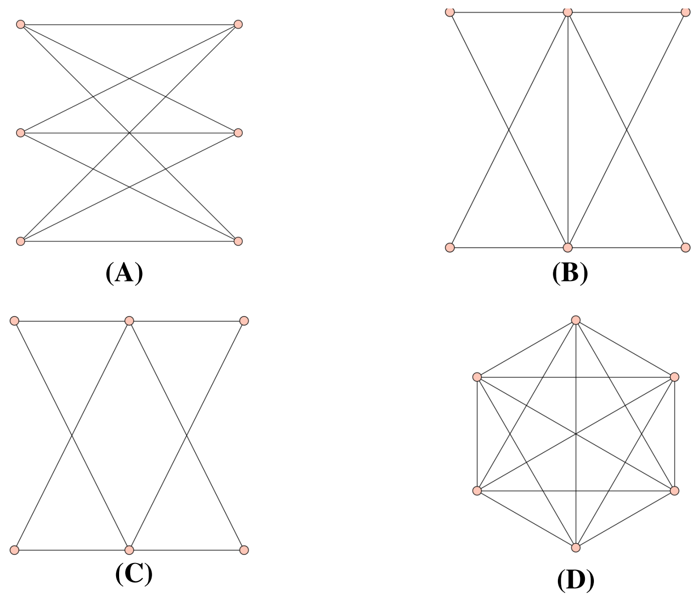


Figure 8.1: $\Gamma_I(R)$ on 6 vertices

We now consider the case when $n = 6$. It follows that $|I| = 2, 3$, or 6. In the case that $|I| = 2$, we have $|V(\Gamma(R/I))| = 3$. We see quickly that classifying the rings up to isomorphism will become unruly as we will have to consider all local rings with 32 elements. In light of the preceding, we will be satisfied simply to classify the graphs of $\Gamma_I(R)$ on 6 vertices up to isomorphism.When $n = 6$, we have $|V(\Gamma(R/I))| = 1, 2,$ or 3. Again, using that we know all zero-divisor graphs on 1,2, and 3 vertices, we may deduce all possible $\Gamma_I(R)$ graphs on 6 vertices using Redmond's construction method. We get the following result.

Proposition 8.2. *Let R be a commutative ring with nonzero identity and I a proper, non-prime, non-zero ideal of R. Then $\Gamma_I(R)$ is a graph on 6 vertices if and only if it is isomorphic to one of the following* **4** *graphs in Figure 8.1.*

Combining this with results from the first chapter, we have classified all possible ideal-based zero-divisor graphs on fewer than 8 vertices. (Notice that the work is done for 7 vertices in Proposition 2.9.) By using [31] and [32], we notice that is possible to give all possible ideal-based zero-divisor graphs on up to 29 vertices. When we consider graphs on 30 vertices, if $|I| = 2$, then $\Gamma(R/I)$ will be a zero-divisor graph on 15 vertices which is not classified in the two papers by Redmond.

Chapter 9

Miscellaneous Results and Future Research

In this chapter, we begin by giving a few small results which do not seem to fit well elsewhere in this dissertation.

Proposition 9.1. *Let R and S be commutative rings with nonzero identity and I a nonzero ideal of R. Then $\Gamma_I(R)$ is complete bipartite if and only if exactly one of the following hold:*

1. *$R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and $|I| = 2$.*

2. *$I = P_1 \cap P_2$, where P_1 and P_2 are prime ideals of R.*

Proof. We begin by proving that (1) and (2) cannot both hold. Assume to the contrary. Then $Z(R/I)^* = \{a+I\}$ and $a^2 \in I = P_1 \cap P_2$ for prime ideals P_1 and P_2 of R. Thus $a^2 \in P_1$ and $a^2 \in P_2$. But since P_1 and P_2 are prime ideals, $a \in P_1 \cap P_2 = I$. However, this is a contradiction as $a+I \in Z(R/I)^*$. Note that the preceding argument shows that (2) implies that R/I is reduced (which is not the case in (1)). (Another way of seeing this is that (2) implies that I is a radical ideal since it is an intersection of prime ideals.) But (1) forces I to be a non-radical ideal as both possibilities for R/I are non-reduced.

We consider two cases: R/I is reduced or non-reduced. If R/I is reduced, then I is a radical ideal. The result then holds by [24, Theorem 3.1(b)].

If R/I is not reduced, then there exists a nonzero $a+I$ in $Z(R/I)$ such that $a^2+I=0+I$. We first claim $\Gamma_I(R)$ complete bipartite implies $Z(R/I)^*=\{a+I\}$. Assume to the contrary. Then there exists a $b+I \in Z(R/I)^*$ such that $a+I$ and $b+I$ are adjacent since zero-divisor graphs are connected. Since I is nonzero, choose $0 \neq i \in I$ and note that $a-a+i-b-a$ is a 3-cycle in $\Gamma_I(R)$. This is impossible as $\Gamma_I(R)$ was assumed to be complete bipartite. Thus in the non-reduced case, $\Gamma_I(R)$ complete bipartite implies that $\Gamma(R/I)$ is a singleton. Therefore $\Gamma_I(R) \cong K^{|I|}$ (by Corollary 2.1) , which is complete bipartite if and only if $|I|=2$. Thus in the non-reduced case, $\Gamma_I(R)$ is complete bipartite if and only $|Z^*(R/I)|=1$ and $|I|=2$, if and only if $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and $|I|=2$ (by Proposition 2.2). □

Proposition 9.2. *Let R be a commutative ring with nonzero identity, I an ideal of R, and X an indeterminate.*

1. *$V(\Gamma_{I[X]}(R[X]))$ has infinite girth if and only if $V(\Gamma_I(R))$ is empty.*

2. *If $V(\Gamma_{I[X]}(R[X]))$ is nonempty, then $gr(\Gamma_{I[X]}(R[X])) \in \{3,4\}$.*

Proof. We prove the forward direction of (1) by contrapositive. Assume that $V(\Gamma_I(R))$ is non-empty. There are two possibilities: $|V(\Gamma(R/I))|=1$ or $|V(\Gamma(R/I))| \geq 2$. In the first case, we must have there exists $a \in V(\Gamma(R/I)$ such that $a^2 \in I$. Then notice that the vertices a, aX, aX^2 are mutually adjacent in $\Gamma_{I[X]}(R[X])$, whence $gr(\Gamma_{I[X]}(R[X]))=3$. In the second case, since zero-divisor graphs are connected (Theorem 1.5), there exists distinct vertices a, b in $\Gamma(R/I)$ that are adjacent. Then $a-b-aX-bX-a$ is a cycle of length 4; hence $gr(\Gamma_{I[X]}(R[X])) \leq 4$. Thus, in both cases, we have that the girth of $V(\Gamma_{I[X]}(R[X]))$ is not infinity as desired.

Property (2) follows from the proof of Property (1).

□

Proposition 9.3. *Let R be a commutative ring with nonzero identity, I a radical ideal of R, and X an indeterminate. Then $gr(\Gamma_{I[X]}(R[X])) = 3$ if and only if $gr(\Gamma_I(R)) = 3$.*

Proof. "$\Rightarrow$" Say that $p(X) - q(X) - r(X) - p(X)$ is a cycle in $\Gamma_{I[X]}(R[X])$. Choose $\overline{p}(X) \in ((R \setminus I \cup \{0\}))[X]$ such that $p(X) + I[X] = \overline{p}(X) + I[X]$. Define $\overline{q}(X)$ and $\overline{r}(X)$ similarly. Notice that the $\overline{p}(X)$,$\overline{q}(X)$,$\overline{r}(X)$ are nonzero. Moreover $\overline{p}(X)$, $\overline{q}(X)$, $\overline{r}(X)$ are mutually distnict as $I = \sqrt{I} \Leftrightarrow I[X] = \sqrt{I[X]}$. To see this, assume two of $\overline{p}(X)$,$\overline{q}(X)$,$\overline{r}(X)$ are equal in $R[X]$. Without loss of generality, assume $\overline{p}(X) = \overline{q}(X)$, then $pq \in I[X]$ implies that $\overline{p}(X)^2 \in I[X]$; the latter is impossible as $I[X] = \sqrt{I[X]}$ and $\overline{p}(X) \notin I[X]$.

Let p, q, r be the leading coefficients of each $\overline{p}(X)$,$\overline{q}(X)$,$\overline{r}(X)$, respectively. Since $\overline{p}(X) \in ((R \setminus I \cup \{0\}))[X]$ and is nonzero, $p \in R \setminus I$. Similarly $r, s \in R \setminus I$. Now $p(X) - q(X) - r(X) - p(X)$ is a cycle implies that $p - q - r - p$ is a cycle in $\Gamma(R/I)$, provided they are distinct. Let us assume that two of the latter are equal; without loss of generality, assume that $p = q$. Then we have that $p^2 = pq \in I$ and $p \in R \setminus I$. But this is impossible as $\sqrt{I} = I$. Therefore $gr(\Gamma_I(R)) = 3$.

"$\Leftarrow$" This implication is evident as a cycle in $\Gamma_I(R)$ will also be a cycle in $\Gamma_{I[X]}(R[X])$. □

We conclude this dissertation by listing a few open questions. We hope to return to these questions in future research.

- When is $\Gamma_I(R)$ an infinite planar graph?
- Assume $\Gamma_I(R) \cong \Gamma_J(S)$, I and J radical ideals of R and S, respectively, with $|I| = |J|$. When do the latter assumptions imply $\Gamma(R/I) \cong \Gamma(S/J)$?
- What relationship exists between $\Gamma_I(R)$ and $\Gamma_{I[X]}(R[X])$? In particular, what can be said about the girth of $\Gamma_{I[X]}(R[X])$?

- Some equivalences for $\Gamma_I(R)$ to be complete bipartite have been explored. What other ring-theoretic properties give complete bipartite ideal-based zero-divisor graphs?

- The concept of the compressed zero-divisor graph has been consider by several authors. Is there a natural way to extend this to $\Gamma_I(R)$?

- What are natural topologies that can be placed on ideal-based zero-divisor graphs and what properties would they have?

Bibliography

[1] Akbari, S., Maimani, H. R., and Yassemi, S. (2003). When a zero-divisor graph is planar or a complete r-partite graph. *J. Algebra*, 270(1):169–180. 31

[2] Anderson, D. D. and Naseer, M. (1993). Beck's coloring of a commutative ring. *J. Algebra*, 159(2):500–514. 13

[3] Anderson, D. F., Axtell, M. C., and Stickles, Jr., J. A. (2011). Zero-divisor graphs in commutative rings. In *Commutative Algebra—Noetherian and non-Noetherian Perspectives*, pages 23–45. Springer, New York. 3, 5

[4] Anderson, D. F. and Badawi, A. (2012). Von Neumann regular and related elements in commutative rings. *Algebra Colloquium*, 19(spec01):1017–1040. 9

[5] Anderson, D. F., Frazier, A., Lauve, A., and Livingston, P. S. (2001). The zero-divisor graph of a commutative ring. II. In *Ideal Theoretic Methods in Commutative Algebra (Columbia, MO, 1999)*, volume 220 of *Lecture Notes in Pure and Appl. Math.*, pages 61–72. Dekker, New York. 26, 31, 34, 80, 81, 83, 88

[6] Anderson, D. F., Levy, R., and Shapiro, J. (2003). Zero-divisor graphs, von Neumann regular rings, and Boolean algebras. *J. Pure Appl. Algebra*, 180(3):221–241. 10, 47, 48, 53, 55, 58, 60, 61

[7] Anderson, D. F. and Livingston, P. S. (1999). The zero-divisor graph of a commutative ring. *J. Algebra*, 217(2):434–447. 4, 5, 13, 30

[8] Anderson, D. F. and Mulay, S. B. (2007). On the diameter and girth of a zero-divisor graph. *J. Pure Appl. Algebra*, 210(2):543–550. 36, 39, 46

[9] Anderson, D. F. and Shirinkam, S. (To Appear). Some remarks on $\Gamma_I(R)$. *Comm. Algebra*. 7, 13, 63, 73, 89

[10] Atiyah, M. and MacDonald, I. G. (1969). *Introduction to Commutative Algebra*. Addison-Wesley. 10

[11] Axtell, M., Coykendall, J., and Stickles, J. (2005). Zero-divisor graphs of polynomials and power series over commutative rings. *Comm. Algebra*, 33(6):2043–2050. 5, 32

[12] Beck, I. (1988). Coloring of commutative rings. *J. Algebra*, 116(1):208–226. 3, 13

[13] Belshoff, R. and Chapman, J. (2007). Planar zero-divisor graphs. *J. Algebra*, 316(1):471–480. 21, 31

[14] Chartrand, G. and Lesniak, L. (1996). *Graphs & Digraphs.* Chapman & Hall/CRC. 8

[15] Corbas, B. and Williams, G. D. (2000a). Rings of order p^5. I. Nonlocal rings. *J. Algebra*, 231(2):677–690. 21

[16] Corbas, B. and Williams, G. D. (2000b). Rings of order p^5. II. Local rings. *J. Algebra*, 231(2):691–704. 21

[17] DeMeyer, F. and Schneider, K. (2002a). Automorphisms and zero divisor graphs of commutative rings. *Int. J. Commut. Rings*, pages 93–106. 5, 13, 32

[18] DeMeyer, F. and Schneider, K. (2002b). Automorphisms and zero divisor graphs of commutative rings. In *Commutative rings*, pages 25–37. Nova Sci. Publ., Hauppauge, NY. 13, 81

[19] Hungerford, T. W. (1974). *Algebra*. Springer. 9

[20] LaGrange, J. D. (2007). Complemented zero-divisor graphs and Boolean rings. *J. Algebra*, 315(2):600–611. 74, 76, 77, 78

[21] Lambek, J. (1966). *Lectures on Rings and Modules.* Blaisdell Publishing Company, Waltham. 76

[22] Levy, R. and Shapiro, J. (2002). The zero-divisor graph of von Neumann regular rings. *Comm. Algebra*, 30(2):745–750. 55

[23] Livingston, P. S. (1997). Structure in zero-divisor graphs of commutative rings. Thesis, Univeristy of Tennessee Knoxville. 4, 5

[24] Maimani, H. R., Pournaki, M. R., and Yassemi, S. (2006). Zero-divisor graph with respect to an ideal. *Comm. Algebra*, 34(3):923–929. 63, 64, 65, 66, 93

[25] Mulay, S. B. (2002). Cycles and symmetries of zero-divisors. *Comm. Algebra*, 30(7):3533–3558. 5, 32

[26] Redmond, S. (2009). Counting zero-divisors. In *Commutative Rings: New Research (Hauppauge, NY, 2009)*, pages 7–12. Nova Science Publisher, New York. 87

[27] Redmond, S. P. (2001). *Generalizations of the zero-divisor graph of a ring.* ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–The University of Tennessee. 4, 5, 6, 55, 59

[28] Redmond, S. P. (2002a). The zero-divisor graph of a non-commutative ring. *Int. J. Commut. Rings*, 1:203–222. 4

[29] Redmond, S. P. (2002b). The zero-divisor graph of a non-commutative ring. In *Commutative rings*, pages 39–47. Nova Sci. Publ., Hauppauge, NY. 4

[30] Redmond, S. P. (2003). An ideal-based zero-divisor graph of a commutative ring. *Comm. Algebra*, 31(9):4425–4443. 5, 6, 33, 34, 66

[31] Redmond, S. P. (2007a). Corrigendum to: "On zero-divisor graphs of small finite commutative rings". *Discrete Math.*, 307(21):2449–2452. 88, 90

[32] Redmond, S. P. (2007b). On zero-divisor graphs of small finite commutative rings. *Discrete Math.*, 307(9-10):1155–1166. 88, 90

[33] Smith, N. O. (2004). *Properties of the Zero-Divisor Graph of a Ring*. dissertation, University of Tennessee Knoxville. 31

[34] Smith, N. O. (2006). Planar zero-divisor graphs. In *Focus on commutative rings research*, pages 177–186. Nova Sci. Publ., New York. 31

[35] Smith, N. O. (2007). Infinite planar zero-divisor graphs. *Comm. Algebra*, 35(1):171–180. 31

[36] Wang, H.-J. (2006). Zero-divisor graphs of genus one. *J. Algebra*, 304(2):666–678. 31

[37] Wright, S. E. (2007). Lengths of paths and cycles in zero-divisor graphs and digraphs of semigroups. *Comm. Algebra*, 35(6):1987–1991. 5, 32

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