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Parity Theorems for Combinatorial Statistics

Mark A. Shattuck
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To the Graduate Council:

I am submitting herewith a dissertation written by Mark A. Shattuck entitled "Parity Theorems for Combinatorial Statistics." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Carl G. Wagner, Major Professor

We have read this dissertation and recommend its acceptance:

John Nolt, Jan Rosinski, Pavlos Tzermias

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

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Graduate Studies

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Parity Theorems for Combinatorial Statistics

A Dissertation
Presented for the
Doctor of Philosophy
Degree
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Mark A. Shattuck

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Abstract

A q-generalization $G_n(q)$ of a combinatorial sequence G_n which reduces to that sequence when $q = 1$ is obtained by q-counting a statistic defined on a sequence of finite discrete structures enumerated by G_n. In what follows, we evaluate $G_n(-1)$ for statistics on several classes of discrete structures, giving both algebraic and combinatorial proofs. For the latter, we define appropriate sign-reversing involutions on the associated structures. We shall call the actual algebraic result of such an evaluation at $q = -1$ a *parity theorem* (for the statistic on the associated class of discrete structures).

Among the structures we study are permutations, binary sequences, Laguerre configurations, derangements, Catalan words, and finite set partitions. As a consequence of our results, we obtain bijective proofs of congruences involving Stirling, Catalan, and Bell numbers. In addition, we modify the ideas used to construct the aforementioned sign-reversing involutions to furnish bijective proofs of combinatorial identities involving sums with alternating signs.

Contents

Notation

$\mathbb{N}$	nonnegative integers
$\mathbb{P}$	positive integers
$[n]$	the set $\{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$ (so $[0] = \varnothing$)
$\subseteq$	containment (perhaps improper) of sets
$\subset$	proper containment of sets
$\lvert S \rvert$	the cardinality (number of elements) of a finite set S
$:=$	equals by definition
0^0	by convention, $0^0 = 1$ throughout
$\lfloor x \rfloor$	greatest integer $\leqslant x$
$\lceil x \rceil$	least integer $\geqslant x$
$\delta_{i,j}$	the Kronecker delta, equal to 1 if $i = j$ and 0 otherwise
$x^{\underline{n}}$	the product $x(x-1)\cdots(x-n+1)$, for $n \in \mathbb{N}$ (so $x^{\underline{0}} = 1$)
n_q	the number $q^{n-1} + q^{n-2} + \cdots + 1$, for $n \in \mathbb{N}$ (so $0_q = 0$)
$n_q^!$	the number $n_q(n-1)_q \cdots 1_q$, for $n \in \mathbb{N}$ (so $0_q^! = 1$)
$\binom{n}{k}_q$	a q-binomial coefficient of order n
$\binom{n}{n_1,\ldots,n_k}_q$	a q-multinomial coefficient of order n
F_q	the finite field of q elements (q implicitly a power of a prime)
F_q^n	the finite n-dimensional vector space over F_q of q^n elements
2^s	the power set (set of all subsets) of a set S
$[k]^{[n]}$	the set of functions from $[n]$ to $[k]$
S_n	the symmetric group on n objects
$S_{n,k}$	the subset of S_n whose members contain k cycles

Introduction

We'll use the following notational conventions: $\mathbb{N} := \{0, 1, 2, \ldots\}$, $\mathbb{P} := \{1, 2, \ldots\}$, $[0] := \varnothing$, and $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{P}$. Empty sums take the value 0 and empty products the value 1, with $0^0 := 1$. The letter q denotes an indeterminate, with $0_q := 0$, $n_q := 1 + q + \cdots + q^{n-1}$ for $n \in \mathbb{P}$, $0_q^! := 1$, $n_q^! := 1_q 2_q \cdots n_q$ for $n \in \mathbb{P}$, and

$$\binom{n}{k}_q := \begin{cases} \dfrac{n_q^!}{k_q^!(n-k)_q^!}, & \text{if } 0 \leqslant k \leqslant n; \\ 0, & \text{if } k < 0 \text{ or } 0 \leqslant n < k. \end{cases} \tag{1}$$

Let Δ be a finite set of discrete structures and $I : \Delta \to \mathbb{N}$, with generating function

$$G(I, \Delta; q) := \sum_{\delta \in \Delta} q^{I(\delta)} = \sum_k |\{\delta \in \Delta : \ I(\delta) = k\}| \, q^k. \tag{2}$$

Of course, $G(I, \Delta; 1) = |\Delta|$. If $\Delta_i := \{\delta \in \Delta : \ I(\delta) \equiv i \pmod 2\}$, then $G(I, \Delta; -1) = |\Delta_0| - |\Delta_1|$. Hence if $G(I, \Delta; -1) = 0$, the set Δ is "balanced" with respect to the parity of I. For example, setting $q = -1$ in the binomial theorem,

$$(1+q)^n = \sum_{S \subseteq [n]} q^{|S|} = \sum_{k=0}^{n} \binom{n}{k} q^k, \tag{3}$$

yields the familiar result that a finite nonempty set has as many subsets of odd cardinality as it has subsets of even cardinality.

When $G(I, \Delta; -1) = 0$ and hence $|\Delta_0| = |\Delta_1|$, it is instructive to identify an I-parity changing involution of Δ. (In what follows, we call the parity of $I(\delta)$ the *I-parity* of δ, and an involution $\delta \mapsto \delta'$ for which δ and δ' have opposite I-parities an *I-parity changing involution*.) For the statistic $|S|$ in (3), the map

$$S \mapsto \begin{cases} S \cup \{1\}, & \text{if } 1 \notin S; \\ S - \{1\}, & \text{if } 1 \in S, \end{cases}$$

furnishes such an involution. More generally, if $G(I, \Delta; -1) = |\Delta_0| - |\Delta_1| = c$, it suffices to identify a subset Δ^* of Δ of cardinality $|c|$ contained completely within Δ_0 or Δ_1 (depending upon the sign of c) and then to define an I-parity changing involution on $\Delta - \Delta^*$. The subset Δ^* thus captures both the sign and magnitude of $G(I, \Delta; -1)$.

Since each member of $\Delta - \Delta^*$ is paired with another of opposite I-parity, we have $|\Delta| \equiv |\Delta^*| \pmod 2$. Thus, the I-parity changing involutions described above, in addition to conveying a visceral understanding of why $G(I, \Delta; -1)$ takes a particular value, also supply combinatorial proofs of congruences of the form $a_n \equiv b_n \pmod 2$. Shattuck [18] has, for example, given such a combinatorial proof of the congruence

$$S(n,k) \equiv \binom{n - \lfloor k/2 \rfloor - 1}{n-k} \pmod 2 \tag{4}$$

for Stirling numbers of the second kind, answering a question posed by Stanley [23, p. 46, Exercise 17b].

In what follows, we undergo a systematic study of the special case $q = -1$, giving both algebraic and combinatorial treatments. In Chapter 1, we establish parity theorems for statistics on multiset permutations and Laguerre configurations, i.e., distributions of labeled balls to unlabeled, contents-ordered urns, algebraically by evaluating q-generating functions at $q = -1$ and combinatorially by identifying appropriate parity changing involutions. In Chapter 2, we perform a similar task for statistics on permutations and Catalan words, i.e., binary sequences with an equal number of 1's and 0's in which no initial segment contains more 1's and 0's. In Chapter 3, we examine the parity of four closely related statistics on partitions of finite sets. As a consequence of our results, we obtain bijective proofs of congruences involving Stirling, Catalan, and Bell numbers.

For a word $w = w_1 w_2 \cdots w_m$ in some alphabet consisting of integers, the inversion and major index statistics are given by

$$inv(w) := |\{(i,j) : \ i < j \text{ and } w_i > w_j\}|$$

and

$$maj(w) := \sum_{i \in D(w)} i, \text{ where } D(w) := \{1 \leqslant i \leqslant m-1 \,:\, w_i > w_{i+1}\}\,.$$

Most of the statistics of the first two chapters involve counting inversions or finding the major index of words used to encode various discrete structures. For the partition statistics studied in the third chapter, one first canonically orders the blocks of a finite partition and then computes various weighted sums involving block cardinalities.

Chapter 1

Parity Theorems for Statistics on Multiset Permutations and Laguerre Configurations

1.1 Introduction

We analyze the parity of two well known statistics on multiset permutations, thereby generalizing results found in [21] for binary words. We also examine the parity of two statistics on what Garsia and Remmel [11] term *Laguerre configurations*, i.e., distributions of labeled balls to unlabeled, contents-ordered urns. The generating functions for the statistics on multiset permutations involve q-multinomial coefficients, while those for the statistics on binary words and Laguerre configurations all involve q-binomial coefficients.

In 1.2, we evaluate q-multinomial coefficients and their sums, when $q = -1$, giving both algebraic and bijective proofs. We also give a bijective proof of a recurrence for sums of q-binomial coefficients, known as *Galois numbers*, furnishing an elementary alternative to Goldman and Rota's proof by the method of linear functionals [12]. In 1.3, we carry out a similar evaluation of the two types of q-Lah numbers that arise as generating functions for the aforementioned Laguerre configuration statistics. In 1.4, we refine a result of the second section by looking at the restriction of one of the statistics to binary words with a fixed number of descents.

1.2 Parity Theorems for Multiset Permutations

We will use the notation $\{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\}$ for the multiset consisting of n_i copies of i for all $i \in [k]$. A *permutation* of a multiset is just a way of listing all of its elements. The number of permutations of the multiset $\{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\}$ is the multinomial coefficient

$$\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{n_1!\, n_2! \cdots n_k!}\,,$$

where $n = n_1 + n_2 + \cdots + n_k$.

Given a permutation $p = p_1 p_2 \cdots p_n$ of a multiset, define the statistics *inv* and *maj* by

$$inv\,(p) := |\{(i,j):\ i<j \text{ and } p_i > p_j\}|$$

and

$$maj\,(p) := \sum_{i \in D(p)} i, \text{ where } D(p) := \{1 \leqslant i \leqslant n-1:\ p_i > p_{i+1}\}\,.$$

The statistics *inv* and *maj* record the number of inversions and the major index, respectively, of a multiset permutation p. The set $D(p)$ is referred to as the *down set* or the *set of descents* of p.

If $(n_1, \ldots, n_k)$ is a sequence of nonnegative integers summing to n, then define the *q-multinomial coefficient* by

$$\binom{n}{n_1, \ldots, n_k}_q := \frac{n_q^!}{{n_1}_q^! \cdots {n_k}_q^!}\,,$$

where $j_q^! := 1_q 2_q \cdots j_q$ and $r_q := 1 + q + \cdots + q^{r-1}$ for $j, r \in \mathbb{N}$.

If $K = \{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\}$ and S_K is the set of multiset permutations of K, then

$$\sum_{p \in S_K} q^{inv\,(p)} = \binom{n}{n_1, \ldots, n_k}_q = \sum_{p \in S_K} q^{maj\,(p)}. \tag{1.1}$$

See [23] and [1]. Let

$$G_q^{(k)}(n) := \sum_{\substack{n_1+\cdots+n_k=n\\ n_i\in\mathbb{N}}} \binom{n}{n_1,\ldots,n_k}_q. \tag{1.2}$$

Then

$$\sum_{\lambda\in[k]^{[n]}} q^{inv\,(\lambda)} = G_q^{(k)}(n) = \sum_{\lambda\in[k]^{[n]}} q^{maj\,(\lambda)}, \tag{1.3}$$

by (1.1), where members of $[k]^{[n]}$ are expressed as words.

Theorem 1.1. *For all $n \in \mathbb{N}$ and sequences $(n_1,\ldots,n_k)$ in $\mathbb{N}$ with $n = n_1+\cdots+n_k$,*

$$\binom{n}{n_1,\ldots,n_k}_{-1} = \begin{cases} \binom{\lfloor n/2\rfloor}{\lfloor n_1/2\rfloor,\ldots,\lfloor n_k/2\rfloor}, & \text{if at most one } n_i \text{ is odd, } i\in[k];\\ 0, & \text{otherwise,}\end{cases} \tag{1.4}$$

and

$$G_{-1}^{(k)}(n) := \sum_{\substack{n_1+\cdots+n_k=n\\ n_i\in\mathbb{N}}} \binom{n}{n_1,\ldots,n_k}_{-1} = k^{\lceil n/2\rceil}. \tag{1.5}$$

Proof. Formula (1.5) follows from (1.4) and the multinomial theorem. To prove (1.4), first assume n is even and take $k = 2$ for simplicity. If n_1 is even, then

$$\binom{n}{n_1,n_2}_{-1} = \lim_{q\to-1}\binom{n}{n_1,n_2}_q = \lim_{q\to-1}\prod_{i=0}^{n_1-1}\frac{(n-i)_q}{(n_1-i)_q} =$$

$$\prod_{\substack{i=0\\ i\text{ even}}}^{n_1-2}\lim_{q\to-1}\left(\frac{q^{n-i}-1}{q^{n_1-i}-1}\right) = \prod_{\substack{i=0\\ i\text{ even}}}^{n_1-2}\frac{n-i}{n_1-i} = \prod_{\substack{i=0\\ i\text{ even}}}^{n_1-2}\frac{n/2-i/2}{n_1/2-i/2} = \binom{n/2}{n_1/2,n_2/2}.$$

If n_1 is odd, then $q=-1$ is a zero of multiplicity one more in the numerator than in the denominator and hence $\binom{n}{n_1,n_2}_{-1} = 0$ in that case. The case for

n odd is handled similarly or is gotten from the even case by taking limits as $q \to -1$ in the recurrence

$$\binom{n}{n_1, n_2}_q = \binom{n-1}{n_1-1, n_2}_q + q^{n_1}\binom{n-1}{n_1, n_2-1}_q.$$

The preceding readily generalizes to the case $k \geqslant 3$.

We now give bijective proofs of formulas (1.5) and (1.4). By (1.3), formula (1.5) asserts that

$$|\Lambda_0(n)| - |\Lambda_1(n)| = k^{\lceil n/2 \rceil}, \tag{1.6}$$

where $\Lambda_i(n) := \{\lambda \in \Lambda(n) : \ inv(\lambda) \equiv i \pmod 2\}$ and $\Lambda(n) := [k]^{[n]}$. Our strategy for proving (1.6) is to identify a subset $\Lambda_0^+(n)$ of $\Lambda_0(n)$ having cardinality $k^{\lceil n/2 \rceil}$, along with an *inv*-parity changing involution of $\Lambda(n) - \Lambda_0^+(n)$.

The set $\Lambda_0^+(n)$ comprises those $\lambda = \lambda_1\lambda_2\cdots\lambda_n \in \Lambda(n)$ such that

$$\lambda_{2j-1} = \lambda_{2j}, \quad 1 \leqslant j \leqslant \lfloor n/2 \rfloor. \tag{1.7}$$

Clearly, $\Lambda_0^+(n) \subseteq \Lambda_0(n)$ and $|\Lambda_0^+(n)| = k^{\lceil n/2 \rceil}$. If $\lambda \in \Lambda(n) - \Lambda_0^+(n)$, let j_0 be the smallest j for which (1.7) fails to hold and let λ' be the result of switching λ_{2j_0-1} and λ_{2j_0} in λ. The map $\lambda \mapsto \lambda'$ is clearly an involution of $\Lambda(n) - \Lambda_0^+(n)$, and the parity of the number of inversions in λ' is opposite the parity of the number of inversions in λ. This establishes (1.6), and hence (1.5).

Let $\Lambda(n;K)$ denote the set of rearrangements of the multiset $K = \{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\}$, where $n_1 + n_2 + \cdots + n_k = n$. By (1.1), formula (1.4) asserts that

$$|\Lambda_0(n;K)| - |\Lambda_1(n;K)| = \begin{cases} \binom{\lfloor n/2 \rfloor}{\lfloor n_1/2 \rfloor, \ldots, \lfloor n_k/2 \rfloor}, & \text{if at most one } n_i \text{ is odd}, i \in [k]; \\ 0, & \text{otherwise}, \end{cases} \tag{1.8}$$

where $\Lambda_i(n;K) := \Lambda_i(n) \cap \Lambda(n;K)$. To show (1.8), let $\Lambda_0^+(n;K) := \Lambda_0^+(n) \cap \Lambda(n;K)$. The cardinality of $\Lambda_0^+(n;K)$ is given by the right-hand side of (1.8), and the restriction of the above map $\lambda \mapsto \lambda'$ to $\Lambda(n;K) - \Lambda_0^+(n;K)$ is again an involution and inherits the parity changing property. This establishes (1.8), and hence (1.4). $\square$

Note that the preceding combinatorial arguments would have worked with the *maj* statistic in place of the *inv* statistic. Thus formulas (1.4) and (1.5) are parity theorems for both the *inv* and *maj* statistics.

Letting $k = n$ and $n_i = 1$ for all $i \in [k]$ in (1.1) yields the well known fact that *inv* and *maj* are equally distributed on the symmetric group S_n. Formula (1.4) then reveals that the *inv* and *maj* statistics are balanced on S_n if $n \geqslant 2$. The bijection for (1.4) in this case amounts to merely switching the first two positions of a permutation of $[n]$.

When $k = 2$, the q-multinomial coefficients are q-binomial coefficients and the numbers $G_q^{(2)}(n)$ are the *Galois numbers* $G_q(n) := \sum_{i=0}^{n} \binom{n}{i}_q$ of Goldman and Rota [12]. We record the $k = 2$ case of Theorem 1.1.

Corollary 1.2. *If* $0 \leqslant i \leqslant n$*, then*

$$\binom{n}{i}_{-1} = \begin{cases} 0, & \text{if } n \text{ is even and } i \text{ is odd;} \\ \binom{\lfloor n/2 \rfloor}{\lfloor i/2 \rfloor}, & \text{otherwise,} \end{cases} \tag{1.9}$$

and

$$G_{-1}(n) := \sum_{i=0}^{n} \binom{n}{i}_{-1} = 2^{\lceil n/2 \rceil}. \tag{1.10}$$

Note that (1.9) also follows upon substituting $q = -1$ into the well known identity [25, pp. 201–202]

$$\sum_{n \geqslant 0} \binom{n}{i}_q x^n = \frac{x^i}{(1-x)(1-qx)\cdots(1-q^i x)}, \quad i \in \mathbb{N}, \tag{1.11}$$

and considering even and odd cases for i.

Formula (1.9) is a parity theorem for both *inv* and *maj* on the set of sequential arrangements of the multiset $\{1^i, 2^{n-i}\}$. Each such sequential arrangement corresponds, geometrically, to a (minimal) lattice path from $(0,0)$ to $(i, n-i)$, with 1 representing a horizontal and 2 a vertical step. Since the number of inversions of a sequential arrangement of $\{1^i, 2^{n-i}\}$ equals the area subtended by the corresponding lattice path [23], one may also view (1.9) and (1.10) as parity theorems for area under lattice paths [21]. Formula (1.10) also follows by induction from the case $q = -1$ of the following recurrence for $G_q(n)$:

Theorem 1.3. *For all* $n \geqslant 1$,

$$G_q(n+1) = 2G_q(n) + (q^n - 1)\, G_q(n-1), \tag{1.12}$$

with $G_q(0) = 1$ *and* $G_q(1) = 2$.

Proof. Let $a(n,i) := \left|\left\{\lambda \in [2]^{[n]} :\ inv(\lambda) = i\right\}\right|$. By (1.3), showing (1.12) is equivalent to showing that

$$\begin{aligned} a(n+1,i) &= 2a(n,i) + a(n-1,i-n) - a(n-1,i) \\ &= a(n,i) + (a\,(n,i) - a\,(n-1,i)) + a\,(n-1,i-n)\,, \end{aligned} \tag{1.13}$$

for all $i \in \mathbb{N}$, where $a(m,j) = 0$ if $m \in \mathbb{N}$ and $j < 0$.

The term $a(n+1,i)$ counts all $\lambda = \lambda_1\lambda_2\cdots\lambda_{n+1} \in [2]^{[n+1]}$ with i inversions. The term $a(n,i)$ counts the subclass of such words for which $\lambda_{n+1} = 2$. The term $a(n,i) - a(n-1,i)$ counts the subclass of such words for which $\lambda_1 = \lambda_{n+1} = 1$. For deletion of λ_1 is a bijection from this subclass to the class of words $u_1u_2\cdots u_n$ with i inversions and $u_n = 1$, and there are clearly $a(n,i) - a(n-1,i)$ words of the latter type. Finally, the term $a(n-1,i-n)$ counts the subclass of words for which $\lambda_1 = 2$ and $\lambda_{n+1} = 1$. For deletion of λ_1 and λ_{n+1} is a bijection from this subclass to the class of words $v_1v_2\cdots v_{n-1}$ with $i-n$ inversions (both classes being empty if $i < n$). □

The above proof provides an elementary alternative to Goldman and Rota's proof of (1.12) using the method of linear functionals [12]. It doesn't appear though that the numbers $G_q^{(k)}(n)$ satisfy a nice recurrence as in (1.12) in general for $k \geqslant 3$.

1.3 Two Statistics on Laguerre Configurations

Let $\mathcal{L}(n,k)$ denote the set of distributions of n balls, labeled $1, 2, \ldots, n$, among k unlabeled, *contents-ordered* urns with no urn left empty. Garsia and Remmel [11] call such distributions *Laguerre configurations*. Members of $\mathcal{L}(n,k)$ will be regarded as partitions of $[n]$ into k blocks, where members of each block are ordered.

If $L(n,k) := |\mathcal{L}\,(n,k)|$, then $L(n,0) = \delta_{n,0}\ \forall\, n \in \mathbb{N}$, $L(n,k) = 0$ if $0 \leqslant n < k$, and

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}, \qquad 1 \leqslant k \leqslant n. \tag{1.14}$$

The numbers $L(n,k)$, called *Lah numbers*, were introduced by Ivo Lah [14] as connection constants in the polynomial identities

$$x(x+1)\cdots(x+n-1)=\sum_{k=0}^{n} L(n,k)x(x-1)\cdots(x-k+1), \quad \forall\, n\in\mathbb{N}. \tag{1.15}$$

In what follows, we analyze the parity of two statistics on Laguerre configurations. The main results of this section (namely Theorems 1.4 and 1.6) appear in [21]. Recall the well known summation formula [23, p. 29],

$$\binom{n}{k}_q = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} q^{0d_0+1d_1+\cdots+kd_k}, \qquad 0\leqslant k\leqslant n. \tag{1.16}$$

1.3.1 The Statistic inv_ρ

Given $\delta\in\mathcal{L}(n,k)$, represent each ordered block by a word in $[n]$ and then arrange these words in a sequence $W_1,\ldots,W_k$, by decreasing order of their least elements. Replace the commas in this sequence by zeros and count inversions in the resulting single word to obtain the value $inv_\rho(\delta)$, i.e.,

$$inv_\rho(\delta) = \operatorname{inv}(W_1 0 W_2 0\cdots 0 W_{k-1} 0 W_k)\,. \tag{1.17}$$

As an illustration, for $\delta = \{3,2,5\},\ \{7,6,8\},\ \{1,4\} \in \mathcal{L}(8,3)$, we have $inv_\rho(\delta)=30$, the number of inversions in the word 7680325014.

The statistic inv_ρ is due to Garsia and Remmel [11], who show that the generating function

$$L_q(n,k) := \sum_{\delta\in\mathcal{L}(n,k)} q^{inv_\rho(\delta)} = q^{k(k-1)}\frac{n_q^!}{k_q^!}\binom{n-1}{k-1}_q, \quad 1\leqslant k\leqslant n, \tag{1.18}$$

which generalizes (1.14). Garsia and Remmel also show that

$$x_q(x+1)_q\cdots(x+n-1)_q = \sum_{k=1}^{n} L_q(n,k)x_q(x-1)_q\cdots(x-k+1)_q, \tag{1.19}$$

where $n\in\mathbb{P}$ and $x_q := (q^x-1)\,/\,(q-1)$. Identity (1.19) has a polynomial version which doesn't seem to have been previously noted:

$$x(qx+1_q)\cdots(q^{n-1}x+(n-1)_q)$$
$$=\sum_{k=1}^{n} L_q(n,k)x\left(\frac{x-1_q}{q}\right)\cdots\left(\frac{x-(k-1)_q}{q^{k-1}}\right), \quad (1.20)$$

which generalizes (1.15).

Theorem 1.4. *If* $1 \leqslant k \leqslant n$, *then*

$$L_{-1}(n,k) = \delta_{n,k}\,. \quad (1.21)$$

Proof. It is obvious from (1.18) that $L_{-1}(n,n) = 1$. If $1 \leqslant k < n$ with n even or k odd, the factor $n_q^!/k_q^! = n_q \cdots (k+1)_q$ in (1.18) is zero since $j_{-1} = 0$ if j is even. In the remaining case, where n is odd and k is even, the factor $\binom{n-1}{k-1}_{-1} = 0$ by (1.9).

For a bijective proof of (1.21), let $\mathcal{L}_i(n,k) := \{\delta \in \mathcal{L}(n,k) :\ inv_\rho(\delta) \equiv i \pmod 2\}$ so that $L_{-1}(n,k) = |\mathcal{L}_0(n,k)| - |\mathcal{L}_1(n,k)|$. Then $L_{-1}(n,n) = 1$ since $\mathcal{L}(n,n)$ consists of a single distribution whose inv_ρ value is $n(n-1)$. If $1 \leqslant k < n$ and $\delta \in \mathcal{L}(n,k)$ has the associated sequence $W_1,\ldots,W_k$, locate the leftmost word W_i containing at least two letters and interchange its first two letters. The resulting map is a parity changing involution of $\mathcal{L}(n,k)$, whence $|\mathcal{L}_0(n,k)| - |\mathcal{L}_1(n,k)| = 0$. □

Note that $\mathcal{L}(n,1) = S_n$, the set of permutations of $[n]$, and so (1.18) generalizes the well known result that

$$\sum_{\pi \in S_n} q^{inv(\pi)} = n_q^!.$$

Formula (1.21) then reflects the fact that among the permutations of $[n]$, if $n \geqslant 2$, there are as many with an odd number of inversions as there are with an even number of inversions. When $n \geqslant 2$ and $k = 1$ in Theorem 1.4 above, the bijection then amounts to switching the first two letters of $\sigma \in S_n$, just as the bijection of Theorem 1.1 did when $n \geqslant 2$ and $n_i = 1$ for all i.

1.3.2 The Statistic $\tilde{w}$

As before, we represent each ordered block of $\delta \in \mathcal{L}(n,k)$ by a word in $[n]$. Having now arranged these words in a sequence $W_1,\ldots,W_k$ by increasing order of their initial elements, we define $\tilde{w}(\delta)$ by the formula

$$\tilde{w}(\delta) = \sum_{i=1}^{k} (i-1)\left(|W_i| - 1\right), \tag{1.22}$$

where $|W_i|$ denotes the length of the word W_i. As an illustration, for $\delta = \{3,2,5\}, \{7,6,8\}, \{1,4\} \in \mathcal{L}(8,3)$, we have W_1, W_2, $W_3 = 14$, 325, 768 and $\tilde{w}(\delta) = 6$. The statistic $\tilde{w}$ is an analogue (see [25]) of a now well known partition statistic first introduced by Carlitz [2].

Theorem 1.5. *The generating function*

$$\tilde{L}_q(n,k) := \sum_{\delta \in \mathcal{L}(n,k)} q^{\tilde{w}(\delta)} = \frac{n!}{k!} \binom{n-1}{k-1}_q, \quad 1 \leqslant k \leqslant n. \tag{1.23}$$

Proof. Given $n_1, \ldots, n_k$ in $\mathbb{P}$ with $\sum n_i = n$, there are $\binom{n}{k}(n-k)!$ members of $\mathcal{L}(n,k)$ whose corresponding words $W_1, \ldots, W_k$, arranged by increasing order of initial elements, satisfy $|W_i| = n_i$ for all $i \in [k]$, as there are $\binom{n}{k}$ ways to choose and place the initial elements and $(n-k)!$ ways to place the remaining elements. By (1.22) and (1.16), it follows that

$$\begin{aligned} \sum_{\delta \in \mathcal{L}(n,k)} q^{\tilde{w}(\delta)} &= \binom{n}{k}(n-k)! \sum_{\substack{n_1+\cdots+n_k=n \\ n_i \in \mathbb{P}}} q^{0(n_1-1)+1(n_2-1)+\cdots+(k-1)(n_k-1)} \\ &= \frac{n!}{k!}\binom{n-1}{k-1}_q. \end{aligned}$$

□

Theorem 1.6. *If* $1 \leqslant k \leqslant n$, *then*

$$\tilde{L}_{-1}(n,k) = \begin{cases} 0, & \text{if } n \text{ is odd and } k \text{ is even;} \\ \frac{n!}{k!}\binom{\lfloor (n-1)/2 \rfloor}{\lfloor (k-1)/2 \rfloor}, & \text{otherwise.} \end{cases} \tag{1.24}$$

Proof. Formula (1.24) is an immediate consequence of (1.23) and (1.9). Alternatively, with $\mathcal{L}_i(n,k) := \{\delta \in \mathcal{L}(n,k) : \tilde{w}(\delta) \equiv i \pmod 2\}$, we have $\tilde{L}_{-1}(n,k) = |\mathcal{L}_0(n,k)| - |\mathcal{L}_1(n,k)|$. To prove (1.24), it thus suffices to identify a subset $\mathcal{L}_0^+(n,k)$ of $\mathcal{L}_0(n,k)$ whose cardinality agrees with the right-hand side of (1.24), along with a $\tilde{w}$-parity changing involution of $\mathcal{L}(n,k) - \mathcal{L}_0^+(n,k)$.

The set $\mathcal{L}_0^+(n,k)$ consists of those distributions whose associated sequences $W_1, W_2, \ldots, W_k$ satisfy

$$|W_{2i-1}| \text{ is odd and } |W_{2i}| = 1, \quad 1 \leqslant i \leqslant \lfloor k/2 \rfloor. \tag{1.25}$$

Clearly, $\mathcal{L}_0^+(n,k) = \varnothing$ if n is odd and k is even. In the remaining cases, the factor $n!/k!$ arises just as in the proof of Theorem 1.5, and

$$\binom{\lfloor (n-1)/2 \rfloor}{\lfloor (k-1)/2 \rfloor} = \Big| \Big\{ (n_1, \ldots, n_k) : \sum n_i = n,\ n_{2i-1} \text{ is odd}, \\ \text{and } n_{2i} = 1,\ 1 \leqslant i \leqslant \lfloor k/2 \rfloor \Big\} \Big| . \tag{1.26}$$

Suppose now that $\delta \in \mathcal{L}(n,k) - \mathcal{L}_0^+(n,k)$ has the associated sequence $W_1, \ldots, W_k$ and that i_0 is the smallest index for which (1.25) fails to hold. If $|W_{2i_0-1}|$ is even, take the last member of W_{2i_0-1} and place it at the end of W_{2i_0}. If $|W_{2i_0-1}|$ is odd, whence $|W_{2i_0-1}| \geqslant 2$, take the last member of W_{2i_0} and place it at the end of W_{2i_0-1}. The resulting map is a parity changing involution of $\mathcal{L}(n,k) - \mathcal{L}_0^+(n,k)$. □

1.4 A Refinement of a Previous Result

Recall that for a multiset permutation $w = w_1 \cdots w_n$, the major index statistic is given by

$$maj(w) := \sum_{i \in D(w)} i, \qquad \text{where } D(w) := \{1 \leqslant i \leqslant n-1 : \ w_i > w_{i+1}\}.$$

Let

$$S(a,b;k) := \{w \in S(a,b) \ : \ |D\,(w)| = k\},$$

where $S(a,b)$ is the set of binary words of length $a+b$ with a 0's.

Given $w \in S(a,b)$, we shall call a (maximal) consecutive sequence of 0's or 1's in w a *run* (of 0's or 1's). So a member of $S(a,b;k)$ will have $k+1$ or k runs of 0's depending on whether or not it starts with a 0 and will have $k+1$ or k runs of 1's depending on whether or not it ends with a 1.

Given $w \in S(a,b)$, associate the sequences $w_a = (a_1, a_2, \ldots)$ and $w_b = (b_1, b_2, \ldots)$, where a_i records the number of 0's in the i^{th} run of 0's and b_i

records the number of 1's in the i^{th} run of 1's. For members of $S(a, b; k)$, it will be convenient to think of the words w_a and w_b as sequences of length $k+1$, where the last entry is 0 if there is no $(k+1)^{\text{st}}$ run. The members of $S(a, b; k)$ are then characterized by sequences w_a and w_b such that

$$\begin{array}{rl} \text{(i)} & a_1 + \cdots + a_{k+1} = a, \quad a_i \in \mathbb{P}, \quad 1 \leqslant i \leqslant k, \quad a_{k+1} \in \mathbb{N}; \\ \text{(ii)} & b_1 + \cdots + b_{k+1} = b, \quad b_i \in \mathbb{P}, \quad 1 \leqslant i \leqslant k, \quad b_{k+1} \in \mathbb{N}. \end{array} \tag{1.27}$$

From (1.27), it follows that

$$|S\,(a, b; k)| = \binom{a}{k}\binom{b}{k}. \tag{1.28}$$

The "problem of the runs" occurring in [8, p. 42] and [17, p. 47] is a closely related problem.

Fürlinger and Hofbauer [10] show that

$$M_q(a, b; k) := \sum_{w \in S(a,b;k)} q^{maj(w)} = q^{k^2} \binom{a}{k}_q \binom{b}{k}_q, \tag{1.29}$$

which is a q-generalization of (1.28).

Theorem 1.7. *For all* $a, b, k \in \mathbb{N}$,

$$M_{-1}(a, b; k) = \begin{cases} 0, & \textit{if } a \textit{ or } b \textit{ is even and } k \textit{ is odd;} \\ (-1)^k \binom{\lfloor a/2 \rfloor}{\lfloor k/2 \rfloor}\binom{\lfloor b/2 \rfloor}{\lfloor k/2 \rfloor}, & \textit{otherwise.} \end{cases} \tag{1.30}$$

Proof. Formula (1.30) is an immediate consequence of (1.29) and (1.9). Alternatively, let $S^{\pm}(a, b; k)$ consist of those members of $S(a, b; k)$ with even or odd major index value, respectively. In each case, we shall identify a subset $S^*(a, b; k)$ of $S(a, b; k)$ whose net weight matches the right-hand side of (1.30) as well as a maj-parity changing involution of $S(a, b; k) - S^*(a, b; k)$.

Given $w \in S(a, b; k)$, consider the two sequences w_a and w_b of length $k+1$ as described above (see (1.27)). First assume k is even. Let $S^*(a, b; k)$ consist of those words w whose associated sequences w_a and w_b satisfy

$$\begin{array}{rl} \text{(a)} & a_{2i-1} = 1, \quad a_{2i} \text{ odd}, \quad 1 \leqslant i \leqslant k/2; \\ \text{(b)} & b_{2i-1} = 1, \quad b_{2i} \text{ odd}, \quad 1 \leqslant i \leqslant k/2. \end{array} \tag{1.31}$$

Checking separately the four cases with regard to the parity of a and b shows that $|S^*(a,b;k)| = \binom{\lfloor a/2\rfloor}{k/2}\binom{\lfloor b/2\rfloor}{k/2}$ in each case with $S^*(a,b;k) \subseteq S^+(a,b;k)$.

Suppose now that $w \in S(a,b;k) - S^*(a,b;k)$ and that i_0 is the smallest index i for which (a) or (b) fails to hold in (1.31). If both (a) and (b) fail when $i = i_0$, consider only (a). In any event, we'll refer to the appropriate pair of disqualifying runs of the same type simply as run $2i_0 - 1$ and run $2i_0$. If run $2i_0$ is of even length, move a single character forward to run $2i_0 - 1$. If run $2i_0$ is of odd length, whence run $2i_0 - 1$ has length of at least two, move a single character from run $2i_0 - 1$ to run $2i_0$.

Next assume k is odd. Let $S^*(a,b;k)$ consist of those words w whose associated sequences w_a and w_b satisfy

$$\begin{array}{ll} \text{(a)} & a_{2i-1} = 1,\ \ a_{2i} \text{ odd},\ \ 1 \leqslant i \leqslant (k-1)/2; \\ \text{(b)} & b_{2i-1} = 1,\ \ b_{2i} \text{ odd},\ \ 1 \leqslant i \leqslant (k-1)/2; \\ \text{(c)} & b_k \text{ odd},\ \ \ b_{k+1} = 0 \text{ with either } a_{k+1} = 0 \text{ or} \\ & a_k = 1,\ \ \ a_{k+1} \text{ odd}. \end{array} \tag{1.32}$$

If b is even, then $S^*(a,b;k) = \varnothing$ as (b) and (c) in (1.32) cannot hold simultaneously. If b is odd and a is even, then $S^*(a,b;k)$ contains $\binom{a/2-1}{(k-1)/2}\binom{(b-1)/2}{(k-1)/2}$ positive and negative members which we pair as follows:

(i) first switch the i^{th} run of 1's with the i^{th} run of 0's for $1 \leqslant i \leqslant k-1$,

(ii) if $a_k = 1$, a_{k+1} odd, merge the two runs and place after the k^{th} run of 1's; if $a_{k+1} = 0$, whence a_k is even, take one of the 0's of the k^{th} run and place it directly in front of the k^{th} run of 1's.

If a and b are both odd, then $S^*(a,b;k)$ contains $\binom{(a-1)/2}{(k-1)/2}\binom{(b-1)/2}{(k-1)/2}$ negative numbers.

Suppose now $w \in S(a,b;k) - S^*(a,b;k)$. First assume w_a fails to satisfy (1.32)(a) or w_b fails to satisfy (1.32)(b). Pair w with another member of $S(a,b;k) - S^*(a,b;k)$ of opposite parity exactly as described above when k was even. So assume w_a satisfies (1.32)(a) and w_b satisfies (1.32)(b). Then (1.32)(c) fails to hold for w_a or w_b. We'll consider two subcases: (I) w_a fails (1.32)(c); (II) w_a satisfies (1.32)(c), while w_b fails (1.32)(c). Under (I), either $a_{k+1} \geqslant 2$ even or a_{k+1} odd, $a_k \geqslant 2$, while under (II), either b_k even or b_k odd, $b_{k+1} \geqslant 1$. For (I), move a single 0 forward from the $(k+1)^{\text{st}}$ run to the k^{th} run of 0's if $a_{k+1} \geqslant 2$ even or move a single 0 from the k^{th} run to the $(k+1)^{\text{st}}$ run if a_{k+1} odd with $a_k \geqslant 2$. For (II), move a single 1 forward from the

$(k+1)^{\text{st}}$ run to the k^{th} run of 1's if b_k is odd or take a single 1 from the k^{th} run and place it at the end of the entire sequence if b_k is even. In all cases, the resulting map is a parity changing involution of $S(a,b;k) - S^*(a,b;k)$. □

Remark. Note that Theorem 1.7 is a refinement of Corollary 1.2. Indeed, summing (1.29) over $k \geqslant 0$ and using the q-Vandermonde identity yields the q-binomial coefficient $\binom{a+b}{a}_q$.

Chapter 2

Parity Theorems for Statistics on Permutations and Catalan Words

2.1 Introduction

Recall that the inversion and major index statistics for a word $w = w_1w_2\cdots w_m$ in some alphabet consisting of integers are given by

$$inv(w) := \left|\{(i,j):\ i<j \text{ and } w_i > w_j\}\right|,$$

and

$$maj(w) := \sum_{i\in D(w)} i, \qquad \text{where } D(w) := \{1\leqslant i\leqslant m-1:\ w_i > w_{i+1}\}.$$

We establish parity theorems for statistics on the symmetric group S_n, the derangements D_n, and the Catalan words C_n, giving both algebraic and bijective proofs. Most of the statistics involve counting inversions or finding the major index of various words.

In 2.2, we establish parity theorems for several permutation statistics defined on all of S_n, algebraically by evaluating q-generating functions at $q = -1$ and combinatorially by identifying appropriate parity changing involutions. In 2.3, we analyze the parity of some statistics on D_n, the set of derangements of $[n]$ (i.e., permutations of $[n]$ having no fixed points).

In Chapter 1, we derived parity theorems for the inversion and major index statistics on binary words of length n with k 1's. In 2.4, we obtain comparable results for C_n, the set of binary words of length $2n$ with n 1's and with no initial segment containing more 1's than 0's (termed *Catalan words*).

Most of the material in this chapter appears in [19] in revised form.

2.2 Permutation Statistics

2.2.1 Some Balanced Permutation Statistics

Let S_n be the set of permutations of $[n]$. A function $f : S_n \to \mathbb{N}$ is called a permutation statistic. Two important permutation statistics are *inv* and *maj*, which record the number of inversions and the major index, respectively, of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, expressed as a word. The statistics *inv* and *maj* have the same q-generating function over S_n:

$$\sum_{\sigma\in S_n} q^{inv(\sigma)} = n_q^! = \sum_{\sigma\in S_n} q^{maj(\sigma)}, \tag{2.1}$$

[23, Corollary 1.3.10] and [1, Corollary 3.8].

Substituting $q = -1$ into (2.1) reveals that $n_{-1}^! = 0$ if $n \geqslant 2$, and hence *inv* and *maj* are both balanced if $n \geqslant 2$. Interchanging σ_1 and σ_2 in $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ changes both the *inv* and *maj* values by one and thus furnishes an appropriate involution. Note that switching the elements 1 and 2 in σ changes the *inv*-parity, but not necessarily the *maj*-parity.

Now express $\sigma \in S_n$ in the *standard cycle form*

$$\sigma = (\alpha_1)(\alpha_2)\cdots,$$

where $\alpha_1, \alpha_2, \ldots$ are the cycles of σ, ordered by increasing smallest elements with each cycle (α_i) written with its smallest element in the first position. Let $S_{n,k}$ denote the set of permutations of $[n]$ with k cycles and $c(n,k) := |S_{n,k}|$, the signless Stirling number of the first kind. The $c(n,k)$ are connection constants in the polynomial identities

$$q(q+1)\cdots(q+n-1) = \sum_{k=0}^{n} c(n,k)q^k. \tag{2.2}$$

Setting $q=-1$ in (2.2) reveals that there are as many permutations of $[n]$ with an even number of cycles as there are with an odd number of cycles if $n \geqslant 2$. Alternatively, breaking apart or merging α_1 and α_2 as shown below, leaving the other cycles undisturbed, changes the parity of the number of cycles:

$$\alpha_1 = (1 \cdots 2 \cdots), \ldots \;\leftrightarrow\; \alpha_1 = (1 \cdots),\; \alpha_2 = (2 \cdots), \ldots.$$

This involution also shows that the statistic recording the number of cycles of σ with even cardinality is balanced if $n \geqslant 2$.

Given $\sigma = (\alpha_1)(\alpha_2)\cdots$, expressed in standard cycle form, let

$$w(\sigma) := \sum_i (i-1)|\alpha_i|.$$

Edelman, Simion, and White [6] show that

$$\sum_{\sigma \in S_n} x^{|\sigma|} q^{w(\sigma)} = \prod_{i=0}^{n-1} (xq^i + i), \tag{2.3}$$

where $|\sigma|$ denotes the number of cycles. Setting $x=1$ in (2.3) yields

$$\sum_{\sigma \in S_n} q^{w(\sigma)} = \prod_{i=0}^{n-1} (q^i + i), \tag{2.4}$$

another q-generalization of $n!$.

Setting $q=-1$ in (2.4) shows that the w statistic is balanced if $n \geqslant 2$. Alternatively, if the last cycle has cardinality greater than one, break off the last member and form a 1-cycle with it; if the last cycle contains a single member, place it at the end of the penultimate cycle.

2.2.2 An Unbalanced Permutation Statistic

Carlitz [2] defines the statistic inv_c on S_n as follows: express $\sigma \in S_n$ in standard cycle form; then remove parentheses and count inversions in the resulting word to obtain $inv_c(\sigma)$. As an illustration, for the permutation $\sigma \in S_7$ given by 3241756, we have $inv_c(\sigma) = 3$, the number of inversions in the word 1342576.

Let

$$c_q(n,k) := \sum_{\sigma \in S_{n,k}} q^{inv_c(\sigma)}, \tag{2.5}$$

where $S_{n,k}$ is the set of permutations of $[n]$ with k cycles. Then $c_q(n,0) = \delta_{n,0},\ c_q(0,k) = \delta_{0,k}$, and

$$c_q(n,k) = c_q(n-1,k-1) + (n-1)_q c_q(n-1,k),\ \forall n,k \in \mathbb{P}, \tag{2.6}$$

since n may go in a cycle by itself or come directly after any member of $[n-1]$ within a cycle.

Using (2.6), it is easy to show that

$$x(x+1_q)\cdots(x+(n-1)_q) = \sum_{k=0}^{n} c_q(n,k)x^k. \tag{2.7}$$

Setting $x=1$ in (2.7) gives

$$c_q(n) := \sum_{k=0}^{n} c_q(n,k) = \sum_{\sigma \in S_n} q^{inv_c(\sigma)} = \prod_{j=0}^{n-1}(1+j_q). \tag{2.8}$$

Theorem 2.1. *For all $n \in \mathbb{N}$,*

$$c_{-1}(n) := \sum_{\sigma \in S_n} (-1)^{inv_c(\sigma)} = 2^{\lfloor n/2 \rfloor}. \tag{2.9}$$

Proof. Put $q=-1$ in (2.8) and note that

$$j_q|_{q=-1} = \begin{cases} 0, & \text{if } j \text{ is even;} \\ 1, & \text{if } j \text{ is odd.} \end{cases}$$

Alternatively, with S_n^+, S_n^- denoting the members of S_n with even or odd inv_c values, respectively, we have $c_{-1}(n) = |S_n^+| - |S_n^-|$. To prove (2.9), it thus suffices to identify a subset S_n^* of S_n^+ such that $|S_n^*| = 2^{\lfloor n/2 \rfloor}$, along with an inv_c-parity changing involution of $S_n - S_n^*$.

First assume n is even. In this case, the set S_n^* consists of those permutations expressible in standard cycle form as a product of 1-cycles and the transpositions $(2i-1,\ 2i)$, $1 \leqslant i \leqslant n/2$. Note that $S_n^* \subseteq S_n^+$ with zero inv_c value for each of its $2^{n/2}$ members.

Before giving the involution on $S_n - S_n^*$, we make a definition: given $\sigma = (\alpha_1)(\alpha_2)\cdots \in S_m$ in standard cycle form and j, $1 \leqslant j \leqslant m$, let $\sigma_{[j]}$ be the permutation of $[j]$ (in standard cycle form) obtained by writing the members of $[j]$ in the order as they appear within the cycles of σ (e.g., if $\sigma = (163)(25)(4)(7) \in S_7$ and $j = 4$, then $\sigma_{[4]} = (13)(2)(4)$ and $\sigma_{[7]} = \sigma$).

Suppose now $\sigma \in S_n - S_n^*$ is expressed in standard cycle form and that i_0 is the smallest integer i, $1 \leqslant i \leqslant n/2$, for which $\sigma_{[2i]} \in S_{2i} - S_{2i}^*$. Then it must be the case for σ that

(i) neither $2i_0 - 1$ nor $2i_0$ starts a cycle, or

(ii) exactly one of $2i_0 - 1$, $2i_0$ starts a cycle with $2i_0 - 1$ and $2i_0$ not belonging to the same cycle.

Switching $2i_0 - 1$ and $2i_0$ within σ, written in standard cycle form, changes the inv_c value by one, and the resulting map is thus a parity changing involution of $S_n - S_n^*$.

If n is odd, let $S_n^* \subseteq S_n^+$ consist of those permutations expressible as a product of 1-cycles and the transpositions $(2i, 2i+1)$, $1 \leqslant i \leqslant \frac{n-1}{2}$. Switch $2i_0$ and $2i_0 + 1$ within $\sigma \in S_n - S_n^*$, where i_0 is the smallest i, $1 \leqslant i \leqslant \frac{n-1}{2}$, for which $\sigma_{[2i+1]} \in S_{2i+1} - S_{2i+1}^*$. □

The preceding parity theorem has the refinement

Theorem 2.2. *For all $n \in \mathbb{N}$,*

$$c_{-1}(n,k) := \sum_{\sigma \in S_{n,k}} (-1)^{inv_c(\sigma)} = \binom{\lfloor n/2 \rfloor}{n-k}, \qquad 0 \leqslant k \leqslant n. \tag{2.10}$$

Proof. Set $q = -1$ in (2.7) to get

$$\sum_{k=0}^{n} c_{-1}(n,k)x^k = x^{\lceil n/2 \rceil}(x+1)^{\lfloor n/2 \rfloor} = \sum_{k=\lceil n/2 \rceil}^{n} \binom{\lfloor n/2 \rfloor}{n-k} x^k.$$

Or let $S_{n,k}^{\pm} := S_{n,k} \cap S_n^{\pm}$ and $S_{n,k}^* := S_{n,k} \cap S_n^*$. Then $S_{n,k}^* \subseteq S_{n,k}^+$ and its cardinality agrees with the right-hand side of (2.10). The restriction of the map used for Theorem 2.1 to $S_{n,k} - S_{n,k}^*$ is again an involution and inherits the parity changing property. □

Remark. The bijection of Theorem 2.2 also proves combinatorially that

$$c(n,k) \equiv \binom{\lfloor n/2 \rfloor}{n-k} \pmod 2, \qquad 0 \leqslant k \leqslant n, \tag{2.11}$$

since off of a set of cardinality $\binom{\lfloor n/2 \rfloor}{n-k}$, each permutation $\sigma \in S_{n,k}$ is paired with another of opposite inv_c-parity. The congruences in (2.11) can also be obtained by taking mod 2 the polynomial identities in (2.2) (cf. [23, p. 46, Exercise 17c]).

2.3 Some Statistics for Derangements

A permutation σ of $[n]$ having no fixed points (i.e., $i \in [n]$ such that $\sigma(i) = i$) is called a derangement. Let D_n denote the set of derangements of $[n]$ and $d_n := |D_n|$. A typical inclusion-exclusion argument gives the well known formula

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \qquad \forall n \in \mathbb{N}. \tag{2.12}$$

Given $\sigma \in D_n$, express it in the form

$$\sigma = (\alpha_1)(\alpha_2)\cdots,$$

where $\alpha_1, \alpha_2, \ldots$ are the cycles of σ arranged as follows:

(i) the cycles $\alpha_1, \alpha_2, \ldots$ are ordered by increasing second smallest elements;

(ii) each cycle (α_i) is written with the second smallest element in the last position.

Garsia and Remmel [11] term this the *ordered cycle factorization* (OCF for brief) of σ.

Define the statistic inv_o on D_n as follows: write out the cycles of $\sigma \in D_n$ in OCF form; then remove parentheses and count inversions in the resulting word to obtain $inv_o(\sigma)$. As an illustration, for the derangement $\sigma \in D_7$ given by 4321756, we have $inv_o(\sigma) = 3$, the number of inversions in the word 2314576.

The statistic inv_o is due to Garsia and Remmel [11], who show that the generating function

$$D_q(n) := \sum_{\sigma \in D_n} q^{inv_o(\sigma)} = n_q^! \sum_{k=0}^{n} \frac{(-1)^k}{k_q^!}, \qquad \forall n \in \mathbb{N}, \tag{2.13}$$

which generalizes (2.12).

Theorem 2.3. *For all* $n \in \mathbb{N}$,

$$D_{-1}(n) = \begin{cases} 1, & \textit{if } n \textit{ is even;} \\ 0, & \textit{if } n \textit{ is odd.} \end{cases} \tag{2.14}$$

Proof. Formula (2.14) is an immediate consequence of (2.13), for

$$\sum_{k=0}^{n} \frac{(-1)^k n_q^!}{k_q^!}\Bigg|_{q=-1} = \sum_{k=0}^{n} (-1)^k \prod_{i=k+1}^{n} i_q \Bigg|_{q=-1} = (-1)^{n-1} n_{-1} + (-1)^n,$$

as

$$j_{-1} = \begin{cases} 0, & \text{if } j \text{ is even;} \\ 1, & \text{if } j \text{ is odd.} \end{cases}$$

Alternatively, let $\sigma = (\alpha_1)(\alpha_2)\cdots \in D_n$ be expressed in OCF form, first assuming n is odd. Locate the leftmost cycle of σ containing at least three members and interchange the first two members of this cycle. Now assume n is even. If σ has a cycle of length greater than two, proceed as in the odd case. If all cycles of σ are transpositions and $\sigma \neq (1,2)(3,4)\cdots(n-1,n)$, let i_0 be the smallest integer i for which the transposition $(2i-1, 2i)$ fails to occur in σ. Switch $2i_0 - 1$ and $2i_0$ in σ, noting that $2i_0 - 1$ and $2i_0$ must both start cycles. Thus whenever n is even, every $\sigma \in D_n$ is paired with another of opposite inv_o-parity except for $(1,2)(3,4)\cdots(n-1,n)$, which has inv_o value zero. □

Now consider the generating function $d_q(n)$ resulting when one restricts inv to D_n, i.e.,

$$d_q(n) := \sum_{\sigma \in D_n} q^{inv(\sigma)}. \tag{2.15}$$

We have been unable to find a simple formula for $d_q(n)$ that generalizes (2.12) or a recurrence satisfied by $d_q(n)$ that generalizes one for d_n. However, we do have the following parity result.

Theorem 2.4. *For all $n \in \mathbb{N}$,*

$$d_{-1}(n) = (-1)^{n-1}(n-1). \tag{2.16}$$

Proof. Equivalently, we show that the numbers $d_{-1}(n)$ satisfy

$$d_{-1}(n) = -d_{-1}(n-1) + (-1)^{n-1}, \qquad \forall n \in \mathbb{P}, \tag{2.17}$$

with $d_{-1}(0) = 1$. Let $n \geqslant 2$, $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in D_n$, and $D_n^* \subseteq D_n$ consist of those derangements σ for which $\sigma_1 = 2$ and $\sigma_2 \geqslant 3$. Define an *inv*-parity changing involution f on $D_n - D_n^* - \{n12\cdots n-1\}$ as follows:

(i) if $\sigma_2 \geqslant 3$, whence $\sigma_1 \geqslant 3$, switch 1 and 2 in σ to obtain $f(\sigma)$;

(ii) if $\sigma_2 = 1$, let k_0 be the smallest integer k, $1 \leqslant k \leqslant \lfloor \frac{n-1}{2} \rfloor$, such that $\sigma_{2k}\sigma_{2k+1} \neq (2k-1)(2k)$; switch $2k_0$ and $2k_0+1$ if $\sigma_{2k_0} = 2k_0 - 1$ or switch $2k_0 - 1$ and $2k_0$ if $\sigma_{2k_0} \geqslant 2k_0 + 1$ to obtain $f(\sigma)$.

Thus,

$$d_{-1}(n) := \sum_{\sigma \in D_n} (-1)^{inv(\sigma)} = \sum_{\sigma \in D_n^* \cup \{n12\cdots n-1\}} (-1)^{inv(\sigma)}. \tag{2.18}$$

One can regard members σ of D_n^* as 2 followed by a derangement of $[n-1]$ since within the terminal segment $\sigma' := \sigma_2\sigma_3\cdots\sigma_n$, we must have $\sigma_2 \neq 1$ and $\sigma_k \neq k$ for all $k \geqslant 3$. Thus,

$$\sum_{\sigma' : \sigma \in D_n^*} (-1)^{inv(\sigma')} = d_{-1}(n-1),$$

from which

$$\sum_{\sigma \in D_n^*} (-1)^{inv(\sigma)} = -d_{-1}(n-1), \tag{2.19}$$

since the initial 2 adds an inversion. The recurrence (2.17) follows immediately from (2.18) and (2.19) upon adding the contribution of $(-1)^{n-1}$ from the singleton $\{n12\cdots n-1\}$. □

Now consider the generating function $r_q(n)$ resulting when one restricts *maj* to D_n, i.e.,

$$r_q(n) := \sum_{\sigma \in D_n} q^{maj(\sigma)}. \tag{2.20}$$

We were unable to find a simple formula for $r_q(n)$ which generalizes (2.12). Yet when $q = -1$ we have

Theorem 2.5. *For all $n \in \mathbb{N}$,*

$$r_{-1}(n) = \begin{cases} (-1)^{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{2.21}$$

Proof. First verify (2.21) for $0 \leqslant n \leqslant 3$. Let $n \geqslant 4$ and $D_n^* \subseteq D_n$ consist of those derangements starting with 2143 when expressed as a word. We define a *maj*-parity changing involution of $D_n - D_n^*$ below. Note that for derangements of the form $\sigma = 2143\sigma_5 \cdots \sigma_n$, the subword $\sigma_5 \cdots \sigma_n$ is itself a derangement on $n-4$ elements. Thus for $n \geqslant 4$,

$$r_{-1}(n) := \sum_{\sigma \in D_n} (-1)^{maj(\sigma)} = \sum_{\sigma \in D_n^*} (-1)^{maj(\sigma)} = r_{-1}(n-4),$$

which proves (2.21).

We now define a *maj*-parity changing involution f of $D_n - D_n^*$ when $n \geqslant 4$. Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in D_n - D_n^*$ be expressed as a word. If possible, pair σ with $\sigma' = f(\sigma)$ according to (I) and (II) below:

(I) first, if both $\sigma_1 \neq 2$ and $\sigma_2 \neq 1$, then switch σ_1 and σ_2 within σ to obtain σ';

(II) if (I) cannot be implemented (i.e., $\sigma_1 = 2$ or $\sigma_2 = 1$) but $\sigma_3 \neq 4$ and $\sigma_4 \neq 3$, then switch σ_3 and σ_4 within σ to obtain σ'.

We now define f for the cases that remain. To do so, consider $S_\sigma := \sigma_1\sigma_2\sigma_3\sigma_4 \cap [4]$, where $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in D_n - D_n^*$ is of a form not covered by rules (I) and (II) above. We consider cases depending upon $|S_\sigma|$. If $|S_\sigma| = 2$ or if $|S_\sigma| = 4$, first multiply σ by the transposition (34) and then exchange the letters in the third and fourth positions to obtain σ'. This corresponds to the pairings

i) $\sigma = a1b3\ldots4\ldots \;\leftrightarrow\; \sigma' = a14b\ldots3\ldots$;

ii) $\sigma = 2ab3\ldots4\ldots \;\leftrightarrow\; \sigma' = 2a4b\ldots3\ldots$;

iii) $\sigma = 2341\ldots \;\leftrightarrow\; \sigma' = 2413\ldots$;

iv) $\sigma = 4123\ldots \;\leftrightarrow\; \sigma' = 3142\ldots$,

where $a, b \geqslant 5$.

If $|S_\sigma| = 3$, then pair according to one of six cases shown below where $a \geqslant 5$, leaving the other letters undisturbed:

i) $\sigma = 314a\ldots \leftrightarrow \sigma' = 41a3\ldots$;

ii) $\sigma = 234a\ldots \leftrightarrow \sigma' = 24a3\ldots$;

iii) $\sigma = a123\ldots \leftrightarrow \sigma' = 2a13\ldots$;

iv) $\sigma = a142\ldots \leftrightarrow \sigma' = 2a41\ldots$;

v) $\sigma = 21a3\ldots4\ldots \leftrightarrow \sigma' = 214a\ldots3\ldots$;

vi) $\sigma = a143\ldots2\ldots \leftrightarrow \sigma' = 2a43\ldots1\ldots$.

It is easy to verify that σ and σ' have opposite *maj*-parity in all cases. ☐

2.4 Statistics for Catalan Words

The Catalan numbers c_n are defined by the closed form

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \in \mathbb{N}, \tag{2.22}$$

as well as by the recurrence

$$c_{n+1} = \sum_{j=0}^{n} c_j c_{n-j}, \qquad c_0 = 1. \tag{2.23}$$

If one defines the generating function

$$f(x) = \sum_{n\geqslant 0} c_n x^n, \tag{2.24}$$

then (2.23) is equivalent to

$$f(x) = 1 + xf(x)^2. \tag{2.25}$$

Due to (2.23), the Catalan numbers enumerate many combinatorial structures, among them the set C_n consisting of words $w = w_1w_2\cdots w_{2n}$ of n 1's and n 0's for which no initial segment contains more 1's than 0's (termed

Catalan words). In this section, we'll look at two q-analogues of the Catalan numbers, one of Carlitz which generalizes (2.25) and another of MacMahon which generalizes (2.22), when $q = -1$. These q-analogues arise as generating functions for statistics on C_n.

If

$$\tilde{C}_q(n) := \sum_{w \in C_n} q^{inv(w)}, \tag{2.26}$$

then

$$\tilde{C}_q(n+1) = \sum_{k=0}^{n} q^{(k+1)(n-k)} \tilde{C}_q(k) \tilde{C}_q(n-k), \qquad \tilde{C}_q(0) = 1, \tag{2.27}$$

upon decomposing a Catalan word $w \in C_{n+1}$ into $w = 0w_1 1 w_2$ with $w_1 \in C_k$, $w_2 \in C_{n-k}$ for some k, $0 \leqslant k \leqslant n$, and noting that the number of inversions of w is given by

$$inv(w) = inv(w_1) + inv(w_2) + (k+1)(n-k).$$

Taking reciprocal polynomials of both sides of (2.27) and writing

$$C_q(n) = q^{\binom{n}{2}} \tilde{C}_{q^{-1}}(n) \tag{2.28}$$

yields the recurrence [10]

$$C_q(n+1) = \sum_{k=0}^{n} q^k C_q(k) C_q(n-k), \qquad C_q(0) = 1. \tag{2.29}$$

If one defines the generating function

$$f(x) = \sum_{n \geqslant 0} C_q(n) x^n, \tag{2.30}$$

then (2.29) is equivalent to the functional equation [3, 10]

$$f(x) = 1 + x f(x) f(qx), \tag{2.31}$$

which generalizes (2.25).

Theorem 2.6. *For all $n \in \mathbb{N}$,*

$$C_{-1}(n) = \begin{cases} \delta_{n,0}, & \textit{if } n \textit{ is even;} \\ (-1)^{\frac{n-1}{2}} c_{\frac{n-1}{2}}, & \textit{if } n \textit{ is odd.} \end{cases} \tag{2.32}$$

Proof. Setting $q = -1$ in (2.31) gives

$$f(x) = 1 + xf(x)f(-x). \tag{2.33}$$

Putting $-x$ for x in (2.33), solving the resulting system in $f(x)$ and $f(-x)$, and noting $f(0) = 1$ yields

$$\begin{aligned} f(x) &= \sum_{n \geqslant 0} C_{-1}(n)x^n \\ &= \frac{(2x-1) + \sqrt{4x^2+1}}{2x} = 1 + \sum_{n \geqslant 1} (-1)^{n-1} \frac{1}{n} \binom{2n-2}{n-1} x^{2n-1}, \end{aligned}$$

which implies (2.32).

Alternatively, note that

$$C_{-1}(n) = (-1)^{\binom{n}{2}} \sum_{w \in C_n} (-1)^{inv(w)},$$

by (2.26) and (2.28). So (2.32) is equivalent to

$$\sum_{w \in C_n} (-1)^{inv(w)} = \begin{cases} \delta_{n,0}, & \text{if } n \text{ is even;} \\ c_{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases} \tag{2.34}$$

To prove (2.34), let C_n^+, $C_n^- \subseteq C_n$ consist of the Catalan words with even or odd *inv* values, respectively, and $C_n^* \subseteq C_n$ consist of those words $w = w_1 w_2 \cdots w_{2n}$ for which

$$w_{2i}w_{2i+1} = 00 \text{ or } 11, \qquad 1 \leqslant i \leqslant n-1. \tag{2.35}$$

Clearly, $C_n^* \subseteq C_n^+$ with cardinality matching the right-hand side of (2.34). Suppose $w \in C_n - C_n^*$ and that i_0 is the smallest index for which (2.35) fails to hold. Switch w_{2i_0} and w_{2i_0+1} in w. The resulting map is a parity changing involution of $C_n - C_n^*$, which proves (2.34) and hence (2.32). $\square$

Another q-Catalan number arises as the generating function for the major index statistic on C_n [15]. If

$$\tilde{c}_q(n) := \sum_{w \in C_n} q^{maj(w)}, \tag{2.36}$$

then there is the closed form (see [10], [15, p. 215])

$$\tilde{c}_q(n) = \frac{1}{(n+1)_q} \binom{2n}{n}_q, \qquad \forall n \in \mathbb{N}, \tag{2.37}$$

which generalizes (2.22).

Theorem 2.7. *For all $n \in \mathbb{N}$,*

$$\tilde{c}_{-1}(n) = \binom{n}{\lfloor n/2 \rfloor}. \tag{2.38}$$

Proof. If n is even, then by (2.37),

$$\begin{aligned} \tilde{c}_{-1}(n) &= \lim_{q \to -1} \tilde{c}_q(n) = \lim_{q \to -1} \frac{1}{(n+1)_q} \prod_{i=0}^{n-1} \frac{(2n-i)_q}{(n-i)_q} \\ &= \prod_{\substack{i=0 \\ i \text{ even}}}^{n-2} \lim_{q \to -1} \left(\frac{q^{2n-i}-1}{q^{n-i}-1} \right) = \prod_{\substack{i=0 \\ i \text{ even}}}^{n-2} \frac{2n-i}{n-i} = \prod_{\substack{i=0 \\ i \text{ even}}}^{n-2} \frac{n-i/2}{n/2-i/2} = \binom{n}{n/2}, \end{aligned}$$

with the odd case handled similarly.

Alternatively, let C_n^+, $C_n^- \subseteq C_n$ consist of the Catalan words with even or odd major index value, respectively, and $C_n^* \subseteq C_n$ consist of those words $w = w_1 w_2 \cdots w_{2n}$ which satisfy the following two requirements:

(i) one can express w as $w = x_1 x_2 \cdots x_n$, where $x_i = 00$, 11, or 01, $1 \leqslant i \leqslant n$;

(ii) for each i, $x_i = 01$ only if the number of 00's in the initial segment $x_1 x_2 \cdots x_{i-1}$ equals the number of 11's. (A word in C_n^* may start with either 01 or 00.)

Clearly, $C_n^* \subseteq C_n^+$ and below it is shown that $|C_n^*| = \binom{n}{\lfloor n/2 \rfloor}$. Suppose $w = w_1 w_2 \cdots w_{2n} \in C_n - C_n^*$ and that i_0 is the smallest integer i, $1 \leqslant i \leqslant n$, such that one of the following two conditions holds:

(i) $w_{2i-1}w_{2i} = 10$, or

(ii) $w_{2i-1}w_{2i} = 01$ and the number of 0's in the initial segment $w_1w_2\cdots w_{2i-2}$ is strictly greater than the number of 1's.

Switching w_{2i_0-1} and w_{2i_0} in w changes the major index by an odd amount and the resulting map is a parity changing involution of $C_n - C_n^*$.

We now show $|C_n^*| = \binom{n}{\lfloor n/2 \rfloor}$ by defining a bijection between C_n^* and the set $\Lambda(n)$ of (minimal) lattice paths from $(0,0)$ to $(\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$. Given $w = x_1x_2\cdots x_n \in C_n^*$ as described in (i) and (ii) above, we construct a lattice path $\lambda_w \in \Lambda(n)$ as follows. Let $j_1 < j_2 < \ldots$ be the set of indices j, possibly empty and denoted $S(w)$, for which $x_j = 01$, with $j_0 := 0$. For $s \geqslant 1$, let step j_s in λ_w be a V (vertical step) if s is odd and an H (horizontal step) if s is even.

Suppose now $i \in [n] - S(w)$ and that t, $t \geqslant 0$, is the greatest integer such that $j_t < i$. If t is even, put a V (resp., H) for the i^{th} step of λ_w if $x_i = 11$ (resp., 00). If t is odd, put a V (resp., H) for the i^{th} step of λ_w if $x_i = 00$ (resp., 11), which now specifies λ_w completely. The map $w \mapsto \lambda_w$ is seen to be a bijection between C_n^* and $\Lambda(n)$; note that $S(w)$ corresponds to the steps of λ_w in which it either rises above the line $y = x$ or returns to $y = x$ from above. □

Note that the preceding supplies a combinatorial proof of the congruence $\frac{1}{n+1}\binom{2n}{n} \equiv \binom{n}{\lfloor n/2 \rfloor} \pmod 2$ for $n \in \mathbb{N}$ since off of a set of cardinality $\binom{n}{\lfloor n/2 \rfloor}$, each Catalan word $w \in C_n$ is paired with another of opposite *maj*-parity. We also have for $n \in \mathbb{P}$ that c_n is even if n is even and $c_n \equiv c_{\frac{n-1}{2}} \pmod 2$ if n is odd, by (2.34). Repeated use of this yields the well known fact [22] that the n^{th} Catalan number c_n is odd if and only if $n = 2^m - 1$ for some $m \in \mathbb{N}$. By the first congruence noted, the same is true for the middle binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$.

Let $P_n \subseteq S_n$ consist of those permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ avoiding the pattern 312, i.e., there are no indices $i < j < k$ such that $\sigma_j < \sigma_k < \sigma_i$ (termed *Catalan permutations*). Knuth [13, p. 238] describes a bijection g between P_n and C_n in which

$$inv(\sigma) = \binom{n}{2} - inv(g(\sigma)), \qquad \forall \sigma \in P_n,$$

and hence

$$C_q(n) := \sum_{w \in C_n} q^{\binom{n}{2} - inv(w)} = \sum_{\sigma \in P_n} q^{inv(\sigma)}. \tag{2.39}$$

By (2.32) and (2.39), we then have the parity result

$$\sum_{\sigma \in P_n} (-1)^{inv(\sigma)} = \begin{cases} \delta_{n,0}, & \text{if } n \text{ is even;} \\ (-1)^{\frac{n-1}{2}} c_{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases} \tag{2.40}$$

The composite map $g^{-1} \circ h \circ g$, where h is the involution establishing (2.34), furnishes an appropriate involution for (2.40).

Chapter 3

Parity Theorems for Partition Statistics

3.1 Introduction

3.1.1 The Partition Statistics $\tilde{w}$, $\hat{w}$, w^*, w

Let $\Pi(n, k)$ denote the set of all partitions of $[n]$ with k blocks and $\Pi(n)$ the set of all partitions of $[n]$. For all $n, k \in \mathbb{N}$, let $S(n, k) := |\Pi(n, k)|$ and $B(n) := |\Pi(n)| = \sum_k S(n, k)$. The numbers $S(n, k)$ are called *Stirling numbers of the second kind* and the numbers $B(n)$ are called *Bell numbers*. Then $S(0, 0) = 1$, $S(n, 0) = S(0, k) = 0$ $\forall n, k \in \mathbb{P}$, and

$$S(n, k) = S(n-1, k-1) + kS(n-1, k), \quad \forall n, k \in \mathbb{P}, \tag{3.1}$$

upon considering whether or not n goes in a block by itself. Then $B(0) = 1$ and

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k), \quad \forall n \in \mathbb{N}, \tag{3.2}$$

since $\binom{n}{k} B(k)$ counts the partitions of $[n+1]$ for which the size of the block containing $n+1$ is $n-k+1$.

Associate to each $\pi \in \Pi(n, k)$ the unique ordered partition $(E_1, \ldots, E_k)$ of $[n]$ comprising the same blocks as Π, arranged in increasing order of their smallest elements, and define statistics $\tilde{w}$, $\hat{w}$, w^*, and w by

$$\begin{aligned}
\tilde{w}(\pi) &:= \sum_{i=1}^{k}(i-1)(|E_i|-1), && (3.3)\\
\hat{w}(\pi) &:= \sum_{i=1}^{k} i(|E_i|-1) = \tilde{w}(\pi)+n-k, && (3.4)\\
w^*(\pi) &:= \sum_{i=1}^{k} i|E_i| = \tilde{w}(\pi)+n+\binom{k}{2}, && (3.5)
\end{aligned}$$

and

$$w(\pi) := \sum_{i=1}^{k}(i-1)|E_i| = \tilde{w}(\pi)+\binom{k}{2}. \tag{3.6}$$

Consider the generating functions (see [2], [16], [24], and [25])

$$\begin{aligned}
\tilde{S}_q(n,k) &:= \sum_{\pi\in\Pi(n,k)} q^{\tilde{w}(\pi)}, && (3.7)\\
\hat{S}_q(n,k) &:= \sum_{\pi\in\Pi(n,k)} q^{\hat{w}(\pi)} = q^{n-k}\tilde{S}_q(n,k), && (3.8)\\
S^*_q(n,k) &:= \sum_{\pi\in\Pi(n,k)} q^{w^*(\pi)} = q^{\binom{k}{2}+n}\tilde{S}_q(n,k), && (3.9)
\end{aligned}$$

and

$$S_q(n,k) := \sum_{\pi\in\Pi(n,k)} q^{w(\pi)} = q^{\binom{k}{2}}\tilde{S}_q(n,k). \tag{3.10}$$

Summing the q-Stirling numbers, $\tilde{S}_q(n,k)$, $\hat{S}_q(n,k)$, $S^*_q(n,k)$, and $S_q(n,k)$, over k yields the respective q-Bell numbers, $\tilde{B}_q(n)$, $\hat{B}_q(n)$, $B^*_q(n)$, and $B_q(n)$. These polynomials reduce to the classical Stirling and Bell numbers when $q = 1$.

In section 3.2, we evaluate these polynomials when $q = -1$, giving both algebraic and bijective proofs. Our algebraic arguments parallel the general scheme presented in [26], though several of our proofs are different. Our bijective proofs are those given in [18]. In 3.3, we carry out a similar evaluation for the polynomials resulting when one restricts the w and w^* statistics to the partitions of $[n]$ whose blocks have cardinality at most two.

3.1.2 Algebraic Preliminaries

In this section, we establish some algebraic properties of the q-Stirling numbers which we'll need in the next section. We first recall a theorem, due to Comtet [5], which greatly facilitates the analysis of many combinatorial arrays.

Theorem 3.1. *Let D be an integral domain. If $(u_n)_{n\geqslant 0}$ is a sequence in D and x is an indeterminate over D, then the following are equivalent characterizations of an array $(U(n,k))_{n,k\geqslant 0}$:*

$$U(n,k) = U(n-1,k-1) + u_k U(n-1,k), \quad \forall n,k \in \mathbb{P}, \tag{3.11}$$

with $U(n,0) = u_0^n$ and $U(0,k) = \delta_{0,k}$ $\forall n,k \in \mathbb{N}$,

$$U(n,k) = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} u_0^{d_0} u_1^{d_1} \cdots u_k^{d_k}, \quad \forall n,k \in \mathbb{N}, \tag{3.12}$$

$$\sum_{n\geqslant 0} U(n,k)x^n = \frac{x^k}{(1-u_0x)(1-u_1x)\cdots(1-u_kx)}, \quad \forall k \in \mathbb{N}, \tag{3.13}$$

and

$$x^n = \sum_{k=0}^{n} U(n,k)p_k(x), \quad \forall n \in \mathbb{N}, \tag{3.14}$$

where $p_0(x) := 1$ and $p_k(x) := (x-u_0)\cdots(x-u_{k-1})$ for $k \in \mathbb{P}$.

Proof. Straightforward algebraic exercise. □

We shall call the numbers $U(n,k)$ the *Comtet numbers associated with the sequence* $(u_n)_{n\geqslant 0}$, as in [25]. By (3.1), the $S(n,k)$ are the Comtet numbers associated with the sequence $(0,1,2,\dots)$.

Theorem 3.2. *The q-Stirling numbers, $\tilde{S}_q(n,k)$, are generated by the recurrence relation*

$$\tilde{S}_q(n,k) = \tilde{S}_q(n-1,k-1) + k_q\tilde{S}_q(n-1,k), \quad \forall n,k \in \mathbb{P}, \tag{3.15}$$

with $\tilde{S}_q(0,0) = 1$ and $\tilde{S}_q(n,0) = \tilde{S}_q(0,k) = 0$, $\forall n,k \in \mathbb{P}$.

Proof. The boundary conditions are obvious. To establish the recurrence (3.15), consider the contribution of the element n to the overall $\tilde{w}$-weight of a member $\pi = (E_1, \ldots, E_k) \in \Pi(n, k)$. The sum of the weights of all members of $\Pi(n, k)$ for which n goes in a block by itself is $\tilde{S}_q(n-1, k-1)$, while the sum of the weights of all members of $\Pi(n, k)$ for which n goes in the i^{th} block E_i, $1 \leqslant i \leqslant k$, together with at least one member of $[n-1]$, is $q^{i-1}\tilde{S}_q(n-1, k)$. Summing over i, $1 \leqslant i \leqslant k$, and noting $k_q = 1+q+\cdots+q^{k-1}$ yields (3.15). □

Recurrence (3.15) reveals that the numbers $\tilde{S}_q(n, k)$ are the Comtet numbers associated with the sequence $(n_q)_{n\geqslant 0}$. By Theorem 3.1, it follows immediately that

$$\tilde{S}_q(n, k) = \sum_{\substack{d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} (1_q)^{d_1}(2_q)^{d_2}\cdots(k_q)^{d_k}, \quad \forall n, k \in \mathbb{N}, \tag{3.16}$$

$$\sum_{n\geqslant 0} \tilde{S}_q(n, k)x^n = \frac{x^k}{(1-1_q x)(1-2_q x)\cdots(1-k_q x)}, \quad \forall k \in \mathbb{N}, \tag{3.17}$$

and

$$x^n = \sum_{k=0}^{n} \tilde{S}_q(n, k)\phi_k(x), \quad \forall n \in \mathbb{N}, \tag{3.18}$$

where $\phi_0(x) := 1$ and $\phi_k(x) := x(x-1_q)\cdots(x-(k-1)_q)$, $\forall k \in \mathbb{P}$.

Variants of (3.15)–(3.18) hold for the other q-Stirling numbers and follow from relations (3.7)–(3.10). For example, we have

$$S^*_q(n, k) = q^k S^*_q(n-1, k-1) + qk_q S^*_q(n-1, k), \quad \forall n, k \in \mathbb{P}, \tag{3.19}$$

and

$$\sum_{n\geqslant 0} S^*_q(n, k)x^n = \frac{q^{\binom{k+1}{2}}x^k}{(1-qx)(1-qx-q^2x)\cdots(1-qx-\cdots-q^k x)}, \quad \forall k \in \mathbb{N}. \tag{3.20}$$

3.2 Parity Theorems for Partition Statistics

In this section, we derive simple expressions for the foregoing q-Stirling and q-Bell numbers when $q = -1$, giving both algebraic and bijective proofs.

Theorem 3.3. *For all* $n \in \mathbb{N}$,

$$\tilde{S}_{-1}(n,k) = \binom{n - \lfloor k/2 \rfloor - 1}{n-k}, \quad 0 \leqslant k \leqslant n. \tag{3.21}$$

Proof. Substituting $q = -1$ into (3.17) and noting

$$i_q\big|_{q=-1} = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even,} \end{cases}$$

yields

$$\begin{aligned} \sum_{n \geqslant 0} \tilde{S}_{-1}(n,k) x^n &= \frac{x^k}{(1-x)^{\lceil k/2 \rceil}} \\ &= \sum_{n \geqslant k} \binom{n - \lfloor k/2 \rfloor - 1}{n-k} x^n. \end{aligned}$$

Note that (3.21) can also be obtained by letting $q = -1$ in (3.16). ☐

Substituting (3.21) into (3.18) at $q = -1$ and applying the binomial theorem yields the orthogonality relation

$$\sum_{k=m}^{\min\{n,2m\}} (-1)^{k-m} \binom{n - \lfloor k/2 \rfloor - 1}{n-k} \binom{\lfloor k/2 \rfloor}{k-m} = \delta_{n,m}, \quad 0 \leqslant m \leqslant n. \tag{3.22}$$

Let $F_0 = F_1 = 1$ with $F_n = F_{n-1} + F_{n-2}$ if $n \geqslant 2$. As is well known,

$$F_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}, \quad \forall n \in \mathbb{N}. \tag{3.23}$$

Theorem 3.4. *For all* $n \in \mathbb{N}$,

$$\tilde{B}_{-1}(n) := \sum_{k=0}^{n} \tilde{S}_{-1}(n,k) = F_n. \tag{3.24}$$

Proof. Clearly, (3.24) holds for $n = 0, 1$. If $n \geqslant 2$, then by (3.21) and (3.23),

$$\begin{aligned}
\tilde{B}_{-1}(n) &= \sum_{k=0}^{n} \binom{n - \lfloor k/2 \rfloor - 1}{n-k} \\
&= \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-i-1}{i} + \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i-1} \\
&= \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} + \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-2-i}{i} \\
&= F_{n-1} + F_{n-2} = F_n,
\end{aligned}$$

proving the theorem. □

Throughout, we'll represent $\pi \in \Pi(n)$ by $(E_1, E_2, \dots)$, the unique ordered partition of $[n]$ comprising the same blocks as π, arranged in increasing order of their smallest elements.

Combinatorial Proof of Theorem 3.4.

Let $\Pi_i(n) := \{\pi \in \Pi(n) : \tilde{w}(\pi) \equiv i \pmod 2\}$ so that $\tilde{B}_{-1}(n) = |\Pi_0(n)| - |\Pi_1(n)|$. To prove (3.24), we'll identify a subset $\tilde{\Pi}(n)$ of $\Pi_0(n)$ such that $|\tilde{\Pi}(n)| = F_n$, along with a $\tilde{w}$-parity changing involution of $\Pi(n) - \tilde{\Pi}(n)$.

The set $\tilde{\Pi}(n)$ consists of those partitions $\pi = (E_1, E_2, \dots)$ whose blocks satisfy the two conditions:

each block of odd index comprises a set of consecutive integers; (3.25a)

each block of even index is a singleton. (3.25b)

Now $|\tilde{\Pi}(n)| = F_n$, as $|\tilde{\Pi}(n)|$ is seen to satisfy the Fibonacci recurrence, upon considering whether or not $\{n\}$ is a block. For if $\{n\}$ is not a block and $n-2$ belongs to an odd-numbered (respectively, even-numbered) block of $\pi \in \tilde{\Pi}(n)$, then $\{n-1, n\}$ constitutes a proper subset of (respectively, all of) the last block of π.

Suppose now that $\pi = (E_1, E_2, \dots)$ belongs to $\Pi(n) - \tilde{\Pi}(n)$ and that i_0 is the smallest of the integers i for which E_{2i-1} fails to satisfy (3.25a) or E_{2i} fails to satisfy (3.25b). Let M be the largest member of $E_{2i_0-1} \cup E_{2i_0}$. If M belongs to E_{2i_0-1}, move it to E_{2i_0}, while if M belongs to E_{2i_0}, move it to E_{2i_0-1} (note that if $|E_{2i_0}| = 1$, then necessarily $M \in E_{2i_0-1}$). The resulting map is a parity changing involution of $\Pi(n) - \tilde{\Pi}(n)$. □

Below, we illustrate the fixed point set $\tilde{\Pi}(n)$ and the pairings of $\Pi(n) - \tilde{\Pi}(n)$ when $n = 4$, wherein the first two members of each row are paired.

$\Pi_0(n) - \tilde{\Pi}(n)$	$\Pi_1(n)$	$\tilde{\Pi}(n)$
$\{1,2,4\},\ \{3\}$	$\{1,2\},\ \{3,4\}$	$\{1,2,3,4\}$
$\{1,3,4\},\ \{2\}$	$\{1,3\},\ \{2,4\}$	$\{1,2,3\},\ \{4\}$
$\{1\},\ \{2,3,4\}$	$\{1,4\},\ \{2,3\}$	$\{1\},\ \{2\},\ \{3,4\}$
$\{1,3\},\ \{2\},\ \{4\}$	$\{1\},\ \{2,3\},\ \{4\}$	$\{1,2\},\ \{3\},\ \{4\}$
$\{1,4\},\ \{2\},\ \{3\}$	$\{1\},\ \{2,4\},\ \{3\}$	$\{1\},\ \{2\},\ \{3\},\ \{4\}$

Note that the above bijection preserves the number of blocks of $\pi \in \Pi(n)$. We'll use its restriction to $\Pi(n,k)$ to supply a

Combinatorial Proof of Theorem 3.3.

Let $\Pi_i(n,k) := \Pi_i(n) \cap \Pi(n,k)$ for $i = 0,1$, $\tilde{\Pi}(n,k) := \tilde{\Pi}(n) \cap \Pi(n,k)$, and $\pi = (E_1,\ldots,E_k) \in \tilde{\Pi}(n,k)$. If k is even, identify each pair of blocks (E_{2i-1}, E_{2i}), $1 \leqslant i \leqslant k/2$, with summands x_i in a composition $x_1+\cdots+x_{k/2} = n$, where each $x_i \geqslant 2$. If k is odd, identify $(E_1,E_2),\ldots,(E_{k-2},E_{k-1}),(E_k)$ with summands x_i in $x_1+\cdots+x_{(k+1)/2} = n$, where $x_i \geqslant 2$ for $1 \leqslant i \leqslant \frac{k-1}{2}$ and $x_{(k+1)/2} \geqslant 1$. The cardinality of $\tilde{\Pi}(n,k)$ is then given by the right-hand side of (3.21), and the restriction of the prior bijection to $\Pi(n,k) - \tilde{\Pi}(n,k)$ is again an involution, and inherits the parity changing property, which proves (3.21). □

From (3.21) along with (3.8), (3.9), and (3.10), we have

$$\hat{S}_{-1}(n,k) = (-1)^{n-k}\binom{n-\lfloor k/2\rfloor-1}{n-k}, \qquad 0 \leqslant k \leqslant n, \tag{3.26}$$

$$S^*_{-1}(n,k) = (-1)^{\binom{k}{2}+n}\binom{n-\lfloor k/2\rfloor-1}{n-k}, \qquad 0 \leqslant k \leqslant n, \tag{3.27}$$

and

$$S_{-1}(n,k) = (-1)^{\binom{k}{2}}\binom{n-\lfloor k/2\rfloor-1}{n-k}, \qquad 0 \leqslant k \leqslant n. \tag{3.28}$$

The bijection establishing (3.21) clearly applies to (3.26)–(3.28) as well.

The bijection of Theorem 3.3 also proves combinatorially that

$$S(n,k) \equiv \binom{n-\lfloor k/2\rfloor-1}{n-k} \pmod 2, \qquad 0 \leqslant k \leqslant n, \tag{3.29}$$

since off of a set of cardinality $\binom{n-\lfloor k/2\rfloor-1}{n-k}$, each partition $\pi \in \Pi(n,k)$ is paired with another of opposite $\tilde{w}$-parity. This furnishes an answer to a question raised by Stanley [23, p. 46, Exercise 17b].

Theorem 3.5. *For all $n \in \mathbb{N}$,*

$$\hat{B}_{-1}(n) := \sum_{k=0}^{n} \hat{S}_{-1}(n,k) = (-1)^{n-1}F_{n-3}, \tag{3.30}$$

where $F_{-3}=-1$, $F_{-2}=1$, and $F_{-1}=0$.

Proof. Clearly, (3.30) holds for $n=0,1$. If $n \geqslant 2$, then by (3.23) and (3.26),

$$\begin{aligned}
\hat{B}_{-1}(n) &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n-\lfloor k/2\rfloor-1}{n-k} \\
&= (-1)^{n-1}\left[\sum_{i=0}^{\lfloor (n-1)/2\rfloor}\binom{(n-1)-i}{i} - \sum_{i=0}^{\lfloor (n-2)/2\rfloor}\binom{(n-2)-i}{i}\right] \\
&= (-1)^{n-1}(F_{n-1}-F_{n-2}) = (-1)^{n-1}F_{n-3}.
\end{aligned}$$

Alternatively, let $n \geqslant 3$, $\tilde{\Pi}(n)$ be as in the proof of Theorem 3.4, and $\hat{\Pi}(n) \subseteq \tilde{\Pi}(n)$ consist of those partitions with an odd number of blocks and whose last block is a singleton. First, $|\hat{\Pi}(n)| = |\tilde{\Pi}(n-3)| = F_{n-3}$ as the removal of $n-2$, $n-1$, and n from $\pi \in \hat{\Pi}(n)$ is seen to be a bijection between $\hat{\Pi}(n)$ and $\tilde{\Pi}(n-3)$. Since $\hat{w}(\pi) = \tilde{w}(\pi) + n - k$ and since every $\pi \in \hat{\Pi}(n)$ has an even $\tilde{w}(\pi)$ value and an odd number of blocks, the $\hat{w}$-parity of each $\pi \in \hat{\Pi}(n)$ is opposite the parity of n. Thus, $\hat{\Pi}(n)$ agrees with the right-hand side of (3.30) in both sign and magnitude.

The $\tilde{w}$-parity changing involution of Theorem 3.4 defined on $\Pi(n)-\tilde{\Pi}(n)$ also changes the $\hat{w}$-parity. We now extend this involution to $\Pi(n)-\hat{\Pi}(n)$ as follows: if the last block of $\pi \in \tilde{\Pi}(n)-\hat{\Pi}(n)$ is $\{n\}$, merge it with the penultimate block; if the last block is not a singleton, take n from this block and form the singleton $\{n\}$. The resulting extension is a $\hat{w}$-parity changing involution of $\Pi(n)-\hat{\Pi}(n)$. ☐

The Bell numbers $B^*_{-1}(n)$ are quite different from the numbers $\tilde{B}_{-1}(n)$ and $\hat{B}_{-1}(n)$, as demonstrated by the following theorem.

Theorem 3.6. *For all* $n \in \mathbb{N}$,

$$B^*_{-1}(n) := \sum_{k=0}^{n} S^*_{-1}(n,k) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 3; \\ -1, & \text{if } n \equiv 1 \pmod 3; \\ 0, & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{3.31}$$

Proof. Using the generating function method and (3.27), we have

$$\begin{aligned} \sum_{n\geqslant 0} B^*_{-1}(n)x^n &= \sum_{n\geqslant 0}\left(\sum_{k=0}^{n} S^*_{-1}(n,k)\right)x^n \\ &= \sum_{n\geqslant 0}\left(\sum_{k=0}^{n}(-1)^{\binom{k}{2}+n}\binom{n-\lfloor\frac{k}{2}\rfloor-1}{n-k}\right)x^n \\ &= \sum_{k\geqslant 0}(-1)^{\binom{k}{2}}\sum_{n\geqslant k}(-1)^n\binom{n-\lfloor\frac{k}{2}\rfloor-1}{n-k}x^n \\ &= 1-\sum_{\substack{k \text{ even}\\ k\geqslant 2}} x^{\frac{k}{2}+1}\sum_{n\geqslant\frac{k}{2}-1}\binom{n}{\frac{k}{2}-1}(-x)^n \\ &\quad -\sum_{k \text{ odd}} x^{\frac{k+1}{2}}\sum_{n\geqslant\frac{k-1}{2}}\binom{n}{\frac{k-1}{2}}(-x)^n \\ &= \sum_{k\geqslant 0}\frac{(-1)^k x^{2k}}{(1+x)^k}-\sum_{k\geqslant 0}\frac{(-1)^k x^{2k+1}}{(1+x)^{k+1}} = \frac{1}{1+x}\sum_{k\geqslant 0}\frac{(-1)^k x^{2k}}{(1+x)^k} \\ &= \frac{1}{1+x+x^2} = \frac{1-x}{1-x^3} = \sum_{m\geqslant 0}(x^{3m}-x^{3m+1}), \end{aligned}$$

which proves (3.31).

Alternatively, let $\Pi_i(n) := \{\pi \in \Pi(n) : \ w^*(\pi) \equiv i \pmod 2\}$ and $\Pi^*(n)$ consist of those partitions $\pi = (E_1, E_2, \dots)$ whose blocks satisfy

$$E_{2i-1} = \{3i-2, 3i-1\}, \quad E_{2i} = \{3i\}, \quad 1 \leqslant i \leqslant \lfloor n/3 \rfloor. \tag{3.32}$$

Then $\Pi^*(n)$ is a singleton contained in $\Pi_0(n)$ if $n \equiv 0 \pmod 3$ or contained in $\Pi_1(n)$ if $n \equiv 1 \pmod 3$. If $n \equiv 2 \pmod 3$, $\Pi^*(n)$ is a doubleton containing two partitions of opposite w^*-parity, which we pair.

Suppose now that $\pi = (E_1, E_2, \dots) \in \Pi(n) - \Pi^*(n)$ and that i_0 is the smallest index for which condition (3.32) fails to hold. Let $n_1 = 3i_0 - 2$, $n_2 = 3i_0 - 1$, $n_3 = 3i_0$ and $V_1 = E_{2i_0-1}$, $V_2 = E_{2i_0}$, $V_3 = E_{2i_0+1}$ (the latter two if they occur). Consider the following four disjoint cases concerning the relative positions of the n_i within the V_i:

(I) $n_2 \in V_2$, $n_3 \in V_3$, and $|V_2 \cup V_3| \geqslant 3$;

(II) Either (a) or (b) holds where (a) $V_2 = \{n_2\}$ and $V_3 = \{n_3\}$, (b) $n_2, n_3 \in V_1$;

(III) $n_2 \in V_2$ and $n_3 \in V_1 \cup V_2$;

(IV) $n_2 \in V_1$, $n_3 \in V_2$, and $|V_1 \cup V_2| \geqslant 4$.

Within each case, we pair partitions of opposite parity as shown below, leaving the other blocks undisturbed:

(i) $V_2 = \{n_2, \dots, M\}$, $V_3 = \{n_3, \dots\} \leftrightarrow V_2 = \{n_2, \dots\}$, $V_3 = \{n_3, \dots, M\}$, where M is the largest member of $V_2 \cup V_3$;

(ii) $V_1 = \{n_1, \dots\}$, $V_2 = \{n_2\}$, $V_3 = \{n_3\} \leftrightarrow V_1 = \{n_1, n_2, n_3, \dots\}$;

(iii) $V_1 = \{n_1, n_3, \dots\}$, $V_2 = \{n_2, \dots\} \leftrightarrow V_1 = \{n_1, \dots\}$, $V_2 = \{n_2, n_3, \dots\}$;

(iv) $V_1 = \{n_1, n_2, \dots, N\}$, $V_2 = \{n_3, \dots\} \leftrightarrow V_1 = \{n_1, n_2, \dots\}$, $V_2 = \{n_3, \dots, N\}$, where N is the largest member of $V_1 \cup V_2$.

The resulting map is a parity changing involution of $\Pi(n) - \Pi^*(n)$, which implies (3.31). □

Below, we illustrate the fixed point set $\Pi^*(n)$ along with the pairings of $\Pi(n) - \Pi^*(n)$ when $n = 4$.

$\Pi_0(n)$	$\Pi_1(n) - \Pi^*(n)$	$\Pi^*(n)$
$\{1,2,3,4\}$	$\{1,4\},\ \{2\},\ \{3\}$	$\{1,2\},\ \{3\},\ \{4\}$
$\{1,2\},\ \{3,4\}$	$\{1,2,4\},\ \{3\}$	
$\{1,3\},\ \{2,4\}$	$\{1\},\ \{2,3,4\}$	
$\{1,4\},\ \{2,3\}$	$\{1,3,4\},\ \{2\}$	
$\{1\},\ \{2,3\},\ \{4\}$	$\{1,3\},\ \{2\},\ \{4\}$	

$\{1\},\ \{2,4\},\ \{3\}$ $\quad\quad$ $\{1\},\ \{2\},\ \{3,4\}$
$\{1\},\ \{2\},\ \{3\},\ \{4\}$ $\quad\quad$ $\{1,2,3\},\ \{4\}$

Note that the bijection above, like the one used for Theorem 3.5, does not always preserve the number of blocks and hence has no meaningful restriction to $\Pi(n,k)$, unlike the bijection of Theorem 3.4.

Remark. In [7], Ehrlich evaluates $\sigma(n) := -\sum_{\pi\in\Pi(n)}(-1)^{\alpha(\pi)}$, where $\alpha(\pi) := \sum_{i \text{ odd}}|E_i|$ for $\pi = (E_1, E_2, \dots) \in \Pi(n)$. The bijection of Theorem 3.6 establishing $B^*_{-1}(n)$ also provides an alternative to Ehrlich's iterative argument establishing his $\sigma(n)$ since

$$\begin{aligned}\sigma(n) &= -\sum_{\pi=(E_1,E_2,\dots)\in\Pi(n)} (-1)^{|E_1|+|E_3|+|E_5|+\cdots} \\ &= -\sum_{\pi=(E_1,E_2,\dots)\in\Pi(n)} (-1)^{|E_1|+2|E_2|+3|E_3|+\cdots} \\ &= -B^*_{-1}(n).\end{aligned}$$

It is easy to verify that one can write (3.31) more compactly as

$$B^*_{-1}(n) = \frac{1}{1-w}w^n - \frac{w}{1-w}w^{2n},$$

where w is a primitive cube root of unity, which yields the pleasing exponential generating function [26]

$$\sum_{n\geqslant 0} B^*_{-1}(n)\frac{x^n}{n!} = \frac{1}{1-w}\mathrm{e}^{wx} - \frac{w}{1-w}\mathrm{e}^{w^2x}. \tag{3.33}$$

Since $S_q(n,k) = q^{-n}S^*_q(n,k)$,

$$B_{-1}(n) := \sum_{k=0}^{n} S_{-1}(n,k) = (-1)^n B^*_{-1}(n),$$

and so by (3.31),

$$B_{-1}(n) = \begin{cases} (-1)^n, & \text{if } n \equiv 0 \pmod 3; \\ (-1)^{n+1}, & \text{if } n \equiv 1 \pmod 3; \\ 0, & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{3.34}$$

with the above bijection clearly showing this. The preceding also supplies a combinatorial proof that $B(n)$, the n^{th} Bell number, is even if and only if $n \equiv 2 \pmod 3$ since every partition of $[n]$ is paired with another of opposite w^*-parity when $n \equiv 2 \pmod 3$ and since all partitions are so paired except for one otherwise (cf. Ehrlich [7, p. 512]).

3.3 A Notable Restriction

Let $\mathcal{B}(n,k) := \{\pi \in \Pi(n,k) : \text{all blocks of } \pi \text{ have cardinality at most } 2\}$, $\mathcal{B}(n) := \cup_k \mathcal{B}(n,k)$, $A(n,k) := |\mathcal{B}(n,k)|$, and $A(n) := |\mathcal{B}(n)|$. The numbers $A(n,k)$, known as *Bessel numbers,* are reparametrized coefficients of Bessel polynomials [4]. By a fairly routine combinatorial argument,

$$A(n,k) = \binom{n}{2n-2k} \frac{(2n-2k)!}{2^{n-k}(n-k)!}, \quad \lceil n/2 \rceil \leqslant k \leqslant n. \tag{3.35}$$

Consider the q-generalizations of $A(n,k)$ and $A(n)$ obtained by restricting the partition statistics w^* and w to $\mathcal{B}(n,k)$ and to $\mathcal{B}(n)$:

$$A_q^*(n,k) := \sum_{\pi \in \mathcal{B}(n,k)} q^{w^*(\pi)}, \tag{3.36}$$

$$A_q(n,k) := \sum_{\pi \in \mathcal{B}(n,k)} q^{w(\pi)} = q^{-n} A_q^*(n,k), \tag{3.37}$$

$$A_q^*(n) := \sum_{\pi \in \mathcal{B}(n)} q^{w^*(\pi)}, \tag{3.38}$$

and

$$A_q(n) := \sum_{\pi \in \mathcal{B}(n)} q^{w(\pi)} = q^{-n} A_q^*(n). \tag{3.39}$$

The numbers $A_q^*(n,k)$ satisfy the boundary conditions $A_q^*(n,0) = \delta_{n,0}$, $A_q^*(0,k) = \delta_{0,k}$, and $A_q^*(1,1) = q$, along with the recurrence

$$A_q^*(n,k) = q^n A_q^*(n-1,k-1) + q^n(n-1)A_q^*(n-2,k-1), \quad n \geqslant 2,\ k \geqslant 1, \tag{3.40}$$

upon considering whether the first block is the singleton $\{1\}$ or a doubleton containing 1.

The recurrence in (3.40) is not of a form covered by Comtet's Theorem. We were unable to find analogues of (3.16)–(3.18) for either $A_q^*(n,k)$ or $A_q(n,k)$, nor were we able to find simple expressions for $A_{-1}^*(n,k)$ or $A_{-1}(n,k)$. However, we do have the following somewhat surprising parity theorem for $\mathcal{B}(n)$.

Theorem 3.7. *For all* $n \in \mathbb{P}$,

$$A_{-1}^*(4n) = A_{-1}^*(4n-1) = \prod_{i=0}^{n-1}(4i+1)(4i+2),$$
$$A_{-1}^*(4n-2) = 0, \tag{3.41}$$

and

$$A_{-1}^*(4n-3) = -\prod_{i=1}^{n-1}(4i+1)(4i-2).$$

Proof. Let $a_m = A_{-1}^*(m)$ and $b_m = A_{-1}(m)$. Conditioning on whether or not the first block is a singleton yields

$$a_m = -b_{m-1} + (m-1)b_{m-2}, \quad m \geqslant 2. \tag{3.42}$$

Since $b_m = (-1)^m a_m$, we have, by (3.42), the second-order recurrence

$$a_m = (-1)^m[a_{m-1} + (m-1)a_{m-2}], \quad m \geqslant 2, \tag{3.43}$$

with $a_0 = 1$, $a_1 = -1$.

To solve (3.43), write out the first several terms of the sequence a_m and conjecture that $a_{4m-2} = 0$, along with $a_{4m-1} = a_{4m}$ if $m \geqslant 1$. Assuming this conjecture to be true for the moment, let $d_m = a_{4m+1}$ and $e_m = a_{4m+3} = a_{4m+4}$ for $m \geqslant 0$ with $e_{-1} := 1$. Then (3.43) implies $e_m = -(4m+2)d_m$ and $d_m = -(4m+1)e_{m-1}$ for $m \geqslant 0$ so that

$$e_m = (4m+2)(4m+1)e_{m-1}, \quad m \geqslant 0,$$

whence

$$e_m = \prod_{i=0}^{m}(4i+1)(4i+2) \quad \text{and} \quad d_m = -\prod_{i=1}^{m}(4i+1)(4i-2), \quad m \geqslant 0.$$

The sequence a_m defined by $a_{4n} = a_{4n-1} = \prod_{i=0}^{n-1}(4i+1)(4i+2)$, $a_{4n-2} = 0$, and $a_{4n-3} = -\prod_{i=1}^{n-1}(4i+1)(4i-2)$, if $n \geqslant 1$ with $a_0 = 1$, is easily shown to satisfy (3.43), which completes the proof. ☐

Conclusion

The formulas of the first three chapters show that the case $q = -1$ is an interesting special case which differs in many respects from the general q-case. In several instances, these formulas arise when either a q-recurrence or q-generating function assumes a particularly nice form for $q = -1$ (e.g., (3.24) in Ch. 3; (3.13) and (3.14) in [20]). The $q = -1$ case may have a nice closed form whereas there is no such closed form known for q general, as seen with (2.32). In a few instances, there seems to be no underlying algebraic development behind a particular parity result (e.g., (2.16), (2.21), and (3.41)). In addition, the author has encountered several instances in which the case $q = -1$ appears to have no simple closed form at all.

One possible way to algebraically extend previous results for $q = -1$ would be to allow q to be an arbitrary complex root of unity. Though in many instances such a generalization doesn't seem possible, in a few, the generalization is straightforward. For example, when $m \geqslant 2$ and $q = e^{2\pi i/m}$, then $n_q^! = 0$ if $n \geqslant m$ since $m_q = 1 + q + \cdots + q^{m-1} = 0$ if $q = e^{2\pi i/m}$. To realize this combinatorially, it suffices to show, by (2.1), that the *inv* (or *maj*) statistic on S_n is balanced in the following sense whenever $n \geqslant m$: the number of permutations of $[n]$ whose *inv* (or *maj*) value is congruent to $i \pmod m$ is equal to the number whose value is congruent to $j \pmod m$ for all i and j, where $0 \leqslant i < j \leqslant m - 1$.

If $n \geqslant m$, then locate the largest member occurring in the first m positions of $\sigma \in S_n$, expressed as a word, and cyclicly shift it through these positions, leaving the relative order of the other letters undisturbed. The m members of S_n so obtained have different *inv* and *maj* values mod m. Since S_n is partitioned into m-member equivalence classes by this procedure, it follows that *inv* and *maj* are both balanced as described. Note that for *inv*, but not *maj*, this can also be realized by locating the positions occupied by the members of $[m]$ within $\sigma \in S_n$, expressed as a word, and shifting m in a cyclic fashion within these m positions, leaving the rest of σ undisturbed.

For another example, we let $q = \rho = \frac{-1+\sqrt{3}i}{2}$, a third root of unity, in the q-binomial coefficient $\binom{n}{k}_q$. By a calculation similar to the one in Theorem 1.1 or by substituting $q = \rho$ into (1.11) and considering cases mod 3, we have the formula

$$\binom{n}{k}_\rho = \begin{cases} \binom{\lfloor n/3 \rfloor}{\lfloor k/3 \rfloor}, & \text{if } n \equiv k \pmod 3 \text{ or } k \equiv 0 \pmod 3; \\ -\rho^2 \binom{\lfloor n/3 \rfloor}{\lfloor k/3 \rfloor}, & \text{if } n \equiv 2 \pmod 3 \text{ and } k \equiv 1 \pmod 3; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The combinatorial arguments of the first chapter used to evaluate $\binom{n}{k}_{-1} = \sum_{w \in \Lambda_{n,k}} (-1)^{inv(w)}$ can be modified to evaluate $\binom{n}{k}_\rho = \sum_{w \in \Lambda_{n,k}} \rho^{inv(w)}$, where $\Lambda_{n,k}$ denotes the set of binary words of length n with k 0's. Instead of pairing members of $\Lambda_{n,k}$ of opposite *inv*-parity, we partition a portion of $\Lambda_{n,k}$ into tripletons each of whose members have different *inv* values mod 3. For each such tripleton $\{\lambda_1, \lambda_2, \lambda_3\}$, we have that $\rho^{inv(\lambda_1)} + \rho^{inv(\lambda_2)} + \rho^{inv(\lambda_3)} = 0$ since $1 + \rho + \rho^2 = 0$.

Let $\Lambda'_{n,k}$ consist of those words $w = w_1 w_2 \cdots w_n$ in $\Lambda_{n,k}$ satisfying

$$w_{3i-2} = w_{3i-1} = w_{3i}, \qquad 1 \leqslant i \leqslant \lfloor n/3 \rfloor. \tag{2}$$

In all cases, the right-hand side of (1) above gives the net contribution of $\Lambda'_{n,k}$ towards the sum $\binom{n}{k}_\rho = \sum_{w \in \Lambda_{n,k}} \rho^{inv(w)}$; note that members of $\Lambda'_{n,k}$ may end in either 01 or 10 if $n \equiv 2 \pmod 3$ and $k \equiv 1 \pmod 3$, hence the $1 + \rho = -\rho^2$ factor in this case.

Suppose now that $w = w_1 w_2 \cdots w_n \in \Lambda_{n,k} - \Lambda'_{n,k}$, with i_0 being the smallest i for which (2) fails to hold. Group the three members of $\Lambda_{n,k} - \Lambda'_{n,k}$ gotten by circularly permuting w_{3i_0-2}, w_{3i_0-1}, and w_{3i_0} within $w = w_1 w_2 \cdots w_n$, leaving the rest of w undisturbed. Note that these three members of $\Lambda_{n,k} - \Lambda'_{n,k}$ have different *inv* values mod 3. The preceding argument works equally well with *maj* in place of *inv*. See also formulas (4.1)–(4.6) in [20] for similar behavior when q is a third root of unity.

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Appendices

Appendix A

A Combinatorial Proof of a q-Binomial Coefficient Identity

Recall the q-binomial theorem [23]

$$(1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q x^k, \tag{1}$$

originally due to Euler. Substituting $-\frac{1}{x}$ for x in (1), multiplying both sides by x^n, and reindexing yields the variant [27]

$$(x-1)(x-q)\cdots(x-q^{n-1}) = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \binom{n}{k}_q x^k. \tag{2}$$

Our aim here is to give a direct combinatorial proof of (2). It suffices to prove equality whenever $x = q^m$, $m \in \mathbb{P}$, and q is a power of some prime. We'll argue that, when $x = q^m$, both sides of (2) count the injective linear transformations from $\mathbb{F}_q^n$ to $\mathbb{F}_q^m$, the left-hand side clearly doing so, where $\mathbb{F}_q^r$ denotes the finite r-dimensional vector space of q^r elements over the finite field $\mathbb{F}_q$ of q elements. To show that the right-hand side of (2) also achieves this, we'll use a weighted, sieve argument described below which starts with the set of all linear transformations.

Let $\mathcal{A}$ denote the set of all ordered pairs $\alpha = (\mathcal{U}, T)$, where T is a linear transformation from $\mathbb{F}_q^n$ to $\mathbb{F}_q^m$ and $\mathcal{U}$ is a subspace of the null space of T. Assign to each $\alpha \in \mathcal{A}$ the weight $(-1)^k q^{\binom{k}{2}}$, where k is the dimension of $\mathcal{U}$.

The right-hand side of (2) at $x = q^m$, written in the form

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{n}{k}_q q^{m(n-k)}, \tag{3}$$

then gives the total weight of all the members of $\mathcal{A}$, according to the dimension of $\mathcal{U}$.

Now let R be a linear transformation from $\mathbb{F}_q^n$ to $\mathbb{F}_q^m$, with null space $\mathcal{W}$ and $\dim(\mathcal{W}) = j$. Then the net weight of all the members of $\mathcal{A}$ with second coordinate R is given by

$$\sum_{i=0}^{j} \sum_{\substack{\mathcal{U} \subseteq \mathcal{W} \\ \dim(\mathcal{U}) = i}} (-1)^i q^{\binom{i}{2}} = \sum_{i=0}^{j} (-1)^i q^{\binom{i}{2}} \binom{j}{i}_q = \delta_{j,0}, \tag{4}$$

the latter equality upon substituting $x = 0$ in the well known identity [25, pp. 201–202]

$$x^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-1)\cdots(x-q^{k-1}), \qquad n \geqslant 0. \tag{5}$$

Thus by (4), the net weight of all the members of $\mathcal{A}$ with second coordinate R is zero when $\dim(\mathcal{W})$ is positive and one when $\dim(\mathcal{W})$ is zero. Therefore, the total weight of $\mathcal{A}$ equals the number of injective linear transformations, which completes the proof.

Appendix B

Bijective Proofs of Alternating Sum Identities

The ideas used to construct the involutions of the first three chapters can be adapted to furnish bijective proofs of various identities involving sums with alternating signs, such as orthogonality relations, connection constant relations, and binomial coefficient identities, which are typically proven using algebraic methods. In this section, we look at some specific identities illustrating how these combinatorial ideas can be applied. As far as providing bijective proofs for these identities, constructing appropriate sign-reversing involutions seems to be more effective than other combinatorial proof techniques, such as direct argument or inclusion-exclusion. The author has used similar sign-reversing involutions in providing bijective proofs for several other combinatorial identities involving sums with alternating signs.

The first identities we'll look at are the well known orthogonality relations for Stirling numbers [23, Proposition 1.4.1],

$$\sum_{j=k}^{n} S(n,j)s(j,k) = \delta_{n,k}, \quad 0 \leqslant k \leqslant n, \tag{1}$$

and

$$\sum_{j=k}^{n} s(n,j)S(j,k) = \delta_{n,k}, \quad 0 \leqslant k \leqslant n, \tag{2}$$

where $S(n,k)$ is the Stirling number of the second kind, $s(n,k) = (-1)^{n-k}$ $c(n,k)$ is the Stirling number of the first kind, and $c(n,k)$ is the signless

Stirling number of the first kind. We rewrite these relations in the more suggestive form

$$\sum_{j=k}^{n}(-1)^j S(n,j)c(j,k) = (-1)^k \delta_{n,k}, \quad 0 \leqslant k \leqslant n, \tag{3}$$

and

$$\sum_{j=k}^{n}(-1)^j c(n,j)S(j,k) = (-1)^n \delta_{n,k}, \quad 0 \leqslant k \leqslant n. \tag{4}$$

We first give a bijective proof of (3). Partition $[n]$ into j blocks, where $k \leqslant j \leqslant n$, writing the members of $[n]$ within a block in ascending order. Now permute these j blocks, ordered lexicographically, according to a permutation on j elements with k cycles in standard cycle form. Let the resulting block arrangement have weight $(-1)^j$, where j is the number of individual blocks. The left-hand side of (3) then gives the total weight of all block arrangements with j individual blocks as j ranges from k to n.

We now pair block arrangements of opposite sign as follows. Let m denote the number of the first cycle encountered, going from left to right within a block arrangement, possessing at least two members of $[n]$ in all and ℓ denote the first, hence the smallest, member of $[n]$ encountered in cycle m. If $\{\ell\}$ is not a block, split off ℓ and start cycle m with the block $\{\ell\}$. If $\{\ell\}$ is the first block of cycle m, hence not the last block, place ℓ at the front of the second block of cycle m. All block arrangements are so paired except when $k = n$, in which case there is a single block arrangement of sign $(-1)^k$.

Similarly for (4), take the j cycles, ordered lexicographically, of $\sigma \in S_{n,j}$, expressed in standard cycle form, and arrange them according to a canonical ordered partition of $[j]$ with k blocks, writing the cycles occurring within a block in ascending order. Let the resulting cycle arrangement have weight $(-1)^j$, where j is the number of individual cycles. Let $c_1, \ldots, c_r$ be the cycles in order which comprise the first block possessing at least two members of $[n]$ in all. If c_r is a 1-cycle, whence $r \geqslant 2$, place the sole member of c_r at the end of c_{r-1}. If $|c_r| \geqslant 2$, break off the last member of c_r and form a 1-cycle within the block.

The $s(n,k)$ $(=(-1)^{n-k}c(n,k))$ are the connection constants in the polynomial identities [23, p. 35]

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)x^k, \qquad n \in \mathbb{N}, \tag{5}$$

where $x^{\underline{n}} := x(x-1)\cdots(x-n+1)$. Equivalently,

$$m^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k}c(n,k)m^k, \qquad n \in \mathbb{N}, \tag{6}$$

for each $m \in \mathbb{N}$.

To prove (6), place the k cycles of $\sigma \in S_{n,k}$, expressed in standard cycle form, into m labeled urns in one of m^k ways, ordering the cycles within an urn lexicographically. Let $\mathcal{A}$ denote the set of all possible arrangements as k varies, $0 \leqslant k \leqslant n$, and $\mathcal{J} \subseteq \mathcal{A}$ those which consist of 1-cycles placed in distinct urns. Let $\operatorname{sign}(\alpha) = (-1)^{n-k}$, where k is the number of cycles in $\alpha \in \mathcal{A}$. Note that $|\mathcal{J}| = m^{\underline{n}}$, with each member of $\mathcal{J}$ having positive sign. Pair members of $\mathcal{A}-\mathcal{J}$ of opposite sign by identifying the first urn possessing at least two members of $[n]$ in all and then either merging or breaking off the last member t of $[n]$ occurring within a cycle in this urn, depending upon whether or not t itself is a 1-cycle.

Next, we generalize a well known orthogonality relation for binomial coefficients (upon taking $m = n$ in (7) below):

$$\sum_{k=r}^{m} (-1)^k \binom{n}{m-k}\binom{k}{r} = (-1)^r \binom{n-r-1}{m-r}, \qquad 0 \leqslant r \leqslant m \leqslant n. \tag{7}$$

Proof of (7)**.** Note that both sides of (7) give the coefficient of x^m in the convolution

$$(1+x)^n \cdot \frac{(-x)^r}{(1+x)^{r+1}} = (-1)^r x^r (1+x)^{n-r-1}.$$

Alternatively, mark $m-k$ members of $[n]$ with red in one of $\binom{n}{m-k}$ ways and consider the first k members of $[n]$ not marked red, where $r \leqslant k \leqslant m$. Choose r of these numbers to be marked blue in one of $\binom{k}{r}$ ways and mark the remaining $k-r$ numbers green. Any remaining members of $[n]$ will be unmarked. Let such a coloring α of $[n]$ have sign $(-1)^k$. The left-hand side of (7) clearly gives the total weight of all possible colorings α.

Now consider the smallest number marked either red or green. Switching to the other option changes the sign of α. This change can be effected provided that the first red or green number does not come later than the last blue number with at least one unmarked number in between. But this can occur only if the smallest red number is greater than $r+1$ with $k=r$, of which there are $\binom{n-r-1}{m-r}$ possibilities each with sign $(-1)^r$. $\square$

We close with a bijective proof of the orthogonality relation (3.22):

$$\sum_{k=m}^{\min\{n,2m\}} (-1)^{k-m} \binom{n-\lfloor k/2 \rfloor - 1}{n-k} \binom{\lfloor k/2 \rfloor}{k-m} = \delta_{n,m}, \qquad 0 \leqslant m \leqslant n. \quad (8)$$

Let $\mathcal{A}_{n,m}$ be the set of "marked compositions" $x = (x_1, x_2, \dots)$ of n satisfying: (a) x itself is a composition of n with $x_i = 1$ for every even index i; (b) x has at least m parts and at most $2m$ parts altogether; (c) if x has k parts, where $m \leqslant k \leqslant 2m$, then $k-m$ pairs of parts of the form (x_{2i-1}, x_{2i}), $i \geqslant 1$, are marked. We'll assign the weight of $(-1)^{k-m}$ to $x \in \mathcal{A}_{n,m}$ possessing $k-m$ designated pairs. Note that the left-hand side of (8) then gives the net weight of all members of $\mathcal{A}_{n,m}$.

If $n = m$, then $\mathcal{A}_{n,m}$ is a singleton with positive sign. So suppose $n > m$ and $x \in \mathcal{A}_{n,m}$, with i_0 the *largest* index $i \geqslant 1$ such that one of the following holds:

(i) (x_{2i-1}, x_{2i}) is marked;

(ii) $x_{2i-1} \geqslant 2$ with x_{2i-1} not part of marked pair.

Note that $x_j = 1$ for all $j \geqslant 2i_0 + 1$ and that the last part of a marked composition with an odd number of parts is, by definition, unmarked. Define a sign-reversing involution of $\mathcal{A}_{n,m}$ by:

(I) if (i) holds, remove the designation and replace (x_{2i_0-1}, x_{2i_0}) with the single part $(x_{2i_0-1} + 1)$;

(II) if (ii) holds, replace x_{2i_0-1} with the two parts $(x_{2i_0-1} - 1, 1)$ and designate the pair $(x_{2i_0-1} - 1, 1)$.

Appendix C

Asymptotics

1 Introduction

Suppose we have a sequence of statistics $I_n : \Delta_n \to \mathbb{N}$, with uniform probability measure on each finite discrete structure Δ_n. The random variable sequence $(I_n)_{n\geqslant 1}$ is said to be *asymptotically normal* if, with $E(I_n) = \mu_n$ and $\mathrm{Var}(I_n) = \sigma_n^2$,

$$\lim_{n\to\infty} P\left(\frac{I_n - \mu_n}{\sigma_n} \leqslant z\right) = \Phi(z), \tag{1}$$

for all z, where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$. For example, this is clearly true if $\Delta_n = 2^{[n]}$ and $I_n(A) = |A|$, by the classical central limit theorem.

In the next section, we'll examine the asymptotic normality of some statistics on discrete structures using a more generalized central limit theorem. The statistics we'll look at all have probability generating functions (pgf's) which factor completely into a product of simpler pgf's. In the last section, we'll briefly look at the asymptotics of the ratio $r_n := G_n(-1)/G_n(1)$, where

$$G_n(q) := \sum_{\delta\in\Delta_n} q^{I_n(\delta)}. \tag{2}$$

In most of the cases we look at, this ratio tends to zero exponentially fast (whenever $G_n(1)$ itself increases exponentially). Thus, it appears that large imbalances for unbalanced statistics are usually rather short lived.

2 Asymptotic Normality

For each $n \geqslant 1$, let $X_{n,1}, X_{n,2}, \ldots, X_{n,n}$ be n mutually independent random variables, which we'll refer to as $X_1, X_2, \ldots, X_n$ by a slight abuse of notation. We assume that the means and variances are finite and put

$$\mu_i = E(X_i), \qquad \sigma_i^2 = \mathrm{Var}(X_i), \quad 1 \leqslant i \leqslant n. \tag{3}$$

The sum $Y_n = X_1 + \cdots + X_n$ then has mean m_n and variance s_n^2 given by

$$m_n = \mu_1 + \cdots + \mu_n, \qquad s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2. \tag{4}$$

The array $\left((X_{n,i})_{i=1}^n\right)_{n\geqslant 1}$ is said to obey the *central limit theorem* if for all z,

$$P\left(\frac{Y_n - m_n}{s_n} \leqslant z\right) \to \Phi(z). \tag{5}$$

The following theorem, a feat of early modern probability theory, gives necessary and sufficient conditions for the central limit theorem to hold (see [9, p. 322]):

Theorem 1 (Lindeberg-Feller). *Suppose that*

$$\lim_{n\to\infty} \frac{max\,\{\sigma_k^2 : \ 1 \leqslant k \leqslant n\}}{s_n^2} = 0. \tag{6}$$

Then the central limit theorem holds if and only if for every $\varepsilon > 0$*, the truncated random variables* U_i*,* $1 \leqslant i \leqslant n$*, defined by*

$$U_i = \begin{cases} X_i - \mu_i, & \text{if} \ \ |X_i - \mu_i| \leqslant \varepsilon s_n; \\ 0, & \text{if} \ \ |X_i - \mu_i| > \varepsilon s_n, \end{cases} \tag{7}$$

satisfy

$$\frac{1}{s_n^2} \sum_{i=1}^{n} E(U_i^2) \to 1. \tag{8}$$

Note that (8) easily implies (6).

Feller [8] uses the sufficiency in Theorem 1 to establish the asymptotic normality for the statistics recording the number of cycles and the number of inversions of a randomly chosen member of S_n. We first outline Feller's proof of asymptotic normality for the number of inversions.

Let $n \geqslant 1$ and X_i, $1 \leqslant i \leqslant n$, be the random variable recording the number of inversions produced by i in $\sigma \in S_n$, expressed as a word. Then $Y_n = X_1 + \cdots + X_n$ records the total number of inversions (i.e., the *inv* value) of $\sigma \in S_n$. The number of inversions produced by i does not depend on the relative order of $1, 2, \ldots, i-1$ within $\sigma \in S_n$, which implies the X_i are mutually independent. The X_i assume the values of $0, 1, \ldots, i-1$, each with probability $\frac{1}{i}$ and therefore $\mu_i = \frac{i-1}{2}$ and $\sigma_i^2 = \frac{i^2-1}{12}$, $1 \leqslant i \leqslant n$, from which one gets $m_n = \frac{n(n-1)}{4}$ and $s_n^2 = \frac{n(n-1)(2n+5)}{72}$.

For large n, we have $U_i = X_i - \mu_i$, $1 \leqslant i \leqslant n$, in (7) as $|X_i - \mu_i| \leqslant n \leqslant \varepsilon s_n$, which ensures that (8) is satisfied. Theorem 1 then gives the asymptotic normality for the *inv* statistic on S_n. Since $m_n \sim \frac{n^2}{4}$ and $s_n^2 \sim \frac{n^3}{36}$, one can conclude, for example, that the number, N_n, of members of S_n whose *inv* value lies between the limits of $\frac{n^2}{4} \pm \frac{\beta n^{3/2}}{6}$ is asymptotically given by $n!(\Phi(\beta) - \Phi(-\beta))$ for each $\beta > 0$.

From the generating function (2.1),

$$\sum_{\sigma \in S_n} q^{inv(\sigma)} = \prod_{i=1}^{n} i_q,$$

one sees that the probability generating function (pgf) for *inv* can be factored into a product of simpler pgf's,

$$\frac{1}{n!} \sum_{\sigma \in S_n} q^{inv(\sigma)} = \frac{1}{1} \left(\frac{1+q}{2} \right) \cdots \left(\frac{1+q+\cdots+q^{n-1}}{n} \right), \tag{9}$$

hence the decomposition of *inv* into the simple independent components given above.

We now apply similar reasoning to other combinatorial statistics. For example, recall that the Carlitz statistic, inv_c, of Section 2.2 has the generating function

$$\sum_{\sigma \in S_n} q^{inv_c(\sigma)} = \prod_{i=0}^{n-1} (1 + i_q),$$

so that the pgf for inv_c on S_n factors as

$$\frac{1}{n!} \sum_{\sigma \in S_n} q^{inv_c(\sigma)} = \frac{1}{1} \cdot \frac{2}{2} \cdot \left(\frac{2+q}{3} \right) \cdots \left(\frac{2+q+\cdots+q^{n-2}}{n} \right). \tag{10}$$

For $n \geqslant 2$, let $(X_i)_{i=1}^n$ be independent and given by $P(X_i = 0) = \frac{2}{i}$, $P(X_i = j) = \frac{1}{i}$, $1 \leqslant j \leqslant i-2$, if $i \geqslant 2$, with $X_1 = 0$. If Y_n records the inv_c value of a randomly chosen member of S_n, then one can express Y_n as $Y_n = X_1 + \cdots + X_n$, where the X_i are as given, by (10). Asymptotic normality for inv_c then follows in much the same way as for inv.

If $m(\sigma)$ denotes the number of left-to-right maxima (i.e., record highs) of $\sigma \in S_n$, expressed as a word, then there is the joint generating function [23, p. 49, Exercise 31]

$$\sum_{\sigma \in S_n} x^{m(\sigma)} q^{inv(\sigma)} = \prod_{i=0}^{n-1} (x + q i_q). \tag{11}$$

Setting $x = q$ in (11) and dividing by $n!$ gives the pgf for Y_n recording the value of $m(\sigma) + inv(\sigma)$ for $\sigma \in S_n$, chosen at random:

$$\frac{1}{n!} \sum_{\sigma \in S_n} q^{m(\sigma)+inv(\sigma)} = \frac{q}{1} \cdot \frac{2q}{2} \cdot \left(\frac{2q+q^2}{3} \right) \cdots \left(\frac{2q+q^2+\cdots+q^{n-1}}{n} \right). \tag{12}$$

For $n \geqslant 2$, define the independent sequence $(X_i)_{i=1}^n$ by $P(X_i = 1) = \frac{2}{i}$, $P(X_i = j) = \frac{1}{i}$, $2 \leqslant j \leqslant i-1$, if $i \geqslant 2$, with $X_1 = 1$. Then $Y_n = X_1 + \cdots + X_n$ records the $m + inv$ value by (12), and the asymptotic normality follows much as before. Even though m and the number of cycles, π, are identically distributed on S_n, we cannot conclude from this that $\pi + inv$ is asymptotically normal since m and π behave differently when considered jointly with inv.

Next, we consider the statistic, denoted $\sigma(S)$, recording the sum of the elements of $S \subset \mathbb{N}$ finite. If $n \geqslant 1$, let Y_n record the subset sum $\sigma(S)$ for a randomly chosen subset S of $\{0, 1, \ldots, n-1\}$. If X_i records the contribution of i towards $\sigma(S)$, $0 \leqslant i \leqslant n-1$, then $Y_n = X_0 + \cdots + X_{n-1}$, where the X_i are independent and given by $P(X_i = 0) = \frac{1}{2} = P(X_i = i)$ if $i \geqslant 1$, with $X_0 = 0$. The asymptotic normality of $\sigma(S)$ then follows from Theorem 1 since s_n^2 is of order n^3 with $0 \leqslant X_i \leqslant n$.

Setting $x = 1$ in the q-binomial theorem [23]

$$\prod_{i=0}^{n-1} (1 + q^i x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q x^k$$

gives

$$\sum_{S \subseteq \{0,\ldots,n-1\}} q^{\sigma(S)} = \prod_{i=0}^{n-1} (1 + q^i) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q. \tag{13}$$

Hence, the statistic on lattice paths given by $\alpha(\lambda)+\binom{k}{2}$, where $\alpha(\lambda)$ is the area subtended by a (minimal) lattice path λ of length n starting from the origin and k denotes the number of vertical steps, is asymptotically normal as $\sigma(S)$ is. We were unable to determine whether or not α itself is asymptotically normal.

Recall that the w statistic on permutations (Section 2.2) has the generating function

$$\sum_{\sigma\in S_n} q^{w(\sigma)} = \prod_{i=0}^{n-1}(i+q^i). \tag{14}$$

If Y_n records the w value of a randomly chosen member of S_n, then, by (14), $Y_n = X_1 + \cdots + X_n$, where the X_i are independent and given by $P(X_i = 0) = \frac{i-1}{i}$, $P(X_i = i-1) = \frac{1}{i}$, if $i \geqslant 2$, with $X_1 = 0$. The X_i fail condition (8) as $\mathrm{Var}(Y_n)$ has order only n^2. As (6) clearly holds, the necessity of Theorem 1 implies that the w statistic on S_n fails to be asymptotically normal. This leaves open the questions of the existence and possible identity of a limiting distribution for w on S_n.

3 The Ratio $\frac{\mathbf{G_n(-1)}}{\mathbf{G_n(1)}}$

In this section, we'll briefly look at the ratio $r_n := \frac{G_n(-1)}{G_n(1)}$ for n large, where

$$G_n(q) := \sum_{\delta\in\Delta_n} q^{I_n(\delta)} \tag{15}$$

and $I_n = I$ is some statistic. Here, we'll choose Δ_n to be the larger discrete structures (indexed by n) of Chapters 1–3 (e.g., $\Pi(n)$, S_n, $2^{[n]}$, C_n, D_n). For the statistics I we've looked at in Chapters 1–3, the ratio r_n tends to zero exponentially (i.e., $|r_n| \leqslant r^n$ for some r, $0 < r < 1$, and all n sufficiently large) in almost all cases for these larger structures.

In many cases, this is either obvious or follows from a short calculation. For instance, take $\Delta_n = C_n$, the set of Catalan words of length $2n$, and $I = maj$. By Theorem 2.7, we have for $n \in \mathbb{N}$,

$$\frac{1}{2^n} = \frac{2^n/(n+1)}{2^{2n}/(n+1)} \leqslant \binom{n}{\lfloor n/2\rfloor} \Big/ \, c_n = r_n$$

$$\leqslant \frac{2^n}{2^{2n}/(n+1)(2n+1)} = \frac{(n+1)(2n+1)}{2^n},$$

as $2^m/(m+1) \leqslant \binom{m}{\lfloor m/2 \rfloor} \leqslant 2^m$, $m \in \mathbb{N}$, so that any r in the interval $(1/2, 1)$ will suffice. From Stirling's formula for $n!$ [8, p. 52], we have in fact $r_n \sim \frac{\sqrt{2}n}{2^n}$.

For another example, we take $\Delta_n = \Pi(n)$, the set of partitions of $[n]$, and $I = \tilde{w}$.

Proposition 1. $\frac{\tilde{B}_{-1}(n)}{\tilde{B}_1(n)}$ *tends to zero exponentially.*

Proof. Note that $B(n) > 2^{n-1}$ for $n \geqslant 3$ since $\Pi(n)$ contains a proper subset which is in 1-1 correspondence with the compositions of n in the obvious way whenever $n \geqslant 3$. Since $F_n \sim c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n$ for large n, we then have, by Theorem 3.4, $\frac{\tilde{B}_{-1}(n)}{\tilde{B}_1(n)} = \frac{F_n}{B(n)} \leqslant 4c_1 \left(\frac{1+\sqrt{5}}{4}\right)^n$ for n sufficiently large, with $\frac{1+\sqrt{5}}{4} < 1$ and c_1 a positive constant. ☐

Next, take $I = w^*$ and $\Delta_n = \mathcal{B}(n)$, the partitions of $[n]$ whose blocks have cardinality at most 2. Recall that

$$A^*_q(n) := \sum_{\pi \in \mathcal{B}(n)} q^{w^*(\pi)}, \qquad n \geqslant 0. \tag{16}$$

Proposition 2. $\frac{A^*_{-1}(n)}{A^*_1(n)}$ *tends to zero.*

Proof. Let $n \geqslant 5$ and first suppose $4|n$. Consider the class of partitions $\mathcal{B}^*(n) \subseteq \mathcal{B}(n)$ consisting of 4 singletons, with the rest of the elements of $[n]$ partitioned into doubletons such that the members of $[n]$ starting doubletons $2i-1$ and $2i$ are the smallest numbers that haven't been used in doubletons $1, 2, \ldots, 2i-2$, $1 \leqslant i \leqslant \frac{n-4}{4}$, or in any of the singletons. Note that

$$|\mathcal{B}^*(n)| = \binom{n}{4} \prod_{i=0}^{\frac{n}{4}-2} (4i+1)(4i+2),$$

so that by Theorem 3.7,

$$\frac{A^*_{-1}(n)}{A_1(n)} \leqslant \frac{A^*_{-1}(n)}{|\mathcal{B}^*(n)|} = \frac{(n-2)(n-3)}{\binom{n}{4}} = \frac{24}{n(n-1)}.$$

If $n \equiv 1 \pmod 4$ or if $n \equiv 3 \pmod 4$, let $\mathcal{B}^*(n)$ be as above except that it now contains 5 singletons or 3 singletons, respectively. If $n \equiv 2 \pmod 4$, then $A^*_{-1}(n) = 0$. ☐

By allowing the members of $\mathcal{B}^*(n)$ to contain arbitrarily many singletons, one can show that the ratio $A^*_{-1}(n)/A^*_1(n)$ tends to zero faster than the reciprocal of any polynomial. Yet it still isn't clear, upon elementary considerations, whether or not this ratio tends to zero exponentially. For example, note that $n^{-\log n}$ tends to zero faster than the reciprocal of any polynomial, yet fails to tend to zero exponentially.

We were able to find an instance in which $G_n(1)$ increases exponentially (i.e., $G_n(1) \geqslant r^n$ for some $r > 1$ and all n sufficiently large), with r_n failing to tend to zero exponentially. Letting $x = q = -1$ in (2.3) and noting $w^*(\sigma) = w(\sigma) + n$ for $\sigma \in S_n$ reveals that

$$\sum_{\sigma \in S_n} (-1)^{w^*(\sigma)-|\sigma|} = \begin{cases} (n-1)!, & \text{if } n \text{ is odd;} \\ -\left((n-1)! + (n-2)!\right), & \text{if } n \text{ is even,} \end{cases} \tag{17}$$

where $n \geqslant 1$. Thus, $r_n \sim \frac{1}{n}$ for large n when $I = w^* - |\sigma|$ and $\Delta_n = S_n$. This imbalance for $w^* - |\sigma|$ is then numerically more significant and persists longer than the imbalances for the other statistics we've studied.

We close with a bijective proof of (17). Let $S_n^{\pm}$ consist of those permutations with even or odd $w^* - |\sigma|$ values, respectively, where members $\sigma = (E_1, E_2, \ldots, E_r)$ are expressed in the standard cycle form. First suppose n is odd and let $S_n^* \subseteq S_n^+$ consist of those permutations with a single cycle. Define a parity changing involution of $S_n - S_n^*$ as follows:

(i) If $|E_r|$ is even, place the last member of E_r after the last member of E_{r-1}.

(ii) If $|E_r|$ is odd and $|E_{r-1}| \geqslant 2$ with the last member of E_{r-1} greater than the first member of E_r, place the last member of E_{r-1} at the end of E_r.

(iii) If $|E_r|$ is odd and $|E_{r-1}| \geqslant 2$ with the last member of E_{r-1} less than the first member of E_r, take the last member of E_{r-1} and form a 1-cycle with it.

(iv) If $|E_r|$ is odd and $|E_{r-1}| = 1$, place the singleton E_{r-1} at the end of E_{r-2}.

Note that $r \geqslant 3$ in (iv) since n is assumed odd. If n is even, let $S_n^* \subseteq S_n^-$ consist of those permutations with a single cycle or of the form $(1)(2i_1 \cdots i_{n-2})$; note that $|S_n^*| = (n-1)! + (n-2)!$. Use the same involution given by (i)–(iv) above, noting that $r \geqslant 3$ in (iv) since $(1)(2i_1 \cdots i_{n-2})$ is now disallowed.

Vita

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