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Fibrator Properties of PL Manifolds

Violeta Vasilevska
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Violeta Vasilevska entitled "Fibrator Properties of PL Manifolds." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Robert J. Daverman, Major Professor

We have read this dissertation and recommend its acceptance:

Jerzy Dydak, Morwen Thistlethwaite, Conrad Plaut, George Siopsis

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Vice Chancellor and
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FIBRATOR PROPERTIES OF PL MANIFOLDS

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Violeta Vasilevska

August, 2004

DEDICATED

to my mother,

Zorka Vasilevska

and to the memory of my father,

Mihajlo Vasilevski

To each of whom

I own more

than I can possibly express

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Poncaré, Julies Henry (1854-1912)

Mathematicians do not study objects, but relations between objects. Thus they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only.

Thought is only a flash between two long nights, but this flash is everything.

In J.R.Newman (ed.) *The World of Mathematics*, New York: Simon and Schuster, 1956.

ABSTRACT

In the early 90s, R.Daverman defined the concept of the PL fibrator ([12]). PL fibrators, by definition, provide detection of PL approximate fibrations. Daverman defines a closed, connected, orientable PL n-manifold to be a *codimension-k PL orientable fibrator* if for all closed, connected, orientable PL $(n+k)$-manifolds M and PL maps $p : M \to B$, where B is a polyhedron, such that each fiber collapses to an n-complex homotopy equivalent to N^n, p is always an approximate fibration.

If N is a codimension-k PL orientable fibrator for all $k > 0$, N is called a *PL orientable fibrator*.

Until now only a few classes of manifolds are known not to be PL fibrators.

Following this concept of Daverman, in this dissertation we attempt to find to what extent such results can be obtained for PL maps $p : M^{n+k} \to B$ between manifolds, such that each fiber has the homotopy type (or more generally the shape) of N, but does not necessarily collapse to an n-complex, which is a severe restriction.

Here we use the following slightly changed PL setting: M is a closed, connected, orientable PL $(n+k)$-manifold, B is a simplicial triangulated manifold (not necessarily PL), $p : M^{n+k} \to B$ a PL proper, surjective map, and N a fixed closed, connected, orientable PL n-manifold.

We call N a *codimension-k shape* $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-*fibrator* if for all orientable, PL $(n+k)$-manifolds M^{n+k} and PL maps $p : M^{n+k} \to B$, such that each fiber is homotopy equivalent to N^n, p is always an approximate fibration. If N is a codimension-k shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrator for all $k > 0$, N is a *shape* $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-*fibrator*.

We are interested in PL manifolds N with $\pi_1(N) \neq 1$, that force every map

$f : N \to N$, with $1 \neq f_\sharp(\pi_1(N)) \lhd \pi_1(N)$, to be a homotopy equivalence. We call PL manifolds N with this property *special manifolds*. There is a similar group theoretic term: a group G is *super Hopfian* if every homomorphism $\phi : G \to G$ with $1 \neq \phi(G) \lhd G$ is an automorphism.

In the first part of the dissertation we study which groups posses this property of being super Hopfian. We find that every non-abelian group of the order pq where p, q are distinct primes is super Hopfian. Also, a free product of non-trivial, finitely generated, residually finite groups at least one of which is not $\mathbb{Z}_2$ is super Hopfian.

Then we give an example of special manifolds to which we apply our main results in the second part of this dissertation.

First we prove that all orientable, special manifolds N with non-cyclic fundamental groups are codimension-2 shape $\mathrm{m_{simpl}o}$-fibrators. Then we find which 3-manifolds have this property.

Next we prove which manifolds are codimension-4 shape $\mathrm{m_{simpl}o}$-fibrators.

Our main result gives that an orientable, special PL n-manifold N with a non-trivial first homology group is a shape $\mathrm{m_{simpl}o}$-fibrators if N is a codimension-2 shape $\mathrm{m_{simpl}o}$-fibrator. The condition of N being a codimension-2 PL shape orientable fibrator can be replaced with N having a non-cyclic fundamental group.

In the last section we list some open questions.

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Chapter 0

Introduction

Topology is the geometric study of continuity; beginning with the continuity of space, or shapes, it generalizes, and then by analogy leads into other kind of continuity. One of the main "tools" used by topologists in this kind of study is that of continuous function. Familiar classes of these continuous maps are fibrations and cell-like maps.

In 1977, Coram and Duvall ([6]) introduced the concept of approximate fibrations as a generalization of both Hurewicz fibrations and cell-like maps. Approximate fibrations are proper mappings that satisfy an approximate version of the homotopy lifting property - the defining property for fibrations. Why study approximate fibrations? Coram and Duvall showed that approximate fibrations have shape theoretic properties analogous to the homotopy theoretic properties of Hurewicz fibrations. For each approximate fibration $p : M \to B$, they showed that all fibers of p are fundamental absolute neighborhood retracts, and moreover that if B is path connected, then any two fibers have the same shape. The most useful property is the existence of an exact sequence involving the homotopy groups of domain, target and shape-theoretical homotopy groups of any point inverse of p. Moreover, Daverman and Husch ([17]) showed that if M is a manifold and B is finite dimensional, B is a generalized manifold. A question that arises is this: how can we detect approximate fibration so we can use these nice properties? Sometimes a proper map defined on an arbitrary manifold of a specific dimension can be recognized as an approximate fibration due to having point inverses all of a certain homotopy type (or shape). So the above question that has been addressed for more than two decades can be restated in a different form:

Which manifolds can detect and recognize approximate fibrations? Or more explicitly: *Which manifolds that appear as the point inverses of a map force the map to be an approximate fibration?*

For addressing these kinds of questions, in 1989, Daverman introduced the concept of *codimension-k (orientable) fibrator* and later the concept of *PL (orientable) fibrator*. Here we introduce a closed, orientable n-manifold, N, called a *codimension-k shape* $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-*fibrator*, that automatically induces approximate fibration, in the sense that all proper, surjective, PL maps $p : M \to B$ from any closed, orientable $(n+k)$-manifold to a triangulated manifold B, such that each point inverse has the same homotopy type (or more generally, the same shape) as N, are approximate fibrations. If this is true for all k, call the manifold N a *shape* $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-*fibrator*. In chapter 2 we discuss more of these concepts, and their relations.

The difference between the two concepts of PL orientable fibrators and shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrators is the following: in the first PL setting the point inverses (fibers) collapse to an n-complex homotopy equivalent to N, while in our PL shape setting we are just asking fibers to have the homotopy type of N but not necessarily to collapse to an n-complex. This property of collapsibility of the fibers in the first category forces the target space of an approximate fibration to be a nice manifold. Since we don't have that in the PL shape category, by definition we take B to be a triangulated (simplicial) manifold.

The main question that we address in this thesis is: *Which manifolds are shape* $\mathrm{m_{simpl}o}$*-fibrators?*

Generally, analysis of fibrator properties applies mostly to Hopfian manifolds N with Hopfian fundamental groups, simply because a map $f : N \to N$ is a homotopy equivalence if and only if the (absolute) degree of f equals 1. A Hopfian manifold is a closed, orientable manifold, such that every degree one self map which induces a π_1-isomorphism is a homotopy equivalence.

The idea of introducing a new concept of a fibrator was motivated by the fact that Daverman showed that most homology n-spheres are codimension-n fibrators, but no homology n-sphere is a codimension-$(n+1)$ fibrator. He also showed that in the PL setting some homology 3-spheres are PL o-fibrators. Here we will change the PL setting slightly, to understand that these homology spheres are NOT shape $\mathrm{m_{simpl}o}$-fibrators, our new category, since they fail to have this property in codimension-5.

In order to address the last question, we introduce first a group theoretical term; we call a group super Hopfian if all self homomorphisms with non-trivial normal image are isomorphisms. Super Hopfian groups are introduced in chapter 3. We show that every non-abelian group of order pq where p, q are distinct primes is super Hopfian. Also the group of rational numbers is a super Hopfian group. In the last section, we discuss which free products of finitely generated groups are super Hopfian, and show that the free product of non-trivial, finitely generated, residually finite groups ($\neq \mathbb{Z}_2 * \mathbb{Z}_2$) are super Hopfian. As a consequence, a free product ($\neq \mathbb{Z}_2 * \mathbb{Z}_2$) of non-trivial, finitely generated, super Hopfian groups is super Hopfian provided that the free product is a Hopfian group. This property is very useful for the problem that we are addressing.

In chapter 4, we introduce a particular type of Hopfian manifold that we call a special manifold: a closed, orientable manifold with a non-trivial fundamental group, for which all self maps with non-trivial normal images on π_1-level are homotopy equivalences. We prove that all closed, orientable surfaces with negative Euler characteristics are special manifolds. Then we address the question: *Which connected sums are special manifolds?* It turns out that all connected sums that are homotopically determined by their fundamental groups, when super Hopfian, are special manifolds.

Special manifolds are of our main interest in the last chapter since the shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrator property can be easily applied to them. First we show that every special n-manifold with a non-cyclic fundamental group is a codimension-2 shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrator. All closed, orientable 3-manifolds whose fundamental group is a non-trivial free product of finitely generated, residually finite groups, which are not all either free or finite groups, are codimension-2 shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrators. Next we show that a closed, connected, orientable PL n-manifold homotopically determined by π_1 with a non-trivial Hopfian fundamental group is a codimension-4 shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrator if it is a codimension-2 fibrator. In the last section we prove our main result: all connected, special PL n-manifolds with non-trivial first homology groups are shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrators if they are codimension-2 shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrators. If the special PL n-manifold has also a non-cyclic fundamental group, the hypothesis of being a codimension-2 shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrator can be omitted. Low-dimensional examples of shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrators are all closed, orientable surfaces with negative Euler characteristics. Next we discuss which connected sums are shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrators and prove that all special manifolds that are connected sums of closed, aspherical, orientable n-manifolds, with a non-trivial first homology group, are shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrators.

At the end we list some open and unsettled problems.

Chapter 1

Approximate Fibrations

In this chapter we will mostly discuss approximate fibrations. First, in section 1, we review several relevant definitions and establish some standard notations used in the thesis. Next, in section 2, we will review the definition of approximate fibration, and then give some of its properties. In section 3, we will discuss a movability condition of maps and see how approximate fibrations can be characterized in terms of movability conditions. In the last section, we will prove a lemma which plays a big role in proving our main results in chapter 5.

1.1 Some definitions and notations

The terminology and definitions that we will use follow standard textbooks such as Munkres' books ([49] and [50]), used for standard material on general and algebraic topology, and Rourke-Sanderson's book ([52]) used for the material on piecewise linear topology.

Symbols $\simeq$, $\approx$, $\cong$, χ and β_i denote homotopy, homeomorphism, isomorphism, Euler characteristic and the i-th Betti number in that order, and homology and cohomology groups will be computed with integer coefficients unless otherwise specified. As usual, space means topological space and maps are continuous functions.

A **(topological) n-manifold** M is a separable, metric space whose points each have a neighborhood homeomorphic to an open subset in $\mathbb{R}^n$. M is called an **n-manifold with boundary** if each of its points has a neighborhood homeomorphic to an open subset of either $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. The **boundary** of an n -manifold M, denoted ∂M, is the set of points of M corresponding to $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n_+$; the **interior** of M, denoted $\mathrm{Int}M$, is $M \backslash \partial M$. By invariance of domain, ∂M is either empty or an $(n-1)$-manifold and $\partial\partial M = \varnothing$. A manifold is said to be **closed** if it is compact and has an empty boundary and is **open** if it has no compact component and has an empty boundary. A manifold M is **aspherical** if $\pi_i(M) = 0$ for all $i > 1$. Superscripted capital letter (e.g. M^n) will denote a manifold of dimension represented by the superscript.

A **generalized k-manifold** is a finite dimensional, locally contractible metric space X, such that $H_*(X, X \backslash \{x\}) \cong H_*(\mathbb{R}^k, \mathbb{R}^k \backslash \{0\})$ for all $x \in X$. A **simplicial homotopy k-manifold** is a triangulated polyhedron K in which the link of each i-simplex has the homotopy type of the $(k-i-1)$-sphere. Unlike polyhedral generalized manifolds, in which vertices possibly fail to have a Euclidean neighborhood, simplicial homotopy manifolds are genuine topological manifolds.

A separable, metric space X is an **absolute neighborhood retract** (ANR) if for any separable, metric space Y, for any closed subset A of Y, and for any map $f : A \to X$ there exists an open set U containing A and an extension map $F : U \to X$ such that $F|_A = f$. If there exists an extension map over all of Y, then X is called an **absolute retract** (AR). All CW complexes and manifolds are ANR's.

A map $p : M \to B$ is **proper** provided $p^{-1}(C)$ is compact for any compact subset C of B. Note that p is proper if and only if p is closed and each point inverse is compact.

A subset $V \subset \mathbb{R}^n$ is a **polyhedron** if each point $a \in A$ has a cone (or stellar) neighborhood $S = v * L = \{\lambda v + \mu l \,|\, l \in L,\, \lambda, \mu \in \mathbb{R},\, \lambda, \mu \geq 0 \,\text{and}\, \lambda + \mu = 1\}$ in V, where L is compact. S is called a **star** of v in V and L a **link** of v in V. All finite simplicial complexes are polyhedra.

A map $f : V \to Y$ between polyhedra is **piecewise-linear** (abbreviated PL) if each point $v \in V$ has a stellar neighborhood $S = v * L$, such that $f(\lambda v + \mu l) = \lambda f(v) + \mu f(l)$, where $l \in L$ and $\lambda, \mu \geq 0$, $\lambda + \mu = 1$. A **PL** n**-manifold** is a polyhedron whose points each have a neighborhood PL homeomorphic to an open subset in $\mathbb{R}^n$.

If B is a simplicial complex, then $B^{(j)}$ denotes the j-skeleton of B and B^j denotes the j-th derived subdivision of B.

A map $f : X \to Y$ is a **shape equivalence** provided for each ANR P, $\tilde{f} : [Y, P] \to [X, P]$ is a bijection of sets, where each $[Y, P]$ and $[X, P]$ denotes the set of homotopy classes of maps and $\tilde{f}([\alpha]) = [\alpha f]$ for $[\alpha] \in [Y, P]$.

A map $f : N \to N'$ between closed, orientable n-manifolds is said to have a (absolute) **degree** d if there are choices of generators $\gamma \in H_n(N) \cong \mathbb{Z}$, $\gamma' \in H_n(N') \cong \mathbb{Z}$, such that $f_*(\gamma) = d\gamma'$, where $d \geq 0$ is an integer.

Proposition 1.1.1. ([27]) *If $\theta : \widetilde{X} \to X$ is a d-fold covering map, $d > 0$, then the degree of θ is d and $\chi(\widetilde{X}) = d\chi(X)$.*

1.2 Properties of Approximate Fibrations

Approximate fibrations were introduced and studied by Coram and Duvall ([6], [7]) as a generalization of Hurewitcz fibrations and cell-like maps. The defining property of a fibration, called the homotopy lifting property (HLP), is a valuable property for a map to have. The next property, which is generalization of HLP, is almost as valuable as the HLP while applying to a larger class of maps.

Definition 1.2.1. ([7]) A surjective map $p : E \to B$ between metric spaces has the **approximate homotopy lifting property** (AHLP) with respect to a space X provided that, given a cover $\mathfrak{U}$ of B and maps $g : X \to E$ and $H : X \times [0,1] \to B$, such that $pg = H_0$, there exists a map $\widetilde{H} : X \times [0,1] \to E$, such that $\widetilde{H}_0 = g$ and $p\widetilde{H}$ and H are $\mathfrak{U}$-close (i.e. for each $z \in X \times [0,1]$, there exists an $U_z \in \mathfrak{U}$ such that $\{H(z), p\widetilde{H}(z)\} \subset U_z$).

From the definition, we can see that AHLP generalizes the usual HLP, whose definition is the same except that $p\widetilde{H} = H$ is required rather than $p\widetilde{H}$, H being $\mathfrak{U}$-close.

Definition 1.2.2. The proper, surjective map $p : E \to B$ between locally compact ANR's is an **approximate fibration** if p satisfies the approximate homotopy lifting property with respect to all spaces.

By definition, fibrations are approximate fibrations. The next example shows that the set of approximate fibrations is larger than the set of fibrations.

Example 1.2.3. ([6]) Let $W = W_1 \cup B$, where $W_1 = \{(0,t) \,|\, -1 \le t \le 1\} \cup \{\big(x, sin(\frac{\pi}{x})\big) \,|\, 0 < x \le 1\}$ and B is an arc which meets W_1 only in the endpoints $(0,0)$ and $(1,0)$. Let x_0 be a base point in the 1-sphere S^1, and let $\pi_2 : S^1 \times S^1 \to S^1$ be the projection map onto the second factor. There is a compactum $A \subset S^1 \times S^1$, such that $A \approx W$ and such that there is a homeomorphism

$$h : (S^1 \times S^1) \backslash A \to S^1 \times (S^1 \backslash \{x_0\}).$$

Then the map given by

$$p(x) = \begin{cases} \pi_2 h(x) & x \in (S^1 \times S^1)\backslash A \\ x_0 & x \in A \end{cases}$$

is continuous, and has a property that $p^{-1}(x_0) = A$ and $p^{-1}(y)$ is a copy of S^1 for each $y \neq x_0$. Moreover, it is an approximate fibration which is not even a weak fibration.

Proposition 1.2.4. ([11] Lemma 2.5) *Suppose that $p : M \to B$ is a proper map defined on an $(n+k)$-manifold M and $q : \widetilde{M} \to M$ is a finite covering. Then p is an approximate fibration if and only if $pq : \widetilde{M} \to B$ is.*

A few of the useful consequences of a map $p : E \to B$ being a fibration are:

(1) ([55]) the property that point inverses are ANR, when E and B are ANR;

(2) ([55] Corollary 2.8.13) the homotopy equivalence of point inverses when the base space is path connected; and

(3) ([55] Theorem 7.2.10) the exact homotopy sequence of fibration, which relates the homotopy groups of the total space E, the base space B and its fibers:

$$\cdots \to \pi_{n+1}(B, b) \to \pi_n\big(p^{-1}(b), e\big) \to \pi_n(E, e) \to \pi_n(B, b) \to \cdots$$

where $e \in p^{-1}(b)$.

Next we will give some reasons why the study of approximate fibrations is important. Approximate fibrations form a particularly useful collection, partially due to the fact that much of the theory of Hurewicz fibrations carries over to the set of approximate fibrations. Coram and Duvall proved analogous theorems about approximate fibrations to the above theorems about fibrations: If $p : E \to B$ is an approximate fibration then:

(1) ([6] Corollary 2.5) each fiber is a fundamental absolute neighborhood retract (FANR) (that is, fibers are ANR in the sense of shape theory, [2]);

(2) ([6] Theorem 2.12) any two fibers have the same shape, provided that B is path connected, ([47] provides the definition of shape);

(3) ([6] Corollary 3.5) there exists an exact sequence:

$$\cdots \to \pi_{n+1}(B,b) \to \underline{\pi}_n\big(p^{-1}(b),e\big) \to \pi_n(E,e) \to \pi_n(B,b) \to \cdots ,$$

where $b \in B$, $e \in p^{-1}(b)$, and

$$\underline{\pi}_n(p^{-1}(b),e) = \varprojlim_j \pi_n(U_j,e)$$

where (U_j, α^j) is an inverse ANR sequence associated with $p^{-1}(b)$ by inclusion.

The last property (3) of approximate fibration, the existence of an exact sequence involving the homotopy groups of domain, target and the shape-theoretic homotopy groups of any point inverse of p, is the most useful property, which we will use frequently. When we work with the PL approximate fibration these properties of an approximate fibration reduce to the usual properties of Hurewicz fibration because the fibers are ANRs, so the i^{th} shape homotopy groups are isomorphic to i^{th} homotopy groups.

1.3 Characterization of Approximate Fibrations in Terms of Movability Conditions

In the late '70s, Coram and Duvall ([7]) studied a movability condition of a map and used it to give a characterization of approximate fibrations. This section will reveal some of the material about characterizing approximate fibrations in terms of this movability condition.

Definition 1.3.1. ([7]) A proper map $p : E \to B$ between ANR's is **completely movable** if for each $b \in B$ and each neighborhood U of the fiber $p^{-1}(b)$, there exists a neighborhood V of $p^{-1}(b)$ in U, such that if $p^{-1}(c)$ is any fiber in V and W is any neighborhood of $p^{-1}(c)$ in V, then there exists a homotopy $H : V \times [0,1] \to U$, such that $H(x,0) = x$ and $H(x,1) \in W$ for each $x \in V$ and $H(x,t) = x$ for all $x \in p^{-1}(c)$ and $t \in [0,1]$.

The following example shows the existence of non-movable maps.

Example 1.3.2. ([7]) Define a map $f : S^1 \times D^2 \to D^2$ by $f(x,y) = \|y\|x$, where D^2 is a 2-disk, $x \in S^1$, $y \in D^2$, and $\|\|$ is the Euclidean norm. Then the fibers consist of all meridional circles of various radii and the center circle of the solid torus $S^1 \times D^2$, an exceptional fiber which is mapped to the center of D^2. f fails to be completely movable at the origin of the disk fiber.

Coram and Duvall proved the following characterization theorem in terms of completely movable maps.

Theorem 1.3.3. ([7] Proposition 3.6) *Let $p : E \to B$ be a proper map. Then p is an approximate fibration if and only if p is completely movable.*

The following theorem is the main tool for efficiently detecting approximate fibrations among proper PL maps.

Theorem 1.3.4. ([12] Lemma 5.1) *A proper, surjective PL map $p : M \to B$, (where M is a $(n+k)$-manifold, and B is a polyhedron), is an approximate fibration if and only if each $v \in B$ has a stellar neighborhood $S = v * L$ whose preimage collapses to $p^{-1}(v)$ via a map $R : p^{-1}S \to p^{-1}(v)$ (the final stage of the collapse), such that, for all $x \in L$, $R|_{p^{-1}(x)} : p^{-1}(x) \to p^{-1}(v)$ is a homotopy equivalence.*

The theorem says that we only need to check the homotopy equivalence condition for the link L rather than the whole star S to see whether a PL map p is a completely movable map, i.e, an approximate fibration according to Theorem 1.3.3.

1.4 Basic Lemma

In this section we will prove the basic lemma which we will use in the last chapter for proving our main results.

Lemma 1.4.1. *Let $p : M \to B$ be a given surjective map and $T \subset B$ closed, such that $\dim T = l$ and for every $t \in T$, $H_i(M, M \backslash p^{-1}(t)) \cong 0$ $(i = 0, 1, ..., r)$, $l \leq r$. Then $H_j(M, M \backslash p^{-1}T) \cong 0$ whenever $j \in \{0, ..., r-l\}$.*

Proof. Clearly, Lemma 1.4.1 is valid when $\dim T = -1$. Assume it to be true for all closed subsets of B of dimension $< l$. Given an $l-$dimensional closed subset T, consider $z \in H_j(M, M \backslash p^{-1}T)$, where $0 \leq j \leq r-l$. We shall show that $z = 0$.

Fix a compact pair $(C', C'') \subset (M, M \backslash p^{-1}T)$ carrying a representative of z. Since $H_j(M, M \backslash p^{-1}(t)) \cong 0$ for every $t \in T$, Theorem 30.5 ([50]) implies that each

$t \in T$ has a neighborhood N_t in T for which the image of z in $H_j(M, M\backslash p^{-1}(N_t))$ is trivial.

Elementary dimension theory properties give a cover $\{C'_i \,|\, i = 1, ..., m\}$ of $p(C') \cap T$ by closed sets, such that $\{C'_i\}$ refines the cover $\{N_t \,|\, t \in p(C') \cap T\}$, the interior (rel$T$) of $\cup C'_i$ contains $p(C') \cap T$, and the frontier of each C'_i has the dimension $\leq l-1$. Now take the cover $\{C_i | i = 1, ..., m\}$, such that $C_i = C'_i \backslash (\cup_{j<i} \text{int } C'_j)$, for all i. Define E_i as $p^{-1}\big(\overline{T\backslash \cup_{j=i+1}^m C_j}\big)$, $(i = 0, ..., m)$. Since E_0 doesn't intersect C', the image of z in $H_j(M, M\backslash E_0)$ is trivial. Inductively, for $E' = E_{i-1}$ and $E'' = p^{-1}(C_i)$, we presume that the image of z in $H_j(M, M\backslash E')$ is trivial and we know it is trivial in $H_j(M, M\backslash E'')$; by construction

$$\dim\big(\overline{T\backslash \cup_{j=i}^m C_j} \cap C_i\big) \leq \dim\big(\text{Fr}C_i\big) \leq l-1.$$

Since E' and E'' are closed subsets of M for which $\dim\big(\overline{T\backslash \cup_{j=i}^m C_j} \cap C_i\big) \leq l-1$, the Mayer-Vietoris sequence for the "excisive couple of pairs" $\big\{(M, M\backslash E'), (M, M\backslash E'')\big\}$ (see [55], p.189) yields an inclusion-induced isomorphism α

$$\begin{array}{rcl} H_{j+1}\big(M, M\backslash(E' \cap E'')\big) & \to & H_j\big(M, M\backslash(E' \cup E'')\big) \xrightarrow{\alpha} \\ \xrightarrow{\alpha} H_j(M, M\backslash E') \oplus H_j(M, M\backslash E'') & \to & H_j\big(M, M\backslash(E' \cap E'')\big) \end{array}$$

because of the inductive assumption that $H_s\big(M, M\backslash(E' \cap E'')\big) = 0$, $(s = j, j+1)$. Therefore, the image of z in $H_j\big(M, M\backslash(E' \cup E'')\big) = H_j(M, M\backslash E_i)$ is trivial. In particular, when $i = m$, this proves that z itself is trivial. $\square$

Chapter 2

Shape Fibrators

Since the late '80s, R. J. Daverman has been addressing the following question:

Which homotopy types of (PL) manifolds, when appearing as the homotopy type of all point inverses of (PL) maps, force these maps to be approximate fibrations?

This chapter will give a short overview of what is answered in terms of this question and will present a new class of manifolds that has the property of detecting approximate fibrations among maps between manifolds. In the first two sections of this chapter we will review concepts of codimension-k (orientable) fibrators and PL fibrators, include examples of them, and in section 3 will introduce the new definition of shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrators.

Note: The concepts of Hopfian groups and Hopfian manifolds that are used in several occasions during this chapter are discussed in the next two chapters.

2.1 Codimension-k (orientable) Fibrators

Throughout this chapter, N is a fixed closed, connected (PL) n-manifold.

A proper, surjective (PL) map $p : M \to B$ defined on a closed, connected, (PL) $(n+k)$-manifold is said to be an N**-shaped (PL) map** if each fiber $p^{-1}(b)$, $b \in B$, (where B is a polyhedron), has the homotopy type (or more generally the shape) of N.

The following lemma gives a useful property of N-shaped PL maps that we use in proving our main results in chapter 5.

Lemma 2.1.1. ([12] Codimension Reduction Lemma 3.1) *Let $p : M \to B$ be an N-shaped PL map defined on the PL $(n+k)$-manifold M. Then each $b \in B$ has a PL neighborhood $S = b * L \subset B$, such that $p^{-1}S$ is a regular neighborhood of $p^{-1}(b)$ in M and $p^{-1}L = \partial(p^{-1}S)$ is a PL $(n+k-1)$-manifold.*

In the late '80s, Daverman introduced the following definition ([11]): N is called a **codimension-k (orientable) fibrator** if, for every N-shaped map $p : M \to B$, where M is a closed, connected (respectively, orientable) $(n+k)$-manifold, and B is finite dimensional, p is an approximate fibration.

The specific requirement of B being finitely dimensional cannot be omitted since an N-shaped map might raise the dimension to infinity ([24]). But if B is finite dimensional the next theorem shows that it must be k-dimensional:

Theorem 2.1.2. ([17] Corollary 1.3 and Theorem 2.1) *Let $p : M \to B$ be an N-shaped map from an $(n+k)$-manifold M onto a finite dimensional space B. Then* $\dim B = k$. *Furthermore, if p is an approximate fibration, then B is a generalized k-manifold.*

Codimension-1 fibrators are well-understood ([10]; [17]).

A lot about codimension-2 fibrators is also known. Daverman, Chinen, Im and Kim gave a rich list of manifolds that are codimension-2 fibrators ([16]; [4]; [5]; [36]; [37]; [41]; [42]; [43]). In his paper [11], Daverman showed that (1) all closed, simply connected manifolds (Corollary 2.4), (2) closed surfaces with a negative Euler characteristic (Theorem 3.1) and (3) real projective n-spaces $\mathbb{R}P^n$ $(n > 1)$ (Theorem 5.1), are codimension-2 fibrators. Moreover, Im and Kim proved that any finite product of closed, orientable surfaces with a genus of at least 2 is a codimension-2 fibrator (Main result, [35]).

Remark 2.1.3. The torus, $T = S^1 \times S^1$, is the only closed, connected, orientable surface which is not a codimension-2 fibrator. The Klein bottle also fails to be a codimension-2 fibrator. S^1 is a codimension-1 o-fibrator but fails to be a codimension-2 o-fibrator ([8]). Daverman also proved that any closed manifold that regularly cyclically covers itself (non-trivially) fails to be a codimension-2 fibrator (Theorem 4.2, [11]).

Daverman also showed that every closed, $(k-1)$-connected n-manifold N $(k > 1)$ (Theorem 2.3, [11]) is a codimension-k fibrator. As an immediate consequence it follows that S^n, $n > 1$ is a codimension-n fibrator (Theorem 4.1, [9]). The next corollary that follows from Theorem 5.12 ([9]) shows that homology n-spheres (i.e. n-manifolds Σ^n for which $H_*(\Sigma^n) \cong H_*(S^n)$) are essentially as effective as S^n at inducing approximate fibrations.

Corollary 2.1.4. *If the homology n-sphere Σ^n is a Hopfian manifold and has a Hopfian fundamental group, then Σ^n is a codimension-n fibrator.*

The following example shows that no homology n-sphere Σ^n is a codimension-$(n+1)$ fibrator.

Example 2.1.5. Let W be the open cone on Σ^n. There is a proper Σ^n-shaped map $p : W \times \Sigma^n \to W$ whose point preimages are either $\{c\} \times \Sigma^n$, c being the cone point of W, or $r\Sigma^n \times \{s\}$, $r\Sigma^n \subset W$ being one of the cone levels and $s \in \Sigma^n$. Clearly, p is not an approximate fibration, since every retraction $W \times \Sigma^n \to \{c\} \times \Sigma^n$ restricts to a degree zero map on all other fibers $p^{-1}(w) = r\Sigma^n \times \{s\}$. Note that W is NOT necessarily an $(n+1)$-manifold: namely, $W \backslash \{c\} \approx \Sigma^n \times (0, 1)$, and the link of c is a homology n-sphere. When Σ^n is a homology n-sphere with a non-trivial fundamental group, then the link of c cannot have the homotopy type of an n-sphere, so W cannot be a topological $(n+1)$-manifold.

2.2 PL Fibrators

Now we restrict our interest to the following PL setting: let N be a fixed closed, connected PL n-manifold, M a closed, connected, PL $(n+k)$-manifold, B a polyhedron and $p : M \to B$ a proper, surjective PL map.

Then p is an N**-like map** ([12]) if each fiber collapses to an n-complex homotopy equivalent to N. By definition, everything provable about N-shaped PL maps is provable for N-like maps.

One of the advantages of N-like PL maps over N-shaped PL maps is given in the next theorem. It shows that N-like approximate fibrations force the base space to be nice space.

Theorem 2.2.1. ([12] Theorem 5.4) *If the N-like map $p : M \to B$ is an approximate fibration, then B is a simplicial homotopy k-manifold.*

Theorem 2.1.2 might not guarantee that the weaker condition of being an N-shaped PL approximate fibration forces the base space to be as nice as an N-like approximate fibration does. The next example will illustrate an approximate fibration whose base is not a manifold, giving a distinction between N-like and N-shaped PL maps.

Example 2.2.2. (Daverman) Let Σ^3 be a non-simply connected homology 3-sphere that bounds a contractible, but not a collapsible PL 4-manifold W^4. Construct a manifold $M = \left(\Sigma^3 \times W^4\right) \cup_\partial \left(\Sigma^3 \times \partial W^4 \times [1, \infty)\right)$, where the attaching is done via the identification $(x, y) \sim (x, y, 1)$ for $y \in \partial W^4$. Consider a map $p : M^7 \to B^4$ with $p^{-1}(b_0) = \left(\Sigma^3 \times W^4\right) \cup_\partial \left(\Sigma^3 \times \partial W^4 \times \{1\}\right) \approx \left(\Sigma^3 \times W^4\right)$ and $p^{-1}(b) = \Sigma^3 \times \{q\} \times \{t\}$, where $q \in \partial W^4$ and $t \in (1, \infty)$. p is a Σ^3-shaped approximate fibration. Also, $B \backslash \{b_0\} \approx \partial W^4 \times (1, \infty)$ (i.e. B is the open cone on Σ^3), and b_0 is a non-manifold point since its link is not simply connected. So p is not a Σ^3-like approximate fibration since the base space of any N-like approximate fibration is necessarily a manifold.

Also, the concept of N-like maps offers the significant homotopy-theoretical relationships given in the next lemma, which are not possessed by a more general N-shaped map.

Lemma 2.2.3. ([14] Lemma 2.4) *Let $p : M \to B$ be an N-like map and $b \in B$. Then, for the PL neighborhood $S = b * L \subset B$ of Lemma 2.1.1,* $\mathrm{incl}_\sharp : \pi_i(p^{-1}L) \to \pi_i(p^{-1}S)$ *is an isomorphism for $1 \leq i \leq k-2$ and an epimorphism for $i = k-1$.*

In the early '90s Daverman defined another concept: N is a **codimension-k (orientable) PL fibrator** if, for every N-like map $p : M \to B$, where M is an (respectively, orientable) PL $(n+k)$-manifold, p is an approximate fibration. Moreover, N is a **PL (orientable) fibrator** if it is a codimension-k (orientable) fibrator for all $k > 0$. In the orientable setting we abbreviate by writing simply that N is a codimension-k PL o-fibrator, or in the extreme case, a PL o-fibrator.

PL fibrators do exist–a large collection within the class of aspherical manifolds. Without being mathematically precise, we can say that "most" manifolds are PL fibrators. But we don't know if a similar statement is true about codimension-k fibrators, $k > 2$, without the PL restriction. For $k = 2$, among PL manifolds there is no known distinction between the sets of codimension-2 fibrators and of codimension-2 PL fibrators. But there are manifolds that are codimension-2 fibrators which fail to be PL fibrators: S^m; $\mathbb{R}P^m$; the orientable S^m-bundle over $\mathbb{R}P^m$; those 3-manifolds N^3 covered by S^3 that arise as coset spaces with respect to the quaternionic group structure of S^3; and Cartesian products involving any of the above as a factor. Among closed surfaces, both the 2-sphere and the projective plane are codimension-2 fibrators but NOT codimension-3 PL fibrators. Excluding the torus, Klein bottle (see Remark 2.1.3), S^2, and $\mathbb{R}P^2$, all other closed surfaces are both codimension-2 fibrators and PL fibrators (Theorem 5.9, [12] and Proposition 4.1, [20]).

In this PL setting, some homology 3-spheres turn out to be PL o-fibrators even though they are not codimension 4-fibrators (example 2.1.5).

Theorem 2.2.4. ([12] Theorem 5.10) *Let Σ^3 denote an aspherical homology 3-sphere, such that $\pi_1(\Sigma^3)$ is Hopfian and Σ^3 admits no (non-trivial) regular covering by another homology 3-sphere. Then Σ^3 is a PL o-fibrator.*

Another big benefit of this PL setting is the opportunity for induction on the codimension. Namely, suppose N is a codimension-$(k-1)$ PL o-fibrator. Take an N-like map $p : M^{n+k} \to B$. To show that N is a codimension-k PL o-fibrator, we need to show that B is a manifold and p is an approximate fibration. As a consequence of the Codimension Reduction Lemma 2.1.1, $p| : p^{-1}L_b \to L_b$ is an approximate fibration and L_b is a closed $(k-1)$-manifold for each vertex $b \in B$. To show that B is a manifold, we must compute homotopy groups of L_b for each vertex $b \in B$ using the homotopy exact sequence of the approximate fibration $p|_{p^{-1}L_b}$. So we need to prove that all links of vertices of B are homotopy $(k-1)$-spheres, and then we can do induction. To prove that p is an approximate fibration, we need to prove that the retraction $R : p^{-1}S_b \to p^{-1}(b)$, restricts to homotopy equivalences $R| : p^{-1}(c) \to p^{-1}(b)$ for all $c \in L_b$, (Theorem 1.3.4). In this case, we can split the restricted retraction $R| : p^{-1}(c) \to p^{-1}(b)$ as:

$$p^{-1}(c) \to p^{-1}L_b \to p^{-1}S_b \to p^{-1}(b)$$

where the first two maps are inclusion and the last one is a strong deformation retraction. We consider two parts. For the first part we can look at the the approximate fibration $p| : p^{-1}L_b \to L_b$, and for the other, the composition of the inclusion incl : $p^{-1}L_b \to p^{-1}S_b$ and the homotopy equivalence (deformation retraction) $R : p^{-1}S_b \to p^{-1}(b)$. When studying the induced homomorphisms on the homotopy groups, dealing with N-like maps allows the results from Lemma 2.2.3, which are not true when we work with N-shaped maps.

2.3 Shape $\mathbf{m}_{\mathbf{simpl}}$o-Fibrators

We already pointed out that when dealing with N-shaped maps we do not have a nice target space, which means that we will have difficulties when looking at the restricted retraction over links (in this case they need not be manifolds) to do induction. Nam ([51]) solved this problem by working with a PL^2 setting, i.e. taking the target space B to be also a PL manifold. In this case links are actual spheres. Nam introduced the following concept (under a slightly different name):

N is a **codimension-k PL^2 shape m-fibrator** (**codimension-k PL^2 shape mo-fibrator**) if for every an N-shaped PL map between PL manifolds, $p : M \to B$, where M is an (respectively, orientable) PL $(n+k)$-manifold. If N is a codimension-k PL^2 shape m-fibrator (codimension-k PL^2 shape mo-fibrator) for all k, then N is called a **PL^2 shape m-fibrator** (**PL^2 shape mo-fibrator**).

In his doctoral thesis ([51]), Nam proved that (1) an aspherical n-manifold with a Hopfian fundamental group that is a codimension-2 PL^2 shape mo-fibrator (Theorem 3.4.2), (2) $\mathbb{Q}P^n$ and $\mathbb{C}P^n$, $n > 1$, (Theorem 3.4.4), and (3) $\mathbb{C}P^2 \sharp \mathbb{C}P^2$ (Theorem 4.2.1), are all PL^2 shape mo-fibrators.

Our main interest in Chapter 5 will be manifolds that can detect approximate fibration in a slightly changed PL setting. Namely, N and M are as before, p is an N-shaped PL map, but B is a triangulated manifold (that is not necessarily a PL manifold). In this case we don't have the collapsibility of fiber to a complex and we don't have a "nice" PL manifold as a target. So in this case we will need different results that will help us solve this problem. We introduce the following concept:

Definition 2.3.1. N is called a **codimension-k shape $\mathbf{m_{simpl}}$-fibrator** if for every PL $(n+k)$-manifold M and N-shaped PL map $p : M \to B$, where B is a simplicial triangulated manifold, p is an approximate fibration.

The abbreviation $\mathrm{m_{simpl}}$ tells us that we have a simplicial triangulated manifold as a target space.

Definition 2.3.2. Similarly, we call N a **codimension-k shape orientable $\mathbf{m_{simpl}}$-fibrator** if for every PL *orientable* $(n+k)$-manifold M and N-shaped PL map p :

$M \to B$, where B is a simplicial triangulated manifold, p is an approximate fibration. We abbreviate this by writing that N is a codimension-k shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrator.

If N is a codimension-k shape $\mathrm{m}_{\mathrm{simpl}}$-fibrator (codimension-$k$ shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrator) for all k, then N is called a **shape $\mathbf{m}_{\mathbf{simpl}}$-fibrator**, (**shape $\mathbf{m}_{\mathbf{simpl}}$o-fibrator**).

It is easy to prove that codimension-k (PL, shape $\mathrm{m}_{\mathrm{simpl}}$) fibrators are necessarily codimension-$(k-1)$ (respectively, PL, shape $\mathrm{m}_{\mathrm{simpl}}$) fibrators as well.

Since the image spaces B in codimension-2 are always manifolds (Theorem 3.6, [22]), there cannot be much difference between codimension-2 PL fibrators and codimension-2 PL shape $\mathrm{m}_{\mathrm{simpl}}$-fibrators. The two classes are precisely the same among Hopfian manifolds with Hopfian fundamental groups.

The main results in the last chapter will provide us with a list of manifolds that are shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrators.

Chapter 3

Super Hopfian Groups

In this chapter we discuss some group theoretical concepts. We list some algebraic results on Hopfian and hyperhopfian groups that we will need for our later reference, then we present the new concept of super Hopfian groups.

In section 1, we recall definitions of Hopfian and hyperhopfian groups and give examples of them. We also list some results on finitely generated groups and residually finite groups. Then, in section 2, we introduce the term super Hopfian group and give examples of it. In section 3, we prove that any free product of non-trivial, finitely generated, residually finite groups, at least one of which is not $\mathbb{Z}_2$, is a super Hopfian group. We give two immediate corollaries of this result.

3.1 Some Results on Finitely Generated, Hopfian and Hyperhopfian Groups

Finitely generated groups are of main interest in the next section, so here we list some results about them that we'll use later. Kurosh proved that every finitely generated group can be written as a free product of indecomposable factors, unique up to isomorphism and the order of factors.

Theorem 3.1.1. (**Grushko**) *Let F be a finitely generated free group, $G = G_1 * G_2$ and let $\phi : F \to G$ be an epimorphism. Then there are subgroups F_1 and F_2 of F, such that $F = F_1 * F_2$ and $\phi(F_i) = G_i$.*

We recall three results from the work of P. Scott and T. Wall ([53]). For a finitely generated group G, $\rho(G)$ denotes the minimal number of generators, i.e. the rank of G. The first result, which we frequently use in section 3, is a consequence of Grushko's theorem:

Corollary 3.1.2. ([53] Corollary 2.1) *If $G = G_1 * G_2$, then $\rho(G) = \rho(G_1) + \rho(G_2)$.*

Corollary 3.1.3. ([53] Corollary 2.2) *If G is a finitely generated group, then $G = G_1 * \cdots * G_n$ for some n, where each G_i is freely indecomposable (i.e. if $G_i = A * B$ then A or B is trivial).*

Theorem 3.1.4. ([53] Theorem 3.11) *If $G = A * B$, where A, B are non-trivial and H is a finitely generated, normal subgroup of G, then H is trivial or has a finite index in G.*

Following P. Hall, we shall say that a group G is **residually finite** if to each non-unit element g in G, there corresponds a homomorphism taking G onto a finite group and g onto a non-unit element of this image group. In other words, G is a residually finite group if every non-trivial element of G is mapped non-trivially in some finite quotient group of G.

All subgroups of residually finite groups also have this property. Also, the direct product of any two residually finite groups, again, is residually finite. Finite groups, free groups ([29]), and fundamental groups of closed surfaces ([32]) are residually finite. Every finitely generated group G with an abelian, normal subgroup N and nilpotent quotient group G/N is residually finite (Theorem 1, [30]). Of course, not all groups have this property. For instance, G.Higman ([34]) gave an example of an infinite group with 4 generators and 4 defining relators, the only finite quotient group of which is the trivial one.

The following result about residually finite groups follows from Gruenberg's work on root properties ([28]):

Theorem 3.1.5. *Every free product of residually finite groups is itself residually finite.*

A group G is called **Hopfian** (after Heinz Hopf, 1894-1971) if every epimorphism $\varphi : G \to G$ is an automorphism. In other words, G is Hopfian if it is not isomorphic to a proper factor of itself. For if G is not isomorphic to a proper factor of itself and $f : G \to G$ is onto, then $\mathrm{Im} f = G \cong G/\ker f$, shows that $\ker f = 1$, which makes f an isomorphism. Conversely, if G is Hopfian and G/H is a factor group of G, such that $G/H \cong G$, then the natural map $f : G \to G/H$ followed by an isomorphism of G/H onto G, is an isomorphism of G onto G. Therefore, $H = 1$.

Every finite group, finitely generated abelian group, and simple group is Hopfian. Also, if G is finitely generated and H is a Hopfian subgroup of G of finite index, then G is Hopfian (Corollary 1, [54]).

Theorem 3.1.6. ([46]) *Any finitely generated, residually finite group is Hopfian.*

In particular, fundamental groups of closed 2-manifolds are Hopfian groups.

Theorem 3.1.7. ([23] Theorem 3.1) *A free product of finitely many, finitely generated Hopfian groups is a Hopfian group.*

In some respects, the property of Hopficity is strange. For example, G. Baumslag and D.Solitar ([1]) have constructed a non-Hopfian group defined by two generators and a single defining relation $G =< a,\, b\,|a^{-1}b^n a = b^l >$, for n, l relatively prime. The non-Hopfian property is preserved under the operation of taking free products; if G is the free product of A and B, and A is a non-Hopfian group, then so is G. For if $G = A * B$, and A is a non-Hopfian group, then there exists an epimorphism $f : A \to A$, such that f is not an isomorphism. Then $f * 1_B : G \to G$ is an epimorphism but not an isomorphism. So if $A * B$ is a Hopfian group, then A and B are also Hopfian.

Another group theoretical concept introduced by Daverman ([13]) is the concept of a finitely presented group being hyperhopfian. Here we make a generalization of the definition in the sense that we omit a condition of the group being finitely presented. So we define a hyperhopfian property for any group. A group G is called **hyperhopfian** if every homomorphism $\varphi : G \to G$ with $\varphi(G) \lhd G$ and $G/\varphi(G)$ cyclic is necessarily an automorphism.

By definition, hyperhopfian groups are Hopfian. Except $\mathbb{Z}_p$, p-prime, all other simple groups are hyperhopfian. A finite fundamental group of a closed 3-manifold is hyperhopfian if and only if it has no cyclic direct factor (Theorem 4.7, [13]). Examples of hyperhopfian groups in the infinite case can be found in:

1. a nontrivial free product of finitely presented, residually finite groups, excluding the anomalous $\mathbb{Z}_2 * \mathbb{Z}_2$ (Theorem 4.11, [13]; Theorem 2.2, [26]);

2. fundamental groups of all compact surfaces with negative Euler characteristics (a class which includes all finitely generated, non-abelian free groups) (Corollary 4.10, [13]);

3. free groups with s generators, $1 < s < \infty$ (Corollary 4.9, [13]).

On the other hand, no group which splits off a cyclic direct factor has this property (Theorem 4.3, [13]). So finitely generated (non-trivial) abelian groups (which are Hopfian) are never hyperhopfian (e.g.: $\phi : \mathbb{Z} \to \mathbb{Z}$, given with $\phi(n) = 2n$ satisfies $\phi(\mathbb{Z}) = 2\mathbb{Z} \lhd \mathbb{Z}$ and $\mathbb{Z}/\phi(\mathbb{Z}) \cong \mathbb{Z}_2$ but ϕ is not an isomorphism).

3.2 Super Hopfian Groups

Now we will define a new group theoretical property. In the next chapter we'll see that this property is very useful for a fundamental group of a manifold.

Definition 3.2.1. A group G is called **super Hopfian** if for all homomorphisms $\varphi : G \to G$ such that $\varphi(G)$ is a non-trivial, normal subgroup of G, φ is an automorphism.

By definition, super Hopfian groups are Hopfian groups. All simple groups are super Hopfian. Every super Hopfian group, except $\mathbb{Z}_p$, p-prime, is hyperhopfian. First we list some examples of super Hopfian groups among finite groups.

Corollary 3.2.2. *Every non-abelian group of the order pq where p, q are distinct primes is super Hopfian.*

Proof. Let G be a non-abelian group of the order pq, with p, q-distinct primes. Suppose $\varphi : G \to G$ is a homomorphism such that $1 \neq \varphi(G) \lhd G$ and $\varphi(G) \neq G$. Using the Lagrange Theorem and the First Isomorphism Theorem, it follows that $|G/\ker\varphi| = \frac{|G|}{|\ker\varphi|} = |\varphi(G)|$, i.e, $pq = |G| = |\ker\varphi|\,|\varphi(G)|$. Then $|G/\varphi(G)| = \frac{|G|}{|\varphi(G)|} = p \,\mathrm{or}\, q$ since p and q are distinct primes. So $G/\varphi(G)$ is cyclic, and since G is hyperhopfian (Corollary 4.4, [13]), it follows that φ is an isomorphism and G is super Hopfian. $\square$

Dihedral groups $D_{2n+1} =< x, y \,|\, x^2 = y^{2n+1} = 1,\, x^{-1}yx = y^{-1} >$ of order $2(2n+1)$, where $2n+1$ is prime, are super Hopfian by Corollary 3.2.2, but $D_{2n} =< x, y \,|\, x^2 = y^{2n} = 1,\, x^{-1}yx = y^{-1} >$ are not super Hopfian (since there exists a homomorphism $\phi : D_{2n} \to D'_{2n} =< y > \lhd D_{2n}$, defined with $x \longmapsto y^n$; $y \longmapsto y^n$, that is not an isomorphism).

The quaternionic group $Q =< c, d \,|\, c^2 = (cd)^2 = d^2 >$, of order 8, is a hyperhopfian group (Section 4, [13]), which is not super Hopfian (since there exists a homomorphism $\phi : Q \to Q' = \{1, c^2\} \lhd Q$ that is not an isomorphism). The following groups fail to be super Hopfian since they are not hyperhopfian (Section 4, [13]):

1. The solvable group of order p^4 (p-prime), $< x, y \,|\, x^{p^2} = y^{p^2} = 1,\, y^{-1}xy = x^{1+p} >$;
2. Group $< a, b \,|\, a^n = b^n = 1,\, b^{-1}ab = a^{-1} >$ of order $4n^2$, $n > 1$.

Now we shift the subject to infinite groups. The group of rational numbers, $\mathbb{Q}$, is an example of a super Hopfian group that is not finitely generated.

3.3 Free Products of Finitely Generated Groups as Super Hopfian Groups

In this section we deal with free products of finitely generated groups. We present some results for detecting instances when free products of finitely generated groups are super Hopfian.

Theorem 3.3.1. *Let G_1, G_2 be non-trivial, finitely generated, residually finite groups and $G_2 \neq \mathbb{Z}_2$. Then $G_1 * G_2$ is a super Hopfian group.*

Proof. Note that the condition $G_2 \neq \mathbb{Z}_2$ is needed since $\mathbb{Z}_2 * \mathbb{Z}_2$ is not hyperhopfian (Theorem 2.2, [26]), and thus is not super Hopfian.

Consider $G = G_1 * G_2$, where G_i $(i = 1, 2)$ has no further non-trivial decomposition as a free product. The general case $G = G_1 * \cdots * G_m$, $m > 2$, is proved by the same means as this special case. Let $\psi : G \to G$ be a homomorphism with $1 \neq H = \psi(G)$ a normal subgroup of G. Here $[G : H] = k < \infty$ by Theorem 3.1.4.

Construct connected 2-complexes X_i with $\pi_1(X_i) \cong G_i$, $(i = 1, 2)$, (Corollary 1.28, [38]) join them with an edge e to form another complex $X \supset X_1 \cup X_2$, and examine the k-fold covering $q : X^* \to X$ corresponding to the subgroup H. The regularity of q ensures that the components of $q^{-1}(X_i)$ are pairwise homeomorphic. Let K_i denote a component of $q^{-1}(X_i)$. Obviously $q|_{K_i}$ gives a regular cover of X_i having an order of k_i, dividing k, with $k_1 \neq 1$ or $k_2 \neq 1$. For definiteness assume $k_2 \neq 1$. Moreover, $\pi_1(X^*)$ is a free product of k/k_1 copies of $\pi_1(K_1)$, k/k_2 copies of $\pi_1(K_2)$ and a free group F.

Examination of certain first homology groups shows F to be trivial: the free part of $H_1(X^*)$ has a rank $\beta_1(X^*)$ equal to the sum of $\frac{k}{k_1}\beta_1(K_1)$, $\frac{k}{k_2}\beta_1(K_2)$ and the rank of the abelianized free group, while similarly $\beta_1(X) = \beta_1(X_1) + \beta_1(X_2)$. Since ψ induces an epimorphism from the abelianization of G to that of H, $\beta_1(X) \geq \beta_1(X^*)$. Being of a finite index in $H_1(X_i)$, $q_*(H_1(K_i))$ has a free part isomorphic to $H_1(X_i)$, so $\beta_1(K_i) \geq \beta_1(X_i)$ for $i = 1, 2$. Hence $F = 1$.

Geometrically, this implies that $q|_{K_1}$ is 1-1 (i.e. $k_1 = 1$), for otherwise one could produce a loop in X^* as a composition of paths $\alpha_1\gamma_1\alpha_2\gamma_2 \ldots \alpha_m\gamma_m$, $m > 1$, where $q(\alpha_i)$ is contained in one of X_1, X_2, $q(\alpha_{i+1})$ is contained in the other, $q(\gamma_i) \subset e$ and the various γ_j are pairwise disjoint. Such a loop would necessarily be carried by the free part of the graph of groups used to describe $\pi_1(X^*)$.

The associated free product representation of $\pi_1(X^*)$ has $k = \frac{k}{k_1}$ copies of $\pi_1(K_1)$ and at least one copy of $\pi_1(K_2)$. $q_\sharp : \pi_1(K_1) \to \pi_1(X_1)$ is an isomorphism since $q| : K_1 \to X_1$ is a bijection.

Clearly $\pi_1(X_2)$ is generated by $q_\sharp\big(\pi_1(K_2)\big)$ and G/H. Then we have

$$\rho\big(\pi_1(K_2)\big) \geq \rho\Big(q_\sharp\big(\pi_1(K_2)\big)\Big) \geq \rho\big(\pi_1(X_2)\big) - \rho(G/H)$$

The first inequality follows from the fact that $q|_\sharp : \pi_1(K_2) \to q_\sharp\big(\pi_1(K_2)\big)$ is an epimorphism. Also $\rho(G/H) < k - 1$, for all $k > 2$ and $\rho(G/H) = 1$ if $k = 2$.

Case 1: $k \geq 3$. Since $\psi : G \to H$ is onto, this case can be ruled out immediately, for it leads to the impossibility:

$$\begin{aligned}
\rho(H) &= k\rho\big(\pi_1(K_1)\big) + (k/k_2)\rho\big(\pi_1(K_2)\big) \\
&\geq k\rho\big(\pi_1(X_1)\big) + \rho\big(\pi_1(K_2)\big) && \text{since } k/k_2 \geq 1 \\
&\geq k\rho\big(\pi_1(X_1)\big) + \rho\big(\pi_1(X_2)\big) - \rho(G/H) \\
&= \rho\big(\pi_1(X_1)\big) + \rho\big(\pi_1(X_2)\big) + (k-1)\rho\big(\pi_1(X_1)\big) - \rho(G/H) \\
&> \rho\big(\pi_1(X_1)\big) + \rho\big(\pi_1(X_2)\big) \\
&= \rho(G)
\end{aligned}$$

Case 2: $k = 2$ (Note that in this case $k_2 = 2$). The same impossibility as in Case 1 occurs if $\rho\big(\pi_1(K_1)\big) > 1$, since:

$$\begin{aligned}
\rho(H) &= 2\rho\big(\pi_1(K_1)\big) + \rho\big(\pi_1(K_2)\big) \\
&= 2\rho\big(\pi_1(X_1)\big) + \rho\big(\pi_1(K_2)\big) \\
&\geq 2\rho\big(\pi_1(X_1)\big) + \rho\big(\pi_1(X_2)\big) - \rho(G/H) \\
&= \rho\big(\pi_1(X_1)\big) + \rho\big(\pi_1(X_2)\big) + \rho\big(\pi_1(X_1)\big) - \rho(G/H) \\
&> \rho\big(\pi_1(X_1)\big) + \rho\big(\pi_1(X_2)\big) && \text{since } \rho(G/H) = 1 \text{ and} \\
& && \rho\big(\pi_1(X_1)\big) > 1 \\
&= \rho(G)
\end{aligned}$$

Now suppose that $\rho\big(\pi_1(K_1)\big) = 1$. In this case we can identify $\pi_1(K_1)$ with the cyclic group $\mathbb{Z}_p$ for the following reasoning (used previously to prove F trivial): suppose $\pi_1(K_1) = \mathbb{Z}$. The free part of $H_1(X^*)$ in this case has a rank $\beta_1(X^*) = 2\beta_1(K_1) + \beta_1(K_2)$, while similarly $\beta_1(X) = \beta_1(X_1) + \beta_1(X_2)$. Since ψ induces an epimorphism from the abelianization of G to that of H, $\beta_1(X) \geq \beta_1(X^*)$. Being of a finite index in $H_1(X_i)$, $q_*(H_1(K_i))$ has a free part isomorphic to $H_1(X_i)$, so $\beta_1(K_i) \geq \beta_1(X_i)$ for $i = 1, 2$. Since $q|_{K_1}$ is 1-1, it follows that $\pi_1(X_1) \cong \pi_1(K_1) \cong \mathbb{Z}$ Therefore, $\beta_1(K_1) = \beta_1(X_1) = 1$. It follows that $k = k_1 = 1$ which is a contradiction with $k = 2$. Accordingly,

$$H_1(X) \cong \mathbb{Z}_p \oplus H_1(X_2) \qquad \text{and} \qquad H_1(X^*) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus H_1(K_2). \tag{3.3.2}$$

Hence $\beta_1(K_2) = \beta_1(X^*) = \beta_1(X) = \beta_1(X_2)$. The middle equality follows from the fact that $\beta_1(X) \geq \beta_1(X^*)$ and $\beta_1(K_2) \geq \beta_1(X_2)$. Since $H_1(K_2)$ surjects to a subgroup of index $k_2 = 2$ in $H_1(X_2)$,

$$\big|\text{torsion}\, H_1(X_2)\big| = 2\big|\text{torsion}\, q_*(H_1(K_2))\big| \leq 2\big|\text{torsion}\, H_1(K_2)\big|. \tag{3.3.3}$$

Since $H_1(X)$ surjects to $H_1(X^*)$, it follows that

$$\big|\text{torsion}\, H_1(X)\big| \geq \big|\text{torsion}\, H_1(X^*)\big|. \tag{3.3.4}$$

Using (3.3.2), (3.3.3) and (3.3.4)we get the following inequality:

$$\begin{aligned} \big|\text{torsion}\, H_1(X^*)\big| = p^2\big|\text{torsion}\, H_1(K_2)\big| &\leq \big|\text{torsion}\, H_1(X)\big| \\ &= p\big|\text{torsion}\, H_1(X_2)\big| \\ &\leq 2p\big|\text{torsion}\, H_1(K_2)\big|, \end{aligned}$$

which reveals $p = 2$. Since $q_\sharp$ is injective, ψ can be lifted to an homomorphism

$$\psi' : G_1 * G_2 \cong \mathbb{Z}_2 * G_2 \to \mathbb{Z}_2 * \mathbb{Z}_2 * \pi_1(K_2) \cong \pi_1(X^*),$$

(defined with $\psi' = q_\sharp^{-1}\psi$), and the composite $\psi' q_\sharp$ provides a homomorphism

$$\psi' q_\sharp : \mathbb{Z}_2 * \mathbb{Z}_2 * \pi_1(K_2) \to \mathbb{Z}_2 * \mathbb{Z}_2 * \pi_1(K_2)$$

whose image has an index ≥ 1. Using the injectivity of $q_\sharp$, we have $\mathbb{Z}_2 * \mathbb{Z}_2 * \pi_1(K_2) \cong H$ and $\psi'(H) \cong q_\sharp(\psi'(H)) < H < G$. Then

$$2 \leq [G : q_\sharp(\psi'(H))] = [G : H]\,[H : q_\sharp(\psi'(H))].$$

So $[\mathbb{Z}_2 * \mathbb{Z}_2 * \pi_1(K_2) : \psi' q_\sharp(\mathbb{Z}_2 * \mathbb{Z}_2 * \pi_1(K_2))] = [H : q_\sharp(\psi'(H))] \geq 1$. Repeating the preceding sort of rank arguments of case 1 and case 2 (with $\rho\big(\pi_1(K_1)\big) > 1$) (regard G_1 as $\mathbb{Z}_2 * \mathbb{Z}_2$), we see that the index cannot be greater or equal to 2. This means that $\psi' q_\sharp$ is an epimorphism. Theorems 3.1.5 and 3.1.6 imply that $\mathbb{Z}_2 * \mathbb{Z}_2 * \pi_1(K_2)$ is Hopfian and $\psi' q_\sharp$ is an automorphism, so ψ' is onto. As a result, the sequence

$$\ker \psi' \to \mathbb{Z}_2 * G_2 \xrightarrow{\psi'} \mathbb{Z}_2 * \mathbb{Z}_2 * \pi_1(K_2)$$

has a direct product splitting. However, this is impossible, since free products are never direct products (Exercises 4.1.13, 4.1.23 and 4.1.24, [45]).

Consequently, $k = 1$. As above, using Theorems 3.1.5 and 3.1.6, $G_1 * G_2$ is Hopfian, implying that $\psi : G_1 * G_2 \to H \cong G_1 * G_2$ is an isomorphism, as required. $\square$

Note that in the conclusion of Case 2 regarding $\mathbb{Z}_2 * \mathbb{Z}_2 * \pi_1(K_2)$ being Hopfian, we might have used Theorems 3.1.6 and 3.1.7. The argument of the theorems establishes variations such as the following.

Corollary 3.3.5. *If G_1, G_2 are non-trivial, finitely generated groups such that G_1 is non-cyclic, and $G_1 * G_2$ is Hopfian, then $G_1 * G_2$ is super Hopfian.*

Corollary 3.3.6. *If G_1, G_2 are non-trivial, finitely generated, super Hopfian groups, with G_1 being non-cyclic and $G_1 * G_2\,(\neq \mathbb{Z}_2 * \mathbb{Z}_2)$ being Hopfian, then $G_1 * G_2$ is super Hopfian.*

So, in principle, the super Hopfian property is closed with respect to free products.

Chapter 4

Special Manifolds

In this chapter, we introduce a new concept of the special manifold. As we will see in the next chapter, shape m_{simpl}o-fibrators can be found among these special PL manifolds.

In section 1, we discuss the concepts of Hopfian manifolds and manifolds determined by π_1, then give examples of them and list some results. In section 2, we present the concept of special manifolds and give examples of them. Finally, in section 3, we see which connected sums are special manifolds.

4.1 Hopfian Manifolds and Manifolds Determined by π_1

In this section we give a list of examples of Hopfian manifolds and manifolds determined by π_1.

A closed, orientable n-manifold N is **Hopfian** ([14]) if every degree one map $f : N \to N$ inducing a π_1-isomorphism is a homotopy equivalence. In what follows we exhibit examples of Hopfian manifolds. The following two propositions present results that are well known:

Proposition 4.1.1. *A closed, simply connected, orientable n-manifold N is Hopfian.*

Proof. Let $f : N \to N$ be a degree one map, which induces a π_1-isomorphism (in this case trivially). Since f has a degree one, Theorem 67.2 ([50]), implies that $f_* : H_j(N) \to H_j(N)$ is an epimorphism for $1 \leq j \leq n$. Since N is a compact manifold, Theorem 6.9.11 ([55]), gives that each $H_j(N)$ is a finitely generated abelian group and hence is a Hopfian group. Thus f_* is an isomorphism for all j. Since N is simply connected, the Whitehead Theorem (Theorem 7.5.9, [55]) implies that $f_\sharp : \pi_j(N) \to \pi_j(N)$ is an isomorphism for all j, and thus, f is a homotopy equivalence. □

Proposition 4.1.2. *A closed, connected, orientable n-manifold N is Hopfian if either*

1. $\pi_1(N)$ is finite, or

2. $n \leq 4$ and $\pi_1(N)$ is Hopfian.

Proof. 1. Let $f : N \to N$ be a degree one map that induces a π_1-isomorphism. Because of the previous proposition we only need to check the result for manifolds N with a non-trivial finite fundamental group. Look at a universal cover $p : \widetilde{N} \to N$. Since $\widetilde{N}$ is simply connected, there exists a lift $\tilde{f} : \widetilde{N} \to \widetilde{N}$ to the composite map $fp : \widetilde{N} \to N$, such that $p\tilde{f} = fp$. The degree of p is a nonzero finite integer since $\pi_1(N)$ is finite non-trivial group and p is a finite cover. Then we have

$$(\deg p)(\deg \tilde{f}) = \deg(p\tilde{f}) = \deg(pf) = (\deg p)(\deg f).$$

It follows that $\deg \tilde{f} = \deg f = 1$. Using the previous proposition and the fact that $\widetilde{N}$ is simply connected, we can conclude that $\tilde{f} : \widetilde{N} \to \widetilde{N}$ is a homotopy equivalence.

This means that $\tilde{f}_\sharp : \pi_j(\widetilde{N}) \to \pi_j(\widetilde{N})$ is an isomorphism for all j. Since p is a universal covering map, Corollary 7.2.11 ([55]) implies that $p_\sharp : \pi_j(\widetilde{N}) \cong \pi_i(N)$ for all $j \geq 2$. Then $f_\sharp : \pi_j(N) \to \pi_j(N)$ is an isomorphism for any $j \geq 2$, since $\tilde{f}_\sharp$, $p_\sharp$ are isomorphisms. Combining the last conclusion with the hypotheses, we conclude that $f_\sharp : \pi_j(N) \to \pi_j(N)$ is an isomorphism for all j, i.e. that f is a homotopy equivalence.

2. That this conditions implies N Hopfian was shown by Hausmann ([31]). □

The next result implies that if the fundamental group of a manifold N is Hopfian, then every degree one self map induces an isomorphism on its fundamental group.

Proposition 4.1.3. ([33] Lemma 15.12) *Let $f : M \to N$ be a degree one map of closed, connected, orientable n-manifolds M, N. Then $f_\sharp : \pi_1(M) \to \pi_1(N)$ is surjective.*

Proof. Consider a covering map $p : N^* \to N$ corresponding to the subgroup $f_\sharp(\pi_1(M))$. Note that N^* is a closed, orientable n-manifold and p has degree $k = [\pi_1(N), \mathrm{Im} f_\sharp]$. Since $\mathrm{Im} f_\sharp \subset p_\sharp(\pi_1(N^*)) = f_\sharp(\pi_1(N))$, we can lift f to a map $\tilde{f} : M \to N^*$, such that $f = p\tilde{f}$. Now consider two cases:

Case 1: $k < \infty$: Then $1 = \deg f = \deg(p\tilde{f}) = (\deg p)\,(\deg \tilde{f})$. This implies that $\deg p = \pm 1$ and $\deg \tilde{f} = \pm 1$. On the other hand (absolute) $\deg p = k$; therefore, $k = 1$, i.e. $\pi_1(N) = \mathrm{Im} f_\sharp$. This implies that $f_\sharp$ is onto.

Case 2: $[\pi_1(N), \mathrm{Im} f_\sharp] = \infty$: In this case $\deg p = 0$. Then $\deg f = (\deg p)\,(\deg \tilde{f})$ $= 0$. This is a contradiction with $\deg f = 1$. So this case is impossible. □

Analysis of fibrator properties applies most readily to Hopfian manifolds with Hopfian fundamental groups. The reason for this is given in the next proposition.

Proposition 4.1.4. *Let N be a closed, connected, orientable manifold with a Hopfian fundamental group. Then N is Hopfian if and only if all degree one maps $f : N \to N$ are homotopy equivalences.*

Proof. The sufficient part is obvious. Necessity follows from Proposition 4.1.3. $\square$

The following theorem of Swarup indicates that any closed, orientable n-manifold N with $\pi_i(N) \cong 0$ for $1 < i < n-1$ is Hopfian.

Theorem 4.1.5. ([56] Lemma 1.1) *Let $f : (M, x) \to (N, f(x))$ be a map of closed, orientable n-manifolds which induces an isomorphism of the fundamental groups. Suppose that $\pi_i(M)$ and $\pi_i(N)$ are trivial for $1 < i < n-1$. Then f is a homotopy equivalence if and only if the (absolute) degree of f is one.*

As a consequence, it follows that any aspherical manifold is Hopfian.

A manifold N is **homotopically determined by** π_1 if every self map $f : N \to N$ that induces a π_1-isomorphism is a homotopy equivalence. Aspherical manifolds are common examples of manifolds determined by π_1. Moreover, an indecomposable connected sum of closed, orientable 3-manifolds is homotopically determined by π_1 if and only if at least one of the summands is aspherical.

Theorem 4.1.6. ([39] Theorem 3.3.1) *The closed, orientable 3-manifold N is homotopically determined by π_1 if $\pi_1(N)$ is not a free product of free and finite groups.*

The next observation follows immediately from the definitions.

Lemma 4.1.7. ([21] Lemma 2.1) *Every closed, orientable manifold homotopically determined by π_1 is Hopfian.*

Theorem 4.1.8. *Suppose N is a Hopfian n-manifold such that $H^n(N)$ is in the subring of $H^*(N)$, generated by $H^1(N)$. Then N is homotopically determined by π_1.*

Proof. Let $f : N \to N$ be a self map that induces a π_1-isomorphism. Since N is a compact manifold, $H^1(N)$ is finitely generated. By the given hypothesis, there exist $\alpha_1, \ldots, \alpha_n$ in $H^1(N)$ such that $\alpha_1 \cup \cdots \cup \alpha_n$ generates $H^n(N) \cong \mathbb{Z}$. We need to prove that the degree of f is one.

Since $f_\sharp : \pi_1(N) \to \pi_1(N)$ is an isomorphism, it follows that $f_* : H_1(N) \to H_1(N)$ is onto. Since $H^1(N) =$ a free part of $H_1(N)$, it follows that $f^* : H^1(N) \to H^1(N)$ is also onto. Then for each α_i in $H^1(N)$, there exists γ_i in $H^1(N)$, such that $f^*(\gamma_i) = \alpha_i$. Therefore, $f^*(\gamma_1 \cup \cdots \cup \gamma_n) = f^*(\gamma_1) \cup \cdots \cup f^*(\gamma_n) = \alpha_1 \cup \cdots \cup \alpha_n$. That is, there exists an element $\gamma_1 \cup \cdots \cup \gamma_n$ in $H^n(N)$, which is mapped into a generator $\alpha_1 \cup \cdots \cup \alpha_n$ of $H^n(N)$. Thus, $f^* : H^n(N) \to H^n(N)$ is an epimorphism between copies of $\mathbb{Z}$. We conclude that f^* is an isomorphism. Then $f_* : H_n(N) \to H_n(N)$ is an isomorphism, and $\deg f = 1$. Since N is a Hopfian manifold, we can conclude that f is a homotopy equivalence. $\square$

The main results of R.J.Daverman and Y.Kim ([21]) assure that a connected sum $N_1 \sharp N_2$ of closed, orientable, PL n-manifolds is homotopically determined by π_1 if $\pi_1(N_i)$ doesn't have a "factor" $\mathbb{Z}$, $i = 1, 2$ and either

1. N_1 is homotopically determined by π_1, $\pi_{n-1}(N_1) \cong 0$, $\beta_1(N_1) > 0$ and $N_1 \sharp N_2$ is a Hopfian manifold (Theorem 3.1); or

2. N_1 is aspherical and $N_1 \sharp N_2$ is a Hopfian manifold (Theorem 3.4).

4.2 Special PL Manifolds

From now on our main interest will be pointed toward a particular class of Hopfian manifolds called special manifolds.

Definition 4.2.1. A closed manifold N is **special** if $\pi_1(N) \neq 1$ and for all $f : N \to N$, such that $1 \neq f_\sharp\big(\pi_1(N)\big) \lhd \pi_1(N)$, f is a homotopy equivalence.

Note that every orientable, special manifold is homotopically determined by π_1, and so is Hopfian.

Remark 4.2.2. Every closed, manifold N which is homotopically determined by $\pi_1(N)$, where $\pi_1(N)$ is super Hopfian, is special.

Proof. Let $f : N \to N$ be a self map such that $1 \neq f_\sharp\big(\pi_1(N)\big) \lhd \pi_1(N)$. Since $\pi_1(N)$ is super Hopfian, it follows that $f_\sharp$ is an isomorphism, and N being homotopicaly determined by π_1 implies that f is a homotopy equivalence. □

In the next chapter we'll describe some special manifolds that are shape m_{simpl}o-fibrators.

Theorem 4.2.3. *Closed, orientable surfaces S with $\chi(S) < 0$ are special manifolds.*

Proof. Suppose $f : S \to S$ is such that $1 \neq f_\sharp\big(\pi_1(S)\big) \lhd \pi_1(S)$.

First note that $\pi_1(S) =< a_1, b_1, ..., a_g, b_g \mid [a_1, b_1][a_2, b_2]...[a_g, b_g] >$, where g is a genus of S and $g > 1$, (since $\chi(S) = 2-2g < 0$). Thus S has a cell structure with one 0-cell, $2g$ 1-cells and one 2-cell. The 1-skeleton is a wedge sum of $2g$ circles, with a fundamental group free on $2g$ generators. The 2-cell is attached along the loop given by the product of the commutators of these generators, $[a_1, b_1][a_2, b_2]...[a_g, b_g]$. Form the covering $p : S^* \to S$ corresponding to $f_\sharp\big(\pi_1(S)\big)$. Now we will consider two cases:

Case 1: $\big[\pi_1(S), f_\sharp\big(\pi_1(S)\big)\big] = k < \infty$. This is impossible if $k > 1$; in this case $\chi(S^*) = k\chi(S)$, i.e. $1 - \beta_1(S^*) + \beta_2(S^*) = k(2-2g)$. So

$$\begin{aligned}
\beta_1(S^*) &= 1 + \beta_2(S^*) - k(2-2g) \\
&> 1 + \beta_2(S^*) - (2-2g) \qquad \text{since}\, k > 1 \,\text{and}\, g > 1 \\
&= 1 + \beta_2(S) - (2-2g) \qquad \text{since}\, \beta_2(S^*) = \beta_2(S) = 1 \\
&= \beta_1(S)
\end{aligned}$$

So we have $\beta_1(S^*) > \beta_1(S)$, which precludes the existence of any epimorphism $\pi_1(S) \to f_\sharp\big(\pi_1(S)\big)$, as the induced homomorphism on abelianizations would yield $\beta_1(S^*) \leq \beta_1(S)$.

Case 2: $\big[\pi_1(S), f_\sharp\big(\pi_1(S)\big)\big] = \infty$. This case is also impossible. Namely, we will get a contradiction by constructing a group G, such that $f_\sharp\big(\pi_1(S)\big) \lhd G < \pi_1(S)$, where G is free, in the following way:

Since $\text{rank}\Big(f_*\big(H_1(S)\big)\Big) \leq \frac{1}{2}\beta_1(S)$ (Claim 4.2.5 below), we can choose $z \in H_1(S)$, such that $nz \notin f_*(H_1(S))$. Now choose $\gamma \in \pi_1(S)$, such that $\gamma \mapsto z$ under the Hurewicz homomorphism. Let G be a group generated with γ and $f_\sharp\big(\pi_1(S)\big)$. $f_\sharp(\pi_1(S))$ is free, because $\pi_1(S^*)$ is free (Claim 4.2.4 below). $\big[\pi_1(S),\, G\big] = \infty$, since $\text{rank}\Big(f_*\big(H_1(S)\big)\Big) \leq \frac{1}{2}\beta_1(S)$ and $g > 1$. Then G is also free (Claim 4.2.4 below). Also, $\big[G,\, f_\sharp(\pi_1(S))\big] = \infty$, since $|G/f_\sharp(\pi_1(S))| = |<\gamma>|$ and γ has infinite order. This is a contradiction with Theorem 3.1.4, since $f_\sharp\big(\pi_1(S)\big)$ is a finitely generated, normal subgroup of the finitely generated, free group G. □

Claim 4.2.4. *If S^* is an $\infty - 1$ cover of S, then $\pi_1(S^*)$ is free.*

Proof. Since S^* is a non-closed surface and non-closed surfaces deformation retract onto graphs, it follows that S^* retracts to a graph X with $\pi_1(S^*) = \pi_1(X)$. But for a connected graph X with a maximal tree T, $\pi_1(X)$ is a free group with a basis the classes $[f_\alpha]$ corresponding to the edges e_α of $X \backslash T$. So $\pi_1(S^*)$ is free. $\square$

Claim 4.2.5. $\operatorname{rank}\Big(f_*\big(H_1(S)\big)\Big) \leq \frac{1}{2}\beta_1(S)$.

Proof. Note that $f_*\big(H_1(S)\big)$ has the same rank as $f^*\big(H^1(S)\big)$, since $Hom(f_*) = f^*$ by duality and $H_1(S) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus ... \oplus \mathbb{Z}}_{2g}$. Also note that $\deg f = 0$ since S^* is $\infty - 1$ covering.

Now suppose that $\operatorname{rank}\Big(f_*\big(H_1(S)\big)\Big) > \frac{1}{2}\beta_1(S)$ and look at the following commutative diagram:

$$\begin{tikzcd} H^1(S) \arrow[r, "\cap\Gamma", "\cong"'] & H_1(S) \arrow[d, "f_*"] \\ H^1(S) \arrow[u, "f^*"] \arrow[r, "\cap d\Gamma"'] \arrow[ur, phantom, "\circlearrowleft"] & H_1(S) \end{tikzcd}$$

where Γ is the orientation class for S and $d = \deg f = 0$.

It follows that $\operatorname{rank}\Big((f_*)(\cap\Gamma)(f^*)\Big)\big(H_1(S)\big)$ is at least 1. On the other hand, since $d = 0$, it follows that $H^1(S) \xrightarrow[\cap d\Gamma]{} H^1(S)$ is the trivial map, which is a contradiction. $\square$

4.3 Connected Sums as Special PL Manifolds

We describe some connected sums which are special in order to derive conclusions about shape fibrators.

Theorem 4.3.1. *Suppose N_1 is an orientable, special PL n-manifold, such that $\pi_{n-1}(N_1) \cong 0$ and $\beta_1(N_1) \neq 0$, N_2 is a closed, orientable PL n-manifold, such that $\pi_1(N_2) \neq 1$ and suppose that $N_1 \sharp N_2$ is Hopfian with a Hopfian fundamental group. Also, assume that $\pi_1(N_i)$ doesn't have a "factor" $\mathbb{Z}$, $i = 1, 2$. Then $N_1 \sharp N_2$ is a special.*

Proof. $N_1 \sharp N_2$ is homotopically determined by $\pi_1(N)$ by Theorem 3.1 ([21]), and $\pi_1(N_1 \sharp N_2)$ is super Hopfian (Theorem 3.3.5), so the previous remark implies that $N_1 \sharp N_2$ is special. □

Also, the following improvement of Theorem 4.3.1 is true, in which $\theta : \pi_{n-1}(N_1) \to H_{n-1}(N_1)$ denotes the Hurewicz homomorphism.

Theorem 4.3.2. *Suppose N_1 is a special PL n-manifold, such that $\beta_1(N_1) \neq 0$ and $\ker\big[\theta : \pi_{n-1}(N_1) \to H_{n-1}(N_1)\big] \cong 0$, and suppose N_2 is a closed, orientable PL n-manifold, such that $\pi_1(N_2) \neq 1$, and $N_1 \sharp N_2$ is Hopfian with a Hopfian fundamental group. Also, assume that $\pi_1(N_i)$ doesn't have a "factor" $\mathbb{Z}$, $i = 1, 2$. Then $N_1 \sharp N_2$ is special.*

Proof. Use Theorem 3.3 ([21]), Theorem 3.3.5 and Remark 4.2.2. □

Theorem 4.3.3. *Suppose N_1 is an aspherical, closed, orientable n-manifold, and N_2 is a closed, orientable n-manifold with $\pi_1(N_2) \neq 1$, such that $N_1 \sharp N_2$ is Hopfian with a Hopfian fundamental group. Then $N_1 \sharp N_2$ is special.*

Proof. Use Theorem 3.4 ([21]), Theorem 3.3.5 and Remark 4.2.2. □

Corollary 4.3.4. *If N_1, N_2 are closed, connected, orientable n-manifolds with non-trivial, residually finite groups, such that N_2 is a Hopfian manifold and $H^n(N_2)$ is in the subring of $H^*(N_2)$ generated by $H^1(N_2)$, and $N_1 \sharp N_2$ is a Hopfian manifold, then $N_1 \sharp N_2$ is special manifold.*

Proof. We only need to prove that (1) $N_1 \sharp N_2$ is homotopically determined by π_1, and (2) has a super Hopfian fundamental group, since then Remark 4.2.2 will give the result.

(1) $N_1 \sharp N_2$ is homotopically determined by π_1 by Theorem 4.1.8, since $H^n(N_1 \sharp N_2) \cong H^n(N_2) = \mathbb{Z}$ is in the subring of $H^*(N_1 \sharp N_2)$ generated by $H^1(N_1 \sharp N_2) = H^1(N_1) \oplus H^1(N_2)$. The latter is true since, by hypothesis, $H^n(N_2)$ is in the subring of $H^*(N_2)$ generated by $H^1(N_2)$. Then since N_2 is compact, i.e. $H^1(N_2)$ is finitely generated, there exist $\alpha_1, \ldots, \alpha_n$ in $H^1(N_2)$, such that $\alpha_1 \cup \cdots \cup \alpha_n$ generates $H^n(N_2) = \mathbb{Z}$. Then $(0, \alpha_1) \cup \cdots \cup (0, \alpha_n)$ generates $H^n(N_1 \sharp N_2)$ as required (see [14]).

(2) Using Theorem 3.3.1 we can conclude that $\pi_1(N_1 \sharp N_2)$ is super Hopfian since $\pi_1(N_1)$, $\pi_1(N_2)$ are non-trivial, finitely generated (since N_1, N_2 are compact manifolds), residually finite groups, and $\pi_1(N_2) \neq \mathbb{Z}_2$ (otherwise $\beta_1(N_2) = 0$ meaning that $H^1(N_2)$ is trivial, so the subring that it generates is trivial which is in contradiction with the hypothesis that $H^n(N_2) \cong \mathbb{Z}$ is in that subring).

Therefore, $N_1 \sharp N_2$ is a special manifold. $\square$

Chapter 5

Special PL Manifolds as Shape $m_{simpl}o$-Fibrators

In this chapter we identify special manifolds that are shape $m_{simpl}o$-fibrators. First we state and prove a Fundamental Theorem, which is a useful tool in proving the later results. In section 2, we see which special manifolds are codimension-2 shape $m_{simpl}o$-fibrators and give a list of examples, and in section 3 we present results about codimension-4 shape $m_{simpl}o$-fibrators.

Next, in section 4, using the Fundamental Theorem we prove the main result, which states that every connected, special PL n-manifold N with a non-trivial first homology group is a shape $m_{simpl}o$-fibrator if it is a codimension-2 shape $m_{simpl}o$-fibrator. The requirement of N being a codimension-2 shape $m_{simpl}o$-fibrator can be omitted if $\pi_1(N)$ is non-cyclic. We give some examples of special PL manifolds that are shape $m_{simpl}o$-fibrators.At the end we present results about connected sums being shape $m_{simpl}o$-fibrators.

In the last section we list some open questions.

5.1 The Fundamental Theorem

In order to prove the Fundamental Theorem we need several facts.

The **continuity set** of a proper PL map $p : M \to B$, usually denoted as C, consists of all points $b \in B$, such that under any retraction $R : p^{-1}U \to p^{-1}b$ defined over a neighborhood $U \subset B$ of b, b has another neighborhood $V_b \subset U$, such that for all $x \in V_b$, $R| : p^{-1}x \to p^{-1}b$ is a degree one map.

The following claim will be used frequently without notice.

Claim 5.1.1. *Suppose N is special and $p^{-1}(b) \simeq N$, for all $b \in B$. Then: $C = B$ if and only if p is an approximate fibration over B.*

Proof. Take $x \in C = B$. It means that under any retraction $R : p^{-1}U \to p^{-1}(x)$ defined over a neighborhood $U \subset B$ of x, x has another neighborhood $V_x \subset U$, such that for all $b \in V_x$, $R| : p^{-1}(b) \to p^{-1}(x)$ is a degree-one map. Then $\big(R|\big)_\sharp : \pi_1(p^{-1}(b)) \to \pi_1(p^{-1}(x))$ is an epimorphism (Proposition 4.1.3), and $\big(R|\big)_\sharp\big(\pi_1(p^{-1}(b))\big) \trianglelefteq \pi_1(p^{-1}(x))$. Since N is special and $p^{-1}(b) \simeq N$, for all $b \in B$, it follows that $R|_{p^{-1}(b)}$ is a homotopy equivalence. Now Coram and Duvall's characterization ([7]) of approximate fibrations in terms of movability properties gives that p is an approximate fibration.

Conversely, if p is an approximate fibration, then Theorem 1.3.3 implies that p is a completely movable map. Take $b \in B$ and any retraction $R : p^{-1}U \to p^{-1}(b)$ defined over a neighborhood $U \subset B$ of b. Since p is completely movable there exists a neighborhood V of b in U such that for every $x \in V$, $R| : p^{-1}(x) \to p^{-1}(b)$ is a homotopy equivalence, and so has a degree one. This implies that $b \in C$. Since b was arbitrary element of B, it follows that $B \subset C$. Therefore, $B = C$. $\square$

Claim 5.1.2. ([17]) *Let $p : M \to B$ be a (not necessarily PL) N-shaped map and T closed in B. Then there exists an open set U, such that $U \cap T \neq \varnothing$ and for all*

$t \in U \cap T$, restrictions $R : V \to p^{-1}(t)$, (where V is a neighborhood of $p^{-1}(t)$), restrict to homotopy equivalences $R| : p^{-1}(t') \to p^{-1}(t)$ for all $t' \in p(V) \cap T$.

Before proving the Fundamental Theorem we will prove a lemma that will be used throughout the remainder of this work.

Lemma 5.1.3. *Let $p : M \to \mathbb{R}^k$, $k \geq 2$, be an N-shaped PL map, M a closed, connected, orientable $(n+k)$-manifold, and N an orientable, special n-manifold. Suppose $T \subset \mathbb{R}^k$ is a closed set, with* $\dim T \leq k-2$*. Then $j_\sharp : \pi_1\big(p^{-1}(\mathbb{R}^k \backslash T)\big) \to \pi_1\big(p^{-1}(\mathbb{R}^k)\big)$ is surjective, where $j : p^{-1}(\mathbb{R}^k \backslash T) \to p^{-1}(\mathbb{R}^k)$ is an inclusion map.*

Proof. First look at the following diagram, where $\theta : \widetilde{p^{-1}\mathbb{R}^k} \to p^{-1}\mathbb{R}^k$ is the universal covering of $p^{-1}\mathbb{R}^k$:

$$\begin{array}{ccc} \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}(p^{-1}T) & \longrightarrow & \widetilde{p^{-1}\mathbb{R}^k} \\ \Big\downarrow{\scriptstyle \theta|} & & \Big\downarrow{\scriptstyle \theta} \\ p^{-1}\mathbb{R}^k\backslash p^{-1}T & \xrightarrow{j} & p^{-1}\mathbb{R}^k \end{array}$$

Take any non-identity element $[\alpha] \in \pi_1\big(p^{-1}\mathbb{R}^k, x_0\big)$, where the base point x_0, without loss of generality, can be chosen not to belong in $p^{-1}T$, since $p^{-1}\mathbb{R}^k$ is path connected.

The loop α in $p^{-1}\mathbb{R}^k$ based at x_0 has a lifted path $\tilde{\alpha}$ in $\widetilde{p^{-1}\mathbb{R}^k}$ such that $\tilde{\alpha}(0) = \tilde{x}_0$, and $\tilde{\alpha}(1) = \tilde{x}'_0$.

We need to prove that $\widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}\big(p^{-1}T\big)$ is path connected: Since the universal cover $\widetilde{p^{-1}\mathbb{R}^k}$ is not necessarily compact, we will use Alexander duality ([55], p.342). Note that $\widetilde{p^{-1}\mathbb{R}^k}$ is an open manifold.

Let $y \in T$ and $f : p^{-1}(y) \to N$ be the given homotopy equivalence. So $f|_\sharp : \pi_1\big(p^{-1}(y)\big) \to \pi_1(N)$ is an isomorphism. Look at $\theta_\sharp\big(\pi_1\big(\theta^{-1}p^{-1}(y)\big)\big) < \pi_1\big(p^{-1}(y)\big)$. Let $\widehat{N}$ be a covering of N corresponding to $G < \pi_1(N)$, such that $G = f_\sharp\Big(\theta_\sharp\big(\pi_1\big(\theta^{-1}p^{-1}(y)\big)\big)\Big)$, and $\Psi : \widehat{N} \to N$ be the corresponding covering map.

To prove that $j_\sharp$ is onto we will consider the following long exact homology sequence of the pair $\big(\widetilde{p^{-1}\mathbb{R}^k}, \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}(p^{-1}T)\big)$.

$$\begin{array}{llll}\longrightarrow & H_i\big(\widetilde{p^{-1}\mathbb{R}^k}, \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}(p^{-1}T)\big) & \longrightarrow & H_{i-1}\big(\widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}(p^{-1}T)\big) \quad \longrightarrow \\ \longrightarrow & H_{i-1}\big(\widetilde{p^{-1}\mathbb{R}^k}\big) & \longrightarrow & H_{i-1}\big(\widetilde{p^{-1}\mathbb{R}^k}, \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}(p^{-1}T)\big)\end{array}$$

By Alexander duality, $H_i\big(\widetilde{p^{-1}\mathbb{R}^k}, \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}(y)\big) \cong H_c^{n+k-i}\big(\theta^{-1}p^{-1}(y)\big)$. Since $\widehat{N}$ and $\theta^{-1}p^{-1}(y)$ belong to the same proper homotopy class by Claim 5.1.4 below, the homotopy axiom of Alexander cohomology implies that $H_c^{n+k-i}\big(\theta^{-1}p^{-1}(y)\big) \cong H_c^{n+k-i}(\widehat{N})$. But $H_c^{n+k-i}(\widehat{N}) \cong 0$ for $i < k$ since $\dim\widehat{N} = n$ and $k \geq 2$.

Since y is an arbitrary element of T (closed subset of $\mathbb{R}^k$), it follows that $H_i\big(\widetilde{p^{-1}\mathbb{R}^k}, \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}(y)\big) \cong 0$, for $i < k$ and all $y \in T$. Then Lemma 1.4.1 implies that $H_j\big(\widetilde{p^{-1}\mathbb{R}^k}, \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}T\big) \cong 0$ for $j = \overline{0, r-l}$, where $r = k-1$ and $\dim T = l \leq k-2$. Since $r - l \geq 1$, it follows that $H_0\big(\widetilde{p^{-1}\mathbb{R}^k}, \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}T\big) \cong 0$ and $H_1\big(\widetilde{p^{-1}\mathbb{R}^k}, \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}T\big) \cong 0$. The exactness of the above sequence shows that $H_0\big(\widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}T\big) \cong H_0(p^{-1}\mathbb{R}^k) \cong \mathbb{Z}$, which proves that $\widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}T$ is path connected.

By our choice of x_0, $\{x_0\}\cap p^{-1}T = \emptyset$ and we have that $\theta^{-1}(x_0)\cap\theta^{-1}(p^{-1}T) = \emptyset$ so that $\theta^{-1}(x_0) \subset \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}T$, and $\tilde{x}_0, \tilde{x}_0' \in \widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}T$. Since $\widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}T$ is path connected, there exists a path $\tilde{\beta}$ in $\widetilde{p^{-1}\mathbb{R}^k}\backslash\theta^{-1}p^{-1}T$ such that $\tilde{\beta}(x_0) = \tilde{x}_0$ and $\tilde{\beta}(x_1) = \tilde{x}_0'$. Then $\tilde{\beta}$ is homotopic to $\tilde{\alpha}$ in $\widetilde{p^{-1}\mathbb{R}^k}$, since $\widetilde{p^{-1}\mathbb{R}^k}$ is simply connected as a universal cover of $p^{-1}\mathbb{R}^k$. Hence $\theta\tilde{\beta}$ is a loop homotopic to $\alpha = \theta\tilde{\alpha}$ in $p^{-1}\mathbb{R}^k$.

Then $\theta\tilde{\beta} \subset p^{-1}\mathbb{R}^k\backslash p^{-1}T$, i.e. $[\theta\tilde{\beta}] \in \pi_1\big(p^{-1}\mathbb{R}^k\backslash p^{-1}T\big)$ and $\theta\tilde{\beta} \simeq \alpha$, so $j_\sharp : \pi_1\big(p^{-1}\mathbb{R}^k\backslash p^{-1}T\big) \to \pi_1(p^{-1}\mathbb{R}^k)$ is an epimorphism. $\square$

Claim 5.1.4. *$\theta^{-1}(p^{-1}(y)) \simeq_p \widehat{N}$.*

Proof. Let $g : N \to p^{-1}(y)$ be the homotopy inverse of f. By construction of $\widehat{N}$ and Lemma 79.1 ([49]), it follows that the map $f\theta| : \theta^{-1}p^{-1}(y) \to N$ has a unique lift $\widetilde{f\theta}| : \theta^{-1}p^{-1}(y) \to \widehat{N}$, such that $\Psi\widetilde{f\theta}| = f\theta|$. Similarly, the map $g\Psi : \widehat{N} \to p^{-1}(y)$ has the unique lift $\widetilde{g\Psi} : \widehat{N} \to \theta^{-1}p^{-1}(y)$, such that $\theta\widetilde{g\Psi} = g\Psi$. Take $\tilde{f} = \widetilde{f\theta}|$ and $\tilde{g} = \widetilde{g\Psi}$. We assert that $\tilde{f}\tilde{g} \simeq 1_{\widehat{N}}$ and $\tilde{g}\tilde{f} \simeq 1_{\theta^{-1}p^{-1}(y)}$.

Let $H : p^{-1}(y) \times I \to p^{-1}(y)$ be the homotopy between $1_{p^{-1}(y)}$ and gf. Define a map $(\theta| \times 1_I) : \theta^{-1}p^{-1}(y) \times I \to p^{-1}(y) \times I$. Then construct a homotopy $H(\theta| \times 1_I) : \theta^{-1}p^{-1}(y) \times I \to p^{-1}(y)$. Note $\big(H(\theta \times 1_I)\big)_0 = \theta| = (\theta|)1_{\theta^{-1}p^{-1}(y)}$. Now we can view this as a homotopy lifting problem for the covering map $\theta| : \theta^{-1}p^{-1}(y) \to p^{-1}(y)$, and we have a lifted homotopy $\widetilde{H} = H\widetilde{\big(\theta| \times 1_I\big)} : \theta^{-1}p^{-1}(y) \times I \to \theta^{-1}p^{-1}(y)$, such that $\widetilde{H}_0 = 1_{\theta^{-1}p^{-1}(y)}$ and $(\theta|)\widetilde{H} = H\big(\theta| \times 1_I\big)$.

Then $\widetilde{H}_1 = \tilde{g}\tilde{f}$. Namely, since $(\theta|)\widetilde{H} = H\big(\theta| \times 1_I\big)$, it follows that $\theta\widetilde{H}(x,1) = H\big(\theta| \times 1_I\big)(x,1) = H\big(\theta(x),1\big) = (gf)(\theta(x)) = g(f\theta)(x) = g(\Psi\tilde{f})(x) = (g\Psi)\tilde{f}(x) = (\theta\tilde{g})\tilde{f}(y) = \theta(\tilde{g}\tilde{f})(x)$. This shows that both $\tilde{g}\tilde{f}$ and $\widetilde{H}_1$ are the lifts of the map $\Big(H\big(\theta \times 1_I\big)\Big)_1$. Because of the uniqueness of lifts, we conclude that $\tilde{H}_1 = \tilde{g}\tilde{f}$. Thus $\widetilde{H}$ is a homotopy between $1_{\theta^{-1}p^{-1}(y)}$ and $\tilde{g}\tilde{f}$.

Similarly, the lift $\widetilde{G} = G\widetilde{\big(\Psi \times 1_I\big)}$ of $G : N \times I \to N$ (where G is a homotopy between 1_N and fg) can be proved to be a homotopy between $1_{\widehat{N}}$ and $\tilde{f}\tilde{g}$.

Thus $\tilde{g}\tilde{f} \simeq 1_{\theta^{-1}p^{-1}(y)}$ and $\tilde{f}\tilde{g} \simeq 1_{\widehat{N}}$ so that $\tilde{f} : \theta^{-1}p^{-1}(y) \to \widehat{N}$ is a homotopy equivalence, i.e. $\theta^{-1}p^{-1}(y) \simeq \widehat{N}$. Moreover, $\widetilde{H}$ is a proper homotopy (proof of Claim 2 to the Proposition 3.2.9, [51]). □

The following is the promised Fundamental Theorem that we will use in proving our main result in the next section.

Theorem 5.1.5. *Let $p : M \to \mathbb{R}^k$, $k > 2$, be an N-shaped PL map, M a closed,* *connected, orientable PL $(n+k)$-manifold, and N a connected, orientable, special manifold, such that $H_1(N) \neq 0$. Suppose $T \subset \mathbb{R}^k$ is closed, with* $\dim T < k-2$, *and such that $p|_{p^{-1}\left(\mathbb{R}^k \backslash T\right)}$ is an approximate fibration. Then p is an approximate fibration.*

Proof. First take $T = \{0\}$. We claim that $p^{-1}(0)$ is a strong deformation retract of M. Properties of ANR's ensure that $p^{-1}(0)$ is a strong deformation retract in M of some neighborhood $\mathbb{R}^k$. An appropriate deformation retraction of $\mathbb{R}^k$ into a neighborhood of 0, fixing a smaller neighborhood of 0 by Proposition 1.5 ([6]), can be lifted to a deformation of M into $\mathbb{R}^k$ while fixing a neighborhood of $p^{-1}(0)$ throughout (first restrict to $M \backslash p^{-1}(0)$; after obtaining the desired lift on the deleted space fill in across $p^{-1}(0)$ with the inclusion). Name this retraction $R : M \to p^{-1}(0)$.

We need to prove that $R|_{p^{-1}(y)}$ is a homotopy equivalence, for all $y \in \mathbb{R}^k \backslash 0$.

Let $y \in \mathbb{R}^k \backslash 0$. Since p is an approximate fibration over the homotopy $(k-1)$ -sphere $\mathbb{R}^k \backslash 0$, the homotopy exact sequence

$$\pi_1\big(p^{-1}(y)\big) \cong \pi_1(N) \xrightarrow{i_\sharp} \pi_1\big(M \backslash p^{-1}(0)\big) \xrightarrow{p|_\sharp} \pi_1\big(\mathbb{R}^k \backslash 0\big) \cong 1$$

gives that $i_\sharp$ is an epimorphism, so

$$i_\sharp\big(\pi_1(N)\big) = \pi_1\big(M \backslash p^{-1}(0)\big). \tag{5.1.6}$$

From the exact sequence of a pair $\big(M, M \backslash p^{-1}(0)\big)$:

$$\pi_1\big(M \backslash p^{-1}(0)\big) \xrightarrow{j_\sharp} \pi_1(M) \xrightarrow{k_\sharp} \pi_1\big(M, M \backslash p^{-1}(0)\big)$$

it follows that $\mathrm{Im} j_\sharp = \ker k_\sharp$. So

$$\mathrm{Im} j_\sharp \lhd \pi_1(M). \tag{5.1.7}$$

Now we have:

$$\pi_1\big(p^{-1}(y)\big) \cong \pi_1(N) \xrightarrow{i_\sharp} \pi_1\big(M \backslash p^{-1}(0)\big) \xrightarrow{j_\sharp} \pi_1(M) \xrightarrow{R_\sharp} \pi_1(p^{-1}(0)) \cong \pi_1(N).$$

Using (5.1.6) and (5.1.7), we can conclude that: $j_\sharp\big(i_\sharp\big(\pi_1(N)\big)\big) = j_\sharp\big(\pi_1\big(M\backslash p^{-1}(0)\big)\big) \lhd \pi_1(M)$. Since $R_\sharp$(normal subgroup) = normal subgroup (being onto), it follows that $R|_\sharp\big(\pi_1\big(p^{-1}(y)\big)\big) = R_\sharp\big(j_\sharp\big(i_\sharp\big(\pi_1(N)\big)\big)\big) \lhd \pi_1(p^{-1}(0))$. Also $R_\sharp j_\sharp i_\sharp\big(\pi_1(N)\big) \neq 1$, since $j_\sharp$ is onto (Lemma 5.1.3) and $R_\sharp$ is an isomorphism (being a strong deformation retract). The special property of N implies that $R|_{p^{-1}(y)}$ is a homotopy equivalence, so Coram and Duvall's characterization ([7]) of approximate fibrations in terms of movability properties gives that p is an approximate fibration.

Now suppose that $T \subset \mathbb{R}^k$ is closed, with $\dim T < k-2$. We only need to prove the case when T is the minimal closed set such that $p|_{p^{-1}\left(\mathbb{R}^k\backslash T\right)}$ is an approximate fibration. Suppose that $T \neq \emptyset$.

By Claim 5.1.2, there exists an open set U, such that $U \cap T \neq \emptyset$ and for all $t \in U \cap T$, a retraction $R : V \to p^{-1}(t)$, (where V is a neighborhood of $p^{-1}(t)$), which restricts to a homotopy equivalence $R| : p^{-1}(t') \to p^{-1}(t)$ for all $t' \in p(V) \cap T$.

Take $t \in U \cap T$. Let $R : V \to p^{-1}(t)$ be such a retraction. $p| : V \to p(V)$ is an N-shaped PL map. Take a connected neighborhood W of t in $p(V)$, such that $W \approx \mathbb{R}^k$. Then $R| : p^{-1}W \to p^{-1}(t)$ is also a retraction, which restricts to a homotopy equivalence $R| : p^{-1}(t') \to p^{-1}(t)$ for all $t' \in W \cap T$. We will show in Claim 5.1.8 below, that $R|_{p^{-1}(x)}$ is a homotopy equivalence for all $x \in W\backslash T$, and then $U \cap T \subset C$. Using Coram and Duvall's characterization ([7]) of approximate fibrations in terms of movability properties, it follows that $p|_{p^{-1}\left(\mathbb{R}^k\backslash(T\backslash U)\right)}$ is an approximate fibration, which is a contradiction to the minimality of T ($T\backslash U$ is a closed subset of $\mathbb{R}^k$). This implies that $T \cap U = T$, i.e. $U \cap T = \emptyset$ which is contradiction with $U \cap T \neq \emptyset$. Then $T = \emptyset$ and the result follows. □

Claim 5.1.8. *$R|_{p^{-1}(x)}$ is a homotopy equivalence for all $x \in W\backslash T$.*

Proof. Since p is an approximate fibration over $W\backslash T$, the homotopy exact sequence

$$\pi_1\big(p^{-1}(x)\big) \cong \pi_1(N) \xrightarrow{i_\sharp} \pi_1\big(p^{-1}(W\backslash T)\big) \xrightarrow{p|_\sharp} \pi_1\big(W\backslash T\big) \longrightarrow 1 \cong \pi_0(N)$$

for all $x \in W\backslash T$, gives $i_\sharp\big(\pi_1(N)\big) \lhd \pi_1\big(p^{-1}(W\backslash T)\big)$.

From the exact sequence of a pair $\big(p^{-1}W, p^{-1}(W\backslash T)\big)$:

$$\pi_1\big(p^{-1}(W\backslash T)\big) \xrightarrow{j_\sharp} \pi_1(p^{-1}W) \xrightarrow{k_\sharp} \pi_1\big(p^{-1}W, p^{-1}(W\backslash T)\big)$$

it follows that $j_\sharp\big(\pi_1(p^{-1}(W\backslash T))\big) \lhd \pi_1(p^{-1}W)$. Moreover by Lemma 5.1.3 it follows that $j_\sharp$ is onto.

The long exact homology sequence of the pair $\big(W, W\backslash T\big)$ gives:

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(W) \arrow[r] \arrow[d, "\cong"] & H_2\big(W, W\backslash T\big) \arrow[r] & H_1\big(W\backslash T\big) \arrow[r] & H_1(W) \arrow[r] \arrow[d, "\cong"] & \cdots \\
& H_2(\mathbb{R}^k) & & & H_1(\mathbb{R}^k) &
\end{tikzcd}$$

Since the end groups of the above exact sequence are trivial, it follows that $H_2(W, W\backslash T) \cong H_1(W\backslash T)$. By Alexander duality ([55], p. 342), $H_2(W, W\backslash T) \cong H_c^{k-2}(T \cap W)$. Hence, $H_1(W\backslash T) \cong H_c^{k-2}(T \cap W)$. Since $H_c^{k-2}(T \cap W) \cong 0$ ($\dim(T \cap W) \leq \dim T < k-2$ and $k > 2$), it follows that $H_1(W\backslash T) \cong 0$, so $\pi_1(W\backslash T)$ is perfect.

Now we can look at this diagram:

$$\begin{tikzcd}
\pi_1(N) \arrow[r, "i_\sharp"] \arrow[d, "\cong"] \arrow[ddr, "i'"'] & \pi_1\big(p^{-1}(W\backslash T)\big) \arrow[r, "p|_\sharp"] \arrow[d, "j_\sharp"] & \pi_1(W\backslash T) \arrow[r] \arrow[ddl, dotted, "s"] & 1 \\
\pi_1(p^{-1}(x)) & \pi_1(p^{-1}W) \arrow[d, "R_\sharp"] & & \\
& \pi_1(N) \cong \pi_1(p^{-1}(t)) & &
\end{tikzcd}$$

where $t \in T \cap U$.

We only need to prove that $i' = R_\sharp j_\sharp i_\sharp$ is not trivial (Claim 5.1.9 below), since then $1 \neq i'(\pi_1(N)) = R_\sharp j_\sharp i_\sharp\big(\pi_1(N)\big) \lhd \pi_1(p^{-1}N)$. Now N being special forces $R|$ to be a homotopy equivalence. $\square$

Claim 5.1.9. *$i' = R_\sharp j_\sharp i_\sharp$ is non-trivial.*

Proof. Suppose that i' is trivial. We claim that $s = R_\sharp j_\sharp (p|_\sharp)^{-1} : \pi_1(W\backslash T) \to \pi_1(p^{-1}W)$ is a well-defined surjection.

<u>s is well defined:</u> Let $s(a_1) = y_1 \neq y_2 = s(a_2)$. Then there exist b_1, b_2 such that $p|_\sharp(b_1) = a_1$ and $p|_\sharp(b_2) = a_2$. Suppose $a_1 = a_2$. Then $p|_\sharp(b_1) = p|_\sharp(b_2)$, i.e. $p|_\sharp(b_1 - b_2) = 0$. Since $b_1 - b_2 \in \ker p|_\sharp = \mathrm{Im} i_\sharp$, it follows that $R_\sharp j_\sharp(b_1 - b_2) = 0$ (since $i' = R_\sharp j_\sharp i_\sharp$ is trivial), so $R_\sharp j_\sharp(b_1) = y_1 = y_2 = R_\sharp j_\sharp(b_2)$, which is a contradiction. It follows that $a_1 \neq a_2$. So s is well defined.

<u>s is onto:</u> Let $y \in \pi_1(p^{-1}(t))$. Since $R_\sharp j_\sharp$ is onto, there exists $b \in \pi_1\big(p^{-1}(W\backslash T)\big)$, such that $R_\sharp j_\sharp(b) = y$. Then $s(p|_\sharp(b)) = y$. So s is onto.

Since $\pi_1(W\backslash T)$ is perfect and $s\big(\pi_1\big(W\backslash T\big)\big) = \pi_1(N)$, it follows that $\pi_1(N)$ is perfect, which is a contradiction to $H_1(N) \neq 0$.

So i' is not trivial. $\square$

5.2 Codimension-2 Shape $\mathbf{m_{simpl}o}$-Fibrators

In this section we present results about special manifolds being codimension-2 shape $\mathrm{m_{simpl}o}$-fibrators.

Let $p : M^{n+1} \to B$ be an N-shaped PL map, where M^{n+1} is a connected orientable PL $(n+1)$-manifold, B a polyhedron, and N a closed, connected PL n-manifold. We will recall two results that we will need to prove Theorem 5.2.1.

Theorem 5.2.1. *Suppose $p : M^{n+k} \to B$ is an N-shaped PL map, for $k = 1, 2$, where M is a closed, connected, orientable PL $(n+k)$-manifold, N a connected, orientable, special n-manifold, such that $\pi_1(N)$ is non-cyclic, and B is a polyhedron. Then p is an approximate fibration.*

Proof. Case 1: Let $k = 1$. By Theorem 3.3' ([10]), it follows that B is a 1-manifold (without boundary) (so B is S^1 or $\mathbb{R}$). Take $b \in B$ and look at the retraction $R : U \to p^{-1}(b)$, where U is a neighborhood of $p^{-1}(b)$ in M. For all $x \in U$ we look at the restriction $R| : p^{-1}([x, b]) \to p^{-1}(b)$, which is a retraction.

Then the inclusion-induced homomorphism $j_* : H_i(p^{-1}(x)) \to H_i\big(p^{-1}([x, b])\big)$ is an isomorphism for all i (Corollary 6.3, [10]), and in particular, $j_* : H_n(p^{-1}(x)) \to H_n\big(p^{-1}([x, b])\big)$ is an isomorphism. Since $R|_* : H_n(p^{-1}([x, b]) \to H_n(p^{-1}(b))$ is also an isomorphism (R being deformation retraction), the (absolute) degree of $R|_{p^{-1}(x)} = jR|$ is one and Proposition 4.1.3 implies that $\big(R|_{p^{-1}(x)}\big)_\sharp$ is onto.

Now the composite map $R|_\sharp : \pi_1(p^{-1}(x)) \xrightarrow{j_\sharp} \pi_1\big(p^{-1}([x, b])\big) \xrightarrow{R_\sharp} \pi_1(p^{-1}(b))$ is an epimorphism for all $x \in U$, and the special property of N implies that $R|_{p^{-1}(x)}$ is a homotopy equivalence. So Coram and Duvall's characterization ([7]) of approximate fibrations in terms of movability properties gives that p is an approximate fibration.

Case 2: Let $k = 2$. By Theorem 3.6 ([22]), it follows that B is 2-manifold (without boundary). In this case, we will use the straightforward observation that for every simplex σ in a complex B, over which p is simplicial, and every $x \in \text{int}\sigma$, there exists a PL homeomorphism $\Psi_\sigma : p^{-1}(\text{int}\sigma) \to p^{-1}x \times \text{int}\sigma$ with $\pi_2\Psi_\sigma = p$, where π_2 is a projection map onto the second factor (Proposition 2.1.2, [51]). Consequently,

$$C \cap \text{int}\sigma \neq \emptyset \qquad \text{implies that} \qquad \text{int}\sigma \subset C. \tag{5.2.2}$$

Take $x \in B^0$ and a stellar neighborhood $S = x * L$ in the expected first derived subdivision as in Lemma 2.1.1. Then $p^{-1}(L) = L'$ is a $(n+1)$-manifold and $p| : L' \to L$ is an approximate fibration by case 1 (since L is a 1-dimensional manifold by Theorem 3.3' ([10]). So the continuity set $C|_L$ of $p|_{L'}$ contains a point of $L \cap \text{int}\sigma$ for all σ such that x is a vertex of σ. Properties of the various Ψ_σ and (5.2.2) imply that $C \supset C|_L$ and yield $C \supset B \backslash B^0$.

To see that $C \supset C|_L$, take $v \in C|_L = L$ and any retraction $R : p^{-1}U \to p^{-1}(v)$, where U is a neighbourhood of v in B. Then $U \cap L = U_L$ is a neighbourhood of v in L and $R| : p^{-1}U_L \to p^{-1}(v)$ is a retraction. Since $v \in C|_L$, it follows that $R|$ restricts to a degree one map for all $y \in U_L$. Now take $y \in U_L \backslash \{v\}$. There exists a unique simplex β in B, such that $y \in \text{int}\beta$. Now, look at the PL homeomorphism $\Psi_\beta : p^{-1}(\text{int}\beta) \to p^{-1}(x) \times \text{int}\beta$. The restriction $\Psi_\beta| : p^{-1}(x) \to p^{-1}(y) \times \{x\}$ is a homeomorphism, so has a degree one. Then the composite map

$$p^{-1}(x) \xrightarrow{\Psi_\beta|} p^{-1}(y) \xrightarrow{R|} p^{-1}(v)$$

is a degree one map. This is true for all x in $\text{int}\beta$, and all β in B, such that $U|_L \cap \text{int}\beta \neq$. Then, $v \in C$.

Therefore $p|_{p^{-1}(B \backslash B^0)}$ is an approximate fibration by Claim 5.1.1. We need to prove that $B^0 \subset C$, i.e, p is an approximate fibration over the vertices of B.

If every $x \in B^0$ belongs to C, then we are done. So suppose that there exists a vertex not in C. Since $B \backslash C \subset B^0$, we can reduce immediately to the situation where B is identical to $\mathbb{R}^2$ and p is an approximate fibration over the complement of the origin, 0.

In this setting, $p^{-1}(0)$ is a strong deformation retract of $p^{-1}(\mathbb{R}^2) = M$ (as in the proof of Theorem 5.1.5). Let $R : M \to p^{-1}(0)$ be this strong deformation retract. So $\pi_1(M) = \pi_1\big(p^{-1}(\mathbb{R}^2)\big) \cong \pi_1(p^{-1}(0)) \cong \pi_1(N)$.

Since $p|$ is an approximate fibration over $\mathbb{R}^2\backslash 0$, the homotopy exact sequence gives

$$\begin{array}{ccccccc}
\pi_1(N) & \xrightarrow{i_\sharp} & \pi_1\big(M^{n+2}\backslash p^{-1}(0)\big) & \xrightarrow{p|_\sharp} & \pi_1(\mathbb{R}^2\backslash 0) & \longrightarrow & \pi_0(N) \\
\big\downarrow\cong & & \| & & \big\downarrow\cong & & \| \\
\pi_1(p^{-1}(y)) & & \pi_1\big(p^{-1}(\mathbb{R}^2\backslash 0)\big) & & \pi_1(S^1)\cong\mathbb{Z} & & 1
\end{array}$$

(where $y \in \mathbb{R}^2\backslash 0$). Then we have $\mathrm{Im} i_\sharp = \mathrm{ker} p|_\sharp$, i.e. $i_\sharp(\pi_1(N)) \lhd \pi_1(p^{-1}\big(\mathbb{R}^2\backslash 0)\big)$. From the exact sequence of the pair $\big(M, M\backslash p^{-1}(0)\big)$:

$$\cdots \to \pi_1\big(M\backslash p^{-1}(0)\big) \xrightarrow{j_\sharp} \pi_1(M) \xrightarrow{k_\sharp} \pi_1\big(M, M\backslash p^{-1}(0)\big) \to \cdots$$

it follows that $j_\sharp\Big(\pi_1\big(M\backslash p^{-1}(0)\big)\Big) \lhd \pi_1(M)$. Now we can look at this diagram:

$$\begin{tikzcd}
\pi_1(N) \arrow[r, "i_\sharp"] \arrow[d, "\cong"] \arrow[dr, "i'"'] & \pi_1\big(p^{-1}(\mathbb{R}^2\backslash 0)\big) \arrow[r, "p|_\sharp"] \arrow[d, "j_\sharp"] & \pi_1(\mathbb{R}^2\backslash 0) \arrow[r] \arrow[d, "\cong"] \arrow[dl, dotted, "s"] & 1 \\
\pi_1(p^{-1}(y)) \neq 0 & \pi_1(p^{-1}(\mathbb{R}^2)) \arrow[d, "\cong"] & \mathbb{Z} & \\
& \pi_1(N) \cong \pi_1(p^{-1}(0)) & &
\end{tikzcd}$$

Lemma 5.1.3 implies that $j_\sharp$ is onto. We want to prove that i' is not trivial. Suppose i' is trivial. Then similar arguments to the proof of Claim 5.1.9 will provide that s is a well defined epimorphism.

Then $s(\mathbb{Z}) = \pi_1(p^{-1}(\mathbb{R}^2))$, so the last group is cyclic, which contradicts the assumption that $\pi_1(N)$ is not cyclic. Therefore i' is a non-trivial map and $1 \neq i'(\pi_1(N) = j_\sharp(i_\sharp(\pi_1(N)) \lhd \pi_1(M) \cong \pi_1(N)$ (since $j_\sharp$(normal subgroup) = normal subgroup). The special property of N implies that $R|$ is a homotopy equivalence. So Coram and Duvall's characterization ([7]) of approximate fibrations in terms of movability properties gives that p is an approximate fibration over the vertices of B, and it follows that p is an approximate fibration over B. $\square$

The next Corollary immediately follows from Theorem 5.2.1.

Corollary 5.2.3. *All orientable, special manifolds N with non-cyclic fundamental groups are codimension-2 shape* $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$*-fibrators.*

All closed, orientable surfaces S with $\chi(S) < 0$ are codimension-2 shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrators.

Corollary 5.2.4. *Suppose that the group G is a free product of non-trivial, finitely generated, residually finite groups, which are not all either free or finite groups, and suppose N^3 is a closed, orientable 3-manifold with $\pi_1(N^3) \cong G$. Then N^3 is a codimension-2 shape* $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$*-fibrator.*

Proof. By Theorem 4.1.6, Theorem 3.3.1 and Remark 4.2.2, it follows that N is special and Corollary 5.2.3 gives the result. □

5.3 Codimension-4 Shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-Fibrators

In this section we will list several results about manifolds that are codimension-4 $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrators.

Theorem 5.3.1. *Suppose N is a closed, connected, orientable PL n-manifold homotopically determined by π_1 with a non-trivial, Hopfian fundamental group, which is a codimension-2 fibrator. Then N is a codimension-4 shape* $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$*-fibrator.*

Proof. We consider first the case $k = 3$. Let $p : M^{n+3} \to B$ be an N-shaped PL map, where M is a closed, connected, orientable PL $(n+k)$-manifold, and B a triangulated manifold. Take any $v \in B$. Specify a star S of v in B, as well as the corresponding link L of v bounding S as in the Lemma 2.1.1. Set $S' = p^{-1}S$ and $L' = p^{-1}L$. Name a strong deformation retraction $R : S' \to p^{-1}(v)$.

By hypothesis $p|_{L'} : L' \to L$ is an approximate fibration since L is a 2-manifold (in particular L is an actual 2-sphere being a homotopy 2-sphere). This approximate fibration yields an exact sequence:

$$\pi_1\big(p^{-1}(c)\big) \cong \pi_1(N) \xrightarrow{i_\sharp} \pi_1(L') \xrightarrow{p|_\sharp} \pi_1(L) \cong \pi_1(S^2) \cong 1$$

for all $c \in L$. So $i_\sharp$ is onto.

Since $L' = p^{-1}L \cong p^{-1}(S^2) = p^{-1}(\mathbb{R}^3\backslash 0)$, we can use Lemma 5.1.3 to conclude that $j_\sharp : \pi_1(L') \cong \pi_1(p^{-1}(\mathbb{R}^3\backslash 0) \to \pi_1(S') \cong \pi_1(\mathbb{R}^3)$ is onto.

Then we have

$$\pi_1\big(p^{-1}(c)\big) \cong \pi_1(N) \xrightarrow{i_\sharp} \pi_1(L') \xrightarrow{j_\sharp} \pi_1(S') \xrightarrow{R_\sharp} \pi_1(p^{-1}(v)) \cong \pi_1(N).$$

Therefore $R|_\sharp = R_\sharp j_\sharp i_\sharp : \pi_(N) \to \pi_1(N)$ is onto. $\pi_1(N)$ being Hopfian forces $R|_\sharp$ to be an isomorphism. Since N is homotopically determined by π_1 we can conclude that $R|_{p^{-1}(c)}$ is a homotopy equivalence, for all $c \in L$. By Theorem 1.3.4, it follows that p is an approximate fibration. So N is a codimension-3 $\mathrm{m}_{\mathrm{simpl}}$o-fibrator.

For $k = 4$, we can proceed with the same argument as for case $k = 3$, since in this case L is a homotopy 3-sphere, and hence a triangulated PL 3-manifold, and $S'\backslash p^{-1}(v) \simeq L'$, since $S'\backslash p^{-1}(v)$ collapses to L'. $\square$

Corollary 5.3.2. *Orientable, special manifolds with Hopfian fundamental groups are codimension-4 shape* $\mathrm{m}_{\mathrm{simpl}}$o*-fibrators if they are codimension-2 fibrators.*

Corollary 5.3.3. *Aspherical, closed, connected, orientable PL n-manifolds with Hopfian fundamental groups are codimension-4 shape* $\mathrm{m}_{\mathrm{simpl}}$o*-fibrators if they are codimension-2 fibrators.*

5.4 Special Manifolds as Shape $\mathrm{m_{simpl}o}$ -Fibrators

Next we present our main result about special manifolds being shape $\mathrm{m_{simpl}o}$-fibrators.

Theorem 5.4.1. *Let N be a connected, orientable, special PL n-manifold, such that $H_1(N) \neq 0$. If N is codimension-2 shape* $\mathrm{m_{simpl}o}$*-fibrator, then N is a shape* $\mathrm{m_{simpl}o}$*-fibrator.*

Proof. Proceed by induction. Suppose N is a codimension-$(k-1)$ shape $\mathrm{m_{simpl}o}$-fibrator. We need to prove that N is a codimension-k shape $\mathrm{m_{simpl}o}$-fibrator.

Suppose $p : M^{n+k} \to B$ is an N-shaped PL map, where M is a closed, connected, orientable, PL $(n+k)$-manifold and B a triangulated manifold. We need to prove that p is an approximate fibration. Actually, we only need to prove that p is an approximate fibration over $B \backslash B^{(k-3)}$ (Claim 5.4.2 below). Then Theorem 5.1.5 implies that p is an approximate fibration.

Therefore, N is a codimension-k shape $\mathrm{m_{simpl}o}$-fibrator. □

Claim 5.4.2. *p is an approximate fibration over $B \backslash B^{(k-3)}$, i.e, $B \backslash B^{(k-3)} \subset C$.*

Proof. We know that $C \supset B \backslash B^{(k-2)}$ (Theorem 3.3, [12]). Thus we only need to show that for arbitrary σ, with dim $\sigma = k-2$, int$\sigma \subset C$. Take $x \in$ intσ. Find $S = x * L$, such that $x \in B^1$ and S in B^2 (as in Lemma 2.1.1). By construction, $S \cap B^{(k-3)} = \emptyset$, so $S \subset B \backslash B^{(k-3)}$, which is a PL manifold by Theorem 3.2 ([12]). Then L is an $(k-1)$-sphere and induction gives that $p : L' = p^{-1}L \to L$ is an approximate fibration (since L' is $(n+k-1)$-manifold by Lemma 3.1, [12]).

Then the continuity set $C|_L$ of $p|_{L'}$ contains a point of $L \cap \text{int}\ \sigma$, so by (5.2.2) int$\sigma \subset C$. Furthermore, properties of various Ψ_σ (as in the proof of Theorem 5.2.1) imply that $C \supset C|_L$. So $B \backslash B^{(k-3)} \subset C$. $\square$

The following example shows that the hypothesis of N having a non-trivial first homology group can't be omitted.

Example 5.4.3. An aspherical, homology 3-sphere Σ^3 that admits no (non-trivial) regular covering by another homology 3-sphere is a special manifold but not a shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrator. Namely, Lemma 5.11 ([12]) gives that Σ^3 has a super Hopfian fundamental group, and Σ^3 is homotopically determined by π_1 (since aspherical), and then Remark 4.2.2 gives that Σ^3 is a special manifold. Also Σ^3 is a codimension-2 shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrator, but it is not a shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrator (since example 5.4.4 below shows that it is not a codimension-5 shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrator).

Example 5.4.4. (Daverman) Here we will generalize the construction from example 2.2.2. Let Σ^n be a non-simply connected homology n-sphere that bounds a contractible but not collapsible PL $(n+1)$-manifold W^{n+1}. This is always true for $n \geq 4$ (Theorem 3, [40]). Note that, while all homology n-spheres DO bound contractible $(n+1)$-manifolds ([25]), certain homology 3-spheres do NOT bound contractible PL 4-manifolds ([48] provides an example of a PL contractible 4-manifold whose boundary is not simply connected homology 3-sphere that is known to be aspherical). Construct a manifold $M = \big(\Sigma^n \times W^{n+1}\big) \cup_\partial \big(\Sigma^n \times \partial W^{n+1} \times [1, \infty)\big)$, where the attaching is done via the identification $(x, y) \sim (x, y, 1)$ for $y \in \partial W^{n+1}$. Note that M is a PL manifold since W is PL manifold. Consider a PL map $p : M^{2n+1} \to B^{n+1}$ (defined as a PL approximation to the abstract quotient map $M \to B = M/\Big(\big(\Sigma^n \times W^{n+1}\big) \cup_\partial \big(\Sigma^n \times \partial W^{n+1} \times \{1\}\big)\Big)$, see Example 2.1, [12]), with $p^{-1}(b_0) = \big(\Sigma^n \times W^{n+1}\big) \cup_\partial \big(\Sigma^n \times \partial W^{n+1} \times \{1\}\big) \approx \big(\Sigma^n \times W^{n+1}\big)$ and $p^{-1}(b) = \{q\} \times \partial W^{n+1} \times \{t\}$, where $q \in \Sigma^n$ and $t \in (1, \infty)$. p is a Σ^n-shaped PL map. Also, $B \backslash \{b_0\} \approx \partial W^{n+1} \times (1, \infty)$ (i.e. B is the open cone of Σ^n), and b_0 is a non-manifold point since its link is not simply connected. Look at the map $p \times 1 : M \times \mathbb{R} \to B \times \mathbb{R}$. It is an Σ^n-shaped map, which fails to be an approximate fibration. Note that $B^{n+1} \times \mathbb{R}$ is a manifold ([3]).

The hypothesis of N being a codimension-2 shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrator in Theorem 5.4.1 can be replaced with the condition of $\pi_1(N)$ being a non-cyclic group by Theorem 5.2.1.

Theorem 5.4.5. *Every connected, orientable, special PL n-manifold N with a non-trivial first homology group and a non-cyclic fundamental group is a shape* $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-*fibrator.*

Theorem 5.4.6. *Suppose N_1 is a connected, orientable, special, PL n-manifold, such that $\pi_{n-1}(N_1) \cong 0$ and $\beta_1(N_1) \neq 0$, and suppose N_2 is a closed, connected, orientable, PL n-manifold with a non-trivial fundamental group, such that $N_1 \sharp N_2$ is Hopfian and $\pi_1(N_1 \sharp N_2)$ is Hopfian. Also, assume that $\pi_1(N_i)$ has no free "factor" $\mathbb{Z}$, $i = 1, 2$. Then $N_1 \sharp N_2$ is a shape* $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-*fibrator.*

Proof. It follows that $N_1 \sharp N_2$ is special from Theorem 4.3.1. Since $\pi_1\big(N_1 \sharp N_2\big)$ is not cyclic, it follows that $N_1 \sharp N_2$ is a shape $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-fibrator by Theorem 5.4.5. □

Theorem 5.4.7. *Suppose N_1 is an aspherical, closed, orientable n-manifold, and N_2 is a closed, orientable n-manifold with $\pi_1(N_2) \neq 1$, such that $N_1 \sharp N_2$ is Hopfian with a Hopfian fundamental group and a non-trivial first homology group. Then $N_1 \sharp N_2$ is a shape* $\mathrm{m}_{\mathrm{simpl}}\mathrm{o}$-*fibrator.*

Examples of shape $\mathbf{m_{simpl}o}$-fibrators:

1. Closed, orientable surfaces S with $\chi(S) < 0$;

2. $N_1\sharp N_2$, where N_1, N_2 are aspherical, closed, orientable n-manifolds, such that $N_1\sharp N_2$ is Hopfian with a Hopfian fundamental group, and $H_1(N_1\sharp N_2) \neq 0$.

5.5 Open problems

In conclusion, we list several open and unsettled problems:

1. Is there any difference between codimension-2 PL fibrators and codimension-2 PL shape fibrators?

2. Which closed 3-manifolds are shape $\mathrm{m}_{\mathrm{simpl}}$-fibrators? Which aspherical manifolds are? (Daverman ([15]) has given a fairly penetrating analysis of which geometric 3-manifolds are PL fibrators.)

3. What are the shape $\mathrm{m}_{\mathrm{simpl}}$-fibrator properties of 4-manifolds that are non-trivial connected sums? (see [19] for related results about the PL fibrator properties.)

4. Which Hopfian manifolds with super Hopfian fundamental groups are shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrators?

5. What are the shape $\mathrm{m}_{\mathrm{simpl}}$-fibrator properties of manifolds $(S^1 \times S^{n-1})\sharp N$, where N is non-trivially covered by S^n? (Here $(S^1 \times S^{n-1})\sharp N$ is not a special manifold, but $\pi_1\Big(S^1 \times S^{n-1})\sharp N\Big)$ is super Hopfian.)

6. Does there exist a homology n-sphere Σ^n which is not a codimension-$(n+1)$ shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrator?

7. Generalize the idea of shape $\mathrm{m}_{\mathrm{simpl}}$o-fibrators to a less restrictive PL setting (e.g., when M is a triangulated manifold).

8. Is the super Hopfian property closed with respect to amalgamated free products?

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