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Differential-Algebraic Approach to Speed and Parameter Estimation of the Induction Motor

Mengwei Li
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Mengwei Li entitled "Differential-Algebraic Approach to Speed and Parameter Estimation of the Induction Motor." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Electrical Engineering.

John Chiasson, Major Professor

We have read this dissertation and recommend its acceptance:

Leon M. Tolbert, J. Douglas Birdwell, Timothy P. Schulze

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Graduate Studies

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Differential-Algebraic Approach to Speed and Parameter Estimation of the Induction Motor

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Mengwei Li
December 2005

Dedication

This dissertation is dedicated to my family

Dianjie Li
Songling Li
Mengqiang Li and family
Chenxi Li and family

Thank you for all the encouragement and support.

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Abstract

This thesis considers a differential-algebraic approach to estimating the speed and rotor time constant of an induction motor using only the measured terminal voltages and currents. It is shown that the induction motor speed satisfies both a second-order and a third-order polynomial equation whose coefficients depend on the stator voltages, stator currents, and their derivatives. Further, it is shown that as long as the stator electrical frequency is nonzero, the speed is uniquely determined by these polynomials. The speed so determined is then used to stabilize a dynamic (Luenberger type) observer to obtain a smoothed speed estimate. With full knowledge of the machine parameters and filtering of the sensor noise, simulations and experiments indicate that this estimator has the potential to provide low speed (including zero speed) control of an induction motor under full load. A differential-algebraic approach is also used to obtain an estimate of the rotor time constant of an induction motor, again using only the measured stator voltages and currents. Experimental results are presented to demonstrate the practical use of the identification method.

Contents

List of Figures

Chapter 1

Introduction

Electric machines play an important role in industry as well as in our day-to-day lives. They are used in power plants to generate electrical power and in industry to provide mechanical work, such as in steel mills, textile mills, and paper mills. They start our automobiles and operate many of our household appliances; an average home in North America uses a dozen or more electric motors daily [2].

The electric machine age can be traced to Faraday's discovery of electromagnetic induction in 1831. Electric machines remained largely a laboratory and demonstration curiosity until the 1870s when Thomas Edison began commercial development of the DC generator to support electrical power distribution. A major milestone in the history of the electric machine was the patent of the three phase induction motor and the concept of alternating currents by Nikola Tesla in 1888. Charles Steinmetz advanced Tesla's concept of alternating current over the next decade so that by 1900 reliable wound-core transformers were available, opening the way for long-distance power transmission to power induction motors. The electrification of the United

States was well under way although the process would take another 30 years to complete with the final rural electric power distribution not being completed until the 1930s. The proliferation of electric machine applications closely tracked the expansion of the electric utility grids.

Electric machines come in forms such as Direct Current (DC) machines, Induction machines, Synchronous machines and some special machines including stepper motors, switched reluctance machines, and brushless DC motors. Electric machines are used in manufacturing facilities, and many electric machines are integrated into appliances, vehicles, and service machines.

In the past, DC machines were used extensively in areas requiring variable-speed operation because the field flux and torque of DC machines can be easily controlled by the field and armature current, respectively. In particular, separately excited DC machines were used mainly for applications where there was a requirement of fast response and four-quadrant operation with high performance. However, DC machines have certain disadvantages due to the existence of the commutator and the brushes, which require periodic maintenance. They cannot be used in explosive or corrosive environments; the commutator limits their capability for high-speed, high-voltage operation. Alternating-current (AC) machines do not have these problems. AC machines have a simple and rugged structure, high maintainability and economy; they are also robust and more immune to heavy overloading.

Variable-speed AC machines have been used in the past to perform important roles in applications which preclude the use of DC machines, either because of the working environment or commutator limits. The low cost of AC machines is a decisive economic factor in multi-motor systems and among the various AC drive systems. Induction machines have a particular cost advantage.

1.1 Induction machines

The induction motor is the most common type of AC motor. It is relatively inexpensive to build, is very rugged, and requires little maintenance. Furthermore, in contrast to the DC machine commutators that corrode and produce small arcs, it can also be used in volatile environments. Single-phase induction motors are used for residential and commercial applications, but industry relies on the three-phase motor for its smoother operation and higher efficiency.

The induction machine can operate both as a motor and as a generator. However, it is seldom used as a generator supplying electrical power to a load (windmills being an exception). The performance characteristics as a generator are not satisfactory for most applications. Thus, the induction machine is extensively used as a motor.

1.2 Control of the induction machine

Much attention has been given to induction motor control for starting, braking, speed reversal and speed change. Open-loop control of the machine with variable frequency may provide a satisfactory variable speed drive if the motor is to operate at constant torque without stringent requirements on speed regulation. When the drive requirements include fast dynamic response and accurate speed or torque control, open-loop control is unsatisfactory. Hence it is necessary to operate the motor in a closed-loop mode, when the dynamic operation of the induction machine drive system has an important effect on the overall performance of the system. The induction motor torque is dependent both on the air-gap flux and the speed, but neither the torque versus flux nor the torque versus speed relationship is linear, which complicates the design of the control system for induction machines.

Several techniques of controlling the induction motor are proposed in literature. These schemes can be classified into two main categories.

1. Scalar control: (a) Voltage/frequency (or V/f) control. (b) Stator current and slip frequency control

2. Vector control: (a) Field oriented control (FOC) [3] [4]. (b) Direct torque and stator flux vector control [5] [6], and [7].

1.2.1 Voltage/frequency (or V/f) control

The V/f control principle adjusts a constant V/Hz ratio of the stator voltage by feedforward control. It serves to maintain the magnetic flux in the machine at a desired level. However, it satisfies only moderate dynamic requirements.

1.2.2 Stator current and slip frequency control

In the current-regulated technique, the three-phase sinusoidal reference currents are compared with the instantaneous values of motor currents. The error is input to the controllers and pulse-width modulated (PWM) logic unit. The amplitude of the current reference is obtained from the function generator block. The stator frequency is obtained from an encoder and the slip frequency signal. The slip frequency is derived either from the output of the speed controller or from an efficiency-optimized slip table as in the case of torque controlled drives. The controllers and PWM Generation block can be either hysteresis controllers or proportional-integral (PI) controllers with PWM.

1.2.3 Vector control

Vector control techniques have made possible the application of induction machines for high-performance applications. The vector control scheme enables the control of the induction machine in the same way as a separately excited DC motor. The key idea of FOC is to transfer to another coordinate system or state variable representation, in which the model resembles that of a separately excited DC machine.

FOC was proposed by Hasse [8] and Blaschke [9]. They showed that, similar to the expression of the electromagnetic torque of a separately excited DC machine, the instantaneous electromagnetic torque of an induction motor can be expressed as the product of a flux-producing current and a torque-producing current. This is done by expressing the dynamic equations in the flux-oriented (field-oriented) reference frame. In this case, the stator current components (which are expressed in the stationary reference frame) are transformed into a rotating reference frame, which rotates with a selected flux-linkage (space) vector. There are in general three possibilities for the selection of the flux-linkage space vector, so that the chosen vector can be either the stator-flux-linkage vector, rotor-flux-linkage vector, or magnetizing-flux-linkage vector.

There is a strong interest by drive manufacturers to replace V/f drives by FOC drives, because FOC has better performance, the technology is becoming mature, and there is only minimal extra cost. The basic difference between the two solutions from the hardware point of view is the number of sensors (currents and speed) that are necessary. There is also a strong interest to implement FOC without any shaft sensor for lower cost and reduced complexity. This thesis concentrates on the topic of speed estimation in induction machines without the use of a speed sensor. The speed estimator is needed for flux estimation in field-oriented drives.

1.3 Sensorless control of induction machine

AC drives based on full digital control have reached the status of a mature technology in a wide range of applications from low-cost to high performance systems. In the last several years, a great effort has been made to speed and/or shaft position-sensorless torque-controlled (vector- and direct-torque-controlled) drives. These drives are usually referred to as “sensorless” drives, although the terminology “sensorless” refers to only the speed sensor.

The ongoing research has been concentrated on the elimination of the speed sensor at the machine shaft without reducing the dynamic performance of the drive control system [10]. Speed estimation is an issue of particular interest with induction motor drives where the mechanical speed of the rotor is generally different from the speed of the revolving magnetic field.

The advantages of speed-sensorless induction motor drives are reduced hardware complexity and lower cost, reduced size of the drive machine, elimination of the sensor cable, better noise immunity, increased reliability, and less maintenance requirements. Most hostile environments require motor operation without a speed sensor [11].

1.4 Goals of the research

Recently, induction motors are used throughout industry in pumps, fans, manufacturing machinery, conveyor belts drives, etc., and there are many situations where a sensorless field-oriented controller would be a distinct advantage. For example, a conveyor belt in a mine that brings out coal in buckets is typically powered by induction motors running in open-loop from a 60 Hz voltage source. When power is lost, the coal buckets must all be emptied to reduce the load so that the motors can bring the conveyor belt system back up to speed. The availability of a sensorless field-oriented controller capable of performing a start up under full load would greatly reduce the down-time. Such reliable sensorless control algorithms would reveal many more industrial applications. Even if a shaft sensor is to be used, one can foresee the need for a control algorithm that is capable of tolerating the failure of the sensor. For example, it would be highly desirable to let an electric vehicle proceed to a garage for service after the shaft sensor has failed.

As mentioned before, a disadvantage of existing, high-performance control algorithms for induction motors is that they require a shaft sensor to estimate the (unmeasured) rotor flux linkages for the "vector control" algorithm. Multiple techniques have been proposed to estimate the speed of an induction motor without a shaft sensor, but none has emerged as being completely satisfactory. This area has a rather large literature, and the reader is referred to [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] for an exposition of many of the existing approaches. In most cases, at low speeds there are speed estimation concerns which are responsible for poor drive performance in that speed range.

Traditional approaches to speed sensorless vector control use the method of flux and slip estimation using stator currents and voltages [20] [21], but this has a large error in speed estimation, particularly in the low-speed range. Model reference adaptive system (MRAS) techniques are also used to estimate the speed of an induction motor [22]. In [22], the open integration is needed which causes the accuracy and drift problems.

The differential-algebraic approach presented in this work is most closely related to the ideas described in [23] [24] [25] [26] [27]. In [23] [24] [25] [26], observability is characterized as being able to reconstruct the unknown state variables as rational functions of the inputs, outputs, and their derivatives (See [24] [25] [26] for a more precise definition). This work obtains an algebraic (polynomial) expression for the rotor speed in terms of the machine inputs, machine outputs, and their derivatives. In the systems theoretic approach considered in [27], the authors have shown that there are indistinguishable trajectories of the induction motor, i.e., pairs of different state trajectories with the same input/output behavior. That is, it is not possible to estimate the speed based on stator measurements for arbitrary trajectories [27]. A similar circumstance is shown here due to the fact that the "coefficients" of the algebraic expression for the speed all happen to be zero for some trajectories. This work characterizes a class of trajectories (or, modes of operation) from which the speed of the machine can be estimated from the stator currents and voltages. It is then shown how this speed estimate can be used in a field-oriented controller with the machine operating at low (including zero) speed under full load.

Implementation of a field-oriented controller requires knowledge of motor parameters, in particular, the rotor time constant T_R to estimate the (unmeasured) rotor fluxes. However, the value of T_R changes due to ohmic heating. The work presented here gives a methodology for identifying the rotor time constant T_R without a shaft sensor.

Chapter 2 is a summary of existing technology of sensorless control of the induction motor.

Chapter 3 explains the differential-algebraic approach to speed estimation of the induction motor.

Chapter 4 shows the experimental results of the differential-algebraic approach to speed estimation of an induction motor.

Chapter 5 explains the differential-algebraic approach to estimation of the rotor time constant T_R of the induction motor, and the open-loop experimental result is presented.

Chapter 6 provides conclusions and future work.

Chapter 2

Background and Literature Survey

In recent years, multiple techniques have been proposed to estimate the speed of an induction motor based exclusively on measured terminal voltages and currents [10] [20] [22] [28] [29] [30] [31] [32] [33]. Open-loop and closed-loop observers differ with respect to accuracy, robustness, and limits of applicability. In this section, a review of some existing methods is presented.

2.1 Mathematical model

Most (but not all) of the speed estimation techniques are based on a nonlinear differential equation model of the motor. A brief summary of the model is now presented. To begin, the construction of a simplistic model of a two-phase induction motor is described (see Chapter 5 of [1]).

Figure 2.1(a) shows a half-cylindrical-shaped loop, which is wound around a cylindrical-shaped iron core, and denoted as loop a. A second identical loop, denoted as loop b, is then wound 90 degrees from loop a as shown in Figure 2.1(b). The

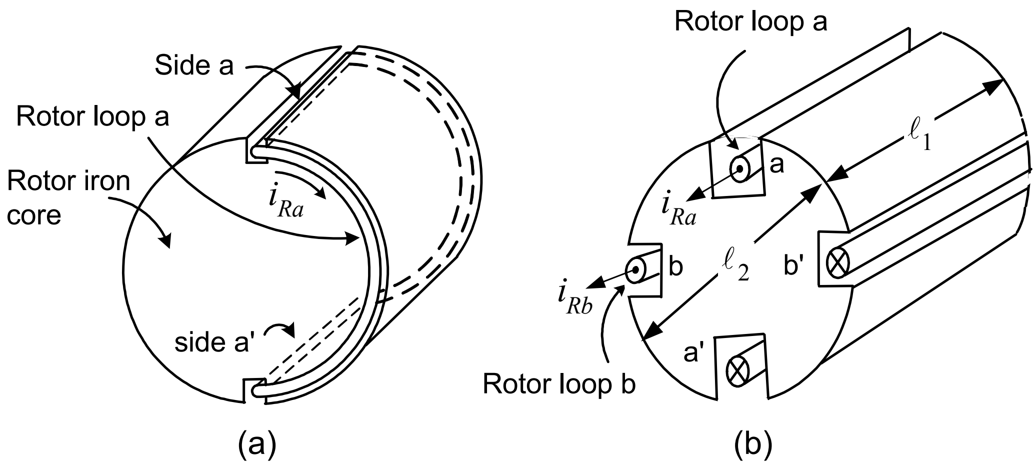

Figure 2.1: Rotor of an induction motor. Ref. [1].

currents in loops a and b are denoted as i_{Ra} and i_{Rb}, respectively. The two loops are electrically isolated.

In Figure 2.1, the notation $\odot$ means that if $i > 0$, the current is coming *out* of the page while $\otimes$ means that if $i > 0$, then the current is going *into* the page.

The stator is constructed similarly. Figure 2.2(a) shows stator loop a which has a half-cylindrical shape and is wound on the inside surface of the stator iron. However, as shown in Figure 2.2(a), it also has a voltage source. Similarly, Figure 2.2(b) shows that stator loop b is identical in form to stator loop a, but is wound 90 degrees from loop a. The applied voltages are denoted as u_{Sa}, u_{Sb} and the corresponding currents are denoted as i_{Sa}, i_{Sb}, respectively. These two loops are electrically isolated.

Combining Figures 2.1 and 2.2, a simple two-phase induction motor is illustrated in Figure 2.3. The position of the rotor is located by a line perpendicular to rotor loop a as shown in Figure 2.3. The flux linkages in the stator and rotor phases in

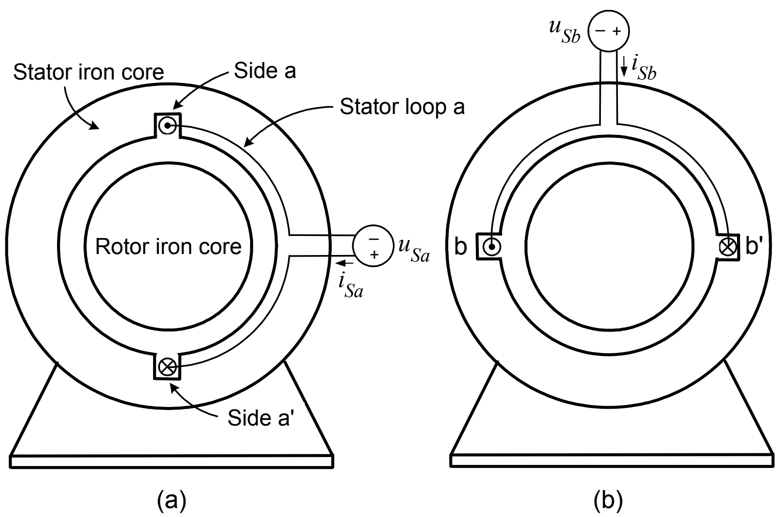

Figure 2.2: Induction motor stator. (a) Stator loop a. (b) Stator loop b. Ref. [1].

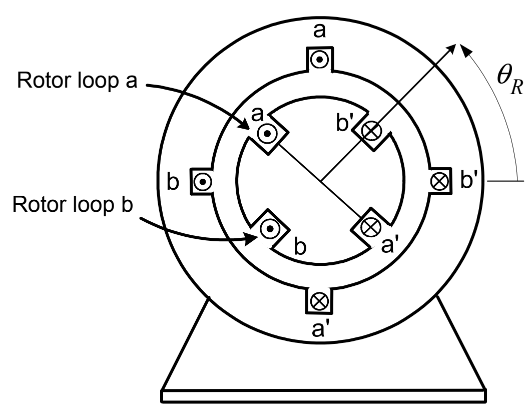

Figure 2.3: Cross-sectional view of a simple two-phase induction motor. Ref. [1].

terms of the currents and rotor position are

$$
\begin{aligned}
\lambda_{Sa}\left(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R\right) &= L_S i_{Sa} + M(i_{Ra}\cos\left(\theta_R\right) - i_{Rb}\sin\left(\theta_R\right)) \\
\lambda_{Sb}\left(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R\right) &= L_S i_{Sb} + M(i_{Ra}\sin\left(\theta_R\right) + i_{Rb}\cos\left(\theta_R\right)) \\
\lambda_{Ra}\left(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R\right) &= L_R i_{Ra} + M(i_{Sa}\cos\left(\theta_R\right) + i_{Sb}\sin\left(\theta_R\right)) \\
\lambda_{Rb}\left(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R\right) &= L_R i_{Rb} + M(-i_{Sa}\sin\left(\theta_R\right) + i_{Sb}\cos\left(\theta_R\right))
\end{aligned} \tag{2.1}
$$

where M is the mutual inductance, L_S and L_R are the stator and rotor inductances, respectively.

By Faraday's and Ohm's laws, the equations describing the electrical dynamics of this system are

$$
\begin{aligned}
-\frac{d\lambda_{Sa}}{dt} - R_S i_{Sa} + u_{Sa} &= 0 \\
-\frac{d\lambda_{Sb}}{dt} - R_S i_{Sb} + u_{Sb} &= 0 \\
-\frac{d\lambda_{Ra}}{dt} - R_R i_{Ra} &= 0 \\
-\frac{d\lambda_{Rb}}{dt} - R_R i_{Rb} &= 0.
\end{aligned}
$$

where R_S and R_R are the stator and rotor resistances. Explicitly, this is

$$
\begin{aligned}
L_S\frac{d}{dt}i_{Sa} + M\frac{d}{dt}(i_{Ra}\cos\left(\theta_R\right) - i_{Rb}\sin\left(\theta_R\right)) + R_S i_{Sa} &= u_{Sa} \\
L_S\frac{d}{dt}i_{Sb} + M\frac{d}{dt}(i_{Ra}\sin\left(\theta_R\right) + i_{Rb}\cos\left(\theta_R\right)) + R_S i_{Sb} &= u_{Sb} \\
L_R\frac{d}{dt}i_{Ra} + M\frac{d}{dt}(i_{Sa}\cos\left(\theta_R\right) + i_{Sb}\sin\left(\theta_R\right)) + R_R i_{Ra} &= 0 \\
L_R\frac{d}{dt}i_{Rb} + M\frac{d}{dt}(-i_{Sa}\sin\left(\theta_R\right) + i_{Sb}\cos\left(\theta_R\right)) + R_R i_{Rb} &= 0.
\end{aligned}
$$

A conservation of energy argument can be used to show that the torque produced by this machine is given by

$$\begin{aligned}\tau_R &= M\left(-i_{Ra}(t)\, i_{Sa}(t)\sin(\theta_R) + i_{Ra}(t)\, i_{Sa}(t)\cos(\theta_R)\right.\\ &\quad \left.-i_{Rb}(t)\, i_{Sa}(t)\cos(\theta_R) - i_{Rb}(t)\, i_{Sb}(t)\sin(\theta_R)\right)\end{aligned}$$

so that the mechanical equation is

$$J\frac{d\omega}{dt} = \tau_R - \tau_L.$$

For a machine with n_p pole-pairs, the model is modified by replacing θ_R by $n_p\theta$ in all the equations and M by n_pM in the torque equation only.

This model can be simplified by a change of variables. To do so, define an equivalent set of rotor flux linkages as

$$\begin{bmatrix}\psi_{Ra}\\ \psi_{Rb}\end{bmatrix} = \begin{bmatrix}\cos(n_p\theta) & -\sin(n_p\theta)\\ \sin(n_p\theta) & \cos(n_p\theta)\end{bmatrix}\begin{bmatrix}\lambda_{Ra}\\ \lambda_{Rb}\end{bmatrix}$$

so that the dynamic model of the induction motor in terms of the state variables ω, ψ_{Ra}, ψ_{Rb}, i_{Sa}, and i_{Sb} is then [34]:

$$\frac{d\omega}{dt} = \frac{n_p M}{J L_R}\left(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}\right) - \frac{\tau_L}{J} \tag{2.2}$$

$$u_{Sa} = R_S i_{Sa} + \sigma L_S di_{Sa}/dt + \frac{M}{L_R}\frac{d\psi_{Ra}}{dt} \tag{2.3}$$

$$u_{Sb} = R_S i_{Sb} + \sigma L_S di_{Sb}/dt + \frac{M}{L_R}\frac{d\psi_{Rb}}{dt} \tag{2.4}$$

$$\frac{d\psi_{Ra}}{dt} = -\frac{1}{T_R}\psi_{Ra} - n_p\omega\psi_{Rb} + \frac{M}{T_R}i_{Sa} \tag{2.5}$$

$$\frac{d\psi_{Rb}}{dt} = -\frac{1}{T_R}\psi_{Rb} + n_p\omega\psi_{Ra} + \frac{M}{T_R}i_{Sb} \tag{2.6}$$

where θ is the position of the rotor, $\omega = d\theta/dt$, n_p is the number of pole pairs, i_{Sa}, i_{Sb} are the (two phase equivalent) stator currents and ψ_{Ra}, ψ_{Rb} are the (two phase equivalent) rotor flux linkages, $T_R = \dfrac{L_R}{R_R}$ is the rotor time constant, $\sigma = 1 - \dfrac{M^2}{L_S L_R}$ is called the total leakage factor, J is the moment of inertia of the rotor and τ_L is the load torque.

The mathematical model can also be written in state-space form as (see [35] [36])

$$\frac{d\omega}{dt} = \frac{n_p M}{J L_R}\left(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}\right) - \frac{\tau_L}{J} \tag{2.7}$$

$$\frac{d\psi_{Ra}}{dt} = -\frac{1}{T_R}\psi_{Ra} - n_p\omega\psi_{Rb} + \frac{M}{T_R}i_{Sa} \tag{2.8}$$

$$\frac{d\psi_{Rb}}{dt} = -\frac{1}{T_R}\psi_{Rb} + n_p\omega\psi_{Ra} + \frac{M}{T_R}i_{Sb} \tag{2.9}$$

$$\frac{di_{Sa}}{dt} = \frac{\beta}{T_R}\psi_{Ra} + \beta n_p\omega\psi_{Rb} - \gamma i_{Sa} + \frac{1}{\sigma L_S}u_{Sa} \tag{2.10}$$

$$\frac{di_{Sb}}{dt} = \frac{\beta}{T_R}\psi_{Rb} - \beta n_p\omega\psi_{Ra} - \gamma i_{Sb} + \frac{1}{\sigma L_S}u_{Sb}. \tag{2.11}$$

The symbols $\beta = \frac{M}{\sigma L_S L_R}$ and $\gamma = \frac{R_S}{\sigma L_S} + \frac{1}{\sigma L_S}\frac{1}{T_R}\frac{M^2}{L_R}$ have been used to simplify the expressions.

2.2 Speed estimators

The system of equations (2.7) – (2.11) (or (2.2) – (2.6)) is the starting point for the model based approaches to speed estimation.

Several techniques [20] [28] [37] estimate the fluxes by pure (unstable) integration. That is, rewriting the equations (2.3) and (2.4) as

$$\dot{\psi}_{Ra} = \frac{L_R}{M}\left(u_{Sa} - R_S i_{Sa} - \sigma L_S di_{Sa}/dt\right) \tag{2.12}$$

$$\dot{\psi}_{Rb} = \frac{L_R}{M}\left(u_{Sb} - R_S i_{Sb} - \sigma L_S di_{Sb}/dt\right), \tag{2.13}$$

the fluxes ψ_{Ra} and ψ_{Rb} can be obtained by integrating equations (2.12) and (2.13).

The angle ϕ of the rotor flux vector is defined as

$$\phi \triangleq \tan^{-1}\left(\frac{\psi_{Rb}}{\psi_{Ra}}\right) \tag{2.14}$$

and its derivative is

$$\frac{d\phi}{dt} = \frac{\dot{\psi}_{Rb}\psi_{Ra} - \dot{\psi}_{Ra}\psi_{Rb}}{\psi_{Ra}^2 + \psi_{Rb}^2} \tag{2.15}$$

Replace $\dot{\psi}_{Ra}$ by (2.5) and $\dot{\psi}_{Rb}$ by (2.6) to obtain

$$\frac{d\phi}{dt} = n_p\omega + \frac{M}{T_R}\frac{\psi_{Ra} i_{Sb} - \psi_{Rb} i_{Sa}}{\psi_{Ra}^2 + \psi_{Rb}^2}. \tag{2.16}$$

Substituting (2.15) into (2.16) and rearranging, the motor speed can be obtained from

$$\omega = \frac{1}{n_p}\left(\frac{\dot{\psi}_{Rb}\psi_{Ra} - \dot{\psi}_{Ra}\psi_{Rb}}{\psi_{Ra}^2 + \psi_{Rb}^2} - \frac{M}{T_R}\frac{\psi_{Ra}i_{Sb} - \psi_{Rb}i_{Sa}}{\psi_{Ra}^2 + \psi_{Rb}^2}\right). \tag{2.17}$$

In (2.17), $\dot{\psi}_{Ra}$ and $\dot{\psi}_{Rb}$ are obtained by (2.12) and (2.13), which are functions of measured voltages and currents. This result indicates that the instantaneous rotor speed can be obtained from the measured voltages and currents.

This method requires knowledge of motor parameters. Calculating ψ_{Ra} and ψ_{Rb} based on equations (2.12) and (2.13) requires the pure integration of sensed variables. This is an unstable estimation and leads to problems with initial conditions and drift.

2.3 Extended Kalman filter techniques

The Kalman filter algorithm [38] has been used for the estimation of parameters of an induction motor [39] [40] [41], and for the speed estimation problem [42] [43].

Kalman filter techniques are based on the complete machine model. The dynamic model for an induction motor in the state space, choosing i_{Sa}, i_{Sb}, ψ_{Ra}, and ψ_{Rb} as the state variables, is given by equations (2.8), (2.9), (2.10), and (2.11). In the dynamic model of an induction motor, if the dimension of the state vector is increased by introducing the mechanical speed as an additional state variable, the state model becomes nonlinear (otherwise, the angular speed of the rotor is considered as a parameter). The extended Kalman filter is used if speed estimation is desired using the nonlinear model. The extended Kalman filter is based on linearizing the nonlinear model about its current operating point. The linearization is performed by assuming the motor speed is constant during the sample time. The corrective inputs to the dynamic models of the stator, the rotor, and the mechanical subsystems are derived based on minimizing a quadratic error function. The error function is evaluated on the basis of predicted state variables, and the noise in the measured signals and in the model parameter deviations should be taken into account.

This approach reduces the error sensitivity and also permits the use of models of lower order than the machine. Experimental verification of this method using machine models of fourth and third order is reported in [43]. This relaxes the extensive computation requirements to some extent; the implementation requires floating-point signal processor hardware [44].

Section 3.8 will discuss speed estimation using extended Kalman filter methods in detail.

2.4 Least-squares method

The least-squares method treats the speed ω as an unknown constant (slowly varying compared to the electrical variables) parameter and finds the estimated speed $\hat{\omega}$ that best fits the measured/calculated data (*i.e.*, u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}, di_{Sa}/dt, di_{Sb}/dt) to the "steady-state" equations of the motor (i.e., $d\omega/dt = 0$) [12] [18] [45]. For this approach, it is convenient to use the mathematical model of the induction motor in the state-space form (2.8), (2.9), (2.10), and (2.11). The least-squares approach is used to find the value of estimated speed $\hat{\omega}$ that best fits the equations (2.8), (2.9), (2.10), and (2.11). However, as the fluxes are not available measurements, the first step is to eliminate the flux linkages ψ_{Ra}, ψ_{Rb} and their derivatives $d\psi_{Ra}/dt$, $d\psi_{Rb}/dt$. The four equations (2.8), (2.9), (2.10), and (2.11) can be used to solve for ψ_{Ra}, ψ_{Rb}, $d\psi_{Ra}/dt$, $d\psi_{Rb}/dt$, but one is left without another set of independent equation to set up a speed estimator. A new set of independent equations is found by differentiating (2.10) and (2.11) (assuming motor speed ω is constant), respectively, to obtain

$$\frac{d^2 i_{Sa}}{dt} = \frac{\beta}{T_R}\frac{d\psi_{Ra}}{dt} + \beta n_p \omega \frac{d\psi_{Rb}}{dt} - \gamma \frac{di_{Sa}}{dt} + \frac{1}{\sigma L_S}\frac{du_{Sa}}{dt} \tag{2.18}$$

$$\frac{d^2 i_{Sb}}{dt} = \frac{\beta}{T_R}\frac{d\psi_{Rb}}{dt} - \beta n_p \omega \frac{d\psi_{Ra}}{dt} - \gamma \frac{di_{Sb}}{dt} + \frac{1}{\sigma L_S}\frac{du_{Sb}}{dt}. \tag{2.19}$$

Substituting $d\psi_{Ra}/dt$, $d\psi_{Rb}/dt$, which are in terms of u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}, di_{Sa}/dt, and di_{Sb}/dt into equations (2.18) and (2.19) results in the least-squares regressor equation given by

$$\sigma L_S \frac{d^2}{dt}\begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix} + \left(R_S + \frac{L_S}{T_R}\right)\frac{d}{dt}\begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix} + \frac{R_S}{T_R}\begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix} - \frac{d}{dt}\begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix} - \frac{1}{T_R}\begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix}$$
$$= n_p \omega \left(\sigma L_S \frac{d}{dt}\begin{bmatrix} -i_{Sb} \\ i_{Sa} \end{bmatrix} + R_S \begin{bmatrix} -i_{Sb} \\ i_{Sa} \end{bmatrix} - \begin{bmatrix} -u_{Sb} \\ u_{Sa} \end{bmatrix}\right). \tag{2.20}$$

The system of equations (2.20) is now in terms of the measured/calculated quantities u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}, di_{Sa}/dt, di_{Sb}/dt, $d^2 i_{Sa}/dt$, $d^2 i_{Sb}/dt$, the known motor parameters, and the unknown speed ω.

The least-squares approach requires that one measure u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}, compute di_{Sa}/dt, di_{Sb}/dt, $d^2 i_{Sa}/dt$, $d^2 i_{Sb}/dt$ over a short time interval where the speed is approximately constant, and take the speed estimate $\hat{\omega}$ that best fits (2.20) for the present and past values of time. This approach assumes that the parameters are known, fixed in time, and the speed is constant.

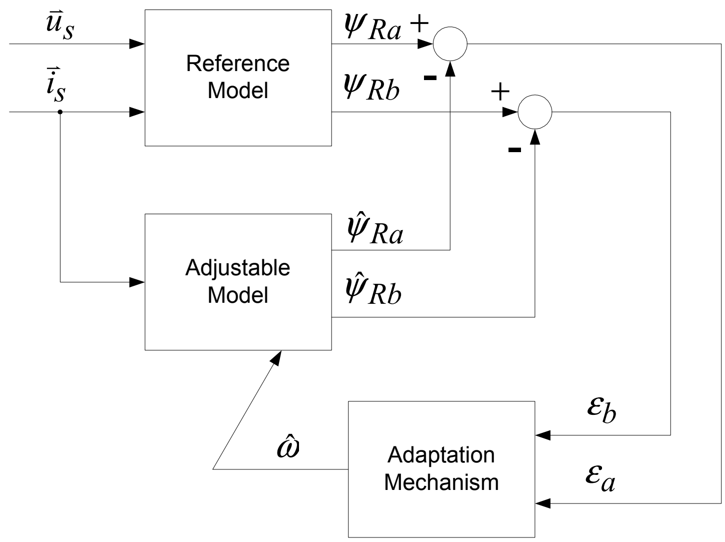

Figure 2.4: MRAS block diagram.

2.5 Model reference adaptive system (MRAS)

The schemes included in this group consider the motor speed as an unknown "constant" parameter and use the techniques of adaptive control to estimate this parameter [10] [22] [31] [32]. This method is based on the comparison between the outputs of two estimators. The estimator that does not involve the quantity to be estimated (the rotor speed ω) is considered as the induction machine reference model given by equations (2.3) , (2.4), and the other estimator, may be regarded as the adjustable model represented by equations (2.5), (2.6). The error between the estimated quantities obtained by the two models is used to drive a suitable adaptation mechanism which generates the estimated rotor speed $\hat{\omega}$. Figure 2.4 shows a general block diagram for this approach. Some schemes compare the rotor fluxes to generate the error signal which is used to estimate the rotor speed.

In general, ω is a variable, and the models are linear time-varying systems. However, for the purpose of deriving an adaptation mechanism, it is valid to initially treat ω as a constant parameter of the reference model.

An estimator for ψ_{Ra} and ψ_{Rb} is defined through

$$\frac{d\hat{\psi}_{Ra}}{dt} = -\frac{1}{T_R}\hat{\psi}_{Ra} - n_p\hat{\omega}\hat{\psi}_{Rb} + \frac{M}{T_R}i_{Sa} \tag{2.21}$$

$$\frac{d\hat{\psi}_{Rb}}{dt} = -\frac{1}{T_R}\hat{\psi}_{Rb} + n_p\hat{\omega}\hat{\psi}_{Ra} + \frac{M}{T_R}i_{Sb}. \tag{2.22}$$

With $\epsilon_a = \psi_{Ra} - \hat{\psi}_{Ra}$, $\epsilon_b = \psi_{Rb} - \hat{\psi}_{Rb}$, the error dynamics are found by subtracting (2.21) and (2.22) from equations (2.5) and (2.6) to obtain

$$\frac{d}{dt}\begin{bmatrix} \epsilon_a \\ \epsilon_b \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_R} & -n_p\hat{\omega} \\ n_p\hat{\omega} & -\frac{1}{T_R} \end{bmatrix}\begin{bmatrix} \epsilon_a \\ \epsilon_b \end{bmatrix} + n_p\left(\omega - \hat{\omega}\right)\begin{bmatrix} -\hat{\psi}_{Rb} \\ \hat{\psi}_{Ra} \end{bmatrix} \tag{2.23}$$

where ψ_{Ra}, ψ_{Rb} are computed from equations (2.3) and (2.4).

As $\hat{\omega}$ is a function of the state error, these equations describe a nonlinear-feedback system. Choosing

$$\hat{\omega} = K_p\left(\psi_{Rb}\hat{\psi}_{Ra} - \psi_{Ra}\hat{\psi}_{Rb}\right) + K_I\int_0^t\left(\psi_{Rb}\hat{\psi}_{Ra} - \psi_{Ra}\hat{\psi}_{Rb}\right)dt.$$

The stability of the system is discussed in [37]. Important issues with this algorithm are the sensitivity to parameter variation, the ability to work under full load torque, and the unstable estimation of ψ_{Ra}, ψ_{Rb}. The method requires knowledge of R_S, σL_S, M, and T_R.

Peng and Fukao [32] modified the approach of [37] so that it does not require pure integration of the flux equations. To do so, one defines

$$\begin{aligned} v_{ma} &\triangleq (M/L_R)d\psi_{Ra}/dt = -R_S i_{Sa} - \sigma L_S di_{Sa}/dt + u_{Sa} \\ v_{mb} &\triangleq (M/L_R)d\psi_{Rb}/dt = -R_S i_{Sb} - \sigma L_S di_{Sb}/dt + u_{Sb}. \end{aligned} \tag{2.24}$$

where v_{ma} and v_{mb} are measured/calculated quantities. Multiplying equations (2.5) and (2.6) by M/L_R, and differentiating them with respect to time, one can obtain:

$$\begin{aligned} dv_{ma}/dt &= -\frac{1}{T_R}v_{ma} - n_p\omega v_{mb} + \frac{1}{T_R}(M^2/L_R)di_{Sa}/dt \\ dv_{mb}/dt &= -\frac{1}{T_R}v_{mb} + n_p\omega v_{ma} + \frac{1}{T_R}(M^2/L_R)di_{Sb}/dt. \end{aligned}$$

An estimator for v_{ma} and v_{mb} is defined through

$$\begin{aligned} d\hat{v}_{ma}/dt &= -\frac{1}{T_R}\hat{v}_{ma} - n_p\hat{\omega}\hat{v}_{mb} + \frac{1}{T_R}(M^2/L_R)di_{Sa}/dt \\ d\hat{v}_{mb}/dt &= -\frac{1}{T_R}\hat{v}_{mb} + n_p\hat{\omega}\hat{v}_{ma} + \frac{1}{T_R}(M^2/L_R)di_{Sb}/dt. \end{aligned} \tag{2.25}$$

Then $\hat{\omega}$ is chosen as

$$\hat{\omega} = K_p\left(v_{ma}\hat{v}_{mb} - v_{mb}\hat{v}_{ma}\right) + K_I\int_0^t \left(v_{ma}\hat{v}_{mb} - v_{mb}\hat{v}_{ma}\right)dt. \tag{2.26}$$

K_p and K_I are adjusted by the designer.

In summary, this method consists in computing v_{ma}, v_{mb} from (2.24), $\hat{v}_{ma}$, $\hat{v}_{mb}$ from (2.25), and then using equation (2.26) to obtain the speed estimate $\hat{\omega}$. This method avoids the integration of rotor flux. Peng and Fukao [32] also modified the approach to make it insensitive to the stator resistance R_S.

2.6 Rotor slot ripple

The schemes included in this group are based on the fact that the rotor speed is obtained by the ripple generated in the stator voltages and currents, due to the reluctance modulation generated by the presence of rotor slots [46] [47]. The rotor slots presence can be taken into account considering the air gap length variable while modelling the induction machine. The air gap modulation is responsible for two harmonic components in the stator voltages and in the stator currents. The speed detection can be performed by measuring the rotor slots harmonics frequency either from the stator currents or from the stator voltages.

The stator currents and voltages are pre-filtered by means of band-pass filters where the center frequency can be tuned on the rotor slots' harmonic. Once the frequency of such harmonic is detected, the rotor speed can be derived. In some schemes, a spectral analysis of one phase current is performed using a Fast Fourier Transform (FFT) method to detect the rotor slots' harmonic frequency, and the pre-filtering stage is performed digitally.

In other schemes the stator voltages are used, and the pre-filtering is performed by means of Switched Capacitor Filter (SCF) in which the center frequency is tuned using the synchronous frequency and the rotor slots' frequency is detected by a Phase-Locked Loop (PLL) [47] or by a frequency-to-voltage converter (FVC) [48].

If a suitable analog electronic detection circuitry is available, the rotor slots ripple method has a behavior which is very close to methods using the measured speed.

However, the rotor slots ripple method has a steady state error of about 1%, independent of the speed, under any load conditions [49]. It also has a dead zone at very low speed (below 50 rpm) due to the limits of the FVC used. It can not properly operate at zero speed.

2.7 Summary

A large variety of sensorless controlled AC drive schemes are used in industrial applications. Open-loop control systems establish the desired machine flux by maintaining the stator voltage-to-frequency ratio at a predetermined level. They are particularly robust at very low and very high speed. However, they satisfy only low or moderate dynamic requirements. Small load-dependent speed deviations can be compensated for by incorporating a speed or rotor frequency estimator.

High-performance vector control schemes require a flux vector estimator to identify the spatial location of the magnetic field. Field-oriented control stabilizes the tendency of induction motors to oscillate during transients, which enables fast control of torque and speed. The robustness of a sensorless drive can be improved by adequate control structures and by parameter identification techniques.

Speed estimators and the MRAS method are based on equations (2.3) and (2.4). The accuracy and drift concerns of an open integration at low frequency exist [28] [37]. These problems can be avoided by using a first-order delay element instead of an integrator, which eliminates an accumulation of the drift error. However, this entails a severe loss of gain in ψ_r at low stator frequency, while the estimated field angle lags considerably behind the actual position of the rotor field. It also makes the integration ineffective in the frequency range around and below the corner frequency of the first-order delay element. If the drive is operated close to zero stator frequency for long periods of time, the estimated flux goes astray, and speed estimation is lost [44]. The modified MRAS method [32] does not require the pure integration, but the slowly changing speed compared to the electrical variables is assumed.

Least-Square methods [12] [18] [45] and the extended Kalman filter method [42] [43] treat the speed ω as an unknown constant (slowly varying compared to the electrical variables) parameter. The convergence of the extended Kalman filter method is not reported in [42].

Other methods such as [50] [51] are very sensitive to the parameters.

Chapter 3

Speed Sensorless Control of the Induction Motor Using Differential-Algebraic Speed Estimator

3.1 Observers

The results in this thesis are a combination of the rather new development of differential-algebraic methods and the classical Luenberger observer methodologies. A brief summary of the Luenberger observer is given and illustrated by estimating the speed in a permanent magnet DC motor. The permanent magnet DC motor is also used to illustrate the differential-algebraic approach to speed estimation.

The theory of state estimation from measured data dates back to the early 1960s. Luenberger considered the problem of estimating (a linear function of) the state in multi-input and multi-output linear systems. The Luenberger observer [52] provides solution for linear systems where a constant system matrix and fixed parameters are assumed.

Luenberger observer

Consider a linear time-invariant system given in state-space form as

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{3.1}$$

$$y(t) = Cx(t) \tag{3.2}$$

where x is $n \times 1$ state vector, u is $r \times 1$ input vector, A is an $n \times n$ system matrix, and B is an $n \times r$ distribution matrix, y is $m \times 1$ output vector, and C is $m \times n$ output matrix.

Assume that only the input vector $u(t)$ and the output vector $y(t)$ can be measured.

Define a state estimator system by

$$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) \tag{3.3}$$

$$\hat{y}(t) = C\hat{x}(t). \tag{3.4}$$

Subtract equation (3.3) from (3.1) , and (3.4) from (3.2) to obtain the estimation error system given by

$$\dot{e}(t) = (A - LC)\, e(t)$$

where $e(t) = x(t) - \hat{x}(t)$. This implies that the error approaches zero asymptotically as $t \to \infty$ provided that eigenvalues of gain matrix $A - LC$ are located in the left-hand side of the complex plane.

It is well known that a gain matrix L can be found that arbitrarily places the eigenvalues of $A - LC$ if and only if the pair (C, A) is observable [53], i.e.,

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{3.5}$$

Example 1 *Consider the following model of a permanent magnet DC motor given by*

$$\frac{d}{dt} \begin{bmatrix} \theta \\ \omega \\ \tau_L/J \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ \tau_L/J \end{bmatrix} + \begin{bmatrix} 0 \\ K_T/J \\ 0 \end{bmatrix} u(t) \tag{3.6}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ \tau_L/J \end{bmatrix} \tag{3.7}$$

where the load torque τ_L *is modelled as an unknown constant parameter. The state of the system is defined as*

$$z \triangleq \begin{bmatrix} \theta \\ \omega \\ \tau_L/J \end{bmatrix}.$$

With the obvious definitions for A, B, *and* C, *the system equations can be written as*

$$\frac{dz}{dt} = Az + Bu(t). \tag{3.8}$$

The output is

$$y(t) = Cz. \tag{3.9}$$

Define a speed and load-torque (Luenberger) observer by

$$\frac{d\hat{z}}{dt} = A\hat{z} + Bu(t) + \ell(y(t) - \hat{y}(t)) \tag{3.10}$$

$$\hat{y}(t) = C\hat{z} \tag{3.11}$$

where

$$\ell \triangleq \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}.$$

Define the estimation error state to be

$$e \triangleq \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \theta - \hat{\theta} \\ \omega - \hat{\omega} \\ \tau_L/J - \hat{\tau}_L/J \end{bmatrix}.$$

Subtract the system (3.10) *from the system* (3.8) *and* (3.11) *from* (3.9) *to obtain the system of equations describing the error dynamics as*

$$\begin{aligned}\frac{de}{dt} &= Ae - \ell\,(y\,(t) - \hat{y}\,(t)) \\ &= (A - \ell C)\,e\end{aligned}$$

where

$$A - \ell C = \begin{bmatrix} -\ell_1 & 1 & 0 \\ -\ell_2 & -f/J & -1 \\ -\ell_3 & 0 & 0 \end{bmatrix}. \tag{3.12}$$

The rank of the observability matrix is

$$rank \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = rank \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -f/J & -1 \end{bmatrix} = 3.$$

This system is observable. A gain matrix ℓ can be found that places the eigenvalues of $A - \ell C$ in the left-hand side of the complex plane [53].

3.2 Differential-algebraic state estimation

The idea of the differential-algebraic approach to state estimation is to find an algebraic equation that the unknown state variable must satisfy. That is, a polynomial equation in the unknown state variable whose coefficients are in term of the inputs, outputs, and a finite number of their derivatives. The following examples are used to illustrate the approach.

Example 2 *Consider the system [23]*

$$\begin{aligned} \dot{x}_1 &= x_2^2 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned}$$

which is in state-space form. Suppose an input-output description, i.e., a description directly relating u *and* y *is desired. The original set of equations is equivalent to*

$$y - x_1 = 0,\ \dot{y} - x_2^2 = 0,\ \dot{x}_2 - u = 0. \tag{3.13}$$

It is now possible to eliminate the derivative of x_2 *by forming*

$$p = \frac{d}{dt}\left(\dot{y} - x_2^2\right) + 2x_2\left(\dot{x}_2 - u\right) = \ddot{y} - 2x_2 u. \tag{3.14}$$

From this construction it follows that $p = 0$ *whenever the equations of* (3.13) *are satisfied. The last equation of* (3.14) *can be replaced by* $p = 0$ *to get the system description*

$$y - x_1 = 0,\ \dot{y} - x_2^2 = 0,\ \ddot{y} - 2x_2 u = 0. \tag{3.15}$$

From (3.14) *it follows that every solution of* (3.13) *also solves* (3.15) *. If, moreover, it is known that* $x_2 \neq 0$*, then the converse is also true and* (3.13) *and* (3.15) *are equivalent. It is now possible to form*

$$x_2\left(\ \ddot{y} - 2x_2 u\right) - 2u\left(\dot{y} - x_2^2\right) = x_2\ddot{y} - 2u\dot{y}, \tag{3.16}$$

and it readily follows that

$$y - x_1 = 0,\ \ddot{y} - 2x_2 u = 0,\ x_2\ddot{y} - 2u\dot{y} = 0 \tag{3.17}$$

is equivalent to (3.15) *if also* $u \neq 0$. *Finally form*

$$\ddot{y}\,(\ddot{y} - 2x_2 u) + 2u\,(x_2\ddot{y} - 2u\dot{y}) = \ddot{y}^2 - 4u^2\dot{y}, \tag{3.18}$$

to conclude that, provided $ux_2 \neq 0$, (3.13) *is equivalent to the following system description*

$$\ddot{y}^2 - 4u^2\dot{y} = 0,\ y - x_1 = 0,\ \ddot{y} - 2x_2 u = 0. \tag{3.19}$$

The left-most equation of (3.19) *is an input-output relation, while the middle and right equations show* x_1 *and* x_2 *can be computed from the input and output. Further, the unmeasured state variable* x_2 *can be estimated by* $x_2 = \ddot{y}/2u$.

Example 3 *A standard mathematical model of a permanent magnet DC motor is given by*

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega + v && (3.20)\\ J\frac{d\omega}{dt} &= K_T i - f\omega - \tau_L && (3.21)\end{aligned}$$

where R, L *represent the resistance and inductance, respectively, of the armature,* K_b *is the back emf constant,* $K_T = K_b$ *is the torque constant,* J, f *are the rotor moment of inertia and the viscous friction coefficients, respectively.* τ_L *is load torque and* v *is* *supply voltage. The load torque is assumed to be constant so that it satisfies*

$$\frac{d\tau_L}{dt} = 0. \tag{3.22}$$

Simply solving (3.20) *for* ω *gives a differential algebraic estimator for the speed as*

$$\omega = \frac{V - Ldi/dt - Ri}{K_b}. \tag{3.23}$$

This differential algebraic estimator has no stability issues associated with it, but it is noisy as the derivative of the current is required. In this example, a Luenberger type observer is combined with a differential-algebraic observer to obtain a smoother estimate compared to that of the previous example. Proceeding, consider an observer defined by

$$J\frac{d\hat{\omega}}{dt} = K_T i - f\hat{\omega} - \hat{\tau}_L + \ell_1(\omega - \hat{\omega}) \tag{3.24}$$

$$\frac{d\hat{\tau}_L}{dt} = \ell_2(\omega - \hat{\omega}) \tag{3.25}$$

where ω *in* (3.24) *and* (3.25) *is obtained from (3.23). Subtracting* (3.24) *from* (3.21) *and* (3.25) *from* (3.22)*, the error system is given by*

$$\frac{d}{dt}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -\frac{f+\ell_1}{J} & -\frac{1}{J} \\ -\ell_2 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{3.26}$$

where $e_1 \triangleq \omega - \hat{\omega}$ *and* $e_2 \triangleq \tau_L - \hat{\tau}_L$*. The gains* ℓ_1 *and* ℓ_2 *can be chosen to make the system stable so that* $e_1 = \omega - \hat{\omega} \longrightarrow 0$*, with* $\hat{\omega}$ *being a smoother estimate than* ω*.*

It is now shown that the differential-algebraic approach does not work for a permanent magnet synchronous motor.

Example 4 *A mathematical model of a permanent magnet synchronous motor is given by [13]*

$$Ls\frac{di_{Sa}}{dt} = -R_S i_{Sa} + K_m \omega \sin(n_p\theta) + u_{Sa} \tag{3.27}$$
$$Ls\frac{di_{Sb}}{dt} = -R_S i_{Sb} - K_m \omega \cos(n_p\theta) + u_{Sb} \tag{3.28}$$
$$J\frac{d\omega}{dt} = K_m\left(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)\right) - f\omega - \tau_L \tag{3.29}$$

where θ is the position of the rotor, $\omega = d\theta/dt$, n_p is the number of pole pairs, L_S is the stator phase self-inductance, R_S is the resistance, J is the moment of inertia, K_m is the torque/back-emf constant, f is the coefficient of viscous friction and τ_L is the load torque.

At zero speed, i.e., $\omega = 0$, the two current equations (3.27) and (3.28) do not contain any information about the rotor angle θ. However, the rotor angle is required to control the motor torque as equation (3.29) shows, and consequently the method is not applicable to the permanent magnet synchronous motor at zero speed.

3.3 Speed estimation in an induction motor

The contribution of this thesis is the development of a new algorithm to estimate the speed of an induction motor [54]. This new approach is now presented.

The starting point for the speed sensorless design proposed in this thesis is a (two-phase equivalent) state-space mathematical model of the induction motor given in the space vector form [13]. Specifically, let $\underline{i}_S = i_{Sa} + ji_{Sb}$, $\underline{\psi}_R = \psi_{Ra} + j\psi_{Rb}$, and $\underline{u}_S = u_{Sa} + ju_{Sb}$. The induction motor model given in (2.7) – (2.11) may be rewritten as

$$\frac{d}{dt}\underline{i}_S = \frac{\beta}{T_R}\left(1 - jn_P\omega T_R\right)\underline{\psi}_R - \gamma\underline{i}_S + \frac{1}{\sigma L_S}\underline{u}_S \tag{3.30}$$

$$\frac{d}{dt}\underline{\psi}_R = -\frac{1}{T_R}\left(1 - jn_P\omega T_R\right)\underline{\psi}_R + \frac{M}{T_R}\underline{i}_S \tag{3.31}$$

$$\frac{d\omega}{dt} = \frac{n_p M}{JL_R}I_m\{\underline{i}_S\underline{\psi}_R^*\} - \frac{\tau_L}{J} \tag{3.32}$$

where θ is the position of the rotor, $\omega = d\theta/dt$, n_p is the number of pole pairs, i_{Sa}, i_{Sb} are the (two phase equivalent) stator currents and ψ_{Ra}, ψ_{Rb} are the (two phase equivalent) rotor flux linkages, R_S and R_R are the stator and rotor resistances, M is the mutual inductance, L_S and L_R are the stator and rotor inductances, J is the moment of inertia of the rotor, and τ_L is the load torque. The symbols

$$T_R = \frac{L_R}{R_R} \qquad \sigma = 1 - \frac{M^2}{L_S L_R}$$

$$\beta = \frac{M}{\sigma L_S L_R} \qquad \gamma = \frac{R_S}{\sigma L_S} + \frac{1}{\sigma L_S}\frac{1}{T_R}\frac{M^2}{L_R}$$

have been used to simplify the expressions. T_R is referred to as the rotor time constant while σ is called the total leakage factor.

3.4 Algebraic speed observer

To develop an algebraic speed estimation, the rotor fluxes and their derivatives must be eliminated from the system equations as they are not measurable. Proceeding, differentiate (3.30) to obtain

$$\frac{d^2}{dt^2}\underline{i}_S = \frac{\beta}{T_R}\left(1 - jn_P\omega T_R\right)\frac{d}{dt}\underline{\psi}_R - jn_P\beta\underline{\psi}_R\frac{d\omega}{dt} - \gamma\frac{d}{dt}\underline{i}_S + \frac{1}{\sigma L_S}\frac{d}{dt}\underline{u}_S. \tag{3.33}$$

Using the complex-valued equations (3.30) and (3.31), one can eliminate $\underline{\psi}_R$ and $\frac{d}{dt}\underline{\psi}_R$ from (3.33) to obtain

$$\begin{aligned}\frac{d^2}{dt^2}\underline{i}_S =& -\frac{1}{T_R}\left(1 - jn_P\omega T_R\right)\left(\frac{d}{dt}\underline{i}_S + \gamma\underline{i}_S - \frac{1}{\sigma L_S}\underline{u}_S\right) + \frac{\beta M}{T_R^2}\left(1 - jn_P\omega T_R\right)\underline{i}_S \\ & - \gamma\frac{d}{dt}\underline{i}_S + \frac{1}{\sigma L_S}\frac{d}{dt}\underline{u}_S - \frac{jn_PT_R}{1 - jn_P\omega T_R}\left(\frac{d}{dt}\underline{i}_S + \gamma\underline{i}_S - \frac{1}{\sigma L_S}\underline{u}_S\right)\frac{d\omega}{dt}.\end{aligned} \tag{3.34}$$

Note from (3.30) that

$$\frac{d}{dt}\underline{i}_S + \gamma\underline{i}_S - \frac{1}{\sigma L_S}\underline{u}_S = \frac{\beta}{T_R}\left(1 - jn_P\omega T_R\right)\underline{\psi}_R$$

and this is zero if and only if $\left|\underline{\psi}_R\right| = 0$. Solving (3.34) for $d\omega/dt$ gives

$$\begin{aligned}\frac{d\omega}{dt} =& -\frac{\left(1 - jn_P\omega T_R\right)^2}{jn_PT_R^2} + \frac{1 - jn_P\omega T_R}{jn_PT_R}\times \\ & \frac{\frac{\beta M}{T_R^2}\left(1 - jn_P\omega T_R\right)\underline{i}_S - \gamma\frac{d}{dt}\underline{i}_S + \frac{1}{\sigma L_S}\frac{d}{dt}\underline{u}_S - \frac{d^2}{dt^2}\underline{i}_S}{\frac{d}{dt}\underline{i}_S + \gamma\underline{i}_S - \frac{1}{\sigma L_S}\underline{u}_S}.\end{aligned} \tag{3.35}$$

If the signals are measured exactly and the motor satisfies its dynamic model, the right-hand side must be real.

Breaking down the right-hand side of (3.35) into its real and imaginary parts, the real part has the form

$$\frac{d\omega}{dt} = a_2\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)\omega^2 + a_1\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)\omega + a_0\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right). \quad (3.36)$$

The expressions for $a_2\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$, $a_1\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$, and $a_0\left(u_{Sa}, u_{Sb},\ i_{Sa}, i_{Sb}\right)$ are given in section 3.7.1. Their steady-state expressions are derived in section 3.7.2. If one considers a speed observer designed by

$$\frac{d\hat{\omega}}{dt} = a_2\hat{\omega}^2 + a_1\hat{\omega} + a_0, \quad (3.37)$$

then its stability must be ascertained, that is, if $\hat{\omega}\left(t_0\right) \neq \omega\left(t_0\right)$, will $\hat{\omega}\left(t\right) \longrightarrow \omega\left(t\right)$? It is shown in section 3.7.5 that (3.36) [equivalently, (3.37)] is never stable in steady state. Consequently, equation (3.37) cannot be used as an observer unless it is stabilized.

On the other hand, the imaginary part of (3.35) must be *identically zero* leading to a second degree polynomial equation in ω of the form

$$q(\omega, t) \triangleq q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega^2 + q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega + q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = 0. \quad (3.38)$$

The expressions for $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$, $q_1\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$, and $q_0\left(u_{Sa}, u_{Sb},\ i_{Sa}, i_{Sb}\right)$ are given in section 3.7.3, and their steady-state expressions are derived in section

3.7.4. There are two solutions to equation (3.38), and at least one of these two solutions must track the motor speed. This equation does not have any stability issues, but a procedure is required to determine which of the two solutions is correct. Further, there are situations when the speed cannot be determined from (3.38). For example, if u_{Sa} = constant and $u_{Sb} = 0$, it turns out that $q_2 = q_1 = q_0 \equiv 0$ and ω is not determinable from (3.38) (see section 3.7.6 where this is shown in detail). On the other hand, if the machine is operated at zero speed ($\omega \equiv 0$) with a load on it, then $q_2 \equiv 0$ and $q_1 \neq 0$, and a unique solution is specified by (3.38) (see section 3.7.4 where this is proved in steady state). In fact, for low speed trajectories, consider equation (3.38) written in the form

$$(q_2\omega + q_1)\,\omega + q_0 = 0. \tag{3.39}$$

At low speeds, defined by $|q_2\omega| \ll |q_1|$, equation (3.39) reduces

$$q_1\omega + q_0 = 0$$

and ω is uniquely determined by $\omega = -q_0/q_1$. Section 3.7.7 shows that, in steady state, $|q_2\omega| \ll |q_1|$ if $(T_R n_p \omega)^2 \ll 1$.

If $q_2 \neq 0$, the correct solution of (3.38) can be determined as follows: Differentiate equation (3.38) to obtain a new independent equation given by

$$(2q_2\omega + q_1)\,\frac{d\omega}{dt} + \dot{q}_2\omega^2 + \dot{q}_1\omega + \dot{q}_0 \equiv 0. \tag{3.40}$$

Next, $d\omega/dt$ is replaced by the right-hand side of equation (3.36) to obtain a polynomial equation in ω given by

$$g(\omega) \triangleq 2q_2a_2\omega^3 + (2q_2a_1 + q_1a_2 + \dot{q}_2)\,\omega^2 + (2q_2a_0 + q_1a_1 + \dot{q}_1)\,\omega + q_1a_0 + \dot{q}_0. \quad (3.41)$$

$g(\omega)$ is a third-order polynomial equation in ω for which the speed of the motor is one of its zeros. Dividing* $g(\omega)$ by $q(\omega)$ from (3.38), $g(\omega)$ may be rewritten in the form

$$\begin{aligned} g(\omega) \equiv (2q_2a_2\omega + 2q_2a_1 - q_2q_1a_2 + \dot{q}_2)\,q(\omega,t) \quad &+ r_1\,(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\,\omega \\ &+ r_0\,(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\,. \end{aligned} \quad (3.42)$$

where

$$r_1\,(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \triangleq 2q_2^2a_0 - q_2q_1a_1 + q_2\dot{q}_1 - 2q_2q_0a_2 + q_1^2a_2 - q_1\dot{q}_2 \quad (3.43)$$

and

$$r_0\,(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \triangleq q_2q_1a_0 + q_2\dot{q}_0 - 2q_2q_0a_1 + q_0q_1a_2 - q_0\dot{q}_2. \quad (3.44)$$

If ω is equal to the speed of the motor, then both $g(\omega) = 0$ and $q(\omega) = 0$, and one obtains

$$r_1\,(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\,\omega + r_0\,(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \equiv 0. \quad (3.45)$$

*Given the polynomials $g(\omega), q(\omega)$ in ω with $\deg\{g(\omega)\} = n_g, \deg\{q(\omega)\} = n_q$, the Euclidean division algorithm ensures that there are polynomials $\gamma(\omega), r(\omega)$ such that $g(\omega) = \gamma(\omega)q(\omega) + r(\omega)$ and $\deg\{r(\omega)\} \leq \deg\{q(\omega)\} - 1 = n_q - 1$. Consequently, if ω_0 is a zero of both $g(\omega)$ and $q(\omega)$, then it must also be a zero of $r(\omega)$.

This is now a first-order polynomial equation in ω with a unique solution as long as $r_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ (the coefficient of ω) is nonzero (It is shown in section 3.7.8 that $r_1 \neq 0$ in steady state if $q_2 \neq 0$). A purely algebraic estimation for the speed is then given by

$$\omega \triangleq \begin{cases} -q_0/q_1 & \text{if } q_2 = 0 \quad \text{See (3.38)} \\ -r_0/r_1 & \text{if } q_2 \neq 0 \quad \text{See (3.45)}. \end{cases}$$

3.5 Stable dynamic speed observer

The coefficients of r_1 and r_0 contain third-order derivatives of the stator currents and second-order derivatives of the stator voltages and, therefore, noise is a concern. Rather than use this purely algebraic estimator, it is now shown that it can be combined with the dynamic model to obtain a smoother (yet stable) speed estimator.

Dividing the right side of the differential equation model (3.36) by $q(\omega, t)$ $(q_2 \neq 0)$, one obtains

$$a_2\omega^2 + a_1\omega + a_0 = \frac{a_2}{q_2}q(\omega, t) + \alpha\omega + \beta \tag{3.46}$$

where

$$\alpha \triangleq a_1 - a_2q_1/q_2 \tag{3.47}$$

and

$$\beta \triangleq a_0 - a_2q_0/q_2. \tag{3.48}$$

Then, as $q(\omega, t) \equiv 0$, equation (3.36) may be rewritten as

$$\frac{d\omega}{dt} = \alpha(t)\omega + \beta(t), \tag{3.49}$$

which is a *linear first-order time-varying* system. With

$$\Phi(t, t_0) \triangleq e^{\int_{t_0}^{t} \alpha(\tau) d\tau}$$

the fundamental solution of (3.49), the full solution is given by (see [55])

$$\omega(t) = \Phi(t, t_0)\omega(0) + \int_{t_0}^{t} \Phi(t, \tau)\beta(\tau) d\tau.$$

Consequently, a sufficient condition for stability is that $\alpha(t) \leq -\kappa < 0$ for some $\kappa > 0$. It is shown in section 3.7.5 that $\alpha > 0$ in steady state, so the system is never stable in steady state.

For the case that $q_2 \neq 0$, consider (3.49) to be the induction motor "model" and the solution ω of algebraic estimator (3.45) to be the "measurement". Then, let an observer be defined by

$$\frac{d\hat{\omega}}{dt} = \alpha(t)\hat{\omega} + \beta(t) + \ell(\omega - \hat{\omega}). \tag{3.50}$$

If $\ell - \alpha(t) > \kappa > 0$ for all t, then the estimator (3.50) is stable with a rate of decay of the error no less than κ. As this estimator is the result of integrating the signals $\alpha(t)$, $\beta(t)$, and ω from (3.45), it is a smoother estimate than the purely algebraic estimate of equation (3.45).

In the case where $q_2 = 0$, the right side of equation (3.36) can be divided by $q_1\omega + q_0 = 0$ to obtain

$$\frac{d\hat{\omega}}{dt} = c(t) + \ell(\omega - \hat{\omega}). \tag{3.51}$$

If $\ell > \kappa > 0$ for all t, then equation (3.51) is stable with a rate of decay of the error no less than κ.

Collecting this together, the estimate of speed proposed here is defined as the solution to the observer

$$\frac{d\hat{\omega}}{dt} \triangleq a_2\hat{\omega}^2 + a_1\hat{\omega} + a_0 + \ell\left(\omega - \hat{\omega}\right) \tag{3.52}$$

where

$$\omega \triangleq \begin{cases} -q_0/q_1 & \text{if } q_2 = 0 \quad \text{See } (3.38) \\ -r_0/r_1 & \text{if } q_2 \neq 0 \quad \text{See } (3.45)\,. \end{cases}$$

In section 3.7.4 it is shown that in steady state $q_1 \neq 0$ if $q_2 = 0$, while in section 3.7.8 it is shown that in steady state $r_1 \neq 0$ if $q_2 \neq 0$.

3.6 Simulation results

As a first look at the viability of the proposed speed sensorless observer, simulations are carried out. Figure 3.1 shows a block diagram of the sensorless control system. In this system, a current command field-oriented controller is used [13] with the induction motor model being equations (3.30), (3.31), and (3.32). The speed observer (3.52) is used with the estimated speed $\hat{\omega}$ fed back to the current command field-oriented controller.

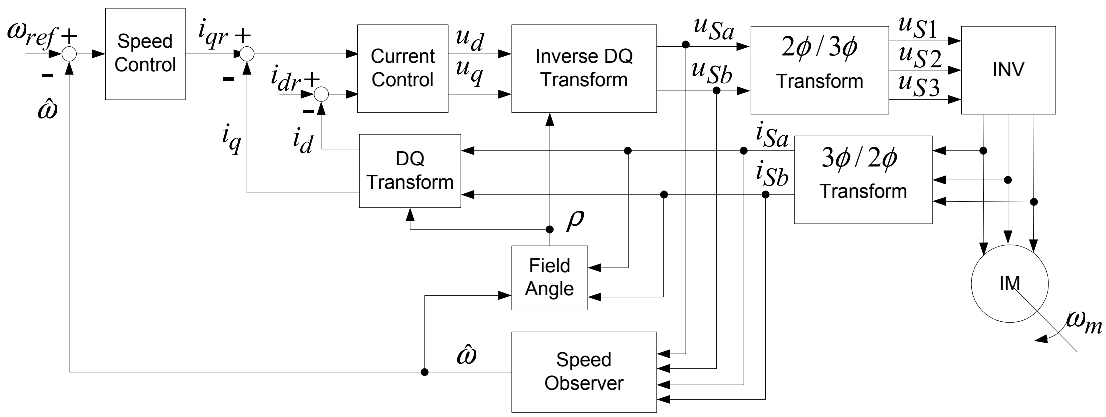

Figure 3.1: Sensorless speed control system.

Here a three-phase (two-phase equivalent) induction motor model is simulated using the machine parameter values

$$n_p = 2,\ R_S = 5.12 \text{ ohms},\ R_R = 2.23 \text{ ohms},\ L_S = L_R = 0.2919 \text{ H},\ M = 0.2768 \text{ H},$$
$$J = 0.0021 \text{ k-gm}^2,\ \tau_{L\_rated} = 2.0337 \text{ N-m},\ I_{\max} = 2.77 \text{ A},\ V_{\max} = 230 \text{ V}.$$

Figure 3.2 shows the simulation results of the motor speed and speed estimator under full load. From $t = 0$ to $t = 0.4$ seconds, a constant u_{Sa} is applied to the motor to build up the flux and the motor is considered to be held with a brake so that $\omega \equiv 0$. At $t = 0.4$ seconds the brake is released, and the machine is running on a low speed trajectory ($\omega_{\max} = 5\ rad/s$) *with full load at the start.* The estimated speed $\hat{\omega}$ is used in the field-oriented controller as shown in Figure 3.1. In this simulation, the observer gain ℓ in equation (3.52) was chosen to be 1000.

Figure 3.3 shows the simulation results of the motor speed and stabilized speed estimator under full load for a zero speed trajectory. From $t = 0$ to $t = 0.4$ seconds, a

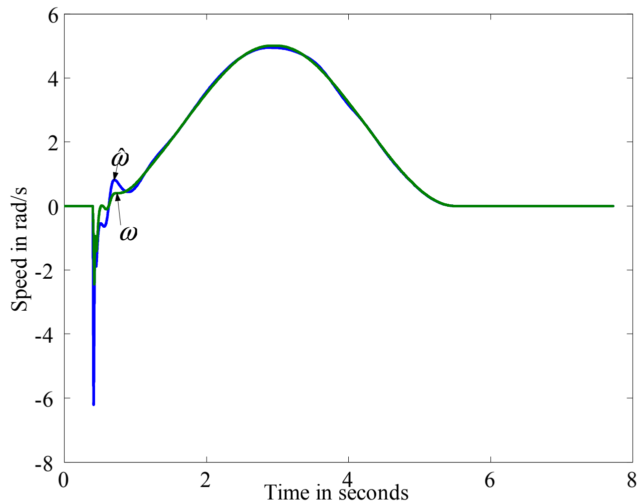

Figure 3.2: Actual speed ω and estimated speed $\hat{\omega}$ (using differential-algebraic method) with the motor tracking a low speed trajectory ($\omega_{\max} = 5\ rad/s$) *with full load at the start*.

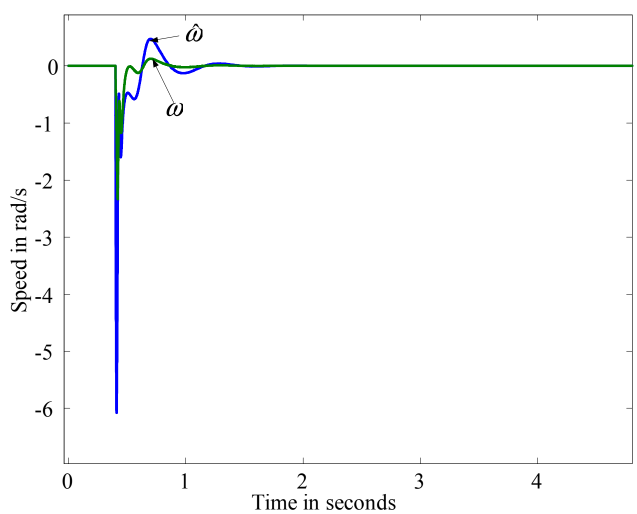

Figure 3.3: Actual speed ω and estimated speed $\hat{\omega}$ (using differential-algebraic method) with the motor tracking a zero speed trajectory ($\omega \equiv 0$) *with full load at the start*.

constant u_{Sa} is applied to the motor to build up the flux and the motor is considered to be held with a brake so that $\omega \equiv 0$. At $t = 0.4$ seconds, the brake is released, and the machine is controlled to a zero speed trajectory ($\omega \equiv 0$) *with full load at the start.* $\hat{\omega}$ is used in the field-oriented controller as shown in Figure 3.1. In this simulation, the observer gain ℓ in equation (3.52) is chosen to be 1000.

Figure 3.4 shows the whole trajectory chosen to have a maximum speed of 5 rad/sec, to do a speed reversal, and to have zero speed at the end.

To consider the effect of noise in the simulation, a 20 kHz PWM inverter is included. Low-pass analog filters (third-order, 100 Hz cutoff) are used (simulated) before the voltages and currents are sampled. Such a filter limits the applicability to speeds where the electrical frequency is below the filter cutoff. The sample period was 1 μs, which is not possible using the standard processor technology in commercial electric drives. However, such a step size is found to be necessary in order to use (simulate) a PWM inverter and then compute third-order derivatives of the currents and second-order derivatives of the voltages to estimate the speed for feedback control.

The interest here is in low-speed sensorless control of the machine with full load. The trajectory is chosen to have a maximum speed of 5 rad/sec, to do a speed reversal, and to have zero speed at the end as shown in Figure 3.5. The induction motor model for the simulation is based on equations (3.30), (3.31), and (3.32). Along with the stator currents, the estimated speed $\hat{\omega}$ is fed back to a current command field-oriented controller [13] [1]. Figure 3.5 shows the simulation results of the motor speed and the stabilized speed estimator under full load. The full load is on the motor from $t = 0.4\,\text{sec}$ to $t = 16\,\text{sec}$, that is, even during the zero speed part of the trajectory. From $t = 0$ to $t = 0.4$ seconds, a constant u_{Sa} is applied to the motor to build up the

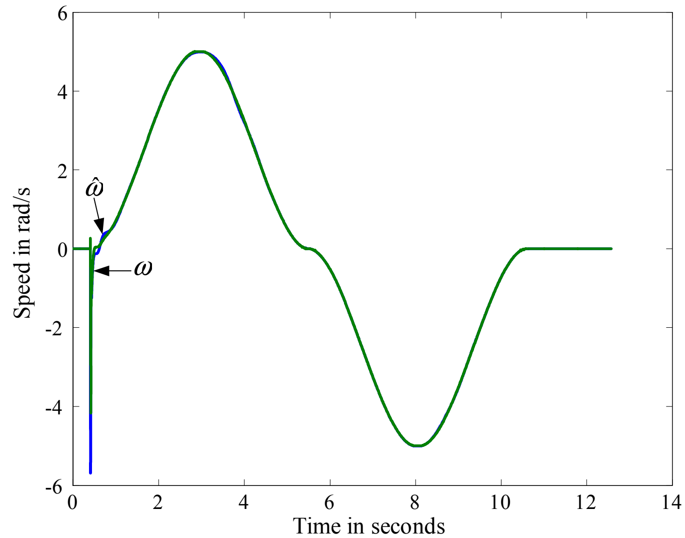

Figure 3.4: Actual speed ω and estimated speed $\hat{\omega}$ (using differential-algebraic method) with full load on the motor.

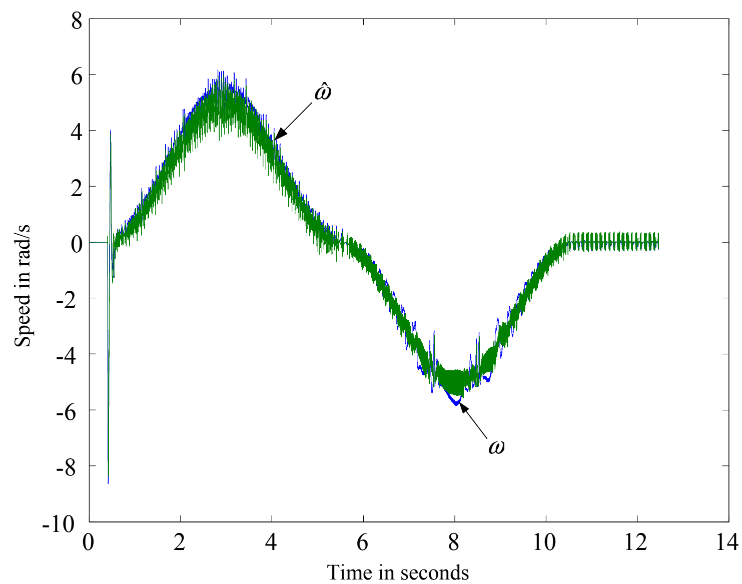

Figure 3.5: Actual speed ω and estimated speed $\hat{\omega}$ (using differential-algebraic method) with full load on the motor drived by PWM inverter.

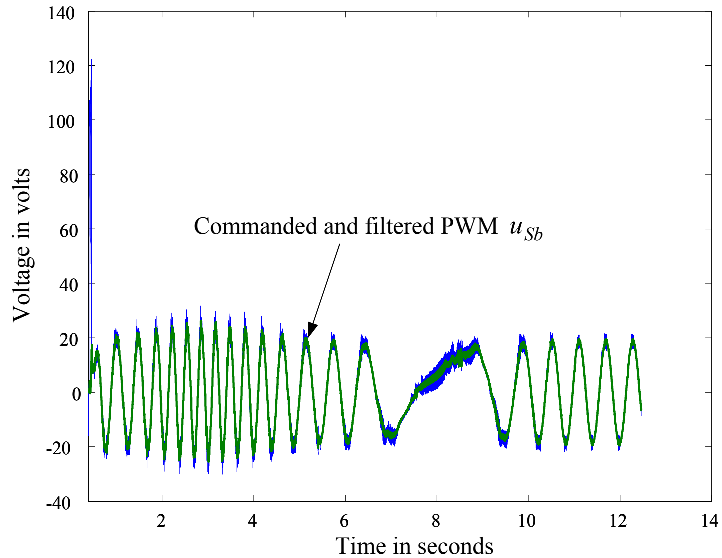

Figure 3.6: Commanded voltage and the filtered PWM voltage.

flux, and the motor is considered to be held with a (mechanical) brake so that $\omega \equiv 0$. Figure 3.5 shows at $t = 0.4$ seconds the brake is released and the machine is running on a low speed trajectory ($\omega_{\max} = 5\ rad/s$) with full load. In this simulation, the observer gain ℓ in equation (3.52) is chosen to be 1000.

Figures 3.6 and 3.7 shows the voltage u_{Sb} and current i_{Sb} corresponding to the trajectory in Figure 3.5.

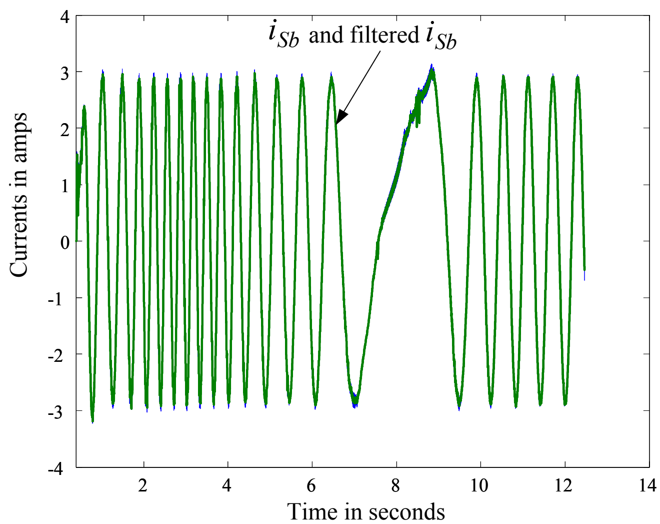

Figure 3.7: The measured current and the filtered current.

3.7 Dynamic and steady-state coefficients of speed estimator and speed estimation proofs in steady state

3.7.1 The dynamic expressions for $a_2\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$, $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ and $a_0\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$

The dynamic expressions for a_2, a_1 and a_0 are given here.

Let

$$
\begin{aligned}
A_1 &\triangleq i_{Sb}\frac{d^2 i_{Sa}}{dt} - i_{Sa}\frac{d^2 i_{Sb}}{dt}, \qquad & A_2 \triangleq i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt},\\
A_3 &\triangleq \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt}, & A_4 \triangleq i_{Sa}^2 + i_{Sb}^2,\\
A_5 &\triangleq u_{Sb} i_{Sa} - u_{Sa} i_{Sb}, & A_6 \triangleq i_{Sa}\frac{du_{Sb}}{dt} - i_{Sb}\frac{du_{Sa}}{dt},\\
A_7 &\triangleq i_{Sa}\frac{d^2 i_{Sa}}{dt} + i_{Sb}\frac{d^2 i_{Sb}}{dt}, & A_8 \triangleq u_{Sa} i_{Sa} + u_{Sb} i_{Sb},\\
A_9 &\triangleq i_{Sa}\frac{du_{Sa}}{dt} + i_{Sb}\frac{du_{Sb}}{dt}, & A_{10} \triangleq u_{Sa}^2 + u_{Sb}^2.
\end{aligned}
\tag{3.53}
$$

such that

$$
a_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \triangleq \frac{-n_p^2 \beta M \left(\frac{1}{\sigma L_S} A_4 A_5 + A_4 A_2\right)}{n_P T_R \left(\left(\frac{1}{2}A_3 + \gamma A_4 - \frac{1}{\sigma L_S} A_8\right)^2 + \left(\frac{1}{\sigma L_S} A_5 + A_2\right)^2\right)} \tag{3.54}
$$

$$\begin{aligned}
a_1\left(u_{Sa},u_{Sb},i_{Sa},i_{Sb}\right) \triangleq\ & \frac{n_P}{n_P T_R\left(\left(\frac{1}{2}A_3+\gamma A_4-\frac{1}{\sigma L_S}A_8\right)^2+\left(\frac{1}{\sigma L_S}A_5+A_2\right)^2\right)} \\
& \times\left(\frac{1}{4}\left(T_R\gamma+2\right)A_3^2+\left(\frac{1}{4}T_R\gamma^2-\frac{\beta M}{2T_R}+\gamma\right)\frac{dA_4^2}{dt}\right. \\
& +\left(\gamma T_R+2\right)A_2^2+\frac{1}{2}T_R A_7 A_3+2\gamma\left(\gamma-\frac{\beta M}{T_R}\right)A_4^2 \\
& +\gamma T_R A_7 A_4+\frac{1}{2}T_R\frac{dA_2^2}{dt}+\frac{1}{\sigma L_S}\left(\gamma T_R+4\right)A_2A_5 \\
& -\frac{1}{\sigma L_S}\left(2+\frac{1}{2}T_R\gamma\right)A_3A_8+\frac{2}{\sigma^2L_S^2}A_4A_{10} \\
& +\frac{2}{\sigma L_S}\left(\frac{\beta M}{T_R}-2\gamma\right)A_4A_8-\frac{1}{\sigma L_S}T_RA_7A_8-\frac{T_R}{\sigma L_S}\frac{1}{2}A_9A_3 \\
& -\frac{T_R}{\sigma L_S}\gamma A_9A_4+\frac{T_R}{\sigma L_S}\frac{1}{\sigma L_S}A_9A_8+\frac{T_R}{\sigma L_S}\frac{1}{\sigma L_S}A_6A_5 \\
& \left.+\frac{T_R}{\sigma L_S}A_6A_2+\frac{T_R}{\sigma L_S}\frac{dA_2}{dt}A_5\right) \qquad (3.55)
\end{aligned}$$

$$\begin{aligned}
a_0(u_{Sa},u_{Sb},i_{Sa},i_{Sb}) \triangleq\ & \frac{1}{n_P T_R\left(\left(\frac{1}{2}A_3+\gamma A_4-\frac{1}{\sigma L_S}A_8\right)^2+\left(\frac{1}{\sigma L_S}A_5+A_2\right)^2\right)} \\
& \times\left(\left(\frac{\beta M}{T_R^2}+\gamma^2\right)A_2A_4-A_7A_2+\frac{1}{2}A_1A_3+\gamma A_1A_4\right. \\
& +\frac{1}{\sigma L_S}\frac{\beta M}{T_R^2}A_4A_5-\frac{1}{\sigma L_S}\frac{1}{2}\gamma A_3A_5-\frac{1}{\sigma L_S}\gamma A_2A_8 \\
& -\frac{1}{\sigma L_S}A_7A_5+\frac{1}{2}\frac{1}{\sigma L_S}A_6A_3+\gamma\frac{1}{\sigma L_S}A_6A_4-\frac{1}{\sigma^2L_S^2}A_6A_8 \\
& \left.+\frac{1}{\sigma^2L_S^2}A_9A_5+\frac{1}{\sigma L_S}A_9A_2-\frac{1}{\sigma L_S}A_1A_8\right). \qquad (3.56)
\end{aligned}$$

3.7.2 The steady-state expressions for $a_2\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$, $a_1($ u_{Sa},u_{Sb}, i_{Sa},$i_{Sb})$ and $a_0\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$

The steady-state expressions for $a_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$, $a_1(u_{Sa}, u_{Sb},\ i_{Sa}, i_{Sb})$ and $a_0(u_{Sa}, u_{Sb}, i_{Sa},\ i_{Sb})$ are now derived.

In steady state, let (see [13])

$$u_{Sa} + j u_{Sb} = \underline{U}_S e^{j\omega_S t}$$

$$i_{Sa} + j i_{Sb} = \underline{I}_S e^{j\omega_S t}.$$

The complex phasors $\underline{U}_S$ and $\underline{I}_S$ are related by

$$\underline{I}_S = \frac{\underline{U}_S}{R_S + j\omega_S L_S \left[\frac{1+j\frac{S}{S_p}}{1+j\frac{S}{\sigma S_p}}\right]}.$$

Here $S_p \triangleq \frac{R_R}{\sigma\omega_S L_R} = \frac{1}{\sigma\omega_S T_R}$ so that

$$\begin{aligned}
\underline{I}_S &= \frac{\underline{U}_S}{R_S + j\omega_S L_S \left[\frac{1+jS\sigma\omega_S T_R}{1+jS\omega_S T_R}\right]} \\
&= \frac{\underline{U}_S}{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right) + j\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}
\end{aligned}$$

$$\underline{U}_S \underline{I}_S^* = |\underline{U}_S|^2 \frac{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right) + j\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}}.$$

The real power is equal to

$$\begin{aligned}P &\triangleq u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\\ &= R_e\left(\underline{U}_S\underline{I}_S^*\right)\\ &= \frac{\left|\underline{U}_S\right|^2\left(R_S + \frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)}{\left(R_S + \frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2 + \frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}}.\end{aligned} \tag{3.57}$$

The reactive power is equal to

$$\begin{aligned}Q &\triangleq u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\\ &= I_m\left(\underline{U}_S\underline{I}_S^*\right)\\ &= \frac{\left|\underline{U}_S\right|^2\frac{\omega_SL_S\left(1+\sigma S^2\omega_S^2T_R^2\right)}{1+S^2\omega_S^2T_R^2}}{\left(R_S + \frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2 + \frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}}.\end{aligned} \tag{3.58}$$

In steady state

$$\begin{aligned}i_{Sa}^2 + i_{Sb}^2 &= \left|\underline{I}_S\right|^2\\ u_{Sa}^2 + u_{Sb}^2 &= \left|\underline{U}_S\right|^2\end{aligned}$$

$$\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} = 0.$$

Let

$$den \triangleq n_p T_R \left(\left(\frac{1}{2} A_3 + \gamma A_4 - \frac{1}{\sigma L_S} A_8 \right)^2 + \left(\frac{1}{\sigma L_S} A_5 + A_2 \right)^2 \right)$$
$$= n_p T_R \left| \underline{\mathrm{U}}_S \right|^4 \frac{\left(\frac{(1-\sigma)}{\sigma T_R} \frac{1+S^2\omega_S^2 T_R^2 - S\omega_S^2 T_R^2}{1+S^2\omega_S^2 T_R^2} \right)^2 + \left(\frac{(1-\sigma)}{\sigma} \frac{\omega_S}{1+S^2\omega_S^2 T_R^2} \right)^2}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2} \right)^2}. \tag{3.59}$$

Remark Recall that it was pointed out following equation (3.35) that $den = 0$ if and only if $\left| \underline{\psi}_R \right| \equiv 0$.

The steady-state expression for $a_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$

As a result, in steady state the two terms of $a_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ are

$$\begin{aligned}
-\frac{1}{den} n_p^2 \beta M \frac{1}{\sigma L_S} A_4 A_5 &= -\frac{1}{den} n_p^2 \beta M \frac{1}{\sigma L_S} \left(i_{Sa}^2 + i_{Sb}^2 \right) \left(u_{Sb} i_{Sa} - u_{Sa} i_{Sb} \right) \\
&= \frac{-n_p^2 \beta M \frac{1}{\sigma L_S} \left| \underline{U}_S \right|^4 \frac{\omega_S L_S \left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}{den \left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2} \right)^2}
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{den} n_p^2 \beta M A_4 A_2 &= -\frac{1}{den} n_p^2 \beta M \left(i_{Sa}^2 + i_{Sb}^2 \right) \left(i_{Sb} \frac{di_{Sa}}{dt} - i_{Sa} \frac{di_{Sb}}{dt} \right) \\
&= -\frac{1}{den} n_p^2 \beta M \left| \underline{I}_S \right|^2 \left(-\omega_S \left| \underline{I}_S \right|^2 \right) \\
&= \frac{n_p^2 \beta M \omega_S \left| \underline{U}_S \right|^4}{den \left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2} \right)^2}.
\end{aligned}$$

Adding these two steady-state expressions to get

$$a_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = \frac{-n_p^2 \left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \times \frac{\omega_S (1-\sigma)^2}{\sigma^2 \left(1+S^2\omega_S^2 T_R^2\right)} \times \frac{1}{den}. \tag{3.60}$$

The steady-state expression for $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$

In steady state, the first, second, fourth, seventh, ninth, thirteenth, and eighteenth terms of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ are all zeros, i.e.

$$\frac{1}{4}\frac{n_p}{den}(T_R\gamma + 2) A_3^2 = \frac{1}{4}\frac{n_p}{den}(T_R\gamma + 2)\left(\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt}\right)^2 = 0$$

$$\frac{n_p}{den}\left(\frac{1}{4}T_R\gamma^2 - \frac{\beta M}{2T_R} + \gamma\right)\frac{dA_4^2}{dt} = \frac{n_p}{den}\left(\frac{1}{4}T_R\gamma^2 - \frac{\beta M}{2T_R} + \gamma\right)\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)^2}{dt} = 0$$

$$\frac{1}{2}\frac{n_p}{den}T_R A_7 A_3 = \frac{1}{2}\frac{n_p}{den}T_R\left(i_{Sa}\frac{d^2 i_{Sa}}{dt} + i_{Sb}\frac{d^2 i_{Sb}}{dt}\right)\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} = 0$$

$$\begin{aligned}
\frac{1}{2}\frac{n_p}{den}T_R\frac{dA_2^2}{dt} &= \frac{1}{2}\frac{n_p}{den}T_R\frac{d\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right)^2}{dt} \\
&= \frac{1}{2}\frac{n_p}{den}T_R\frac{d\left(-\omega_S\left|\underline{I}_S\right|^2\right)^2}{dt} = 0
\end{aligned}$$

$$\begin{aligned}
-\frac{n_p}{den}\frac{1}{\sigma L_S}\left(2 + \frac{1}{2}T_R\gamma\right)A_3 A_8 &= -\frac{n_p}{den}\frac{\left(2 + \frac{1}{2}T_R\gamma\right)}{\sigma L_S}\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt}\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) \\
&= 0
\end{aligned}$$

$$-\frac{1}{2}\frac{n_p}{den}\frac{T_R}{\sigma L_S}A_9A_3 = -\frac{1}{2}\frac{n_p}{den}\frac{T_R}{\sigma L_S}\left(i_{Sa}\frac{du_{Sa}}{dt}+i_{Sb}\frac{du_{Sb}}{dt}\right)\frac{d\left(i_{Sa}^2+i_{Sb}^2\right)}{dt} = 0$$

$$\frac{n_p}{den}\frac{T_R}{\sigma L_S}\frac{dA_2}{dt}A_5 = \frac{n_p}{den}\frac{T_R}{\sigma L_S}\frac{d\left(i_{Sb}\frac{di_{Sa}}{dt}-i_{Sa}\frac{di_{Sb}}{dt}\right)}{dt}\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right) = 0.$$

The third term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{n_p}{den}\left(\gamma T_R+2\right)A_2^2 &= \frac{n_p}{den}\left(\gamma T_R+2\right)\left(i_{Sb}\frac{di_{Sa}}{dt}-i_{Sa}\frac{di_{Sb}}{dt}\right)^2 \\
&= \frac{n_p}{den}\left(\gamma T_R+2\right)\omega_S^2\left|\underline{I}_S\right|^4 \\
&= \frac{n_p\left(\gamma T_R+2\right)\omega_S^2\left|\underline{U}_S\right|^4}{den\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The fifth term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}2n_p\gamma\left(\gamma-\frac{\beta M}{T_R}\right)A_4^2 &= \frac{1}{den}2n_p\gamma\left(\gamma-\frac{\beta M}{T_R}\right)\left(i_{Sa}^2+i_{Sb}^2\right)^2 \\
&= \frac{1}{den}2n_p\gamma\left(\gamma-\frac{\beta M}{T_R}\right)\left|\underline{I}_S\right|^4 \\
&= \frac{2n_p\gamma\left(\gamma-\frac{\beta M}{T_R}\right)\left|\underline{U}_S\right|^4}{den\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The sixth term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}n_p\gamma T_RA_7A_4 &= \frac{1}{den}n_p\gamma T_R\left(i_{Sa}\frac{d^2i_{Sa}}{dt}+i_{Sb}\frac{d^2i_{Sb}}{dt}\right)\left(i_{Sa}^2+i_{Sb}^2\right) \\
&= -\frac{1}{den}n_p\gamma T_R\omega_S^2\left|\underline{I}_S\right|^4 \\
&= \frac{-n_p\gamma T_R\omega_S^2\left|\underline{U}_S\right|^4}{den\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The eighth term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}\frac{1}{\sigma L_S}n_p\left(\gamma T_R+4\right)A_2A_5 &= \frac{1}{den}\frac{\gamma T_R+4}{\sigma L_S}n_p\left(i_{Sb}\frac{di_{Sa}}{dt}-i_{Sa}\frac{di_{Sb}}{dt}\right)\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right)\\
&= \frac{1}{den}\frac{1}{\sigma L_S}n_p\left(\gamma T_R+4\right)\left(-\omega_S\left|\underline{I}_S\right|^2\right)\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right)\\
&= \frac{-\frac{1}{\sigma L_S}n_p\left(\gamma T_R+4\right)\omega_S\left|\underline{U}_S\right|^4\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2T_R^2\right)}{1+S^2\omega_S^2T_R^2}}{den\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The tenth term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}\frac{2n_p}{\sigma^2L_S^2}A_4A_{10} &= \frac{1}{den}\frac{2n_p}{\sigma^2L_S^2}\left(i_{Sa}^2+i_{Sb}^2\right)\left(u_{Sa}^2+u_{Sb}^2\right)\\
&= \frac{1}{den}\frac{2n_p}{\sigma^2L_S^2}\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2\\
&= \frac{\frac{2n_p}{\sigma^2L_S^2}\left|\underline{U}_S\right|^4}{den\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)}.
\end{aligned}$$

The eleventh term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}\frac{2n_p}{\sigma L_S}\left(\frac{\beta M}{T_R}-2\gamma\right)A_4A_8 &= \frac{1}{den}\frac{2n_p}{\sigma L_S}\left(\frac{\beta M}{T_R}-2\gamma\right)\left(i_{Sa}^2+i_{Sb}^2\right)\left(u_{Sa}i_{Sa}+u_{Sb}i_{Sb}\right)\\
&= \frac{\frac{2n_p}{\sigma L_S}\left(\frac{\beta M}{T_R}-2\gamma\right)\left|\underline{U}_S\right|^4\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)}{den\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The twelfth term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-\frac{1}{den}\frac{n_p}{\sigma L_S}T_R A_7 A_8 &= -\frac{1}{den}\frac{n_p T_R}{\sigma L_S}\left(i_{Sa}\frac{d^2 i_{Sa}}{dt} + i_{Sb}\frac{d^2 i_{Sb}}{dt}\right)\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) \\
&= -\frac{1}{den}\frac{n_p T_R}{\sigma L_S}\left(-\omega_S^2\left|\underline{I}_S\right|^2\right)\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) \\
&= \frac{\frac{n_p}{\sigma L_S}T_R\omega_S^2\left|\underline{U}_S\right|^4\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The fourteenth term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-\frac{1}{den}\frac{n_p T_R}{\sigma L_S}\gamma A_9 A_4 &= -\frac{1}{den}\frac{n_p T_R}{\sigma L_S}\gamma\left(i_{Sa}\frac{du_{Sa}}{dt} + i_{Sb}\frac{du_{Sb}}{dt}\right)\left(i_{Sa}^2 + i_{Sb}^2\right) \\
&= \frac{1}{den}\frac{n_p T_R}{\sigma L_S}\gamma\omega_S\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right)\left|\underline{I}_S\right|^2 \\
&= \frac{\frac{n_p T_R}{\sigma L_S}\gamma\omega_S\left|\underline{U}_S\right|^4\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The fifteenth term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}\frac{n_p T_R}{\sigma^2 L_S^2}A_9 A_8 &= -\frac{1}{den}\frac{n_p T_R}{\sigma^2 L_S^2}\omega_S\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right)\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) \\
&= -\frac{\frac{n_p T_R}{\sigma^2 L_S^2}\omega_S\left|\underline{U}_S\right|^4\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The sixteenth term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}\frac{n_p T_R}{\sigma^2 L_S^2} A_6 A_5 &= \frac{1}{den}\frac{n_p T_R}{\sigma^2 L_S^2}\left(i_{Sa}\frac{du_{Sb}}{dt} - i_{Sb}\frac{du_{Sa}}{dt}\right)(u_{Sb} i_{Sa} - u_{Sa} i_{Sb}) \\
&= \frac{1}{den}\frac{n_p T_R}{\sigma^2 L_S^2}\omega_S (u_{Sa} i_{Sa} + u_{Sb} i_{Sb})(u_{Sb} i_{Sa} - u_{Sa} i_{Sb}) \\
&= \frac{\frac{n_p T_R}{\sigma^2 L_S^2}\omega_S \left|\underline{U}_S\right|^4 \left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The seventeenth term of $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}\frac{n_p T_R}{\sigma L_S} A_6 A_2 &= \frac{1}{den}\frac{n_p T_R}{\sigma L_S}\left(i_{Sa}\frac{du_{Sb}}{dt} - i_{Sb}\frac{du_{Sa}}{dt}\right)\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right) \\
&= -\frac{1}{den}\frac{n_p T_R}{\sigma L_S}\omega_S^2 (u_{Sa} i_{Sa} + u_{Sb} i_{Sb})\left|\underline{I}_S\right|^2 \\
&= \frac{-\frac{n_p T_R}{\sigma L_S}\omega_S^2 \left|\underline{U}_S\right|^4 \left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

Finally, substituting these steady-state expressions into the expression for $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$, one obtains

$$\begin{aligned}
a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \quad = \quad & \frac{n_p \left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \\
& \times \frac{2\omega_S^2 (1-\sigma)^2 (1-S)}{\sigma^2 \left(1+S^2\omega_S^2 T_R^2\right)} \times \frac{1}{den}.
\end{aligned} \tag{3.61}$$

The steady-state expression for $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$

Similar to above analysis, in steady state, the third, fourth, sixth, ninth, and fourteenth terms of $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ are all zeros, i.e.

$$\frac{1}{2}\frac{1}{den}A_1A_3 = \frac{1}{2}\frac{1}{den}\left(i_{Sb}\frac{d^2 i_{Sa}}{dt} - i_{Sa}\frac{d^2 i_{Sb}}{dt}\right)\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} = 0$$

$$\frac{1}{den}\gamma A_1A_4 = \frac{1}{den}\gamma\left(i_{Sb}\frac{d^2 i_{Sa}}{dt} - i_{Sa}\frac{d^2 i_{Sb}}{dt}\right)\left(i_{Sa}^2 + i_{Sb}^2\right) = 0$$

$$-\frac{1}{den}\frac{1}{\sigma L_S}\frac{1}{2}\gamma A_3A_5 = -\frac{1}{den}\frac{1}{\sigma L_S}\frac{1}{2}\gamma\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt}\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right) = 0$$

$$\frac{1}{2}\frac{1}{den}\frac{1}{\sigma L_S}A_6A_3 = \frac{1}{2}\frac{1}{den}\frac{1}{\sigma L_S}\left(i_{Sa}\frac{du_{Sb}}{dt} - i_{Sb}\frac{du_{Sa}}{dt}\right)\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} = 0$$

$$-\frac{1}{den}\frac{1}{\sigma L_S}A_1A_8 = -\frac{1}{den}\frac{1}{\sigma L_S}\left(i_{Sb}\frac{d^2 i_{Sa}}{dt} - i_{Sa}\frac{d^2 i_{Sb}}{dt}\right)\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) = 0.$$

The first term of $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}\left(\frac{\beta M}{T_R^2} + \gamma^2\right)A_2A_4 &= \frac{1}{den}\left(\frac{\beta M}{T_R^2} + \gamma^2\right)\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right)\left(i_{Sa}^2 + i_{Sb}^2\right) \\
&= -\frac{1}{den}\left(\frac{\beta M}{T_R^2} + \gamma^2\right)\omega_S\left|\underline{I}_S\right|^4 \\
&= \frac{-\left(\frac{\beta M}{T_R^2} + \gamma^2\right)\omega_S\left|\underline{U}_S\right|^4}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The second term of $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
-\frac{1}{den}A_7A_2 &= -\frac{1}{den}\left({}_{Sa}\frac{d^2 i_{Sa}}{dt} + i_{Sb}\frac{d^2 i_{Sb}}{dt}\right)\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right) \\
&= -\frac{1}{den}\omega_S^3 \left|\underline{I}_S\right|^4 \\
&= \frac{-\omega_S^3 \left|\underline{U}_S\right|^4}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}
$$

The fifth term of $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
\frac{1}{den}\frac{1}{\sigma L_S}\frac{\beta M}{T_R^2}A_4A_5 &= \frac{1}{den}\frac{1}{\sigma L_S}\frac{\beta M}{T_R^2}\left(i_{Sa}^2 + i_{Sb}^2\right)\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right) \\
&= \frac{1}{den}\frac{1}{\sigma L_S}\frac{\beta M}{T_R^2}\left|\underline{I}_S\right|^2\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right) \\
&= \frac{\frac{1}{\sigma L_S}\frac{\beta M}{T_R^2}\left|\underline{U}_S\right|^4 \frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}
$$

The seventh term of $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
-\frac{1}{den}\frac{1}{\sigma L_S}\gamma A_2A_8 &= -\frac{1}{den}\frac{1}{\sigma L_S}\gamma\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right)\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) \\
&= \frac{1}{den}\frac{1}{\sigma L_S}\gamma\omega_S\left|\underline{I}_S\right|^2\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) \\
&= \frac{\frac{1}{\sigma L_S}\gamma\omega_S\left|\underline{U}_S\right|^4\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}
$$

The $8th$ term of $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-\frac{1}{den}\frac{1}{\sigma L_S}A_7A_5 &= -\frac{1}{den}\frac{1}{\sigma L_S}\left(i_{Sa}\frac{d^2 i_{Sa}}{dt} + i_{Sb}\frac{d^2 i_{Sb}}{dt}\right)\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right) \\
&= \frac{1}{den}\frac{1}{\sigma L_S}\omega_S^2 \left|\underline{I}_S\right|^2 \left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right) \\
&= \frac{\frac{1}{\sigma L_S}\omega_S^2 \left|\underline{U}_S\right|^4 \frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The $10th$ term of $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}\gamma\frac{1}{\sigma L_S}A_6A_4 &= \frac{1}{den}\gamma\frac{1}{\sigma L_S}\left(i_{Sa}\frac{du_{Sb}}{dt} - i_{Sb}\frac{du_{Sa}}{dt}\right)\left(i_{Sa}^2 + i_{Sb}^2\right) \\
&= \frac{1}{den}\gamma\frac{1}{\sigma L_S}\left|\underline{I}_S\right|^2 \omega_S \left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) \\
&= \frac{\frac{1}{\sigma L_S}\gamma\omega_S \left|\underline{U}_S\right|^4 \left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The $11th$ term of $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-\frac{1}{den}\frac{1}{\sigma^2 L_S^2}A_6A_8 &= -\frac{1}{den}\frac{1}{\sigma^2 L_S^2}\left(i_{Sa}\frac{du_{Sb}}{dt} - i_{Sb}\frac{du_{Sa}}{dt}\right)\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) \\
&= -\frac{1}{den}\frac{1}{\sigma^2 L_S^2}\omega_S \left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right)^2 \\
&= \frac{-\frac{1}{\sigma^2 L_S^2}\omega_S \left|\underline{U}_S\right|^4 \left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The twelfth term of $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}\frac{1}{\sigma^2 L_S^2} A_9 A_5 &= \frac{1}{den}\frac{1}{\sigma^2 L_S^2}\left(i_{Sa}\frac{du_{Sa}}{dt} + i_{Sb}\frac{du_{Sb}}{dt}\right)\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right) \\
&= -\frac{1}{den}\frac{1}{\sigma^2 L_S^2}\omega_S\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right)^2 \\
&= \frac{-\frac{1}{\sigma^2 L_S^2}\omega_S\left|\underline{U}_S\right|^4\left(\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}\right)^2}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The thirteenth term of $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\frac{1}{den}\frac{1}{\sigma L_S} A_9 A_2 &= \frac{1}{den}\frac{1}{\sigma L_S}\left(i_{Sa}\frac{du_{Sa}}{dt} + i_{Sb}\frac{du_{Sb}}{dt}\right)\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right) \\
&= \frac{1}{den}\frac{1}{\sigma L_S}\omega_S^2\left|\underline{I}_S\right|^2\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right) \\
&= \frac{\frac{1}{\sigma L_S}\omega_S^2\left|\underline{U}_S\right|^4\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}{den\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

Finally, substituting these steady-state expressions into the expression for $a_0\left(u_{Sa}, u_{Sb},\right.$ $\left.i_{Sa}, i_{Sb}\right)$, one obtains

$$\begin{aligned}
a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = &\frac{-\left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \\
&\times \frac{\omega_S^3\left(1-\sigma\right)^2\left(1-S\right)^2}{\sigma^2\left(1+S^2\omega_S^2 T_R^2\right)} \times \frac{1}{den}.
\end{aligned} \tag{3.62}$$

3.7.3 The dynamic expressions for $q_2\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$, $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ and $q_0\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$

The dynamic expressions for q_2, q_1 and q_0 are give here.

$$\begin{aligned} q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \;\; &\triangleq \;\; n_p^2 \times \left(\frac{1}{4}\sigma L_S T_R^2 A_3^2 - T_R^2 A_3 A_8 + \frac{T_R^2}{\sigma L_S} A_4 A_{10}\right. \\ &+ \left(-\frac{\beta M}{T_R} + 2\gamma\right)\frac{1}{4}\sigma L_S T_R^2 \frac{dA_4^2}{dt} + \sigma L_S T_R^2 A_2^2 + 2T_R^2 A_2 A_5 \\ &+ \left.\left(-\frac{\beta M}{T_R} + \gamma\right)\sigma L_S \gamma T_R^2 A_4^2 + \left(\frac{\beta M}{T_R} - 2\gamma\right) T_R^2 A_4 A_8\right) \end{aligned} \tag{3.63}$$

$$\begin{aligned} q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \triangleq & n_p \times \left(-T_R^2 \frac{1}{2}\gamma A_3 A_5 - T_R^2 \frac{1}{\sigma L_S} A_6 A_8 + \frac{1}{2} T_R^2 A_3 A_6\right. \\ &+ \gamma T_R^2 A_4 A_6 + \frac{1}{2}\sigma L_S T_R^2 \frac{dA_2}{dt} A_3 + \sigma L_S \gamma T_R^2 \frac{dA_2}{dt} A_4 \\ &+ T_R \sigma L_S \left(2\frac{\beta M}{T_R} + T_R \gamma^2\right) A_2 A_4 - T_R^2 \gamma A_2 A_8 \\ &- \sigma L_S T_R^2 A_2 A_7 - T_R^2 A_5 A_7 + 2\beta M A_4 A_5 \\ &\left. - T_R^2 \frac{dA_2}{dt} A_8 + T_R^2 A_2 A_9 + \frac{T_R^2}{\sigma L_S} A_5 A_9\right) \end{aligned} \tag{3.64}$$

$$
\begin{aligned}
q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \triangleq & -\frac{1}{2}\sigma L_S T_R A_3 A_7 - \sigma L_S T_R \gamma A_4 A_7 + T_R A_7 A_8 - \frac{1}{\sigma L_S} A_4 A_{10} \\
& + \frac{1}{2} T_R A_3 A_9 + T_R \gamma A_4 A_9 - \frac{1}{\sigma L_S} T_R A_8 A_9 + \left(-\frac{\beta M}{T_R} + 2\gamma\right) A_4 A_8 \\
& - (T_R \gamma + 1) \frac{1}{4} \sigma L_S A_3^2 + \left(\frac{1}{2} T_R \gamma + 1\right) A_3 A_8 \\
& + \left(\frac{\beta M}{T_R} - 2\gamma - T_R \gamma^2\right) \frac{1}{4} \sigma L_S \frac{d A_4^2}{dt} + \left(\frac{\beta M}{T_R} - \gamma\right) \sigma L_S \gamma A_4^2 \\
& - (T_R \gamma + 1) \sigma L_S A_2^2 - (\gamma T_R + 2) A_2 A_5 - \sigma L_S T_R A_1 A_2 \\
& - T_R A_1 A_5 - T_R A_2 A_6 - \frac{T_R}{\sigma L_S} A_5 A_6. \qquad (3.65)
\end{aligned}
$$

where A_1 to A_{10} are expressed in (3.53) .

3.7.4 The steady-state expressions for $q_2\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$, $q_1($ $u_{Sa}, u_{Sb},\ i_{Sa}, i_{Sb})$ and $q_0\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$

The steady-state expressions for $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$, $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$, and $q_0(u_{Sa}, u_{Sb}, i_{Sa},\ i_{Sb})$ are now derived. These expressions are then used to show that $q_2 > 0$ for $\omega \neq 0$, $q_2 \equiv 0$ for $\omega = 0$, and $q_1 \neq 0$ if $q_2 \equiv 0$.

Steady-state expression for $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$

In steady state, the first, second and fourth terms of $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ are all zeros, i.e.,

$$
\frac{1}{4} n_p^2 \sigma L_S T_R^2 A_3^2 = \frac{1}{4} n_p^2 \sigma L_S T_R^2 \left(\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt}\right)^2 = 0
$$

$$
-n_p^2 T_R^2 A_3 A_8 = -n_p^2 T_R^2 \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} \left(u_{Sa} i_{Sa} + u_{Sb} i_{Sb}\right) = 0
$$

$$n_p^2\left(-\frac{\beta M}{T_R}+2\gamma\right)\frac{1}{4}\sigma L_S T_R^2\frac{dA_4^2}{dt}$$
$$=n_p^2\left(-\frac{\beta M}{T_R}+2\gamma\right)\frac{1}{4}\sigma L_S T_R^2\frac{d\left(i_{Sa}^2+i_{Sb}^2\right)^2}{dt}=0.$$

The third term of $q_2(u_{Sa},u_{Sb},i_{Sa},i_{Sb})$ is

$$\begin{aligned}
\frac{n_p^2T_R^2}{\sigma L_S}A_4A_{10} &= \frac{n_p^2T_R^2}{\sigma L_S}\left(i_{Sa}^2+i_{Sb}^2\right)\left(u_{Sa}^2+u_{Sb}^2\right)\\
&= \frac{n_p^2T_R^2}{\sigma L_S}\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2\\
&= \frac{n_p^2T_R^2}{\sigma L_S}\frac{\left|\underline{U}_S\right|^4}{\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}}.
\end{aligned}$$

The fifth term of $q_2(u_{Sa},u_{Sb},i_{Sa},i_{Sb})$ is

$$\begin{aligned}
n_p^2\sigma L_ST_R^2A_2^2 &= n_p^2\sigma L_ST_R^2\left(i_{Sb}\frac{di_{Sa}}{dt}-i_{Sa}\frac{di_{Sb}}{dt}\right)^2\\
&= n_p^2\sigma L_ST_R^2\left(-\omega_S\left|\underline{I}_S\right|^2\right)^2\\
&= \frac{n_p^2\sigma L_ST_R^2\omega_S^2\left|\underline{U}_S\right|^4}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The sixth term of $q_2(u_{Sa},u_{Sb},i_{Sa},i_{Sb})$ is

$$\begin{aligned}
2n_p^2T_R^2A_2A_5 &= 2n_p^2T_R^2\left(i_{Sb}\frac{di_{Sa}}{dt}-i_{Sa}\frac{di_{Sb}}{dt}\right)\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right)\\
&= \frac{-2n_p^2T_R^2\omega_S\left|\underline{U}_S\right|^4\frac{\omega_SL_S\left(1+\sigma S^2\omega_S^2T_R^2\right)}{1+S^2\omega_S^2T_R^2}}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The seventh term of $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$n_p^2 \left(-\frac{\beta M}{T_R} + \gamma\right) \sigma L_S \gamma T_R^2 A_4^2 = n_p^2 \left(-\frac{\beta M}{T_R} + \gamma\right) \sigma L_S \gamma T_R^2 \left(i_{Sa}^2 + i_{Sb}^2\right)^2$$
$$= \frac{n_p^2 \left(\frac{R_s^2}{\sigma L_s} + \frac{(1-\sigma)R_s}{\sigma T_R}\right) T_R^2 \left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.$$

The eighth term of $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$n_p^2 \left(\frac{\beta M}{T_R} - 2\gamma\right) T_R^2 A_4 A_8 = n_p^2 \left(\frac{\beta M}{T_R} - 2\gamma\right) T_R^2 \left(i_{Sa}^2 + i_{Sb}^2\right) \left(u_{Sa} i_{Sa} + u_{Sb} i_{Sb}\right)$$
$$= \frac{-n_p^2 \left(\frac{2R_s}{\sigma L_s} + \frac{1-\sigma}{\sigma T_R}\right) T_R^2 \left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}$$
$$\times \left(R_S + \frac{(1-\sigma) S\omega_S^2 L_S T_R}{1 + S^2 \omega_S^2 T_R^2}\right).$$

Finally, substituting these steady-state expressions into the expression for $q_2\left(u_{Sa}, u_{Sb},\right.$ $\left.i_{Sa}, i_{Sb}\right)$, one obtains

$$q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = \frac{n_p^2 T_R^2 \left|\underline{U}_S\right|^4}{\sigma \left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}$$
$$\times \frac{\omega_S^2 L_S \left(1-\sigma\right)^2 \left(1-S\right)}{1 + S^2 \omega_S^2 T_R^2}. \tag{3.66}$$

With $\omega \neq 0$, it is seen that $q_2 > 0$, and $q_2 = 0$ if and only if $S = 1$ (which is equivalent to $\omega = 0$).

Steady-state expression for $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$

Similar to the above analysis, in steady state, the first, third, fifth, sixth, and twelfth terms of $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ are all zeros, i.e.,

$$-\frac{1}{2}\gamma n_p T_R^2 A_3 A_5 = -\frac{1}{2}\gamma n_p T_R^2 \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt}\left(u_{Sb} i_{Sa} - u_{Sa} i_{Sb}\right) = 0$$

$$\frac{1}{2} n_p T_R^2 A_3 A_6 = \frac{1}{2} n_p T_R^2 \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt}\left(i_{Sa}\frac{du_{Sb}}{dt} - i_{Sb}\frac{du_{Sa}}{dt}\right) = 0$$

$$\frac{1}{2} n_p \sigma L_S T_R^2 \frac{dA_2}{dt} A_3 = \frac{1}{2} n_p \sigma L_S T_R^2 \left(i_{Sb}\frac{d^2 i_{Sa}}{dt} - i_{Sa}\frac{d^2 i_{Sb}}{dt}\right)\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} = 0$$

$$\begin{aligned} n_p \sigma L_S \gamma T_R^2 \frac{dA_2}{dt} A_4 &= n_p \sigma L_S \gamma T_R^2 \left(i_{Sb}\frac{d^2 i_{Sa}}{dt} - i_{Sa}\frac{d^2 i_{Sb}}{dt}\right)\left(i_{Sa}^2 + i_{Sb}^2\right) \\ &= n_p \sigma L_S \gamma T_R^2 \frac{d\left(-\omega_S \left|\underline{I}_S\right|^2\right)}{dt}\left(i_{Sa}^2 + i_{Sb}^2\right) = 0 \end{aligned}$$

$$\begin{aligned} -n_p T_R^2 \frac{dA_2}{dt} A_8 &= -n_p T_R^2 \left(i_{Sb}\frac{d^2 i_{Sa}}{dt} - i_{Sa}\frac{d^2 i_{Sb}}{dt}\right)\left(u_{Sa} i_{Sa} + u_{Sb} i_{Sb}\right) \\ &= -n_p T_R^2 \frac{d\left(-\omega_S \left|\underline{I}_S\right|^2\right)}{dt}\left(u_{Sa} i_{Sa} + u_{Sb} i_{Sb}\right) \\ &= 0. \end{aligned}$$

The second term of $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
-\frac{n_p T_R^2}{\sigma L_S} A_6 A_8 &= -\frac{n_p T_R^2}{\sigma L_S} \left(i_{Sa} \frac{du_{Sb}}{dt} - i_{Sb} \frac{du_{Sa}}{dt} \right) (u_{Sa} i_{Sa} + u_{Sb} i_{Sb}) \\
&= -\frac{n_p \omega_S T_R^2}{\sigma L_S} (u_{Sa} i_{Sa} + u_{Sb} i_{Sb})^2 \\
&= \frac{-n_p \omega_S T_R^2 \left| \underline{U}_S \right|^4 \left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2 \omega_S^2 T_R^2} \right)^2}{\sigma L_S \left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2 \omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2 \omega_S^2 T_R^2\right)^2} \right)^2}.
\end{aligned}
$$

The fourth term of $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
n_p \gamma T_R^2 A_4 A_6 &= n_p \gamma T_R^2 \left(i_{Sa}^2 + i_{Sb}^2 \right) \left(i_{Sa} \frac{du_{Sb}}{dt} - i_{Sb} \frac{du_{Sa}}{dt} \right) \\
&= n_p \gamma T_R^2 \omega_S \left| \underline{I}_S \right|^2 (u_{Sa} i_{Sa} + u_{Sb} i_{Sb}) \\
&= \frac{n_p \gamma T_R^2 \omega_S \left| \underline{U}_S \right|^4 \left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2 \omega_S^2 T_R^2} \right)}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2 \omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2 \omega_S^2 T_R^2\right)^2} \right)^2}.
\end{aligned}
$$

The seventh term of $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
& n_p T_R \sigma L_S \left(2 \frac{\beta M}{T_R} + T_R \gamma^2 \right) A_2 A_4 \\
&= n_p T_R \sigma L_S \left(2 \frac{\beta M}{T_R} + T_R \gamma^2 \right) \left(i_{Sb} \frac{di_{Sa}}{dt} - i_{Sa} \frac{di_{Sb}}{dt} \right) \left(i_{Sa}^2 + i_{Sb}^2 \right) \\
&= \frac{-n_p \omega_S T_R \sigma L_S \left(2 \frac{\beta M}{T_R} + T_R \gamma^2 \right) \left| \underline{U}_S \right|^4}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2 \omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2 \omega_S^2 T_R^2\right)^2} \right)^2}.
\end{aligned}
$$

The eighth term of $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
-n_p T_R^2 \gamma A_2 A_8 &= -n_p T_R^2 \gamma \left(i_{Sb} \frac{di_{Sa}}{dt} - i_{Sa} \frac{di_{Sb}}{dt} \right) (u_{Sa} i_{Sa} + u_{Sb} i_{Sb}) \\
&= n_p T_R^2 \gamma \omega_S \left| \underline{I}_S \right|^2 (u_{Sa} i_{Sa} + u_{Sb} i_{Sb}) \\
&= \frac{n_p T_R^2 \gamma \omega_S \left| \underline{U}_S \right|^4 \left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2 \omega_S^2 T_R^2} \right)}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2 \omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2 \omega_S^2 T_R^2\right)^2} \right)^2}.
\end{aligned}
$$

The ninth term of $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
-n_p \sigma L_S T_R^2 A_2 A_7 &= -n_p \sigma L_S T_R^2 \left(i_{Sb} \frac{di_{Sa}}{dt} - i_{Sa} \frac{di_{Sb}}{dt} \right) \left(i_{Sa} \frac{d^2 i_{Sa}}{dt} + i_{Sb} \frac{d^2 i_{Sb}}{dt} \right) \\
&= -n_p \sigma L_S T_R^2 \omega_S \left| \underline{I}_S \right|^2 \omega_S^2 \left| \underline{I}_S \right|^2 \\
&= \frac{-n_p \omega_S^3 \sigma L_S T_R^2 \left| \underline{U}_S \right|^4}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2 \omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2 \omega_S^2 T_R^2\right)^2} \right)^2}.
\end{aligned}
$$

The tenth term of $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
-n_p T_R^2 A_5 A_7 &= -n_p T_R^2 (u_{Sb} i_{Sa} - u_{Sa} i_{Sb}) \left(i_{Sa} \frac{d^2 i_{Sa}}{dt} + i_{Sb} \frac{d^2 i_{Sb}}{dt} \right) \\
&= \frac{n_p T_R^2 \omega_S^2 \left| \underline{U}_S \right|^4 \frac{\omega_S L_S \left(1+\sigma S^2 \omega_S^2 T_R^2\right)}{1+S^2 \omega_S^2 T_R^2}}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2 \omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2 \omega_S^2 T_R^2\right)^2} \right)^2}.
\end{aligned}
$$

The eleventh term of $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
2n_p\beta M A_4 A_5 &= 2n_p\beta M\left(i_{Sa}^2+i_{Sb}^2\right)\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right)\\
&= \frac{2n_p\frac{1-\sigma}{\sigma}\left|\underline{U}_S\right|^4\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2T_R^2\right)}{1+S^2\omega_S^2T_R^2}}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}
$$

The thirteenth term of $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
n_pT_R^2A_2A_9 &= n_pT_R^2\left(i_{Sb}\frac{di_{Sa}}{dt}-i_{Sa}\frac{di_{Sb}}{dt}\right)\left(i_{Sa}\frac{du_{Sa}}{dt}+i_{Sb}\frac{du_{Sb}}{dt}\right)\\
&= n_pT_R^2\left(-\omega_S\left|\underline{I}_S\right|^2\right)\left(-\omega_S\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right)\right)\\
&= \frac{n_pT_R^2\omega_S^2\left|\underline{U}_S\right|^4\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2T_R^2\right)}{1+S^2\omega_S^2T_R^2}}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}
$$

The fourteenth term of $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$
\begin{aligned}
\frac{n_pT_R^2}{\sigma L_S}A_5A_9 &= \frac{n_pT_R^2}{\sigma L_S}\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right)\left(i_{Sa}\frac{du_{Sa}}{dt}+i_{Sb}\frac{du_{Sb}}{dt}\right)\\
&= \frac{-n_p\omega_ST_R^2\left|\underline{U}_S\right|^4\left(\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2T_R^2\right)}{1+S^2\omega_S^2T_R^2}\right)^2}{\sigma L_S\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}
$$

Finally, substituting these steady-state expressions into the expression for $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$, one obtains

$$q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = \frac{n_p \omega_S \left|\underline{U}_S\right|^4}{\sigma \left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2} \right)^2} \times \frac{L_S (1-\sigma)^2 \left(1-\omega_S^2 T_R^2 (1-S)^2\right)}{1+S^2\omega_S^2 T_R^2}. \tag{3.67}$$

If $\omega = 0$, then $S = 1$ and $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \neq 0$.

Steady-state expression for $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$

Similarly, in steady state, the first, fifth, ninth, tenth, eleventh, fifteenth and sixteenth terms of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ are all zero, i.e.,

$$-\frac{1}{2}\sigma L_S T_R A_3 A_7 = -\frac{1}{2}\sigma L_S T_R \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} \left(i_{Sa}\frac{d^2 i_{Sa}}{dt} + i_{Sb}\frac{d^2 i_{Sb}}{dt} \right) = 0$$

$$\frac{1}{2}T_R A_3 A_9 = \frac{1}{2}T_R \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} \left(i_{Sa}\frac{du_{Sa}}{dt} + i_{Sb}\frac{du_{Sb}}{dt} \right) = 0$$

$$-\left(T_R\gamma + 1\right)\frac{1}{4}\sigma L_S A_3^2 = -\left(T_R\gamma + 1\right)\frac{1}{4}\sigma L_S \left(\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} \right)^2 = 0$$

$$\left(\frac{1}{2}T_R\gamma + 1\right) A_3 A_8 = \left(\frac{1}{2}T_R\gamma + 1\right) \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} \left(u_{Sa} i_{Sa} + u_{Sb} i_{Sb}\right) = 0$$

$$\left(\frac{\beta M}{T_R} - 2\gamma - T_R\gamma^2\right)\frac{1}{4}\sigma L_S \frac{dA_4^2}{dt} = \left(\frac{\beta M}{T_R} - 2\gamma - T_R\gamma^2\right)\frac{1}{4}\sigma L_S \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)^2}{dt} = 0$$

$$-\frac{1}{2}\sigma L_S T_R \frac{dA_2^2}{dt} = -\frac{1}{2}\sigma L_S T_R \frac{d\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right)^2}{dt} = 0$$

$$-T_R A_1 A_5 = -T_R \left(i_{Sb}\frac{d^2 i_{Sa}}{dt} - i_{Sa}\frac{d^2 i_{Sb}}{dt} \right) \left(u_{Sb} i_{Sa} - u_{Sa} i_{Sb}\right) = 0.$$

The second term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-\sigma L_S T_R \gamma A_4 A_7 &= -\sigma L_S T_R \gamma \left(i_{Sa}^2 + i_{Sb}^2\right) \left(i_{Sa} \frac{d^2 i_{Sa}}{dt} + i_{Sb} \frac{d^2 i_{Sb}}{dt}\right) \\
&= -\sigma L_S T_R \gamma \left|\underline{I}_S\right|^2 \left(-\omega_S^2 \left|\underline{I}_S\right|^2\right) \\
&= \frac{\sigma L_S T_R \gamma \omega_S^2 \left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The third term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
T_R A_7 A_8 &= T_R \left(i_{Sa} \frac{d^2 i_{Sa}}{dt} + i_{Sb} \frac{d^2 i_{Sb}}{dt}\right) \left(u_{Sa} i_{Sa} + u_{Sb} i_{Sb}\right) \\
&= -T_R \omega_S^2 \left|\underline{I}_S\right|^2 \left(u_{Sa} i_{Sa} + u_{Sb} i_{Sb}\right) \\
&= \frac{-T_R \omega_S^2 \left|\underline{U}_S\right|^4 \left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The fourth term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-\frac{1}{\sigma L_S} A_4 A_{10} &= -\frac{1}{\sigma L_S} \left(i_{Sa}^2 + i_{Sb}^2\right) \left(u_{Sa}^2 + u_{Sb}^2\right) \\
&= -\frac{1}{\sigma L_S} \left|\underline{I}_S\right|^2 \left|\underline{U}_S\right|^2 \\
&= -\frac{1}{\sigma L_S} \frac{\left|\underline{U}_S\right|^4}{\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}}.
\end{aligned}$$

The sixth term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
T_R\gamma A_4 A_9 &= T_R\gamma\left(i_{Sa}^2+i_{Sb}^2\right)\left(i_{Sa}\frac{du_{Sa}}{dt}+i_{Sb}\frac{du_{Sb}}{dt}\right)\\
&= -T_R\gamma\left|\underline{I}_S\right|^2\omega_S\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right)\\
&= \frac{-T_R\gamma\omega_S\left|\underline{U}_S\right|^4\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2T_R^2\right)}{1+S^2\omega_S^2T_R^2}}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The seventh term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-\frac{1}{\sigma L_S}T_R A_8 A_9 &= -\frac{1}{\sigma L_S}T_R\left(u_{Sa}i_{Sa}+u_{Sb}i_{Sb}\right)\left(i_{Sa}\frac{du_{Sa}}{dt}+i_{Sb}\frac{du_{Sb}}{dt}\right)\\
&= \frac{T_R\omega_S}{\sigma L_S}\left(u_{Sa}i_{Sa}+u_{Sb}i_{Sb}\right)\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right)\\
&= \frac{T_R\omega_S}{\sigma L_S}\frac{\left|\underline{U}_S\right|^4\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2T_R^2\right)}{1+S^2\omega_S^2T_R^2}}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The eighth term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\left(-\frac{\beta M}{T_R}+2\gamma\right)A_4A_8 &= \left(-\frac{\beta M}{T_R}+2\gamma\right)\left(i_{Sa}^2+i_{Sb}^2\right)\left(u_{Sa}i_{Sa}+u_{Sb}i_{Sb}\right)\\
&= \left(-\frac{\beta M}{T_R}+2\gamma\right)\left|\underline{I}_S\right|^2\left(u_{Sa}i_{Sa}+u_{Sb}i_{Sb}\right)\\
&= \frac{\left(-\frac{\beta M}{T_R}+2\gamma\right)\left|\underline{U}_S\right|^4\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The twelfth term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
\left(\frac{\beta M}{T_R} - \gamma\right) \sigma L_S \gamma A_4^2 &= \left(\frac{\beta M}{T_R} - \gamma\right) \sigma L_S \gamma \left(i_{Sa}^2 + i_{Sb}^2\right)^2 \\
&= \left(\frac{\beta M}{T_R} - \gamma\right) \sigma L_S \gamma \left|\underline{I}_S\right|^2 \left|\underline{I}_S\right|^2 \\
&= \frac{\left(\frac{\beta M}{T_R} - \gamma\right) \sigma L_S \gamma \left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The thirteenth term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-\left(T_R\gamma + 1\right) \sigma L_S A_2^2 &= -\left(T_R\gamma + 1\right) \sigma L_S \left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right)^2 \\
&= -\left(T_R\gamma + 1\right) \sigma L_S \omega_S^2 \left|\underline{I}_S\right|^4 \\
&= \frac{-\left(T_R\gamma + 1\right) \sigma L_S \omega_S^2 \left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The fourteenth term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-\left(\gamma T_R + 2\right) A_2 A_5 &= -\left(\gamma T_R + 2\right) \left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right) \left(u_{Sb} i_{Sa} - u_{Sa} i_{Sb}\right) \\
&= \left(\gamma T_R + 2\right) \omega_S \left|\underline{I}_S\right|^2 \left(u_{Sb} i_{Sa} - u_{Sa} i_{Sb}\right) \\
&= \frac{\left(\gamma T_R + 2\right) \omega_S \left|\underline{U}_S\right|^4 \frac{\omega_S L_S \left(1+\sigma S^2 \omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.
\end{aligned}$$

The seventeenth term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-T_R A_2 A_6 &= -T_R \left(i_{Sb} \frac{di_{Sa}}{dt} - i_{Sa} \frac{di_{Sb}}{dt} \right) \left(i_{Sa} \frac{du_{Sb}}{dt} - i_{Sb} \frac{du_{Sa}}{dt} \right) \\
&= T_R \omega_S^2 \left| \underline{I}_S \right|^2 \left(u_{Sa} i_{Sa} + u_{Sb} i_{Sb} \right) \\
&= \frac{T_R \omega_S^2 \left| \underline{U}_S \right|^4 \left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2} \right)}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2 \right)^2}{\left(1+S^2\omega_S^2 T_R^2 \right)^2} \right)^2}.
\end{aligned}$$

The eighteenth term of $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ is

$$\begin{aligned}
-\frac{T_R}{\sigma L_S} A_5 A_6 &= -\frac{T_R}{\sigma L_S} \left(u_{Sb} i_{Sa} - u_{Sa} i_{Sb} \right) \left(i_{Sa} \frac{du_{Sb}}{dt} - i_{Sb} \frac{du_{Sa}}{dt} \right) \\
&= -\frac{T_R \omega_S}{\sigma L_S} \left(u_{Sb} i_{Sa} - u_{Sa} i_{Sb} \right) \left(u_{Sa} i_{Sa} + u_{Sb} i_{Sb} \right) \\
&= -\frac{T_R \omega_S}{\sigma L_S} \frac{\left| \underline{U}_S \right|^4 \left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2} \right) \frac{\omega_S L_S \left(1+\sigma S^2 \omega_S^2 T_R^2 \right)}{1+S^2\omega_S^2 T_R^2}}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2 \right)^2}{\left(1+S^2\omega_S^2 T_R^2 \right)^2} \right)^2}.
\end{aligned}$$

Finally, substituting these steady-state expressions into the expression for $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$, one obtains

$$\begin{aligned}
q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \quad = \quad & \frac{-\left| \underline{U}_S \right|^4}{\sigma \left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2 \right)^2}{\left(1+S^2\omega_S^2 T_R^2 \right)^2} \right)^2} \\
& \times \frac{\omega_S^2 L_S \left(1-\sigma \right)^2 \left(1-S \right)}{1+S^2\omega_S^2 T_R^2} \qquad (3.68)
\end{aligned}$$

3.7.5 Steady-state value for α

The purpose of this section is to show that steady-state value for α is always positive.

To compute the steady-state value of α, note that by (3.47)

$$\alpha = a_1 - a_2 q_1 / q_2.$$

It is then easily seen that $a_1 > 0$, $a_2 q_1 < 0$, and $q_2 > 0$, so that in steady state $\alpha > 0$. That is, the system (3.49) is never stable in steady state.

3.7.6 If u_{Sa} = constant and $u_{Sb} = 0$, ω is not determinable from $q_2\omega^2 + q_1\omega + q_0 = 0$

The purpose of this section is to show if u_{Sa} = constant and $u_{Sb} = 0$, it turns out that $q_2 = q_1 = q_0 \equiv 0$ and ω is not determinable from (3.38) . In this case,

$$\begin{aligned}
A_1 &= i_{Sb}\frac{d^2 i_{Sa}}{dt} - i_{Sa}\frac{d^2 i_{Sb}}{dt} = 0, \qquad A_2 = i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt} = 0, \\
A_3 &= \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt} = 0, \qquad A_4 = \left(i_{Sa}^2 + i_{Sb}^2\right) = i_{Sa}^2, \\
A_5 &= u_{Sb} i_{Sa} - u_{Sa} i_{Sb} = 0, \qquad A_6 = i_{Sa}\frac{du_{Sb}}{dt} - i_{Sb}\frac{du_{Sa}}{dt} = 0, \\
A_7 &= i_{Sa}\frac{d^2 i_{Sa}}{dt} + i_{Sb}\frac{d^2 i_{Sb}}{dt} = 0, \qquad A_8 = u_{Sa} i_{Sa} + u_{Sb} i_{Sb} = u_{Sa} i_{Sa}, \\
A_9 &= i_{Sa}\frac{du_{Sa}}{dt} + i_{Sb}\frac{du_{Sb}}{dt} = 0, \qquad A_{10} = u_{Sa}^2 + u_{Sb}^2 = u_{Sa}^2.
\end{aligned}$$

$$
\begin{aligned}
q_2 &= n_p^2 \left(\left(-\frac{\beta M}{T_R} + \gamma \right) \sigma L_S \gamma T_R^2 A_4^2 + \frac{T_R^2}{\sigma L_S} A_4 A_{10} + \left(\frac{\beta M}{T_R} - 2\gamma \right) T_R^2 A_4 A_8 \right) \\
&= n_p^2 T_R^2 i_{Sa}^4 \left(\left(-\frac{\beta M}{T_R} + \gamma \right) \sigma L_S \gamma T_R^2 + \frac{T_R^2}{\sigma L_S} R_S^2 + \left(\frac{\beta M}{T_R} - 2\gamma \right) T_R^2 R_S \right) \\
&\equiv 0
\end{aligned}
$$

$$
q_1 = n_p \times 0 \equiv 0
$$

$$
\begin{aligned}
q_0 &= \left(-\frac{\beta M}{T_R} + 2\gamma \right) A_4 A_8 - \frac{1}{\sigma L_S} A_4 A_{10} + \left(\frac{\beta M}{T_R} - \gamma \right) \sigma L_S \gamma A_4^2 \\
&= \left(-\frac{\beta M}{T_R} + 2\gamma \right) R_S i_{Sa}^4 - \frac{R_S^2}{\sigma L_S} i_{Sa}^4 + \left(\frac{\beta M}{T_R} - \gamma \right) \sigma L_S \gamma i_{Sa}^4 \\
&= \left(\frac{R_S}{\sigma L_S} + \gamma \right) R_S i_{Sa}^4 - \frac{R_S^2}{\sigma L_S} i_{Sa}^4 - \frac{R_S}{\sigma L_S} \sigma L_S \gamma i_{Sa}^4 \\
&\equiv 0.
\end{aligned}
\tag{3.69}
$$

Since $q_2 = q_1 = q_0 \equiv 0$ in this case, ω is not determinable from (3.38).

3.7.7 $(T_R n_p \omega)^2 \ll 1 \Longrightarrow |q_2 \omega| \ll |q_1|$

The purpose of this section is to show that in steady state, $|q_2\omega| \ll |q_1|$ if $(T_R n_p \omega)^2 \ll 1$. That is $|q_2\omega| \ll |q_1|$ at low speed.

In steady state,

$$
\begin{aligned}
|q_2\omega| =& \frac{n_p |\omega_S| L_S (1-\sigma)^2 |\underline{U}_S|^4}{\left(\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2} \right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2} \right)^2} \\
&\times \frac{(T_R n_p \omega)^2}{\sigma \left(1 + S^2 \omega_S^2 T_R^2\right)}
\end{aligned}
$$

and

$$|q_1| = \frac{n_p |\omega_S| L_S (1-\sigma)^2 |\underline{U}_S|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \times \frac{\left|\left(1-(T_R n_p \omega)^2\right)\right|}{\sigma\left(1+S^2\omega_S^2 T_R^2\right)}.$$

Their ratio is then

$$\frac{|q_2 \omega|}{|q_1|} = \left|\frac{(T_R n_p \omega)^2}{1-(T_R n_p \omega)^2}\right|$$

which shows that $(T_R n_p \omega)^2 \ll 1 \Longrightarrow |q_2 \omega| \ll |q_1|$.

3.7.8 The steady-state expressions for $r_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ and $r_0(u_{Sa},\ \ u_{Sb}, i_{Sa}, i_{Sb})$

It is now shown that the steady-state value of r_1 in (3.43) is nonzero.

Substituting the steady-state values of q_2, q_1, q_0, a_2, a_1, and a_0 (noting that $\dot{q}_1 = 0$ and $\dot{q}_2 = 0$ in steady state) into (3.43) gives

$$r_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = \frac{-|\underline{U}_S|^{12}}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^6} \times \left(\frac{1}{1+S^2\omega_S^2 T_R^2}\right)^3 \times \frac{n_p^4 (1-\sigma)^6 \omega_S^3 L_S^2}{\sigma^4} \times \left(1+T_R^2\omega_S^2 (1-S)^2\right)^2 \times \frac{1}{den}$$

where den is given by (3.59) in section 3.7.2. It is then seen that $r_1 \neq 0$ in steady state.

Substituting the steady-state values of q_2, q_1, q_0, a_2, a_1, and a_0 (noting that $\dot{q}_0 = 0$ and $\dot{q}_2 = 0$ in steady state) into (3.44) gives

$$
\begin{aligned}
r_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \quad = \quad & \frac{|\underline{U}_S|^{12}}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^6} \\
& \times \left(\frac{1}{1+S^2\omega_S^2 T_R^2}\right)^3 \frac{n_p^3 (1-\sigma)^6 \omega_S^4 L_S^2 (1-S)}{\sigma^4} \\
& \times \left(1 + \omega_S^2 T_R^2 \times (1-S)^2\right)^2 \frac{1}{den}
\end{aligned}
$$

where den is given by (3.59) in section 3.7.2.

In steady state, according to (3.45) the motor speed ω can be found by

$$\omega = -\frac{r_0}{r_1} = \frac{\omega_S (1-S)}{n_p}.$$

3.7.9 Steady-state speed

Substituting the steady-state values of a_2, a_1, and a_0, it is seen that $a_1^2 - 4a_2a_0 = 0$, so that the steady-state value of the right-hand side of (3.36) may be rewritten as

$$a_2\omega^2 + a_1\omega + a_0 = a_2\left(\omega + a_1/(2a_2)\right)^2 = 0$$

where a_2 is nonzero by (3.54). In steady state the motor speed can be solved by

$$\omega = -a_1/(2a_2).$$

On the other hand, the steady-state solutions of (3.38) are

$$\omega_1 \triangleq \frac{-q_1 + \sqrt{q_1^2 - 4q_2q_0}}{2q_2} = \omega \tag{3.70}$$

$$\omega_2 \triangleq \frac{-q_1 - \sqrt{q_1^2 - 4q_2q_0}}{2q_2} = -1/\left(T_R^2 n_p^2 \omega\right). \tag{3.71}$$

That means in steady state motor speed can be uniquely determined and equal to ω_1.

3.8 Speed estimation of induction motor using extended Kalman filter (EKF)

The Kalman filter algorithm has been used both for the parameter estimation of the induction motor [39] [40] [41] and for speed estimation [42] [43]. Here, the extended Kalman filter approach to speed estimation is presented to compare with the differential-algebraic method.

3.8.1 Extended Kalman filter algorithm

The extended Kalman filter algorithm [38] is calculated using a microprocessor so that a discrete-time model of the motor is needed. Equations (3.72) and (3.73) represent the discrete-state model and output model respectively,

$$x(k+1) = f[x(k), u(k)] + w(k) \tag{3.72}$$

and

$$z(k) = h[x(k)] + v(k) \tag{3.73}$$

where

$$\begin{aligned} w(t) &= \text{noise matrix of state model (system noise)} \\ v(t) &= \text{noise matrix of output model (measurement noise).} \end{aligned}$$

The process noise $w(k)$ is characterized by

$$\begin{aligned} E\{w(k)\} &= 0 \\ E\{w(k)\, w(j)^T\} &= Q\delta_{kj} \quad Q \geqslant 0. \end{aligned}$$

The measurement noise $v(k)$ is characterized by

$$\begin{aligned} E\{v(k)\} &= 0 \\ E\{v(k)\, v(j)^T\} &= R\delta_{kj} \quad R \geqslant 0. \end{aligned}$$

The initial state is characterized by

$$\begin{aligned} E\{x(0)\} &= \hat{x}_0 \\ E\{(x(0) - \hat{x}_0)(x(0) - \hat{x}_0)^T\} &= P_0. \end{aligned}$$

where $E(\cdot)$ denotes the expected value and

$$\delta_{kj} = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases}$$

In this model $f\left[x\left(k\right), u\left(k\right)\right]$ is the state-space model of the motor. The extended Kalman filter relinearizes the nonlinear-state model about each new estimate of the state as it becomes available. From the above dynamic model, the rotor speed can be estimated by the extended Kalman filter algorithm as follows

1. Prediction of State

$$\hat{x}\left(k+1|k\right) = f\left[\hat{x}\left(k/k\right), u\left(k\right)\right] \tag{3.74}$$

2. Estimation of Error Covariance Matrix

$$P\left(k+1|k\right) = \Phi\left(k+1,k\right) P\left(k|k\right) \Phi^{T}\left(k+1,k\right) + Q \tag{3.75}$$

where

$$\Phi\left(k+1,k\right) = e^{F(k)T_S} \tag{3.76}$$

$$T_S = \text{ sampling time}$$

$$F\left[k\right] = \frac{\partial f\left[x\left(k\right), u\left(k\right)\right]}{\partial x}.$$

3. Computation of Kalman Filter Gain

$$K\left(k+1\right) = P\left(k+1|k\right) H^{T}\left(k+1\right)\left[H\left(k+1\right) P\left(k+1|k\right) H^{T}\left(k+1\right) + R\left(k+1\right)\right]^{-1} \tag{3.77}$$

where

$$H\left[k+1\right] = \frac{\partial h\left[x\left(k\right)\right]}{\partial x}.$$

4. Update of the Error Covariance Matrix

$$P\left(k+1|k+1\right) = \left[I - K\left(k+1\right)H\left(k+1\right)\right]P\left(k+1|k\right). \tag{3.78}$$

5. State Estimation

$$\hat{x}\left(k+1|k+1\right) = \hat{x}\left(k+1|k\right) + K\left(k+1\right)\left(z\left(k+1\right) - h\left[\hat{x}\left(k+1|k\right), k+1\right]\right). \tag{3.79}$$

3.8.2 Dynamic model of an induction motor

A (two-phase equivalent) dynamic mathematical model of an induction motor is given by equations (2.7) – (2.11) and repeated here

$$\begin{aligned}
\frac{d\omega}{dt} &= \frac{n_p M}{J L_R}\left(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}\right) - \frac{\tau_L}{J} \\
\frac{di_{Sa}}{dt} &= \frac{\beta}{T_R}\psi_{Ra} + \beta n_p \omega \psi_{Rb} - \gamma i_{Sa} + \frac{1}{\sigma L_S}u_{Sa} \\
\frac{di_{Sb}}{dt} &= \frac{\beta}{T_R}\psi_{Rb} - \beta n_p \omega \psi_{Ra} - \gamma i_{Sb} + \frac{1}{\sigma L_S}u_{Sb} \\
\frac{d\psi_{Ra}}{dt} &= -\frac{1}{T_R}\psi_{Ra} - n_p \omega \psi_{Rb} + \frac{M}{T_R}i_{Sa} \\
\frac{d\psi_{Rb}}{dt} &= -\frac{1}{T_R}\psi_{Rb} + n_p \omega \psi_{Ra} + \frac{M}{T_R}i_{Sb}
\end{aligned}$$

The symbols

$$T_R = \frac{L_R}{R_R},\ \sigma = 1 - \frac{M^2}{L_S L_R},\ \beta = \frac{M}{\sigma L_S L_R},\ \gamma = \frac{R_S}{\sigma L_S} + \frac{1}{\sigma L_S}\frac{1}{T_R}\frac{M^2}{L_R}$$

have been used to simplify the expressions.

3.8.3 Rotor speed estimation by EKF

Here the extended Kalman filter is used to estimate the rotor speed of the induction motor. Let the state variables be defined as follows

$$x(k) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} i_{Sa} \\ i_{Sb} \\ \psi_{Ra} \\ \psi_{Rb} \\ \omega \end{bmatrix}.$$

Then, the mathematical model of induction motor can be described as

$$x(k+1) = f[x(k), u(k)] + w(k).$$

So the extended Kalman filter prediction equation is given by

$$\begin{aligned}
x_1(k+1) &= (1 - T_S\gamma)\, x_1(k) + T_S\frac{\beta}{T_R} x_3(k) + T_S n_p \beta x_4(k)\, x_5(k) + \frac{T_S}{\sigma L_s} u_{Sa}(k) \\
x_2(k+1) &= (1 - T_S\gamma)\, x_2(k) - T_S n_p \beta x_3(k)\, x_5(k) + T_S\frac{\beta}{T_R} x_4(k) + \frac{T_S}{\sigma L_s} u_{Sb}(k) \\
x_3(k+1) &= T_S\frac{M}{T_R} x_1(k) + \left(1 - \frac{T_S}{T_R}\right) x_3(k) - T_S n_p x_4(k)\, x_5(k) \\
x_4(k+1) &= T_S\frac{M}{T_R} x_2(k) + T_S n_p x_3(k)\, x_5(k) + \left(1 - \frac{T_S}{T_R}\right) x_4(k) \\
x_5(k+1) &= x_5(k)
\end{aligned}$$

where T_S is sampling time.

In addition, the output matrix is as follows

$$z(k) = h[x(k)] + v(k)$$

where

$$z(k) = \begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix}$$

$$h[x(k)] = \begin{bmatrix} i_{Sa}(k) \\ i_{Sb}(k) \end{bmatrix}.$$

This model is nonlinear. Therefore, the extended Kalman filter has to be used to estimate the speed. In this case, the angular speed of the rotor is considered as an unknown constant (slowly varying compared to the electrical variables) parameter.

The estimation of the error covariance matrix is given by

$$P(k+1|k) = \Phi(k+1,k)\, P(k|k)\, \Phi^{T}(k+1,k) + Q$$

where

$$\Phi(k+1,k) = e^{F(k)T_S}$$

$$
\begin{aligned}
F[k] &= \frac{\partial f[x(k), u(k)]}{\partial x} \\
&= \begin{bmatrix} -\gamma & 0 & \frac{\beta}{T_R} & \beta n_p x_5(k) & \beta x_4(k) \\ 0 & -\gamma & -\beta n_p x_5(k) & \frac{\beta}{T_R} & -\beta x_3(k) \\ \frac{M}{T_R} & 0 & -\frac{1}{T_R} & -n_p x_5(k) & -x_4(k) \\ 0 & \frac{M}{T_R} & n_p x_5(k) & -\frac{1}{T_R} & x_3(k) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{3.80}
$$

The Kalman gain $K(k+1)$ is given by

$$
K(k+1) = P(k+1|k) H^T(k+1) \left[H(k+1) P(k+1|k) H^T(k+1) + R(k+1)\right]^{-1}
$$

where $H[k+1]$ is:

$$
\begin{aligned}
H[k+1] &= \frac{\partial h[x(k)]}{\partial x} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

The updated covariance is given by

$$
P(k+1|k+1) = [I - K(k+1) H(k+1)] P(k+1|k).
$$

The state estimation is given by

$$
\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K(k+1)(z(k+1) - h[\hat{x}(k+1|k), k+1]).
$$

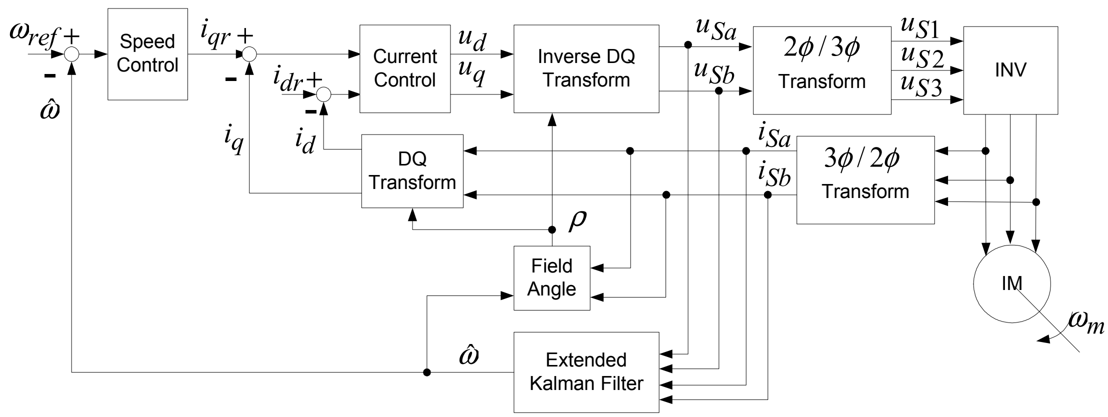

Figure 3.8: Sensorless speed control system using extended Kalman filter.

3.8.4 Simulation results of on-line speed estimation using EKF

To verify the extended Kalman filter speed estimation method, simulations are carried out. Here, a three-phase (two-phase equivalent) induction motor model is simulated using the machine parameter values

$$
\begin{aligned}
n_p &= 2,\ R_S = 5.12 \text{ ohms},\ R_R = 2.23 \text{ ohms},\ L_S = L_R = 0.2919 \text{ H}, \\
M &= 0.2738 \text{ H},\ J = 0.0021 \text{ k-gm}^2,\ \tau_{L\_rated} = 2.0337 \text{ N-m}, \\
I_{\max} &= 2.77 \text{ A},\ V_{\max} = 230 \text{ V}.
\end{aligned}
$$

Figure 3.8 shows a block diagram of sensorless speed control system using an extended Kalman filter. In this system, a current command field-oriented controller is used [13] with the induction motor model being equations (3.30) , (3.31) , and (3.32) .

Estimated speed $\hat{\omega}$ (using extended Kalman filter) is used here and fed back to the current command field-oriented controller. The system noise $w(k)$, white noise with covariance $Q = 0.0001$, is added to the input voltage u_{Sa} and u_{Sb}. The measurement noise $v(t)$, white noise with covariance $R = 0.0001$, is added to the measurement i_{Sa} and i_{Sb}. The initial value P_0 is chosen to be $10I$.

Figure 3.9 shows the simulation results of the motor speed and speed estimator under full load. From $t = 0$ to $t = 0.4$ seconds, a constant u_{Sa} is applied to the motor to build up the flux, and the motor is considered to be held with a brake so that $\omega \equiv 0$. At $t = 0.4$ seconds the brake is released and the machine is running on a low speed trajectory ($\omega_{\max} = 5\ rad/s$) *with full load at the start.* The estimated speed $\hat{\omega}$ is used in the field-oriented controller as shown in Figure 3.8.

Figure 3.10 shows the simulation results of the motor speed and stabilized speed estimator under full load. From $t = 0$ to $t = 0.4$ seconds, a constant u_{Sa} is applied to the motor to build up the flux and the motor is considered to be held with a brake so that $\omega \equiv 0$. At $t = 0.4$ seconds the brake is released and the machine is controlled to a zero speed trajectory ($\omega \equiv 0$) *with full load at the start.* $\hat{\omega}$ is used in the field-oriented controller as shown in Figure 3.8.

Figure 3.11 shows the whole trajectory chosen to have a maximum speed of 5 rad/sec, to do a speed reversal, and to have zero speed at the end.

To consider the effect of PWM noise in the simulation, a 4 kHz PWM inverter is included. Low-pass analog filters (third-order, 500 Hz cutoff) are used (simulated) before the voltages and currents are sampled. The interest here is in low-speed sensorless control of the machine with full load. The trajectory is chosen to have a maximum speed of 5 rad/sec, to do a speed reversal, and to have zero speed at the end as

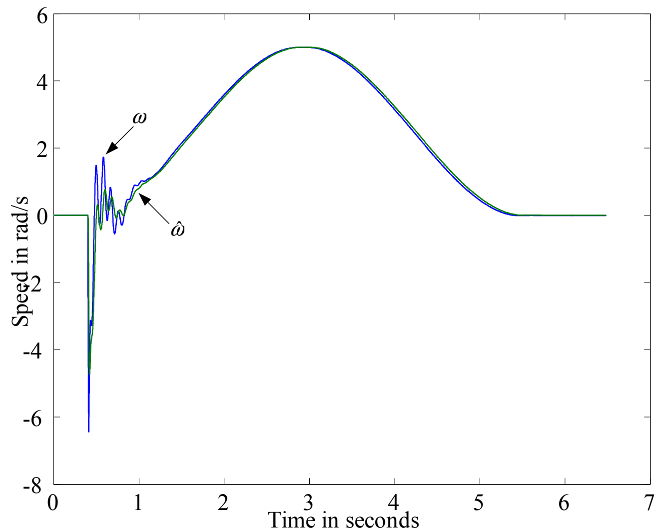

Figure 3.9: Actual speed ω and estimated speed $\hat{\omega}$ (using EKF) with the motor tracking a low speed trajectory ($\omega_{\max} = 5$ rad/s) *with full load at the start.*

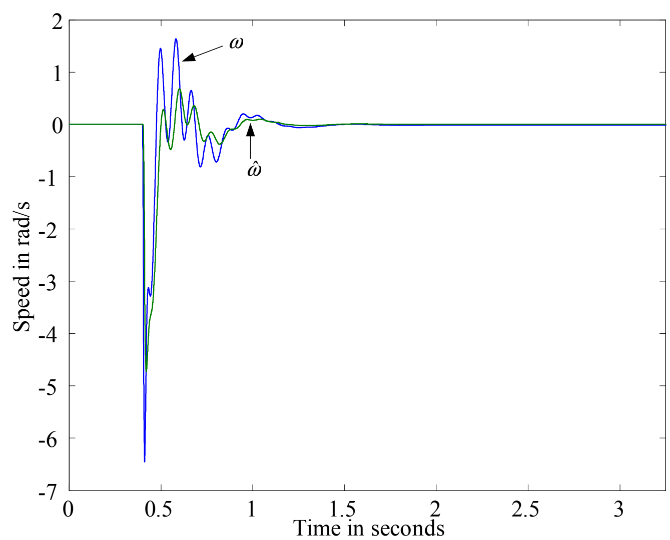

Figure 3.10: Actual speed ω and estimated speed $\hat{\omega}$ (EKF) with the motor tracking a zero speed trajectory ($\omega \equiv 0$) *with full load at the start.*

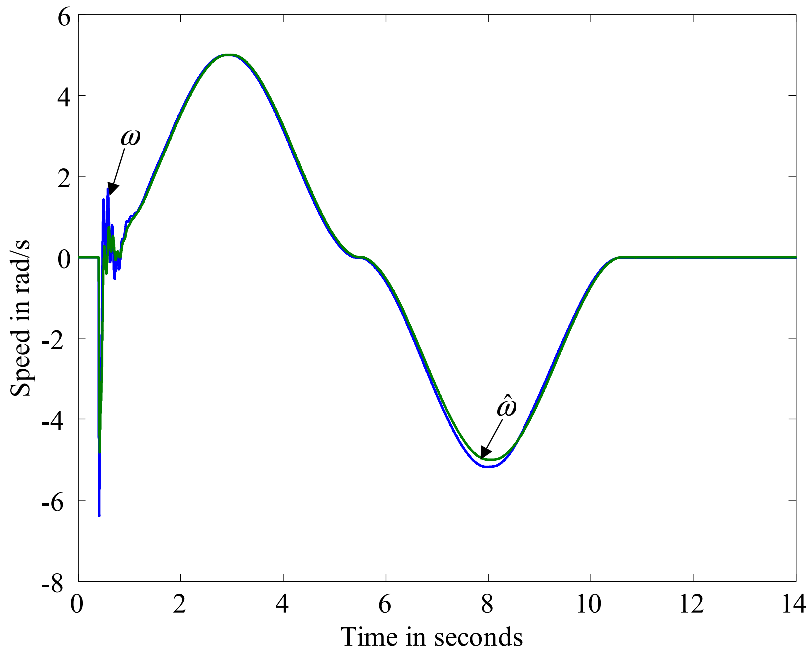

Figure 3.11: Actual speed ω and estimated speed $\hat{\omega}$ (using EKF) with full load on the motor.

shown in Figure 3.12. The full load is on the motor from $t = 0.4\,\mathrm{sec}$ to $t = 13\,\mathrm{sec}$, that is, even during the zero speed part of the trajectory. From $t = 0$ to $t = 0.4$ seconds, a constant u_{Sa} is applied to the motor to build up the flux, and the motor is considered to be held with a (mechanical) brake so that $\omega \equiv 0$. Figure 3.12 shows at $t = 0.4$ seconds the brake is released and the machine is running on a low speed trajectory ($\omega_{\max} = 5\ rad/s$) with full load. The estimated speed $\hat{\omega}$ is fed back to a current command field-oriented controller.

The extended Kalman filter also works in simulation on a low trajectory with full load. However, the extended Kalman filter assumes the motor speed is "slowly varying" parameter compared to the electrical variables while the differential-algebraic method does not make this assumption. Further, the design premise of the extended Kalman filter assumes all noises are uncorrelated white noises, which is never true in a induction motor drive system. The differential-algebraic method neglects the noise. This is an issue with the method. The extended Kalman filter is also based on linearizing a nonlinear mathematical model of the induction motor about its current operation point. Thus there is an inherent assumption of "small" perturbations and, as such, there is no guarantee that the extended Kalman filter converges. The differential-algebraic method is a stable speed estimator.

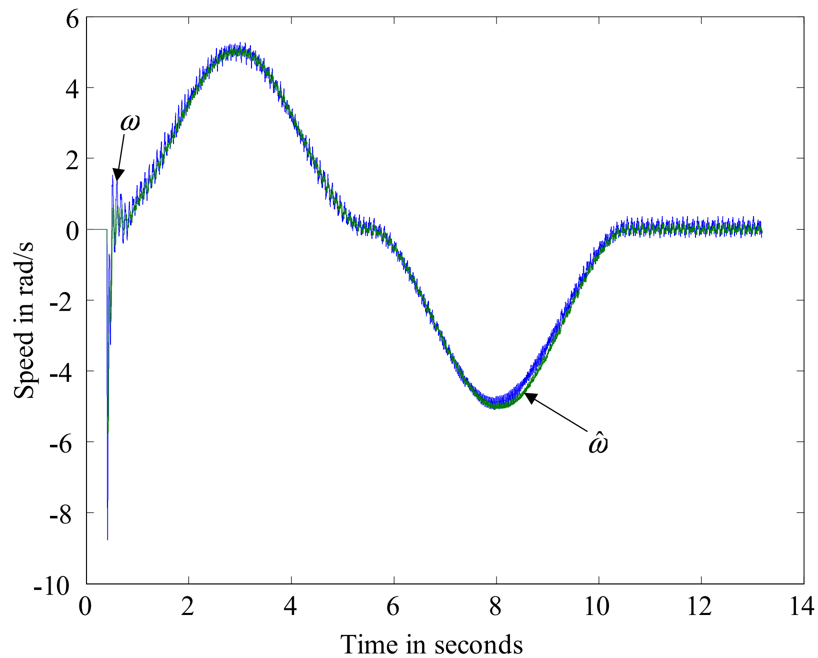

Figure 3.12: Actual speed ω and estimated speed $\hat{\omega}$ (using EKF) with full load on the motor driven by PWM inverter.

3.9 Summary

This chapter introduces a new differential-algebraic approach to speed estimation of an induction motor based on the measured stator voltages, currents, and their derivatives. This method entails using an algebraic estimate of the motor speed to stabilize a dynamic speed observer. It also shows that some trajectories are indistinguishable because the "coefficients" of the algebraic expression for the speed all happen to be zero. This new observer does not requires any sort of "slowly varying" speed assumption and is stable. Simulation results show that this method has the potential for speed estimation at low speeds under full load.

Chapter 4

Experimental Results of Differential-Algebraic Speed Estimator

To verify the differential-algebraic method, experiments are carried out. A three-phase, 0.5 hp, 1735 rpm ($n_p = 2$ pole-pair) induction motor is used for the experiment. The parameters of the induction motor are

$$n_p = 2,\ R_S = 5.12 \text{ ohms},\ R_R = 2.23 \text{ ohms},\ L_S = L_R = 0.2919 \text{ H},\ M = 0.2768 \text{ H},$$
$$J = 0.0021 \text{ k-gm}^2,\ \tau_{L\_rated} = 2.0337 \text{ N-m},\ I_{\max} = 2.77 \text{ A},\ V_{\max} = 230 \text{ V}.$$

4.1 Open-loop experiments with PWM inverter

4.1.1 Experimental setup

Figure 4.1 shows the experimental setup. An ALLEN-BRADLEY AB1305 PWM inverter shown in Figure 4.2 is used to drive the induction motor shown in Figure 4.3, and a 4096 pulse/rev optical encoder in Figure 4.3 is attached to the motor for position measurements. The sensor board shown in Figure 4.4 is used to measure the stator voltages and the stator currents. This board provides 3 voltage and 3 current measurements. The measurements are electrically isolated from the drive. The real-time computing system RTLAB shown in Figure 4.5 from OPAL-RT with a fully integrated hardware and software system is used to collect data [56].

4.1.2 Open-loop experimental results without load

Simulation results show that the speed observer (3.52) requires small step sizes (1 μs) to accommodate the derivatives of the measurement. The RTLAB could only sample the data at 100 μs without computational overruns.

Two solutions of equation (3.38)

$$q(\omega, t) \triangleq q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega^2 + q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega + q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = 0.$$

are

$$\omega_1 = \frac{-q_1 + \sqrt{q_1^2 - 4q_2q_0}}{2q_2} \text{ and } \omega_2 = \frac{-q_1 - \sqrt{q_1^2 - 4q_2q_0}}{2q_2}. \tag{4.1}$$

Simulations show that if the induction motor runs open-loop without load on it, $\omega = \omega_1$ in equation (4.1) is the motor speed.

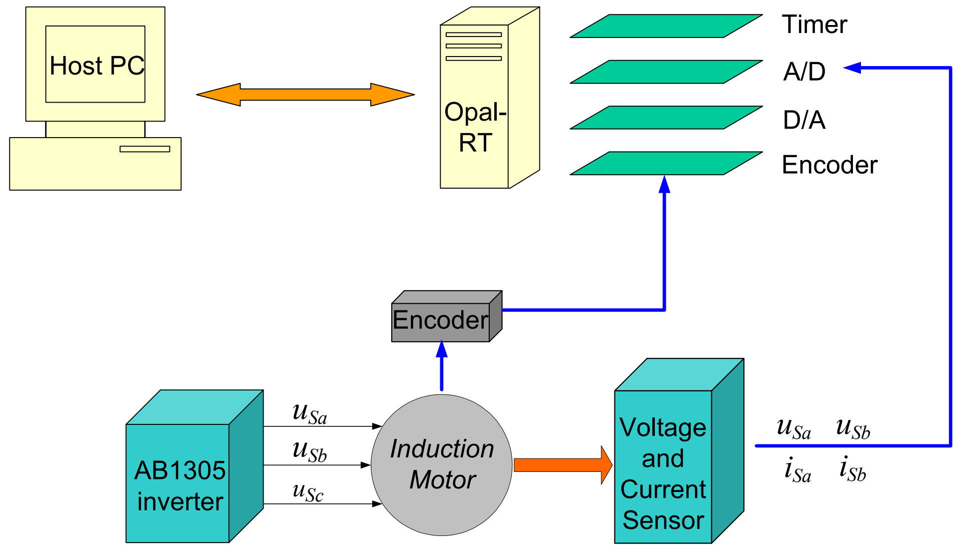

Figure 4.1: Open-loop experimental setup.

Figure 4.2: Experimental setup of AB1305 inverter.

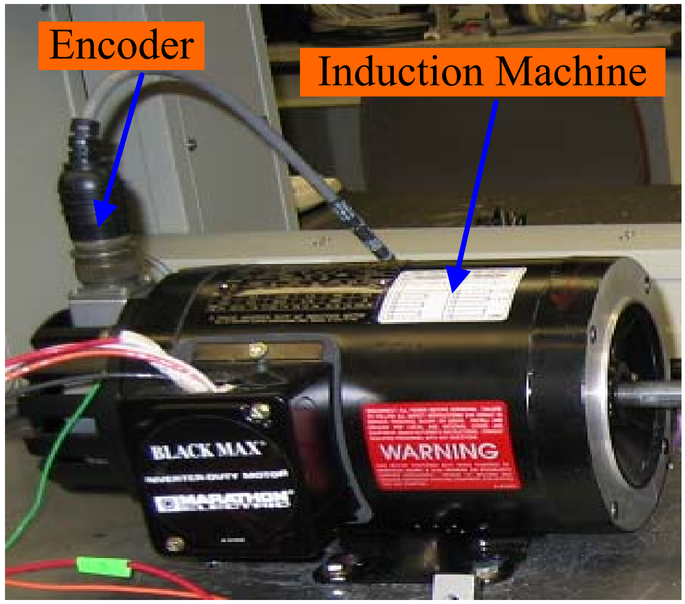

Figure 4.3: Experimental setups of induction machine and encoder.

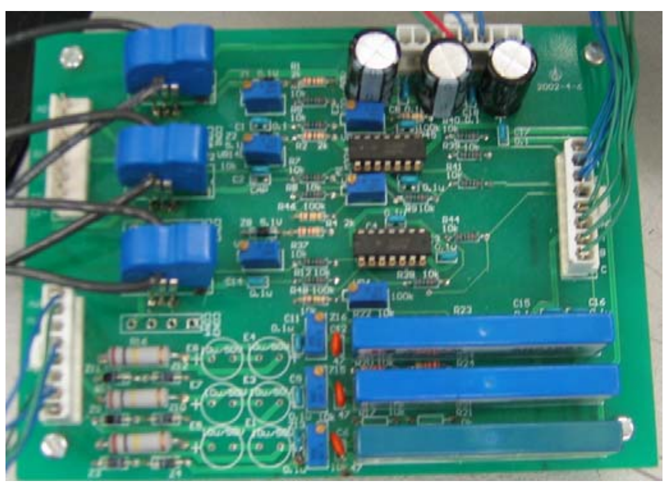

Figure 4.4: Voltage and current sensor board.

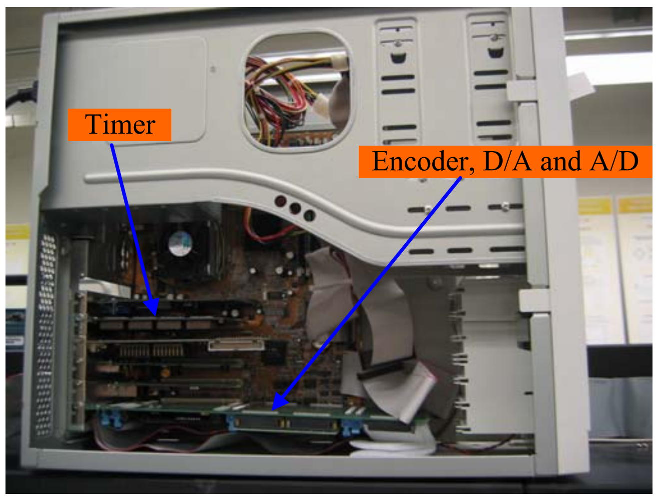

Figure 4.5: Experimental setup of RTLAB machine.

Also, it is shown in section 3.7.9 that in steady state, ω_1 is the correct solution. ω_1 contains only second-order derivatives of stator voltages and currents so it has less noise.

In the observer

$$\frac{d\hat{\omega}}{dt} \triangleq a_2\hat{\omega}^2 + a_1\hat{\omega} + a_0 + \ell\left(\omega - \hat{\omega}\right) \tag{4.2}$$

with ℓ large enough, $\ell\left(\omega - \hat{\omega}\right)$ dominates. So equation (4.2) may be simplified to

$$\frac{d\hat{\omega}}{dt} \triangleq \ell\left(\omega - \hat{\omega}\right).$$

Collecting this together, the estimate of speed used for the open-loop experiment without load is defined as the solution to the observer

$$\frac{d\hat{\omega}}{dt} \triangleq \ell\left(\omega - \hat{\omega}\right) \tag{4.3}$$

where

$$\omega = \omega_1 = \frac{-q_1 + \sqrt{q_1^2 - 4q_2q_0}}{2q_2}.$$

The stator voltages and currents along with the rotor position are sampled at 10 kHz ($100\mu s$). Filtered differentiation (using digital filters) is used for calculating the motor speed and the derivatives of the stator voltages and currents.

Figure 4.6 shows the experimental result of motor speed and estimated speed when the machine is driven by the inverter to the rated speed $\omega_{rate} = 2\pi 60/n_p = 188$ rad/s.

Figure 4.7 shows the experimental result of the motor speed and the estimated speed while the machine tracks a high speed trajectory. In this experiment, at approximately $t = 0.5$ sec, the inverter drives the induction motor from zero speed to

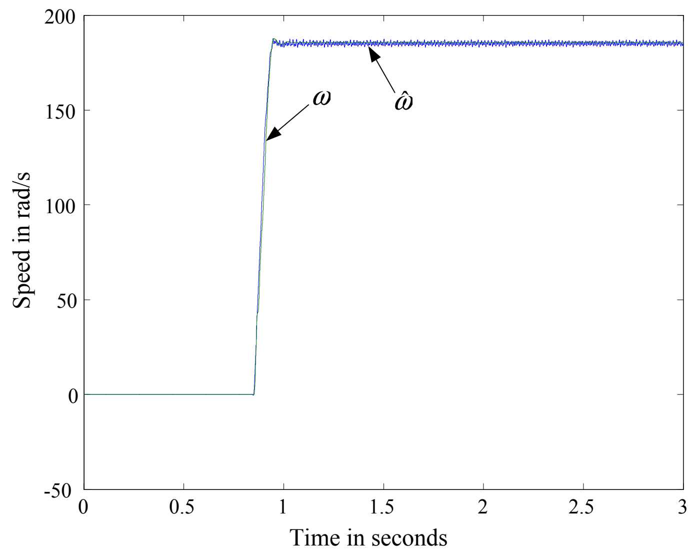

Figure 4.6: Actual speed ω and estimated speed $\hat{\omega}$ when the motor tracks step speed command open-loop.

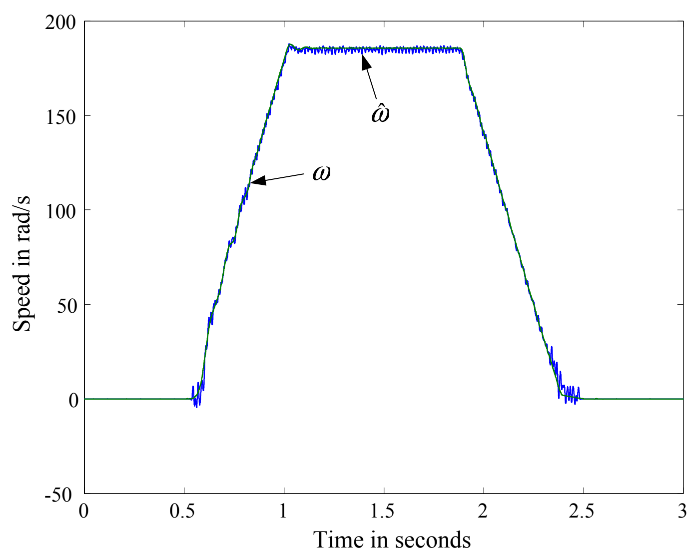

Figure 4.7: Actual speed ω and estimated speed $\hat{\omega}$ when the machine tracks the high speed trajectory open-loop.

rated speed. From $t = 1$ to 1.9 sec, the inverter maintains the motor at constant rated speed, and then decelerates the machine to zero speed from $t = 1.9$ to 2.4 sec.

4.2 Closed-loop experiments with field-oriented control

4.2.1 Experimental setup

Figure 4.8 shows the experimental setup for field-oriented control. A 10 kHz PWM inverter from SEMIKRON is used to drive the induction motor, and a PWM generator board is designed and built to drive the inverter (shown in Figure 4.9). A 4096 pulse/rev optical encoder is attached to the motor for position measurements. A 0.5 hp BALDOR BC202 DC motor is coupled to the induction motor as a mechanical load. This drive can vary the maximum motor torque as a function of the control signal voltage. The real-time computing system RTLAB from OPAL-RT with a fully integrated hardware and software system is used to provide the control signal [56]. Figure 4.10 shows the connection between the induction motor and DC motor.

4.2.2 Closed-loop experiment with field-oriented control at high speed trajectory

Figure 4.11 shows the experimental result of the motor speed and speed estimator with the motor under full load. From $t = 0$ to $t = 0.4$ seconds, $u_{Sa} = 10$ V and $u_{Sb} = 0$ is applied to the motor to build up the flux. At $t = 0.4$ seconds, the field-oriented controller is used to control the machine running on a high-speed trajectory

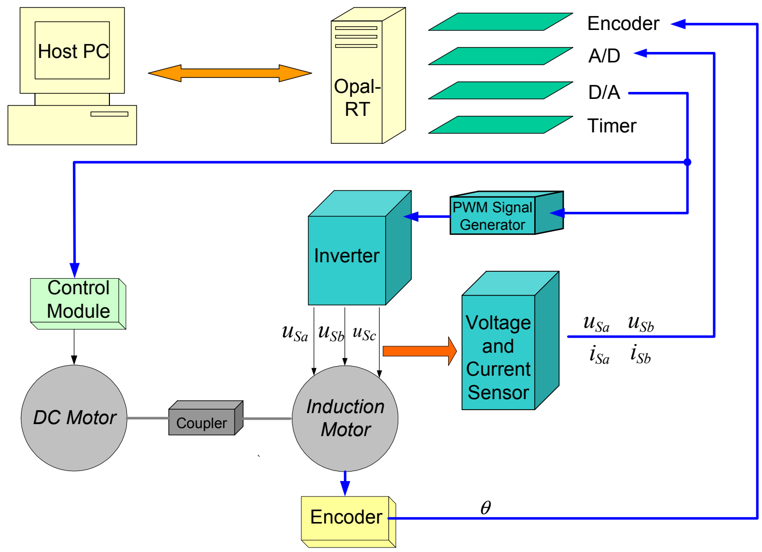

Figure 4.8: Closed-loop experimental setup.

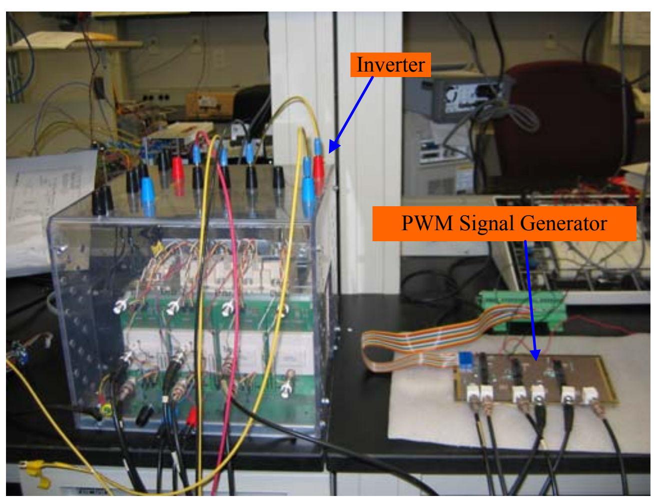

Figure 4.9: Experimental setups of PWM signal generator and inverter.

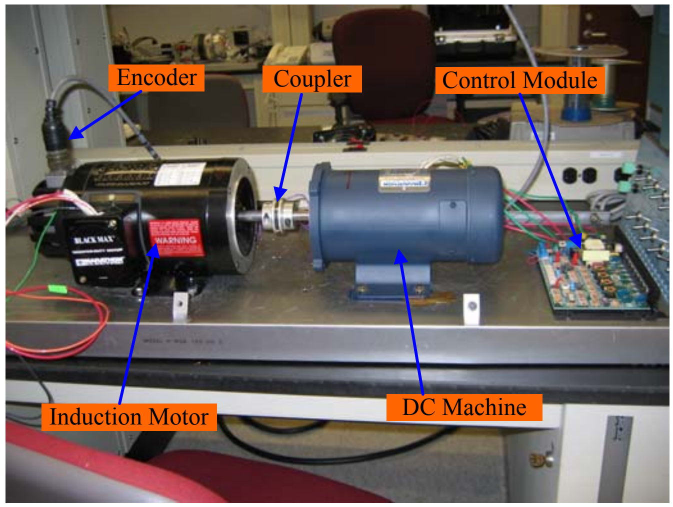

Figure 4.10: Experimental setups of induction motor and DC motor.

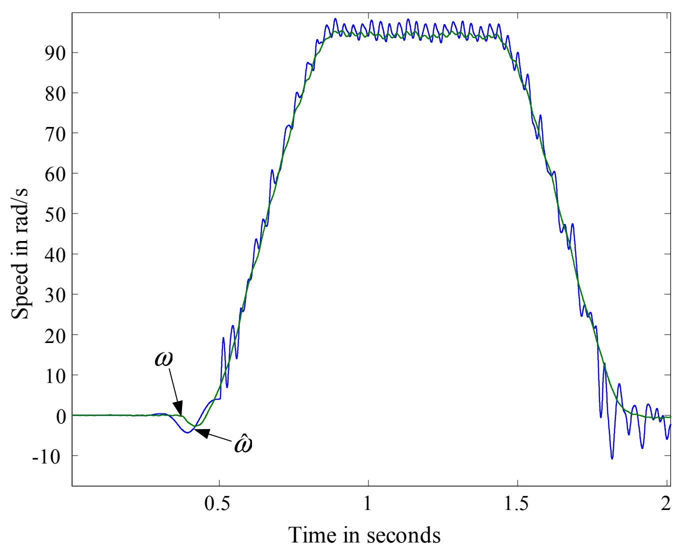

Figure 4.11: Motor speed ω and estimated speed $\hat{\omega}$ with the motor tracking a high speed trajectory ($\omega_{\max} = 94$ *rad/s*) *with full load at the start.*

($\omega_{\max} = 2\pi 15 = 94\ rad/s$) with *full* load. In this experiment, the motor speed ω is fed back to the field-oriented controller rather than $\hat{\omega}$. The stator voltages and currents are collected and sampled at 120 μs.

During the period $t \in [0.4\ 0.5]$, a third-order Butterworth filter with cutoff frequency of 20 Hz is used to filter the measured stator voltages and currents, which have a low frequency less than 2 Hz. The estimated speed $\hat{\omega}$ is calculated according to equation (3.52) repeated here

$$\frac{d\hat{\omega}}{dt} \triangleq a_2\hat{\omega}^2 + a_1\hat{\omega} + a_0 + \ell\left(\omega - \hat{\omega}\right)$$

where

$$\omega \triangleq \begin{cases} -q_0/q_1 & \text{if } q_2 = 0 \quad \text{See } (3.38) \\ -r_0/r_1 & \text{if } q_2 \neq 0 \quad \text{See } (3.45)\,. \end{cases}$$

During the period $t \in [0.5\ 1.9]$, a third-order Butterworth filter with cutoff frequency of 60 Hz is used to filter the measured stator voltages and currents. The estimated speed $\hat{\omega}$ is calculated by equation (4.3).

During the period $t \in [1.9\ 2.0]$, a third-order Butterworth filter with cutoff frequency of 20 Hz is used to filter the measured stator voltages and currents, which have a low frequency less than 2 Hz. The estimated speed $\hat{\omega}$ is calculated by equation (3.52).

Simulation results show that when the machine runs with full load at the beginning, there is a short time (0.05 sec), the actual motor speed alternates between ω_1 and ω_2, which are defined by equation (4.1). So between $t = 0.4$ and $t = 0.5$, a speed observer (3.52) has to be used. However, this observer (3.52) does not track a high speed trajectory because (3.52) requires a small step size ($1\mu s$) but the data are sampled at 120 μs. For high speed, the stator voltages and currents change much faster than at low speed so the third-order derivatives of stator currents is too noisy to be used to estimate the speed.

4.2.3 Closed-loop experiment with field-oriented control at low and zero speed trajectory

Figure 4.12 shows the experimental results of the motor speed and speed estimator with the motor under full load. From $t = 0$ to $t = 0.4$ seconds, $u_{Sa} = 10$ V and $u_{Sb} = 0$ is applied to the motor to build up the flux. At $t = 0.4$ seconds, the field-oriented controller is used to control the machine running on a low-speed trajectory ($\omega_{\max} = 5\ rad/s$) with *full* load. In this experiment, the motor speed ω is fed back to the field-oriented controller rather than $\hat{\omega}$. The stator voltages and currents are collected and sampled at 120 μs. A third-order Butterworth filter with cutoff frequency of 30 Hz is used to filter the measured stator voltages and currents, which have frequencies less than 2 Hz for this low speed trajectory. The speed observer (3.52) is used to obtain the estimated speed $\hat{\omega}$. Simulations indicate a significant improvement in tracking if faster sampling rates are possible.

Figure 4.13 shows the experimental results of the motor speed and speed estimator with the motor under full load. From $t = 0$ to $t = 0.4$ seconds, $u_{Sa} = 10$ V and $u_{Sb} = 0$ is applied to the motor to build up the flux. At $t = 0.4$ seconds, the field-oriented control is used to control the machine running at a zero speed trajectory ($\omega \equiv 0$) with *full* load. The motor speed ω is fed back to the field-oriented controller. The stator voltages and currents along with the rotor position are collected and sampled at 120 μs. A third-order Butterworth filter with cutoff frequency of 30 Hz is used to filter

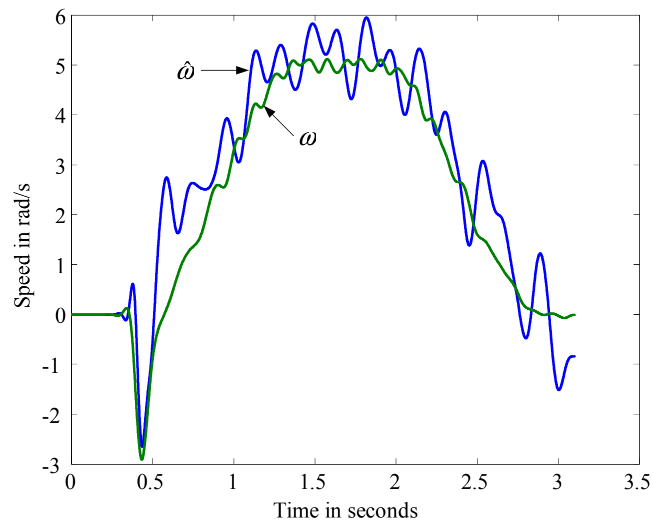

Figure 4.12: Motor speed ω and estimated speed $\hat{\omega}$ with the motor tracking a low speed trajectory ($\omega_{\max} = 5\ rad/s$) *with full load at the start.*

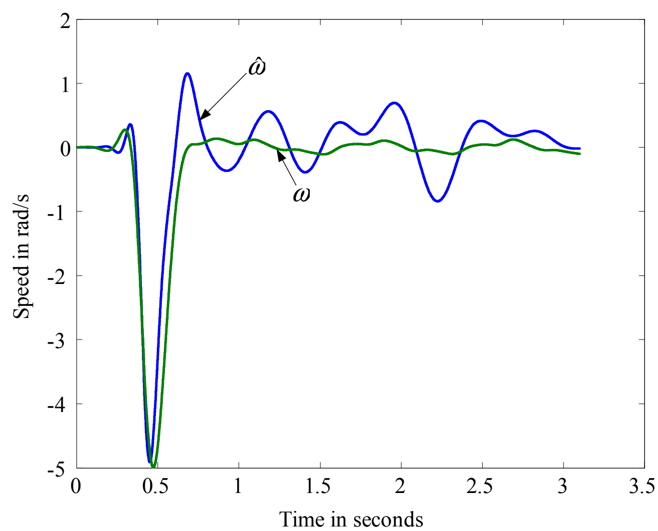

Figure 4.13: Motor speed ω and estimated speed $\hat{\omega}$ with the motor tracking a zero speed trajectory ($\omega \equiv 0$) *with full load at the start.*

the measured stator voltages and currents, which have frequencies less than 2 Hz for the zero speed trajectory. A speed observer (3.52) is used to obtain the estimated speed $\hat{\omega}$. Simulations indicate a significant improvement in tracking if faster sampling rates are possible.

4.3 Summary

In this chapter, experimental results are presented to verify the differential-algebraic approach to speed estimation.

When the machine runs open-loop without load on it, the simplified speed observer (4.3)

$$\frac{d\hat{\omega}}{dt} \triangleq \ell\left(\omega - \hat{\omega}\right)$$

where

$$\omega = \omega_1 = \frac{-q_1 + \sqrt{q_1^2 - 4q_2q_0}}{2q_2}$$

can be used. This observer does not require a small step size. However, when the induction machine runs with the load on it, this method does not work because the motor speed is not always equal to ω_1, it alternates between ω_1 and ω_2 which are defined by equation (4.1).

If the induction motor runs at the low speed trajectory with load on it, the speed observer (3.52)

$$\frac{d\hat{\omega}}{dt} \triangleq a_2\hat{\omega}^2 + a_1\hat{\omega} + a_0 + \ell\left(\omega - \hat{\omega}\right)$$

where

$$\omega \triangleq \begin{cases} -q_0/q_1 & \text{if } q_2 = 0 \quad \text{See (3.38)} \\ -r_0/r_1 & \text{if } q_2 \neq 0 \quad \text{See (3.45)} \end{cases}$$

has to be used. This observer requires small step size (1 μs) which is not possible using the standard processor technology in commercial electric drives.

Chapter 5

Differential-Algebraic Approach to Speed Sensorless Estimation of T_R

This chapter presents a differential-algebraic method to rotor time constant T_R estimation of an induction motor, using measured stator voltages and currents. That is, this method does not require speed information. Experimental results are presented to demonstrate the practical use of the identification method.

5.1 Introduction

The speed sensorless controller proposed in Chapter 3 requires the value of T_R, which can vary due to Ohmic heating. A method is now presented to estimate T_R without a speed sensor. Multiple techniques have been proposed to estimate T_R without a speed sensor. Refs. [17] [57] combine parameter identification and speed estimation. A least-squares method is used to identify T_R and ω simultaneously in [17] and is based on the transfer function of the induction motor, which is only valid during constant speed operation. Ref. [57] also assumes constant speed operation. The work presented here uses a differential-algebraic approach to identify the rotor time constant T_R without the motor speed information. It also shows that during the steady state, T_R is not identifiable, i.e., $d\omega/dt \neq 0$ is required to identify T_R .

5.2 Differential-algebraic approach to T_R estimation

The T_R estimation is based on equations (3.38) and (3.45), which are repeated in (5.1) and (5.2) below,

$$q(\omega) \triangleq q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega^2 + q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega + q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = 0, \quad (5.1)$$

$$r_1\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)\omega + r_0\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right) = 0, \quad (5.2)$$

where

$$r_1\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right) \triangleq 2q_2^2 a_0 - q_2 q_1 a_1 + q_2 \dot{q}_1 - 2q_2 q_0 a_2 + q_1^2 a_2 - q_1 \dot{q}_2 \quad (5.3)$$

and

$$r_0\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right) \triangleq q_2 q_1 a_0 + q_2 \dot{q}_0 - 2q_2 q_0 a_1 + q_0 q_1 a_2 - q_0 \dot{q}_2. \quad (5.4)$$

Equation (5.2) is a first-order polynomial equation in ω with a unique solution as long as r_1 (the coefficient of ω) is nonzero (It is shown in section 3.7.8 that $r_1 \neq 0$ in steady state if $q_2 \neq 0$). Solving ω from equation (5.2) to obtain

$$\omega = -r_0/r_1. \tag{5.5}$$

Replace ω in equation (5.1) by (5.5) to obtain

$$q_2 r_0^2 - q_1 r_0 r_1 + q_0 r_1^2 = 0. \tag{5.6}$$

This turns out to be a twelfth-order polynomial equation of T_R, which can be rewritten as

$$\sum_{i=0}^{12} C_i \left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right) T_R^i = 0. \tag{5.7}$$

Solving equation (5.7) gives T_R. The coefficients $C_i \left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$ of (5.7) contain third-order derivatives of the stator currents and second-order derivatives of the stator voltages and, therefore, noise is a concern. For time intervals for which T_R does not vary, equation (5.7) must hold identically. In order to smooth out the noise in C_i, (5.7) integrated the data collection for the interval $[t_1\ t_2]$ to obtain

$$\sum_{i=0}^{12} \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} C_i \left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right) \right) T_R^i = 0. \tag{5.8}$$

There are 12 solutions satisfying equation (5.8) , but the simulation results show that there are always 10 conjugate solutions. The other two solutions include the correct value of T_R, while the other one is either negative or close to zero. This will be illustrated in the experimental section, section 5.4.

5.3 T_R is not identifiable in steady state

5.3.1 In steady state, T_R is not identifiable by differential-algebraic approach

In steady state, solving equation (5.2)

$$r_1\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)\omega + r_0\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right) \equiv 0$$

gives the motor speed

$$\omega = -\frac{r_0}{r_1} = \frac{\omega_S\left(1-S\right)}{n_p} \tag{5.9}$$

as

$$\begin{aligned} r_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = & \frac{-\left|\underline{U}_S\right|^{12}}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^6} \\ & \times \left(\frac{1}{1+S^2\omega_S^2 T_R^2}\right)^3 \times \frac{n_p^4\left(1-\sigma\right)^6\omega_S^3 L_S^2}{\sigma^4} \\ & \times \left(1 + T_R^2\omega_S^2\left(1-S\right)^2\right)^2 \times \frac{1}{den} \end{aligned}$$

$$\begin{aligned} r_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \quad = \quad & \frac{\left|\underline{U}_S\right|^{12}}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^6} \\ & \times \left(\frac{1}{1+S^2\omega_S^2 T_R^2}\right)^3 \frac{n_p^3\left(1-\sigma\right)^6\omega_S^4 L_S^2\left(1-S\right)}{\sigma^4} \\ & \times \left(1 + \omega_S^2 T_R^2 \times \left(1-S\right)^2\right)^2 \times \frac{1}{den} \end{aligned}$$

in steady state (see section 3.7.8). Replacing ω in equation (5.1)

$$q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega^2 + q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega + q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = 0$$

by (5.9) to obtain

$$q_2 \frac{\omega_S^2 (1-S)^2}{n_p^2} + q_1 \frac{\omega_S (1-S)}{n_p} + q_0 = 0. \tag{5.10}$$

The steady-state expressions of q_2, q_1, and q_0 are given by equations (3.66), (3.67), and (3.68) in section 3.7.4 and repeated below,

$$\begin{aligned} q_2 &= \frac{n_p^2 T_R^2 \left|\underline{U}_S\right|^4}{\sigma\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2} \\ &\quad \times \frac{\omega_S^2 L_S (1-\sigma)^2 (1-S)}{1+S^2\omega_S^2T_R^2} \end{aligned}$$

$$\begin{aligned} q_1 &= \frac{n_p \omega_S \left|\underline{U}_S\right|^4}{\sigma\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2} \\ &\quad \times \frac{L_S (1-\sigma)^2 \left(1-\omega_S^2T_R^2(1-S)^2\right)}{1+S^2\omega_S^2T_R^2} \end{aligned}$$

$$\begin{aligned} q_0 &= \frac{-\left|\underline{U}_S\right|^4}{\sigma\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2} \\ &\quad \times \frac{\omega_S^2 L_S (1-\sigma)^2 (1-S)}{1+S^2\omega_S^2T_R^2}. \end{aligned}$$

Replacing q_2, q_1, and q_0 in (5.10) by their steady-state expressions, one obtains

$$\begin{aligned} & T_R^2\left(\omega_S^2\left(1-S\right)^2-\omega_S^2\left(1-S\right)^2\right)+1-1=0 \\ \Rightarrow \;& T_R^2\times 0+0=0 \end{aligned}$$

That is, in steady state (5.1) and (5.2) hold independent of the value of T_R.

5.3.2 In steady state, T_R is not identifiable by the least-squares method

Vélez-Reyes [16] [17] [18] have used least-squares methods for simultaneous parameter and speed identification in induction machines. In this approach $d\omega/dt$ is taken to be zero. At constant speed, a linear (in the parameters) regressor model can be found. Specifically, consider the mathematical model of the induction motor in (3.34) repeated here

$$\begin{aligned} \frac{d^2}{dt^2}\underline{i}_S = & -\frac{1}{T_R}\left(1-jn_P\omega T_R\right)\left(\frac{d}{dt}\underline{i}_S+\gamma\underline{i}_S-\frac{1}{\sigma L_S}\underline{u}_S\right)+\frac{\beta M}{T_R^2}\left(1-jn_P\omega T_R\right)\underline{i}_S \\ & -\gamma\frac{d}{dt}\underline{i}_S+\frac{1}{\sigma L_S}\frac{d}{dt}\underline{u}_S-\frac{jn_PT_R}{1-jn_P\omega T_R}\left(\frac{d}{dt}\underline{i}_S+\gamma\underline{i}_S-\frac{1}{\sigma L_S}\underline{u}_S\right)\frac{d\omega}{dt}. \end{aligned}$$

In steady state, $d\omega/dt=0$ so that this equation reduces to

$$\begin{aligned} \frac{d^2}{dt^2}\underline{i}_S = & -\frac{1}{T_R}\left(1-jn_P\omega T_R\right)\left(\frac{d}{dt}\underline{i}_S+\gamma\underline{i}_S-\frac{1}{\sigma L_S}\underline{u}_S\right)+\frac{\beta M}{T_R^2}\left(1-jn_P\omega T_R\right)\underline{i}_S \\ & -\gamma\frac{d}{dt}\underline{i}_S+\frac{1}{\sigma L_S}\frac{d}{dt}\underline{u}_S \end{aligned} \tag{5.11}$$

where $\underline{i}_S = i_{Sa} + ji_{Sb}$ and $\underline{u}_S = u_{Sa} + ju_{Sb}$. Decomposing equation (5.11) into its real and imaginary parts gives

$$\begin{aligned}\frac{d^2 i_{Sa}}{dt} =& \frac{1}{T_R}\left(-\frac{di_{Sa}}{dt} - \frac{R_S}{\sigma L_S} i_{Sa} + \frac{1}{\sigma L_S} u_{Sa}\right) \\ &+ n_p\omega\left(-\frac{di_{Sb}}{dt} - \frac{R_S}{\sigma L_S} i_{Sb} + \frac{1}{\sigma L_S} u_{Sb}\right) - \gamma\frac{di_{Sa}}{dt} + \frac{1}{\sigma L_S}\frac{du_{Sa}}{dt}\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}\frac{d^2 i_{Sb}}{dt} =& \frac{1}{T_R}\left(-\frac{di_{Sb}}{dt} - \frac{R_S}{\sigma L_S} i_{Sb} + \frac{1}{\sigma L_S} u_{Sb}\right) \\ &- n_p\omega\left(-\frac{di_{Sa}}{dt} - \frac{R_S}{\sigma L_S} i_{Sa} + \frac{1}{\sigma L_S} u_{Sa}\right) - \gamma\frac{di_{Sb}}{dt} + \frac{1}{\sigma L_S}\frac{du_{Sb}}{dt}\end{aligned} \tag{5.13}$$

where

$$\gamma = \frac{R_S}{\sigma L_S} + \frac{1}{\sigma L_S}\frac{1}{T_R}\frac{M^2}{L_R} = \frac{R_S}{\sigma L_S} + \frac{1-\sigma}{\sigma}\frac{1}{T_R}.$$

The goal here is to estimate T_R without knowledge of ω. So, it is now assumed the motor parameters are all known except for T_R. The set of equations (5.12) and (5.13) may then be rewritten in regressor form as

$$y(t) = W(t)K. \tag{5.14}$$

Here $K \in \mathbb{R}^2$, $W \in \mathbb{R}^{2\times 2}$, and $y \in \mathbb{R}^2$ are given by

$$K \triangleq \begin{bmatrix} 1/T_R \\ n_p\omega \end{bmatrix}$$

$$W(t) \triangleq \begin{bmatrix} L_S \frac{di_{Sa}}{dt} - u_{Sa} + R_S i_{Sa} & \sigma L_S \frac{di_{Sb}}{dt} - u_{Sb} + R_S i_{Sb} \\ L_S \frac{di_{Sb}}{dt} - u_{Sb} + R_S i_{Sb} & -\sigma L_S \frac{di_{Sa}}{dt} + u_{Sa} - R_S i_{Sa} \end{bmatrix}$$

and

$$y(t) \triangleq \begin{bmatrix} \frac{du_{Sa}}{dt} - \sigma L_S \frac{d^2 i_{Sa}}{dt} - R_S \frac{di_{Sa}}{dt} \\ \frac{du_{Sb}}{dt} - \sigma L_S \frac{d^2 i_{Sb}}{dt} - R_S \frac{di_{Sb}}{dt} \end{bmatrix}.$$

The regressor system (5.14) is linear in the parameters. The standard least-squares approach is to collect data at times $t = 0, T, 2T, \cdots, NT$ and compute the solution to

$$R_W K = R_{YW} \tag{5.15}$$

where

$$R_W \triangleq \sum_{n=1}^{N} W^T(nT) W(nT)$$

$$R_{YW} \triangleq \sum_{n=1}^{N} W^T(nT) y(nT).$$

A unique solution to (5.15) requires R_W to be invertible. However, R_W is never invertible in steady state! To show this, let

$$D(t) = \begin{bmatrix} i_{Sb}(t) & -i_{Sa}(t) \\ i_{Sa}(t) & i_{Sb}(t) \end{bmatrix}.$$

In steady state, the determinant of $D(t)$ is

$$\det(D(t)) = i_{Sa}^2(t) + i_{Sb}^2(t) = |\underline{\mathrm{I}}_S|^2 .$$

and a simple computation shows

$$D(t)^T D(t) = |\underline{\mathrm{I}}_S|^2 I_{2\times 2}.$$

Multiply both sides of equation (5.14) on the left by $D(t)$ to obtain

$$\begin{array}{rcl} D(t)\,y(t) & = & D(t)\,W(t)\,K \\ \begin{bmatrix} R_S\omega_S |\underline{\mathrm{I}}_S|^2 - \omega_S P \\ \sigma L_S \omega_S^2 |\underline{\mathrm{I}}_S|^2 - \omega_S Q \end{bmatrix} & = & \begin{bmatrix} -\omega_S L_S |\underline{\mathrm{I}}_S|^2 + Q & R_S |\underline{\mathrm{I}}_S|^2 - P \\ R_S |\underline{\mathrm{I}}_S|^2 - P & \sigma L_S \omega_S |\underline{\mathrm{I}}_S|^2 - Q \end{bmatrix} K \end{array} \tag{5.16}$$

where P and Q are given by (3.58) and (3.58) in section 3.7.2 and repeated here

$$\begin{array}{rcl} P & \triangleq & u_{Sa} i_{Sa} + u_{Sb} i_{Sb} \\ & = & R_e\left(\underline{U}_S \underline{I}_S^*\right) \\ & = & \dfrac{|\underline{U}_S|^2 \left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)}{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}} \end{array} \tag{5.17}$$

$$
\begin{aligned}
Q &\triangleq u_{Sb} i_{Sa} - u_{Sa} i_{Sb} \\
&= I_m \left(\underline{U}_S \underline{I}_S^* \right) \\
&= \frac{\left|\underline{U}_S\right|^2 \frac{\omega_S L_S \left(1+\sigma S^2 \omega_S^2 T_R^2\right)}{1+S^2 \omega_S^2 T_R^2}}{\left(R_S + \frac{(1-\sigma) S \omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2 \left(1+\sigma S^2 \omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}}.
\end{aligned}
\tag{5.18}
$$

Further, in steady state, it is also true that (see section 3.7.2)

$$
\begin{aligned}
i_{Sb} \frac{di_{Sa}}{dt} - i_{Sa} \frac{di_{Sb}}{dt} &= -\omega_S \left|\underline{\mathrm{I}}_S\right|^2 \\
\frac{d\left(i_{Sb}^2 + i_{Sa}^2\right)}{dt} &= 0 \\
i_{Sa}^2 + i_{Sb}^2 &= \left|\underline{\mathrm{I}}_S\right|^2 .
\end{aligned}
$$

Use (5.17) and (5.18) to replace P and Q in (5.16) to obtain

$$
D(t) W(t) = -\frac{\left|\underline{\mathrm{I}}_S\right|^2 (1-\sigma) \omega_S L_S}{1+S^2 \omega_S^2 T_R^2} \begin{bmatrix} S^2 \omega_S^2 T_R^2 & S \omega_S T_R \\ S \omega_S T_R & 1 \end{bmatrix} \tag{5.19}
$$

$$
D(t) y(t) = -\omega_S \frac{\left|\underline{\mathrm{I}}_S\right|^2 (1-\sigma) \omega_S L_S}{1+S^2 \omega_S^2 T_R^2} \begin{bmatrix} S \omega_S T_R \\ 1 \end{bmatrix} \tag{5.20}
$$

That is, $D(t) W(t)$ and $D(t) y(t)$ are constant matrices.

Further, it is easily seen that the determinant of $D(t) W(t)$ is zero, i.e.,

$$
\det\left(D(t) W(t)\right) \equiv 0.
$$

In steady state, $D(t)\, W(t)$ is a constant matrix whose determinant equal to zero. Also

$$\begin{aligned} R_{DW} &\triangleq \sum_{n=1}^{N} \left(D(nT)\, W(nT)\right)^{T} \left(D(nT)\, W(nT)\right)) \\ &= \sum_{n=1}^{N} W^{T}(nT) D^{T}(nT)\, D(nT)\, W(nT) \\ &= \left|\underline{\mathrm{I}}_{S}\right|^{2} \sum_{n=1}^{N} W^{T}(nT) W(nT) \\ &= \left|\underline{\mathrm{I}}_{S}\right|^{2} R_{W}. \end{aligned}$$

Because $\det(R_{DW}) = 0$, it follows that $\det(R_{W}) = 0$ showing that R_{W} is never invertible using steady-state data.

Also,

$$\begin{aligned} R_{DWY} &\triangleq \sum_{n=1}^{N} \left(D(nT)\, W(nT)\right)^{T} \left(D(nT)\, y(nT)\right)) \\ &= \sum_{n=1}^{N} W^{T}(nT) D^{T}(nT)\, D(nT)\, y(nT) \\ &= \left|\underline{\mathrm{I}}_{S}\right|^{2} \sum_{n=1}^{N} W^{T}(nT) y(nT) \\ &= \left|\underline{\mathrm{I}}_{S}\right|^{2} R_{YW}. \end{aligned}$$

Further, it can be shown

$$\begin{aligned} R_W &= R_{DW}/\left|\underline{\mathrm{I}}_S\right|^2 \\ &= N\left(D\left(0\right)W\left(0\right)\right)^T\left(D\left(0\right)W\left(0\right)\right)/\left|\underline{\mathrm{I}}_S\right|^2 \\ &= \frac{N\left|\underline{\mathrm{I}}_S\right|^2\left(1-\sigma\right)^2\omega_S^2L_S^2}{1+S^2\omega_S^2T_R^2}\begin{bmatrix} S^2\omega_S^2T_R^2 & S\omega_ST_R \\ S\omega_ST_R & 1 \end{bmatrix}, \end{aligned}$$

where $D\left(0\right)W\left(0\right)$ is taken from (5.19).

Also,

$$\begin{aligned} R_{YW} &= R_{DWY}/\left|\underline{\mathrm{I}}_S\right|^2 \\ &= N\left(D\left(0\right)W\left(0\right)\right)^T\left(D\left(0\right)y\left(0\right)\right))/\left|\underline{\mathrm{I}}_S\right|^2 \\ &= \omega_S\frac{N\left|\underline{\mathrm{I}}_S\right|^2\left(1-\sigma\right)^2\omega_S^2L_S^2}{1+S^2\omega_S^2T_R^2}\begin{bmatrix} S\omega_ST_R \\ 1 \end{bmatrix}. \end{aligned}$$

where $D\left(0\right)W\left(0\right)$ and $D\left(0\right)y\left(0\right)$ are taken from (5.19) and (5.20).

By inspection, $K=\left[0\ \omega_S\right]^T$ is one solution to (5.15). The null space of R_W is generated by

$$\begin{bmatrix} -1/T_R \\ S\omega_S \end{bmatrix} \tag{5.21}$$

so that all possible solution are given by

$$\begin{bmatrix} 0 \\ \omega_S \end{bmatrix}+\alpha\begin{bmatrix} -1/T_R \\ S\omega_S \end{bmatrix}$$

for some $\alpha\in\mathbb{R}$.

Remark 1 *The Morse-Penrose (pseudo-inverse) solution [58] to (5.15)*

$$R_W K = R_{YW}$$

is

$$K = \frac{\omega_S}{1 + S^2\omega_S^2 T_R^2}\begin{bmatrix} S\omega_S T_R \\ 1 \end{bmatrix}. \tag{5.22}$$

the solution K in (5.22) *can be written in the form*

$$\begin{bmatrix} 0 \\ \omega_S \end{bmatrix} + \alpha \begin{bmatrix} -1/T_R \\ S\omega_S \end{bmatrix}$$

for

$$\alpha = -\frac{S\omega_S^2 T_R^2}{1 + S^2\omega_S^2 T_R^2}.$$

In summary, solving (5.15) using steady-state data leads to an infinite set of solution and so T_R is not identifiable by this method.

5.4 Experimental results

To demonstrate the viability of the T_R estimator (5.8), experiments are carried out. A three-phase, 0.5 hp, 1735 rpm ($n_p = 2$ pole-pair) induction motor model is used for the experiment. An Allen-Bradley PWM inverter is used to drive the induction motor. Given a speed command to the inverter, the inverter will produce PWM voltage which drives the induction motor to track the speed trajectory. Here a step speed trajectory

$$\omega_{ref} = \begin{cases} 0 & t < 0 \\ 2\pi 60/n_p = 188 \text{ rad/s} & t \geq 0 \end{cases}$$

is chosen. The stator currents and voltages are sampled at 10 kHz. The real-time computing system RTLAB from OPAL-RT with a fully integrated hardware and software system is used to collect data [56]. Filtered differentiation (using digital filters) is used for the derivatives of the voltages and currents. Specifically, the signals are filtered with a third-order Butterworth filter with cutoff frequency of 100 Hz. The voltages and currents are put through a $3-2$ transformation to obtain the two phase equivalent voltages u_{Sa}, u_{Sb} which are plotted in Figures 5.1, and the corresponding two phase equivalent currents i_{Sa}, i_{Sb} are plotted in Figure 5.2.

Using the data $\{u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\}$ collected between 0.84 sec to 0.91 sec, the quantities du_{Sa}/dt, du_{Sa}/dt, di_{Sa}/dt, di_{Sb}/dt, $d^2 i_{Sa}/dt^2$, $d^2 i_{Sb}/dt^2$, $d^3 i_{Sa}/dt^3$, $d^3 i_{Sb}/dt^3$ are calculated and used to evaluate the coefficients in equation (5.8). Solving equation (5.8) one obtains the 12 solutions

$$\begin{array}{cc} T_{R1} = 0.1064 & T_{R2} = -0.0186 \\ T_{R3} = -0.0576 + j0.0593 & T_{R4} = -0.0576 - j0.0593 \\ T_{R5} = -0.0037 + j0.0166 & T_{R6} = -0.0037 - j0.0166 \\ T_{R7} = -0.0072 + j0.0103 & T_{R8} = -0.0072 - j0.0103 \\ T_{R9} = 0.0125 + j0.0077 & T_{R10} = 0.0125 - j0.0077 \\ T_{R11} = 0.0065 + j0.0018 & T_{R12} = 0.0065 - j0.0018. \end{array}$$

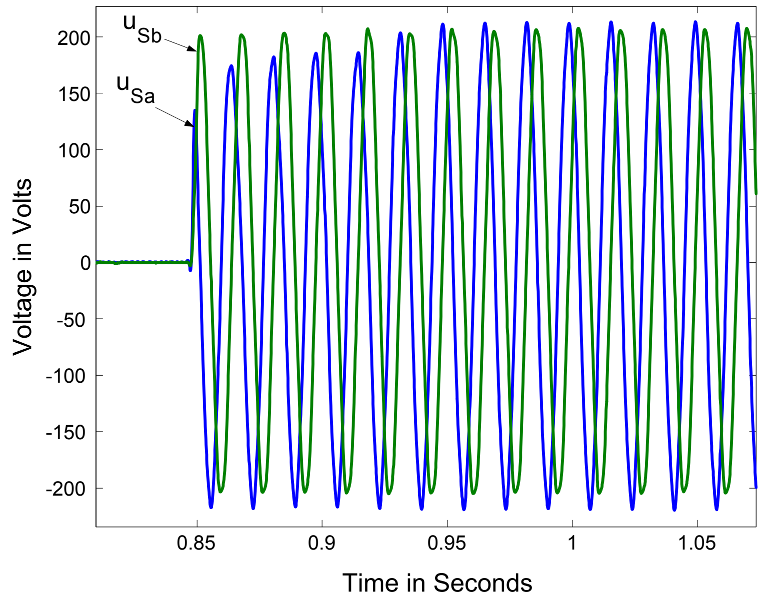

Figure 5.1: Sampled two phase equivalent voltages u_{Sa}, u_{Sb}.

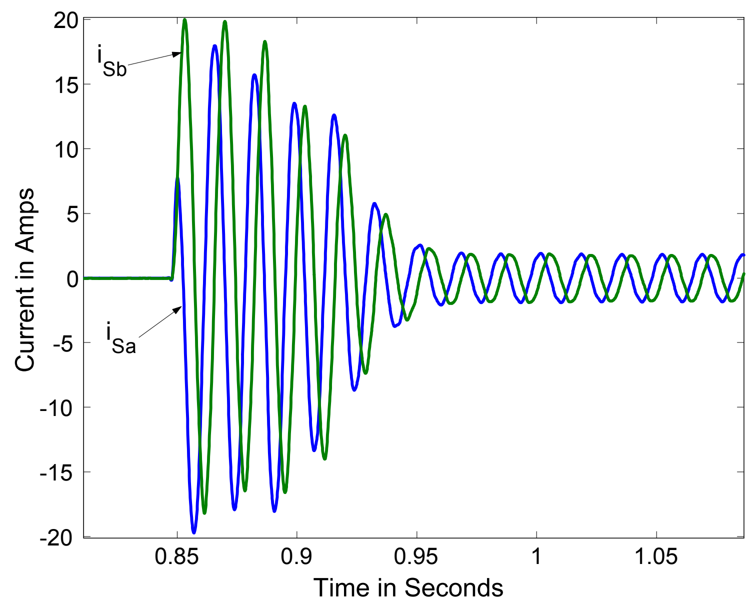

Figure 5.2: Sampled two phase equivalent currents i_{Sa}, i_{Sb}.

T_R should be a real positive number, so $T_R = 0.1064$ is the only possible choice. This compares favorably with the value of $T_R = 0.11$ obtained using the method developed by Wang et al [59] where a speed sensor is also used.

To illustrate the identified T_R, a simulation induction motor model is used with the measured voltages as input. Then the simulation's output (stator currents) are used to compare with the measured (stator currents) outputs. Figure 5.3 shows the sampled two phase equivalent current i_{Sb} and its simulated response i_{Sb-sim}. The phase a current i_{Sa} is similar, but shifted by $\pi/(2n_p)$. The resulting phase b current i_{Sb-sim} from the simulation corresponds well with the actual measured current i_{Sb}.

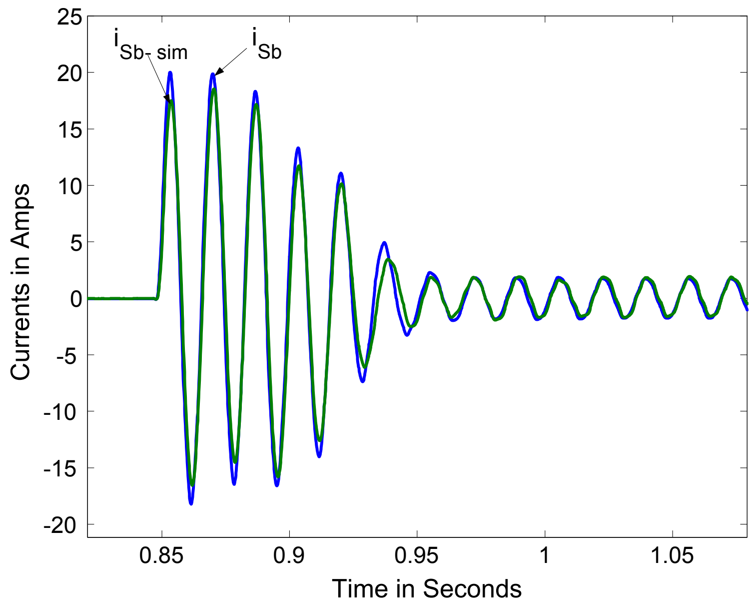

Figure 5.3: Phase b current i_{Sb} and its simulated response i_{Sb-sim}.

5.5 Summary

This chapter presents a differential-algebraic approach to estimating the rotor time constant without a speed sensor. The experimental results demonstrate the practical use of this method. This method is not applicable in steady state. It is also shown that a standard least-squares approach is not applicable in steady state.

Chapter 6

Conclusion and Future Work

6.1 Conclusion

This research produced a differential-algebraic approach to speed and parameter estimation of an induction motor. The speed estimator entails using an algebraic estimate of the speed to stabilize a dynamic speed observer. This work presented a characterization of the observability of the rotor speed of an induction motor based on input and output measurements (stator voltages and currents). This was done in terms of the speed being the solution to some polynomial equations whose coefficients were functions of the input/output measurements and their derivatives. The singularities of these algebraic equations (i.e., whether or not the leading coefficient is zero) were characterized under steady-state conditions. The algebraic estimate of the speed was then used to stabilize a dynamic (Luenberger type) speed observer. The new observer does not require any sort of "slowly varying" speed assumption and is stable. This sensorless speed controller shows potential for speed estimation at low speeds under full load.

The speed observer requires the value of T_R, which can vary due to Ohmic heating. Differential-algebraic approach was also used in the rotor time constant T_R estimation. It shows the potential for the estimation of T_R without motor speed information. This parameter estimation method does not require any sort of "slowly varying" speed assumption, but rather requires $d\omega/dt \neq 0$.

6.2 Future work

The differential-algebraic approach to speed and rotor time constant T_R estimation is computationally intense. Also, the algebraic speed observer and T_R estimation includes the third-order derivative of the stator current. Future possible work includes

- Finding a way to simplify the computation and reduce the order of the derivative of the stator current.
- Studying on-line estimation of T_R and using the estimated T_R in the speed estimator.
- Closed-loop experiment where the estimated speed is fed back to the field-oriented controller.

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