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An ARIMA Supply Chain Model with a Generalized Ordering Policy

Vuttichai Chatpattananan
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Vuttichai Chatpattananan entitled "An ARIMA Supply Chain Model with a Generalized Ordering Policy." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Management Science.

Kenneth Gilbert, Major Professor

We have read this dissertation and recommend its acceptance:

Mellissa Bowers, Mandyam Srinivasan, Hamparsum Bozdogan

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Vice Chancellor and
Dean of Graduate Studies

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An ARIMA Supply Chain Model with a Generalized Ordering Policy

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Vuttichai Chatpattananan

December 2004

To My Mom

Acknowledgments

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Abstract

This dissertation develops models to understand and mitigate the bullwhip effect across supply chains. The models explain the bullwhip effect that is caused by using the up to target ordering policy in standard Material Requirement Planning (MRP) systems. In the up to target ordering policy, the orders are directly driven by actual demand oscillations. We develop the models in AutoRegressive Integrated Moving Average (ARIMA) forms for a single demand item in a tandem line supply chain model. Different from supply chain models in current literature that are based on the assumption of the up to target ordering policy with some specific ARIMA models and specific numbers of stages in supply chain, the up to target ordering policy models in this dissertation can be applied to any ARIMA demand, any ordering lead time, and any number of stages in supply chains to derive the closed form expressions of the variation in inventory and the variation in orders. In addition, we propose the generalized ordering policy in which the up to target ordering policy is a special case. The generalized ordering policy permits manufacturers to smooth orders with the guaranteed stationary inventory in which smoothing orders is regarded as an effective way to mitigate the bullwhip effect. With the generalized ordering policy, manufacturers can control the tradeoffs between the variation in inventory and the variation in differencing orders which is stationary due to differencing. The generalized order models can be applied to any ARIMA demand, any ordering lead time, and any smoothing period. Two special cases of the generalized ordering policy are also illustrated. One is the previously mentioned up to target ordering policy that minimizes the variation in inventory. Another is the smoothing ordering policy that minimizes the variation in differencing orders. We also provide generic formulas to determine the optimal smoothing weights in the smoothing ordering policy for ARIMA$(p, 0, q)$ and ARIMA$(p, 1, q)$ orders. Finally, this dissertation introduces the bounded MRP following the rate based planning concept. We propose a simulation based technique to set the bounds into standard MRP systems for exponential smoothing or ARIMA(0,1,1) demand. With this bounded MRP, we can mitigate the bullwhip effect and reduce the conflict between production planning and infeasible capacity planning.

Keywords and Phrases: *Bullwhip Effect; Supply Chain Modeling; Production Planning; Capacity Planning; Material Requirement Planning; Rate Based Planning; Generalized Ordering Policy; Infinite Loading; Order up to a Target; Smoothing Production; AutoRegressive Integrated Moving Average; and Exponential Smoothing.*

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Chapter 1

Introduction

Supply chain management mainly deals with the management of materials and information across the supply chain, from suppliers to manufacturers to distribution (warehouses and retailers), and ultimately to the consumer. The objective of supply chain management is to provide a flow of relevant information that will enable suppliers to provide an uninterrupted and precisely timed flow of materials to customers. In other words, the goal of any effective supply chain management system is to reduce inventory with the assumption that products are available when needed. Figure 1.1 shows a diagram of information flow and materials flow in a tandem line supply chain.

The purpose of supply chain modeling is to gain an understanding of the dynamics of flows in supply chains. The ultimate goal is to apply these models in designing and managing supply chains.

On the manufacturing floor, a major barrier to reduced inventory and bet-

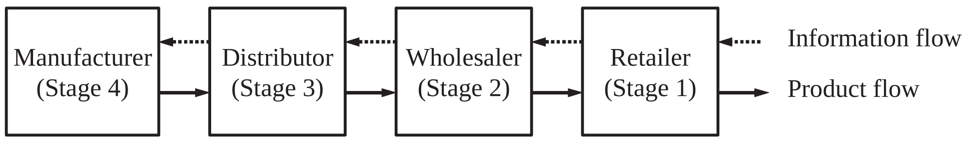

Figure 1.1: Flows of Supply Chain Diagram

ter flow is congestion, i.e. some form of queuing due to variation, capacity constraints and batching. Factory physics (Hopp and Spearman 2000) provides theoretical models of the impact of these factors on the flow of products in processes. The discipline has reached the level of maturity needed to become a valuable tool in designing and managing manufacturing and service processes.

In supply chains, additional factors such as time delays and imperfect information become important. For example, there will typically be a lead time between the time when an order is placed and the time that the order is available to meet customer demand. This means that the order is placed to meet future demand and therefore is made with imperfect information. In this environment, a major barrier to reduced inventory and better flow is the bullwhip effect, a phenomenon in which variation in customer demand results in progressively larger variation in orders and inventory at the upstream stages of the supply chain.

Supply chain models have provided some basic insights into the dynamics of the supply chain. For example, they have shown that better inventory accuracy, improved forecasting and information sharing alone are incomplete.

(Many supply chain strategies are focused primarily on improving the quality of information.) To eliminate the bullwhip effect, it is also necessary to reduce lead times.

However there are many other unanswered questions about dynamics of supply chains. The goal of this dissertation is to develop theoretical models to provide additional insights into these questions.

Typically, in lean manufacturing, final assembly produces (in the short term) at a constant rate and serves as the drumbeat for the entire manufacturing operation. A finished goods inventory (or alternately an order queue) buffers final assembly from the day to day variations in customer demand. Operations upstream from this drumbeat process are driven by pull signals.

In planning the rate of this drumbeat for future weeks, there are several goals. First, the changes in the drumbeat should be gradual since it would be difficult for the manufacturing process to accommodate quick and drastic changes. Second, the drumbeat should be relatively close to the rate that has been planned (i.e. forecasted) earlier, so that the internal operations and the external suppliers can prepare in advance for changes. Third, there must be adequate flexibility to change the rate to adjust quickly to changes in the level of customer demand. Obviously, the third objective may conflict with the first two.

Rate based planning is a method for planning the rate of the drumbeat while attempting to achieve a satisfactory tradeoff among these three goals. In rate based planning, a plan of weekly production rates is specified, typically for twelve weeks into the future. This plan is updated weekly.

Rather than freeze this plan (as is often done in traditional planning and scheduling methods), rate based planning allows the actual production to vary within specified ranges around the plan. Let P_t denote the actual production in week t and let $\hat{P}_t(l)$ denote the production for week $t+l$ that is planned in week t. In rate based planning, constraints are imposed in the form of $(1-c_s)*\hat{P}_{t-s}(s) \leq P_t \leq (1+c_s)*\hat{P}_{t-s}(s)$, $s = 1, 2, 3, ..., S$. S is the number of periods in the planning horizon. The sequence c_s, $s = 1, 2, 3, ..., S$ is called the flexibility requirements profile, where $c_s < 1$ (see Srinivasan 2004).

The flexibility requirements profile gives guaranteed restrictions on deviations of actual production from the planned values given in earlier weeks (i.e., guaranteed production forecast accuracy) and the same amount of freedom to flex in response to variation in customer demand. However, there have been no mathematical models of the relationships between the flexibility requirements profile and other performance measures such as schedule smoothness and inventory variation.

Clearly understanding these relationships would be a valuable aid in defining the flexibility requirements profile, since errors in defining this profile can be costly. For example, if the flexibility profile is too tight, is creates delays in responding to changes in demand and therefore creates a bullwhip effect. On the other hand if it is to loose it allows production to vary in response to random week-to-week variation in customer demand and therefore results in an erratic schedule. Toney 2004 has done numerical simulations to study these relationships. However there has been no previous work of developing theoretical models of these relationships.

One goal of this dissertation is to provide insights into the tradeoffs among production forecast accuracy, schedule smoothness and inventory variation. We define a model that has a generalized ordering policy. Using this model we can, for example, optimize the parameters of the ordering policy to minimize the week-to-week variation in production and observe the resulting inventory variation and production forecast accuracy.

The following sections clarify the previously introduced topics in details.

1.1 The Bullwhip Effect in Supply Chains

The problem in unplanned demand oscillations in the supply chains that causes inventory overstock or stock-outs and creates distortions interrupting the flow of the supply chain is known as the "Bullwhip Effect." This is caused by a disturbance or lump of demand oscillates back through the supply chain often resulting in huge and costly disturbances at the supplier end of the chain. Often, these demand oscillations cost manufacturing to acquire and expedite more raw materials and reschedule production in order to avoid inventory overstock or stock-outs.

Figure 1.2 shows a well known picture of the bullwhip effect from the studies of Lee, Padmanahan and Whang 1997. The studies show the bullwhip

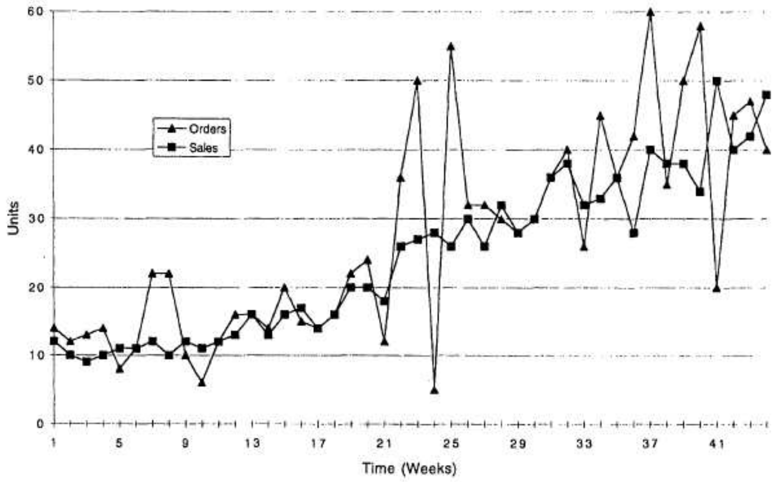

Figure 1.2: Bullwhip Effect in Diaper Industry (Lee, Padmanahan and Whang 1997)

effect in the diaper industry between diaper demand and the diaper supply chain although the demand should be easy to estimate directly from the number of new born babies in a region.

In the bullwhip effect, orders to the supplier tend to have a larger variance than sales to the buyer and this distortion propagates upstream in an amplified form. The distortion of demand data implies that the manufacturer who only observes its immediate order data will be misled by the amplified demand patterns. Theoretically, let σ_C, σ_R, σ_M, and σ_S denote the standard deviations of weekly order sizes by the consumer, retailer, manufacturer, and supplier, respectively. The bullwhip effect means that

$$\sigma_C < \sigma_R < \sigma_M < \sigma_S.$$

That is, small changes in customer orders result in moderate spikes in retailer orders, creating large spikes in wholesaler orders, which finally results in even larger spikes in manufacturer orders to suppliers. Even when consumer demand is stable, order sizes are highly variable and the variability increases as one moves upstream. This has serious cost implications. For example, the manufacturer incurs excess raw material cost due to unplanned purchases of suppliers, additional manufacturing expenses created by excess capacity, inefficient utilization and overtime, excess warehousing expenses and additional transportation costs due to inefficient scheduling and premium shipping rates. This leads to huge inefficiencies, as each part of supply chain stocks inventory to prepare for variability. The costs of the bullwhip effect ranged form $14 billion for the food service industry (Troyer 1996) to $30 billion for the grocery industry (Kurt 1993).

Lee, Padmanahan and Whang 1997 shows how four rational causes create the bullwhip effect:

1. Demand signal processing: if demand increases, retailers order more than the actual demand needed in anticipations of future supply shortages.

2. Rationing game: supply shortages cause retailers to order more than the actual forecasts in the hope of receiving larger shares.

3. Order batching: high ordering setup costs motivate retailers to order in large batches.

4. Manufacturer price variation: low prices cause retailers to order in large quantities to stock inventory to handle the price uncertainties.

Lee, Padmanahan and Whang 1997 also suggests several ways to react to the bullwhip effect:

1. Avoid multiple demand forecast updates. Point of sale demand data or information sharing across the supply chain by passing demand data from downstream through upstream can reduce highly variable demand and long resupply lead times.

2. Eliminate gaming in shortage situations. Suppliers can allocate product based on past sales records, rather than on orders, so customers don't exaggerate their orders. Another way is penalty on return policies, so retailers are less likely to cancel orders.

3. Break order batches. Electronic data interchange can reduce the cost of placing orders.

4. Stabilize prices. Everyday low price can reduce the frequency and level of wholesale price discounting to prevent customers from stockpiling.

From the four sources of the bullwhip effect proposed by Lee, Padmanahan and Whang 1997, we distinct our mathematical models in this dissertation to the demand signal processing aspect. The other aspects in rationing game, order batching, and price variation can be developed in future research.

1.2 Material Requirement Planning and Capacity Planning

Material Requirement Planning (MRP) is used to tell when to order and when to manufacture, based on demand. MRP goals are to minimize inventory levels and maintain delivery schedules.

MRP-based production is often referred to as a *push-type* production because job orders are initiated according to schedules generated by MRP systems that push the jobs from one operation to the next throughout the process. This is in contrast to *pull-type* production where jobs are initiated from downstream operations, pulling from one operation to the next. Although *pull-type* production, such as Just In Time (JIT) or lean production, has advantages over *push-type* production in reducing waste, it is a tool only for a short-range production planning. MRP-based systems are still needed as tools for medium-range and long-range production planning.

Standard MRP systems assume materials or capacities to be infinite loading. In practice, MRP systems may not be feasible because of the capacity limitations unless manufacturers have excess capacities or manufacturers can easily flex production capacity in response to lumpy demand. Standard MRP systems do not provide capacity planning. Traditionally, manufacturers use rough cut capacity planning (RCCP) to check the feasibility of the capacities. If the capacities are not feasible, the problems must be solved manually by changing the timing of production requirements or updating MRP which is a time-consuming process for most manufacturers of any size or complexity.

Hence, both *push-type* production and *pull-type* production have difficulties to encounter the bullwhip effect since *push-type* production would have a conflict between production planning and capacity planning while *pull-type* production is suitable only for a stable and predictable production process not for a volatile production process driven by lumpy demand.

1.3 Rate Based Planning

Rate based planning is a way to level the production that can be done by smoothing the productions. Under rate based planing, demand variation is accommodated with changes in capacities. In this way, manufacturers are warned in advance so that planning should significantly reduce the level of surprises at or near the build date. Hence, rate based planning mitigates the bullwhip effect in the supply chain and alleviates the conflict between production planning and infeasible capacity planning in using MRP.

Rate-based planning can be managed by rates and bounds. The bounds for rate based planning reflect day-to-day or week-to-week production quantities that will not be an exact set amount. Rather, the production will vary around a rate but within a range of production forecasts.

Figure 1.3 shows a flexibility requirements profile for rate base planning in which the flexible capacity boundaries are planned as a function of lead time. The amount of flexible capacity depends on its planning horizon in

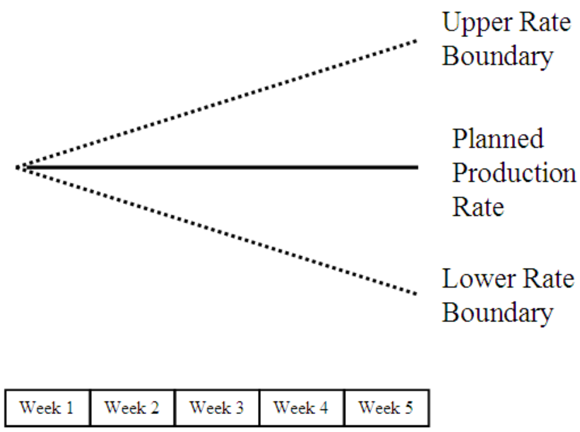

Figure 1.3: Flexibility Requirements Profile

which the more planning time manufacturers have, the greater their ability to respond to variations in production. The additional lead time provides time to increase or decrease capacity in more significant amounts. As the execution time period draws nearer, the rate-based plan constrains the production rate into more narrow boundaries. Thus, at execution, the actual productions are inside the near-term rate-based execution band.

1.4 Time Series Forecasting Techniques

Manufacturers need forecasting techniques for demand modeling in the production planning. This dissertation uses the univariate time series forecasting techniques for demand modeling since we consider only a single demand item. The demand models in this dissertation are developed following the Box-Jenkins methodology called the univariate autoregressive integrated moving average (ARIMA). Some ARIMA models are closely related to the exponential smoothing models.

A time series is a set of ordered observations equally spaced over time or space

$$Z_1, Z_2, Z_3, ..., Z_{t-1}, Z_t, Z_{t+1}, ...$$

The purpose of the time series modeling is to find a model that accurately represents the past and future pattern of time series

$$Z_t = Pattern + e_t \tag{1.1}$$

where Z_t can be the observed demand at time t. The pattern can be random, seasonal, trend, cyclical, intervention, or all combinations. Hence, a time series Z_t is a linear function of the past actual values and random shocks (i.e., error terms) in the form:

$$Z_t = f[Z_{t-i}, e_{t-i}] + e_t, \text{ where } i > 0.$$

The purpose of time series analysis is to extract all possible information (pattern) from a time series so that e_t's are distributed as white noise. By definition, white noise is identical and independently distributed (IID), having no patterns with a zero mean and an error variance that is lower than the variance of Z_t.

A major aspect of time series modeling is the consideration of the autocorrelated pattern of the time series. While correlation measures the degree of dependence or association between two variables, autocorrelation means that the value of a series in one period is related to the value of itself in previous periods. With autocorrelation, there is an automatic correlation between observations in a series. Frequently, this autocorrelation results from the momentum of a series.

The ARIMA model has gained enormous popularity in many areas and research practice due to its power and flexibility (Hoff 1983, Pankratz 1983, Vandaele 1983). However, it is a complex technique which is not easy to use and requires a great deal of experience (Bails and Peppers 1982). The ARIMA method is appropriate only for a time series that is stationary (i.e., its mean, variance, and autocorrelation should be approximately constant through time) and it is recommended that there are at least 50 observations in the input data. It is also assumed that the values of the estimated parameters are constant throughout the series.

The specific number and type of ARIMA parameters to be estimated need to be predetermined in the model identification phase. The major tools used in the identification phase are plots of the series, correlograms of auto correlation (ACF), and partial autocorrelation (PACF). The decision is not straightforward and in less typical cases requires not only experience but also a good deal of experimentation with alternative models as well as the technical parameters of ARIMA.

An ARIMA(p, d, q) model following Box, Jenkins, Reinsel and Jenkins, 1994 has three types of estimated parameters in the ARIMA model; which are: the autoregressive parameters (p), the differencing parameters (d), and moving average parameters (q). Due to the complexity of ARIMA modeling, the degrees of p, d, q commonly used in practice are no more than 2.

There are several different methods for estimating the parameters in which all of these methods should produce very similar estimates but may be more or less efficient for any given model. In general, a function minimization algorithm (such as quasi-Newton method which is a nonlinear estimation method, is used during the parameter estimation phase by maximizing the likelihood of the observed series given the parameter values.

A seasonal ARIMA$(p, d, q)(ps, ds, qs)$ model is a generalization and extension of the simple ARIMA(p, d, q) model in which a pattern repeats seasonally over time. For a seasonal ARIMA model, six types of parameters need to be estimated. In addition to the non-seasonal parameters (p, d, q), seasonal parameters (ps, ds, qs) also need to be estimated. Analogous to the simple ARIMA parameters, these seasonal parameters are: seasonal autoregressive parameters ps, seasonal differencing parameters ds, and seasonal moving average parameters qs. For example, the model (0,1,2)(0,1,1) describes a model that includes no autoregressive parameters, 1 differencing parameter, 2 moving average parameters, no seasonal autoregressive parameters, 1 seasonal differencing parameter, 1 seasonal moving average parameter.

Exponential smoothing models are closely related to ARIMA(p, d, q) models. The model ARIMA(0,1,1) is the single exponential smoothing model.

The model ARIMA(0,2,2) is the double exponential smoothing model. The model ARIMA(0,3,3) is the triple exponential smoothing model. The single exponential smoothing model assumes that the series displays a time-varying mean with no trend. The double exponential smoothing model assumes that the series displays a time-varying level with a linear trend. The triple exponential smoothing model assumes that the series displays a time-varying level with a quadratic trend.

Exponential smoothing models are widely used as a time series forecasting method. The simplest form of exponential smoothing is single exponential smoothing. In single exponential smoothing, the *Pattern* in (1.1) is computed using a moving average, where the current and immediately preceding observations are assigned greater weight than the respective older observations. The specific formula for single exponential smoothing is

$$\hat{Z}_t = \lambda Z_t + (1 - \lambda)\hat{Z}_{t-1}.$$

When applied recursively to each successive observation in the series, each new smoothed value, or forecast value $\hat{Z}_t$, is computed as the weighted average of the current observation Z_t and the previous smoothed observation $\hat{Z}_{t-1}$. The previous smoothed observation was computed in turn from the previous observed value and the smoothed value before the previous observation, and so on. Thus, in effect, each smoothed value is the weighted average of the previous observations, where the weights decrease exponentially depending on the value of parameter λ.

1.5 Organization of the Dissertation

This dissertation consists of six chapters.

Chapter 2 is an overview of the current mathematical modeling techniques that explain the bullwhip effect in the supply chain.

Chapter 3 describes the ARIMA supply chain models for the up to target ordering policy used in standard MRP systems (Gilbert 2004). The models can be applied to any ARIMA demand, any ordering lead time, and any number of stages in supply chains.

Chapter 4 proposes the generalized ordering policy that permits manufacturers to control the tradeoffs between the variation in inventory and the variation in differencing orders which is stationary by differencing. The generalized ordering policy can be applied to any ARIMA demand, any ordering lead time, and any predetermined smoothing period. The generalized ordering policy includes the up to target ordering policy introduced in chapter 3 that minimizes the variation in inventory and the smoothing ordering policy that minimizes the variation in differencing orders. This chapter also provides generic formulas to determine the optimal smoothing weights in the smoothing orders when the demand models are ARIMA$(p, 0, q)$ and ARIMA$(p, 1, q)$.

Chapter 5 illustrates the unbounded MRP tables using the up to target ordering policy and the smoothing policy. This chapter also proposes the bounded MRP using the rate based planning concept by proposing a simulation-based technique to set the bounds that can be incorporated into standard (order up to target) MRP tables for the single exponential smoothing or ARIMA(0,1,1) demand.

Chapter 6 gives the conclusion and suggests directions for future research.

1.6 Contributions

This dissertation makes several contributions to the supply chain management field.

First, all supply chain models in current literature are based on the assumption that up to target ordering policy is used. We propose the generalized ordering policy that includes the up to target ordering policy as a special case. The generalized ordering policy permits manufacturers to smooth orders arbitrarily to mitigate the bullwhip effect by controlling the tradeoffs between the variation in inventory and the variation in differencing orders (which is stationary by differencing) by changing the smoothing weights with the guaranteed stationary inventory. The generalized order models can be applied to any type of demand, any ordering lead time, and any desired smoothing period. With the generalized order models, manufacturers can know the variation in inventory and the variation in orders theoretically such that manufacturers can set the safety stock corresponding to the variation in inventory or set the production plan corresponding to the variation in orders.

Second, supply chain models in current literature that explain how the bullwhip effect propagates across supply chains only applied to some specific ARIMA models and specific numbers of stages in supply chain, the order up to target in chapter 3 (Gilbert 2004) can be responded to any ARIMA demand, any ordering lead time, and any number of stages in supply chains. We also show that the up to target ordering policy is a special case of the generalized ordering policy that minimizes the variation in inventory.

Third, we describe the smoothing ordering policy which is a special case of the generalized ordering policy that minimizes the variation in differencing orders. We also provide generic formulas to determine the optimal smoothing weights for the smoothing ordering policy for any ARIMA$(p, 0, q)$ and ARIMA$(p, 1, q)$ demand.

Finally, we propose the bounded MRP system corresponding to the rate based planning concept for single exponential smoothing or ARIMA(0,1,1) demand. We provide a guideline using a simulation based technique for manufacturers to set the bounds such that the variation in week-to-week orders is significantly reduced compared with the standard MRP tables without bounds. Hence, the bounded MRP not only mitigates the bullwhip effect but also reduces the conflict between production planning and infeasible capacity planning. We also provide a MATLAB program that automatically sets the bounds for a given ARIMA(0,1,1) demand data for any ordering lead time and any smoothing period.

Chapter 2

Previous Works and Literature Review

Mathematical models of supply chains have provided some very practical managerial insights about supply chain dynamics. These models have lead to an understanding of the bullwhip effect, a phenomenon in which the variation in orders and inventory grow at successive stages of the supply chain.

Forrester 1961 documented case studies and computer simulations of the bullwhip effect. Sterman 1989 discussed the bullwhip effect using the beer game which is an experiential supply chain simulation. Both papers provides the understanding of the causes and managerial implications of the bullwhip effect. Sterman 1989 proposed, through illustration in the beer game, that irrational decisions lead to bullwhip effect. Lee, Padmanahan and Whang 1997 argued that rational actions still result in bullwhip effect.

Lee, Padmanahan and Whang 1997 proposed four sources of the bullwhip effect: demand signal processing, rationing game, order batching, and price variation. The paper also proposed the actions that can be taken to mitigate the impact of the four sources of the bullwhip effect. The paper analyzed mathematical models and shows that with a single stage, AR(1) demand, and lead time is 1, the variation in orders exceeds the variation in demand. Lee, So, and Tang 2000 used an AR(1) demand process and arbitrary lead times to explicitly model the orders and inventory in a two-stage supply chain. They applied these results in determining the value of information in a two-stage supply chain.

Chen et. al., 2000 quantified the bullwhip effect by using two level system with order up to policy. The demand is forecast with p-period moving average. Where

$$\text{Order quantity} = \text{Mean demand during lead time} + Z_{.95}(\text{Standard deviation of lead time demand forecast error})$$

then

$$\frac{\text{Variance(Retailer order quantity)}}{\text{Variance(Customer order quantity)}} \geq 1 + 2L/p + 2(L/p)^2$$

where L is the lead time between when an order is placed and when it is delivered and p is the period used to forecast demand. The paper proposed that variability increases with lead time, decreases with forecast horizon and decreases with correlation in demand.

Graves 1999 used an ARIMA(0,1,1) time series of demand and an arbitrary lead time to derive the ARIMA(0,0,1) series of orders and the distribution of inventory. He used these results to recursively model multi-stage system. Graves 1999 developed a model of a multistage supply chain, in which the order at a given stage becomes the demand for the upstream stage. The lead time at each stage is permitted to be an arbitrary number of periods. Graves assumes that the customer demand is ARIMA(0,1,1) or exponential smoothing.

These supply chain models have been based on a particular time series model for demand, for example, independent identically distributed, autoregressive or exponential smoothing. These models have been single stage models, i.e., given the time series of demand, they modeled the order and inventory at a single stage. In many instances, the lead time was assumed to be one period.

Two recent supply chain models have used more general demand models. Aviv has developed a single stage supply chain model which uses a stage space approach to model demand. Gilbert 2004 has developed s multiple stage supply chain model where the demand is assumed to be any ARIMA time series.

Aviv et. al., 2002 proposed a methodology for assessing the benefits of various types of information sharing agreements between members of a supply chain. A Kalman filter was used to model demand and the method was applied to a two-stage supply chain.

Gilbert 2004 proposed a supply chain model, based on ARIMA time series models. The model can be applied with any given lead time and any ARIMA demand and explicitly provides the ARIMA models of orders and inventory. It applies to supply chain with any numbers of stages, under the assumption that the order for a given stage becomes the demand for the stage immediately upstream.

All of these previous models are based on the assumption of the up to target ordering policy used in standard MRP systems. This dissertation introduces a generalized ordering policy, having infinite loading ordering policy as a special case. This model can be used to derive the optimal smoothing ordering policy, i.e., the policy that minimizes the week to week variation in production. In particular, we develop closed form expressions to determine the optimal smoothing weights for the smoothing ordering policy for ARIMA$(p, 0, q)$ and ARIMA$(p, 1, q)$ demand models. We demonstrate how this policy can be used to set bounds into standard MRP tables for ARIMA(0,1,1) or exponential smoothing demand.

Chapter 3

The Standard MRP Ordering Policy

This chapter introduces a method to explain the bullwhip effect in supply chains generated from demand oscillations provided that the order up to a target is used in the ordering policy (Gilbert 2004). The method assumes that demand can be modeled with the autoregressive integrated moving average (ARIMA) methodology. The orders are generated using an order up to a target which is the standard MRP/ERP system for production planning in manufacturing. The method provides a means of quantifying the impact of lead-time reduction on required inventory levels, on required manufacturing flex capacity and on the variability of orders to upstream suppliers.

Specifically, this chapter gives a method for predicting the range of variation in inventory as a means to determine the target safety stock and the range of variation in orders as a means to determine the flex capacity requirements.

Since the inventory absorbs the variation in demand over the entire lead-time, the variation in inventory will be larger than the standard error of the demand forecast. Orders absorb not only the change in demand but also the change in the forecast over the lead-time. Therefore the variation in orders will be larger than the variation in customer demand. This multiplication of variation in orders is sometimes called the bullwhip effect. (See Forester 1961, Sterman 1988, Lee, Padmanabhan and Whang 1997, and Gilbert 2004.)

3.1 Introduction

With a customer demand for an item sold by a retailer (or manufacturer or distributor), the retailer sends weekly orders for these items to the supplier, but there is a lead-time that elapses between the placement of the order and delivery.

The retailer would like to know how to choose a safety stock inventory target to get some desired level (say three-sigma) of protection against stock-outs. The retailer would also like to be able to provide the upstream supplier long-term forecasts of future orders, along with a commitment that the actual orders will not deviate from their forecast by more than a specified range. (See Tsay 1999, and Tsay and Lovejoy 1999) for a discussion of this type of agreement.)

Thus the retailers would like to know:

- What is the predicted range of variation in ending inventory?
- What is the predicted range of variation in actual orders around the long-term forecasts?

The results of this chapter apply with any given any lead time L and any ARIMA(p, d, q) demand series and explicitly provides the ARIMA model of orders and inventory. It applies to supply chains with any numbers of stages, under the assumption that the orders for a given stage become the demand for the stage immediately upstream.

3.2 The Assumptions of the Method

This section introduces the autoregressive integrated moving average (ARIMA) methodology following the notation of Box, Jenkins, Reinsel and Jenkins 1994 as a way for demand modeling. With the demand in an ARIMA form, we can derive its corresponding forecast demand. Then we present the interrelationship between the inventory, demand, and orders in which we propose the standard MRP/ERP ordering policy as the order policy.

3.2.1 Demand

We will use the general class of ARIMA time series models to model the demand in our supply chain model, and then derive ARIMA models of the time series of inventory and orders. Many of the specific models (i.e., the autoregressive model, the exponential smoothing model and the independent identically distributed (IID) model) that have been used by previous research to model demand are special cases of the ARIMA model.

We follow the notation of Box, Jenkins, Reinsel and Jenkins 1994. We will assume that the time series of demand, if it is stationary, can be represented by an autoregressive moving average model, ARMA(p, q) where p is the number of autoregressive terms and q is the number of moving average terms:

$$\begin{aligned} Z_t = \mu + \phi_1(Z_{t-1} - \mu) + \phi_2(Z_{t-2} - \mu) + ... + \phi_p(Z_{t-p} - \mu) \\ + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - ... - \theta_q a_{t-q}. \end{aligned} \tag{3.1}$$

Where Z_t is the demand at time t. μ is the process average. a_t is a time series of independent identically distributed random variables with expected value $E(a_t) = 0$ and variance $V(a_t) = \sigma_a^2$. The series of random shocks, a_t, is referred to as the noise series.

B will denote the time series backshift operator where

$$BZ_t = Z_{t-1}; \text{ hence } B^n Z_t = Z_{t-n}.$$

Then the stationary time series can be written as:

$$\phi(B)(Z_t - \mu) = \theta(B)a_t.$$

where

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p$$

is called the autoregressive operator of order p.

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - ... - \theta_q B^q$$

is called the moving average operator of order q.

We will assume that the time series of demand, if it is nonstationary, can be modeled by differencing to obtain a stationary series.

∇ will be used to denote the difference operator:

$$\nabla Z_t = (1 - B)Z_t \text{ and } \nabla^d Z_t = (1 - B)^d Z_t.$$

A nonstationary demand series Z_t will then be represented by an autoregressive integrated moving average or ARIMA(p, d, q) series:

$$\phi(B)\nabla^d Z_t = \theta(B)a_t$$
$$\varphi(B)Z_t = \theta(B)a_t$$

where

$$\varphi(B) = \phi(B)\nabla^d$$

is called the autoregressive integrated operator of order $p + d$.

The time series Z_t can also be represented as a linear transfer function of the noise series

$$Z_t = \mu + \psi(B)a_t$$

where

$$\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + ...$$

$\psi(B)$ can be computed as $\psi(B) = \theta(B)/\varphi(B)$. Therefore, the ψ weights of the ARIMA process can be determined recursively by equating coefficients of B as follows:

$$\begin{aligned}
\psi_1 &= \varphi_1\psi_0 - \theta_1 \\
\psi_2 &= \varphi_1\psi_1 + \varphi_2\psi_0 - \theta_2 \\
\psi_3 &= \varphi_1\psi_2 + \varphi_2\psi_1 + \varphi_3\psi_0 - \theta_3 \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots \\
\psi_j &= \varphi_1\psi_{j-1} + \varphi_2\psi_{j-2} + ... + \varphi_{p+d}\psi_{j-p-d} - \theta_j
\end{aligned} \tag{3.2}$$

where $\psi_0 = 1$, $\psi_j = 0$ for $j < 0$ and $\theta_j = 0$ for $j > q$.

For $j > max\{p + d - 1, q\}$ the ψ's satisfy the difference equation

$$\psi_j = \varphi_1\psi_{j-1} + \varphi_2\psi_{j-2} + ... + \varphi_{p+d}\psi_{j-p-d}.$$

Then, Z_t can be expressed in random shock form as

$$Z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + ... \tag{3.3}$$

3.2.2 Forecast Demand

Suppose that the lead-time L is the number of periods that elapse between the time an order is placed and the time the items ordered are received in inventory. This lead-time could be due to order processing, manufacturing flow time transportation time or any other types of delays. At origin time t, we are to make a forecast $\hat{Z}_t(L)$ of Z_{t+L}. Z_{t+L} in the random shock form is

$$Z_{t+L} = \sum_{j=0}^{\infty} \psi_j a_{t+L-j}.$$

Let denote $E[Z_{t+L}|Z_t, Z_{t-1}, ...]$, which is the conditional expectation of Z_{t+L} given knowledge of all the Z's up to time t, be $E_t[Z_{t+L}]$. Since a_t are a sequence of independent random variables with mean zero and variance σ_a^2, then $E[a_{t+j}|Z_t, Z_{t-1}, ...] = 0$, $j > 0$. Thus,

$$\hat{Z}_t(L) = \psi_L a_t + \psi_{L+1} a_{t-1} + ... = E_t[Z_{t+L}].$$

We then have

$$\begin{aligned} Z_{t+L} &= (a_{t+L} + \psi_1 a_{t+L-1} + ... + \psi_{L-1} a_{t+1}) + (\psi_L a_t + \psi_{L+1} a_{t-1} + ...) \\ &= e_t(L) + \hat{Z}_t(L). \end{aligned}$$

where $e_t(L)$ is the error of the forecast $\hat{Z}_t(L)$ at lead time L.

In summary, let j be a nonnegative integer, $\hat{Z}_t(j)$ can be calculated from the conditional expectations of Z_{t+j}, following these rules:

1. $E_t[Z_{t-j}] = Z_{t-j}$; $j = 0, 1, 2, ...$

   The Z_{t-j} $(j = 0, 1, 2, ...)$, which have already happened at origin t, are left unchanged.

2. $E_t[Z_{t+j}] = \hat{Z}_t(j)$; $j = 1, 2, 3, ...$

   The Z_{t+j} $(j = 0, 1, 2, ...)$, which have not yet happened, are replaced by their forecasts $\hat{Z}_t(j)$ at origin t.

3. $E_t[a_{t-j}] = a_{t-j} = Z_{t-j} - \hat{Z}_{t-j-1}(1)$; $j = 0, 1, 2, ...$

   The a_{t-j} $(j = 0, 1, 2, ...)$, which have happened, are available from $Z_{t-j} - \hat{Z}_{t-j-1}(1)$.

4. $E_t[a_{t+j}] = 0$; $j = 1, 2, 3, ...$

   The a_{t+j} $(j = 0, 1, 2, ...)$, which have not yet happened, are replaced by zeros.

3.2.3 Inventory

We assume that the inventory of this item is managed by placing orders at the end of each period. The order placed in period t arrives in period $t+L$, where L is the lead-time. Thus in each period current period's demand is taken from inventory, the order placed L periods in the past is received into inventory and a new order is placed to be received L periods into the future.

The inventory, demand and orders are related by the equation:

$$I_t = I_{t-1} + O_{t-L} - Z_t \tag{3.4}$$

where I_t is the ending inventory of period t and O_t is the order placed at the end of period t.

(3.4) assumes that any unmet demand is backordered, i.e., the inventory is not restricted to non-negative values.

I_t can be expressed in another form by substituting

$$I_{t-1} = I_{t-2} + O_{t-L-1} - Z_{t-1}$$

into (3.4). By substituting recursively, we have

$$I_t = \sum_{j=0}^{\infty} O_{t-L-j} - \sum_{j=0}^{\infty} Z_{t-j}. \tag{3.5}$$

3.2.4 Order

We assume the order policy is the standard MRP/ERP ordering policy: The order is equal to the safety stock target minus the current ending inventory plus the sum of the forecast over the lead-time minus the outstanding orders:

$$\begin{aligned} O_t = T - I_t + \hat{Z}_t(1) + \hat{Z}_t(2) + ... + \hat{Z}_t(L) \\ - O_{t-1} - O_{t-2} - ... - O_{t-L+1} \end{aligned} \tag{3.6}$$

where T is the safety stock inventory target.

Simply stated, the ordering policy is to place the order in time period t, the quantity needed to bring the expected inventory to the target level T in time period $t + L$. The actual inventory will differ from the target by the sum of the forecast errors over the lead-time. This ordering policy can be viewed as an "order up to a target" policy (Veinott 1965).

In manufacturing viewpoint, this order policy is an "infinite loading" policy because it assumes that the manufacturer has an unlimited capacity to fulfill the order requested by the retailer.

3.3 The Method

Under the assumptions given in the previous section, the following results due to Gilbert (2002) hold:

- The time series of inventory I_t is ARIMA$(0, 0, L-1)$ or MA$(L-1)$.
- The time series of orders O_t is ARIMA$(p, d, max\{p+d, q-L\})$.

The parameters of both of these models can be explicitly determined from the parameters of the original ARIMA(p, d, q) model of the demand time series Z_t.

Theorem 1 *I_t is an MA(L-1) with mean T and standard deviation*
$\sigma_I = \sqrt{1 + (1+\psi_1)^2 + (1+\psi_1+\psi_2)^2 + ... + (1+\psi_1+\psi_2+...+\psi_{L-1})^2}\sigma_a$.

<u>*Proof*</u>

I_t differs from the inventory target T by the sum of the errors of the forecasts made at time $t-L$

$$I_t = T - e_{t-L}(1) - e_{t-L}(2) - ... - e_{t-L}(L).$$

These errors are given by

$$\begin{aligned}
e_{t-L}(1) &= Z_{t-L+1} - \hat{Z}_{t-L}(1) = a_{t-L+1} \\
e_{t-L}(2) &= Z_{t-L+2} - \hat{Z}_{t-L}(2) = a_{t-L+2} + \psi_1 a_{t-L+1} \\
e_{t-L}(3) &= Z_{t-L+3} - \hat{Z}_{t-L}(3) = a_{t-L+3} + \psi_1 a_{t-L+2} + \psi_2 a_{t-L+1} \\
&\vdots \qquad \vdots \qquad \vdots \\
e_{t-L}(L) &= Z_{t-L+L} - \hat{Z}_{t-L}(L) = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + ... + \psi_{L-1} a_{t-L+1}.
\end{aligned}$$

Combining the above terms gives

$$\begin{aligned}
I_t = T - a_t - (1+\psi_1)a_{t-1} - (1+\psi_1+\psi_2)a_{t-2} - ... \\
- (1+\psi_1+\psi_2+...+\psi_{L-1})a_{t-L+1}.
\end{aligned} \tag{3.7}$$

To write I_t in a standard ARIMA(p,d,q) form in (3.1), we define $a_t^{(I)} = a_t$ to give

$$I_t = T - a_t^{(I)} - \theta_1^{(I)} a_{t-1}^{(I)} - \theta_2^{(I)} a_{t-2}^{(I)} - ... - \theta_{L-1}^{(I)} a_{t-L+1}^{(I)}.$$

Thus I_t can be seen to be MA$(L-1)$ with parameters

$$\theta_i^{(I)} = 1 + \psi_1 + \psi_2 + ... + \psi_i;\ i = 1, 2, 3, ..., L-1$$

which is an MA$(L-1)$.

The expected value of the inventory $E[I_t] = T$.

The standard deviation of the inventory is

$$\sigma_I = \sqrt{1 + (1+\psi_1)^2 + (1+\psi_1+\psi_2)^2 + ... + (1+\psi_1+\psi_2+...+\psi_{L-1})^2}\sigma_a.$$

End of proof

Theorem 2 *O_t is an ARIMA$(p, d, max\{p+d, q-L\})$ with the form: $\phi(B)\nabla^d(O_t - \mu) = \theta^{(O)}(B)a_t^{(O)}$, or $\varphi(B)(O_t - \mu) = \theta^{(O)}(B)a_t^{(O)}$ with $\theta^{(O)}(B)$ of order $q^{(O)} = max\{p+d, q-L\}$ and $\sigma_{a^{(O)}} = K\sigma_a$ where $K = 1+\psi_1+\psi_2+...+\psi_L$.*

Proof

The theorem says that O_t has the same autoregressive operator as $\phi(B)$ and difference operator $\nabla^d(B)$ as the demand series Z_t, and in the stationary case has the same mean as the demand series. However, Z_t has a different moving average operator $\theta^{(O)}(B)$ and has a noise series that is a multiple of K times the original noise series of the demand process.

We will first write the O_t as a linear transfer function of the original noise series a_t. We then scale a_t to give O_t in standard ARIMA linear transfer function form $O_t = \psi^{(O)}(B)a_t^{(O)}$.

Then we observe that the coefficients of $\psi^{(O)}(B)a_t^0$ exhibit the same long term pattern as those of $\psi(B)$. Hence the ARIMA model for O_t written in difference equation form differs from that of Z_t only in the form of the MA terms. However, since $\psi^{(O)}(B)$ and $\varphi(B)$ are known we can solve for $\theta^{(O)}$

using the relationships of (3.2).

We first express the order time series O_t as a transfer function of the original noise series. In time period $t-1$, O_{t-1} was computed so that if the forecasts $\hat{Z}_{t-1}(1)$, $\hat{Z}_{t-1}(2)$, ..., $\hat{Z}_{t-1}(L)$, were perfect, then I_{t+L-1} would be exactly equal to the target T. In this case settings, O_t equal to $\hat{Z}_{t-1}(L)$ would make I_{t+L} also equal to the target T. However since the forecasts are not perfect, O_t must also account for the changes in the forecasts that occur between time $t-1$ and t, and account the difference between the actual demand Z_t and the forecast $\hat{Z}_{t-1}(1)$.

Therefore we write O_t as the difference between the actual demand and the forecast made one period earlier, plus the changes in the forecast for the next $L-1$ periods, plus the forecast of demand for period $t+L$.

$$\begin{aligned} O_t = & (Z_t - \hat{Z}_{t-1}(1)) + (\hat{Z}_t(1) - \hat{Z}_{t-1}(2)) + (\hat{Z}_t(2) - \hat{Z}_{t-1}(3)) + ... \\ & + (\hat{Z}_t(L-1) - \hat{Z}_{t-1}(L)) + \hat{Z}_t(L) \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} Z_t - \hat{Z}_{t-1}(1) &= a_t \\ \hat{Z}_t(1) - \hat{Z}_{t-1}(2) &= \psi_1 a_t \\ \hat{Z}_t(2) - \hat{Z}_{t-1}(3) &= \psi_2 a_t \\ \vdots \qquad & \vdots \qquad \vdots \\ \hat{Z}_t(L-1) - \hat{Z}_{t-1}(L) &= \psi_{L-1} a_t \end{aligned}$$

and

$$\hat{Z}_t(L) = \mu + \psi_L a_t + \psi_{L+1} a_{t-1} + \psi_{L+2} a_{t-2} + ...$$

Thus we obtain

$$O_t = \mu + (1 + \psi_1 + \psi_2 + ... + \psi_L) a_t + \psi_{L+1} a_{t-1} + \psi_{L+2} a_{t-2} + ... \qquad (3.9)$$

To check the validity of (3.7) and (3.9) with the assumption of (3.6), we substitute I_t, $\hat{Z}_t(i)$; $i = 1, 2, 3, ..., L$, and O_{t-i}; $i = 1, 2, 3, ..., L-1$ into (3.6). Then, we have

$$\begin{aligned} O_t &= T - I_t + \hat{Z}_t(1) + \hat{Z}_t(2) + ... + \hat{Z}_t(L) - O_{t-1} - O_{t-2} - ... - O_{t-L+1} \\ &= \mu + (1 + \psi_1 + \psi_2 + ... + \psi_L) a_t + \psi_{L+1} a_{t-1} + \psi_{L+2} a_{t-2} + ... \end{aligned}$$

which is the same as (3.9).

To establish a standard ARIMA transfer function form of O_t first define

$$K = 1 + \psi_1 + \psi_2 + ... + \psi_L$$

and define a noise process

$$a_t^{(O)} = K a_t$$

It follows that

$$\sigma_{a^{(O)}} = K \sigma_a.$$

Then it is possible to write O_t in standard linear transfer function form

$$\begin{aligned} O_t - \mu &= a_t^{(O)} + (\psi_{L+1}/K)a_{t-1}^{(O)} + (\psi_{L+2}/K)a_{t-2}^{(O)} + ... \\ &= \psi^{(O)}(B)a_t^{(O)} \end{aligned} \tag{3.10}$$

where

$$\psi_j^{(O)} = \psi_{L+j}/K,\ j = 1, 2, 3, ...$$

From the above equation, we observe that the coefficients of $\psi^{(O)}(B)$, adjusted by a time lag L, are proportional to the coefficients of $\psi(B) : \psi^{(O)}(B) = B^L\psi(B)/K$. Thus the difference equation form of O_t differs from that of Z_t only in the form of the MA operator $\theta^{(O)}$.

Multiply both sides of the above equation by $\varphi(B)$, we get

$$\varphi(B)(O_t - \mu) = \theta^{(O)}a_t^{(O)}.$$

If we solve for $\theta^{(O)}(B) = \varphi(B)\psi^{(O)}(B)$ using (3.2), we find $\theta^{(O)}(B)$ as follows

$$\begin{aligned} \theta_0^{(O)} &= 1 \\ \theta_1^{(O)} &= -\psi_{L+1}/K + \varphi_1 \\ \theta_2^{(O)} &= -\psi_{L+2}/K + \varphi_1\psi_{L+1}/K + \varphi_2 \\ \theta_3^{(O)} &= -\psi_{L+3}/K + \varphi_1\psi_{L+2}/K + \varphi_2\psi_{L+1}/K + \varphi_3 \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \\ \theta_j^{(O)} &= -\psi_{L+j}/K + \varphi_1\psi_{L+j-1}/K + \varphi_2\psi_{L+j-2}/K + ... + \varphi_{j-1}\psi_{L+1}/K + \varphi_j. \end{aligned}$$

To prove that the MA operator $\theta^{(O)}$ has the order $q^{(O)} = max\{p+d, q-L\}$, let $j = max\{p+d, q-L\}$.

<u>Case 1</u>: let $j = p + d$ or $p + d \geq q \geq q - L$.

From Z_t, the ψ's are

$$
\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \varphi_1\psi_0 - \theta_1 \\
\psi_2 &= \varphi_1\psi_1 + \varphi_2\psi_0 - \theta_2 \\
\psi_3 &= \varphi_1\psi_2 + \varphi_2\psi_1 + \varphi_3\psi_0 - \theta_3 \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \\
\psi_{q-L} &= \varphi_1\psi_{q-L-1} + \varphi_2\psi_{q-L-2} + ... + \varphi_{q-L}\psi_0 - \theta_{q-L} \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \\
\psi_{q-1} &= \varphi_1\psi_{q-2} + \varphi_2\psi_{q-3} + ... + \varphi_{q-1}\psi_0 - \theta_{q-1} \\
\psi_q &= \varphi_1\psi_{q-1} + \varphi_2\psi_{q-2} + ... + \varphi_q\psi_0 - \theta_q \\
\psi_{q+1} &= \varphi_1\psi_{q+1} + \varphi_2\psi_{q+2} + ... + \varphi_{q+1}\psi_0 \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \\
\psi_{p+d-1} &= \varphi_1\psi_{p+d-2} + \varphi_2\psi_{p+d-3} + ... + \varphi_{p+d-1}\psi_0 \\
\psi_{p+d} &= \varphi_1\psi_{p+d-1} + \varphi_2\psi_{p+d-2} + ... + \varphi_{p+d}\psi_0 \\
\psi_{p+d+1} &= \varphi_1\psi_{p+d} + \varphi_2\psi_{p+d-1} + ... + \varphi_{p+d}\psi_1 \\
\vdots \quad & \quad \vdots \qquad\quad \vdots
\end{aligned}
$$

hence, for $i \geq 0$

$$\psi_{p+d+i} = \varphi_1 \psi_{p+d+i-1} + \varphi_2 \psi_{p+d+i-2} + ... + \varphi_{p+d} \psi_i. \qquad (3.11)$$

For O_t, the $\theta^{(O)}$'s are

$$\begin{aligned}
\theta_1^{(O)} &= -\psi_{L+1}/K + \varphi_1 \\
\theta_2^{(O)} &= -\psi_{L+2}/K + \varphi_1 \psi_{L+1}/K + \varphi_2 \\
\theta_3^{(O)} &= -\psi_{L+3}/K + \varphi_1 \psi_{L+2}/K + \varphi_2 \psi_{L+1}/K + \varphi_3 \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots \\
\theta_{p+d}^{(O)} &= -\psi_{L+p+d}/K + \varphi_1 \psi_{L+p+d-1}/K + ... + \varphi_{p+d-1} \psi_{L+1}/K + \varphi_{p+d} \\
\theta_{p+d+1}^{(O)} &= -\psi_{L+p+d+1}/K + \varphi_1 \psi_{L+p+d}/K + ... + \varphi_{p+d} \psi_{L+1}/K \\
\theta_{p+d+2}^{(O)} &= -\psi_{L+p+d+2}/K + \varphi_1 \psi_{L+p+d+1}/K + ... + \varphi_{p+d} \psi_{L+2}/K \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots
\end{aligned}$$

hence, from (3.11), we have $\theta_{p+d+i}^{(O)} = 0$ for $i \geq 1$. In this case, the MA operator $\theta^{(O)}$ has the order $q^{(O)} = p+d$ when $j = max\{p+d, q-L\} = p+d$.

Case 2: let $j = q - L$ or $q - L \geq p + d$.

From Z_t, the ψ's are

$$
\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \varphi_1\psi_0 - \theta_1 \\
\psi_2 &= \varphi_1\psi_1 + \varphi_2\psi_0 - \theta_2 \\
\psi_3 &= \varphi_1\psi_2 + \varphi_2\psi_1 + \varphi_3\psi_0 - \theta_3 \\
\vdots \quad & \quad \vdots \quad\quad \vdots \\
\psi_{p+d-1} &= \varphi_1\psi_{p+d-2} + \varphi_2\psi_{p+d-3} + ... + \varphi_{p+d-1}\psi_0 - \theta p + d - 1 \\
\psi_{p+d} &= \varphi_1\psi_{p+d-1} + \varphi_2\psi_{p+d-2} + ... + \varphi_{p+d}\psi_0 - \theta_{p+d} \\
\psi_{p+d+1} &= \varphi_1\psi_{p+d} + \varphi_2\psi_{p+d-1} + ... + \varphi_{p+d}\psi_1 - \theta_{p+d+1} \\
\vdots \quad & \quad \vdots \quad\quad \vdots \\
\psi_{q-L} &= \varphi_1\psi_{q-L-1} + \varphi_2\psi_{q-L-2} + ... + \varphi_{p+d}\psi_{q-p-d-L} - \theta_{q-L} \\
\vdots \quad & \quad \vdots \quad\quad \vdots \\
\psi_{q-1} &= \varphi_1\psi_{q-2} + \varphi_2\psi_{q-3} + ... + \varphi_{p+d}\psi_{q-p-d-1} - \theta_{q-1} \\
\psi_q &= \varphi_1\psi_{q-1} + \varphi_2\psi_{q-2} + ... + \varphi_{p+d}\psi_{q-p-d} - \theta_q \\
\psi_{q+1} &= \varphi_1\psi_{q+1} + \varphi_2\psi_{q+2} + ... + \varphi_{p+d}\psi_{q-p-d+1} \\
\psi_{q+2} &= \varphi_1\psi_{q+2} + \varphi_2\psi_{q+1} + ... + \varphi_{p+d}\psi_{q-p-d+2} \\
\vdots \quad & \quad \vdots \quad\quad \vdots
\end{aligned}
$$

hence, for $i \geq 1$

$$\psi_{q+i} = \varphi_1\psi_{q+i-1} + \varphi_2\psi_{q-i-2} + ... + \varphi_{p+d}\psi_{q-p-d+i}. \tag{3.12}$$

For O_t, the $\theta^{(O)}$'s are

$$
\begin{aligned}
\theta_1^{(O)} &= -\psi_{L+1}/K + \varphi_1 \\
\theta_2^{(O)} &= -\psi_{L+2}/K + \varphi_1\psi_{L+1}/K + \varphi_2 \\
\theta_3^{(O)} &= -\psi_{L+3}/K + \varphi_1\psi_{L+2}/K + \varphi_2\psi_{L+1}/K + \varphi_3 \\
\vdots \quad & \quad \vdots \qquad \quad \vdots \\
\theta_{p+d}^{(O)} &= -\psi_{L+p+d}/K + \varphi_1\psi_{L+p+d-1}/K + ... + \varphi_{p+d-1}\psi_{L+1}/K + \varphi_{p+d} \\
\theta_{p+d+1}^{(O)} &= -\psi_{L+p+d+1}/K + \varphi_1\psi_{L+p+d}/K + ... + \varphi_{p+d}\psi_{L+1}/K \\
\theta_{p+d+2}^{(O)} &= -\psi_{L+p+d+2}/K + \varphi_1\psi_{L+p+d+1}/K + ... + \varphi_{p+d}\psi_{L+2}/K \\
\vdots \quad & \quad \vdots \qquad \quad \vdots \\
\theta_{q-L}^{(O)} &= -\psi_q/K + \varphi_1\psi_{q-1}/K + ... + \varphi_{p+d}\psi_{q-p-d}/K \\
\theta_{q-L+1}^{(O)} &= -\psi_{q+1}/K + \varphi_1\psi_q/K + ... + \varphi_{p+d}\psi_{q-p-d+1}/K \\
\theta_{q-L+2}^{(O)} &= -\psi_{q+2}/K + \varphi_1\psi_{q+1}/K + ... + \varphi_{p+d}\psi_{q-p-d+2}/K \\
\vdots \quad & \quad \vdots \qquad \quad \vdots
\end{aligned}
$$

hence, from (3.12), we have $\theta_{q-L+i}^{(O)} = 0$ for $i \geq 1$. In this case, the MA operator $\theta^{(O)}$ has the order $q^{(O)} = q-L$ when $j = max\{p+d, q-L\} = q-L$.

From case 1 and case 2, $q^{(O)} = max\{p+d, q-L\}$. Thus O_t is ARIMA$(p, d, max\{p+d, q-L\})$ having the same autoregressive operator $\varphi(B)$ and difference operator $\nabla^d(B)$ as Z_t but having a moving average operator $\theta^{(O)}(B)$ of order $q^{(O)} = max\{p+d, q-L\}$ with parameters $\theta_j^{(O)}(B)$, $j = 1, 2, ..., q^{(O)}$ as defined above. The underlying noise series has $\sigma_{a^{(O)}} = K\sigma_a$.

End of proof

3.4 The Model for Multistage Supply Chain

The model provides a measure of the bullwhip effect and insights into the parameters that determine the magnitude of the bullwhip effect. It provides a means of determining the cumulative impact of lead times in a multistage supply chain.

A measure of the bullwhip effect is $K = 1+\psi_1+\psi_2+...+\psi_L$, the multiplier that translates a shock in demand into a shock in orders: $a_t^{(1)} = Ka_t$. The multiplier K tells how the error in the order forecast grows as the forecast horizon L increases. This additional forecast error is due to changes in the forecast demand $\hat{Z}$ during the forecast horizon.

Under the assumptions of the model, the effect of lead times in a multistage supply chain (in which the order at each stage become the demand for the stage immediately upstream) are additive. For example, the multiplier of N stages having lead-times $L^{(1)}$, $L^{(2)}$, $L^{(3)}$, ..., $L^{(N)}$ is the same as the multiplier of a single stage having lead-time

$$L = L^{(1)} + L^{(2)} + L^{(3)} + ... + L^{(N)}.$$

For example, consider a two-stage supply chain with lead times $L^{(1)}$ and

$L^{(2)}$. The transfer function for the demand at the first stage is

$$Z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + ...$$

The multiplier for the first stage is

$$K^{(1)} = 1 + \psi_1 + \psi_2 + ... + \psi_L.$$

The noise process for the first stage is

$$a_t^{(1)} = K^{(1)} a_t.$$

The transfer function for the order for the first stage is

$$O_t^{(1)} = \mu + a_t^{(1)} + (\psi_{L^{(1)}+1}/K^{(1)}) a_{t-1}^{(1)} + (\psi_{L^{(1)}+2}/K^{(1)}) a_{t-2}^{(1)} + ...$$

The demand for the upstream stage two is the order of the adjacent downstream stage one or $Z_t^{(2)} = O_t^{(1)}$.

Therefore,the multiplier for the stage two is

$$K^{(2)} = 1 + \psi_1/K^{(1)} + \psi_2/K^{(1)} + ... + \psi_{L^{(1)}+L^{(2)}}/K^{(1)}.$$

The noise process for the stage two is

$$a_t^{(2)} = K^{(2)} a_t^{(1)}.$$

The transfer function for the order at stage two is

$$O_t^{(2)} = \mu + a_t^{(2)} + \big((\psi_{L^{(1)}+L^{(2)}+1}/K^{(1)})/K^{(2)}\big)a_{t-1}^{(2)}\big((\psi_{L^{(1)}+L^{(2)}+2}/K^{(1)})/K^{(2)}\big)a_{t-2}^{(2)} + ...$$

Substituting

$$a_t^{(2)} = K^{(2)}a_t^{(1)} = K^{(2)}K^{(1)}a_t = (1 + \psi_1 + \psi_2 + ... + \psi_{L^{(1)}+L^{(2)}})a_t$$

into the above equation, we obtain

$$O_t^{(2)} = \mu + (1 + \psi_1 + \psi_2 + ... + \psi_{L^{(1)}+L^{(2)}})a_t + \psi_{L^{(1)}+L^{(2)}+1}a_{t-1} + \psi_{L^{(1)}+L^{(2)}+2}a_{t-2} + ...$$

The above expression can be seen to be the multiplier of a single stage having a lead-time of $L^{(1)} + L^{(2)}$.

Since $Z_t^{(2)} = O_t^{(1)}$ is ARIMA$(p, d, q^{(1)})$ where $q^{(1)} = max\{p + d, q - L^{(1)}\}$, by theorem 2, $O_t^{(2)}$ is ARIMA$(p, d, q^{(2)})$ where $q^{(2)} = max\{p + d, q^{(1)} - L^{(2)}\}$.

3.5 Sample Models

This section exemplifies the generic formulas from the previous section when the demand series are ARIMA(0,1,1), ARIMA(0,2,2), ARIMA(0,3,3), and ARIMA(1,0,0). The model ARIMA(0,1,1) or IMA(1,1) is the single exponential smoothing model (McKenzie (1984)). The model ARIMA(0,2,2) or IMA(2,2) is the double exponential smoothing model. The model ARIMA(0,3,3) or IMA(3,3) is the triple exponential smoothing model. These exponential smoothing models are commonly used in forecast method.

The single exponential smoothing method, or simple exponential smoothing model, uses the exponentially weighted average of recent data and the forecast. The model assumes that the data displays a time-varying mean without a consistent trend.

The double exponential smoothing model, or linear exponential smoothing (Brown's or Holt's), is used when the data displays a time-varying linear trend as well as a time-varying level (Brown's uses 1 parameter, Holt's uses separate smoothing parameters for level and trend).

The triple exponential smoothing model, or quadratic exponential smoothing, is used when the data displays a time-varying quadratic trend as well as a time-varying level.

The models in AR(p) are intuitively appealing as descriptions of nature. Much of classical physics can be written as low order differential equations. In an AR(p) model, the forecast is made from a set of exponentially decay weights of the past data.

3.5.1 ARIMA(0,1,1)

This section illustrates the applications of theorem 1 and 2 when the demand series is ARIMA(0,1,1). These results correspond to those given in Graves (1999). One interesting observation of Graves is that if consumer demand is IMA(1,1) then the orders at all upstream stages are also IMA(1,1), which is stated in theorem 2 that O_t is an ARIMA$(p, d, max\{p+d, q-L\})$. Since L can be an integer greater than zero, then $max\{p+d, q-L\}$ is 1 and O_t is always IMA(1,1), given Z_t is IMA(1,1).

The general form of the IMA(1,1) model is

$$Z_t = Z_{t-1} + a_t - \theta a_{t-1} \tag{3.13}$$

or

$$\varphi(B)Z_t = \theta(B)a_t$$

where $\varphi(B) = 1 - B$, i.e., $\varphi_1 = 1$ and $\varphi_j = 0$ for $j > 1$. $\theta(B) = 1 - \theta B$, i.e., $\theta_1 = \theta$ and $\theta_j = 0$ for $j > 1$. Thus, we can compute

$$\psi(B) = \theta(B)/\varphi(B) = 1 + (1-\theta)B + (1-\theta)B^2 + (1-\theta)B^3 + ...$$

i.e., $\psi_i = 1 - \theta$, $i = 1, 2, 3, ...$

Hence, Z_t in random shock form is

$$Z_t = a_t + (1-\theta)a_{t-1} + (1-\theta)a_{t-2} + (1-\theta)a_{t-3} + ... \tag{3.14}$$

By theorem 1, I_t is MA$(L-1)$ with

$$\begin{aligned}
\theta_1^{(I)} &= 1 + (1-\theta) = 2-\theta \\
\theta_2^{(I)} &= 1 + (1-\theta) + (1-\theta) = 3-2\theta \\
\theta_3^{(I)} &= 1 + (1-\theta) + (1-\theta) + (1-\theta) = 4-3\theta \\
\vdots \quad & \quad \vdots \qquad \quad \vdots \\
\theta_{L-1}^{(I)} &= L-(L-1)\theta.
\end{aligned}$$

Thus,

$$I_t = T - a_t - (2-\theta)a_{t-1} - (3-2\theta)a_{t-2} - ... - (L-(L-1)\theta)a_{t-L+1}.$$

The standard deviation of the inventory is given by

$$\sigma_I = \sqrt{1 + (2-\theta)^2 + (3-2\theta)^2 + ... + \big(L-(L-1)\theta\big)^2}\sigma_a$$

By theorem 2, O_t is IMA(1,1)

$$(1-B)O_t = (1-\theta_1^{(O)}B)a_t^{(O)}$$

where

$$\theta_1^{(O)} = -\psi_{L+1}/K + \varphi_1$$

$$a_t^{(O)} = Ka_t$$

$$K = 1 + L(1-\theta).$$

Therefore

$$\theta_1^{(O)} = \frac{-(1-\theta)}{1+L(1-\theta)} + 1 = \frac{1+(L-1)(1-\theta)}{1+L(1-\theta)}.$$

By theorem 2 we also have

$$\sigma_{a^{(O)}} = K\sigma_a = (1+L(1-\theta))\sigma_a.$$

Rewritten following the ARIMA(p,d,q) form in (3.1), we have

$$\begin{aligned}(1-B)O_t &= \Big(1 - \frac{1+(L-1)(1-\theta)}{1+L(1-\theta)}B\Big)a_t^{(O)} \\ &= \Big(\big(1+L(1-\theta)\big) - \big(1+(L-1)(1-\theta)\big)B\Big)a_t\end{aligned}$$

or in the random shock form in (3.9)

$$\begin{aligned}O_t = &\ \big(1+L(1-\theta)\big)a_t + (1-\theta)a_{t-1} + (1-\theta)a_{t-2} \\ &+ (1-\theta)a_{t-3} + \ldots.\end{aligned} \tag{3.15}$$

3.5.2 ARIMA(0,2,2)

The general form of the IMA(2,2) model is

$$Z_t = 2Z_{t-1} - Z_{t-2} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

or

$$\varphi(B)Z_t = \theta(B)a_t$$

where $\varphi(B) = 1 - \varphi_1 B - \varphi_2 B^2 = 1 - 2B + B^2$, $\theta(B) = 1 - \theta_1 B - \theta_2 B^2$. Thus, we can compute ψ's using (3.2)

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \varphi_1\psi_0 - \theta_1 = 2 - \theta_1 \\
\psi_2 &= \varphi_1\psi_1 + \varphi_2\psi_0 - \theta_2 = 2(2-\theta_1) - 1 - \theta_2 = 3 - 2\theta_1 - \theta_2 \\
\psi_3 &= \varphi_1\psi_2 + \varphi_2\psi_1 + \varphi_3\psi_0 - \theta_3 = \varphi_1\psi_2 + \varphi_2\psi_1 \\
&= 2(3 - 2\theta_1 - \theta_2) - 1(2 - \theta_1) = 4 - 3\theta_1 - 2\theta_2 \\
\psi_4 &= \varphi_1\psi_3 + \varphi_2\psi_2 + \varphi_3\psi_1 + \varphi_4\psi_0 - \theta_4 = \varphi_1\psi_3 + \varphi_2\psi_2 \\
&= 2(4 - 3\theta_1 - 2\theta_2) - 1(3 - 2\theta_1 - \theta_2) = 5 - 4\theta_1 - 3\theta_2 \\
\psi_5 &= \varphi_1\psi_4 + \varphi_2\psi_3 = 2(5 - 4\theta_1 - 3\theta_2) - 1(4 - 3\theta_1 - 2\theta_2) = 6 - 5\theta_1 - 4\theta_2 \\
\vdots & \qquad \vdots \qquad\qquad \vdots \\
\psi_j &= \varphi_1\psi_{j-1} + \varphi_2\psi_{j-2} = (j+1) - j\theta_1 - (j-1)\theta_2.
\end{aligned}$$

Thus,

$$\psi(B) = \theta(B)/\varphi(B) = 1 + (2-\theta_1)B + (3 - 2\theta_1 - \theta_2)B^2 + (4 - 3\theta_1 - 2\theta_2)B^3 + \ldots.$$

i.e., $\psi_i = (i+1) - i\theta_1 - (i-1)\theta_2$, $i = 1, 2, 3, ...$

By theorem 1, I_t is MA$(L-1)$ with

$$\begin{aligned}
\theta_1^{(I)} &= 1 + (2-\theta_1) = 3 - \theta_1 \\
\theta_2^{(I)} &= 1 + (2-\theta_1) + (3 - 2\theta_1 - \theta_2) = 6 - 3\theta_1 - \theta_2 \\
\theta_3^{(I)} &= 1 + (2-\theta_1) + (3 - 2\theta_1 - \theta_2) + (4 - 3\theta_1 - 2\theta_2) = 10 - 6\theta_1 - 3\theta_2 \\
\vdots \quad & \quad \vdots \qquad \quad \vdots \\
\theta_{L-1}^{(I)} &= (L/2)(L+1) - \big((L-1)/2\big)L\theta_1 - \big((L-2)/2\big)(L-1)\theta_2.
\end{aligned}$$

Thus,

$$I_t = T - a_t - (3-\theta_1)a_{t-1} - (6-3\theta_1-\theta_2)a_{t-2} - ... - \Big(\sum_{i=1}^{L} i - \sum_{i=1}^{L-1} i\theta_1 - \sum_{i=1}^{L-2} i\theta_2\Big)a_{t-L+1}.$$

The standard deviation of the inventory is given by

$$\sigma_I = \sqrt{1 + (3-\theta_1)^2 + (6 - 3\theta_1 - \theta_2)^2 + ... + \Big(\sum_{i=1}^{L} i - \sum_{i=1}^{L-1} i\theta_1 - \sum_{i=1}^{L-2} i\theta_2\Big)^2}\,\sigma_a.$$

By theorem 2, O_t is IMA(2,2)

$$(1 - 2B + B^2)O_t = (1 - \theta_1^{(O)}B - \theta_2^{(O)}B^2)a_t^{(O)}$$

where

$$
\begin{aligned}
\theta_1^{(O)} &= -\psi_{L+1}/K + \varphi_1 \\
\theta_2^{(O)} &= -\psi_{L+2}/K + \varphi_1\psi_{L+1}/K + \varphi_2 \\
\theta_j^{(O)} &= -\psi_{L+j}/K + \varphi_1\psi_{L+j-1}/K + \varphi_2\psi_{L+j-2}/K = 0;\ j \geq 3 \\
a_t^{(O)} &= Ka_t \\
K &= \sum_{i=1}^{L+1} i - \sum_{i=1}^{L} i\theta_1 - \sum_{i=1}^{L-1} i\theta_2 \\
\sigma_{a^{(O)}} &= K\sigma_a = \Big(\sum_{i=1}^{L+1} i - \sum_{i=1}^{L} i\theta_1 - \sum_{i=1}^{L-1} i\theta_2\Big)\sigma_a.
\end{aligned}
$$

3.5.3 ARIMA(0,3,3)

The general form of the IMA(3,3) model is

$$Z_t = 3Z_{t-1} - 3Z_{t-2} + Z_{t-3} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \theta_3 a_{t-3}$$

or

$$\varphi(B)Z_t = \theta(B)a_t$$

where

$$
\begin{aligned}
\varphi(B) &= 1 - \varphi_1 B - \varphi_2 B^2 - \varphi_3 B^3 = 1 - 3B + 3B^2 - B^3 \\
\theta(B) &= 1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3.
\end{aligned}
$$

Thus, we can compute ψ's using (3.2)

$$\psi_0 = 1$$

$$\psi_1 = \varphi_1\psi_0 - \theta_1 = 3 - \theta_1$$

$$\psi_2 = \varphi_1\psi_1 + \varphi_2\psi_0 - \theta_2 = 3(3-\theta_1) - 3 - \theta_2 = 6 - 3\theta_1 - \theta_2$$

$$\begin{aligned}
\psi_3 &= \varphi_1\psi_2 + \varphi_2\psi_1 + \varphi_3\psi_0 - \theta_3 \\
&= 3(6-3\theta_1-\theta_2) - 3(3-\theta_1) - \theta_3 = 9 - 6\theta_1 - 3\theta_2 - \theta_3 \\
\psi_4 &= \varphi_1\psi_3 + \varphi_2\psi_2 + \varphi_3\psi_1 + \varphi_4\psi_0 - \theta_4 = \varphi_1\psi_3 + \varphi_2\psi_2 + \varphi_3\psi_1 \\
&= 3(9-6\theta_1-3\theta_2-\theta_3) - 3(6-3\theta_1-\theta_2) + (3-\theta_1) = 12 - 9\theta_1 - 6\theta_2 - 3\theta_3 \\
\psi_5 &= \varphi_1\psi_4 + \varphi_2\psi_3 + \varphi_3\psi_2 \\
&= 3(12-9\theta_1-6\theta_2-3\theta_3) - 3(9-6\theta_1-3\theta_2-\theta_3) + (6-3\theta_1-\theta_2) \\
&= 15 - 12\theta_1 - 9\theta_2 - 6\theta_3 \\
\vdots \quad & \quad \vdots \quad \quad \vdots \\
\psi_j &= \varphi_1\psi_{j-1} + \varphi_2\psi_{j-2} + \varphi_3\psi_{j-3} = 3j - 3(j-1)\theta_1 - 3(j-2)\theta_2 - 3(j-3)\theta_3.
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi(B) &= \theta(B)/\varphi(B) \\
&= 1 + (3-\theta_1)B + (6-3\theta_1-\theta_2)B^2 + (9-6\theta_1-3\theta_2-\theta_3)B^3 \\
&\quad + (12-9\theta_1-6\theta_2-3\theta_3)B^4 + ...
\end{aligned}$$

i.e., $\psi_i = 3j - 3(j-1)\theta_1 - 3(j-2)\theta_2 - 3(j-3)\theta_3$, $i = 1, 2, 3, ...$

By theorem 1, I_t is MA$(L-1)$ with

$$\begin{aligned}
\theta_1^{(I)} &= 1 + (3 - \theta_1) = 4 - \theta_1 \\
\theta_2^{(I)} &= 1 + (3 - \theta_1) + (6 - 3\theta_1 - \theta_2) = 10 - 4\theta_1 - \theta_2 \\
\theta_3^{(I)} &= 1 + (3 - \theta_1) + (6 - 3\theta_1 - \theta_2) + (9 - 6\theta_1 - 3\theta_2 - \theta_3) = 19 - 10\theta_1 - 4\theta_2 - \theta_3 \\
\theta_4^{(I)} &= 1 + (3 - \theta_1) + (6 - 3\theta_1 - \theta_2) + (9 - 6\theta_1 - 3\theta_2 - \theta_3) + (12 - 9\theta_1 - 6\theta_2 - 3\theta_3) \\
&= 31 - 19\theta_1 - 10\theta_2 - 4\theta_3 \\
\vdots & \qquad \vdots \qquad \vdots \\
\theta_{L-1}^{(I)} &= \Big(\frac{3(L-1)L}{2} + 1\Big) - \Big(\frac{3(L-2)(L-1)}{2} + 1\Big)\theta_1 \\
&\quad - \Big(\frac{3(L-3)(L-2)}{2} + 1\Big)\theta_2 - \Big(\frac{3(L-4)(L-3)}{2} + 1\Big)\theta_3.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_t = T &- a_t - (4 - \theta_1)a_{t-1} - (10 - 4\theta_1 - \theta_2)a_{t-2} \\
&- (19 - 10\theta_1 - 4\theta_2 - \theta_3)a_{t-3} - (31 - 19\theta_1 - 10\theta_2 - 4\theta_3)a_{t-4} \\
&- ... - \Bigg(\Big(1 + \sum_{i=1}^{L-1} 3i\Big) - \Big(1 + \sum_{i=1}^{L-2} 3i\Big)\theta_1 - \Big(1 + \sum_{i=1}^{L-3} 3i\Big)\theta_2 - \Big(1 + \sum_{i=1}^{L-4} 3i\Big)\theta_3\Bigg)a_{t-L+1}.
\end{aligned}$$

The standard deviation of the inventory is given by

$$\sigma_I = \sqrt{1 + (\theta_1^{(I)})^2 + (\theta_2^{(I)})^2 + (\theta_3^{(I)})^2 + ... + (\theta_{L-1}^{(I)})^2}\sigma_a.$$

By theorem 2, O_t is IMA(2,2)

$$(1-3B+3B^2-B^3)O_t = (1-\theta_1^{(O)}B-\theta_2^{(O)}B^2-\theta_3^{(O)}B^3)a_t^{(O)}$$

where

$$\begin{aligned}
\theta_1^{(O)} &= -\psi_{L+1}/K+\varphi_1 \\
\theta_2^{(O)} &= -\psi_{L+2}/K+\varphi_1\psi_{L+1}/K+\varphi_2 \\
\theta_3^{(O)} &= -\psi_{L+3}/K+\varphi_1\psi_{L+2}/K+\varphi_2\psi_{L+1}/K+\varphi_3 \\
\theta_j^{(O)} &= -\psi_{L+j}/K+\varphi_1\psi_{L+j-1}/K+\varphi_2\psi_{L+j-2}/K+\varphi_3\psi_{Lj-3}/K = 0;\ j \geq 4 \\
a_t^{(O)} &= Ka_t \\
K &= \Big(1+\sum_{i=1}^{L}3i\Big)-\Big(1+\sum_{i=1}^{L-1}3i\Big)\theta_1-\Big(1+\sum_{i=1}^{L-2}3i\Big)\theta_2-\Big(1+\sum_{i=1}^{L-3}3i\Big)\theta_3 \\
\sigma_{a^{(O)}} &= K\sigma_a.
\end{aligned}$$

3.5.4 ARIMA(1,0,0)

When demand is ARIMA(1,0,0) or AR(1)

$$Z_t-\mu = \phi(Z_{t-1}-\mu)+a_t$$

or

$$\varphi(B)(Z_t-\mu) = \theta(B)a_t$$

where $\varphi = \phi(B) = 1 - \phi$, with $-1 < \phi < 1$ and $\theta(B) = 1$. Thus, we can compute

$$\psi(B) = \theta(B)/\varphi(B) = 1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + ...$$

i.e., $\psi_1 = \phi^i$,for $i = 1, 2, 3, ...$

By theorem 1, I_t is MA$(L-1)$ with

$$\begin{aligned}
\theta_1^{(I)} &= 1 + \phi = (1-\phi^2)/(1-\phi) \\
\theta_2^{(I)} &= 1 + \phi + \phi^2 = (1-\phi^3)/(1-\phi) \\
\theta_3^{(I)} &= 1 + \phi + \phi^2 + \phi^3 = (1-\phi^4)/(1-\phi) \\
&\vdots \qquad \vdots \qquad \vdots \\
\theta_{L-1}^{(I)} &= 1 + \phi + \phi^2 + \phi^3 + ... + \phi^{L-1} = (1-\phi^L)/(1-\phi).
\end{aligned}$$

Thus,

$$I_t = T - a_t - \Big((1-\phi^2)/(1-\phi)\Big)a_{t-1} - \Big((1-\phi^3)/(1-\phi)\Big)a_{t-2} - ... - \Big((1-\phi^L)/(1-\phi)\Big)a_{t-L+1}.$$

The standard deviation of the inventory is given by

$$\sigma_I = \sqrt{1 + \Big((1-\phi^2)/(1-\phi)\Big)^2 + \Big((1-\phi^3)/(1-\phi)\Big)^2 + ... + \Big((1-\phi^L)/(1-\phi)\Big)^2}\sigma_a.$$

By theorem 2, O_t is ARIMA(1,0,1) or ARMA(1,1)

$$(1-\phi B)(O_t-\mu)=(1-\theta_1^{(O)}B)a_t^{(O)}$$

where

$$\theta_1^{(O)}=-\psi^{L+1}/K+\varphi_1$$

$$a_t^{(O)}=Ka_t$$

$$K=1+\phi+\phi^2+\phi^3+...+\phi^L=(1-\phi^{L+1})/(1-\phi).$$

Therefore

$$\theta_1^{(O)}=\frac{-\phi^{L+1}}{(1-\phi^{L+1})/(1-\phi)}+\phi=\frac{\phi(1-\phi^L)}{1-\phi^{L+1}}.$$

By theorem 2 we also have

$$\sigma_{a^{(O)}}=K\sigma_a=\frac{1-\phi^{L+1}}{1-\phi}\sigma_a.$$

Rewritten following the form in (3.1), we have

$$\begin{aligned}(1-\phi B)O_t&=\Big(1-\frac{\phi(1-\phi^L)}{1-\phi^{L+1}}B\Big)a_t^{(O)}\\&=\Big(\frac{1-\phi^{L+1}}{1-\phi}-\frac{\phi(1-\phi^L)}{1-\phi}B\Big)a_t\end{aligned}$$

or in the random shock form in (3.9)

$$O_t=(1+\phi+\phi^2+\phi^3+...+\phi^L)a_t+\phi^{L+1}a_{t-1}+\phi^{L+2}a_{t-2}+\phi^{L+3}a_{t-3}+....$$

The lag one autocorrelation for AR(1) demand is ϕ. As was pointed out in Lee, So, and Tang (2000) the bullwhip effect in orders does not exist when the autocorrelation (ϕ) is zero. If $\phi = 0$, then $K = 1$, $\theta^{(O)} = 0$, and $O_t = Z$, i.e. the ordering process is the white noise series which is a pure pull.

For a given lead time, K grows as ϕ increases. As ϕ approaches 1, (i.e. demand approaches a random walk), the value of K approaches $L + 1$.

3.6 Applications of Sample Models

This section gives the applications of the sample models compared to various current literatures in 1) multistage supply chain for ARIMA(0,1,1) demand model used by Sternman 1989, 2) ARIMA(0,1,1) demand model used by Tsay 1999, and 3) ARIMA(1,0,0) demand model used by Lee, So, and Tang 2000.

3.6.1 Multistage Supply Chain ARIMA(0,1,1) Application

In a multistage supply chain in which the consumer demand follows an exponential smoothing model, the orders at the upstream stages also follow an exponential smoothing model, but with successively smaller values of the smoothing parameter and successively larger values of the standard error. If the stages are numbered consecutively with stage 1 being closest to the consumer and the lead time for each stage i is denoted $L^{(i)}$. The IMA(0,1,1) model written as an exponential smoothing model is

$$\hat{Z}_t = \alpha Z_t + (1-\alpha)\hat{Z}_{t-1} \tag{3.16}$$

with $\alpha = 1-\theta$, Then

$$\alpha^{(O^{(i)})} = \frac{\alpha}{1+(L^{(1)}+L^{(2)}+...+L^{(i)})\alpha}$$

$$\sigma_{O^{(i)}} = \big(1+(L^{(1)}+L^{(2)}+...+L^{(i)})\alpha\big)\sigma_a$$

where $\alpha^{O^{(i)}}$ is the exponential smoothing parameter for forecasting the orders at stage i. $\sigma^{O^{(i)}}$ is the standard error of the forecast of orders at stage i.

An exponential smoothing model can be thought of as a random walk buried in noise (see Box, Jenkins, Reinsel and Jenkins 1994). The parameter α indicates the fraction of a_t, the single period forecast error that is due to changes in the level of consumer demand and $1-\alpha$ is the fraction of the error due to noise. The expressions above say that if the lead time is long then the variation in orders is large and most of the variation is due to noise. A small α provides a lot of smoothing. While a large α provides a fast response to the recent changes in the time series and a smaller amount of smoothing. For example, when $\alpha = 1-\theta = 0$, the demand series is $Z_t = \mu + a_t$, which is a white noise series ARIMA(0,0,0). When $\alpha = 1-\theta = 1$, the demand series is $Z_t = Z_{t-1} + a_t$, which is a random walk series ARIMA(0,1,0).

Consider a four-stage supply chain, with each stage having a lead time of four weeks as in the beer game (Sternman 1989). Suppose that demand is generated by a random walk ($\alpha = 1$) with a standard error of 1 ($\sigma_a = 1$).

Then $\alpha^{O^{(1)}} = 1/5$, $\alpha^{O^{(1)}} = 1/9$, $\alpha^{O^{(1)}} = 1/13$, $\alpha^{O^{(1)}} = 1/17$ and $\sigma_{O^{(1)}} = 5$, $\sigma_{O^{(2)}} = 9$, $\sigma_{O^{(3)}} = 13$, $\sigma_{O^{(1)}} = 17$.

3.6.2 ARIMA(0,1,1) Application

As a practical tool, ARIMA supply chain models provide a means to determine the inventory levels and manufacturing flex capabilities required to meet customer demand. And they provide a means of explicitly quantifying the impact of lead-time reduction on required inventory levels, on manufacturing flex capacity required and on the ability to reduce variation for upstream suppliers.

Suppose for example, that the customer demand for an item sold by a retailer is given by a time series Z_t, where Z_t is the demand in week t. Suppose that Z_t has been modeled with an exponential smoothing forecast model $\big($ARIMA(0,1,1)$\big)$ with $\alpha = 1 - \theta = 0.2$ and $\sigma_a = 10$. Suppose also that the current level of the process is $\hat{Z}_t = 100$.

Suppose the supplier delivers the items to the retailer $L = 4$ weeks after the order is placed. The retailer also gives the supplier a forecast of the order $F = 10$ weeks before it is placed along with a commitment that the actual order will not deviate from this forecast by more than a specified range (Tsay 1999).

Thus the retailer would like to know how to compute an inventory target

T (safety stock level) to get specified (say a three-sigma) level of protection against stock-outs. The retailer would also like to compute the (again say 3-sigma) range of variation in actual orders will not deviate from their forecast by more than a specified range (Tsay 1999).

The formula's below can all be derived from the results of this chapter, using standard time series techniques.

The standard deviation of the inventory from theorem 1 is

$$\begin{aligned}\sigma_I &= \sqrt{1 + (2-\theta)^2 + (3-2\theta)^2 + ... + \big(L-(L-1)\theta\big)^2}\sigma_a \\ &= \sqrt{1 + (1+\alpha)^2 + (1+2\alpha)^2 + ... + (1+(L-1)\alpha)^2}\sigma_a \\ &= \sqrt{1 + 1.2^2 + 1.4^2 + 1.6^2} \cdot 10 \approx 26.\end{aligned}$$

To maintain a three-sigma level of protection against stock-outs the safety stock would be set at $\pm 3\sigma_I = \pm 3 * 26 = \pm 78$. If the lead time could be reduced to one week then the three-sigma safety stock level could be reduced to $\pm 3 * \sqrt{1} * 10 = \pm 30$.

The forecast of the orders is given by

$$\hat{O}_t(F) = E[O_{t+F}] = (1-\theta)(a_t + a_{t-1} + a_{t-2} + ...) = \hat{Z}_t$$

where $\hat{O}_t(F)$ is the forecast of O_{t+F} made at time t.

From (3.9), the standard error of the order forecast, i.e. the standard deviations of $(O_{t+F}) - \hat{O}_t(F))$ can be derived as

$$\begin{aligned} O_{t+F} - \hat{O}_t(F) &= \big(1 + L(1-\theta)\big)a_{t+F} + (1-\theta)(a_{t-F-1} + a_{t-F-2} + ... + a_{t+1}) \\ s(F) &= \sqrt{O_{t+F} - \hat{O}_t(F)} = \sqrt{(1+L\alpha)^2 + (F-1)\alpha^2}\sigma_a \\ s(10) &= \sqrt{(1+4*0.2)^2 + 9*0.2^2} * 10 \approx 19. \end{aligned}$$

Thus the three-sigma range of variation in actual orders around the 10-week forecast is $\pm 3 * 19 = \pm 57$. If the lead time could be reduced to one week the range could be reduced to $\pm 3 * \sqrt{(1+1*0.2)^2 + 9*0.2^2} * 10 \approx \pm 40$.

3.6.3 AR(1) Application

Lee, So, and Tang 2000 give an expression for the time series of orders when demand is AR(1). Here we show that the expression that they derive for orders, is equivalent to the ARMA(1,1) process derived above.

To show the equivalence we first must make adjustments for notational differences. The demand model in equation 2.1 of Lee, So, and Tang 2000 is

$$D_t = d + \rho D_{t-1} + \epsilon_t$$

where D_t is the actual demand in period t. ϵ_t is i.i.d. normally distributed with zero and variance σ^2. $-1 < \rho < 1$.

This is equivalent to the AR(1) demand model used above

$$Z_t - \mu = \phi(Z_{t-1} - \mu) + a_t$$

where

$$\mu = d/(1-\theta).$$

Lee, So, and Tang 2000 define a lead-time l in the order $l+1$ periods after the order is placed. Therefore the relationship between the lead-time L used in this paper and the lead time l used by Lee, So, and Tang 2000 is $L = l + 1$.

With these notational changes the expression given in equation 3.6 of Lee, So, and Tang 2000

$$Y_{t+1} = d + \rho Y_t + \frac{1-\rho^{l+2}}{1-\rho}\epsilon_{t+1} - \frac{\rho(1-\rho^{l+1})}{1-\rho}\epsilon_t$$

where Y_t is the orders, can be written as

$$O_t - \mu = \phi(O_{t-1} - \mu) + \frac{1-\phi^{L+1}}{1-\phi}a_t + \frac{\phi(1-\phi^L)}{1-\phi}a_{t-1}$$

if we define

$$a_t^{(O)} = \frac{1-\phi^{L+1}}{1-\phi}a_t = Ka_t$$

then we get

$$O_t - \mu = \phi(O_{t-1} - \mu) + a_t^{(O)} + \frac{\phi(1-\phi^L)}{1-\phi^{L+1}}a_{t-1}^{(O)}$$

which is the ARMA(1,1) model for orders given above.

3.7 Insights

These results provide mathematical insights as the followings.

1. The magnitude of the bullwhip effect. The bullwhip effect depends only on the lead-time and the autocorrelation in the demand. From theorem 1, the bullwhip effect on inventory is

$$K_I = \sqrt{1 + (1 + \psi_1)^2 + (1 + \psi_1 + \psi_2)^2 + ... + (1 + \psi_1 + \psi_2 + ... + \psi_{L-1})^2}.$$

From theorem 2 the bullwhip effect on the order is

$$K_O = 1 + \psi_1 + \psi_2 + ... + \psi_L.$$

From K_I and K_O, when the lead-time (L) is long and the autocorrelation (ψ) is high, most of the variation in orders and inventory is due to the bullwhip effect, rather than variation in demand.

2. The impact of the number of stages. In multistage supply chains the bullwhip effect on orders depends only on the total of the lead times, not on the number of stages. In a supply chain having N stages with lead times $L^{(1)}$, $L^{(2)}$, $L^{(3)}$, ..., $L^{(N)}$, the standard error of orders at the N^{th} stage, is the same of the standard error of orders of a single stage supply chain having the same demand series and a lead time of $L^{(1)} + L^{(2)} + L^{(3)} + ... + L^{(N)}$.

3. The existence of the bullwhip effect with the optimal forecast model. The results shown above are based on an assumption that the minimum mean

square error forecasting model is being used. Yet the bullwhip effect exists if the lead time is nonzero and the demand is autocorrelated. In practical terms this says that improving forecast accuracy can reduce, but does not eliminate, the bullwhip effect.

4. The existence of the bullwhip effect with shared of point of sale demand data. From the transfer functions of demand and orders made at time t forecasted for time $t+L$ periods in the future, it can be shown that $E_t[O_{t+L}] = E_t[Z_{t+L}]$. This means that an upstream customer who receives the orders as demand, will make the same forecast from the time series of the orders as he would make from point of sale data. This result is not surprising since, by assumption, each stage has the optimal forecast model.

The practical implications of these results is that the bullwhip effect exists even under the most optimistic assumptions regarding point of sale information sharing and forecasting models. Thus a supply chain integration strategy based solely on better data accuracy, better forecasting models and point of sale information sharing will not guarantee successful supply chain integration. A reduction in lead-time is necessary in reducing the bullwhip effect.

Chapter 4

The Generalized Ordering Policy and The Smoothing Ordering Policy

This chapter introduces the generalized ordering policy that permits manufacturers to control the tradeoffs between the variation in inventory and the variation in differencing orders. The standard MRP ordering policy introduced in chapter 3 is a special case of the generalized ordering policy that has the minimum variation in inventory (point A in figure 4.1). Where the smoothing ordering policy introduced later in this chapter has the minimum variation in differencing orders (point B in figure 4.1). Figure 4.1 exemplifies the indifference curve of the sum of the variation in inventory and the variation in differencing orders. For single exponential smoothing or

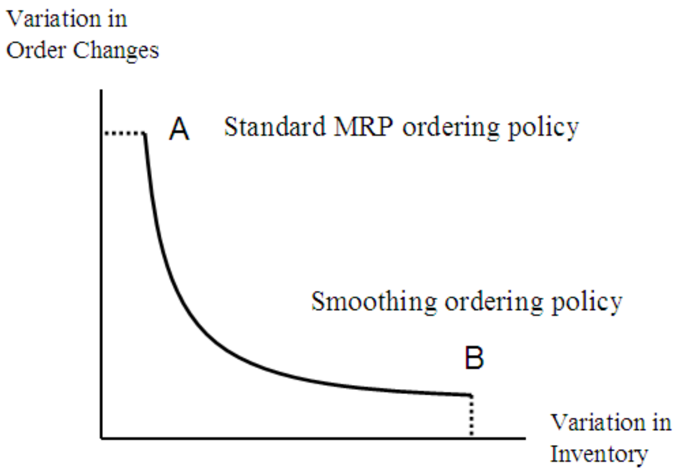

Figure 4.1: Indifference Curve of the Sum of Variations for Single Exponential Smoothing Demand

ARIMA(0,1,1), the variation in differencing orders is the variation in order changes from one period to the next.

With this generalized ordering policy, manufacturers can smooth orders which is regarded as an effective way to mitigate the bullwhip effect. The generalized order models can be applied to any ARIMA demand, any ordering lead time, and any smoothing period. We also provide generic formulas to determine the optimal smoothing weights in the smoothing ordering policy for ARIMA$(p, 0, q)$ and ARIMA$(p, 1, q)$ orders.

4.1 Introduction

With an incoming retailer order, a supplier may not take an immediate response to the changes in retailer orders but gradually react to these changes since both supplier and retailer do not instantly believe that the fluctuation in incoming customer demand (that drives the variation in retailer's orders) is an indicative of a permanent change. By taking slow reactions to the order changes, they can cancel out insignificant order variation by smoothing the order changes.

Hence, instead of using the standard MRP ordering policy to immediately bring the inventory back to target at the ordering lead time (L) periods in the future, a retailer may place orders gradually to control the variation in order changes. Smoothing out the orders instead of directly placing the current order to bring the inventory level back to target is regarded as an effective way to mitigate the bullwhip effect.

However, the retailer also needs a warranty that the order in a given period will not cause under/overstock on the inventory level. For example, the retailer may level the production rate that would not be changed (its variation in productions is zero), then the production may not be able to react to the demand needed. As a result, the inventory level will be either overstocked (if the production rate is set too high) or stock-outs (if the production rate is set too low). Thus, an implemented ordering policy must have a stationary inventory. The generalized ordering policy guarantees that such ordering policy has a stationary inventory.

We also introduce the smoothing ordering policy which is a special case of the generalized ordering policy that minimizes the variation in differencing orders. The differencing order is needed because it has a finite term in the

ARIMA random shock form. Hence, its variance is finite which is a requirement for optimization. With the gain in minimizing variation in differencing orders, the smoothing ordering policy increases the variation in inventory and the safety stock is increased compared with those of the standard MRP ordering policy.

4.2 Generalized Ordering Policy

The transfer function of the up to target order in (3.9) is

$$\begin{aligned} O_t - \mu &= (1 + \psi_1 + \psi_2 + ... + \psi_L)a_t + \psi_{L+1}a_{t-1} + \psi_{L+2}a_{t-2} + ... \\ &= K^{(O)}a_t + \sum_{i=1}^{\infty} \psi_{L+i}a_{t-i}. \end{aligned}$$

The term $K^{(O)}a_t$ represents the change between $t-1$ and t of the forecast for period $t, t+1, t+2, ..., t+L$. The term $\sum_{i=1}^{\infty} \psi_{L+i}a_{t-i}$ represents the forecast of demand for period $t+L$ that was made in the period $t-1$. This is infinite loading because it absorbs all of the changes in the forecast in a single period.

A generalized model uses S periods to adjust to the changes in the forecast. Let S be the smoothing period. L is the lead time. Let $\mu = 0$, we can define the generalized order in random shock form

$$
\begin{aligned}
O_t = \beta_0 a_t + \beta_1 a_{t-1} + \beta_2 a_{t-2} + ... + \beta_{S-1} a_{t-S+1} \\
+ (1 + \psi_1 + \psi_2 + ... + \psi_{S+L} - \sum_{i=0}^{S-1} \beta_i) a_{t-S} \\
+ \psi_{S+L+1} a_{t-S-1} + \psi_{S+L+2} a_{t-S-2} + ...
\end{aligned}
\tag{4.1}
$$

where $\beta_i \in \Re;\ i = 0, 1, 2, ..., S-1$ is the smoothing weight.

This order is a weights sum of the $S+1$ most recent shocks (rather than only the one most recent shock as in the infinite loading model) plus the $\hat{Z}_{t-S}(S+L)$, the forecast of demand in period $t+L$ made in period $t-S$.

The noise series order function in (4.1) can be rewritten in the forms of demand Z's and forecast demand $\hat{Z}$ where

$$
\begin{aligned}
a_{t-j} &= Z_{t-j} - \hat{Z}_{t-j-1}(1);\ j = 0, 1, 2, ... \\
\hat{Z}_{t-S}(L) &= \psi_{S+L} a_{t-S} + \psi_{S+L+1} a_{t-S-1} + \psi_{S+L+2} a_{t-S-2} + ... \\
\hat{Z}_{t-S-1}(L) &= \psi_{S+L+1} a_{t-S-1} + \psi_{S+L+2} a_{t-S-2} + \psi_{S+L+3} a_{t-S-3} + ...
\end{aligned}
$$

Thus

$$
\begin{aligned}
O_t = \beta_0 \big(Z_t - \hat{Z}_{t-1}(1)\big) + \beta_1 \big(Z_{t-1} - \hat{Z}_{t-2}(1)\big) \\
+ \beta_2 \big(Z_{t-2} - \hat{Z}_{t-3}(1)\big) + ... + \beta_{S-1} \big(Z_{t-S+1} - \hat{Z}_{t-S}(1)\big) \\
+ \Big(1 + \psi_1 + \psi_2 + ... + \psi_{S+L-1} - \sum_{i=0}^{S-1} \beta_i\Big) \big(Z_{t-S} - \hat{Z}_{t-S-1}(1)\big) \\
+ \hat{Z}_{t-S}(L)
\end{aligned}
\tag{4.2}
$$

or

$$
\begin{aligned}
O_t = {} & \beta_0\big(Z_t - \hat{Z}_{t-1}(1)\big) + \beta_1\big(Z_{t-1} - \hat{Z}_{t-2}(1)\big) \\
& + \beta_2\big(Z_{t-2} - \hat{Z}_{t-3}(1)\big) + ... + \beta_{S-1}\big(Z_{t-S+1} - \hat{Z}_{t-S}(1)\big) \\
& + \Big(1 + \psi_1 + \psi_2 + ... + \psi_{S+L} - \sum_{i=0}^{S-1} \beta_i\Big)\big(Z_{t-S} - \hat{Z}_{t-S-1}(1)\big) \\
& + \hat{Z}_{t-S-1}(L).
\end{aligned}
\tag{4.3}
$$

We next show that in order to have a stationary inventory (see the proof in theorem 3), it is necessary that

$$
\begin{aligned}
\sum_{i=0}^{S} \beta_i &= K \\
\text{where } \beta_S &= K - \sum_{i=0}^{S-1} \beta_i \\
K &= 1 + \psi_1 + \psi_2 + ... + \psi_{S+L}.
\end{aligned}
\tag{4.4}
$$

Thus, K represents the magnitude of the bullwhip effect, the multiplier applied to each shock in demand in creating the order.

In the infinite loading model, the shocks get multiplied all in one period. In the generalized ordering policy, the multiplication can be multiplied. From the proof in theorem 3, the inventory needs to be stationary, that is

$$
I_t = \sum_{j=0}^{\infty} O_{t-L-j} - \sum_{j=0}^{\infty} Z_{t-j}
$$

which converges only when

$$\sum_{i=0}^{S} \beta_i = K = 1 + \psi_1 + \psi_2 + ... + \psi_{S+L}.$$

(4.1) is in fact a generalized form of the ordering policy. For example, if we can adjust the coefficient weight β by letting $\beta_0 = 1 + \psi_1 + \psi_2 + ... + \psi_L$ and $\beta_i = \psi_{L+i}$, for $i = 1, 2, ..., S-1$ then (4.1) is the standard MRP ordering policy. In this case, $S = 0$, hence, S can be interpreted as the β_{S-1}'s weight that is different from $\psi_{S-1}^{(O)}$'s weight given in (3.6).

Under the generalized ordering policy in (4.1), the following results hold:

- The time series of inventory I_t is MA($S + L - 1$).
- The time series of order O_t is ARIMA($p, d, max\{p+d+S, q-L+S\}$). $\nabla^d O_t$, the stationary part of O_t, is ARMA($p, max\{p+d+S, q-L+S\}$).

The parameters of both of these models can be explicitly determined from the parameters of the original ARIMA(p, d, q) model of the demand time series Z_t.

Theorem 3 *I_t is an MA(S+L-1) with mean T and standard deviation* $\sigma_I = \sqrt{1 + (\theta_1^{(I)})^2 + (\theta_2^{(I)})^2 + ... + (\theta_{S+L-1}^{(I)})^2}\sigma_a$ *where* $\theta_i^{(I)} = 1 + \psi_1 + \psi_2 + ... + \psi_i - \beta_0 - \beta_1 - ...\beta_{i-L};\ i = 0, 1, 2, ..., L + S - 1.$

Corollary 1 *The generalized ordering policy that has the minimum variation in inventory is the standard MRP ordering policy.*

Proof

To prove that inventory I_t under the ordering policy in (4.1) is stationary, from (3.5)

$$I_t = \sum_{j=0}^{\infty} O_{t-L-j} - \sum_{j=0}^{\infty} Z_{t-j}$$

from (3.3)

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + ...$$

Thus, we have

$$\begin{aligned}
O_{t-L} - Z_t &= -a_t - \psi_1 a_{t-1} - \psi_2 a_{t-2} - ... - \psi_{L-1} a_{t-L+1} + (\beta_0 - \psi_L) a_{t-L} \\
&+ (\beta_1 - \psi_{L+1}) a_{t-L-1} + ... + (\beta_{S-1} - \psi_{S+L-1}) a_{t-L-S+1} \\
&+ (1 + \psi_1 + \psi_2 + ... + \psi_{S+L-1} - \sum_{i=0}^{S-1} \beta_i) a_{t-S-L} \\
O_{t-L-1} - Z_{t-1} &= -a_{t-1} - \psi_1 a_{t-2} - \psi_2 a_{t-3} - ... - \psi_{L-1} a_{t-L} + (\beta_0 - \psi_L) a_{t-L-1} \\
&+ (\beta_1 - \psi_{L+1}) a_{t-L-2} + ... + (\beta_{S-1} - \psi_{S+L-1}) a_{t-L-S} \\
&+ (1 + \psi_1 + \psi_2 + ... + \psi_{S+L-1} - \sum_{i=0}^{S-1} \beta_i) a_{t-S-L-1} \\
O_{t-L-2} - Z_{t-2} &= -a_{t-2} - \psi_1 a_{t-3} - \psi_2 a_{t-4} - ... - \psi_{L-1} a_{t-L-1} + (\beta_0 - \psi_L) a_{t-L-2} \\
&+ (\beta_1 - \psi_{L+1}) a_{t-L-3} + ... + (\beta_{S-1} - \psi_{S+L-1}) a_{t-L-S-1} \\
&+ (1 + \psi_1 + \psi_2 + ... + \psi_{S+L-1} - \sum_{i=0}^{S-1} \beta_i) a_{t-S-L-2} \\
\vdots \qquad & \vdots \qquad \vdots
\end{aligned}$$

combining the above terms gives

$$
\begin{aligned}
I_t &= \sum_{j=0}^{\infty} O_{t-L-j} - \sum_{j=0}^{\infty} Z_{t-j} \\
&= -a_t - (1+\psi_1)a_{t-1} - (1+\psi_1+\psi_2)a_{t-2} - ... \\
&\quad - (1+\psi_1+\psi_2+...+\psi_{L-1})a_{t-L+1} \\
&\quad - (1+\psi_1+\psi_2+...+\psi_L-\beta_0)a_{t-L} \\
&\quad - (1+\psi_1+\psi_2+...+\psi_{L+1}-\beta_0-\beta_1)a_{t-L-1} \\
&\quad - ... - \big(1+\psi_1+\psi_2+...+\psi_{S+L-1}-\sum_{i=0}^{S-1}\beta_i\big)a_{t-L-S+1}.
\end{aligned}
\tag{4.5}
$$

To write I_t in a standard ARIMA(p,d,q) form in (3.1), we define $a_t^{(I)} = a_t$ to give

$$I_t = T - a_t^{(I)} - \theta_1^{(I)}a_{t-1}^{(I)} - \theta_2^{(I)}a_{t-2}^{(I)} - ... - \theta_{S+L-1}^{(I)}a_{t-L-S+1}^{(I)}.$$

Thus I_t can be seen to be MA$(S+L-1)$ with parameters

$$\theta_i^{(I)} = 1+\psi_1+\psi_2+...+\psi_i-\beta_0-\beta_1-...\beta_{i-L};\ i=0,1,2,...,L+S-1$$

which is an MA$(S+L-1)$.

The expected value of the inventory $E[I_t]=T$.

The standard deviation of the inventory is

$$\sigma_I = \sqrt{1+(\theta_1^{(I)})^2+(\theta_2^{(I)})^2+...+(\theta_{S+L-1}^{(I)})^2}\sigma_a.$$

If we substitute the coefficient weight β's of the generalized ordering policy in (4.5) with $\beta_0 = 1+\psi_1+\psi_2+...+\psi_L$ and $\beta_i = \psi_{L+i}$, for $i = 1, 2, ..., S-1$ then we have the minimum variation in inventory and its standard variation is

$$\sigma_I = \sqrt{1 + (\theta_1^{(I)})^2 + (\theta_2^{(I)})^2 + ... + (\theta_{L-1}^{(I)})^2}\sigma_a$$

which is the standard variation in inventory using standard MRP ordering policy. Hence, the generalized ordering policy that has the minimum variation in inventory is the standard MRP ordering policy as stated in corollary 1.

End of proof

Theorem 4 *O_t is an ARIMA(p,d,max{p+d+S,q-L+S}) with the form: $\theta(B)\nabla^d(O_t - \mu) = \theta^{(O)}a_t^{(O)}$, or $\varphi(B)(O_t - \mu) = \theta^{(O)}a_t^{(O)}$ with $\theta^{(O)}(B)$ of order $q^{(O)} = max\{p + d + S, q - L + S\}$. Its stationary part of O_t is $\nabla^d O_t$ with the ARMA(p,max{p+d+S,q-L+S}) form: $\phi(B)(\nabla^d O_t - \nabla^d\mu) = \theta^{(O)}a_t^{(O)}$ with $\theta^{(O)}(B)$ of order $q^{(O)} = max\{p + d + S, q - L + S\}$.*

Proof

For a standard MRP order O_t in ARIMA$(p, d, q^{(O)})$ form

$$\varphi(B)(O_t - \mu) = \theta^{(O)}a_t^{(O)}$$

where $q^{(O)}$, $\theta^{(O)}$, and $a_t^{(O)}$ follow the notations in section 3.3 or, given the demand Z_t is ARIMA(p, d, q)

$$
\begin{aligned}
q^{(O)} &= max\{p + d, q - L\} \\
K &= 1 + \psi_1 + \psi_2 + \psi_3 + ... + \psi_L \\
a_t^{(O)} &= K a_t \\
\theta_j^{(O)} &= -\psi_{L+j}/K + \varphi_1 \psi_{L+j-1}/K + \varphi_2 \psi_{L+j-2}/K + ... + \varphi_{j-1}\psi_{L+1}/K + \varphi_j
\end{aligned}
$$

or in the random shock form in (3.6)

$$
\begin{aligned}
O_t - \mu &= \psi^{(O)}(B) a_t^{(O)} = (1 + \psi_1^{(O)} B + \psi_1^{(O)} B^2 + ...) a_t^{(O)} \\
&= a_t^{(O)} + (\psi_{L+1}/K) a_{t-1}^{(O)} + (\psi_{L+2}/K) a_{t-2}^{(O)} + ... \\
&= (1 + \psi_1 + \psi_2 + ... + \psi_L) a_t + \psi_{L+1} a_{t-1} + \psi_{L+2} a_{t-2} + ...
\end{aligned}
$$

We can obtain its stationary order, $\nabla^d O_t$, by differencing ARIMA$(p, d, q^{(O)})$ O_t to be in ARMA$(p, q^{(O)})$ form

$$
\phi(B)(\nabla^d O_t - \nabla^d \mu) = \theta^{(O)} a_t^{(O)}.
$$

$\nabla^d O_t$ can be expressed in the random shock form

$$
\begin{aligned}
\nabla^d O_t &= \psi^{(\nabla^d O)}(B) a_t^{(O)} \\
&= \big(1 + \psi_1^{(\nabla^d O)} B + \psi_2^{(\nabla^d O)} B^2 + \psi_3^{(\nabla^d O)} B^3 + ...\big) a_t^{(O)}.
\end{aligned}
$$

The weights $\psi_j^{(\nabla^d O)}$ are determined from $\phi(B)\psi^{(\nabla^d O)}(B) = \theta^{(O)}(B)$ to satisfy

$$\psi_j^{(\nabla^d O)} = \phi_1\psi_{j-1}^{(\nabla^d O)} + \phi_2\psi_{j-2}^{(\nabla^d O)} + ... + \phi_p\psi_{j-p}^{(\nabla^d O)} - \theta_j^{(O)}$$

with $\psi_0^{(\nabla^d O)} = 1$, $\psi_j^{(\nabla^d O)} = 0$ for $j < 0$, $\phi_j = 0$ for $j > p$, and $\theta_j^{(O)} = 0$ for $j < q^{(O)}$. Since the $\psi^{(\nabla^d O)}$ weights are absolutely summable and the process $\nabla^d O_t$ itself is stationary, so we can obtain a constant mean and variance of $\nabla^d O_t$.

For the generalized ordering policy, we replace the first S $\psi^{(O)}$'s weights in (3.6) with β_i; $i = 0, 1, 2, ..., S-1$, hence, the moving average operator for the generalized order will shift S more terms. Let

$$K = 1 + \psi_1 + \psi_2 + \psi_3 + ... + \psi_{S+L}$$
$$a_t^{(O)} = \beta_0 a_t.$$

Then the generalized order in random shock form

$$
\begin{aligned}
O_t - \mu = \psi^{(O)}(B)a_t^{(O)} &= (1 + \psi_1^{(O)}B + \psi_1^{(O)}B^2 + ...)a_t^{(O)} \\
&= a_t^{(O)} + (\beta_1/\beta_0)a_{t-1}^{(O)} + ... + (\beta_{S-1}/\beta_0)a_{t-S+1}^{(O)} \\
&\quad + \big((K - \sum_{i=0}^{S-1}\beta_i\big)/\beta_0)a_{t-S}^{(O)} + (\psi_{S+L+1}/\beta_0)a_{t-S-1}^{(O)} \\
&\quad + (\psi_{S+L+2}/\beta_0)a_{t-S-2}^{(O)} + (\psi_{S+L+2}/\beta_0)a_{t-S-3}^{(O)} + ... \\
&= \beta_0 a_t + \beta_1 a_{t-1} + ... + \beta_{S-1}a_{t-S+1} + (K - \sum_{i=0}^{S-1}\beta_i)a_{t-S} \\
&\quad + \psi_{S+L+1}a_{t-S-1} + \psi_{S+L+2}a_{t-S-2} + \psi_{S+L+3}a_{t-S-3} + ...
\end{aligned}
$$

Multiply both sides of the above equation by $\varphi(B)$, we get

$$\varphi(B)(O_t - \mu) = \theta^{(O)}a_t^{(O)}.$$

If we solve for $\theta^{(O)}(B) = \varphi(B)\psi^{(O)}(B)$ using (3.2), we find $\theta^{(O)}(B)$ as follows

$$
\begin{aligned}
\theta_0^{(O)} &= 1 \\
\theta_1^{(O)} &= -\beta_1/\beta_0 + \varphi_1 \\
\theta_2^{(O)} &= -\beta_2/\beta_0 + \varphi_1\beta_1/\beta_0 + \varphi_2 \\
\vdots \quad & \quad \vdots \quad \quad \vdots \\
\theta_{S-1}^{(O)} &= -\beta_{S-1}/\beta_0 + \varphi_1\beta_{S-2}/\beta_0 + ... + \varphi_{S-2}\beta_1/\beta_0 + \varphi_{S-1} \\
\theta_S^{(O)} &= -(K - \sum_{i=0}^{S-1}\beta_i)/\beta_0 + \varphi_1\beta_{S-1}/\beta_0 + ... + \varphi_{S-1}\beta_1/\beta_0 + \varphi_S \\
\theta_{S+1}^{(O)} &= -\psi_{L+S+1}/\beta_0 + \varphi_1(K - \sum_{i=0}^{S-1}\beta_i)/\beta_0 + \varphi_2\beta_{S-1}/\beta_0 + ... + \varphi_S\beta_1/\beta_0 + \varphi_{S+1} \\
\theta_{S+2}^{(O)} &= -\psi_{L+S+2}/\beta_0 + \varphi_1\psi_{L+S+1}/\beta_0 + \varphi_2(K - \sum_{i=0}^{S-1}\beta_i)/\beta_0 + \varphi_3\beta_{S-1}/\beta_0 + ... \\
&\quad + \varphi_{S+1}\beta_1/\beta_0 + \varphi_{S+2} \\
\vdots \quad & \quad \vdots \quad \quad \vdots \\
\theta_{S+j}^{(O)} &= -\psi_{L+S+j}/\beta_0 + \varphi_1\psi_{L+S+j-1}/\beta_0 + ... + \varphi_{j-1}\psi_{L+S+1}/\beta_0 + \varphi_j(K - \sum_{i=0}^{S-1}\beta_i)/\beta_0 \\
&\quad + \varphi_{j+1}\beta_{S-1}/\beta_0 + ... + \varphi_{S+j-1}\beta_1/\beta_0 + \varphi_{S+j}.
\end{aligned}
$$

To prove that the MA operator $\theta^{(O)}$ of the generalized ordering policy has the order $q^{(O)} = max\{p+d+S, q-L+S\}$, let $j = max\{p+d+S, q-L+S\}$.

<u>Case 1</u>: let $j = p + d + S$ or $p + d \geq q \geq q - L$.

From Z_t, the ψ's in (3.11) are

$$\psi_{p+d+i} = \varphi_1 \psi_{p+d+i-1} + \varphi_2 \psi_{p+d-i-2} + ... + \varphi_{p+d} \psi_i; \; i \geq 0$$

however, since we arbitrarily change the first S ψ's weights in O_t, then

$$\psi_{S+p+d+i} = \varphi_1 \psi_{S+p+d+i-1} + \varphi_2 \psi_{S+p+d-i-2} + ... + \varphi_{p+d} \psi_{S+i}; \; i \geq 0.$$

For the generalized order O_t, the $\theta^{(O)}$'s are

$$
\begin{aligned}
\theta_0^{(O)} &= 1 \\
\theta_1^{(O)} &= -\beta_1/\beta_0 + \varphi_1 \\
\theta_2^{(O)} &= -\beta_2/\beta_0 + \varphi_1\beta_1/\beta_0 + \varphi_2 \\
\vdots \quad & \quad \vdots \qquad \vdots \\
\theta_{S-1}^{(O)} &= -\beta_{S-1}/\beta_0 + \varphi_1\beta_{S-2}/\beta_0 + ... + \varphi_{S-2}\beta_1/\beta_0 + \varphi_{S-1} \\
\theta_S^{(O)} &= -(K - \sum_{i=0}^{S-1}\beta_i)/\beta_0 + \varphi_1\beta_{S-1}/\beta_0 + ... + \varphi_{S-1}\beta_1/\beta_0 + \varphi_S \\
\theta_{S+1}^{(O)} &= -\psi_{L+S+1}/\beta_0 + \varphi_1(K - \sum_{i=0}^{S-1}\beta_i)/\beta_0 + \varphi_2\beta_{S-1}/\beta_0 + ... + \varphi_S\beta_1/\beta_0 + \varphi_{S+1} \\
\vdots \quad & \quad \vdots \qquad \vdots \\
\theta_{S+p+d-1}^{(O)} &= -\psi_{L+S+p+d-1}/\beta_0 + \varphi_1\psi_{L+S+p+d-2}/\beta_0 + ... + \varphi_{p+d-2}\psi_{L+S+1}/\beta_0 \\
&\quad + \varphi_{p+d-1}(K - \sum_{i=0}^{S-1}\beta_i)/\beta_0 + \varphi_{p+d} \\
\theta_{S+p+d}^{(O)} &= -\psi_{L+S+p+d}/\beta_0 + \varphi_1\psi_{L+S+p+d-1}/\beta_0 + ... + \varphi_{p+d-1}\psi_{L+S+1}/\beta_0 + \varphi_{p+d} \\
\theta_{S+p+d+1}^{(O)} &= -\psi_{L+S+p+d+1}/\beta_0 + \varphi_1\psi_{L+S+p+d}/\beta_0 + ... + \varphi_{p+d}\psi_{L+S+1}/\beta_0 \\
\theta_{S+p+d+2}^{(O)} &= -\psi_{L+S+p+d+2}/\beta_0 + \varphi_1\psi_{L+S+p+d+1}/\beta_0 + ... + \varphi_{p+d}\psi_{L+S+2}/\beta_0 \\
\vdots \quad & \quad \vdots \qquad \vdots
\end{aligned}
$$

hence, from Z_t, we have $\theta_{p+d+S+i}^{(O)} = 0$ for $i \geq 1$. In this case, the MA operator $\theta^{(O)}$ has the order $q^{(O)} = p + d + S$ when $j = max\{p + d + S, q - L + S\} = p + d + S$.

Case 2: let $j = q - L$ or $q - L \geq p + d$.

From Z_t, the ψ's in (3.12) are

$$\psi_{q+i} = \varphi_1 \psi_{q+i-1} + \varphi_2 \psi_{q-i-2} + ... + \varphi_{p+d} \psi_{q-p-d+i};\ i \geq 1$$

however, since we arbitrarily change the first S ψ's weights in O_t, then

$$\psi_{S+q+i} = \varphi_1 \psi_{S+q+i-1} + \varphi_2 \psi_{S+q-i-2} + ... + \varphi_{p+d} \psi_{S+q-p-d+i};\ i \geq 1.$$

For the generalized order O_t, the $\theta^{(O)}$'s are

$$
\begin{aligned}
\theta_0^{(O)} &= 1 \\
\theta_1^{(O)} &= -\beta_1/\beta_0 + \varphi_1 \\
\theta_2^{(O)} &= -\beta_2/\beta_0 + \varphi_1\beta_1/\beta_0 + \varphi_2 \\
\vdots \quad & \quad \vdots \quad \quad \vdots \\
\theta_{S-1}^{(O)} &= -\beta_{S-1}/\beta_0 + \varphi_1\beta_{S-2}/\beta_0 + ... + \varphi_{S-2}\beta_1/\beta_0 + \varphi_{S-1} \\
\theta_S^{(O)} &= -(K - \sum_{i=0}^{S-1}\beta_i)/\beta_0 + \varphi_1\beta_{S-1}/\beta_0 + ... + \varphi_{S-1}\beta_1/\beta_0 + \varphi_S \\
\theta_{S+1}^{(O)} &= -\psi_{L+S+1}/\beta_0 + \varphi_1(K - \sum_{i=0}^{S-1}\beta_i)/\beta_0 + \varphi_2\beta_{S-1}/\beta_0 + ... + \varphi_S\beta_1/\beta_0 + \varphi_{S+1} \\
\vdots \quad & \quad \vdots \quad \quad \vdots \\
\theta_{S+p+d-1}^{(O)} &= -\psi_{L+S+p+d-1}/\beta_0 + \varphi_1\psi_{L+S+p+d-2}/\beta_0 + ... + \varphi_{p+d-2}\psi_{L+S+1}/\beta_0 \\
&\quad + \varphi_{p+d-1}(K - \sum_{i=0}^{S-1}\beta_i)/\beta_0 + \varphi_{S+p+d-1} \\
\theta_{S+p+d}^{(O)} &= -\psi_{L+S+p+d}/\beta_0 + \varphi_1\psi_{L+S+p+d-1}/\beta_0 + ... + \varphi_{p+d-1}\psi_{L+S+1}/\beta_0 + \varphi_{S+p+d} \\
\theta_{S+p+d+1}^{(O)} &= -\psi_{L+S+p+d+1}/\beta_0 + \varphi_1\psi_{L+S+p+d}/\beta_0 + ... + \varphi_{p+d}\psi_{L+S+1}/\beta_0 \\
\theta_{S+p+d+2}^{(O)} &= -\psi_{L+S+p+d+2}/\beta_0 + \varphi_1\psi_{L+S+p+d+1}/\beta_0 + ... + \varphi_{p+d}\psi_{L+S+2}/\beta_0 \\
\vdots \quad & \quad \vdots \quad \quad \vdots \\
\theta_{S+q-L}^{(O)} &= -\psi_{S+q}/\beta_0 + \varphi_1\psi_{S+q-1}/\beta_0 + ... + \varphi_{p+d}\psi_{S+q-p-d}/\beta_0 \\
\theta_{S+q-L+1}^{(O)} &= -\psi_{S+q+1}/\beta_0 + \varphi_1\psi_{S+q}/\beta_0 + ... + \varphi_{p+d}\psi_{S+q-p-d+1}/\beta_0 \\
\theta_{S+q-L+2}^{(O)} &= -\psi_{S+q+2}/\beta_0 + \varphi_1\psi_{S+q+1}/\beta_0 + ... + \varphi_{p+d}\psi_{S+q-p-d+2}/\beta_0 \\
\vdots \quad & \quad \vdots \quad \quad \vdots
\end{aligned}
$$

hence, from Z_t, we have $\theta^{(O)}_{q-L+S+i} = 0$ for $i \geq 1$. In this case, the MA operator $\theta^{(O)}$ has the order $q^{(O)} = q - L + S$ when $j = max\{p + d + S, q - L + S\} = q - L + S$.

From case 1 and case 2, $q^{(O)} = max\{p + d + S, q - L + S\}$. Thus O_t is ARIMA$(p, d, max\{p + d + S, q - L + S\})$. Also, its differencing generalized order is ARMA$(p, max\{p + d + S, q - L + S\})$

$$\phi(B)(\nabla^d O_t - \nabla^d \mu) = \theta^{(O)} a^{(O)}_t.$$

$\nabla^d O_t$ can be expressed in the random shock form

$$\begin{aligned} \nabla^d O_t &= \psi^{(\nabla^d O)}(B) a^{(O)}_t \\ &= \left(1 + \psi^{(\nabla^d O)}_1 B + \psi^{(\nabla^d O)}_2 B^2 + \psi^{(\nabla^d O)}_3 B^3 + ...\right) a^{(O)}_t \end{aligned} \tag{4.6}$$

$$\text{where, } \psi^{(\nabla^d O)}_j = \phi_1 \psi^{(\nabla^d O)}_{j-1} + \phi_2 \psi^{(\nabla^d O)}_{j-2} + ... + \phi_p \psi^{(\nabla^d O)}_{j-p} - \theta^{(O)}_j$$

with $\psi^{(\nabla^d O)}_0 = 1$, $\psi^{(\nabla^d O)}_j = 0$ for $j < 0$, $\phi_j = 0$ for $j > p$, and $\theta^{(O)}_j = 0$ for $j < q^{(O)}$.

The standard deviation of $\nabla^d O_t$ is absolutely summable in the form

$$\sigma_{\nabla^d O_t} = \sqrt{1 + \left(\psi^{(\nabla^d O)}_1\right)^2 + \left(\psi^{(\nabla^d O)}_2\right)^2 + \left(\psi^{(\nabla^d O)}_3\right)^2 + ...}\ \sigma_{a^{(O)}_t} \tag{4.7}$$

where $\sigma_{a^{(O)}_t} = \beta_0 \sigma_a$.

End of Proof

4.3 Sample Models of the Generalized Ordering Policy

This section exemplifies the smoothing ordering policy for ARIMA(0,1,1), ARIMA(0,2,2), ARIMA(0,3,3), and ARIMA(1,0,0) demand.

4.3.1 ARIMA(0,1,1)

This section illustrates the application of theorem 3 and 4 when the demand series, Z_t, is ARIMA(0,1,1)

$$(1-B)Z_t = (1-\theta B)a_t$$

or in random shock form

$$Z_t = a_t + (1-\theta)a_{t-1} + (1-\theta)a_{t-2} + (1-\theta)a_{t-3} + ...$$

The standard MRP order in (3.15) is

$$(1-B)O_t = \Big(\big(1+L(1-\theta)\big) - \big(1+(L-1)(1-\theta)\big)B\Big)a_t$$

or in random shock form

$$O_t = \big(1+L(1-\theta)\big)a_t + (1-\theta)a_{t-1} + (1-\theta)a_{t-2} + (1-\theta)a_{t-3} + ...$$

Its differencing order is an MA(2)

$$\nabla^d O_t = \big(1 + L(1-\theta)\big)a_t - \big(1 + (L-1)(1-\theta)\big)a_{t-1}.$$

The generalized order with S smoothing period in random shock form is

$$\begin{aligned} O_t &= \beta_0 a_t + \beta_1 a_{t-1} + \beta_2 a_{t-2} + ... + \beta_{S-1} a_{t-S+1} \\ &+ \Big(1 + (S+L)(1-\theta) - \sum_{i=0}^{S-1} \beta_i\Big) a_{t-S} \\ &+ (1-\theta)a_{t-S-1} + (1-\theta)a_{t-S-2} + (1-\theta)a_{t-S-3} + ... \end{aligned} \tag{4.8}$$

Rewritten in the forms of demand Z's and forecast demand $\hat{Z}$ where

$$\begin{aligned} a_{t-j} &= Z_{t-j} - \hat{Z}_{t-j-1}(1);\ j = 0, 1, 2, ... \\ \hat{Z}_{t-S-1}(L) &= (1-\theta)a_{t-S-1} + (1-\theta)a_{t-S-2} + (1-\theta)a_{t-S-3} + ... \end{aligned}$$

Thus

$$\begin{aligned} O_t &= \beta_0\big(Z_t - \hat{Z}_{t-1}(1)\big) + \beta_1\big(Z_{t-1} - \hat{Z}_{t-2}(1)\big) \\ &+ \beta_2\big(Z_{t-2} - \hat{Z}_{t-3}(1)\big) + ... + \beta_{S-1}\big(Z_{t-S+1} - \hat{Z}_{t-S}(1)\big) \\ &+ \Big(1 + (S+L)(1-\theta) - \sum_{i=0}^{S-1} \beta_i\Big)\big(Z_{t-S} - \hat{Z}_{t-S-1}(1)\big) \\ &+ \hat{Z}_{t-S-1}(L). \end{aligned} \tag{4.9}$$

By theorem 3, I_t is MA$(S+L-1)$ with

$$\begin{aligned}
\theta_1^{(I)} &= 2-\theta \\
\theta_2^{(I)} &= 3-2\theta \\
\theta_3^{(I)} &= 4-3\theta \\
\vdots \quad & \quad \vdots \\
\theta_{L-1}^{(I)} &= L-(L-1)\theta \\
\theta_L^{(I)} &= (L+1)-L\theta-\beta_0 \\
\theta_{L+1}^{(I)} &= (L+2)-(L+1)\theta-\beta_0-\beta_1 \\
\vdots \quad & \quad \vdots \qquad \quad \vdots \\
\theta_{L+S-1}^{(I)} &= (L+S)-(L+S-1)\theta-\sum_{i=0}^{S-1}\beta_i.
\end{aligned}$$

Thus

$$\begin{aligned}
I_t = T &- a_t-(2-\theta)a_{t-1}-...-\big(L-(L-1)\theta\big)a_{t-L+1} \\
&-\big(L+1-L\theta-\beta_0\big)a_{t-L} \\
&-\big(L+2-(L+1)\theta-\beta_0-\beta_1\big)a_{t-L+1}-... \\
&-\Big(L+S-(L+S-1)\theta-\sum_{i=0}^{S-1}\beta_i\Big)a_{t-L+S+1}.
\end{aligned} \tag{4.10}$$

The standard deviation of the inventory is given by

$$\sigma_I = \Big(1 + (2-\theta)^2 + (3-2\theta)^2 + ... + \big(L - (L-1)\theta\big)^2$$
$$+ \big(L+1-L\theta-\beta_0\big)^2 + \big(L+2-(L+1)\theta-\beta_0-\beta_1\big)^2 + ...$$
$$+ \big(L+S-(L+S-1)\theta - \sum_{i=0}^{S-1}\beta_i\big)^2\Big)^{1/2}\sigma_a.$$

By theorem 4, O_t is IMA(1,1+S)

$$(1-B)O_t = (1-\theta_1^{(O)}B - \theta_2^{(O)}B^2 - ... - \theta_{S+1}^{(O)}B^{S+1})a_t^{(O)}$$

where

$$a_t^{(O)} = \beta_0 a_t$$

$$K = 1 + (S+L)(1-\theta)$$

and

$$
\begin{aligned}
\theta_0^{(O)} &= 1 \\
\theta_1^{(O)} &= -\beta_1/\beta_0 + 1 = (\beta_0 - \beta_1)/\beta_0 \\
\theta_2^{(O)} &= -\beta_2/\beta_0 + \beta_1/\beta_0 = (\beta_1 - \beta_2)/\beta_0 \\
\vdots \qquad & \quad \vdots \qquad\qquad \vdots \\
\theta_{S-1}^{(O)} &= -\beta_{S-1}/\beta_0 + \beta_{S-2}/\beta_0 = (\beta_{S-2} - \beta_{S-1})/\beta_0 \\
\theta_S^{(O)} &= -(K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 + \beta_{S-1}/\beta_0 = \Big(- \big(1 + (S+L)(1-\theta)\big) + \sum_{i=0}^{S-1} \beta_i + \beta_{S-1} \Big)/\beta_0 \\
\theta_{S+1}^{(O)} &= -\psi_{L+S+1}/\beta_0 + (K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 = \big(1 + (S+L-1)(1-\theta) - \sum_{i=0}^{S-1} \beta_i\big)/\beta_0.
\end{aligned}
$$

Rewritten following the ARIMA(p, d, q) form, we have

$$
\begin{aligned}
(1-B)O_t = a_t^{(O)} &- \big((\beta_0 - \beta_1)/\beta_0\big) a_{t-1}^{(O)} \\
&- \big((\beta_1 - \beta_2)/\beta_0\big) a_{t-2}^{(O)} - \ldots - \big((\beta_{S-2} - \beta_{S-1})/\beta_0\big) a_{t-S+1}^{(O)} \\
&- \bigg(\Big(- \big(1 + (S+L)(1-\theta)\big) + \sum_{i=0}^{S-1} \beta_i + \beta_{S-1} \Big)/\beta_0 \bigg) a_{t-S}^{(O)} \\
&- \Big(\big(1 + (S+L-1)(1-\theta) - \sum_{i=0}^{S-1} \beta_i\big)/\beta_0 \Big) a_{t-S-1}^{(O)} \\
= \beta_0 a_t &- (\beta_0 - \beta_1) a_{t-1} - (\beta_1 - \beta_2) a_{t-2} - \ldots - (\beta_{S-2} - \beta_{S-1}) a_{t-S+1} \\
&- \Big(- \big(1 + (S+L)(1-\theta)\big) + \sum_{i=0}^{S-1} \beta_i + \beta_{S-1} \Big) a_{t-S} \\
&- \big(1 + (S+L-1)(1-\theta) - \sum_{i=0}^{S-1} \beta_i\big) a_{t-S-1}.
\end{aligned}
$$

Its differencing generalized order is an MA(1+S)

$$\begin{aligned}
\nabla^d O_t = &\beta_0 a_t - (\beta_0 - \beta_1)a_{t-1} \\
&- (\beta_1 - \beta_2)a_{t-2} - ... - (\beta_{S-2} - \beta_{S-1})a_{t-S+1} \\
&- \Big(- \big(1 + (S+L)(1-\theta)\big) + \sum_{i=0}^{S-1} \beta_i + \beta_{S-1} \Big) a_{t-S} \\
&- \big(1 + (S+L-1)(1-\theta) - \sum_{i=0}^{S-1} \beta_i\big) a_{t-S-1}.
\end{aligned} \tag{4.11}$$

The standard deviation of $\nabla^d O_t$ following (4.7) is

$$\begin{aligned}
\sigma_{\nabla^d O_t} = &\Big(\beta_0^2 + (\beta_1 - \beta_0)^2 + (\beta_2 - \beta_1)^2 + ... + (\beta_{S-2} - \beta_{S-1})^2 \\
&+ \big(1 + (S+L)(1-\theta) - \sum_{i=0}^{S-1} \beta_i - \beta_{S-1}\big)^2 \\
&+ \big(1 + (S+L-1)(1-\theta) - \sum_{i=0}^{S-1} \beta_i\big)^2\Big)^{1/2} \sigma_a.
\end{aligned} \tag{4.12}$$

4.3.2 ARIMA(0,2,2)

The IMA(2,2) demand model in ARIMA(p, d, q) form is

$$(1-B)^2 Z_t = (1 - \theta_1 B - \theta_2 B^2) a_t$$

or in random shock form

$$Z_t = a_t + (2 - \theta_1)a_{t-1} + (3 - 2\theta_1 - \theta_2)a_{t-2} + (4 - 3\theta_1 - 2\theta_2)a_{t-3} + ...$$

The standard MRP order is

$$(1-2B+B^2)O_t = (1-\theta_1^{(O)}B-\theta_2^{(O)}B^2)a_t^{(O)}$$

or in random shock form

$$O_t = Ka_t + \psi_{L+1}a_{t-1} + \psi_{L+2}a_{t-2} + \psi_{L+3}a_{t-3} + ...$$

where

$$K = \sum_{i=1}^{L+1} i - \sum_{i=1}^{L} i\theta_1 - \sum_{i=1}^{L-1} i\theta_2$$

$$\psi_i = (i+1) - i\theta_1 - (i-1)\theta_2.$$

Its differencing order is an MA(2)

$$\nabla^d O_t = (1-\theta_1^{(O)}B-\theta_2^{(O)}B^2)a_t^{(O)}.$$

The generalized order with S smoothing period in random shock form is

$$O_t = \beta_0 a_t + \beta_1 a_{t-1} + \beta_2 a_{t-2} + ... + \beta_{S-1}a_{t-S+1} + (K - \sum_{i=0}^{S-1}\beta_i)a_{t-S}$$

$$+ \psi_{S+L+1}a_{t-S-1} + \psi_{S+L+2}a_{t-S-2} + \psi_{S+L+3}a_{t-S-3} + ...$$

By theorem 3, I_t is MA$(S+L-1)$ with

$$\begin{aligned}
\theta_1^{(I)} &= 3 - \theta_1 \\
\theta_2^{(I)} &= 6 - 3\theta_1 - \theta_2 \\
\theta_3^{(I)} &= 10 - 6\theta_1 - 3\theta_2 \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \\
\theta_{L-1}^{(I)} &= \frac{L(L+1)}{2} - \frac{(L-1)L}{2}\theta_1 - \frac{(L-2)(L-1)}{2}\theta_2 \\
\theta_L^{(I)} &= \frac{(L+1)(L+2)}{2} - \frac{L(L+1)}{2}\theta_1 - \frac{(L-1)(L)}{2}\theta_2 - \beta_0 \\
\theta_{L+1}^{(I)} &= \frac{(L+2)(L+3)}{2} - \frac{(L+1)(L+2)}{2}\theta_1 - \frac{L(L+1)}{2}\theta_2 - \beta_0 - \beta_1 \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \\
\theta_{L+S-1}^{(I)} &= \frac{(L+S)(L+S+1)}{2} - \frac{(L+S-1)(L+S)}{2}\theta_1 \\
&\quad - \frac{(L+S-2)(L+S-1)}{2}\theta_2 - \sum_{i=0}^{S-1}\beta_i.
\end{aligned}$$

Thus

$$I_t = T - a_t^{(I)} - \theta_1^{(I)}a_{t-1}^{(I)} - \theta_2^{(I)}a_{t-2}^{(I)} - ... - \theta_{S+L-1}^{(I)}a_{t-L-S+1}^{(I)}.$$

The standard deviation of the inventory is given by

$$\sigma_I = \sqrt{1 + (\theta_1^{(I)})^2 + (\theta_2^{(I)})^2 + ... + (\theta_{S+L-1}^{(I)})^2}\sigma_a.$$

By theorem 4, O_t is IMA(2,2+S)

$$(1-B)^2 O_t = (1 - \theta_1^{(O)} B - \theta_2^{(O)} B^2 - ... - \theta_{S+1}^{(O)} B^{S+1} - \theta_{S+2}^{(O)} B^{S+2}) a_t^{(O)}$$

where

$$a_t^{(O)} = \beta_0 a_t$$

$$K = \sum_{i=1}^{S+L+1} i - \sum_{i=1}^{S+L} i\theta_1 - \sum_{i=1}^{S+L-1} i\theta_2$$

and

$$\begin{aligned}
\theta_0^{(O)} &= 1 \\
\theta_1^{(O)} &= -\beta_1/\beta_0 + 2 = (2\beta_0 - \beta_1)/\beta_0 \\
\theta_2^{(O)} &= -\beta_2/\beta_0 + 2\beta_1/\beta_0 - 1 = (-\beta_0 + 2\beta_1 - \beta_2)/\beta_0 \\
\theta_3^{(O)} &= -\beta_3/\beta_0 + 2\beta_2/\beta_0 - \beta_1/\beta_0 = (-\beta_3 + 2\beta_2 - \beta_1)/\beta_0 \\
\vdots \quad & \quad \vdots \qquad \quad \vdots \\
\theta_{S-1}^{(O)} &= -\beta_{S-1}/\beta_0 + 2\beta_{S-2}/\beta_0 - \beta_{S-3}/\beta_0 = (-\beta_{S-1} + 2\beta_{S-2} - \beta_{S-3})/\beta_0 \\
\theta_S^{(O)} &= -(K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 + 2\beta_{S-1}/\beta_0 - \beta_{S-2}/\beta_0 \\
\theta_{S+1}^{(O)} &= -\psi_{L+S+1}/\beta_0 + 2(K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 - \beta_{S-1})/\beta_0 \\
\theta_{S+2}^{(O)} &= -\psi_{L+S+2}/\beta_0 + 2\psi_{L+S+1}/\beta_0 - (K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 \\
\theta_{S+j}^{(O)} &= -\psi_{L+S+j}/\beta_0 + 2\psi_{L+S+j-1}/\beta_0 - (\psi_{L+S+j-2}/\beta_0 = 0;\ j \geq 3.
\end{aligned}$$

Rewritten following the ARIMA(p, d, q) form, we have

$$(1 - 2B + B^2)O_t = (1 - \theta_1^{(O)} B - \theta_2^{(O)} B^2 - \theta_3^{(O)} B^3 - ... - \theta_{S+2}^{(O)} B^{S+2}) a_t^{(O)}.$$

Its differencing generalized order is an MA(2+S)

$$\nabla^d O_t = (1 - \theta_1^{(O)} B - \theta_2^{(O)} B^2 - \theta_3^{(O)} B^3 - ... - \theta_{S+2}^{(O)} B^{S+2}) a_t^{(O)}.$$

The standard deviation of $\nabla^d O_t$ following (4.7) is

$$\begin{aligned}
\sigma_{\nabla^d O_t} = \Big(&\beta_0^2 + (2\beta_1 - \beta_0)^2 + (\beta_2 - 2\beta_1 + \beta_0)^2 + ... + (\beta_{S-1} - 2\beta_{S-2} + \beta_{S-3})^2 \\
&+ \big(K - \sum_{i=0}^{S-1} \beta_i - 2\beta_{S-1} + \beta_{S-2}\big)^2 \\
&+ \big(\psi_{L+S+1} - 2(K - \sum_{i=0}^{S-1} \beta_i) + \beta_{S-1}\big)^2 \\
&+ \big(\psi_{L+S+2} - 2\psi_{L+S+1} + (K - \sum_{i=0}^{S-1} \beta_i)\big)^2\Big)^{1/2} \sigma_a
\end{aligned}$$

where

$$K = \sum_{i=1}^{L+1} i - \sum_{i=1}^{L} i\theta_1 - \sum_{i=1}^{L-1} i\theta_2$$

$$\psi_i = (i + 1) - i\theta_1 - (i - 1)\theta_2.$$

4.3.3 ARIMA(0,3,3)

The IMA(3,3) demand model in ARIMA(p, d, q) form is

$$(1 - B)^3 Z_t = (1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3) a_t$$

or in random shock form

$$\begin{aligned} Z_t &= a_t + (3 - \theta_1) a_{t-1} + (6 - 3\theta_1 - \theta_2) a_{t-2} \\ &\quad + (9 - 6\theta_1 - 3\theta_2 - \theta_3) a_{t-3} + (12 - 9\theta_1 - 6\theta_2 - 3\theta_3) + ... \end{aligned}$$

The standard MRP order is

$$(1 - 3B + 3B^2 - B^3) O_t = (1 - \theta_1^{(O)} B - \theta_2^{(O)} B^2 - \theta_3^{(O)} B^3) a_t^{(O)}$$

or in random shock form or in random shock form

$$O_t = K a_t + \psi_{L+1} a_{t-1} + \psi_{L+2} a_{t-2} + \psi_{L+3} a_{t-3} + ...$$

where

$$K = \Big(1 + \sum_{i=1}^{L} 3i\Big) - \Big(1 + \sum_{i=1}^{L-1} 3i\Big)\theta_1 - \Big(1 + \sum_{i=1}^{L-2} 3i\Big)\theta_2 - \Big(1 + \sum_{i=1}^{L-3} 3i\Big)\theta_3$$

$$\psi_i = 3i - 3(i-1)\theta_1 - 3(i-2)\theta_2 - 3(i-3)\theta_3.$$

Its differencing order is an MA(2)

$$\nabla^d O_t = (1 - \theta_1^{(O)} B - \theta_2^{(O)} B^2 - \theta_3^{(O)} B^3) a_t^{(O)}.$$

The generalized order with S smoothing period in random shock form is

$$O_t = \beta_0 a_t + \beta_1 a_{t-1} + \beta_2 a_{t-2} + ... + \beta_{S-1} a_{t-S+1} + (K - \sum_{i=0}^{S-1} \beta_i) a_{t-S}$$

$$+ \psi_{S+L+1} a_{t-S-1} + \psi_{S+L+2} a_{t-S-2} + \psi_{S+L+3} a_{t-S-3} + ...$$

By theorem 3, I_t is MA$(S+L-1)$ with

$$
\begin{aligned}
\theta_1^{(I)} &= 4-\theta_1 \\
\theta_2^{(I)} &= 10-4\theta_1-\theta_2 \\
\theta_3^{(I)} &= 19-10\theta_1-4\theta_2-\theta_3 \\
\theta_4^{(I)} &= 31-19\theta_1-10\theta_2-4\theta_3 \\
\vdots \quad & \quad \vdots \qquad \quad \vdots \\
\theta_{L-1}^{(I)} &= \Big(1+\sum_{i=1}^{L-1}3i\Big)-\Big(1+\sum_{i=1}^{L-2}3i\Big)\theta_1-\Big(1+\sum_{i=1}^{L-3}3i\Big)\theta_2-\Big(1+\sum_{i=1}^{L-4}3i\Big)\theta_3 \\
\theta_{L}^{(I)} &= \Big(1+\sum_{i=1}^{L}3i\Big)-\Big(1+\sum_{i=1}^{L-1}3i\Big)\theta_1-\Big(1+\sum_{i=1}^{L-2}3i\Big)\theta_2-\Big(1+\sum_{i=1}^{L-3}3i\Big)\theta_3-\beta_0 \\
\theta_{L+1}^{(I)} &= \Big(1+\sum_{i=1}^{L+1}3i\Big)-\Big(1+\sum_{i=1}^{L}3i\Big)\theta_1-\Big(1+\sum_{i=1}^{L-1}3i\Big)\theta_2-\Big(1+\sum_{i=1}^{L-2}3i\Big)\theta_3-\beta_0-\beta_1 \\
\vdots \quad & \quad \vdots \qquad \quad \vdots \\
\theta_{L+S-1}^{(I)} &= \Big(1+\sum_{i=1}^{L+S-1}3i\Big)-\Big(1+\sum_{i=1}^{L+S-2}3i\Big)\theta_1-\Big(1+\sum_{i=1}^{L+S-3}3i\Big)\theta_2 \\
&\quad -\Big(1+\sum_{i=1}^{L+S-4}3i\Big)\theta_3-\sum_{i=0}^{S-1}\beta_i.
\end{aligned}
$$

Thus

$$I_t = T - a_t^{(I)} - \theta_1^{(I)}a_{t-1}^{(I)} - \theta_2^{(I)}a_{t-2}^{(I)} - \dots - \theta_{S+L-1}^{(I)}a_{t-L-S+1}^{(I)}.$$

The standard deviation of the inventory is given by

$$\sigma_I = \sqrt{1 + (\theta_1^{(I)})^2 + (\theta_2^{(I)})^2 + ... + (\theta_{S+L-1}^{(I)})^2}\sigma_a.$$

By theorem 4, O_t is IMA(3,3+S)

$$(1-B)^3 O_t = (1 - \theta_1^{(O)} B - \theta_2^{(O)} B^2 - ... - \theta_{S+1}^{(O)} B^{S+1} - \theta_{S+3}^{(O)} B^{S+3}) a_t^{(O)}$$

where

$$a_t^{(O)} = \beta_0 a_t$$

$$K = \Big(1 + \sum_{i=1}^{L+S} 3i\Big) - \Big(1 + \sum_{i=1}^{L+S-1} 3i\Big)\theta_1 - \Big(1 + \sum_{i=1}^{L+S-2} 3i\Big)\theta_2 - \Big(1 + \sum_{i=1}^{L+S-3} 3i\Big)\theta_3$$

and

$$
\begin{aligned}
\theta_0^{(O)} &= 1 \\
\theta_1^{(O)} &= -\beta_1/\beta_0 + 3 = (3\beta_0 - \beta_1)/\beta_0 \\
\theta_2^{(O)} &= -\beta_2/\beta_0 + 3\beta_1/\beta_0 - 3 = (-3\beta_0 + 3\beta_1 - \beta_0)/\beta_0 \\
\theta_3^{(O)} &= -\beta_3/\beta_0 + 3\beta_2/\beta_0 - 3\beta_1/\beta_0 + 1 = (-\beta_3 + 3\beta_2 - 3\beta_1 + \beta_0)/\beta_0 \\
\theta_4^{(O)} &= -\beta_4/\beta_0 + 3\beta_3/\beta_0 - 3\beta_2/\beta_0 + \beta_1/\beta_0 = (-\beta_4 + 3\beta_3 - 3\beta_2 + \beta_1)/\beta_0 \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \\
\theta_{S-1}^{(O)} &= -\beta_{S-1}/\beta_0 + 3\beta_{S-2}/\beta_0 - 3\beta_{S-3}/\beta_0 + \beta_{S-4}/\beta_0 \\
&= (-\beta_{S-1} + 3\beta_{S-2} - 3\beta_{S-3} + \beta_{S-4})/\beta_0 \\
\theta_S^{(O)} &= -(K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 + 3\beta_{S-1}/\beta_0 - 3\beta_{S-2}/\beta_0 + \beta_{S-3}/\beta_0 \\
\theta_{S+1}^{(O)} &= -\psi_{L+S+1}/\beta_0 + 3(K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 - 3\beta_{S-1})/\beta_0 + \beta_{S-2}/\beta_0 \\
\theta_{S+2}^{(O)} &= -\psi_{L+S+2}/\beta_0 + 3\psi_{L+S+1}/\beta_0 - (K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 + \beta_{S-1}/\beta_0 \\
\theta_{S+3}^{(O)} &= -\psi_{L+S+3}/\beta_0 + 3\psi_{L+S+2}/\beta_0 - 3\psi_{L+S+2}/\beta_0 - (K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 \\
\theta_{S+j}^{(O)} &= -\psi_{L+S+j}/\beta_0 + 3\psi_{L+S+j-1}/\beta_0 - 3(\psi_{L+S+j-2}/\beta_0 + \psi_{L+S+j-3}/\beta_0 = 0;\ j \geq 4.
\end{aligned}
$$

Rewritten following the ARIMA(p, d, q) form, we have

$$(1 - 3B + 3B^2 - B^3)O_t = (1 - \theta_1^{(O)} B - \theta_2^{(O)} B^2 - \theta_3^{(O)} B^3 - ... - \theta_{S+3}^{(O)} B^{S+3}) a_t^{(O)}.$$

Its differencing generalized order is an MA(3+S)

$$\nabla^d O_t = (1 - \theta_1^{(O)} B - \theta_2^{(O)} B^2 - \theta_3^{(O)} B^3 - ... - \theta_{S+3}^{(O)} B^{S+3}) a_t^{(O)}.$$

The standard deviation of $\nabla^d O_t$ following (4.7) is

$$\sigma_{\nabla^d O_t} = \sqrt{(1 + (\theta_1^{(O)})^2 + (\theta_2^{(O)})^2 + (\theta_3^{(O)})^2 + ... + (\theta_{S+3}^{(O)})^2}\ \sigma_a.$$

4.3.4 ARIMA(1,0,0)

The AR(1) demand model in ARIMA(p, d, q) form is

$$(1 - \phi B) Z_t = a_t$$

or in random shock form

$$Z_t = a_t + \phi a_{t-1} + \phi^2 a_{t-1} + \phi^3 a_{t-3} + ...$$

The standard MRP order is

$$(1 - \phi B) O_t = \Big(1 - \frac{\phi(1 - \phi^L)}{1 - \phi^{L+1}} B\Big) a_t^{(O)}$$

or in random shock form

$$O_t = (1 + \phi + \phi^2 + \phi^3 + ... + \phi^L) a_t + \phi^{L+1} a_{t-1} + \phi^{L+2} a_{t-2} + \phi^{L+3} a_{t-3} + ...$$

Since $d = 0$, its differencing order is the same as the order O_t.

The generalized ordering policy with S smoothing period in random shock form is

$$O_t = \beta_0 a_t + \beta_1 a_{t-1} + \beta_2 a_{t-2} + ... + \beta_{S-1} a_{t-S+1}$$
$$+ (K - \sum_{i=0}^{S-1} \beta_i) a_{t-S} + \phi^{S+L+1} a_{t-S-1} + \phi^{S+L+2} a_{t-S-2} + \phi^{S+L+3} a_{t-S-3} + ...$$

where

$$K = 1 + \phi + \phi^2 + \phi^3 + ... + \phi^{S+L} = \frac{1 - \phi^{S+L+1}}{1 - \phi}.$$

By theorem 3, I_t is MA$(S + L - 1)$ with

$$\begin{aligned}
\theta_1^{(I)} &= (1 - \phi^2)/(1 - \phi) \\
\theta_2^{(I)} &= (1 - \phi^3)/(1 - \phi) \\
\theta_3^{(I)} &= (1 - \phi^4)/(1 - \phi) \\
&\vdots \\
\theta_{L-1}^{(I)} &= (1 - \phi^L)/(1 - \phi) \\
\theta_L^{(I)} &= (1 - \phi^{L+1})/(1 - \phi) - \beta_0 \\
\theta_{L+1}^{(I)} &= (1 - \phi^{L+2})/(1 - \phi) - \beta_0 - \beta_1 \\
&\vdots \\
\theta_{L+S-1}^{(I)} &= (1 - \phi^{L+S})/(1 - \phi) - \sum_{i=0}^{S-1} \beta_i.
\end{aligned}$$

Thus

$$I_t = T - a_t^{(I)} - \theta_1^{(I)} a_{t-1}^{(I)} - \theta_2^{(I)} a_{t-2}^{(I)} - ... - \theta_{S+L-1}^{(I)} a_{t-L-S+1}^{(I)}.$$

The standard deviation of the inventory is given by

$$\sigma_I = \sqrt{1 + (\theta_1^{(I)})^2 + (\theta_2^{(I)})^2 + ... + (\theta_{S+L-1}^{(I)})^2} \sigma_a.$$

By theorem 4, O_t is ARMA(1,1+S)

$$(1 - \phi B) O_t = (1 - \theta_1^{(O)} B - \theta_2^{(O)} B^2 - ... - \theta_{S+1}^{(O)} B^{S+1}) a_t^{(O)}$$

where

$$a_t^{(O)} = \beta_0 a_t$$

$$K = 1 + \phi + \phi^2 + \phi^3 + ... + \phi^{S+L} = \frac{1 - \phi^{S+L+1}}{1 - \phi}$$

and

$$
\begin{aligned}
\theta_0^{(O)} &= 1 \\
\theta_1^{(O)} &= -\beta_1/\beta_0 + \phi = (\phi\beta_0 - \beta_1)/\beta_0 \\
\theta_2^{(O)} &= -\beta_2/\beta_0 + \phi\beta_1/\beta_0 = (\phi\beta_1 - \beta_2)/\beta_0 \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots \\
\theta_{S-1}^{(O)} &= -\beta_{S-1}/\beta_0 + \phi\beta_{S-2}/\beta_0 = (\phi\beta_{S-2} - \beta_{S-1})/\beta_0 \\
\theta_S^{(O)} &= -(K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 + \phi\beta_{S-1}/\beta_0 \\
\theta_{S+1}^{(O)} &= -\phi^{L+S+1}/\beta_0 + \phi(K - \sum_{i=0}^{S-1} \beta_i)/\beta_0.
\end{aligned}
$$

Rewritten following the ARIMA(p, d, q) form, we have

$$
(1 - \phi B)O_t = (1 - \theta_1^{(O)} B - \theta_2^{(O)} B^2 - \theta_3^{(O)} B^3 - ... - \theta_{S+1}^{(O)} B^{S+1}) a_t^{(O)}.
$$

Its differencing generalized order is the same as the generalized order. From (4.6)

$$
\begin{aligned}
\nabla^d O_t &= \psi^{(\nabla^d O)}(B) a_t^{(O)} \\
&= \big(1 + \psi_1^{(\nabla^d O)} B + \psi_2^{(\nabla^d O)} B^2 + \psi_3^{(\nabla^d O)} B^3 + ...\big) a_t^{(O)}
\end{aligned}
$$

where the $\psi^{(\nabla^d O)}$'s are

$$\begin{aligned}
\psi_0^{(\nabla^d O)} &= 1 \\
\psi_1^{(\nabla^d O)} &= \varphi_1 - \theta_1^{(\nabla^d O)} = \beta_1/\beta_0 \\
\psi_2^{(\nabla^d O)} &= \varphi_1 \psi_1^{(\nabla^d O)} - \theta_2^{(\nabla^d O)} = \beta_2/\beta_0 \\
\psi_3^{(\nabla^d O)} &= \varphi_1 \psi_2^{(\nabla^d O)} - \theta_3^{(\nabla^d O)} = \beta_3/\beta_0 \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \\
\psi_{S-1}^{(\nabla^d O)} &= \varphi_1 \psi_{S-2}^{(\nabla^d O)} - \theta_{S-1}^{(\nabla^d O)} = \beta_{S-1}/\beta_0 \\
\psi_S^{(\nabla^d O)} &= \varphi_1 \psi_{S-1}^{(\nabla^d O)} - \theta_S^{(\nabla^d O)} = (K - \sum_{i=0}^{S-1} \beta_i)/\beta_0 \\
\psi_{S+1}^{(\nabla^d O)} &= \varphi_1 \psi_S^{(\nabla^d O)} - \theta_{S+1}^{(\nabla^d O)} = \phi^{L+S+1}/\beta_0 \\
\psi_{S+2}^{(\nabla^d O)} &= \varphi_1 \psi_{S+1}^{(\nabla^d O)} = \phi^{L+S+2}/\beta_0 \\
\psi_{S+3}^{(\nabla^d O)} &= \varphi_1 \psi_{S+2}^{(\nabla^d O)} = \phi^{L+S+3}/\beta_0 \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \\
\psi_{S+j}^{(\nabla^d O)} &= \varphi_1 \psi_{S+j-1}^{(\nabla^d O)} = \phi^{L+S+j}/\beta_0.
\end{aligned}$$

The standard deviation of $\nabla^d O_t$ following (4.7) is

$$\begin{aligned}
\sigma_{\nabla^d O_t} &= \Big(\beta_0^2 + \beta_1^2 + \beta_2^2 + ... + \beta_{S-1}^2 + \big(K - \sum_{i=0}^{S-1} \beta_i\big)^2 \\
&\quad + \phi^{2(L+S+1)} + \phi^{2(L+S+2)} + \phi^{2(L+S+3)} + ...\Big)^{1/2} \sigma_a \qquad (4.13) \\
&= \left(\beta_0^2 + \beta_1^2 + ... + \beta_{S-1}^2 + \left(K - \sum_{i=0}^{S-1} \beta_i\right)^2 + \frac{\phi^{2(L+S+1)}}{1 - \phi^2}\right)^{1/2} \sigma_a.
\end{aligned}$$

4.4 The Smoothing Ordering Policy for ARIMA(p,0,q) and ARIMA(p,1,q)

Instead of using the standard MRP ordering policy introduced in chapter 3 that minimizes the variation in inventory I_t, we can control the tradeoffs between the variation in inventory I_t and the variation in differencing generalized orders $\nabla^d O_t$ by adjusting the weight β's. The minimum variation in differencing generalized orders $\nabla^d O_t$, the so called smoothing orders, is obtained by minimizing the variation of $\nabla^d O_t$ respected to $\{\beta_i;\ i = 0, 1, 2, ..., S-1\}$.

This section provides generic formulas to determine the β's weights for the smoothing orders for ARIMA$(p, 0, q)$ and ARIMA$(p, 1, q)$ demand models. By using these generic formulas, a retalier/supplier can obtain an MRP plan that has the smallest changes in the ordering plan from period to period. However, the tradeoff for the retalier/supplier in using the smoothing ordering policy is the need in the increased amount of inventory compared with using the up to target ordering policy. These generic formulas can be applied to any ordering lead time L and any smoothing period S. Under the smoothing ordering policy, the following theorems hold:

Theorem 5 *For an ARIMA$(p, 0, q)$ order model, the smoothing order that has the minimum variation in orders has the set of optimal smoothing weights* $\beta_i = \dfrac{K}{S+1}$ *for* $i = 0, 1, 2, ..., S$ *where* $K = 1 + \psi_1 + \psi_2 + \psi_3 + ... + \psi_{S+L}$ *and* $\beta_S = K - \sum_{i=0}^{S-1} \beta_i$.

<u>*Proof*</u>

The smoothing ordering policy for ARIMA$(p, 0, q)$ is to minimize $\sigma_{\nabla^d O_t}$ respected to $\{\beta_i;\ i = 0, 1, 2, ..., S-1\}$. However, the differencing order for ARIMA$(p, 0, q)$ is the same as the generalized order in (4.1)

$$\nabla^d O_t = O_t = \beta_0 a_t + \beta_1 a_{t-1} + \beta_2 a_{t-2} + ... + \beta_{S-1} a_{t-S+1} + (K - \sum_{i=0}^{S-1} \beta_i) a_{t-S} \\ + \psi_{S+L+1} a_{t-S-1} + \psi_{S+L+2} a_{t-S-2} + ...$$

where $K = 1 + \psi_1 + \psi_2 + ... + \psi_{S+L}$ and its standard deviation $\sigma_{\nabla^d O_t} = \sigma_{O_t}$ is

$$\sigma_{\nabla^d O_t} = \big(\beta_0^2 + \beta_1^2 + \beta_2^2 + ... + \beta_{S-1}^2 + (K - \sum_{i=0}^{S-1} \beta_i)^2 \\ + \psi_{S+L+1}^2 + \psi_{S+L+2}^2 + ...\big)^{1/2} \sigma_a$$

where $\sum_{i=S+L+1}^{\infty} \psi_i^2$ is absolutely summable.

Thus, the system of linear equations of $\frac{d}{d\beta} Var(\nabla^0 O_t) = \frac{d}{d\beta} Var(O_t) = 0$

is

$$
\begin{aligned}
\frac{d}{d\beta_0} &= 2\beta_0 - 2\big(K - \sum_{i=0}^{S-1} \beta_i\big) = 0 \\
\frac{d}{d\beta_1} &= 2\beta_1 - 2\big(K - \sum_{i=0}^{S-1} \beta_i\big) = 0 \\
\frac{d}{d\beta_2} &= 2\beta_2 - 2\big(K - \sum_{i=0}^{S-1} \beta_i\big) = 0 \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots \\
\frac{d}{d\beta_{S-1}} &= 2\beta_{S-1} - 2\big(K - \sum_{i=0}^{S-1} \beta_i\big) = 0.
\end{aligned}
$$

Hence, the set of optimal smoothing weights β_i for $i = 0, 1, 2, ..., S-1$ for the smoothing ordering policy for that minimizes the variation in differencing order for ARIMA$(p, 0, q)$ model is

$$\beta_i = K - \sum_{i=0}^{S-1} \beta_i$$

where $i = 0, 1, 2, ..., S-1$ and $K = 1 + \psi_1 + \psi_2 + ... + \psi_{S+L}$, hence,

$$\beta_0 = \beta_1 = \beta_2 = ... = \beta_S = \frac{K}{S+1} = \frac{1 + \psi_1 + \psi_2 + ... + \psi_{S+L}}{S+1}. \tag{4.14}$$

<u>*End of Proof*</u>

For AR(1) model, substitute $K = 1+\phi+\phi^2+\phi^3+...+\phi^{S+L} = \dfrac{1-\phi^{S+L+1}}{1-\phi}$

into (4.14), we have

$$\beta_0 = \beta_1 = \beta_2 = ... = \beta_S = \frac{K}{S+1} = \frac{1-\phi^{S+L+1}}{(S+1)(1-\phi)}. \tag{4.15}$$

Theorem 6 *For an ARIMA(p,1,q) order model, the smoothing order that has the minimum variation in week-to-week order changes has the set of optimal smoothing weights*

$$\beta_i = \frac{(i+1)(-2S+3i)}{(S+2)(S+3)}\psi_{S+L+1} + \frac{(i+1)6(S-i+1)}{(S+1)(S+2)(S+3)}K$$

for $i = 0, 1, 2, ..., S$ *where* $K = 1 + \psi_1 + \psi_2 + \psi_3 + ... + \psi_{S+L}$ *and* $\beta_S = K - \sum_{i=0}^{S-1} \beta_i$.

Proof

From (4.1), the generalized order

$$O_t = \beta_0 a_t + \beta_1 a_{t-1} + ... + \beta_{S-1} a_{t-S+1} + (K - \sum_{i=0}^{S-1} \beta_i) a_{t-S}$$
$$+ \psi_{S+L+1} a_{t-S-1} + \psi_{S+L+2} a_{t-S-2} + ...$$

where

$$K = 1 + \psi_1 + \psi_2 + ... + \psi_{S+L}.$$

The first differencing order $\nabla^d O_t = O_t - O_{t-1}$

$$\begin{aligned} O_t - O_{t-1} = &\beta_0 a_t + (\beta_1 - \beta_0)a_{t-1} + (\beta_2 - \beta_1)a_{t-2} + ... \\ &+ (\beta_{S-1} - \beta_{S-2})a_{t-S+1} \\ &+ \big(K - \sum_{i=0}^{S-1} \beta_i - \beta_{S-1}\big)a_{t-S} \\ &+ \Big(\psi_{S+L+1} - \big(K - \sum_{i=0}^{S-1} \beta_i\big)\Big)a_{t-S-1} \\ &+ \big(\psi_{S+L+2} - \psi_{S+L+1}\big)a_{t-S-2} \\ &+ \big(\psi_{S+L+3} - \psi_{S+L+2}\big)a_{t-S-3} + ... \end{aligned} \tag{4.16}$$

The variation in first differencing order is

$$\begin{aligned} Var(O_t - O_{t-1}) = &\beta_0^2 + (\beta_1 - \beta_0)^2 + (\beta_2 - \beta_1)^2 + ... \\ &+ (\beta_{S-1} - \beta_{S-2})^2 \\ &+ \big(K - \sum_{i=0}^{S-1} \beta_i - \beta_{S-1}\big)^2 \\ &+ \big(K - \psi_{S+L+1} - \sum_{i=0}^{S-1} \beta_i\big)^2 \\ &+ (\psi_{S+L+2} - \psi_{S+L+1})^2 \\ &+ (\psi_{S+L+3} - \psi_{S+L+2})^2 + ... \end{aligned} \tag{4.17}$$

where $\sum_{i=S+L+2}^{\infty}(\psi_i - \psi_{i-1})^2$ is finite by the property in the variance in differencing ARIMA model.

We obtain the minimum variation in differencing orders by minimizing the variation in $\nabla^d O_t$ respected to $\{\beta_i;\ i=0,1,2,...,S-1\}$. The system of linear equations of $\frac{d}{d\beta}Var(O_t - O_{t-1}) = 0$ is

$$\frac{d}{d\beta_0} = 2\beta_0 - 2(\beta_1 - \beta_0) - 2\big(K - \sum_{i=0}^{S-2}\beta_i - 2\beta_{S-1}\big) - 2\big(K - \psi_{S+L+1} - \sum_{i=0}^{S-1}\beta_i\big) = 0$$

$$= 8\beta_0 + 2\beta_1 + 4\beta_2 + 4\beta_3 + ... + 4\beta_{S-2} + 6\beta_{S-1} - 4K + 2\psi_{S+L+1}$$

$$\frac{d}{d\beta_1} = 2(\beta_1 - \beta_0) - 2(\beta_2 - \beta_1) - 2\big(K - \sum_{i=0}^{S-2}\beta_i - 2\beta_{S-1}\big)$$

$$- 2\big(K - \psi_{S+L+1} - \sum_{i=0}^{S-1}\beta_i\big) = 0$$

$$= 2\beta_0 + 8\beta_1 + 2\beta_2 + 4\beta_3 + 4\beta_4 + ... + 4\beta_{S-2} + 6\beta_{S-1} - 4K + 2\psi_{S+L+1}$$

$$\frac{d}{d\beta_2} = 2(\beta_2 - \beta_1) - 2(\beta_3 - \beta_2) - 2\big(K - \sum_{i=0}^{S-2}\beta_i - 2\beta_{S-1}\big)$$

$$- 2\big(K - \psi_{S+L+1} - \sum_{i=0}^{S-1}\beta_i\big) = 0$$

$$= 4\beta_0 + 2\beta_1 + 8\beta_2 + 2\beta_3 + 4\beta_4 + 4\beta_5 + ... + 4\beta_{S-2} + 6\beta_{S-1}$$

$$- 4K + 2\psi_{S+L+1}$$

$$\frac{d}{d\beta_3} = 2(\beta_3 - \beta_2) - 2(\beta_4 - \beta_3) - 2\big(K - \sum_{i=0}^{S-2}\beta_i - 2\beta_{S-1}\big)$$

$$- 2\big(K - \psi_{S+L+1} - \sum_{i=0}^{S-1}\beta_i\big) = 0$$

$$= 4\beta_0 + 4\beta_1 + 2\beta_2 + 8\beta_3 + 2\beta_4 + 4\beta_5 + 4\beta_6 + ... + 4\beta_{S-2} + 6\beta_{S-1}$$

$$- 4K + 2\psi_{S+L+1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{d}{d\beta_{S-4}} = 2(\beta_{S-4} - \beta_{S-5}) - 2(\beta_{S-3} - \beta_{S-2}) - 2\big(K - \sum_{i=0}^{S-2} \beta_i - 2\beta_{S-1}\big)$$
$$- 2\big(K - \psi_{S+L+1} - \sum_{i=0}^{S-1} \beta_i\big)$$
$$= 4\beta_0 + 4\beta_1 + ... + 4\beta_{S-6} + 2\beta_{S-5} + 8\beta_{S-4} + 2\beta_{S-3} + 4\beta_{S-2} + 6\beta_{S-1}$$
$$- 4K + 2\psi_{S+L+1}$$
$$\frac{d}{d\beta_{S-3}} = 2(\beta_{S-3} - \beta_{S-4}) - 2(\beta_{S-2} - \beta_{S-3}) - 2\big(K - \sum_{i=0}^{S-2} \beta_i - 2\beta_{S-1}\big)$$
$$- 2\big(K - \psi_{S+L+1} - \sum_{i=0}^{S-1} \beta_i\big)$$
$$= 4\beta_0 + 4\beta_1 + ... + 4\beta_{S-5} + 2\beta_{S-4} + 8\beta_{S-3} + 2\beta_{S-2} + 6\beta_{S-1}$$
$$- 4K + 2\psi_{S+L+1}$$
$$\frac{d}{d\beta_{S-2}} = 2(\beta_{S-2} - \beta_{S-3}) - 2(\beta_{S-1} - \beta_{S-2}) - 2\big(K - \sum_{i=0}^{S-2} \beta_i - 2\beta_{S-1}\big)$$
$$- 2\big(K - \psi_{S+L+1} - \sum_{i=0}^{S-1} \beta_i\big)$$
$$= 4\beta_0 + 4\beta_1 + ... + 4\beta_{S-4} + 2\beta_{S-3} + 8\beta_{S-2} + 4\beta_{S-1} - 4K + 2\psi_{S+L+1}$$
$$\frac{d}{d\beta_{S-1}} = 2(\beta_{S-1} - \beta_{S-2}) - 4\big(K - \sum_{i=0}^{S-2} \beta_i - 2\beta_{S-1}\big) - 2\big(K - \psi_{S+L+1} - \sum_{i=0}^{S-1} \beta_i\big)$$
$$= 6\beta_0 + 6\beta_1 + ... + 6\beta_{S-3} + 4\beta_{S-2} + 12\beta_{S-1} - 6K + 2\psi_{S+L+1}.$$

The system of linear equations can be expressed in the matrix form shown in table 4.1. The step 1 to step 4 for solving the generic formula for β_i; $i = 0, 1, 2, ..., S-1$ of table 4.1 are shown as the followings.

Step 1: The row operations of table 4.1 using Eq.i-(2/3)*Eq.S, i=1,2,3,...,S-1

Table 4.1: The System of Linear Equations in Matrix Form

Eq.	β_0	β_1	β_2	β_3	…	β_{S-6}	β_{S-5}	β_{S-4}	β_{S-3}	β_{S-2}	β_{S-1}	K	ψ_{S+L+1}
1	8	2	4	4	…	4	4	4	4	4	6	-4	2
2	2	8	2	4	…	4	4	4	4	4	6	-4	2
3	4	2	8	2	…	4	4	4	4	4	6	-4	2
4	4	4	2	8	…	4	4	4	4	4	6	-4	2
5	4	4	4	2	…	4	4	4	4	4	6	-4	2
6	4	4	4	4	…	4	4	4	4	4	6	-4	2
…	…	…	…	…	…	…	…	…	…	…	…	…	…
S-4	4	4	4	4	…	2	8	2	4	4	6	-4	2
S-3	4	4	4	4	…	4	2	8	2	4	6	-4	2
S-2	4	4	4	4	…	4	4	2	8	2	6	-4	2
S-1	4	4	4	4	…	4	4	4	2	8	4	-4	2
S	6	6	6	6	…	6	6	6	6	4	12	-6	2

is shown in table 4.2.

Step 2: Solve the Block Diagonal.

The block diagonal matrix is shown in table 4.3.

Step 2.1: LU Factorization of the Block Diagonal Matrix.

By setting $A = LU$, where A is the block diagonal matrix in table 4.3. We decompose matrix A into a product of a lower triangular matrix L and an upper triangular matrix U, shown in table 4.4. From table 4.4, we have

$$l_i = \frac{-(i-1)}{i};\ i = 2, 3, 4, ..., S-3$$

$$u_i = \frac{2(i+1)}{i};\ i = 1, 2, 3, ..., S-3.$$

Proof by induction that

$$l_n = -2/u_{n-1} = -(n-1)/n$$

$$u_n = 4 + 2l_n = 4 - 2(n-1)/n = 2(n+1)/n.$$

Given

Table 4.2: Step 1: Row Operations of the System of Linear Equations

Eq.	β_0	β_1	β_2	β_3	…	β_{S-6}	β_{S-5}	β_{S-4}	β_{S-3}	β_{S-2}	β_{S-1}	K	ψ_{S+L+1}
1	4	-2			…					4/3	-2		2/3
2	-2	4	-2		…					4/3	-2		2/3
3		-2	4	-2	…					4/3	-2		2/3
4			-2	4	…					4/3	-2		2/3
5				-2	…					4/3	-2		2/3
…	…	…	…	…	…	…	…	…	…	…	…	…	…
S-4					…	-2	4	-2		4/3	-2		2/3
S-3					…		-2	4	-2	4/3	-2		2/3
S-2					…			-2	4	-2/3	-2		2/3
S-1					…				-2	16/3	-4		2/3
S	6	6	6	6	…	6	6	6	6	4	12	-6	2

Table 4.3: Step 2: Block Diagonal Matrix of the System of Linear Equations

Eq.	β_0	β_1	β_2	β_3	…	β_{S-6}	β_{S-5}	β_{S-4}
1	4	-2			…			
2	-2	4	-2		…			
3		-2	4	-2	…			
4			-2	4	…			
5				-2	…			
…	…	…	…	…	…	…	…	…
S-4					…	-2	4	-2
S-3					…		-2	4

Table 4.4: Step 2.1: LU Factorization of the Block Diagonal Matrix.

A =

Eq.	β_0	β_1	β_2	β_3	…	β_{S-6}	β_{S-5}	β_{S-4}
1	4	-2			…			
2	-2	4	-2		…			
3		-2	4	-2	…			
…	…	…	…	…	…	…	…	…
S-5					…	4	-2/3	
S-4					…	-2	16/3	-4
S-3					…		4	12

L =

Eq.	β_0	β_1	β_2	β_3	…	β_{S-6}	β_{S-5}	β_{S-4}
1	1				…			
2	l_2	1			…			
3		l_3	1		…			
…	…	…	…	…	…	…	…	…
S-5						1		
S-4						l_{S-4}	1	
S-3							l_{S-3}	1

U =

Eq.	β_0	β_1	β_2	β_3	…	β_{S-6}	β_{S-5}	β_{S-4}
1	u_1	-2			…			
2		u_2	-2		…			
3			u_3	-2	…			
…	…	…	…	…	…	…	…	…
S-5						u_{S-5}	-2	
S-4							u_{S-4}	-2
S-3								u_{S-3}

$u_1 = 4 = 2(1+1)/1.$

Let $n = 2$

$l_2 = -2/u_1 = -2/4 = -1/2 = -(n-1)/n = -1/2 = -(2-1)/2$

$u_2 = 4 + 2l_2 = 4 + 2(-1/2) = 3 = 2(n+1)/n = 2\big((2+1)/2\big).$

Let $n = k$

$l_k = -(k-1)/k$

$u_k = 2(k+1)/k.$

Let $n = k+1$

$l_{k+1} = -2/u_k = -2/\big(2(k+1)/k\big) = -k/(k+1) = -\big((k+1)-1\big)/(k+1)$

$$u_{k+1} = 4 + 2l_{k+1} = 4 + 2(-k/(k+1)) = 2(k+2)/(k+1)$$
$$= 2\big((k+1)+1\big)/(k+1).$$

The solution for the matrix lower tridiagonal matrix L and upper tridiagonal matrix U is shown in table 4.5.

Step 2.2: Solve for y from $Ly = b$.

y is the vector of $[y_1, y_2, ..., y_{S-3}]^T$ and b is the $S-3$ by 4 matrix of $\beta_{S-3}, \beta_{S-2}, \beta_{S-1}, \psi_{S+L+1}$ from Eq. 1 to $S-3$. The matrix $Ly = b$ is shown in table 4.6.

From $Ly = b$ shown in table 4.6, we have $y = \beta_{S-3} + ky^T[\beta_{S-2}, \beta_{S-1}, \psi_{S+L+1}]$ where $ky = [ky_1, ky_2, ..., ky_{S-4}, ky_{S-3}]^T$. We have $ky_i = (i+1)/2;\ i = 1, 2, 3, ..., S-3$.

Proof by induction that

$ky_n = 1 - l_n ky_{n-1} = 1 - \big(-(n-1)/n\big) ky_{n-1} = (n+1)/2.$

Given

$l_n = -(n-1)/n$

Table 4.5: Step 2.1: LU Solution of the Block Diagonal Matrix

L =

Eq.	β_0	β_1	β_2	β_3	…	β_{S-6}	β_{S-5}	β_{S-4}
1	1							
2	-1/2	1						
3		-2/3	1					
…	…	…	…	…	…	…	…	…
S-5						1		
S-4						-(S-5)/(S	1	
S-3							-(S-4)/(S	1

U =

Eq.	β_0	β_1	β_2	β_3	…	β_{S-6}	β_{S-5}	β_{S-4}
1	4	-2			…			
2		3	-2		…			
3			8/3	-2	…			
…	…	…	…	…	…	…	…	…
S-5						2(S-4)/(S-5)	-2	
S-4							2(S-3)/(S-4)	-2
S-3								2(S-2)/(S-3)

Table 4.6: Step 2.2: The Matrix $Ly = b$

Eq.	β_0	β_1	β_2	…	β_{S-7}	β_{S-6}	β_{S-5}	β_{S-4}	β_{S-3}	β_{S-2}	β_{S-1}	K	ψ_{S+L+1}
1	1			…						4/3	-2		2/3
2	-1/2	1		…						4/3	-2		2/3
3		-2/3	1	…						4/3	-2		2/3
4			-3/4	…						4/3	-2		2/3
…	…	…	…	…	…	…	…	…	…	…	…	…	…
S-6				…	1					4/3	-2		2/3
S-5					-(S-6)/(S-5)	1				4/3	-2		2/3
S-4				…		-(S-5)/(S-4)	1			4/3	-2		2/3
S-3				…	…		-(S-4)/(S-3)	1	-2	4/3	-2		2/3

Table 4.7: Step 2.2: Solve for y from $Ly = b$

	Eq.	β_{S-3}	ky	β_{S-2}	β_{S-1}	K	ψ_{S+L+1}
	1		2/2	4/3	-2		2/3
	2		3/2	4/3	-2		2/3
	3		4/2	4/3	-2		2/3
y =	…	…	…	…	…	…	…
	S-5		(S-4)/2	4/3	-2		2/3
	S-4		(S-3)/2	4/3	-2		2/3
	S-3	-2	(S-2)/2	4/3	-2		2/3

$$ky_1 = 1 = (1+1)/2.$$

Let $n = 2$

$$ky_2 = 1 - l_2 ky_1 = 1 - (-1/2)1 = 3/2 = (2+1)/2.$$

Let $n = k$

$$ky_k = (k+1)/2\ .$$

Let $n = k+1$

$$ky_{k+1} = 1 - l_{k+1} ky_k = 1 - \big(-k/(k+1)\big)\big((k+1)/2\big) = (k+2)/2.$$

The solution for y is shown in table 4.7.

Step 2.3: Solve for x from $Ux = y$.

x is the vector of $[x_1, x_2, ..., x_{S-3}]^T$ and y is the $S-3$ by 4 matrix of $\beta_{S-3}, \beta_{S-2}, \beta_{S-1}, \psi_{S+L+1}$ from Eq. 1 to $S-3$ shown in table 4.7. The matrix $Ux = y$ is shown in table 4.8.

From $Ux = y$ shown in table 4.8, we have $kx_i = (S-i-2)(i/4);\ i = 1, 2, 3, ..., S-3$ and $\beta_{i,S-3} = -i/(S-2);\ i = 1, 2, 3, ..., S-3$.

Proof by induction that

$$kx_n = (2kx_{n+1} + ky_n)/u_n = (S-n-2)(n/4).$$

Table 4.8: Step 2.3: The Matrix $Ux = y$

Eq.	β_0	β_1	β_2	…	β_{S-6}	β_{S-5}	β_{S-4}	β_{S-3}	ky	β_{S-2}	β_{S-1}	K	ψ_{S+L+1}
1	4	-2		…					2/2	4/3	-2		2/3
2		3	-2	…					3/2	4/3	-2		2/3
3			8/3	…					4/2	4/3	-2		2/3
…	…	…	…	…	…	…	…	…	…	…	…	…	…
S-6				…	-2				(S-5)/2	4/3	-2		2/3
S-5					2(S-4)/(S-5)	-2			(S-4)/2	4/3	-2		2/3
S-4				…		2(S-3)/(S-4)	-2		(S-3)/2	4/3	-2		2/3
S-3				…			2(S-2)/(S-3)	-2	(S-2)/2	4/3	-2		2/3

Given

$$u_n = 2(n+1)/n$$

$$ky_n = 1 = (n+1)/2$$

$$kx_{S-3} = \frac{ky_{S-3}}{u_{S-3}} = \frac{(S-2)/2}{2(S-2)/(S-3)}$$
$$= (S-3)/4 = \big(S-(S-3)-2\big)(S-3)/4.$$

Let $n = S-4$

$$kx_{S-4} = \frac{2kx_{S-3} + ky_{S-4}}{u_{S-4}} = \frac{\big(2(S-3)/4\big) + (S-3)/2}{2(S-3)/(S-4)}$$
$$= (S-4)/2 = \big(S-(S-4)-2\big)(S-4)/4.$$

Let $n = k$

$$kx_k = (S-k-2)(k/4).$$

Let $n = k+1$

$$kx_{k-1} = \frac{2kx_k + ky_{k-1}}{u_{k-1}} = \frac{\big(2(S-k-2)(k/4)\big) + k/2}{2k/(k-1)}$$
$$= (S-k-1)(k-1)/4 = \big(S-(k-1)-2\big)(k-1)/4.$$

Proof by induction that

$$\beta_{n,S-3} = 2\beta_{n+1,S-3}/u_n = -n/(S-2).$$

Table 4.9: Step 2.3: Solve for x from $Ux = y$

	Eq.	β_{S-3}	kx	β_{S-2}	β_{S-1}	K	ψ_{S+L+1}
	1	-1/(S-2)	(S-3)/4	4/3	-2		2/3
	2	-2/(S-2)	(S-4)*2/4	4/3	-2		2/3
	3	-3/(S-2)	(S-5)*3/4	4/3	-2		2/3
x =	…	…	…	…	…	…	…
	S-5	-(S-5)/(S-2)	3(S-5)/4	4/3	-2		2/3
	S-4	-(S-4)/(S-2)	2(S-4)/4	4/3	-2		2/3
	S-3	-(S-3)/(S-2)	(S-3)/4	4/3	-2		2/3

Given

$$u_n = 2(n+1)/n$$

$$\beta_{S-3,S-3} = 2\beta_{S-2,S-3}/u_{S-3} = -2.$$

Let $n = S - 4$

$$\beta_{S-4,S-3} = 2\beta_{S-3,S-3}/u_{S-4} = \frac{2(S-3)/(S-2)}{2(S-3)/(S-4)} = (S-4)/(S-2).$$

Let $n = k$

$$\beta_{k,S-3} = -k/(S-2).$$

Let $n = k - 1$

$$\beta_{k-1,S-3} = 2\beta_{k,S-3}/u_{k-1} = \frac{2\big(-k/(S-2)\big)}{2k/(k-1)} = -(k-1)/(S-2).$$

The solution for x is shown in table 4.9.

Step 3: Solve for Eq. S-3 to Eq. S-1 to diagonalize β_{S-4}.

The solution from solving Eq. S-3 to Eq. S-1 to diagonalize β_{S-4} is shown in table 4.10.

Step 4: Solve Eq. 1 to Eq. S to diagonalize β_{S-3}, β_{S-2}, and β_{S-1}.

After step 1 - 3, the final matrix is shown in table 4.11.

Table 4.10: Step 3: Solve Eq. S-3 to S-1 to diagonalize β_{S-4}

Eq.	β_{S-4}	β_{S-3}	β_{S-2}	β_{S-1}	K	ψ_{S+L+1}
S-3	1	-(S-3)/(S-2)	(S-3)/3	-(S-3)/2		(S-3)/6
S-2	-2	4	-2/3	-2		2/3
S-1		-2	16/3	-4		2/3

Eq.	β_{S-4}	β_{S-3}	β_{S-2}	β_{S-1}	K	ψ_{S+L+1}
S-3	1	-(S-3)/(S-2)	(S-3)/3	-(S-3)/2		(S-3)/6
S-2		1	(S-2)(S-4)/[3(S-1)]	-(S-2)/2		(S-2)/6
S-1			1	-3(S-1)/(2*S)		(S-1)/(2*(S+2))

Table 4.11: Step 4: Solve Eq. 1 to S to diagonalize β_{S-3} and β_{S-2}

Eq.	β_0	β_1	β_2	…	β_{S-6}	β_{S-5}	β_{S-4}	β_{S-3}	β_{S-2}
1	1			…				-1/(S-2)	(S-3)/3
2		1		…				-2/(S-2)	(S-4)*2/3
3			1	…				-3/(S-2)	(S-5)*3/3
…	…	…	…	…	…	…	…	…	…
S-5				…	1			-(S-5)/(S-2)	3(S-5)/3
S-4				…		1		-(S-4)/(S-2)	2(S-4)/3
S-3				…			1	-(S-3)/(S-2)	(S-3)/3
S-2				…				1	(S-2)(S-4)/[3(S-1)]
S-1				…					1
S	6	6	6	…	6	6	6	6	4

Eq.	β_{S-1}	K	ψ_{S+L+1}
1	-(S-3)/2		(S-3)/6
2	-(S-4)*2/2		(S-4)*2/6
3	-(S-5)*3/2		(S-5)*3/6
…	…	…	…
S-5	-3(S-5)/2		3(S-5)/6
S-4	-2(S-4)/2		2(S-4)/6
S-3	-(S-3)/2		(S-3)/6
S-2	-(S-2)/2		(S-2)/6
S-1	-3(S-1)/(2*L)		(S-1)/(2*(S+2))
S	12	-6	2

Table 4.12: Step 4.1: Solve Eq. 1 to S to diagonalize β_{S-3}

Eq.	β_0	β_1	β_2	…	β_{S-6}	β_{S-5}	β_{S-4}	β_{S-3}	β_{S-2}
1	1			…					(1/3)*[(S-4)/(S-1)+(S-3)]
2		1		…					(2/3)*[(S-4)/(S-1)+(S-4)]
3			1	…					(3/3)*[(S-4)/(S-1)+(S-5)]
…	…	…	…	…	…	…	…	…	…
S-4				…		1			[(S-4)/3]*[(S-4)/(S-1)+2]
S-3				…			1		[(S-3)/3]*[(S-4)/(S-1)+1]
S-2				…				1	[(S-2)/3]*[(S-4)/(S-1)]
S-1				…					1
S	1	1	1	…	1	1	1	1	2/3

Eq.	β_{S-1}	K	ψ_{S+L+1}
1	-(S-2)/2		(S-2)/6
2	-(S-3)*2/2		(S-3)*2/6
3	-(S-4)*3/2		(S-4)*3/6
…	…	…	…
S-4	-3(S-4)/2		3(S-4)/6
S-3	-2(S-3)/2		2(S-3)/6
S-2	-(S-2)/2		(S-2)/6
S-1	-3(S-1)/(2*S)		(S-1)/(2*(S+2))
S	2	-1	1/3

Step 4.1: Solve Eq. 1 to S to diagonalize β_{S-3} (shown in table 4.12).

Step 4.2: Solve Eq. 1 to S to diagonalize β_{S-2} (shown in table 4.13).

Step 4.3: Solve Eq. 1 to S to diagonalize β_S.

Table 4.13: Step 4.2: Solve Eq. 1 to S to diagonalize β_{S-2}

Eq.	β_0	β_1	β_2	…	β_{S-3}	β_{S-2}	β_{S-1}
1	1			…			[1/2]*[[(S-4)/(S-1)+(S-3)]*(S-1)/S-(S-2)]
2		1		…			[2/2]*[[(S-4)/(S-1)+(S-4)]*(S-1)/S-(S-3)]
3			1	…			[3/2]*[[(S-4)/(S-1)+(S-5)]*(S-1)/S-(S-4)]
…	…	…	…	…	…	…	…
S-5				…			[(S-5)/2]*[[(S-4)/(S-1)+3]*(S-1)/(S-4)]
S-4				…			[(S-4)/2]*[[(S-4)/(S-1)+2]*(S-1)/(S-3)]
S-3				…			[(S-3)/2]*[[(S-4)/(S-1)+1]*(S-1)/(S-2)]
S-2				…	1		[(S-2)/2]*[[(S-4)/(S-1)]*(S-1)/(S-1)]
S-1				…		1	-3(S-1)/(2*S)
S	1	1	1	…	1	2/3	2

Eq.	K	ψ_{S+L+1}
1		-[1/6]*[[(S-4)/(S-1)+(S-3)]*(S-1)/(S+2)-(S-2)]
2		-[2/6]*[[(S-4)/(S-1)+(S-4)]*(S-1)/(S+2)-(S-3)]
3		-[3/6]*[[(S-4)/(S-1)+(S-5)]*(S-1)/(S+2)-(S-4)]
…	…	…
S-5		-[(S-5)/6]*[[(S-4)/(S-1)+3]*(S-1)/(S+2)-4]
S-4		-[(S-4)/6]*[[(S-4)/(S-1)+2]*(S-1)/(S+2)-3]
S-3		-[(S-3)/6]*[[(S-4)/(S-1)+1]*(S-1)/(S+2)-2]
S-2		-[(S-2)/6]*[[(S-4)/(S-1)]*(S-1)/(S+2)-1]
S-1		(S-1)/(2*(S+2))
S	-1	1/3

By solving step 4.2 Eq. 1 to Eq. S, we have

$$\left(2-\frac{2}{3}\frac{-3(S-1)}{2S}-\sum_{i=1}^{S-2}\frac{i}{2}\Big(\big(\frac{S-4}{S-1}+S-i-2\big)\frac{S-1}{S}-(S-i-1)\Big)\right)\beta_{S-1}$$
$$+\left(\frac{1}{3}-\frac{2}{3}\frac{S-1}{2(S+2)}-\sum_{i=1}^{S-2}-\frac{i}{6}\Big(\big(\frac{S-4}{S-1}+S-i-2\big)\frac{S-1}{S+2}-(S-i-1)\Big)\right)\psi_{S+L+1}$$
$$-K=0$$

then,

$$\frac{(S+1)(S+2)(S+3)}{12S}\beta_{S-1}-\frac{S^2-2S-3}{12}\psi_{S+L+1}-K=0$$

hence,

$$\beta_{S-1}=\frac{S(S-3)}{((S+2)(S+3)}\psi_{S+L+1}+\frac{12S}{(S+1)(S+2)(S+3)}K.$$

By solving step 3 Eq. S-1, we have

$$\beta_{S-2}-\frac{3(S-1)}{2S}\beta_{S-1}+\frac{S-1}{2(S+2)}\psi_{S+L+1}=0$$

hence,

$$\beta_{S-2}=\frac{(S-1)(S-6)}{(S+2)(S+3)}\psi_{S+L+1}+\frac{18(S-1)}{(S+1)(S+2)(S+3)}K.$$

By solving step 1 Eq. S-1, we have

$$-2\beta_{S-3} + \frac{16}{3}\beta_{S-2} - 4\beta_{S-1} + \frac{2}{3}\psi_{S+L+1}$$

hence,

$$\beta_{S-3} = \frac{(S-2)(S-9)}{(S+2)(S+3)}\psi_{S+L+1} + \frac{24(S-2)}{(S+1)(S+2)(S+3)}K.$$

By solving step 1 Eq. S-2, we have

$$-2\beta_{S-4} + 4\beta_{S-3} - \frac{2}{3}\beta_{S-2} - 2\beta_{S-1} + \frac{2}{3}\psi_{S+L+1}$$

hence,

$$\beta_{S-4} = \frac{(S-3)(S-12)}{(S+2)(S+3)}\psi_{S+L+1} + \frac{30(S-3)}{(S+1)(S+2)(S+3)}K.$$

By solving step 1 Eq. 2 to Eq. S-3, we have

$$-2\beta_{S-i} + 4\beta_{S-i+1} - 2\beta_{S-i+2} + \frac{4}{3}\beta_{S-2} - 2\beta_{S-1} + \frac{2}{3}\psi_{S+L+1}$$

for i=5,6,7,...S, hence,

$$\beta_{S-i} = \frac{(S-i+1)(S-3i)}{(S+2)(S+3)}\psi_{S+L+1} + \frac{(S-i+1)6(i+1)}{(S+1)(S+2)(S+3)}K$$

where $i = 5, 6, 7, ..., S$.

Also

$$\beta_S = K - \sum_{i=0}^{S-1} \beta_i = \frac{-2S\psi_{S+L+1} + 6K}{(S+2)(S+3)}.$$

Hence, the value of $\beta_i;\ i = 0, 1, 2, ..., S$

$$\beta_i = \frac{(i+1)(-2S+3i)}{(S+2)(S+3)}\psi_{S+L+1} + \frac{(i+1)6(S-i+1)}{(S+1)(S+2)(S+3)}K \tag{4.18}$$

where $K = 1 + \psi_1 + \psi_2 + ... + \psi_{S+L}$ and the ψ's weights following (3.2).

Its variation in first differencing generalized order, $Var\big(O_t - O_{t-1}\big)$, can be found by substituting the β's weights following (4.18), K, and ψ's weights into (4.17) where $\sum_{i=S+L+2}^{\infty}(\psi_i - \psi_{i-1})^2$ is absolutely summable.

End of proof

For ARIMA(0,1,1) model, substitute $\psi_{S+L+1} = 1 - \theta$ and $K = 1 + \psi_1 + \psi_2 + \psi_3 + ... + \psi_{S+L} = 1 + (S+L)(1-\theta)$ into (4.18), we have the smoothing weights's $\beta_i;\ i = 0, 1, 2, ..., S$

$$\begin{aligned}\beta_i = &\frac{(i+1)\big(4S^2 + (10-3i)S + (6-3i) + 6(S-i+1)L\big)}{(S+1)(S+2)(S+3)} \\ &- \frac{(i+1)\big(4S^2 + (4-3i)S + 3i + 6(S-i+1)L\big)}{(S+1)(S+2)(S+3)}\theta.\end{aligned} \tag{4.19}$$

Its variation can be expressed in the function of S, L, and θ by substituting

β's, K, and ψ's into (4.18), then

$$
\begin{aligned}
Var(O_t - O_{t-1}) = \Bigg(& \frac{(4S^2 + 10S + 6) + 12\big(L^2 + (S+1)L\big)}{(S+1)(S+2)(S+3)} \\
& - \frac{8(S^2 + S) + 24(L^2 + SL)}{(S+1)(S+2)(S+3)}\theta \\
& + \frac{(4S^2 - 2S + 6) + 12(L^2 + SL - 1)}{(S+1)(S+2)(S+3)}\theta^2 \Bigg) \sigma_a^2.
\end{aligned} \tag{4.20}
$$

4.5 Insights

This section provides the insights for the generalized ordering policy and the smoothing ordering policy.

1. Stationary inventory condition for the generalized ordering policy. In order to have a stationary inventory variance, $\sum_{i=0}^{S} \beta i = K$ (where $K = 1 + \psi_1 + \psi_2 + \psi_3 + ... + \psi_{S+L}$) represents the magnitude of the bullwhip effect. In the infinite loading policy, the (forecast error) shocks are summed up in one period, In the smoothing policy, the shocks are extended to smoothing $S + 1$ periods such that it minimizes the variation in differencing orders. However, the sum of (forecast error) shocks for both ordering policies are the same.

2. The distribution of the optimal smoothing weight β for the smoothing policy. For ARIMA$(p, 0, q)$ order model, the ratio of β_i for $i = 0, 1, 2, ..., S$ over the magnitude of the bullwhip effect K is

$$
\frac{\beta_i}{K} = \frac{K/(S+1)}{K} = \frac{1}{S+1} \text{ for } i = 0, 1, 2, ..., S
$$

That is, the optimal smoothing weights β for ARIMA$(p, 0, q)$ order model are the average of K over $S + 1$ periods.

For ARIMA$(p, 1, q)$ order model, the shape of the ratio of β_i for $i = 0, 1, 2, ..., S$ over the magnitude of the bullwhip effect K depends on the autocorrelation in the demand (ψ's weights). The shape is moving from a bell shape when the autocorrelation is low to a left tail shape when the autocorrelation is high, which is shown below. That means the smoothing ordering policy tends to react less to the most recent shocks in order to smooth the week to week order variation.

To illustrate the shape of the β's weights, we consider the model ARIMA(0,1,1) demand model in (3.14)

$$Z_t = a_t + (1 - \theta)a_{t-1} + (1 - \theta)a_{t-2} + (1 - \theta)a_{t-3} + ...$$

The infinite loading orders following (3.15)

$$O_t = \big(1 + L(1 - \theta)\big)a_t + (1 - \theta)a_{t-1} + (1 - \theta)a_{t-2} + (1 - \theta)a_{t-3} + ...$$

For simplicity, let $L = 0$, when $\theta = 1$, ARIMA(0,1,1) demand is the noise series $Z_t = a_t$ and its infinite loading orders $O_t = Z_t$. For the smoothing orders, $K = 1$. The β's weights following (4.19) is

$$\beta_i = \frac{6(i + 1)(S - i + 1)}{(S + 1)(S + 2)(S + 3)} \text{ for } i = 0, 1, 2, ..., S.$$

Although $lim_{S \to \infty} \beta_i = 0$, asymptotically, the shape of β_i can be roughly estimated from

$$\begin{aligned}\beta_0 : \beta_{S/2} : \beta_S &= \frac{6}{(S+2)(S+3)} : \frac{6}{4}\frac{S+2}{(S+1)(S+3)} : \frac{6}{(S+2)(S+3)} \\ &\approx \frac{1}{S^2} : \frac{1}{S} : \frac{1}{S^2}\end{aligned}$$

which is approximately a bell shape.

When $\theta = 0.5$, $K = 1 + \frac{S}{2}$ for the smoothing orders. The β's weights following (4.19) is

$$\beta_i = \frac{(i+1)(4S - 3i + 4)}{(S+1)(S+2)^2} \text{ for } i = 0, 1, 2, ..., S.$$

Also $lim_{S \to \infty} \beta_i = 0$. Asymptotically, the shape of β_i can be roughly estimated from

$$\begin{aligned}\beta_0 : \beta_{S/2} : \beta_S &= \frac{4}{(S+2)^2} : \frac{5}{4}\frac{5S+8}{(S+1)(S+2)} : \frac{S+4}{(S+2)^2} \\ &\approx \frac{1}{S^2} : \frac{1}{S} : \frac{1}{S}\end{aligned}$$

which is approximately a left tail shape.

When $\theta = 0$, ARIMA(0,1,1) demand model is the random walk series $Z_t = Z_{t-1} + a_t$ and its infinite loading orders $O_t = Z_t$. For the smoothing orders, $K = 1 + S$. The β's weights following (4.19) is

$$\beta_i = \frac{(i+1)(4S - 3i + 6)}{(S+1)(S+2)(S+3)} \text{ for } i = 0, 1, 2, ..., S.$$

Table 4.14: The β's Weights for ARIMA(0,0,0), $L = 0$, $S = 0$ to 6

S	β_0/K	β_1/K	β_2/K	β_3/K	β_4/K	β_5/K	β_6/K
0	1						
1	0.5	0.5					
2	0.3	0.4	0.3				
3	0.2	0.3	0.3	0.2			
4	0.1429	0.2286	0.2571	0.2286	0.1429		
5	0.1071	0.1786	0.2143	0.2143	0.1786	0.1071	
6	0.0833	0.1429	0.1786	0.1905	0.1786	0.1429	0.0833

Also $lim_{S\to\infty}\beta_i = 0$. Asymptotically, the shape of β_i can be roughly estimated from

$$\begin{aligned}\beta_0 : \beta_{S/2} : \beta_S &= \frac{4S+6}{(S+1)(S+2)(S+3)} : \frac{(S+2)(5S+12)}{8(S+1)(S+2)(S+3)} : \frac{S+6}{(S+2)(S+3)} \\ &\approx \frac{1}{S^2} : \frac{1}{S} : \frac{1}{S}\end{aligned}$$

which is approximately a left tail shape. Table 4.14 shows the ratio β_i/K for S from 0 to 6 when $\theta = 1$. Table 4.15 shows the ratio β_i/K for S from 0 to 6 when $\theta = 0.5$. Table 4.16 shows the ratio β_i/K for S from 0 to 6 when $\theta = 0$.

4.6 Applications of Smoothing Policy for Supplier Contracts

This section illustrates applications of using the smoothing policy for supplier contracts with specific ARIMA models. With constraints in supplier

Table 4.15: The β_i/K for ARIMA(0,1,1) $\theta = 0.5$, $L = 0$, $S = 0$ to 6

S	β_0/K	β_1/K	β_2/K	β_3/K	β_4/K	β_5/K	β_6/K
0	1						
1	0.4444	0.5556					
2	0.25	0.375	0.375				
3	0.16	0.26	0.3	0.28			
4	0.1111	0.1889	0.2333	0.2444	0.2222		
5	0.0816	0.1429	0.1837	0.2041	0.2041	0.1837	
6	0.0625	0.1116	0.1473	0.1696	0.1786	0.1741	0.1563

Table 4.16: The β_i/K for ARIMA(0,1,0), $L = 0$, $S = 0$ to 6

S	β_0/K	β_1/K	β_2/K	β_3/K	β_4/K	β_5/K	β_6/K
0	1						
1	0.4167	0.5833					
2	0.2333	0.3667	0.4				
3	0.15	0.25	0.3	0.3			
4	0.1048	0.181	0.2286	0.2476	0.2381		
5	0.0774	0.1369	0.1786	0.2024	0.2083	0.1964	
6	0.0595	0.1071	0.1429	0.1667	0.1786	0.1786	0.1667

contracts, a manufacturer requires the production level to be within specific tolerance bounds that suppliers agree to follow. The manufacturer also wants to hold the smallest level of inventory and the small variation in week-to-week variation in production. Section 4.6.1 illustrates the supplier contracts when the demand is AR(1). Section 4.6.2 illustrates the supplier contracts when the demand is ARIMA(0,1,1).

4.6.1 AR(1) Demand Model

Suppose that the weekly demand for a product is AR(1) with $\phi = .5$, $\mu = 100$, $\sigma_a = 5$. The company has a zero lead time, $L = 0$, but has supplier contracts that require the production level to be within 5% of the amount forecast 4 weeks earlier, within 10% of the forecast of 8 weeks earlier and within 15% of the forecast made 12 weeks earlier.

The company has four considerations: 1) It must meet the commitments to its suppliers. 2) It wants small variation in week to week variation in production. 3) It wants to hold as little inventory as possible. 4) It wants to meet customer demand.

The production plan for this supplier contracts is to find the smallest number of smoothing period, S, for which the optimal smoothing policy meets constraint 1. By making the number of smoothing periods as small as possible, we minimize the variation in inventory. Then by using the optimal smoothing policy, we find the policy that has the smallest week to week variation of all policies that smooth over that number of weeks. Then we set the inventory target needed to meet customer demand. In other words, we find the optimal smoothing production policy that minimizes the variation in inventory subject to the constraint that the actual production will be within the required ranges within high 3-sigma probability.

For supplier contracts, we set

$$3\sigma_{O_t-\hat{O}_{t-4}} \leq 5\%\mu = 5$$
$$3\sigma_{O_t-\hat{O}_{t-8}} \leq 10\%\mu = 10$$
$$3\sigma_{O_t-\hat{O}_{t-12}} \leq 15\%\mu = 15.$$

For an AR(1) model, the generalized order following (4.1) is

$$\begin{aligned} O_t &= \beta_0 a_t + \beta_1 a_{t-1} + ... + \beta_{S-1} a_{t-S+1} + \beta_S a_{t-S} + \psi_{S+L+1} a_{t-S-1} + \psi_{S+L+2} a_{t-S-2} + ... \\ &= \beta_0 a_t + \beta_1 a_{t-1} + ... + \beta_{S-1} a_{t-S+1} + \beta_S a_{t-S} + \phi^{S+L+1} a_{t-S-1} + \phi^{S+L+2} a_{t-S-2} + ... \end{aligned}$$

and the standard deviation of the forecast error

$$\begin{aligned} \sigma_{O_t-\hat{O}_{t-i}} &= \left(\beta_0^2 + \beta_1^2 + \beta_2^2 + ... + \beta_{i-1}^2\right)^{1/2} \sigma_a,\ i \leq S+1 \\ &= \left(\beta_0^2 + \beta_1^2 + ... + \beta_S^2 + \phi^{2(S+L+1)} + \phi^{2(S+L+2)} + ... + \phi^{2(i-S-1)}\right)^{1/2} \sigma_a,\ i \geq S+2 \end{aligned}$$

where the optimal beta weight following theorem 5 is

$$\beta_i = \frac{K}{S+1} = \frac{1 + \psi_1 + \psi_2 + ... + \psi_{S+L}}{S+1} = \frac{1}{S+1} \frac{1-\phi^{S+L+1}}{1-\phi} \text{ for } i = 0, 1, 2, ..., S.$$

The standard deviation of the inventory following theorem 3 is

$$\sigma_I = \Big(1 + (\theta_1^{(I)})^2 + (\theta_2^{(I)})^2 + ... + (\theta_{S+L-1}^{(I)})^2\Big)^{1/2}\sigma_a$$
$$= \Big(1 + (\frac{1-\phi^2}{1-\phi})^2 + (\frac{1-\phi^3}{1-\phi})^2 + ... + (\frac{1-\phi^L}{1-\phi})^2$$
$$+ (\frac{1-\phi^{L+1}}{1-\phi} - \beta_0)^2 + (\frac{1-\phi^{L+2}}{1-\phi} - \beta_0 - \beta_1)^2 + ... + (\frac{1-\phi^{L+S}}{1-\phi} - \sum_{i=0}^{S-1}\beta_i)^2\Big)^{1/2}\sigma_a.$$

Also, the standard deviation of the production following theorem 4 is

$$\sigma_{O_t} = \Big(\beta_0^2 + \beta_1^2 + ... + \beta_S^2 + \phi^{2(S+L+1)} + \phi^{2(S+L+2)} + \phi^{2(1S+L+3)} + ...\Big)^{1/2}\sigma_a$$
$$= \big(\beta_0^2 + \beta_1^2 + ... + \beta_S^2 + \frac{\phi^{2(S+L+1)}}{1-\phi^2}\big)^{1/2} * \sigma_a.$$

Hence, we find the optimal smoothing production by varying the values of smoothing periods S from 0 to 12. The result is shown in table 4.17. From the table, the optimal smoothing period is 11 with $\beta_i = .1666$ for $i = 0, 1, ...11$ where the target inventory is set as $3\sigma_I = 44.31$ and the 3-sigma range of the variation in production is $3\sigma_{O_t} = 8.66$. Compared with the loading policy, the smoothing policy reduces the 3-sigma range of the variation in production from 17.32 to 8.66 while it increases the inventory target needed from 0 to 44.31.

4.6.2 ARIMA(0,1,1) Demand Model

In this section, suppose that the weekly demand for a product is ARIMA(0,1,1) with $\theta = .7$, lead time, $L = 0$, $\sigma_a = 5$. The supplier contracts require the

Table 4.17: Supplier Contracts for AR(1) Model with $L = 0$, $\phi = .5$, $S = 0$ to 12

S	0	1	2	3	4	5	6	7	8	9	10	11	12
β_i	1	0.75	0.5833	0.4688	0.3875	0.3281	0.2835	0.2490	0.2218	0.1998	0.1817	0.1666	0.1538

$3\sigma_{O_t-\hat{O}_{t-i}}$

S	0	1	2	3	4	5	6	7	8	9	10	11	12
i = 4	17.287	16.453	15.271	14.063	11.625	9.844	8.504	7.471	6.654	5.994	5.452	4.999	4.615
i = 8	17.320	16.488	15.309	14.104	13.008	12.059	11.251	10.565	9.410	8.477	7.710	7.069	6.526
i = 12	17.321	16.489	15.309	14.104	13.008	12.059	11.251	10.565	9.981	9.478	9.041	8.658	7.993

S	0	1	2	3	4	5	6	7	8	9	10	11	12
$3\sigma_I$	0	3.75	8.00	12.70	17.44	22.02	26.34	30.41	34.21	37.78	41.14	44.31	47.32

S	0	1	2	3	4	5	6	7	8	9	10	11	12
$3\sigma_{Ot}$	17.32	16.49	15.31	14.10	13.01	12.06	11.25	10.57	9.98	9.48	9.04	8.66	8.32

quantity changes in production level to be within 10 units of the forecast 4 weeks earlier, within 15 units of the forecast 8 weeks earlier, and within 20 units of the forecast 12 weeks earlier. Hence, for supplier contracts, we set

$$3\sigma_{O_t-\hat{O}_{t-4}} \leq 10$$
$$3\sigma_{O_t-\hat{O}_{t-8}} \leq 15$$
$$3\sigma_{O_t-\hat{O}_{t-12}} \leq 20.$$

For an ARIMA(0,1,1) model, the generalized order following (4.1) is

$$\begin{aligned} O_t &= \beta_0 a_t + \beta_1 a_{t-1} + ... + \beta_{S-1} a_{t-S+1} + \beta_S a_{t-S} + \psi_{S+L+1} a_{t-S-1} + \psi_{S+L+2} a_{t-S-2} + ... \\ &= \beta_0 a_t + \beta_1 a_{t-1} + ... + \beta_{S-1} a_{t-S+1} + \beta_S a_{t-S} + (1-\theta) a_{t-S-1} + (1-\theta) a_{t-S-2} + ... \end{aligned}$$

and the standard deviation of the forecast error

$$\begin{aligned} \sigma_{O_t - \hat{O}_{t-i}} &= \big(\beta_0^2 + \beta_1^2 + \beta_2^2 + ... + \beta_{i-1}^2\big)^{1/2} \sigma_a, \; i \leq S+1 \\ &= \Big(\beta_0^2 + \beta_1^2 + ... + \beta_S^2 + (1-\theta)^2 (i - S - 1)\Big)^{1/2} \sigma_a, \; i \geq S+2 \end{aligned}$$

where the optimal beta weight following theorem 6 is

$$\begin{aligned} \beta_i = & \frac{(i+1)\big(4S^2 + (10-3i)S + (6-3i) + 6(S-i+1)L\big)}{(S+1)(S+2)(S+3)} \\ & - \frac{(i+1)\big(4S^2 + (4-3i)S + 3i + 6(S-i+1)L\big)}{(S+1)(S+2)(S+3)} \theta \end{aligned}$$

for $i = 0, 1, 2, ..., S$.

The standard deviation of the inventory following theorem 3 is

$$\begin{aligned} \sigma_I &= \Big(1 + (\theta_1^{(I)})^2 + (\theta_2^{(I)})^2 + ... + (\theta_{S+L-1}^{(I)})^2\Big)^{1/2} \sigma_a \\ &= \Big(1 + (2-\theta)^2 + (3-2\theta)^2 + ... + \big(L - (L-1)\theta\big)^2 \\ &\quad + \big((L+1) - L\theta - \beta_0\big)^2 + \big((L+2) - (L+1)\theta - \beta_0 - \beta_1\big)^2 + ... \\ &\quad + \big((L+S) - (L+S-1)\theta - \sum_{i=0}^{S-1} \beta_i\big)^2\Big)^{1/2} \sigma_a. \end{aligned}$$

Also, the standard deviation of the first differencing production following theorem 4 is

$$\begin{aligned}\sigma_{\nabla^d O_t} &= \sigma_{O_t - O_{t-1}} \\ &= \Big(\beta_0^2 + (\beta_1-\beta_0)^2 + (\beta_2-\beta_1)^2 + ... + (\beta_S-\beta_{S-1})^2 + (1-\theta-\beta_S)^2\Big)^{1/2}\sigma_a.\end{aligned}$$

The optimal smoothing production is found by varying the values of smoothing periods S from 0 to 12, the result is shown in table 4.18. From the table, the optimal smoothing period is 11 with the target inventory set as $3\sigma_I = 36.42$ and the 3-sigma range of the variation in production changes is $3\sigma_{O_t - O_{t-1}} = 3.05$. Compared with the loading policy, the smoothing policy reduces the 3-sigma range of the variation in production changes from 18.31 to 3.05 while it increases the inventory target needed from 0 to 36.42.

From the table, we can see that the values of $3\sigma_{O_t - \hat{O}_{t-12}}$ do not decrease when the forecasting periods increase since the distribution of the smoothing β's weights for ARIMA$(p,1,q)$ model is not uniform distributed as those in ARIMA$(p,0,q)$ model.

Table 4.18: Supplier Contracts for ARIMA(0,1,1) Model with $L = 0$, $\theta = .7$, $S = 0$ to 12

S	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}
0	1												
1	0.6	0.7											
2	0.42	0.61	0.57										
3	0.32	0.51	0.57	0.5									
4	0.257	0.431	0.523	0.531	0.457								
5	0.214	0.371	0.471	0.514	0.5	0.429							
6	0.183	0.325	0.425	0.483	0.5	0.475	0.408						
7	0.16	0.288	0.385	0.45	0.483	0.485	0.455	0.393					
8	0.142	0.259	0.351	0.418	0.461	0.478	0.471	0.439	0.382				
9	0.127	0.235	0.322	0.389	0.436	0.464	0.471	0.458	0.425	0.373			
10	0.115	0.214	0.297	0.363	0.413	0.446	0.463	0.463	0.447	0.414	0.365		
11	0.105	0.197	0.275	0.34	0.39	0.427	0.45	0.459	0.455	0.437	0.405	0.359	
12	0.097	0.183	0.256	0.319	0.369	0.408	0.435	0.451	0.455	0.447	0.428	0.397	0.354

$3\sigma_{O_t - \hat{O}_{t-i}}$

S	0	1	2	3	4	5	6	7	8	9	10	11	12
i = 4	16.904	15.223	14.723	14.523	13.484	12.284	11.159	10.168	9.308	8.567	7.924	7.366	6.876
i = 8	19.151	17.685	17.256	17.086	17.017	16.999	17.009	17.037	16.705	16.178	15.562	14.918	14.278
i = 12	21.160	19.843	19.462	19.311	19.250	19.234	19.244	19.268	19.303	19.345	19.391	19.442	19.289

S	0	1	2	3	4	5	6	7	8	9	10	11	12
$3\sigma_I$	0	6.00	9.60	12.76	15.74	18.66	21.55	24.45	27.38	30.35	33.36	36.42	39.53

S	0	1	2	3	4	5	6	7	8	9	10	11	12
$3\sigma_{\nabla O_t}$	18.31	10.92	8.04	6.49	5.51	4.84	4.35	3.97	3.67	3.43	3.22	3.05	2.90

Chapter 5

Bounded MRP for Exponential Smoothing Demand

This chapter illustrates the implementations of the up to target ordering policy (in chapter 3) and the smoothing ordering policy (in chapter 4) into MRP tables. Then it introduces the bounded MRP system which is an up to target MRP table with an enforced set of upper/lower bounds. This set of bounds is served as a flex quantity profile in rate based planning that aids manufacturers in reducing the conflict between production planning and infeasible capacity planning in standard MRP systems. The set of bounds also help to mitigate the bullwhip by limiting the changes in orders instead of letting the orders be oscillated uncontrollably by actual lumpy demand.

The objective of the bounded MRP is to control the variation in week-to-week production rate to match the flexibility in capacity changes with a stationary inventory. If the set of bounds is too wide, the bounded MRP will behave like an unbounded standard MRP since maximum planned orders will be at the level that the planned inventory hit the target before the planned orders hit the bounds. If the set of bounds is too narrow, the orders in the bounded MRP will be exploded after the enforced bounded periods, hence, the orders after the bounded periods will be unusually large or small to recover the inventory stock-outs or overstocks. Figure 5.1 illustrates the production rate in the bounded MRP when the set of bounds is too wide. Figure 5.2 illustrates the production rate in the bounded MRP when the set of bounds is too narrow.

This chapter also provides a simulation based technique to determine the optimal bound widths in bounded MRP tables for exponential smoothing or ARIMA(0,1,1) demand. We considers only single exponential smoothing demand for several reasons. First, single exponential is one of the simplest and most widely used forecasting techniques. Second, it has a stationary order changes from one period to the next, hence it has a finite variance which is a prerequisite for optimization. Finally, its smoothing orders with the minimum variation in order changes from one period to the next follow the goal of capacity planning that desires the minimum changes of production plan.

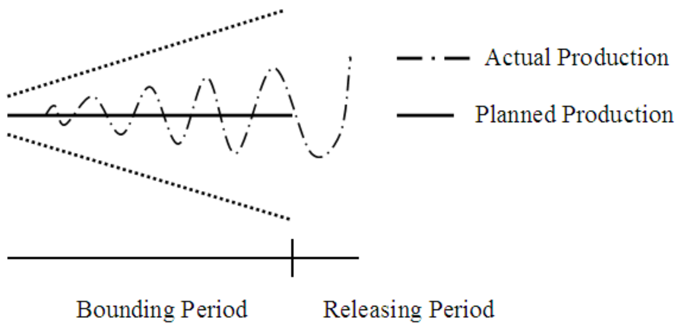

Figure 5.1: The Set of Bounds is too Wide in an Bounded MRP

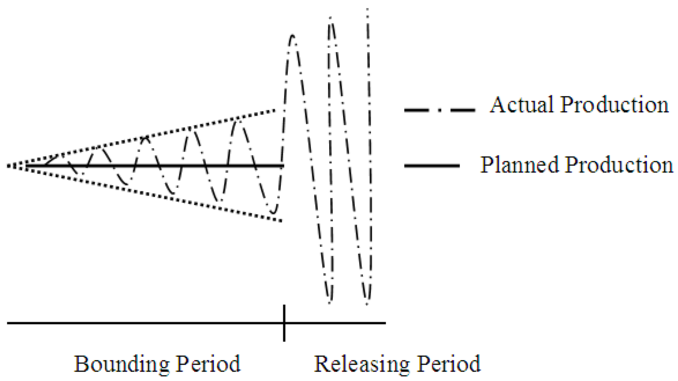

Figure 5.2: The Set of Bounds is too Narrow in an Bounded MRP

5.1 Introduction

In a *pull-type* production, the production and delivery of materials are driven by the drumbeat process from the upstream processes. The drumbeat process maintains a constant production rate that creates a flat demand pattern for the internal upstream processes and the external suppliers. Scheduling is done only at the drumbeat process and all the other processes respond to pull signals from the process immediately downstream. This *pull-type* production is well served in the short term. In the longer term, however, manufacturers need a *push-type* production such as MRP systems to set a plan for future production planning that can handle the changing level of demand and the rate of the drumbeat.

However, the traditional MRP system is not very well suited for production planning. If the MRP schedule is frozen for several weeks into the future, it does not allow quick enough response variation in customer demand. When the MRP schedule is not frozen, the result is “schedule nervousness”, that is, the plan is constantly changing.

Manufacturers need a tool for planning the production rate, or the drumbeat, called rate-based planning. Rate-based planning gives a plan of the future rate of the drumbeat, but with a pre-specified amount of flexibility to vary from this plan. To understand the rationale for rate-based planning it is necessary to first understand the three primary planning and scheduling goals.

These goals can be articulated from a functional perspective of the rate-based planning:

- Manufacturing would like production rate that never changes.
- Marketing would like instantaneous changes in production rate in response to changes in demand, to avoid late deliveries, stock-outs and overstocks.
- Purchasing and the external suppliers would like a firm long-term commitment to future production rates in order to know future material needs.

For the first goal, it is impossible to maintain a production rate that never changes. However, it is necessary to smooth demand to insulate the production processes and the suppliers from the short-term variation in demand. The smoothing technique to satisfy this goal is the smoothing ordering policy introduced in section 5.2.2 that can be applied to the MRP system.

The second goal is to respond quickly to true changes in the level of demand. A slow reaction time (lead time) creates high variation in inventory, which in turn results in stock-outs and overstocks. The production and inventory levels were generated using standard infinite loading MRP computations. This is the standard MRP ordering policy introduced in section 5.2.1.

By using the standard MRP ordering policy, the inventory variation is much larger that of the demand. In addition, each change in demand is followed by a much larger change in production due to its lead time. This is the bullwhip effect that the reaction time causes the variation in production to be much larger than the variation in demand. Since there is a lead time delay in reacting to variation, the change in the production level must cover not only the variation in current week's demand but also the change in the forecasted level of demand over lead time. Since in reality no production process could economically accommodate such large swings in the production rate, most of the variation created by the lead time will be absorbed by the inventory. Thus the variation in inventory will in actuality be much larger than that the standard MRP production plan resulting in some combination of late shipments, lost sales, overstocks and other failures.

The third goal is to reduce the uncertainty for manufacturing and the upstream suppliers in planning future capacity and materials requirements. This objective would suggest the weekly rates of the drumbeat should be determined and fixed (frozen) for several weeks into the future. However this objective conflicts with the first and second objectives. The period of the freeze in the schedule represents a delay in responding to changes in the level of demand, i.e., lead time. The slow response time will create large swings in the inventory resulting in stock-outs and overstocks. Although the freeze will make production levels predictable, the production levels will be highly variable due the bullwhip effect.

The Flexibility Requirements Profile for rate-based planning does not use a frozen schedule. Rather it produces a plan but permits deviations, within specified ranges, from that plan. This compromise gives manufacturing and the suppliers a forecast of future production, with a guaranteed level of accuracy, while retaining some flexibility to respond quickly to changes in the level of demand. Since the amount of change from the plan is constrained, rate-based planning also has a smoothing effect on production.

Rate-based planning begins with a predetermined weekly profile of flexibility that has been agreed upon by manufacturing and the suppliers. This is the flexibility requirements profile that gives the amount (usually expressed as a percentage of the planned drumbeat) by which the actual drumbeat can deviate from the current plan for future weeks (see figure 1.3).

There will typically be smaller amounts of deviation permitted for weeks in the near future than for weeks in the more distant future. For example, the flexibility requirements profile may require that the actual drumbeat for the first four weeks not deviate from the plan by more than five percent. For weeks five through eight the allowable deviation may be ten percent and for weeks nine though twelve the allowable deviation may be fifteen percent. The rate-based planning for an MRP system is introduced by the bounded MRP ordering policy in section 5.3.

5.2 The Unbounded MRP

This section illustrates the implementations of the up to target ordering policy (introduced in chapter 3) and the smoothing ordering policy (introduced in chapter 4) into unbounded MRP tables.

5.2.1 MRP Mechanism

Let L be the ordering lead time, in which the receipt at time t or R_t is the order placed at time $t-L$ or O_{t-L}. F is the forecast period. An example of the unbounded MRP at period t and at period $t+1$ is shown in table 5.1 and table 5.2.
For ARIMA(0,1,1), the demand Z_t:

$$Z_t = a_t + (1-\theta)(a_{t-1} + a_{t-2} + a_{t-3} + ...).$$

Forecast demand $\hat{Z}_t$ from section 3.5.1:

$$\begin{aligned}\hat{Z}_t(F) &= (1-\theta)(a_t + a_{t-1} + a_{t-2} + ...) \text{ for } F = 1,2,3,... \\ &= (1-\theta)Z_t + \theta\hat{Z}_{t-1}(F) \qquad (5.1) \\ &= \hat{Z}_t.\end{aligned}$$

Receipt:

$$\text{Receipt} = \text{Order placed } L \text{ periods in the past.}$$

Table 5.1: Standard MRP at Period t

Period	t	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	…	$t+F\text{-}1$	$t+F$
Demand	Z_t	$\hat{Z}_t(1)$	$\hat{Z}_t(2)$	$\hat{Z}_t(3)$	$\hat{Z}_t(4)$	$\hat{Z}_t(5)$	…	$\hat{Z}_t(F-1)$	$\hat{Z}_t(F)$
Receipts	R_t	$\hat{R}_t(1)$	$\hat{R}_t(2)$	$\hat{R}_t(3)$	$\hat{R}_t(4)$	$\hat{R}_t(5)$	…	$\hat{R}_t(F-1)$	$\hat{R}_t(F)$
Inventory	I_t	$\hat{I}_t(1)$	$\hat{I}_t(2)$	$\hat{I}_t(3)$	$\hat{I}_t(4)$	$\hat{I}_t(5)$	…	$\hat{I}_t(F-1)$	$\hat{I}_t(F)$
Order	O_t	$\hat{O}_t(1)$	$\hat{O}_t(2)$	$\hat{O}_t(3)$	$\hat{O}_t(4)$	$\hat{O}_t(5)$	…	$\hat{O}_t(F-1)$	$\hat{O}_t(F)$

Table 5.2: Standard MRP at Period $t+1$

Period	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	…	$t+F$	$t+F+1$
Demand	Z_{t+1}	$\hat{Z}_{t+1}(1)$	$\hat{Z}_{t+1}(2)$	$\hat{Z}_{t+1}(3)$	$\hat{Z}_{t+1}(4)$	$\hat{Z}_{t+1}(5)$	…	$\hat{Z}_{t+1}(F-1)$	$\hat{Z}_{t+1}(F)$
Receipts	R_{t+1}	$\hat{R}_{t+1}(1)$	$\hat{R}_{t+1}(2)$	$\hat{R}_{t+1}(3)$	$\hat{R}_{t+1}(4)$	$\hat{R}_{t+1}(5)$	…	$\hat{R}_{t+1}(F-1)$	$\hat{R}_{t+1}(F)$
Inventory	I_{t+1}	$\hat{I}_{t+1}(1)$	$\hat{I}_{t+1}(2)$	$\hat{I}_{t+1}(3)$	$\hat{I}_{t+1}(4)$	$\hat{I}_{t+1}(5)$	…	$\hat{I}_{t+1}(F-1)$	$\hat{I}_{t+1}(F)$
Order	O_{t+1}	$\hat{O}_{t+1}(1)$	$\hat{O}_{t+1}(2)$	$\hat{O}_{t+1}(3)$	$\hat{O}_{t+1}(4)$	$\hat{O}_{t+1}(5)$	…	$\hat{O}_{t+1}(F-1)$	$\hat{O}_{t+1}(F)$

or

$$
\begin{aligned}
R_t &= \hat{R}_{t-1}(1) \\
\hat{R}_t(i) &= \hat{R}_{t-1}(i+1)\ for\ i = 1, 2, 3, ..., L-1 \\
\hat{R}_t(L) &= O_t \\
\hat{R}_t(L+i) &= \hat{O}_t(i)\ for\ i = 1, 2, 3, ..., F-L.
\end{aligned} \tag{5.2}
$$

The inventory I_t from (3.4):

Ending Inventory = Beginning Inventory + Receipt - Demand

or

$$
\begin{aligned}
I_t &= I_{t-1} + O_{t-L} - Z_t = I_{t-1} + R_t - Z_t \\
\hat{I}_t(1) &= I_t + \hat{R}_t(1) - \hat{Z}_t(1) \\
\hat{I}_t(i) &= \hat{I}_t(i-1) + \hat{R}_t(i) - \hat{Z}_t(i)\ for\ i = 2, 3, 4, ..., F.
\end{aligned} \tag{5.3}
$$

5.2.2 Standard MRP Ordering Policy

In the standard MRP ordering policy introduced in chapter 3, the order at time t, O_t, is the quantity needed to bring the inventory back to the target level T in time period $t + L$. This is the up to target ordering policy that minimizes the variation in inventory. Since this ordering policy assumes that the order can be filled up indefinitely with the assumption in unlimited materials and capacity to bring the inventory up to the target, it is also called the infinite loading policy. Thus, Net Requirement = Target Ending Inventory - Prior $(L-1)$ Weeks Ending Inventory + Forecast Demand

or

$$\begin{aligned}
O_t &= T - I_{t-1} + \hat{Z}_t(L), \text{ if } L = 0 \\
O_t &= T - \hat{I}_t(L-1) + \hat{Z}_t(L), \text{ if } L \geq 1 \\
\hat{O}_t(i) &= T - \hat{I}_t(L-1+i) + \hat{Z}_t(L+i);\ \text{for } i = 1, 2, 3, ..., F-L \\
\hat{O}_t(i) &= \hat{O}_t(F-L);\ \text{for } i = F-L+1, F-L+2, F-L+3, ..., F.
\end{aligned} \tag{5.4}$$

An example of a simulated order up to target policy MRP uses 52 weeks with a_t is assumed to be randomly Normal distributed $N(\mu, \sigma) = N(0, 10)$. The initial Z_0 is 100. F = 11. L = 4. The demand Z_t is ARIMA(0,1,1) where $Z_t = Z_{t-1} + a_t - \theta a_{t-1}$ and $\theta = 0.7$. The forecast demand $\hat{Z}_t(F) = (1-\theta)Z_t + \theta \hat{Z}_{t-1}(F)$ for $F \geq 1$. The noise a_t following section 3.2.2 is $a_{t-i} = Z_{t-i} - \hat{Z}_{t-i-1}(1);\ i = 0, 1, 2, ...$ The generated 52 weeks noise a_t and demand Z_t are shown in table 5.3.

The inventory target T is set to $3\sigma_I$ since T can vary within $\pm 3\sigma_I$ since a_t is generated from a normal distribution. From theorem 1,

$$\sigma_I = \sqrt{1 + (2-.7)^2 + (3-2*.7)^2 + (4-3*.7)^2} * 10 = 29.77$$

hence $T = 3\sigma_I = 89.31$. Table 5.4 shows the standard MRP at period 11. Table 5.5 shows the standard MRP at period 12.

Table 5.3: 52 Weeks Generated Demand Data

Week	a_t	Z_t	Week	a_t	Z_t	Week	a_t	Z_t	Week	a_t	Z_t
1	-5.20	94.80	14	-8.44	99.00	27	5.50	107.52	40	15.92	112.54
2	4.45	102.89	15	-7.41	97.50	28	-12.19	91.48	41	2.93	104.32
3	9.36	109.13	16	6.47	109.16	29	9.67	109.68	42	5.00	107.27
4	3.15	105.73	17	9.15	113.78	30	-3.64	99.27	43	3.94	107.71
5	-17.29	86.23	18	0.97	108.35	31	24.31	126.13	44	-3.63	101.32
6	2.57	100.91	19	-1.90	105.77	32	2.19	111.30	45	9.75	113.61
7	-7.59	91.52	20	4.23	111.33	33	6.73	116.50	46	-1.41	105.38
8	3.11	99.95	21	1.40	109.77	34	-14.61	97.18	47	2.56	108.93
9	-1.74	96.03	22	-0.71	108.08	35	-20.74	86.66	48	13.33	120.47
10	-1.30	95.95	23	-8.42	100.16	36	-12.24	88.95	49	4.74	115.87
11	-0.67	96.19	24	-1.20	104.85	37	-6.45	91.06	50	-10.86	101.69
12	25.58	122.24	25	-2.84	102.85	38	-10.23	85.35	51	12.44	121.73
13	10.37	114.70	26	-9.38	95.45	39	13.69	106.20	52	-16.40	96.62

Table 5.4: Standard MRP at Period 11

Week	11	12	13	14	15	16	17	18	19	20	21	22
Demand	96.19	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66
Receipts	82.42	103.68	93.94	94.40	95.39	96.66	96.66	96.66	96.66	96.66	96.66	96.66
Inventory	88.52	95.54	92.83	90.57	89.30	89.30	89.30	89.30	89.30	89.30	89.30	89.30
Order	95.39	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66

Table 5.5: Standard MRP at Period 12

Week	12	13	14	15	16	17	18	19	20	21	22	23
Demand	122.24	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33
Receipts	103.68	93.94	94.40	95.39	152.93	104.33	104.33	104.33	104.33	104.33	104.33	104.33
Inventory	69.97	59.58	49.64	40.70	89.30	89.30	89.30	89.30	89.30	89.30	89.30	89.30
Order	152.93	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33

The forecast demand at week 10, $\hat{Z}_{10}$, is 96.86 and the demand at week 11, Z_{11}, is 96.19. At week 11, the forecast demand at week 11 following (5.1) is $\hat{Z}_{11} = (1-\theta)Z_{11} + \theta\hat{Z}_{10} = 0.3 * 96.19 + 0.7 * 96.86 = 96.66$.

The orders made at week 7, 8, 9, and 10, or O_7, O_8, O_9, O_{10}, are 82.42, 103.68, 93.94, and 94.40. At week 11, the receipts following (5.2) are $R_{11} = O_7 = 82.42$, $\hat{R}_{11}(1) = O_8 = 103.68$, $\hat{R}_{11}(2) = O_9 = 93.94$, and $\hat{R}_{11}(3) = O_{10} = 94.40$.

The inventory at week 10 is 102.29. At week 11, the inventory following (5.3) is $I_{11} = I_{10} + R_{11} - Z_{11} = 102.29 + 82.42 - 96.19 = 88.52$. $\hat{I}_{11}(1) = I_{11} + \hat{R}_{11}(1) - \hat{Z}_{11}(1) = 88.52 + 103.68 - 96.66 = 95.54$. $\hat{I}_{11}(2) = \hat{I}_{11}(1) + \hat{R}_{11}(2) - \hat{Z}_{11}(2) = 95.54 + 93.94 - 96.66 = 92.83$. $\hat{I}_{11}(3) = \hat{I}_{11}(2) + \hat{R}_{11}(3) - \hat{Z}_{11}(3) = 92.82 + 94.40 - 96.66 = 90.57$.

By using the standard MRP ordering policy, at week 11 following (5.4), $O_{11} = T - \hat{I}_{11}(3) + \hat{Z}_{11}(4) = 89.31 - 90.57 + 96.66 = 95.39$. From (5.2), $\hat{R}_{11}(4) = O_{11} = 95.39$. From (5.3), $\hat{I}_{11}(4) = \hat{I}_{11}(3) + \hat{R}_{11}(4) - \hat{Z}_{11}(4) = 90.57 + 95.39 - 96.66 = 89.30$. From (5.4), $\hat{O}_{11}(1) = T - \hat{I}_{11}(4) + \hat{Z}_{11}(5) = 89.30 - 89.30 + 96.66 = 96.66$. Repeating these calculations, we have the results shown in table 5.4. Also from (5.4), we let $\hat{O}_{11}(8)$, $\hat{O}_{11}(9)$, $\hat{O}_{11}(10)$, $\hat{O}_{11}(11)$ be $\hat{O}_{11}(7)$, which is 96.66.

At week 12, the demand at week 12, Z_{12}, is 122.24. The forecast demand at week 12 is $\hat{Z}_{12} = (1-\theta)Z_{11} + \theta\hat{Z}_{11} = 0.3 * 122.24 + 0.7 * 96.66 = 104.33$. From (5.2), $R_{12} = \hat{R}_{11}(1) = 103.68$, $\hat{R}_{12}(1) = \hat{R}_{11}(2) = 93.94$, $\hat{R}_{12}(2) = \hat{R}_{11}(3) = 94.40$, and $\hat{R}_{12}(3) = \hat{R}_{11}(4) = 95.39$. The inventory following (5.3) is $I_{12} = I_{11} + R_{12} - Z_{12} = 88.52 + 103.68 - 122.24 = 69.97$. The order following (5.4), $O_{12} = T - \hat{I}_{12}(3) + \hat{Z}_{12}(4) = 89.31 - 40.70 + 104.33 = 152.93$. Then the rest of the calculations for the MRP table for week 12 can be repeated as those in period 11.

5.2.3 Smoothing Ordering Policy

The smoothing ordering policy, which minimizes the variation in order changes from one period to the next, can also be applied to the above MRP by changing the orders O_t and the order forecasts $\hat{O}_t(i)$; $i = 1, 2, 3, ..., F$ (F is the forecast period) in an MRP table at a given time period t, and all mechanisms to calculate demand, receipt, and inventory are the same as a standard MRP table in section 5.2.2. From (4.9), the smoothing order at week t

$$
\begin{aligned}
O_t = \beta_0\big(Z_t - \hat{Z}_{t-1}(1)\big) + \beta_1\big(Z_{t-1} - \hat{Z}_{t-2}(1)\big) + \beta_2\big(Z_{t-2} - \hat{Z}_{t-3}(1)\big) \\
+ ... + \beta_{S-1}\big(Z_{t-S+1} - \hat{Z}_{t-S}(1)\big) + \beta_S\big(Z_{t-S} - \hat{Z}_{t-S-1}(1)\big) + \hat{Z}_{t-S-1}(L).
\end{aligned}
$$

Since $a_{t-i} = Z_{t-i} - \hat{Z}_{t-i-1}(1)$; $i = 0, 1, 2, ...,$ the smoothing order at week t can be rewritten as

$$
\begin{aligned}
O_t &= \beta_0 a_t + \beta_1 a_{t-1} + ... + \beta_{S-1} a_{t-S+1} + \beta_S a_{t-S} + \hat{Z}_{t-S-1}(L) \\
\hat{O}_t(i) &= \beta_i a_t + \beta_{i+1} a_{t-1} + ... + \beta_S a_{t-S} + \hat{Z}_{t-S-1}(L); \text{ when } i \leq S \qquad (5.5) \\
&= \hat{Z}_t(L); \text{ when } S + 1 \leq i \leq F, \text{ which is the infinite loading.}
\end{aligned}
$$

Table 5.6: β's Weights for $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$

β	β₀	β₁	β₂	β₃	β₄	β₅	β₆	β₇	β₈	β₉	β₁₀
Value	0.1615	0.2983	0.4101	0.4972	0.5594	0.5969	0.6094	0.5972	0.5601	0.4983	0.4115

Where the β's weights following theorem 6 and (4.19) are

$$\begin{aligned}\beta_i = & \frac{(i+1)\big(4S^2 + (10-3i)S + (6-3i) + 6(S-i+1)L\big)}{(S+1)(S+2)(S+3)} \\ & - \frac{(i+1)\big(4S^2 + (4-3i)S + 3i + 6(S-i+1)L\big)}{(S+1)(S+2)(S+3)}\theta\end{aligned} \tag{5.6}$$

for $i = 0, 1, 2, ..., S$ and β_S can be found from either (5.6) or

$$\beta_S = \Big(1 + (S+L)(1-\theta) - \sum_{i=0}^{S-1} \beta_i\Big).$$

With lead times $L = 4$ weeks, smoothing period $S = 10$ weeks, forecast period $F = 11$ weeks, $\theta = 0.7$, the β's weights are show in table 5.6.

The inventory target T is set to $3\sigma_I$ since T can vary within $\pm 3\sigma_I$. From theorem 3,

$$\begin{aligned}\sigma_I = & \Big(1 + (2-.7)^2 + (3-2*.7)^2 + (4-3*.7)^2 \\ & + (5-4*.7-.1615)^2 + (6-5*.7-.1615-.2983)^2 + ... \\ & + (14-13*.7-.1615-.2983-...-.4983)^2\Big)^{1/2} * 10 = 63.52\end{aligned}$$

hence $T = 3\sigma_I = 160.57$.

Table 5.7: Smoothing MRP at Period 11

Week	11	12	13	14	15	16	17	18	19	20	21	22
Demand	96.19	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66	96.66
Receipts	98.74	97.65	96.47	95.30	94.29	93.56	92.39	91.98	93.10	95.14	95.39	96.13
Inventory	183.40	184.40	184.21	182.85	180.49	177.39	173.12	168.43	164.87	163.35	162.08	161.54
Order	94.29	93.56	92.39	91.98	93.10	95.14	95.39	96.13	96.03	96.38	96.58	96.66

Table 5.8: Smoothing MRP at Period 12

Week	12	13	14	15	16	17	18	19	20	21	22	23
Demand	122.24	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33	104.33
Receipts	97.65	96.47	95.30	94.29	97.69	100.02	102.47	105.81	109.44	110.65	111.71	111.31
Inventory	158.82	150.96	141.93	131.89	125.25	120.93	119.07	120.55	125.66	131.98	139.36	146.34
Order	97.69	100.02	102.47	105.81	109.44	110.65	111.71	111.31	110.71	109.33	107.18	104.33

Table 5.7 shows the smoothing MRP at period 11. Table 5.8 shows the smoothing MRP at period 12.

The demand pattern in this section is the same as the demand pattern shown in table 5.3. The forecast demand at week 10, $\hat{Z}_{10}$, is 96.86 and the demand at week 11, Z_{11}, is 96.19. At week 11, the forecast demand at week 11 following (5.1) is $\hat{Z}_{11} = (1-\theta)Z_{10}+\theta\hat{Z}_{10} = 0.3*96.19+0.7*96.86 = 96.66$.

The data for the week t demand Z_t, forecast demand $\hat{Z}_t$, and the noise a_t are shown in table 5.9.

Let $\hat{Z}_i = \hat{Z}_0$ for $i < 0$, then $a_i = 0$ for $i \leq 0$. From (5.5), the smoothing

Table 5.9: Z_t, $\hat{Z}_t$, and a_t for $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$

t	0	1	2	3	4	5	6	7	8	9	10	11	12
Z_t		94.80	102.89	109.13	105.73	86.23	100.91	91.52	99.95	96.03	95.95	96.19	122.24
$\hat{Z}_t$	100	98.44	99.77	102.58	103.53	98.34	99.11	96.83	97.77	97.25	96.86	96.66	104.33
a_t		-5.20	4.45	9.36	3.15	-17.29	2.57	-7.59	3.11	-1.74	-1.30	-0.67	25.58

orders at week 7, 8, 9, and 10 are shown below.

$$O_7 = \beta_0 a_7 + \beta_1 a_6 + \beta_2 a_5 + ... + \beta_9 a_{-2} + \beta_{10} a_{-3} + \hat{Z}_{-4}$$
$$= 0.1615 * (-7.59) + 0.2983 * (2.57) + 0.4101 * (-17.29) + ...$$
$$+ 0.6094 * (-5.20) + 100 = 98.74$$
$$O_8 = \beta_0 a_8 + \beta_1 a_7 + \beta_2 a_6 + ... + \beta_9 a_{-1} + \beta_{10} a_{-2} + \hat{Z}_{-3}$$
$$= 0.1615 * (3.11) + 0.2983 * (-7.59) + 0.4101 * (2.57) + ...$$
$$+ 0.5972 * (-5.20) + 100 = 97.65$$
$$O_9 = \beta_0 a_9 + \beta_1 a_8 + \beta_2 a_7 + ... + \beta_9 a_0 + \beta_{10} a_{-1} + \hat{Z}_{-2}$$
$$= 0.1615 * (-1.74) + 0.2983 * (3.11) + 0.4101 * (-7.59) + ...$$
$$+ 0.5601 * (-5.20) + 100 = 96.47$$
$$O_{10} = \beta_0 a_{10} + \beta_1 a_9 + \beta_2 a_8 + ... + \beta_9 a_1 + \beta_{10} a_0 + \hat{Z}_{-1}$$
$$= 0.1615 * (-1.74) + 0.2983 * (-1.74) + 0.4101 * (3.11) + ...$$
$$+ 0.4982 * (-5.20) + 100 = 95.30.$$

Hence, the smoothing orders made at week 7, 8, 9, and 10, or O_7, O_8, O_9,

O_{10}, are 98.74, 97.65, 96.47, and 95.30. At week 11, the receipts following (5.2) are $R_{11} = O_7 = 98.74$, $\hat{R}_{11}(1) = O_8 = 97.65$, $\hat{R}_{11}(2) = O_9 = 96.47$, and $\hat{R}_{11}(3) = O_{10} = 95.30$.

The inventory at week 10 is 180.86. At week 11, the inventory following (5.3) is $I_{11} = I_{10} + R_{11} - Z_{11} = 180.86 + 98.74 - 96.19 = 183.40$. $\hat{I}_{11}(1) = I_{11} + \hat{R}_{11}(1) - \hat{Z}_{11}(1) = 183.40 + 97.65 - 96.66 = 184.40$. $\hat{I}_{11}(2) = \hat{I}_{11}(1) + \hat{R}_{11}(2) - \hat{Z}_{11}(2) = 184.40 + 96.47 - 96.66 = 184.21$. $\hat{I}_{11}(3) = \hat{I}_{11}(2) + \hat{R}_{11}(3) - \hat{Z}_{11}(3) = 184.21 + 95.30 - 96.66 = 182.85$.

By using the smoothing ordering policy, the smoothing order following (5.5) at week 11

$$
\begin{aligned}
O_{11} &= \beta_0 a_{11} + \beta_1 a_{10} + \beta_2 a_9 + ... + \beta_9 a_2 + \beta_{10} a_1 + \hat{Z}_0 \\
&= 0.1615 * (-0.67) + 0.2983 * (-1.30) + 0.4101 * (-1.74) + ... \\
&\quad + 0.4983 * (4.45) + 0.4115 * (-5.20) + 100 = 94.29.
\end{aligned}
$$

From (5.2), $\hat{R}_{11}(4) = O_{11} = 94.29$. From (5.3), $\hat{I}_{11}(4) = \hat{I}_{11}(3) + \hat{R}_{11}(4) -$

$\hat{Z}_{11}(4) = 182.85 + 94.29 - 96.66 = 180.49$. From (5.5),

$$\begin{aligned}
\hat{O}_{11}(1) &= \beta_1 a_{11} + \beta_2 a_{10} + ... + \beta_{10} a_2 + \hat{Z}_1 \\
&= 0.2983 * (-0.67) + 0.4101 * (-1.30) + ... + 0.4115 * (4.45) + 98.44 = 93.56 \\
\hat{O}_{11}(2) &= \beta_2 a_{11} + \beta_3 a_{10} + ... + \beta_{10} a_3 + \hat{Z}_2 \\
&= 0.4101 * (-0.67) + 0.4972 * (-1.30) + ... + 0.4115 * (9.36) + 99.77 = 92.39 \\
\vdots \quad & \quad \vdots \quad \quad \vdots \\
\hat{O}_{11}(9) &= \beta_9 a_{11} + \beta_{10} a_{10} + \hat{Z}_9 = 0.4983 * (-0.67) + 0.4115 * (-1.30) + 97.25 = 96.38 \\
\hat{O}_{11}(10) &= \beta_{10} a_{11} + \hat{Z}_{10} = 0.4115 * (-0.67) + 96.86 = 96.58 \\
\hat{O}_{11}(11) &= \hat{Z}_{11} = 96.66.
\end{aligned}$$

Repeating these calculations, we have the results shown in table 5.7.

At week 12, the demand at week 12, Z_{12}, is 122.24. The forecast demand at week 12 is $\hat{Z}_{12} = (1-\theta)Z_{11} + \theta \hat{Z}_{11} = 0.3 * 122.24 + 0.7 * 96.66 = 104.33$. From (5.2), $R_{12} = \hat{R}_{11}(1) = 97.65$, $\hat{R}_{12}(1) = \hat{R}_{11}(2) = 96.47$, $\hat{R}_{12}(2) = \hat{R}_{11}(3) = 95.30$, and $\hat{R}_{12}(3) = \hat{R}_{11}(4) = 94.29$. The inventory following (5.3) is $I_{12} = I_{11} + R_{12} - Z_{12} = 183.40 + 97.65 - 122.24 = 158.82$. The order following (5.4), the smoothing order

$$\begin{aligned}
O_{12} &= \beta_0 a_{12} + \beta_1 a_{11} + \beta_2 a_{10} + ... + \beta_9 a_3 + \beta_{10} a_2 + \hat{Z}_1 \\
&= 0.1615 * (25.58) + 0.2983 * (-0.67) + 0.4101 * (-1.30) + ... \\
&\quad + 0.4983 * (9.36) + 0.4115 * (4.45) + 98.44 = 97.69.
\end{aligned}$$

Then the rest of the calculations for the MRP table for week 12 can be repeated as those in period 11.

The comparison between standard MRP inventory and smoothing MRP inventory using 52 weeks generated demand data is shown in figure 5.3. The comparison between standard MRP $(1-B)O_t$ and smoothing MRP $(1-B)O_t$, including the demand changes from one period to the next $(Z_t - Z_{t-1})$ using 52 weeks generated demand data is shown in figure 5.4.

From the data, the variation in demand changes $Z_t - Z_{t-1}$ is 91.73. The variation in standard MRP $O_t - O_{t-1}$ is 751.14 while the variation in smoothing MRP $O_t - O_{t-1}$ is 8.72. For the variations in inventory, the variation in standard MRP I_t is 886.47 while the variation in smoothing MRP $O_t - O_{t-1}$ is 1212.51. Hence, using the smoothing policy can reduce the variation in order changes $O_t - O_{t-1}$ 99 percent compared with using the standard MRP ordering policy with the compensation in the increase in the variation in inventory 37 percent.

5.3 The Bounded MRP

This section proposes a simulation based technique to set upper/lower bounds to an order up to target MRP table. The width of the bounds should be set such that it will not be too wide then it has no effect to control the variation in production that creates the bullwhip effect (see figure 5.1), or it will not too narrow then the orders after the enforced bounded will be

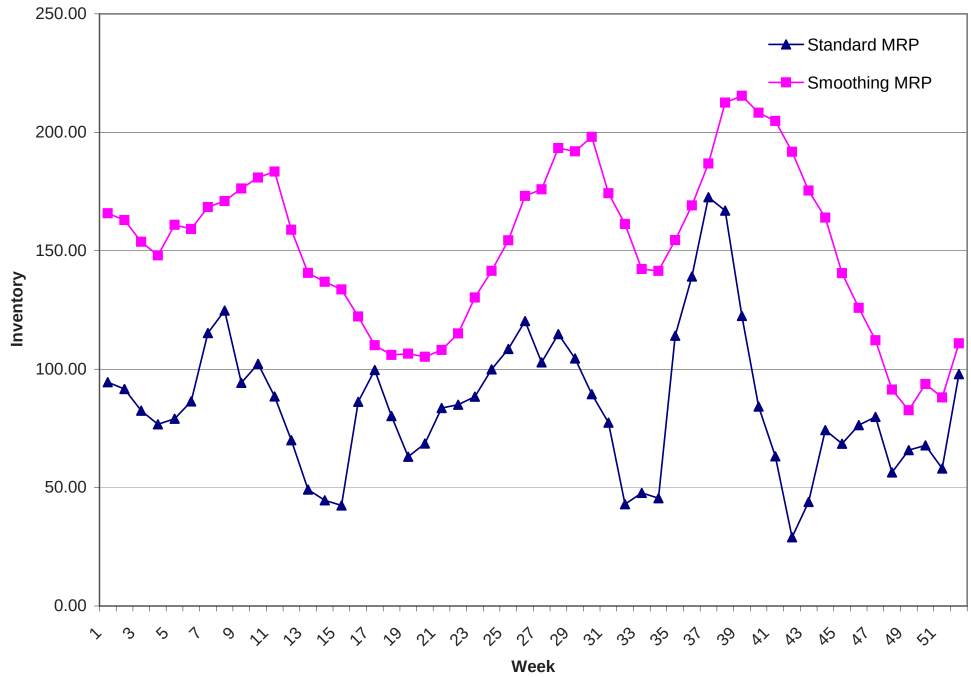

Figure 5.3: Unbounded MRP Inventory for 52 weeks with $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$

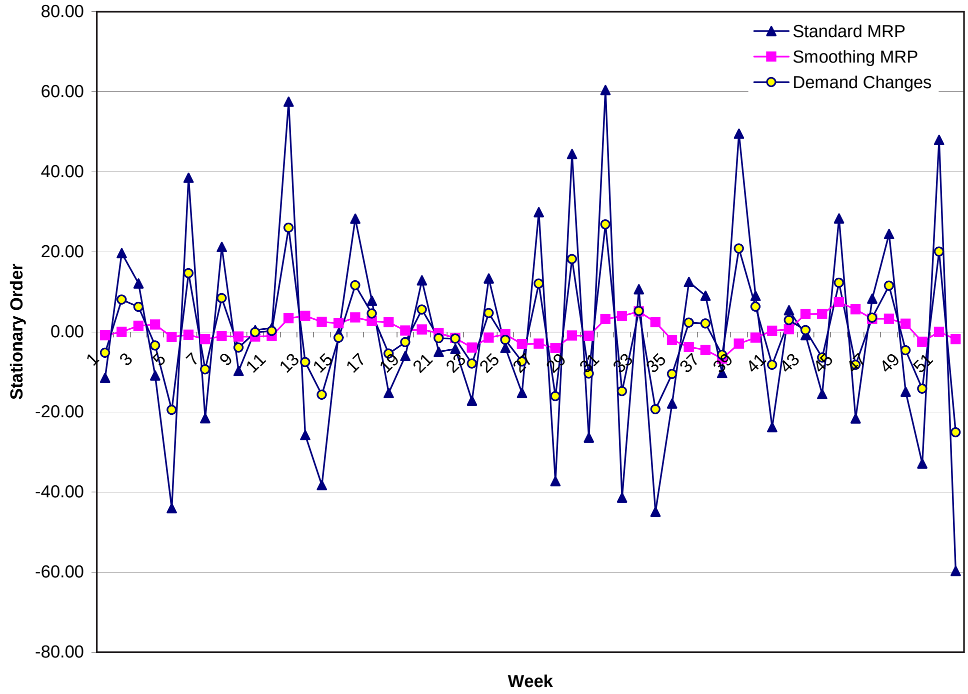

Figure 5.4: Unbounded MRP $(1 - B)O_t$ for 52 weeks with $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$

exploded (see figure 5.2). The optimal width of the bounds is set into an unbounded standard MRP table to enforce its variation in order changes to have the closet variation as if using an unbounded smoothing MRP table.

5.3.1 Bounded MRP Mechanism

Let L be the ordering lead time. S is the smoothing period. F is the forecast period. An example of the bounded MRP at period t and at period $t+1$ is shown in table 5.10 and table 5.11.
At period t, the upper bound is set as

$$U_t(i) = \hat{O}_t(i) + b_t(i)$$

and lower bound is set as

$$L_t(i) = \hat{O}_t(i) - b_t(i)$$

for $i = 1, 2, 3, ..., F$. The demand Z_t and $\hat{Z}_t$, receipt R_t,inventory I_t, order O_t and $\hat{O}_t(i)$ are calculated the same as the standard MRP table. However, O_t cannot exceed the lower bound and upper bounds set in the previous period. For example, at period $t+1$,

$$L_t(1) \leq O_{t+1} \leq U_t(1)$$

Table 5.10: Bounded MRP at Period t

t	Flex. Qty.	$b_t(1)$	$b_t(2)$	$b_t(3)$	$b_t(4)$	$b_t(5)$	…	$b_t(F-1)$	$b_t(F)$
Period	t	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	…	$t+F-1$	$t+F$
Demand	Z_t	$\hat{Z}_t$	$\hat{Z}_t$	$\hat{Z}_t$	$\hat{Z}_t$	$\hat{Z}_t$	…	$\hat{Z}_t$	$\hat{Z}_t$
Receipts	R_t	$\hat{R}_t(1)$	$\hat{R}_t(2)$	$\hat{R}_t(3)$	$\hat{R}_t(4)$	$\hat{R}_t(5)$	…	$\hat{R}_t(F-1)$	$\hat{R}_t(F)$
Inventory	I_t	$\hat{I}_t(1)$	$\hat{I}_t(2)$	$\hat{I}_t(3)$	$\hat{I}_t(4)$	$\hat{I}_t(5)$	…	$\hat{I}_t(F-1)$	$\hat{I}_t(F)$
Order	O_t	$\hat{O}_t(1)$	$\hat{O}_t(2)$	$\hat{O}_t(3)$	$\hat{O}_t(4)$	$\hat{O}_t(5)$	…	$\hat{O}_t(F-1)$	$\hat{O}_t(F)$
Upper Bound		$U_t(1)$	$U_t(2)$	$U_t(3)$	$U_t(4)$	$U_t(5)$	…	$U_t(F-1)$	$U_t(F)$
Lower Bound		$L_t(1)$	$L_t(2)$	$L_t(3)$	$L_t(4)$	$L_t(5)$	…	$L_t(F-1)$	$L_t(F)$

Table 5.11: Bounded MRP at Period $t+1$

$t+1$	Flex. Qty.	$b_{t+1}(1)$	$b_{t+1}(2)$	$b_{t+1}(3)$	$b_{t+1}(4)$	$b_{t+1}(5)$	…	$b_{t+1}(F-1)$	$b_{t+1}(F)$
Period	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	…	$t+F$	$t+F+1$
Demand	Z_{t+1}	$\hat{Z}_{t+1}$	$\hat{Z}_{t+1}$	$\hat{Z}_{t+1}$	$\hat{Z}_{t+1}$	$\hat{Z}_{t+1}$	…	$\hat{Z}_{t+1}$	$\hat{Z}_{t+1}$
Receipts	R_{t+1}	$\hat{R}_{t+1}(1)$	$\hat{R}_{t+1}(2)$	$\hat{R}_{t+1}(3)$	$\hat{R}_{t+1}(4)$	$\hat{R}_{t+1}(5)$	…	$\hat{R}_{t+1}(F-1)$	$\hat{R}_{t+1}(F)$
Inventory	I_{t+1}	$\hat{I}_{t+1}(1)$	$\hat{I}_{t+1}(2)$	$\hat{I}_{t+1}(3)$	$\hat{I}_{t+1}(4)$	$\hat{I}_{t+1}(5)$	…	$\hat{I}_{t+1}(F-1)$	$\hat{I}_{t+1}(F)$
Order	O_{t+1}	$\hat{O}_{t+1}(1)$	$\hat{O}_{t+1}(2)$	$\hat{O}_{t+1}(3)$	$\hat{O}_{t+1}(4)$	$\hat{O}_{t+1}(5)$	…	$\hat{O}_{t+1}(F-1)$	$\hat{O}_{t+1}(F)$
Upper Bound		$U_{t+1}(1)$	$U_{t+1}(2)$	$U_{t+1}(3)$	$U_{t+1}(4)$	$U_{t+1}(5)$	…	$U_{t+1}(F-1)$	$U_{t+1}(F)$
Lower Bound		$L_{t+1}(1)$	$L_{t+1}(2)$	$L_{t+1}(3)$	$L_{t+1}(4)$	$L_{t+1}(5)$	…	$L_{t+1}(F-1)$	$L_{t+1}(F)$

also,

$$L_{t+1}(i) \geq L_t(i+1)$$

and

$$U_{t+1}(i) \leq U_t(i+1)$$

for $i = 1, 2, 3, ..., F-1$. If the bound width $b_t(i)$ is too wide $(-\infty, +\infty)$, O_{t+1} will be exactly the order up to target (infinite loading) policy because O_{t+1} will hit the inventory target first before it will hit the bound width $b_t(i)$ made at time t. However, If the bound width $b_t(i)$ is too narrow, the orders in the bounded MRP will be exploded after the bounded periods in order to get the inventory back to the target.

We can set the bound directly to standard unbounded MRP by setting the bound width $b_t(i)$ from the standard deviation of the smoothing $O_t - O_{t-i}(i)$, or $\sigma_{O_t - O_{t-i}(i)}$, times a constant c. $\sigma_{O_t - O_{t-i}(i)}$ can be interpreted as the standard deviation of the error of the actual smoothing order at time t and the prior forecast smoothing orders made at time $t-i$ projected for time t. We use the constant c to adjust the width of the bounds to enforce the bounded MRP to have the closet variation as the smoothing unbounded MRP $\sigma_{O_t - O_{t-i}(i)}$. The motivations in setting the bounds width $b_t(i)$ as $c * \sigma_{O_t - O_{t-i}(i)}$

is provided in section 5.3.2. Thus the bound width $b_t(i)$ is set as

$$
\begin{aligned}
& b_t(i) = c * \text{ Smoothing}\sqrt{Var\big(O_t - O_{t-i}(i)\big)} \\
& \text{when } i \leq S+1 \\
& b_t(i) = c\big(\beta_0^2 + \beta_1^2 + \beta_2^2 + ... + \beta_{i-1}^2\big)^{1/2}\sigma_a \\
& \text{when } S+2 \leq i \leq F \\
& b_t(i) = c\Big(\beta_0^2 + \beta_1^2 + ... + \beta_{S-1}^2 + \beta_S^2 + (1-\theta)^2(i-S-1)\Big)^{1/2}\sigma_a.
\end{aligned}
\tag{5.7}
$$

By setting the bounds width $b_t(i)$ this way, new bounds are computed at the *flex fences*, that is, the period at which the bounds are first imposed or become tighter. The bounds $b_t(i)$ is the *flexibility requirements profiles* that gives the amount by which the actual order at period $t+i$, O_{t+i}, can deviate from the current planned orders made at the period t, $\hat{O}_t(i)$. This bound width $b_t(i)$ can be expressed as a percentage of the planned orders following the notation in table 5.10, where $i = 1, 2, 3, ..., F$

$$
\begin{aligned}
\frac{b_t(i)}{\hat{O}_t(i)} * 100 &= \frac{U_t(i) - \hat{O}_t(i)}{\hat{O}_t(i)} * 100 \\
&= \frac{\hat{O}_t(i) - L_t(i)}{\hat{O}_t(i)} * 100 \\
&= \frac{c * \text{Smoothing}\sqrt{Var\big(O_t - O_{t-i}(i)\big)}}{\hat{O}_t(i)} * 100.
\end{aligned}
\tag{5.8}
$$

To the supplier, the lower bound on the flex requirements profile represents a commitment from the customer guaranteeing the minimum quantity

Table 5.12: Bound Widths $b_t(i)$ for $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$

i	1	2	3	4	5	6	7	8	9	10	11
b_t(i)	1.62	3.39	5.32	7.28	9.18	10.95	12.53	13.88	14.97	15.78	16.31

of materials that will be bought in the future weeks. The upper bound represents a commitment from the supplier to the customer guaranteeing the ability to ramp up beyond the plan by the specified amount. Indeed, the flexibility requirements profile does not have to be symmetric, the ramp up and ramp down requirements could be different for a given week.

To illustrate, we use the same 52 weeks demand data shown in table 5.3 with ordering lead time $L = 4$, smoothing period $S = 10$, forecast period $F = 11$. Suppose that we let the inventory target T to be $3\sigma_I$ following theorem 3, then $T = 160.57$.

Suppose we let $c = 1$. The values of the smoothing β's weights is shown in table 5.6. The values of the bounds width $b_t(i)$, where $i = 1, 2, 3, ...11$ following (5.7) is shown in table 5.12. Table 5.13 shows the bounded MRP at period 1. Table 5.14 shows the bounded MRP at period 2.

In the bounded MRP, the demand, receipt, inventory, and order are calculated the same as the standard MRP in section 5.2.2 except that the order can not exceed the lower bound and upper bound.

At week 1, the order O_1 following (5.4) is $O_1 = T - \hat{I}_1(3) + \hat{Z}_1(4) = 160.57 - 170.45 + 98.44 = 88.55$. $\hat{O}_1(1) = T - \hat{I}_1(4) + \hat{Z}_1(5) = 160.57 - 160.57 +$

Table 5.13: Bounded MRP at Period 1

1	$b_t(i)$	1.62	3.39	5.32	7.28	9.18	10.95	12.53	13.88	14.97	15.78	16.31
Week	1	2	3	4	5	6	7	8	9	10	11	12
Demand	94.80	98.44	98.44	98.44	98.44	98.44	98.44	98.44	98.44	98.44	98.44	98.44
Receipts	100.00	100.00	100.00	100.00	88.55	98.44	98.44	98.44	98.44	98.44	98.44	98.44
Inventory	165.77	167.33	168.89	170.45	160.57	160.57	160.57	160.57	160.57	160.57	160.57	160.57
Order	88.55	98.44	98.44	98.44	98.44	98.44	98.44	98.44	98.44	98.44	98.44	98.44
LB		96.82	95.05	93.12	91.16	89.26	87.49	85.90	84.55	83.47	82.66	82.13
UB		100.05	101.83	103.76	105.72	107.62	109.39	110.97	112.32	113.41	114.22	114.75

Table 5.14: Bounded MRP at Period 2

2	$b_t(i)$	1.62	3.39	5.32	7.28	9.18	10.95	12.53	13.88	14.97	15.78	16.31
Week	2	3	4	5	6	7	8	9	10	11	12	13
Demand	102.89	99.77	99.77	99.77	99.77	99.77	99.77	99.77	99.77	99.77	99.77	99.77
Receipts	100.00	100.00	100.00	88.55	100.05	101.83	103.76	101.90	99.77	99.77	99.77	99.77
Inventory	162.88	163.11	163.33	152.11	152.39	154.45	158.44	160.57	160.57	160.57	160.57	160.57
Order	100.05	101.83	103.76	101.90	99.77	99.77	99.77	99.77	99.77	99.77	99.77	99.77
LB		100.22	100.37	96.58	92.49	90.59	88.82	87.24	85.89	84.80	84.00	83.47
UB		101.83	103.76	105.72	107.06	108.96	110.73	112.31	113.41	114.22	114.75	116.08

$98.44 = 98.44$. The lower bound is $L_1(1) = \hat{O}_1(1) - b_1(1) = 98.44 - 1.62 = 96.82$. The upper bound $U_1(1) = \hat{O}_1(1) + b_1(1) = 98.44 + 1.62 = 100.05$. $\hat{O}_1(2) = T - \hat{I}_1(5) + \hat{Z}_1(6) = 160.57 - 160.57 + 98.44 = 98.44$. The lower bound is $L_1(2) = \hat{O}_1(2) - b_1(2) = 98.44 - 3.39 = 95.05$. The upper bound $U_1(2) = \hat{O}_1(2) + b_1(2) = 98.44 + 3.39 = 101.83$. Repeating these calculations, we have the results shown in table 5.13.

At week 2, the order O_2 following (5.4) is $O_2 = T - \hat{I}_2(3) + \hat{Z}_2(4) = 160.57 - 152.11 + 99.77 = 108.23$. However, O_2 must stay within the bounds $[L_1(1), U_1(1)]$ or $[96.82, 100.05]$, hence $O_2 = 100.05$. $\hat{O}_2(1) = T - \hat{I}_2(4) + \hat{Z}_2(5) = 160.57 - 152.39 + 99.77 = 107.95$. However, O_2 must stay within the bounds $[L_1(2), U_1(2)]$ or $[95.05, 101.83]$, hence $\hat{O}_2(1) = 101.83$. The lower bound is $L_2(1) = \hat{O}_2(1) - b_2(1) = 101.83 - 1.62 = 100.22$. Since $L_2(1) > L_1(2)$, the lower bound is $L_2(1) = 100.22$. The upper bound $U_2(1) = \hat{O}_2(1) + b_2(1) = 101.83 + 1.62 = 103.45$. Since $U_2(1) > U_1(2)$, the upper bound $U_2(1)$ is $U_1(2) = 101.83$.

Also, $\hat{O}_2(2) = T - \hat{I}_2(5) + \hat{Z}_2(6) = 160.57 - 154.45 + 99.77 = 105.89$. However, $\hat{O}_2(2)$ must stay within the bounds $[L_1(3), U_1(3)]$ or $[93.12, 103.76]$, hence $O_2 = 103.76$. The lower bound is $L_2(2) = \hat{O}_2(2) - b_2(2) = 103.76 - 3.39 = 100.37$. Since $L_2(2) > L_1(3)$, the lower bound is $L_2(2) = 100.37$. The upper bound $U_2(2) = \hat{O}_2(2) + b_2(2) = 103.76 + 3.39 = 107.15$. Since $U_2(2) > U_1(3)$, the upper bound $U_2(2)$ is $U_1(3) = 103.76$. Repeating these calculations, we have the results shown in table 5.14 where the lower bound $L_2(11) = \hat{O}_2(11) - b_2(11) = 99.77 - 16.31 = 83.47$ and the upper bound

$U_2(11) = \hat{O}_2(11) + b_2(11) = 99.77 + 16.31 = 116.08.$

5.3.2 Motivations in Setting $b_t(i)$ as $c * \sigma_{O_t - O_{t-i}(i)}$

From the information from the order at time t, O_t, we want to make a forecast of the order at time $t + i$ periods in the future by using the past information in O_t and its forecast values $\hat{O}_t(i)$. The objective is to set the bounds $b_t(i)$ around the current orders O_t or $\hat{O}_t(i)$ such that the bounds is wide enough to guarantee that its inventory will not be exploded, in other words, the inventory is stationary. Also, we want to set the bounds tight enough to be effective in controlling the rate of orders changes. Following the generalized ordering policy proposed in chapter 4, the order O_t in (4.1).

$$O_t = \beta_0 a_t + \beta_1 a_{t-1} + ... + \beta_{S-1} a_{t-S+1} + (1 + \psi_1 + \psi_2 + ... + \psi_{S+L} - \sum_{i=0}^{S-1} \beta_i) a_{t-S}$$
$$+ \psi_{S+L+1} a_{t-S-1} + \psi_{S+L+2} a_{t-S-2} + ...$$

where S is the smoothing period, L is the ordering lead time, β's is the smoothing weights.

For exponential smoothing demand (ARIMA(0,1,1)), the generalized order O_t in (4.8) is

$$O_t = \beta_0 a_t + \beta_1 a_{t-1} + ... + \beta_{S-1} a_{t-S+1} + \Big(1 + (S+L)(1-\theta) - \sum_{i=0}^{S-1} \beta_i\Big) a_{t-S}$$
$$+ (1-\theta) a_{t-S-1} + (1-\theta) a_{t-S-2} + (1-\theta) a_{t-S-3} + ...$$

Let $B_{t-i}(i)$ be a the proposed (upper/lower) bounds modified from the generalized orders O_t to be used with the bounded MRP. $B_{t-i}(i)$ is interpreted as the bounds made at the past $t-i$ periods projected for i periods ahead. At period t, suppose that O_t in table 5.12 hit the upper bound $U_{t-i}(i)$ in table 5.12 that is previously set at period $t-i$, then $O_t = U_{t-i}(i) = B_{t-i}(i)$. Vice versa, suppose that O_t hit the lower bound $L_{t-i}(i)$, then $O_t = L_{t-i}(i) = B_{t-i}(i)$. We want to set $B_{t-i}(i)$ such that it can absorb the future orders that makes the inventory at time t be stationary. Two proposed forms of bounds $B_{t-i}(i)$ for any ARIMA demand are:

1. The bounds $B_{t-i}(i)$ is the sum of the smoothing weight β's up to period i modified from the generalized orders in (4.1).

2. The bounds $B_{t-i}(i)$ is to lengthen lead time i more periods modified from the generalized orders in (4.1).

The two mathematical forms of the bounds $B_{t-i}(i)$ are shown below:

<u>Form 1</u>: The bounds $B_{t-i}(i)$ is the sum of the smoothing weight β's up to

period i modified from the generalized orders in (4.1).

When $i < S$

$$B_{t-i}(i) = (\beta_0 + \beta_1 + ... + \beta_i)a_{t-i} + \beta_{i+1}a_{t-i-1}$$
$$+ \beta_{i+2}a_{t-i-2} + ... + \beta_{S-1}a_{t-S+1}$$
$$+ \Big(1 + \psi_1 + \psi_2 + ... + \psi_{S+L} - \sum_{i=0}^{S-1}\beta_i\Big)a_{t-S} \tag{5.9}$$
$$+ \psi_{S+L+1}a_{t-S-1} + \psi_{S+L+2}a_{t-S-2} + \psi_{S+L+3}a_{t-S-3} + ...$$

When $i \geq S$

$$B_{t-i}(i) = \big(1 + \psi_1 + \psi_2 + ... + \psi_{i+L}\big)a_{t-i}$$
$$+ \psi_{i+L+1}a_{t-i-1} + \psi_{i+L+2}a_{t-i-2} + ...$$

In case 2, the bound $B_{t-i}(i)$ is simply the standard MRP ordering policy for lead time $i + L$.

To prove that the inventory I_t in (5.9) is stationary. Suppose that the inventory target $T = 0$ and the level $\mu = 0$.

Proof

From (3.5),

$$I_t = \sum_{j=0}^{\infty} O_{t-L-j} - \sum_{j=0}^{\infty} Z_{t-j}.$$

Case 1: $i \leq S - 1$.

From (5.9),

$$B_{t-i}(i) = (\beta_0 + \beta_1 + ... + \beta_i)a_{t-i} + \beta_{i+1}a_{t-i-1} + \beta_{i+2}a_{t-i-2}$$

$$+ ... + \beta_{S-1}a_{t-S+1} + \left(1 + \psi_1 + \psi_2 + ... + \psi_{S+L} - \sum_{i=0}^{S-1}\beta_i\right)a_{t-S}$$

$$+ \psi_{S+L+1}a_{t-S-1} + \psi_{S+L+2}a_{t-S-2} + \psi_{S+L+3}a_{t-S-3} + ...$$

From (3.3),

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + ...$$

Suppose that O_t at period t in table 5.12 hits the bound $B_{t-i}(i)$ made at period $t-i$, then $O_t = B_{t-i}(i)$. Thus, we have

$$O_{t-L} - Z_t = -a_t - \psi_1 a_{t-1} - \psi_2 a_{t-2} - ... - \psi_{L+i-1}a_{t-L-i+1}$$

$$+ (\beta_0 + \beta_1 + ... + \beta_i - \psi_{L+i})a_{t-L-i} + (\beta_{i+1} - \psi_{L+i+1})a_{t-L-i-1}$$

$$+ (\beta_{i+2} - \psi_{L+i+2})a_{t-L-i-2} + ... + (\beta_{S-1} - \psi_{L+S-1})a_{t-L-S+1}$$

$$+ \left(1 + \psi_1 + \psi_2 + ... + \psi_{L+S} - \sum_{i=0}^{S-1}\beta_i\right)a_{t-L-S}$$

$$+ \psi_{L+S+1}a_{t-L-S-1} + \psi_{L+S+2}a_{t-L-S-2} + ...$$

$$
\begin{aligned}
O_{t-L-1} - Z_{t-1} = & -a_{t-1} - \psi_1 a_{t-2} - \psi_2 a_{t-3} - ... - \psi_{L+i-1} a_{t-L-i} \\
& + (\beta_0 + \beta_1 + ... + \beta_i - \psi_{L+i}) a_{t-L-i-1} + (\beta_{i+1} - \psi_{L+i+1}) a_{t-L-i-2} \\
& + (\beta_{i+2} - \psi_{L+i+2}) a_{t-L-i-3} + ... + (\beta_{S-1} - \psi_{L+S-1}) a_{t-L-S} \\
& + \Big(1 + \psi_1 + \psi_2 + ... + \psi_{L+S} - \sum_{i=0}^{S-1} \beta_i\Big) a_{t-L-S-1} \\
& + \psi_{L+S+1} a_{t-L-S-2} + \psi_{L+S+2} a_{t-L-S-3} + ...
\end{aligned}
$$

$$
\begin{aligned}
O_{t-L-2} - Z_{t-2} = & -a_{t-2} - \psi_1 a_{t-3} - \psi_2 a_{t-4} - ... - \psi_{L+i-1} a_{t-L-i-1} \\
& + (\beta_0 + \beta_1 + ... + \beta_i - \psi_{L+i}) a_{t-L-i-2} + (\beta_{i+1} - \psi_{L+i+1}) a_{t-L-i-3} \\
& + (\beta_{i+2} - \psi_{L+i+2}) a_{t-L-i-4} + ... + (\beta_{S-1} - \psi_{L+S-1}) a_{t-L-S-1} \\
& + \Big(1 + \psi_1 + \psi_2 + ... + \psi_{L+S} - \sum_{i=0}^{S-1} \beta_i\Big) a_{t-L-S-2} \\
& + \psi_{L+S+1} a_{t-L-S-3} + \psi_{L+S+2} a_{t-L-S-4} + ...
\end{aligned}
$$

$$
\vdots \qquad \vdots \qquad \vdots
$$

combining the above terms gives

$$
\begin{aligned}
I_t &= \sum_{j=0}^{\infty} O_{t-L-j} - \sum_{j=0}^{\infty} Z_{t-j} \\
&= -a_t - (1+\psi_1)a_{t-1} - (1+\psi_1+\psi_2)a_{t-2} \\
&\quad - ... - (1+\psi_1+\psi_2+...+\psi_{L+i-1})a_{t-L-i+1} \\
&\quad - \big(1+\psi_1+\psi_2+...+\psi_{L+i} - \beta_0 - \beta_1 - ... - \beta_i\big)a_{t-L-i} \\
&\quad - \big(1+\psi_1+\psi_2+...+\psi_{L+i+1} - \beta_0 - \beta_1 - ... - \beta_{i+1}\big)a_{t-L-i-1} \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad - \Big(1+\psi_1+\psi_2+...+\psi_{L+S-1} - \sum_{i=0}^{S-1}\beta_i\Big)a_{t-S-L-1}.
\end{aligned}
$$

For ARIMA(0,1,1) demand,

$$
\begin{aligned}
I_t &= -a_t - (2-\theta)a_{t-1} - (3-2\theta)a_{t-2} - ... - (L+i-(L+i-1)\theta)a_{t-L-i+1} \\
&\quad - \big(1+(L+i)(1-\theta) - \beta_0 - \beta_1 - ... - \beta_i\big)a_{t-L-i} \\
&\quad - \big(1+(L+i+1)(1-\theta) - \beta_0 - \beta_1 - ... - \beta_{i+1}\big)a_{t-L-i-1} \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad - \Big(1+(L+S-1)(1-\theta) - \sum_{i=0}^{S-1}\beta_i\Big)a_{t-L-S+1}.
\end{aligned}
$$

Case 2: $i \geq S$.

From (5.9),

$$B_{t-i}(i) = \big(1+\psi_1+\psi_2+...+\psi_{i+L}\big)a_{t-i} + \psi_{i+L+1}a_{t-i-1} + \psi_{i+L+2}a_{t-i-2} + ...$$

From (3.3),

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + ...$$

Suppose that O_t at period t in table 5.12 hits the bound $B_{t-i}(i)$ made at period $t-i$, then $O_t = B_{t-i}(i)$. Thus, we have

$$\begin{aligned}
O_{t-L} - Z_t &= -a_t - \psi_1 a_{t-1} - \psi_2 a_{t-2} - ... - \psi_{L+i-1}a_{t-L-i+1} \\
&\quad + \big(1+\psi_1+\psi_2+...+\psi_{i+L-1}\big)a_{t-L-i} \\
O_{t-L-1} - Z_{t-1} &= -a_{t-1} - \psi_1 a_{t-2} - \psi_2 a_{t-3} - ... - \psi_{L+i-1}a_{t-L-i} \\
&\quad + \big(1+\psi_1+\psi_2+...+\psi_{i+L-1}\big)a_{t-L-i-1} \\
O_{t-L-2} - Z_{t-2} &= -a_{t-2} - \psi_1 a_{t-3} - \psi_2 a_{t-4} - ... - \psi_{L+i-1}a_{t-L-i-1} \\
&\quad + \big(1+\psi_1+\psi_2+...+\psi_{i+L-1}\big)a_{t-L-i-2} \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots
\end{aligned}$$

combining the above terms gives

$$\begin{aligned}
I_t &= \sum_{j=0}^{\infty} O_{t-L-j} - \sum_{j=0}^{\infty} Z_{t-j} \\
&= -a_t - (1+\psi_1)a_{t-1} - (1+\psi_1+\psi_2)a_{t-2} \\
&\quad - ... - (1+\psi_1+\psi_2+...+\psi_{L+i-1})a_{t-L-i+1}.
\end{aligned}$$

For ARIMA(0,1,1) demand,

$$I_t = -a_t - (2-\theta)a_{t-1} - (3-2\theta)a_{t-2} - ... - (L+i-(L+i-1)\theta)a_{t-L-i+1}.$$

<u>*End of proof*</u>

<u>Form 2</u>: The bounds $B_{t-i}(i)$ is to lengthen lead time i more periods modified from the generalized orders in (4.1).

$$\begin{aligned} B_{t-i}(i) &= \beta_0 a_{t-i} + \beta_1 a_{t-i-1} + \beta_2 a_{t-i-2} + ... + \beta_{S-1} a_{t-i-S+1} \\ &\quad + \Big(1 + \psi_1 + \psi_2 + ... + \psi_{S+L+i} - \sum_{i=0}^{S-1} \beta_i\Big) a_{t-i-S} \\ &\quad + \psi_{S+L+i+1} a_{t-i-S-1} + \psi_{S+L+i+2} a_{t-i-S-2} + ... \end{aligned} \tag{5.10}$$

To prove that the inventory I_t in (5.10) is stationary. Suppose that the inventory target $T = 0$ and the level $\mu = 0$.

<u>*Proof*</u>

From (3.5),

$$I_t = \sum_{j=0}^{\infty} O_{t-L-j} - \sum_{j=0}^{\infty} Z_{t-j}.$$

From (5.10),

$$\begin{aligned} B_{t-i}(i) &= \beta_0 a_{t-i} + \beta_1 a_{t-i-1} + \beta_2 a_{t-i-2} + ... + \beta_{S-1} a_{t-i-S+1} \\ &\quad + \Big(1 + \psi_1 + \psi_2 + ... + \psi_{S+L+i} - \sum_{i=0}^{S-1} \beta_i\Big) a_{t-i-S} \\ &\quad + \psi_{S+L+i+1} a_{t-i-S-1} + \psi_{S+L+i+2} a_{t-i-S-2} + ... \end{aligned}$$

From (3.3),

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + ...$$

Suppose that O_t at period t in table 5.12 hits the bound $B_{t-i}(i)$ made at period $t-i$, then $O_t = B_{t-i}(i)$. Thus, we have

$$\begin{aligned} O_{t-L} - Z_t = & -a_t - \psi_1 a_{t-1} - \psi_2 a_{t-2} - ... - \psi_{L+i-1} a_{t-L-i+1} \\ & - (\psi_{L+i} - \beta_0) a_{t-L-i} - (\psi_{L+i+1} - \beta_1) a_{t-L-i-1} \\ & - ... - (\psi_{L+i+S-1} - \beta_{S-1}) a_{t-L-i-S+1} \\ & + \Big(1 + \psi_1 + \psi_2 + ... + \psi_{L+i+S} - \sum_{i=0}^{S-1} \beta_i\Big) a_{t-L-S-i} \\ & + \psi_{L+i+S+1} a_{t-L-S-i-1} + \psi_{L+i+S+2} a_{t-L-S-i-2} + ... \end{aligned}$$

$$\begin{aligned} O_{t-L-1} - Z_{t-1} = & -a_{t-1} - \psi_1 a_{t-2} - \psi_2 a_{t-3} - ... - \psi_{L+i-1} a_{t-L-i} \\ & - (\psi_{L+i} - \beta_0) a_{t-L-i-1} - (\psi_{L+i+1} - \beta_1) a_{t-L-i-2} \\ & - ... - (\psi_{L+i+S-1} - \beta_{S-1}) a_{t-L-i-S} \\ & + \Big(1 + \psi_1 + \psi_2 + ... + \psi_{L+i+S} - \sum_{i=0}^{S-1} \beta_i\Big) a_{t-L-S-i-1} \\ & + \psi_{L+i+S+1} a_{t-L-S-i-2} + \psi_{L+i+S+2} a_{t-L-S-i-3} + ... \end{aligned}$$

$$
\begin{aligned}
O_{t-L-2} - Z_{t-2} = & -a_{t-2} - \psi_1 a_{t-3} - \psi_2 a_{t-4} - ... - \psi_{L+i-1} a_{t-L-i-1} \\
& - (\psi_{L+i} - \beta_0) a_{t-L-i-2} - (\psi_{L+i+1} - \beta_1) a_{t-L-i-3} \\
& - ... - (\psi_{L+i+S-1} - \beta_{S-1}) a_{t-L-i-S-1} \\
& + \Big(1 + \psi_1 + \psi_2 + ... + \psi_{L+i+S} - \sum_{i=0}^{S-1} \beta_i\Big) a_{t-L-S-i-2} \\
& + \psi_{L+i+S+1} a_{t-L-S-i-3} + \psi_{L+i+S+2} a_{t-L-S-i-4} + ... \\
& \vdots \qquad \vdots \qquad \vdots
\end{aligned}
$$

combining the above terms gives

$$
\begin{aligned}
I_t = & \sum_{j=0}^{\infty} O_{t-L-j} - \sum_{j=0}^{\infty} Z_{t-j} \\
= & -a_t - (1 + \psi_1) a_{t-1} - (1 + \psi_1 + \psi_2) a_{t-2} \\
& - ... - (1 + \psi_1 + \psi_2 + ... + \psi_{L+i-1}) a_{t-L-i+1} \\
& - \big(1 + \psi_1 + \psi_2 + ... + \psi_{L+i} - \beta_0\big) a_{t-L-i} \\
& - \big(1 + \psi_1 + \psi_2 + ... + \psi_{L+i+1} - \beta_0 - \beta_1\big) a_{t-L-i-1} \\
& \quad \vdots \qquad \vdots \qquad \vdots \\
& - \Big(1 + \psi_1 + \psi_2 + ... + \psi_{L+i+S-1} - \sum_{i=0}^{S-1} \beta_i\Big) a_{t-L-i-S+1}.
\end{aligned}
$$

For ARIMA(0,1,1) demand,

$$
\begin{aligned}
I_t = & -a_t - (2-\theta)a_{t-1} - (3-2\theta)a_{t-2} - ... - (L+i-(L+i-1)\theta)a_{t-L-i+1} \\
& - \big(1+(L+i)(1-\theta) - \beta_0\big)a_{t-L-i} \\
& - \big(1+(L+i+1)(1-\theta) - \beta_0 - \beta_1\big)a_{t-L-i-1} \\
& \quad \vdots \qquad \vdots \qquad \vdots \\
& - \Big(1+(L+i+S-1)(1-\theta) - \sum_{i=0}^{S-1}\beta_i\Big)a_{t-L-i-S+1}.
\end{aligned}
$$

End of proof

We can see that the bound width $b_t(i) = c * \sigma_{O_t - O_{t-i}(i)}$ in (5.7) is closely related to the two forms of $B_{t-i}(i)$ in (5.9) and (5.10). Although both forms of $B_{t-i}(i)$ guarantee stationary inventory, they don't guarantee that the bound width will be tighter as the forecasting period is nearer due to the random noise $a_(t)$ that can be positive or negative. Since our objective in setting the bounds in bounded MRP is to enforce the bounded MRP to have the closet properties in order and inventory variations as the smoothing MRP does, we propose to use the value of the bounds width $b_t(i) = c * \sigma_{O_t - O_{t-i}(i)}$ in (5.7) as a flex quantity profile to the standard MRP systems instead of finding the complicated forms of $B_{t-i}(i)$. Where the c value is served as an adjusting factor to enforce the properties in order and inventory variations. c value can be found by simulated the value of c to any specific demand time series data. By using one determined c value across all forecast periods, we can guarantee that the bound width will be tighter as the forecasting period is nearer.

5.3.3 Set $b_t(i)$ by Varying the c Value

In this section, we propose a simulation based technique to set the bounds by determining the c value in (5.7). If the bounds are too narrow, the variation in order changes $O_t - O_{t-1}$ will be small, but the variation in inventory I_t will be too large. On the other hand, If the bounds are too wide, the variation in order changes $O_t - O_{t-1}$ will be too large, but the variation in inventory I_t will be close to variation in inventory using the standard MRP ordering policy.

Since our objective in setting the bounds is to reduce the variation in standard MRP order changes $O_t - O_{t-1}$ to be close to the variation in order changes $O_t - O_{t-1}$ by using the smoothing policy, we can find the bounds by varying the c values.

By using 52 weeks generated demand data, with $\theta = 0.7$, lead time $L = 4$, smoothing period $S = 10$, forecast period $F = 11$, the comparison between the variance of (unbounded) smoothing MRP $(1 - B)O_t$ and variance of bounded MRP $(1 - B)O_t$ by varying the c values is shown in figure 5.5. The comparison between the variance of (unbounded) smoothing MRP inventory and the variance of bounded MRP inventory by varying the c values is shown in figure 5.6.

From figure 5.5 and figure 5.6, we can choose the value of c between 0.5 to 1.5. If the c value chosen is 0.61, the variation in bounded MRP $O_t - O_{t-1}$

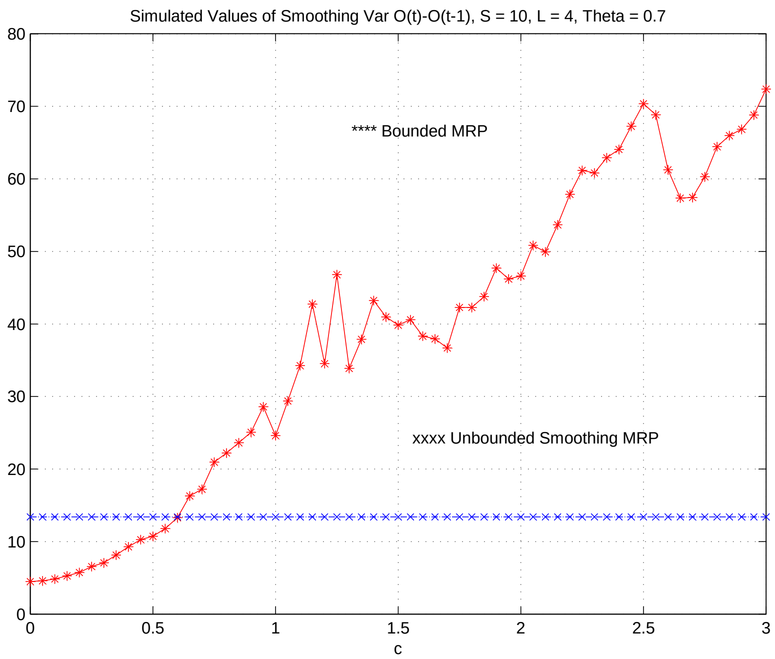

Figure 5.5: Comparison of $(1-B)O_t$ Variations between Bounded MRP v.s. (Unbounded) Smoothing MRP for 52 weeks with $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$

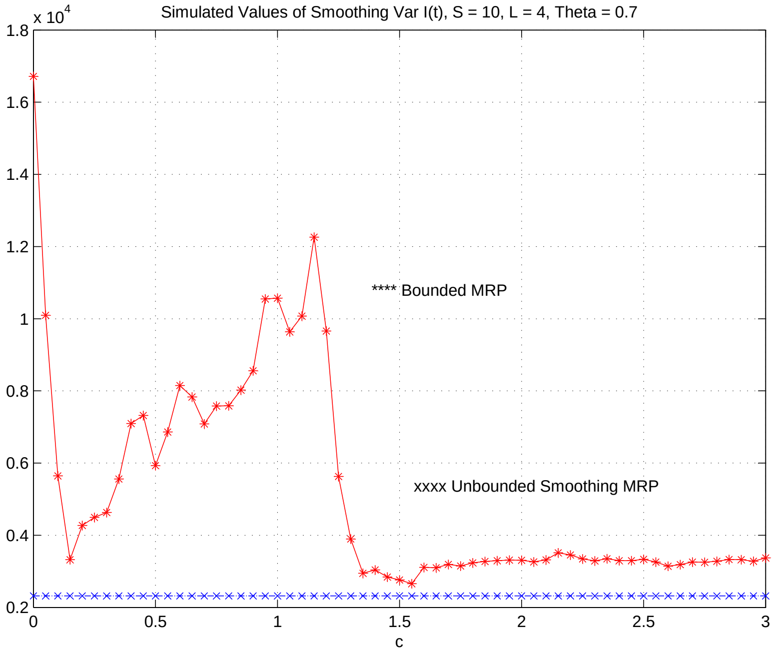

Figure 5.6: Comparison of Inventory Variations between Bounded MRP v.s. (Unbounded) Smoothing MRP for 52 weeks with $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$

Table 5.15: Comparison of I_t and $O_t - O_{t-1}$ Variations for $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$

	Variation	
	Order Changes	Inventory
Demand Changes	114.22	
Ordering Policy		
Standard MRP	633.37	1238.46
Smoothing MRP	13.39	2320.89
Bounded MRP, c = 0.61	13.48	8399.53
Bounded MRP, c = 1.53	38.06	2607.45

is approximately the same as the variation in (unbounded) smoothing MRP $O_t - O_{t-1}$. If the c value chosen is 1.53, the variation in bounded MRP I_t is closet to the variation in (unbounded) smoothing MRP I_t.

By the MATLAB program provided in appendix A, The result is shown in table 5.15.

The variation in demand changes, $Var(Z_t - Z_{t-1})$, is 114.02. The variation in order changes by using the standard MRP ordering policy, $Var(O_t - O_{t-1})$, is 633.37 in which the bullwhip effect multiplier, $Var(O_t - O_{t-1})/Var(Z_t - Z_{t-1})$, is 5.55. The variation in inventory by using the standard MRP ordering policy, $Var(I_t)$, is 1238.46.

The variation in order changes by using the smoothing ordering policy, $Var(O_t - O_{t-1})$, is 13.39. The bullwhip multiplier, $Var(O_t - O_{t-1})/Var(Z_t - Z_{t-1})$, is reduced to 0.11. However, the variation in inventory by using the smoothing ordering policy, $Var(I_t)$, is 2320.89 then the ratio of the inventory variation by using smoothing MRP over standard MRP is 1.87.

By using the bounded MRP with $c = 0.61$, the variation in order changes, $Var(O_t - O_{t-1})$, is 13.48. The variation in inventory $Var(I_t)$ is 8399.53. The variation in order changes by using the bounded MRP is close to the variation by using the (unbounded) smoothing MRP. However, the ratio of the variation in inventory by using bounded MRP over (unbounded) smoothing MRP is 3.8. Figure 5.7 shows the comparison of the variations in order changes $O_t - O_{t-1}$ when the ordering policy are (unbounded) standard MRP, (unbounded) smoothing MRP, and bounded MRP at $c = 0.61$. Figure 5.8 shows the comparison of the variations in the inventory when the ordering policy are (unbounded) standard MRP, (unbounded) smoothing MRP, and bounded MRP at $c = 0.61$.

By using the bounded MRP with $c = 1.53$, the variation in order changes, $Var(O_t - O_{t-1})$, is 38.06. The variation in inventory $Var(I_t)$ is 2607.45. The variation in inventory by using the bounded MRP is close to the variation by using the (unbounded) smoothing MRP. However, the bullwhip multiplier is increased from 0.11 to 0.33. Figure 5.9 shows the comparison of the variations in order changes $O_t - O_{t-1}$ when the ordering policy are (unbounded) standard MRP, (unbounded) smoothing MRP, and bounded MRP at $c = 1.53$. Figure 5.10 shows the comparison of the variations in the inventory when the ordering policy are (unbounded) standard MRP, (unbounded) smoothing MRP, and bounded MRP at $c = 1.53$.

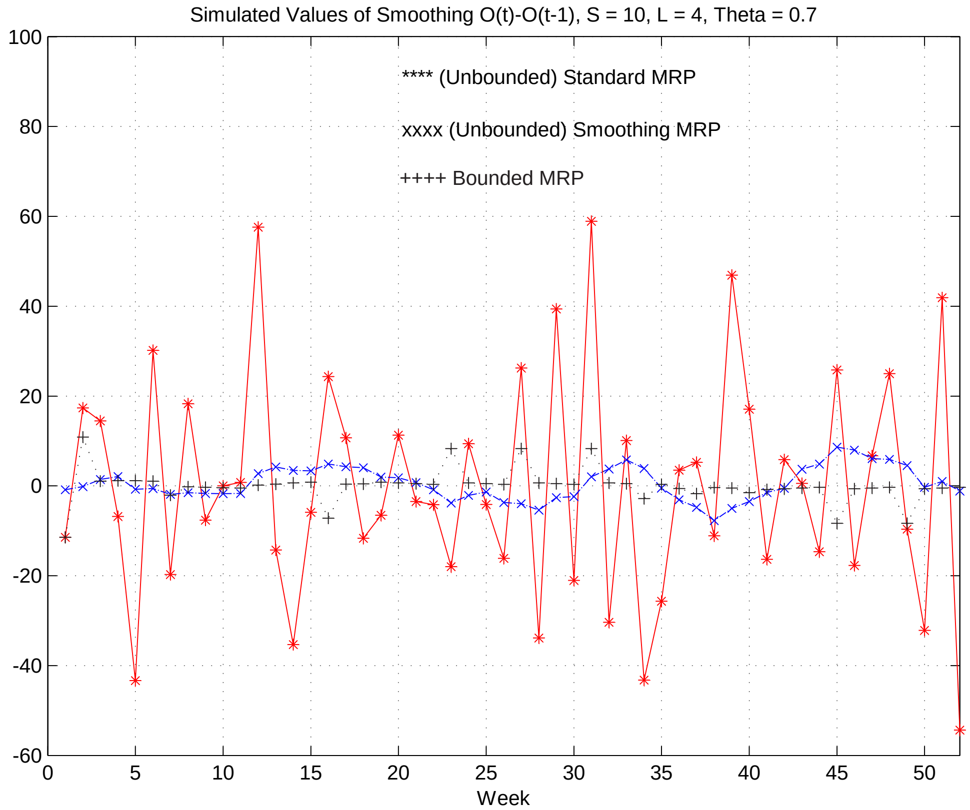

Figure 5.7: Bounded MRP $O_t - O_{t-1}$ for 52 weeks with $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$, $c = 0.61$

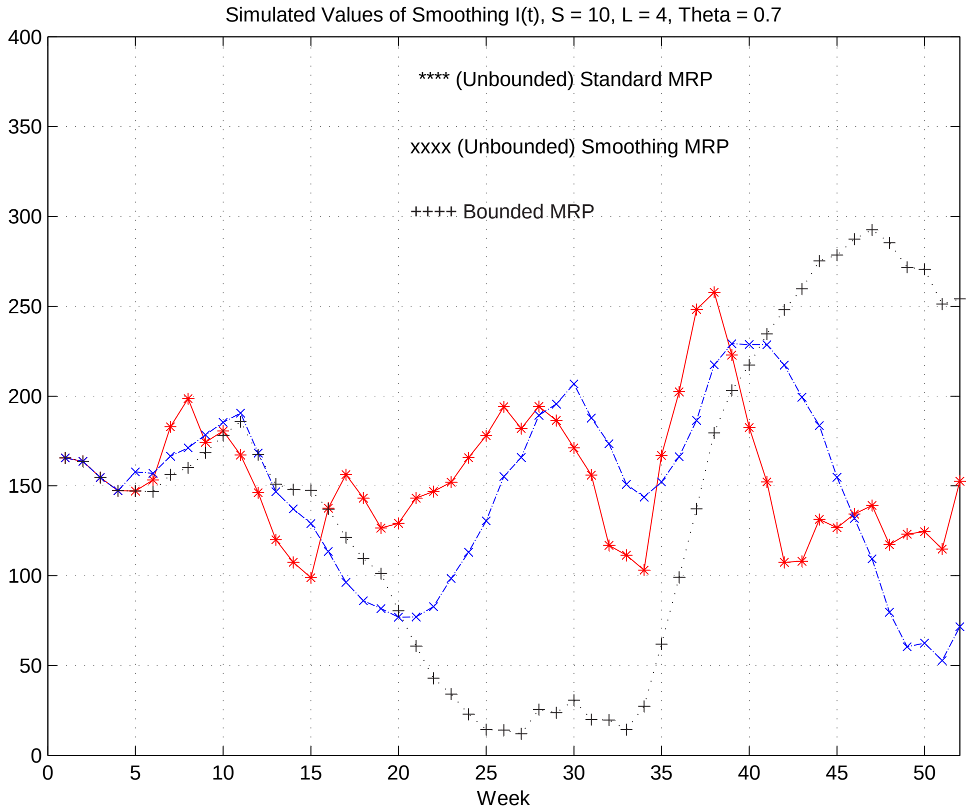

Figure 5.8: Bounded MRP I_t for 52 weeks with $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$, $c = 0.61$

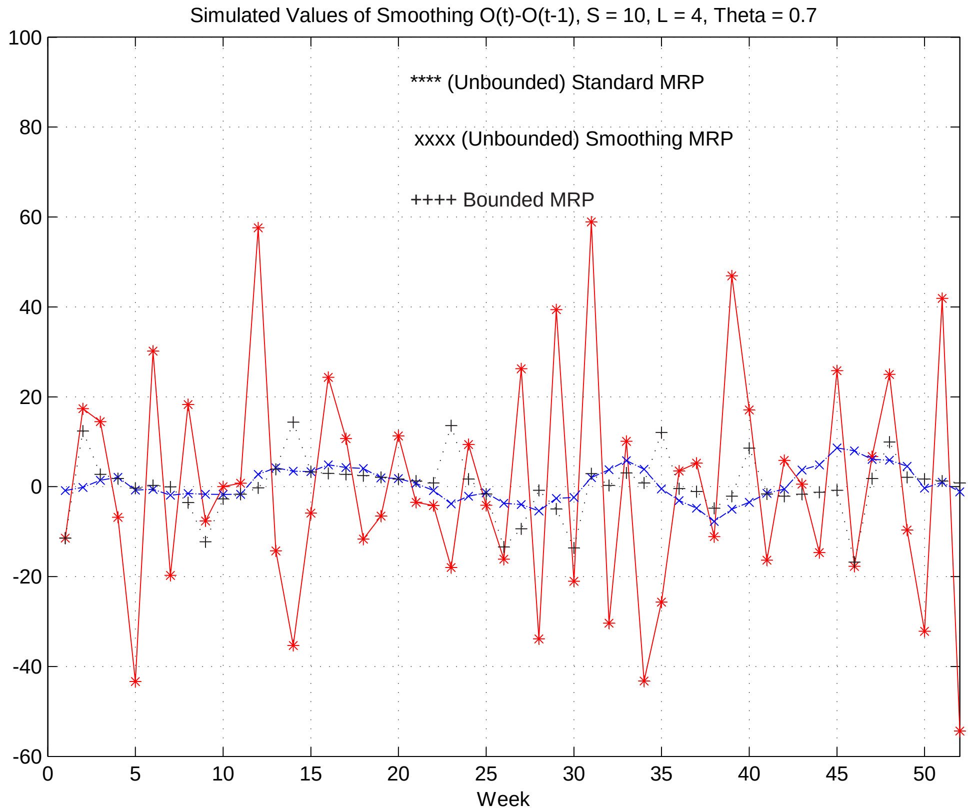

Figure 5.9: Bounded MRP $O_t - O_{t-1}$ for 52 weeks with $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$, $c = 1.53$

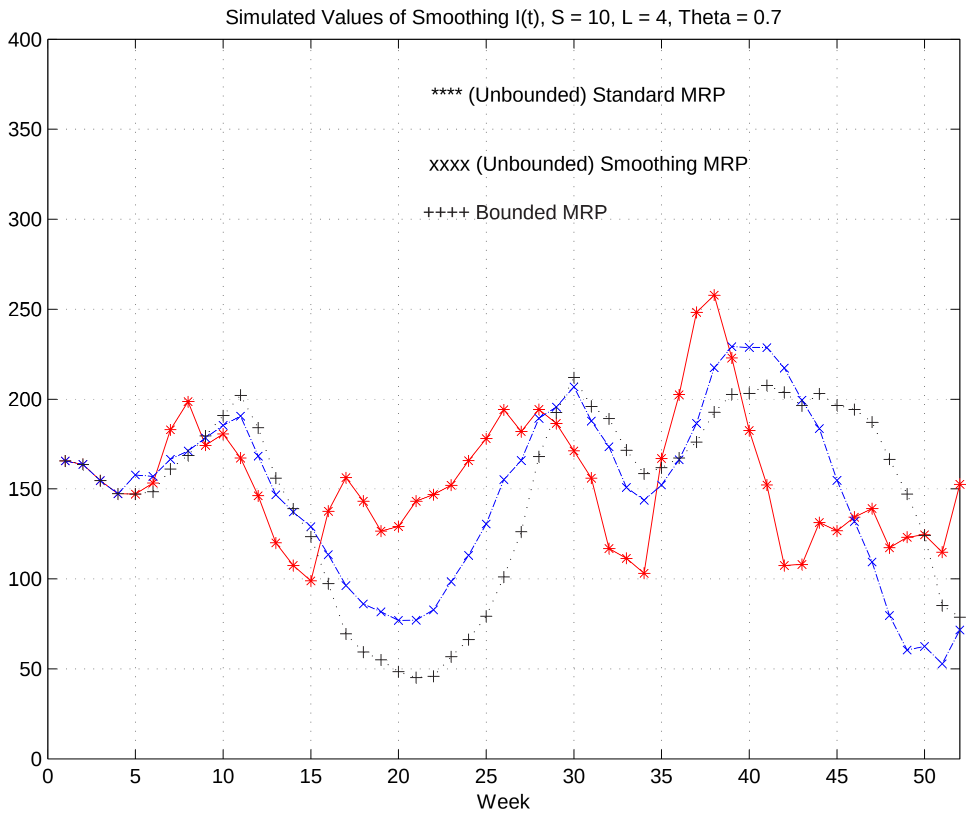

Figure 5.10: Bounded MRP I_t for 52 weeks with $L = 4$, $S = 10$, $F = 11$, $\theta = 0.7$, $c = 1.53$

From the simulation results in varying the c value, we can see that the lower bound and upper bound should be set at $c = 1.53$ where the variation in bounded MRP I_t is closet to the variation in (unbounded) smoothing MRP I_t.

5.3.4 Steps for Setting the Bounds

This section provides a guideline for manufacturers in setting bounds for standard MRP tables. The MATLAB code to determine the bounds for a given demand series for a standard MRP table is also provided in the appendix A.

The steps in setting bounds for a standard MRP table are shown below.

Step 1: Fit the demand data Z_t to ARIMA(0,1,1) following (3.13)

$$Z_t = Z_{t-1} + a_t - \theta a_{t-1}.$$

This can be done using any time series statistical software. From this step, we obtain the θ value.

Step 2: Calculate the forecast demand $\hat{Z}_t$ following the form of the exponential smoothing demand in (3.16)

$$\hat{Z}_t = \alpha Z_t + (1 - \alpha)\hat{Z}_{t-1}, \text{ where } \alpha = 1 - \theta.$$

Step 3: Let N be the number of observations of demand data. Calculate σ_a, the standard deviation of the white noise series, where a_t is calculated from

$$a_{t-j} = Z_{t-j} - \hat{Z}_{t-j-1}(1);\ j = 0, 1, 2, ..., N-1.$$

Step 4: With a given order lead time L, smoothing period S, forecast period F, calculate the smoothing weight β_i where i=0,1,2,...,S following (5.6)

$$\beta_i = \frac{(i+1)\big(4S^2 + (10-3i)S + (6-3i) + 6(S-i+1)L\big)}{(S+1)(S+2)(S+3)} - \frac{(i+1)\big(4S^2 + (4-3i)S + 3i + 6(S-i+1)L\big)}{(S+1)(S+2)(S+3)}\theta.$$

Step 5: With the standard MRP table following table 5.10, we can set the bound width $b_t(i)$ where $i = 1, 2, 3, ..., F$ following (5.7).

$$b_t(i) = c * \text{ Smoothing}\sqrt{Var\big(O_t - O_{t-i}(i)\big)}$$

when $i \leq S+1$

$$b_t(i) = c\big(\beta_0^2 + \beta_1^2 + \beta_2^2 + ... + \beta_{i-1}^2\big)^{1/2}\sigma_a$$

when $S+2 \leq i \leq F$

$$b_t(i) = c\Big(\beta_0^2 + \beta_1^2 + ... + \beta_{S-1}^2 + \beta_S^2 + (1-\theta)^2(i-S-1)\Big)^{1/2}\sigma_a.$$

Where the constant c is determined by ranging the c values applied to set the widths of the bounds. the simulated c value can be from 0 to 3. Thus a retailer/supplier can choose the value of c where the variation in bounded MRP inventory meets the (unbounded) smoothing MRP inventory.

Also, the (unbounded) smoothing MRP can be determined following table

5.1 where the smoothing order at period t, O_t, in table 5.1 is calculated following (5.5)

$$O_t = \beta_0 a_t + \beta_1 a_{t-1} + \beta_2 a_{t-2} + ... + \beta_{S-1} a_{t-S+1} + \beta_S a_{t-S} + \hat{Z}_{t-S-1}.$$

The MATLAB program to determine the bounds for a given demand series for a standard MRP table is provided in the appendix A. The information required for a user are:

1. .txt file demand data.
2. θ.
3. Ordering lead time L.
4. Smoothing period S.
5. Forecasting period F.
6. Initial value of the demand μ.
7. c_I value from 0 to 3 to set the inventory target at $c_I \sigma_I$.
8. $R_{O/I}$ value from 0 to 1;
    - If $R_{O/I} = 1$, the bound is set where the bounded MRP week-to-week order variance equal to the (unbounded) smoothing MRP week-to-week order variance.

- If $R_{O/I} = 0$, the bounds is set where the bounded MRP inventory variance equal to the (unbounded) smoothing MRP inventory variance.

5.4 Insights

This section provides the insights for the bounded MRP for exponential smoothing demand.

1. The shape of the flex quantity profile. From section 4.5, we know that the distribution of the smoothing weights β for ARIMA(0,1,1) order model is moving from a bell shape when the autocorrelation is low to a left tail shape when the autocorrelation is high. Hence, the shape of the flex quantity profile is directly determined by the β's weights as shown in (5.7). For a high autocorrelation demand, the widths of the flex quantity profile are wider those of a lower autocorrelation demand. Although the widths of the bounds are wider for the longer forecast periods, the rate in increasing widths are not constant across all forecast periods depending on the distribution of the smoothing weights β.

This can be seen from the table 5.16 and table 5.17. Both tables show the values of β_i for $L = 0$, S from 0 to 6. In table 5.16, when $\theta = 1$, the model is a noise series ARIMA(0,0,0) with low autocorrelation. In table 5.17, when $\theta = 0$, the model is a random walk series ARIMA(0,1,0) with high autocorrelation. Those tables also show the values of the flex quantity

Table 5.16: The β, $b(i)$, $b(i) - b(i-1)$ for $\theta = 1$ ARIMA(0,0,0), $L = 0$, $S =$ 0 to 6

S	β_0	β_1	β_2	β_3	β_4	β_5	β_6
0	1.0000						
1	0.2500	0.2500					
2	0.0900	0.1600	0.0900				
3	0.0400	0.0900	0.0900	0.0400			
4	0.0204	0.0523	0.0661	0.0523	0.0204		
5	0.0115	0.0319	0.0459	0.0459	0.0319	0.0115	
6	0.0069	0.0204	0.0319	0.0363	0.0319	0.0204	0.0069

S	b(1)	b(2)	b(3)	b(4)	b(5)	b(6)	b(7)
0	1.0000						
1	0.5000	0.7071					
2	0.3000	0.5000	0.5831				
3	0.2000	0.3606	0.4690	0.5099			
4	0.1429	0.2696	0.3725	0.4371	0.4598		
5	0.1071	0.2083	0.2988	0.3677	0.4088	0.4226	
6	0.0833	0.1654	0.2434	0.3091	0.3570	0.3845	0.3935

S	b(2)-b(1)	b(3)-b(2)	b(4)-b(3)	b(5)-b(4)	b(6)-b(5)	b(7)-b(6)
1	0.2071					
2	0.2000	0.0831				
3	0.1606	0.1085	0.0409			
4	0.1267	0.1029	0.0645	0.0228		
5	0.1012	0.0906	0.0689	0.0411	0.0138	
6	0.0821	0.0780	0.0657	0.0479	0.0275	0.0089

Table 5.17: The β, $b(i)$, $b(i) - b(i-1)$ for $\theta = 0$ ARIMA(0,1,0), $L = 0$, $S =$ 0 to 6

S	β_0	β_1	β_2	β_3	β_4	β_5	β_6
0	1.0000						
1	0.6946	1.3610					
2	0.4899	1.2102	1.4400				
3	0.3600	1.0000	1.4400	1.4400			
4	0.2746	0.8190	1.3064	1.5326	1.4173		
5	0.2157	0.6747	1.1483	1.4748	1.5620	1.3886	
6	0.1735	0.5621	1.0006	1.3617	1.5630	1.5630	1.3617

S	b(1)	b(2)	b(3)	b(4)	b(5)	b(6)	b(7)
0	1.0000						
1	0.8334	1.4337					
2	0.6999	1.3039	1.7720				
3	0.6000	1.1662	1.6733	2.0591			
4	0.5240	1.0458	1.5492	1.9831	2.3130		
5	0.4644	0.9436	1.4278	1.8744	2.2529	2.5425	
6	0.4165	0.8576	1.3176	1.7601	2.1589	2.4948	2.7542

S	b(2)-b(1)	b(3)-b(2)	b(4)-b(3)	b(5)-b(4)	b(6)-b(5)	b(7)-b(6)
1	0.6003					
2	0.6040	0.4682				
3	0.5662	0.5071	0.3858			
4	0.5218	0.5035	0.4339	0.3299		
5	0.4792	0.4842	0.4466	0.3785	0.2896	
6	0.4411	0.4600	0.4424	0.3988	0.3359	0.2594

profiles following the bound widths in (5.7) with $c = 1$ for illustration. Also the increase rates of the bound widths, $b(i) - b(i-1)$, are also shown in the tables to show the shape of the flex profile.

Chapter 6

Conclusion and Future Research

6.1 Conclusion

The main goal of this dissertation is to develop ARIMA supply chain models to understand and mitigate the bullwhip effect across supply chains caused by actual lumpy demand. In distinction of supply chain models in current literature that made assumption in the up to target ordering policy and are limited to specific ARIMA demand models, we propose the generalized ordering policy that includes the up to target ordering policy as a special case and can be applied to any ARIMA demand, any ordering lead time, and any desired smoothing period with the guaranteed stationary inventory. With the generalized ordering policy, manufacturers can smooth the orders arbitrarily to mitigate the bullwhip effect by controlling the tradeoffs between the variation in inventory and the variation in differencing orders (which is stationary by differencing) by changing the smoothing weights. We also provide generic formulas to determine the optimal smoothing weights for the smoothing ordering policy for ARIMA$(p, 0, q)$ and ARIMA$(p, 1, q)$ demand, hence, manufacturers can achieve the minimum variation in orders when the demand is ARIMA$(p, 0, q)$ demand or the minimum variation in order changes from one period to the next when the demand is ARIMA$(p, 1, q)$.

This dissertation also exemplifies the implementation of the smoothing policy into MRP tables for exponential smoothing ARIMA$(0, 1, 1)$ demand. We also propose the bounded MRP system corresponding to the rate based planning concept for single exponential smoothing or ARIMA(0,1,1) demand. The bounded MRP can be directly implemented into standard (order up to target) MRP tables. With this bounded MRP, manufacturers can mitigate the bullwhip effect and reduce the conflict between production planning and infeasible capacity planning.

6.2 Future Works

There are several directions that we can further explore the research.

First, the smoothing ordering policy and the bounded MRP can achieve only an optima point for each single stage in the proposed tandem line supply chain model. The future work can be developed to achieve the optimal point across the tandem line supply chain model.

Second, we provide the optimal smoothing weight formulas in the smoothing ordering policy only for ARIMA$(p, 0, q)$ and ARIMA$(p, 1, q)$ demand. The optimal smoothing weight formulas for ARIMA$(p, 2, q)$ and ARIMA$(p, 3, q)$ demand can be derived in future research. Hence, the optimal smoothing weight formulas can be applied to the family of exponential smoothing models that cover no trend (ARIMA(0,1,1)), linear trend (ARIMA(0,2,2)), and quadratic trend (ARIMA(0,3,3)) time series data.

Third, we consider only ARIMA models without seasonality (ARIMA(p, d, q)). The seasonal ARIMA models (ARIMA$(p, d, q)(ps, qs, ds)$) can be explored since seasonality plays a major role in supply chain management field due to actual consumer behavior.

Fourth, this dissertation can only be applied to a single demand item that uses univariate ARIMA modeling techniques. Future works can consider multiple items in a supply chain that requires multivariate ARIMA modeling techniques such as vector autoregression (VAR).

Fifth, the supply chain model used in this work is the tandem line supply chain model which is the simplest form of the supply chain models. A network of more complex supply chain models such as multiple retailers and multiple suppliers can be investigated.

Sixth, following the four causes that create the bullwhip effect proposed by Lee, Padmanahan, and Whang 1997, we consider only the demand signal processing. The other aspects in rationing game, order batching, and price variation can be developed in future research.

Seventh, other time series forecasting techniques can be explored to study the bullwhip effect in supply chains. Examples of those techniques are autoregressive conditional heteroskedastic (ARCH), generalized autoregressive conditional heteroskedastic (GARCH), and nonlinear time-series.

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Appendix

Appendix A: Determining the Bounds for a given demand time series

1. Main Function

```
function MRPBoundInfC
%
% A MATLAB file to determine the upper and lower bound for a
%   bounded MRP table, given demand series modelled to be an
%   ARIMA(0,1,1) or exponential smoothing demand
% Require functions: findbeta.m, findinvtarget.m, findunbORI.m,
%   setabsFbnd.m, findinfbndORI.m
% Usage: MRPBoundInfC
% Input:
%   1) A .txt file of N observations demand series Z(t)
%   2) theta: from 0 to 1
%   3) L: ordering lead time
%   4) S: smoothing period
%   5) F: forecasting period
%   6) Zbar: The initial value of the demand Z(0)
%   7) cInv: Inventory target at cInv*std(sigma Inventory)
%   8) RatioVarOI:
%          If RatioVarOI = 1, optimal is chosen where the bounded
%             MRP week-to-week order variance equal to the smoothing
```

```
%             unbounded MRP week-to-week order variance
%          If RatioVarOI = 0, optimal is chosen where the bounded
%             MRP inventory variance equal to the smoothing
%             unbounded MRP inventory variance
% Output:
%   1) c: optimal value of the bound
%   2) b: 2 by F-1 matrix of the upper and lower bound
%   3) beta: the smoothing weight
% Vuttichai Chatpattananan

% Input
Z = []; Zhat = []; a = [];
Zfilename = input('Enter the demand
file name, i.e. Z52.txt: ','s');
Z = load(Zfilename); theta =
input('Enter the theta value [0,1], i.e., 0.7: ','s');
theta =
str2num(theta);
L = input('Enter the ordering lead time, i.e., 4:
','s');
L = str2num(L);
S = input('Enter the smoothing period,
i.e., 10: ','s');
```

```
S = str2num(S);
F = input('Enter the forecast
period, i.e., 12: ','s');
F = str2num(F); Zbar = input('Enter the
initial value of the demand Z(0),
    i.e., 100: ','s');
    Zbar = str2num(Zbar);
cInv = input('Enter the Inventory target at cInv*std
    (sigma Inventory), i.e., 3: ','s'); cInv = str2num(cInv);
disp('Criteria to choose optimal c');
disp('   0: bounded MRP
inventory variance equal to the smoothing
    unbounded MRP inventory variance');
disp('   1: bounded MRP week-to-week order variance equal to the
    smoothing unbounded MRP week-to-week order variance');
RatioVarOI = input('   Enter 0 or 1 , i.e., 0: ','s');
RatioVarOI = str2num(RatioVarOI);
Zhat(1) = (1-theta)*Z(1)+theta*Zbar;
% Calculate the forecast demand Zhat
for i = 2:length(Z)
    Zhat(i) = (1-theta)*Z(i)+theta*Zhat(i-1);
end
a(1) = Z(1)-Zbar;
```

```
% Calculate the noise series a(t) and its standard deviation sigmaa
for i = 2:length(Z)
    a(i) = Z(i)-Zhat(i-1);
end
sigmaa = std(a);
% Unbounded MRP
betainf = []; OInf = []; RInf = []; IInf = []; InfLoadT = 0;
betaopt = []; OOpt = []; ROpt = []; IOpt = []; SmoothT = 0;
betainf = findbeta(S,L,theta,'I');
% Option for smoothing: I = Infinite Loading, S = Smoothing
betaopt = findbeta(S,L,theta,'S');
% Option for smoothing: I = Infinite Loading, S = Smoothing
[VarInvI,VarInvS,InfLoadT,SmoothT] =
     findinvtarget(S,L,theta,sigmaa,cInv,betaopt);
     % Set the inventory target
[OOpt,ROpt,IOpt,OOptDev,TOOptDev] =
     findunbORI(S,L,theta,Zbar,sigmaa,a,Z,Zhat,betaopt,SmoothT);
     % Smoothing Unbounded MRP
cBM = []; OOptDevMat = []; IOptMat = []; SmoothODevbMat = [];
SmoothiMat = []; VarOchk = 10^9; VarIchk = 10^9; cBMin = 0; BMin =
[]; OOptMin = 0; IOptMin = 0; VarOI = []; cBi = 0;
for cBOpt
=0:.01:3
```

```
cBi = cBi+1; cBM(cBi) = cBOpt;
% Bounded MRP
bopt = []; BOpt = []; SmoothUB = [];
SmoothLB = []; SmoothB = []; % Smoothing MRP Bounds
MRPd = []; Smoothr = []; Smoothi = []; Smootho = [];
% Generate Smoothing Bounded MRP
Smoothr = []; Smoothi = []; Smootho = [];
% Generate Smoothing Bounded MRP
[bopt,BOpt] = setabsFbnd(S,L,F,theta,Zbar,a,sigmaa,Zhat,'S');
% Set the bound for the Smoothing
BOpt = cBOpt*BOpt;
[MRPd,Smootho,Smoothr,Smoothi,SmoothB,SmoothUB,SmoothLB,
    SmoothODevb,TSmoothODevb] = findinfbndORI(S,L,F,theta,
    Zbar,sigmaa,a,Z,Zhat,SmoothT,bopt,BOpt);
VarOsim = abs((std(SmoothODevb))^2-(std(OOptDev))^2)
    /(std(OOptDev))^2;
VarIsim = abs((std(Smoothi(1:length(a),1)))^2-(std(IOpt))^2)
    /(std(IOpt))^2;
if RatioVarOI*VarOsim+(1-RatioVarOI)*VarIsim<RatioVarOI*VarOchk
    +(1-RatioVarOI)*VarIchk ...
    VarOchk = abs((std(SmoothODevb))^2-(std(OOptDev))^2)
        /(std(OOptDev))^2;
    VarIchk = abs((std(Smoothi(1:length(a),1)))^2-
```

```
            (std(IOpt))^2)/(std(IOpt))^2;
        cBMin = cBOpt;
        BMin = BOpt;
        OOptMin = (std(SmoothODevb))^2;
        IOptMin = (std(Smoothi(1:length(a),1)))^2;
    end
    OOptDevMat(cBi) = (std(OOptDev))^2;
    IOptMat(cBi) = (std(IOpt))^2;
    SmoothODevbMat(cBi) = (std(SmoothODevb))^2;
    SmoothiMat(cBi) = (std(Smoothi(1:length(a),1)))^2;
    VarOI(cBi) = RatioVarOI*VarOchk+(1-RatioVarOI)*VarIchk;
end
% Screen Output Variance Summary
ZDev = []; % Z(t)-Z(t-1)
ZDev(1) = Z(1)-Zbar; for i = 2:length(a)
    ZDev(i) = Z(i)-Z(i-1);
end
disp(sprintf('\n%s%6.2f%s%6.2f%s%6.2f','S = ',S,', L = ',L,',
    theta = ',theta));
disp(sprintf('%s%f','Inventory Target: ',SmoothT));
disp(sprintf('%s%f','Simulated Var Z(t)-Z(t-1): ',(std(ZDev))^2));
    % Demand
disp(sprintf('\n%s%f','Optimal Bound: Constant number = ',cBMin));
```

```
    % Optimal Bound
disp(sprintf('\t%s%s','Bound: ',num2str(BMin)));
disp(sprintf('\n%s','Simulated Var O(t)-O(t-1), Bounded MRP,
Unbounded Smoothing MRP, Unbounded Standard MRP')); % Var O(t)-O(t-1)
disp(sprintf('%s%f%s%f%s%f','Smoothing Bounded MRP: ',OOptMin,',
',(std(OOptDev))^2,', ',(std(OInfDev))^2));
disp(sprintf('\n%s','Simulated Var I(t), Bounded MRP, Unbounded
Smoothing MRP, Unbounded Standard MRP')); % Var I(t)
disp(sprintf('%s%f%s%f%s%f','Smoothing Bounded MRP: ',IOptMin,',
',(std(IOpt))^2,', ',(std(IInf))^2));

% Plot
figure % Var O(t)-O(t-1)
plot(cBM,SmoothODevbMat,'r--');
title(sprintf('%s%d%s%d%s%s','Simulated Values of Smoothing
    Var O(t)-O(t-1), S = ',S,', L = ',L,', Theta = ',num2str(theta)));
xlabel('c'); grid on; gtext('----- Bounded MRP'); hold on;
plot(cBM,OOptDevMat,'b-.'); gtext('-.-.- Unbounded MRP'); hold off
figure % Var I(t)
plot(cBM,SmoothiMat,'r--');
title(sprintf('%s%d%s%d%s%s','Simulated Values of Smoothing
    Var I(t), S = ',S,', L = ',L,', Theta = ',num2str(theta)));
xlabel('c'); grid on; gtext('----- Bounded MRP'); hold on;
```

```
plot(cBM,IOptMat,'b-.'); gtext('-.-.- Unbounded MRP'); hold off
```

2. Find the Smoothing Weights beta

```
function beta = findbeta(S,L,theta,method)
%
% Find the smoothing beta weights
%
if method == 'I'
   betainf = []; % Infinite Loading
   betainf(1) = 1+L*(1-theta);
   betainf(2:S) = 1-theta;
   beta = betainf;
else
    betaopt = []; % Optimal beta
    for j = 0:S-1
       betaopt(j+1) = (j+1)*(4*S^2+(10-3*j)*S+6-3*j+6*(S-j+1)*L)
          /(S+1)/(S+2)/(S+3)-(j+1)*(4*S^2+(4-3*j)*S+3*j+6*
          (S-j+1)*L)/(S+1)/(S+2)/(S+3)*theta;
    end
    beta = betaopt;
end
```

3. Find the Inventory Target

```
function [VarInvI,VarInvS,InfLoadT0,SmoothT0] =
    findinvtarget(S,L,theta,sigmaa,cInv,betaopt,method)
%
% Find the Inventory target
%
VarInvI = 0; VarInvS = 0; % Inventory Target
for i = 1:L
    VarInvI = VarInvI+(1+(i-1)*(1-theta))^2;
end
sum1 = 0; for i = 0:S-1
    sum1 = sum1+betaopt(i+1);
    VarInvS = VarInvS+(sum1-1-(L+i)*(1-theta))^2;
end
InfLoadT0 = cInv*sqrt(VarInvI)*sigmaa; VarInvS =
VarInvS+VarInvI;
SmoothT0 = cInv*sqrt(VarInvS)*sigmaa;
```

4. **Find the Unbounded MRP**

```
function [O,R,I,ODev,TODev] = findunbORI(S,L,theta,Zbar,sigmaa,
    a,Z,Zhat,beta,InvTarget)
%
% Find the unbounded smoothing MRP
%
```

```
O = []; % Order
for i = 0:S-1
    sum1 = 0;
    for j = 0:i
        sum1 = sum1+beta(j+1)*a(i-j+1);
    end
    O(i+1) = sum1+Zbar;
end
   sum1 = 0;
   for j = 0:S-1
   sum1 = sum1+beta(j+1)*a(S+1-j);
end
O(S+1) = sum1+(1+(S+L)*(1-theta)-sum(beta))*a(1)+Zbar;
for i =
S+2:length(a)
    sum1 = 0;
    for j = 0:S-1
        sum1 = sum1+beta(j+1)*a(i-j);
    end
    O(i) = sum1+(1+(S+L)*(1-theta)-sum(beta))*a(i-S)+Zhat(i-S-1);
end
R = []; % Receipts
for i = 1:L
```

```
   R(i) = Zbar;
end for i = L+1:length(a);
   R(i) = O(i-L);
end
I = []; % Inventory
I(1) = InvTarget+R(1)-Z(1); for i = 2:length(a)
    I(i) = I(i-1)+R(i)-Z(i);
end
ODev = []; % O(t)-O(t-1)
ODev(1) = O(1)-Zbar; for i = 2:length(a)
    ODev(i) = O(i)-O(i-1);
end
TODev = beta(1)^2; % Theoretical value of Var O(t)-O(t-1)
for i = 2:S
    TODev = TODev+(beta(i)-beta(i-1))^2;
end
TODev =
TODev+(1+(S+L)*(1-theta)-sum(beta)-beta(S))^2+(1+(S+L-1)
    *(1-theta)-sum(beta))^2;
TODev = TODev*sigmaa^2;
```

5. Set the Bounds from the Var[O(t)-O(t-i)(i)]

```
function [bnd,BND] =
```

```
    setabsFbnd(S,L,F,theta,Zbar,a,sigmaa,Zhat,method)
%
% Set the bounds by using the variance of week-to-week order
%    Var[O(t)-O(t-1)]
%
bnd = []; % Infinite Loading Bound
for i = 1:F-1
    bnd = [bnd findbeta(S,L,theta,method)'];
end
BND = []; sum1 = 0; for i = 0:S-1
    if i<F-1
        sum1 = sum1+bnd(i+1,1)^2*sigmaa^2;
        BND(i+1) = sqrt(sum1);
    end
end
if S<F-1
    sum1 = sum1+(1+(S+L)*(1-theta)-sum(bnd(:,1)))^2*sigmaa^2;
    BND(S+1) = sqrt(sum1);
end
for i = S+2:F-1
    sum1 = sum1+(1-theta)^2*sigmaa^2;
    BND(i) = sqrt(sum1);
end
```

6. **Find the Bounded MRP**

```
function [MRPd,bndo,bndr,bndi,bndB,bndUB,bndLB,ODevb,TODevb]
    = findinfbndORI(S,L,F,theta,Zbar,sigmaa,a,Z,Zhat,bndT,bnd,BND)
%
% Find the bounded MRP
%
for i = 1:length(a)
    MRPd(i,1) = Z(i); MRPd(i,2:F) = Zhat(i);
    for j = 1:L
        if i==1
            bndr(i,j) = Zbar;
            if j==1
                bndi(i,j) = bndT+bndr(i,j)-MRPd(i,j);
            else
                bndi(i,j) = bndi(i,j-1)+bndr(i,j)-MRPd(i,j);
            end
        else
            bndr(i,j) = bndr(i-1,j+1);
            if j==1
                bndi(i,j) = bndi(i-1,j)+bndr(i,j)-MRPd(i,j);
            else
                bndi(i,j) = bndi(i,j-1)+bndr(i,j)-MRPd(i,j);
            end
```

```
    end
end
for j = L+1:F
    if L==0
        if j==1
            if i==1
                bndo(i,j) = MRPd(i,j);
            else
                bndo(i,j) = bndT-bndi(i-1,j)+MRPd(i,j);
                if bndo(i,j)>bndUB(i-1,j)
                    bndo(i,j) = bndUB(i-1,j);
                elseif bndo(i,j)<bndLB(i-1,j)
                    bndo(i,j) = bndLB(i-1,j);
                end
            end
        elseif j-L<length(bnd(1,:))+2
            bndo(i,j-L) = bndT-bndi(i,j-1)+MRPd(i,j);
            bndB(i,j-L-1) = BND(j-L-1);
            bndUB(i,j-L-1) = bndo(i,j-L)+bndB(i,j-L-1);
            bndLB(i,j-L-1) = bndo(i,j-L)-bndB(i,j-L-1);
            if (i>1)&(j-L<length(bnd(1,:))+1)
                if bndo(i,j-L)>bndUB(i-1,j-L)
                    bndo(i,j-L) = bndUB(i-1,j-L);
```

```
            elseif bndo(i,j-L)<bndLB(i-1,j-L)
                bndo(i,j-L) = bndLB(i-1,j-L);
            end
            bndUB(i,j-L-1) = bndo(i,j-L)+bndB(i,j-L-1);
            bndLB(i,j-L-1) = bndo(i,j-L)-bndB(i,j-L-1);
            bndUB(i,j-L-1) =
                min(bndUB(i,j-L-1),bndUB(i-1,j-L));
            bndLB(i,j-L-1) =
                max(bndLB(i,j-L-1),bndLB(i-1,j-L));
        end
    else
        bndo(i,j-L) = bndT-bndi(i,j-1)+MRPd(i,j);
    end
else
    bndo(i,j-L) = bndT-bndi(i,j-1)+MRPd(i,j);
    if j-L==1
        if i>1
            if bndo(i,j-L)>bndUB(i-1,j-L)
                bndo(i,j-L) = bndUB(i-1,j-L);
            elseif bndo(i,j-L)<bndLB(i-1,j-L)
                bndo(i,j-L) = bndLB(i-1,j-L);
            end
        end
```

```
elseif j-L<length(bnd(1,:))+2
    bndB(i,j-L-1) = BND(j-L-1);
    bndUB(i,j-L-1) = bndo(i,j-L)+bndB(i,j-L-1);
    bndLB(i,j-L-1) = bndo(i,j-L)-bndB(i,j-L-1);
    if (i>1)&(j-L<length(bnd(1,:))+1)
        if bndo(i,j-L)>bndUB(i-1,j-L)
            bndo(i,j-L) = bndUB(i-1,j-L);
        elseif bndo(i,j-L)<bndLB(i-1,j-L)
            bndo(i,j-L) = bndLB(i-1,j-L);
        end
        bndUB(i,j-L-1) = bndo(i,j-L)+bndB(i,j-L-1);
        bndLB(i,j-L-1) = bndo(i,j-L)-bndB(i,j-L-1);
        bndUB(i,j-L-1) =
            min(bndUB(i,j-L-1),bndUB(i-1,j-L));
        bndLB(i,j-L-1) =
            max(bndLB(i,j-L-1),bndLB(i-1,j-L));
    end
    if bndo(i,j-L)>bndUB(i,j-L-1)
        bndo(i,j-L) = bndUB(i,j-L-1);
    elseif bndo(i,j-L)<bndLB(i,j-L-1)
        bndo(i,j-L) = bndLB(i,j-L-1);
    end
else
```

```
            bndo(i,j-L) = bndT-bndi(i,j-1)+MRPd(i,j);
        end
    end
    bndr(i,j) = bndo(i,j-L);
    if i==1
        if j==1
            bndi(i,j) = bndT+bndr(i,j)-MRPd(i,j);
        else
            bndi(i,j) = bndi(i,j-1)+bndr(i,j)-MRPd(i,j);
        end
    else
        if j==1
            bndi(i,j) = bndi(i-1,j)+bndr(i,j)-MRPd(i,j);
        else
            bndi(i,j) = bndi(i,j-1)+bndr(i,j)-MRPd(i,j);
        end
    end
end
bndo(i,F-L+1:F) = bndo(i,F-L);
for j = F+1:F+L
    if j-L<length(bnd(1,:))+2
        bndB(i,j-L-1) = BND(j-L-1);
        bndUB(i,j-L-1) = bndo(i,j-L)+bndB(i,j-L-1);
```

```
            bndLB(i,j-L-1) = bndo(i,j-L)-bndB(i,j-L-1);
            if (i>1)&(j<F+L)
                if bndo(i,j-L)>bndUB(i-1,j-L)
                    bndo(i,j-L) = bndUB(i-1,j-L);
                elseif bndo(i,j-L)<bndLB(i-1,j-L)
                    bndo(i,j-L) = bndLB(i-1,j-L);
                end
                bndUB(i,j-L-1) = bndo(i,j-L)+bndB(i,j-L-1);
                bndLB(i,j-L-1) = bndo(i,j-L)-bndB(i,j-L-1);
                bndUB(i,j-L-1) = min(bndUB(i,j-L-1),bndUB(i-1,j-L));
                bndLB(i,j-L-1) = max(bndLB(i,j-L-1),bndLB(i-1,j-L));
            end
        end
     end
end
ODevb = []; TODevb = 0; % O(t)-O(t-1)
ODevb(1) = bndo(1,1)-Zbar;
for i = 2:length(a)
    ODevb(i) = bndo(i,1)-bndo(i-1,1);
end
TODevb = bnd(1,1)^2; % Theoretical value of Var O(t)-O(t-1)
for i = 2:S
    TODevb = TODevb+(bnd(i,1)-bnd(i-1,1))^2;
```

```
end
TODevb = TODevb+(1+(S+L+1)*(1-theta)-sum(bnd(:,1))-bnd(S,1))^2
    +(1+(S+L)*(1-theta)-sum(bnd(:,1)))^2;
TODevb = TODevb*sigmaa^2;
```

Vita

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