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New approaches to Risk Management and Scenario Approximation in Financial Optimization

Maksym Bychkov
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Maksym Bychkov entitled "New approaches to Risk Management and Scenario Approximation in Financial Optimization." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Management Science.

Chanaka Edirisinghe, Major Professor

We have read this dissertation and recommend its acceptance:

Charles Noon, Phillip Daves, Halima Bensmail

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

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NEW APPROACHES TO RISK MANAGEMENT AND SCENARIO APPROXIMATION IN FINANCIAL OPTIMIZATION

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Maksym Bychkov

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Dedication

This dissertation is dedicated to my parents, Larisa Bychkova and Victor Bychkov, and my sister Valeriya Bychkova, for their great help, support, and encouragement to reach my goal and to find my way in science.

Acknowledgments

I wish to thank all those who helped me to complete my Philosophy Doctor degree in Management Science. I would like to thank Dr. Edirisinghe for his guidance and his advise while directing my research. I would like to thank Dr. Patterson for his careful reading of the thesis that significantly improved the quality. I would like to thank Dr. Noon, Dr. Daves, and Dr. Bensmail for serving in my committee. Their advise and guidance were very valuable. I would also like to thank Dr. Rockafellar and Dr. Uryasev from the University of Florida, Gainesville for making me familiar with VaR/CVaR concept and discussions during research.

Last but not least, I would like to thank my family and friends, whose encouragement made this work possible.

Abstract

The first part of the thesis addresses the problem of risk management in financial optimization modeling. Motivation for constructing a new concept of risk measurement is given through the history of development: utility theory, risk/return trade-off, and coherent risk measures. The process of describing investor's preferences is presented through the proposed collection of Rational Level Sets (RLS). Based on RLS, a new concept termed Rational Risk Measures (RRM) for financial optimization models is defined. The advantages of RRM over coherent risk measures are discussed.

Approximation of a given set of scenarios using tail information is addressed in the second part of the thesis. Motivation for the scenario approximation problem, as a way of reducing computation time and preserving solution accuracy, is given through examples of financial optimization and asset allocation models. Using the basic ideas of Conditional Value at Risk (CVaR), this thesis develops a new methodology for scenario approximation for stochastic portfolio optimization. First, the concepts termed Scenarios-at-Risk (SaR) and Scenarios-at-Gain (SaG) are proposed as for the purpose of partitioning the underlying multivariate domain for a fixed investment portfolio and a fixed probability level of CVaR. Then, under a given set of CVaR values, a two-stage method is developed for determining a smaller, and discrete, set of scenarios over which CVaR risk control is satisfied for all portfolios of interest. Convergence of the method is shown and numerical results are presented to validate the proposed technique.

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Chapter 1

Introduction

1.1 Financial investment problem

This thesis explores the class of problems called financial investment optimization. A decision maker or an investor wishes to allocate an initial budget (a fixed sum of money) in certain financial assets (to construct a portfolio). The future returns of the assets are unknown or uncertain at the time of allocation. Among other objectives, the investor seeks to maximize his/her expectations on the future portfolio profit. The investor may also have a future cash inflow to be invested over time, and may possibly face future liabilities as well. The return on investments for some of the assets may be stochastic (eg. stocks, options) or it can be known in advance (eg. bonds). In addition to the requirement that portfolio wealth must be as large as possible, the decision maker may also be subject to regulatory and policy restrictions that affect portfolio management. For instance, portfolios such as mutual funds or retirement funds must be formed by trading in assets (stocks or bonds) that the fund owns, i.e. via long positions. Creating short positions (sell stocks that the fund does not currently own) is prohibited. There are other restrictions imposed by either the investor himself or a brokerage firm. Broker's restrictions usually guard the investor from very risky investment policies, see SEC rule 15c-1a in [74] for restrictions on option portfolios. For an example of restrictions on margins for option portfolios, see [35]. Investor's own restrictions reflect his/her recourse limitations and risk attitude; for instance, limited budget, margin rules, and aversion to investments with large chances of bankruptcy in the future; also see [13] for an asset allocation model with such restrictions.

The second application area of financial investment optimization is pricing future contracts, such as options. The option seller hedges a cash outflow stream in the future that is contingent upon the prevailing market conditions. The problem of finding an appropriate initial payment for the future cash outflow is termed the *option pricing* problem. The seller's goal is to create an investment portfolio that will finance the future cash outflow completely or partially. The initial investments for such an investment portfolio is the option price (or a bound on it). This price guarantees the replication of the future uncertain payments for the appropriate premium paid to the seller now. This problem, in spirit, is similar to the portfolio optimization problem described earlier. Rather than maximizing the future wealth, the minimum current wealth that guarantees hedging the future uncertainty is sought. This problem belongs to the general class of models for financing contingent claims, see [11], [12], [24], [25], [27], [60], and [72].

The above problems can be formulated as deterministic mathematical programming models (linear or non-linear) only if complete information about future events is available. However, information about future is almost surely never available before decisions are made. Typically, future performance of financial assets is forecasted with some degree of uncertainty and random variables may be used to model this uncertainty. Therefore, mathematical programming models of investment decision problems must incorporate the latter uncertainty explicitly. How does one maximize the future profit if it is random and what is an appropriate objective function for such maximization? How does randomness affect the quality of investment decisions? These questions are still subjects of active research today. The randomness of the future performance of financial assets and the way it affects the financial investment decisions are the primary foci of the research in this thesis.

1.2 Decision tree as a basis for decision making

Before considering uncertainty, the structure of the investment mathematical model needs to be defined. As mentioned before, investment decisions are made without the knowledge about future events, hence, they are called "*here and now*" decisions.

However, if the investment decision is concerned with a certain future period of time, it is possible to adjust the "*here and now*" decision by making future adaptive (recourse) decisions (i.e., periodically change the asset allocation). The subsequent investment decisions will depend on the new information that will become available over time. This decision structure can be presented in a tree diagram where a decision is made at each node of the tree. This approach to decision making is described in the famous work of von Neumann and Morgenstern [98], see Chapter II, Section 9: "The set-theoretical description of a game".

The underlying premise of a decision tree is two-fold. First, the future is observed (and acted upon) at finite time intervals. The number of decisions to be made is finite. That allows constructing a finite and discrete time decision tree. Second, at a given time epoch, having followed a sequence of realizations (of observed random events), partial information about the conditional future is available. Consequently, an information filtration that corresponds to the specified time partition may be defined on a decision tree. A sequence of realizations (of random events) through the decision tree up to a given decision node is termed a *scenario* at the node. Each node in the tree reflects a particular situation investor will face if the appropriate *scenario* will be realized. von Neumann and Morgenstern argue that a decision tree as a part of game description can be constructed based on the game rules. Player needs to make a decision at each node of the tree. The difficulty in making such decisions is due to the absence of information about both the future events and decisions of the other players. However, the specific decision will affect decisions of the other players and possibly future events. This is the *cycle problem* according to von Neumann and Morgenstern [98].

The financial investment problem does not have the difficulty of multiple players. The investor makes decisions "here and now", see [89] for definition of decisions without knowledge about future (*nonanticipativity*) and discussion of the *nonanticipativity* role in the modeling process. The second simplification is that investor's decisions do not influence the future uncertainty. Thus, the financial investment models of the kind considered in this thesis do not encounter the *cycle problem*. By adapting a mathematical programming model to each node of the above decision tree, one obtains the so-called multistage stochastic programming models of decision making.

1.3 Stochastic programming decision modeling

Stochastic programming is a branch within optimization modeling, where random variables are used in the model formulation. The model becomes multistage if the decision tree has more than one decision node. A simple (one period) mathematical programming problem is embedded at each node of the decision tree. These mathematical programming problems are nested according to the information filtering depicted on the tree, see Figure 1.1. Stochastic programming models have been explored since the early 50's, see Beale [6] and Dantzig [17]. Various solution techniques have been developed for stochastic programming models, see for instance [89] and [99], using results of functional analysis, see [56], [61], and [87]. A significant progress has been made in the area of linear stochastic programming.

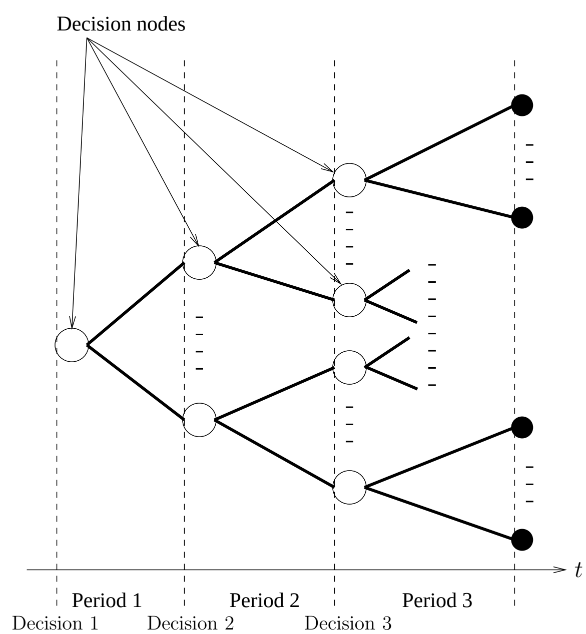

Figure 1.1. Decision tree structure for financial investment model

Linear stochastic programming is a modeling tool where models have linear formulation in each node of the decision tree, see [18], [21], and [55], for detailed descriptions. Solution techniques for linear stochastic programming are based on the theory of linear programming, see [4], [18], and [103], and convex analysis, see [37] and [84]. This area was originated by Dantzig [15], [16], and [18]. The major idea behind stochastic linear programming is to deal with random variables in the problem formulation. This area has been developed significantly in the past several decades. The most efficient technique for solving multistage stochastic linear programming is Bender's L-shaped decomposition or Dantzig-Wolf decomposition (the dual version of the L-shaped), see [7] and [94]. Both schemes require partitioning of the original problem into a framework of master and sub-problems. The decomposition technique allows solving large stochastic linear programming problems more efficiently than the pure simplex or barrier methods, [28] and [52]. Many other solution techniques are also proposed for multistage formulations, see for example [10], [28], [41], [53], and [99].

The non-linear stochastic programming (NSP) is a problem with non-linear objective and/or constraints at each node of the decision tree. In this case, algorithms such as Newton-Rapson, Hooke and Jeeves, and Lagrangian relaxation, must be applied, see [5] for details. Unfortunately, such algorithms are slow to converge and they can guarantee a global optimum only for special functions. Thus, there is no general-purpose efficient method that can be applied for solving NSP with an arbitrary objective function and constraints.

Specification of an appropriate objective function is an important aspect in stochastic programming models. This function has to link decisions at each node with the available information at the node. The objective function must also reflect portfolio future performance conditional upon the available information. Therefore, the objective function transforms all available information about the past as well as the future random events into a single (nodal) value for optimization. Such a transformation reflects investor's attitude towards risk and/or wealth preferences. In this context, attitude towards risk recognizes investor's concerns about future uncertainty, an area of study termed utility theory. Utility theory was developed in the mid 50's, however, the concept of risk preferences in decision making under uncertainty is still a subject of active research, see [38] and [83].

1.4 Utility functions and risk measures

The first formal axiomatic treatment of utility as a numerical function was given by von Neumann and Morgenstern in [98]. Then, Keeney and Raiffa in [58] develop an approach to measure utility. Their development is based on defining utility as a function of wealth and future uncertainty. Markowitz in [69] and [70] proposes an efficient set for expressing utility. Recently, Artzner et al. [2] have raised the question of separating the wealth valuation from risk attitude. This landmark paper of Artzner et al. [2] describes the principles of measuring risk attitude, where the authors propose the concept of coherent risk measures that allows numerical expression of risk attitude. According to this concept, coherent risk measures scale linearly if the underlying uncertainty is changed. Because of linearity, coherent risk measures do not cover all risk measures that are useful in applications, see [87] for examples of such risk measures. The question of risk measures that are non-linear with respect to the underlying uncertainty will be addressed in Chapter 3 of this thesis.

One interesting method of risk measurement is to use the conditional value-at-risk (CVaR), see for example Rockafellar and Uryasev [85] and Ogryczak and Ruszczynski [77]. The risk measures based on mean and CVaR are coherent, see [87]. However, relationship between the risk measures based on CVaR and utility functions has escaped attention. Extension of the existing CVaR concept and exploration of the relation between utility functions and risk measures based on CVaR are the subjects of research in Chapter 4 of this thesis.

The difficulty with CVaR and other risk measures that do not have closed-form expressions is that a discrete sample of outcomes of the random variable is required for its numerical evaluation when embedded within financial investment mathematical programming models. In stochastic programming, a sample of random variables is necessary at each decision node of the decision tree. This sample is used in the model formulation. Thus, determining an appropriate sample is a very important issue in stochastic programming modeling.

1.5 Scenario generation and approximation

A useful method of presenting a random variable in financial investment mathematical programming models is to create a finite sample of the underlying random variable. There are several restrictions on the applicability of this method. The first restriction is the requirement of full or partial knowledge about the true distribution. It may be possible to make an *a priori* assumption about the true distribution (normal, log-normal, etc.). Another possibility is to use the historical data for constructing a histogram, assuming that history represents the underlying random variable correctly. However, even with this knowledge, sample generation is difficult because a sample needs to preserve information about the underlying random variable. That means a sample is required to provide sufficient information about uncertainty for the mathematical programming model. Also, the distribution parameters need to be estimated. Sampling techniques are well developed for simple distributions: normal, log-normal, uniform, see Billingsley [9] for theoretical foundations, and *Numerical Recipes* [82] for efficient computer implementations.

While it may be impossible to determine the correct distribution of the future asset returns, the information about first and second moments of this distribution can often be estimated with some accuracy through historical data analysis. Then the questions of generating a sample based on the given moment information and incorporating the sample into financial investment decision making model are important issues. But the more important issue is how well a given sample approximates the original financial investment model. Quality of such approximations is frequently investigated via the so-called Generalized Moment Problem (GMP). The GMP was first addressed by Kemperman [59] in the context of developing bounds on expectation of a function. Such bounds are readily applicable within stochastic programming. This area gained much attention because constructing bounds on financial investment stochastic programming models is computationally more efficient than solving the original problem, see [10], [22], [29], [32], [33], and [44].

The next difficulty is determining an appropriate size for the sample. Law of large numbers [9] states that the mean of large samples is asymptotically close to the true mean of the underlying distribution. But, large samples lead to stochastic programming models that are extremely large in size. Thus, they are difficult to solve [29]. Then, the question of approximating the large sample with a smaller one is critically important. The problem of sample approximations is widely addressed in the literature, see [10], [22], and [30]. Usually, such schemes allow approximating a given sample based on the limited moment information and then constructing lower and upper bounds for the original problem using Kemperman [59] result or Madansky upper bound [65] and Jensen's lower bound [9]. Such methods can not be applied for approximating the sample tails because the proposed methods rely on the moment information rather than tail information. The proper approximation of the given sample while preserving tail information is also the subject for research in this thesis, as presented in Chapter 5.

1.6 Outline of research

The thesis is organized in six chapters. The topics covered in each chapter of this thesis proposal are outlined below:

- Chapter 2 provides the theoretical background in the form of literature review for stochastic programming, with particular emphasis to utility/risk measures.
- Chapter 3 discusses axioms and the definition for the proposed new concept termed rational risk measures, along with its properties. The question of decomposition of a risk measure is explored. Examples of rational risk measures are given.
- Chapter 4 provides the theoretical basis for CVaR and the proposed conditional value-at-gain (CVaG). It also explores the question of connection between risk measures based on CVaR/CVaG and risk averse utility functions.
- Chapter 5 develops the theory for approximation of a given large sample with a smaller one whilst preserving CVaR/CVaG information. An algorithm for such an approximation is developed.

- Chapter 6 presents a computational study of the proposed concepts in financial investments. The experiments for testing the effectiveness of CVaR/CVaG approximation algorithm will be presented. Quantitative performance analyses of different risk measures are also included.

The research in this thesis are anticipated to benefit both researchers and practitioners alike. For instance:

1. Managers of hedge, mutual, and pension funds will find the results regarding rational level sets and risk measures useful as an approach to formalize risk attitude and incorporate them into financial investment decision making models. Moreover, as a new approach for constructing risk measures, rational risk measures are expected to gain attention for further research.

2. The extended treatment of CVaR/CVaG provides better insights for understanding the tail risk measures and their connection to utility theory. The extended treatment of CVaR/CVaG is also a basis for constructing tail approximation schemes. This would be particularly appealing for the insurance industry where risks due to extremal (tail) events are important in risk control.

3. CVaR approximation algorithm is valuable for investors who use multiperiod stochastic programming models specified with large number of financial assets. The algorithm is computationally efficient and it allows constructing bounds on the original stochastic programming model. Such an efficient algorithm also allows sensitivity analyses with respect to distributional assumptions.

Chapter 2

Mathematical Models of Financial Investments

2.1 Preliminaries

Consider a single-period static financial investment problem at a particular decision node of the underlying multiperiod decision tree. Such problems at decision nodes are then nested to form the full multiperiod problem. Complete formulation of the problem will be discussed later. In this chapter, the nodal problem is discussed in isolation to set the notation and discuss the modeling of risk-return aspects of the problem.

Let $\mathcal{X}$ denote the random N-vector (i.e. a vector of random variables - *r.v.*) representing future returns of a given set of N financial instruments at a particular decision node. The random vector $\mathcal{X}$ is characterized by three parameters $\{\Omega, \mathcal{F}(\Omega), \mathcal{P}(\mathcal{F}(\Omega))\}$, see Billingsley [9], where:

- $\Omega \subseteq \mathbb{R}^N$ is the domain of random variable values (space on which random vector takes on values).
- $\mathcal{F}$ is the σ-field on Ω.
- $\mathcal{P}$ is a probability measure defined on $\mathcal{F}$.

A set $\mathcal{F}$ of Ω subsets is called a σ-field if it contains Ω itself and is closed under the formation of complements, finite, and infinite unions, see Kolmogorov [61] for a detailed description of σ-fields. $\mathbb{R}^N$ is the N-dimensional real space.

Only one element $x \in \Omega$ will be realized in the future. The investor needs to make a decision about how best to allocate a given budget among the N securities. The set of all possible investment decisions is denoted by $\Omega^* \subseteq \mathbb{R}^N$. The investor will choose a decision vector $y \in \Omega^*$ for implementation. The combination of the realized return

x and the decision y produces the portfolio return (i.e. profit) R that is a mapping:

$$R: \Omega \times \Omega^* \longrightarrow \mathbb{R}^1. \tag{2.1}$$

The investor cannot determine *a priori* the specific vector x of future returns. Instead, the investor has full or partial information about random vector $\mathcal{X}$. Therefore, the portfolio profit is also a *r.v.* from the investor's perspective. However, it is impossible to use a random function as an objective for optimization. Thus, the investor must construct a performance measure. Such a measure is supposed to include the information about future portfolio profit as well as investor's attitude to risk. The function that combines the random vector $\mathcal{X}$, decision y, and investors preferences into a performance measure is denoted by $f(\mathcal{X}, y)$, where:

$$f: \{\Omega, \mathcal{F}(\Omega), \mathcal{P}(\mathcal{F}(\Omega))\} \times \Omega^* \longrightarrow \mathbb{R}^1. \tag{2.2}$$

Several performance measures have been proposed in the literature - expectation of utility function, see [58] and [98], combination of first and second moments of the portfolio profit, see [69], [70], and [71], combination of the expected portfolio profit with an appropriate deviation function, see [39] and [70], for instance.

Decision maker usually has restrictions on the investment decision y. The restrictions are imposed due to regulations or specified by the investor. Such restrictions are generally expressed in terms of the underlying random vector $\mathcal{X}$ and the decision y. For example, the budget restriction involves only the decision y. However, broker's restrictions on option portfolios, see SEC rule 15c-1a in [74], depend on both the underlying random vector $\mathcal{X}$ and the decision y. A second example of a restriction that depends on both the random vector $\mathcal{X}$ and decision y is the CVaR restriction, see [85]. Such restrictions are denoted by functions $g_i(\mathcal{X}, y)$, which is greater or equal to zero, where i is the index for the restriction, $i = 1, ..., I$.

The mathematical model for financial investment problem at a particular node of the decision tree can then be formulated as follows:

$$\max_{y} \quad f(\mathcal{X}, y) \tag{2.3}$$

$$s.t. \quad g_i(\mathcal{X}, y) \geqslant 0; i = 1, ..., I \tag{2.4}$$

$$y \in \Omega^* \tag{2.5}$$

where $\mathcal{X}$ is a given and fixed random vector specified prior to model construction. This is a static stochastic programming model because it has only the decision vector of the current time period. This model is used for one period decision making. It is easy to extend the model to multi-period investments because for each node of the decision tree, a value function representing future decision nodes (descendant from the current node) can be embedded in the model above.

The objective function $f(\mathcal{X}, \cdot)$ of the single-stage financial investment model (2.3)-(2.5) drives the optimization process. This function is a performance measure that incorporates investor's attitude to risk.

2.2 Utility functions and risk measures

According to Keeney and Raiffa [58], it is difficult to specify the function $f(\mathcal{X}, \cdot)$ because this function incorporates investor's attitude towards both uncertainty and value of future return. Even an attempt to clarifying investor's attitude towards the value of future return is a complex procedure. Utility theory provides a theoretical description of investor's preferences towards future outcomes.

2.2.1 Utility functions

von Neumann and Morgenstern [98] founded the axiomatic treatment of utility in their landmark work in 1943. According to their construction, if preference structure satisfies the set of axioms, then utility can be expressed in a certain numerical form. A short verbal description of these axioms is the following:

- Investor can always compare any two deterministic alternatives $x_1, x_2 \in \Omega$ for a fixed decision $y \in \Omega^*$.
- Investor can always compare the outcomes of stochastic and deterministic alternatives for a fixed decision $y \in \Omega^*$. The first alternative is a lottery of outcomes $x_1, x_2 \in \Omega$ with corresponding probabilities $p_1, p_2 \geqslant 0;\ p_1 + p_2 = 1$. The second alternative is a deterministic outcome $x \in \Omega$.

Then utility function $u(R(x, y))$ for a fixed decision y is defined as a mapping:

$$u \colon \Omega \longrightarrow \mathbb{R}^1. \tag{2.6}$$

The requirements for utility function $u(x)$ are formulated as follows:

- Consistency in ordering:

$$x_1 \succ x_2 \Longrightarrow u(R(x_1, y)) > u(R(x_2, y)) \tag{2.7}$$

- Consistency in risk attitude:

$$u(R(\alpha x_1 + (1-\alpha)x_2, y)) \geqslant \alpha u(R(x_1, y)) + (1-\alpha)u(R(x_2, y)) \tag{2.8}$$

$$0 \leqslant \alpha \leqslant 1.$$

The above axioms constitute the basis for constructing numerical utility functions in financial investment problems. However, utility functions do not incorporate information about investor's attitude toward the future uncertainty. For the financial investment model in (2.3)-(2.5), the function $f(\mathcal{X},\ y)$ for some fixed $y \in \Omega^*$ and deterministic $\mathcal{X}$ is a utility function. Keeney and Raiffa [58] explore the question of defining utility in applications. They define a procedure to describe investor's preferences. The major difficulty that Keeney and Raiffa were faced with during experimentations is inconsistent answers given by decision makers. However, Keeney and Raiffa clarify the concept of rational behavior and provide relationships between utility theory and rational behavior. According to Keeney and Raiffa [58], rational investors always prefer a deterministic profit to a random profit with mean equal to the fixed profit. Mathematically, it is expressed as follows:

$$E[f(\mathcal{X}, y)] \leqslant f(E[\mathcal{X}], y) \tag{2.9}$$

where $E[\mathcal{X}]$ is the first moment of the random variable $\mathcal{X}$. The *rational* investors are also called *risk averse* investors. The *concavity* of the function $f(\,\cdot\,, y)$ implies inequality (2.9). Convex and concave properties of a function are defined in [84] as follows:

Definition 2.1. *Function $g\colon \Omega \to \mathbb{R}^1$ is called **convex** if:*

$$g(\alpha\, x_1 + (1-\alpha)\, x_2) \leqslant \alpha\, g(x_1) + (1-\alpha)\, g(x_2)$$

$$x_1, x_2 \in \Omega \qquad\qquad 0 \leqslant \alpha \leqslant 1.$$

Definition 2.2. *Function $g\colon \Omega \to \mathbb{R}^1$ is called **concave** if $-g$ is **convex**.*

It follows directly from the above definitions that a *concave* utility function satisfies the inequality (2.9) and thus, *concavity* implies *risk aversion*.

Utility functions are applied in financial investment modeling, see [46], [63], [64], and [91]. However, financial investment models then become non-linear and difficult to solve. Also, utility functions indirectly incorporate information about investor's attitude toward the future uncertainty. The assumption is that the expectation of a utility function reflects investor's attitude towards risk and serves as a performance measure in financial investment model. An improvement is to incorporate the information about future uncertainty directly into a performance measure for the financial investment problem.

2.2.2 Risk/return trade-off

Markowitz in [69], [70], and [71] addresses the question of risk control based on rational behavior. The key principle is the balance between risk and return. This balance is different for different investors. Thus, Markowitz derived the set of efficient portfolios. He proved that a rational investor picks a portfolio from an efficient set. Markowitz proposes to present risk control mechanism in two parts. The first part is the expected profit or mean of the future profit (*return*). The second part is *risk* as measured by deviations from the mean. Different investors pick different levels of trade-off between *risk* and *return*. Markowitz considers several types of risk:

- Variance of final outcome.
- Semi-deviation of final outcome.
- Semi-variance of final outcome.

The investment problem can be formulated either as a quadratic programming or as a linear programming problem for the above risk measures.

Quadratic programming is appropriate if *risk* is expressed in the form of variance or semi-variance. In case of variance, we have:

$$\max_{y} \quad y' \cdot m - \lambda \cdot y' \, Q \, y \tag{2.10}$$

$$s.t. \qquad y \in \Omega^* \tag{2.11}$$

where $m = E[\mathcal{X}]$, $Q = \text{Var}[\mathcal{X}]$, and y' is the transposed decision vector.

Linear programming is appropriate if risk is expressed in the form of deviation or semi-deviation:

$$\max_{y} \quad y' \cdot m - \lambda \cdot E\Big[y' \cdot [m - \mathcal{X}]^{+}\Big] \tag{2.12}$$

$$s.t. \qquad y \in \Omega^{*}, \tag{2.13}$$

where

$$[x]^{+} = \left\{ \begin{array}{l} [x_1]^{+} \\ \vdots \\ [x_N]^{+} \end{array} \right\} \tag{2.14}$$

$$[x_i]^{+} = \begin{cases} x_i, & x_i > 0 \\ 0, & x_i \leqslant 0 \end{cases} \tag{2.15}$$

Then the efficient set can be constructed by solving the above problem for different λ values. The graph of efficient set is concave, see Figure 2.1.

In Markowitz's risk/return trade-off, the investor picks only one portfolio from the efficient set and the investor can not improve return without increasing risk or decrease risk without decreasing the return.

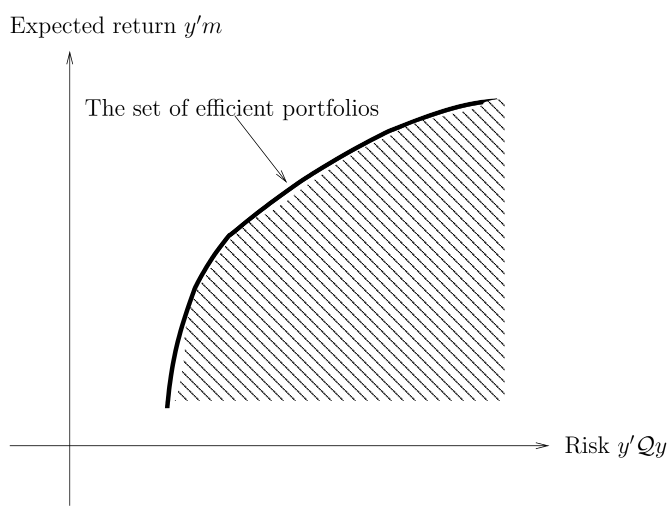

Figure 2.1. Concavity of Risk/Return Efficient frontier.

Unfortunately Markowitz's risk/return trade-off can generate inappropriate results. Consider, for example, two portfolios y_1 and y_2 that generate the outcomes given in Table 2.1 below. A rational investor always picks the portfolio y_1 because its profit is always greater than profit of portfolio y_2. However, Markowitz model with Mean/Variance trade-off and coefficient $\lambda > 0.00075$ picks the portfolio y_2 with deterministic profit because of the larger objective value, see Table 2.2 below. In general, such issues as consistency in choice can be addressed by the stochastic dominance approach.

2.2.3 Stochastic dominance

Stochastic dominance is a way of ordering random variables, see [39], [47], [48], [75], [76], [90], and [101]. The basic idea behind stochastic dominance is the following. For two random vectors $\mathcal{X}_1$ and $\mathcal{X}_2$, if $\mathcal{X}_1 \geqslant \mathcal{X}_2$ (element by element, almost surely i.e., with probability 1) then $\mathcal{X}_1$ is said to stochastically dominate $\mathcal{X}_2$, expressed as $\mathcal{X}_1 \succcurlyeq_{\text{FSD}} \mathcal{X}_2$. This is called the *First order Stochastic Dominance (FSD)* relation.

Table 2.1. Example of portfolio returns

Portfolio	Probability	Profit
y_1	0.5	\$6,000
	0.5	\$2,000
y_2	1.0	\$1,000

Table 2.2. Markowitz's Mean/Variance model performance with two example portfolios

Portfolio	Mean	Variance	Objective value		
			$\lambda = 0.0$	$\lambda = 0.001$	$\lambda = 0.002$
y_1	\$4,000	4,000,000	4,000	0	-4,000
y_2	\$1,000	0.0	1,000	1,000	1,000

Suppose that the random variable that presents the future portfolio profit of financial investment has the domain of variation as $\Psi \in \mathbb{R}^1$. Then the *cumulative distribution function* (c.d.f.) $F_1^{\mathcal{Z}}(\eta)$ of the future profit $\mathcal{Z}$ is defined as follows:

$$F_1^{\mathcal{Z}}(\eta) = \mathbb{P}\{\mathcal{Z} \leq \eta \,|\, \eta \in \Psi\} \tag{2.16}$$

Hadar and Russel [47] introduce First order Stochastic Dominance (FSD) based on *cumulative distribution functions (c.d.f.).*

Definition 2.3. *First order stochastic dominance (FSD). Random variable* $\mathcal{Z}_1$ *is FSD to random variable* $\mathcal{Z}_2$ *(*$\mathcal{Z}_2 \preccurlyeq_{\mathrm{FSD}} \mathcal{Z}_1$*) if and only if* $F_1^{\mathcal{Z}_1}(\eta) \leqslant F_1^{\mathcal{Z}_2}(\eta)$ *for* $\forall \eta \in \mathbb{R}^1$*.*

Inverse cumulative distribution functions are defined as follows:

Definition 2.4.

Left continuous inverse cumulative distribution function for r.v. $\mathcal{Z}$ *is defined as follows:*

$$F_1^{\mathcal{Z},-1}(p) = \inf_{\eta}\{\eta | F_1(\mathcal{Z}, \eta) \geqslant p\} \qquad \text{for } 0 < p \leqslant 1.$$

This function is also known as the *first quantile function*, see for instance Ogryczak et al. [75]. The *first quantile function* is useful for analyzing tail risk measures later in this chapter.

Rothschild and Stiglitz [90] introduce second order performance function in order to weaken the FSD condition. Second order performance function $F_2^{\mathcal{Z}}(\eta)$ of *r.v.* $\mathcal{Z}$ is defined as follows:

$$F_2^{\mathcal{Z}}(\eta) = \int_{-\infty}^{\eta} F_1^{\mathcal{Z}}(\xi) d\xi \tag{2.17}$$

which is the area under the *c.d.f.* curve. Then *Second order Stochastic Dominance (SSD)* can be defined as follows:

Definition 2.5. *(Rothschild and Stiglitz [90]) Second order stochastic dominance (SSD). Random variable* $\mathcal{Z}_1$ *is SSD to random variable* $\mathcal{Z}_2$ *(*$\mathcal{Z}_2 \preccurlyeq_{\mathrm{SSD}} \mathcal{Z}_1$*) if and only if*

$$F_2^{\mathcal{Z}_1}(\eta) \leqslant F_2^{\mathcal{Z}_2}(\eta), \forall x \in \mathbb{R}^1$$

SSD is weaker than FSD in that FSD implies SSD but not vise-versa. The reason is that for some $\eta \in \mathbb{R}^1$, FSD condition can be violated but SSD condition can be valid for $\forall \eta \in \mathbb{R}^1$, see Figure 2.2 for example.

Fishburn [39] derives a one way relationship between stochastic dominance and utility functions. Fishburn formulates his results in the form of the following Lemma:

Lemma 2.6. *(Fishburn [39]) If $\mathcal{X}_1$ FSD $\mathcal{X}_2$, then $E[\mathcal{X}_1] \geqslant E[\mathcal{X}_2]$ and $E[u(\mathcal{X}_1)] \geqslant E[u(\mathcal{X}_2)]$ for every nondecreasing real valued function u. Furthermore, if $\mathcal{X}_1 \succcurlyeq_{\text{SSD}} \mathcal{X}_2$ then $E[\mathcal{X}_1] \geqslant E[\mathcal{X}_2]$ and $E[u(\mathcal{X}_1)] \geqslant E[u(\mathcal{X}_2)]$ for every nondecreasing **concave** real valued function u.*

This lemma shows that *risk averse* behavior implies the ordering of decisions based on Second order Stochastic Dominance (SSD). Thus, decisions based on a *concave* utility functions are consistent with decisions based on SSD ordering.

The decision rule based on FSD or SSD is straightforward. All possible future profits are sorted in descending order (under FSD or SSD ordering) and the decision with the highest rank is picked for implementation. The application of such a decision rule in financial investments is possible, see [102], but it has some problems in implementation. Under FSD and SSD ordering, a large number of future portfolio profits (random variables) are left unordered because for FSD or SSD one random variable

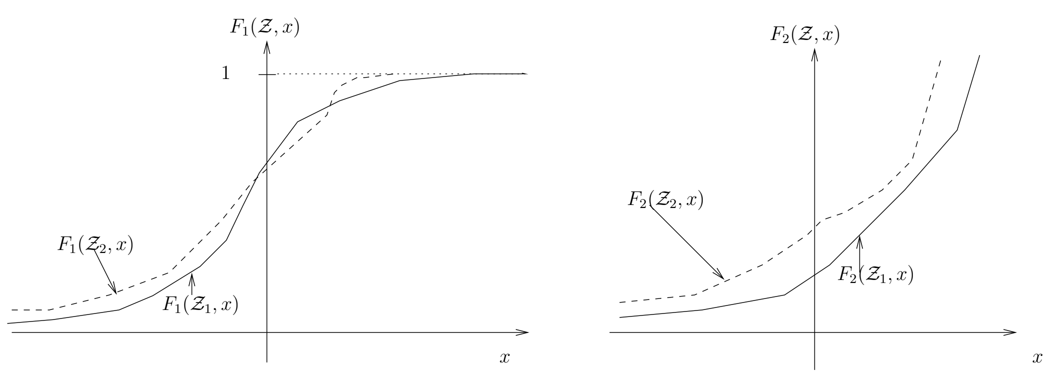

Figure 2.2. Example of $\mathcal{Z}_2 \preccurlyeq_{\text{SSD}} \mathcal{Z}_1$ and $\mathcal{Z}_2 \not\preccurlyeq_{\text{FSD}} \mathcal{Z}_1$

must dominate another almost surely (with probability 1). For example of unordered future portfolio profits see Table 2.3.

Thus, it is impossible to define the decision with the highest rank because of incomplete ordering of underlying future portfolio profits. However, FSD or SSD can be useful as a consistency check for objective functions in financial investment model.

2.2.4 Acceptance sets and coherent risk measures

Suppose that a combination of random returns (of financial instruments) $\mathcal{X}$ and a decision y results in the random future profit $\mathcal{Z}$:

$$f(\mathcal{X}, y) \rightarrow \mathcal{Z}: \{\mathbb{R}^1, \mathcal{F}(\mathbb{R}^1), \mathcal{P}(\mathcal{F}(\mathbb{R}^1))\}$$

Then the question of how to choose a decision y can be answered by ordering all possible future random variables of profit $\mathcal{Z}$ (for all possible decisions y) and then picking the best. Artzner, Delbaen, Eber, and Heath in [2] propose to use this approach in financial investment problems. They define the concept of an acceptance set through a set of axioms and propose to use the acceptance set as a basis for decision making. These axioms are given next with short comments about each:

Axiom 1. *The acceptance set* $\mathcal{A}$ *contains* $L^+ = \{\mathcal{Z} \,|\, \forall z \in \mathbb{R}^1, z > 0\}$.

If the future profit is always non-negative then this profit will be accepted. This is rational because everybody will accept a random profit with non-negative values, for instance a free lottery ticket.

Axiom 2. The acceptance set $\mathcal{A}$ does not intersect $L^- = \{\mathcal{Z} \,|\, \forall z \in \mathbb{R}^1, z < 0\}$.

Table 2.3. Example of unordered random portfolio profits.

Probability	Portfolio Profit $\mathcal{Z}_1$	Portfolio Profit $\mathcal{Z}_2$
0.5	\$600	\$750
0.5	-\$200	-\$250

If future profit under any circumstance is negative, then this random variable is definitely rejected. This is rational behavior because investors will never accept decisions where money is lost for sure.

Axiom 3. *The acceptance set $\mathcal{A}$ is a convex set.*

Convex combination of any number of accepted outcomes is also acceptable. If the investor decides to split his resource among decisions which are individually acceptable then a convex combination of the decisions must also be acceptable.

Axiom 4. *The acceptance set $\mathcal{A}$ is a positively homogeneous cone.*

No verbal explanation can be found for this axiom in the literature.

The rationality for the first three axioms is justified as indicated above from an investor's viewpoint. However, the last axiom implies an independence of the level of risk aversion from the invested amount. Suppose, for example, that there are two future portfolio profits (see Table 2.4) with $\mathcal{Z}_2 = 5 * \mathcal{Z}_1$. Then the following implications are correct if the investor uses the acceptance set approach:

$$\mathcal{Z}_1 \in \mathcal{A} \Rightarrow \mathcal{Z}_2 \in \mathcal{A} \tag{2.18}$$

$$\mathcal{Z}_2 \in \mathcal{A} \Rightarrow \mathcal{Z}_1 \in \mathcal{A}, \tag{2.19}$$

due to the positive homogeneity axiom. However, suppose the investor accepts portfolio profit only if investor's potential loss will not exceed some predefined level (say, \$100). Then, the implication (2.18) is invalid because under the specified decision rule the investor accepts $\mathcal{Z}_1$ but rejects $\mathcal{Z}_2$. Thus, the assumption about independence

Table 2.4. Two random profits for acceptance sets

Probability	Portfolio profit	
	$\mathcal{Z}_1$	$\mathcal{Z}_2$
0.5	\$100	\$500
0.5	-\$50	-\$250

of the level of risk aversion from the invested amount is not true for several efficient risk control tools, for instance Markowitz's risk/return trade-off. The properties of acceptance sets without the last axiom and a possible generalization of the first three axioms is a subject for research in this thesis. This is presented in Chapter 3 under the proposed term: *Rational Risk Measures*.

An acceptance set contains random outcomes that are acceptable to the investor. Thus, acceptance sets reflect investor's risk attitude. Artzner et al. [2] use acceptance sets to define "coherent risk measures". A risk measure ρ according to [2] is a function of a random variable $\mathcal{Z}$ (in our case, it is the future portfolio profit):

$$\rho\colon \left\{\mathbb{R}^1, \mathcal{F}(\mathbb{R}^1), \mathcal{P}(\mathcal{F}(\mathbb{R}^1))\right\} \longrightarrow \mathbb{R}^1 \tag{2.20}$$

The risk measure $\rho(\,\cdot\,)$ is coherent if it satisfies the following axioms:

Axiom T. *Translation invariance*. For all *r.v.* $\mathcal{Z}$ and $\alpha \in \mathbb{R}$

$$\rho(\mathcal{Z}+\alpha) = \rho(\mathcal{Z}) - \alpha$$

Axiom S. *Sub-additivity*:

$$\rho(\mathcal{Z}_1 + \mathcal{Z}_2) \leqslant \rho(\mathcal{Z}_1) + \rho(\mathcal{Z}_2)$$

Axiom PH. *Positive homogeneity*:

$$\rho(\lambda\mathcal{Z}) = \lambda\rho(\mathcal{Z}), \qquad \forall \lambda \geq 0$$

Axiom M. *Monotonicity*:

$$\mathcal{Z}^1 \leqslant \mathcal{Z}^2 \Longrightarrow \rho(\mathcal{Z}^1) \geqslant \rho(\mathcal{Z}^2), \qquad\qquad \forall\, \mathcal{Z}^1, \mathcal{Z}^2$$

Artzner et al. [2] prove a result regarding one-to-one correspondence between acceptance sets $\mathcal{A}$ and coherent risk measures $\rho(\,\cdot\,)$, see Proposition 2.3 in [2]. This proposition establishes the following relationships:

$$\rho(\mathcal{Z}) = \inf_{C \in \mathbb{R}} \{C \,|\, \mathcal{Z} + C \in \mathcal{A}\} \tag{2.21}$$

$$\mathcal{A} = \{\mathcal{Z} \,|\, \rho(\mathcal{Z}) \leq 0\} \tag{2.22}$$

These show the equivalence between the coherent risk measure $\rho(\,\cdot\,)$ and the acceptance set $\mathcal{A}$. The formulas (2.21) and (2.22) are the foundation for using coherent risk measures as a tool in financial investment mathematical modeling if the acceptance set concept of decision making is confirmed by the investor. The rationale for each axiom of the coherent risk measures follows from the corresponding axiom in the definition of acceptance set, see [2] for details.

Rockafellar et al. in [87] extend the coherency axioms with the expectation bounded condition:

Axiom EB. Expectation bounded:

$$\rho(\mathcal{Z}) > -E[\mathcal{Z}];\ \ \text{for stochastic}\,\mathcal{Z} \tag{2.23}$$

$$\rho(\mathcal{Z}) = -\mathcal{Z};\ \ \text{for constant}\,\mathcal{Z} \tag{2.24}$$

This axiom is used for extending the concept of coherent risk measures into deviation measures and risk envelopes, see section 2.2.5 of this chapter.

Ogryczak and Ruszczynski [76] explore the correspondence between expectation bounded coherent risk measures and SSD. They introduce the concept of SSD safety consistent risk measures. Risk measure ρ is SSD α-safety consistent if, for any two random variables $\mathcal{Z}_1$ and $\mathcal{Z}_2$, the following conditions are satisfied:

$$\mathcal{Z}_1 \succcurlyeq_{\text{SSD}} \mathcal{Z}_2 \Rightarrow E[\mathcal{Z}_1] \geq E[\mathcal{Z}_2] \tag{2.25}$$

$$\mathcal{Z}_1 \succcurlyeq_{\text{SSD}} \mathcal{Z}_2 \Rightarrow E[\mathcal{Z}_1] - \alpha\cdot\rho(\mathcal{Z}_1) \geq E[\mathcal{Z}_2] - \alpha\cdot\rho(\mathcal{Z}_2) \tag{2.26}$$

SSD 1-safety consistent measures are termed SSD safety consistent measures for brevity.

Ogryczak et al. [75] derive the correspondence between coherent risk measures and SSD safety consistent measures. Ogryczak starts with the intermediate result:

Theorem 2.7. *(Ogryczak et al. [75],Theorem 2)*

Let $\rho(\mathcal{Z}) \geq 0$ be a convex, positively homogeneous and translation invariant risk measure. If this measure is also SSD safety consistent then the corresponding performance function $C(\mathcal{Z}) = \rho(\mathcal{Z}) - E[\mathcal{Z}]$ fulfills the coherency axioms (T, S, PH, M).

This result shows that every SSD safety consistent risk measure is a coherent risk measure. Then Ogryczak et al. [75] establish one-to-one correspondence between coherent risk measures and SSD safety consistent risk measures. Authors state this correspondence as the following theorem:

Theorem 2.8. *(Ogryczak et al. [75], Theorem 5) Let* $\rho(\mathcal{Z}) \geq 0$ *be a convex, positively homogeneous and translation invariant risk measure in the form:*

$$\rho(\mathcal{Z}) = f(E[g(\mathcal{Z})]), \quad g - \text{convex}, f - \text{increasing}$$

Then measure ρ *is SSD safety consistent if and only if it is expectation bounded:*

$$\mathcal{Z} \geq 0 \Longrightarrow \rho(\mathcal{Z}) < E[\mathcal{Z}]$$

If a risk measure is coherent and expectation bounded then the risk measure is SSD safety consistent. Any SSD safety consistent and expectation bounded risk measure is coherent. This is a very strong result because it fully describes the relationship between SSD and coherent risk measures.

Further developments in the concept of coherent risk measures are decomposition of the coherent risk measures into independent parts and exploration of sensitivity of decisions with respect to the probability measure of the underlying random variable $\mathcal{Z}$.

2.2.5 Deviation measures and risk envelopes

Rockafellar et al. in [87] use the expectation bounded condition (**Axiom EB**, page 22) to develop the concept of deviation measures and risk envelopes. Deviation measures are an extended development of the concept of coherent risk measures. The idea is to measure the deviation of the random variable. Axioms for deviation measures are stated as follows:

Axiom 1. $D(\mathcal{Z}+C) = D(\mathcal{Z})$, $\forall \mathcal{Z}$ and constant C

Axiom 2. $D(0)=0, \text{and}\, D(\lambda\mathcal{Z})=\lambda D(\mathcal{Z}),\, \forall\mathcal{Z}\, \text{and}\, \lambda>0$

Axiom 3. $D(\mathcal{Z}_1+\mathcal{Z}_2)\leqslant D(\mathcal{Z}_1)+D(\mathcal{Z}_2), \forall\mathcal{Z}_1, \mathcal{Z}_2$

Axiom 4. $D(\mathcal{Z})>0$ for non-constant $\mathcal{Z}$

Axiom 5. $D(\mathcal{Z})\leq E[\mathcal{Z}]-\inf\mathcal{Z}$

Deviation measures are related to the coherent risk measures as follows:

$$\rho(\mathcal{Z})=D(\mathcal{Z})-E[\mathcal{Z}] \tag{2.27}$$

$$D(\mathcal{Z})=\rho(\mathcal{Z}-E[\mathcal{Z}]) \tag{2.28}$$

The formulae (2.27) and (2.28) give insight into the process of decision making using expectation bounded coherent risk measures. The risk term is constructed using two parts. The first part is an expected value of portfolio and the second part is the volatility of portfolio.

Risk envelopes are an extension of the risk measure concept. The idea is to define all possible variations of the probability measure $\mathcal{P}(\mathcal{F}(\mathbb{R}^1))$ under which the acceptance set remains the same. Risk envelope is the set of probability measures that satisfy the following axioms:

Q1. $\mathcal{Q}$ is a closed, convex set containing 1 (constant).

Q2. every $Q\in\mathcal{Q}$ has $E[Q]=1$.

Q3. There is no $\mathcal{Z}\in\mathcal{L}^2$ such that $E[\mathcal{Z}Q]\leq E[\mathcal{Z}]$ for all $Q\in\mathcal{Q}$.

Q4. $Q\geq 0$ for all $Q\in\mathcal{Q}$.

$$E[\mathcal{Z}Q]=\int_{\mathbb{R}^1}\mathcal{Z}(x)Q(x)d\,P(x)$$

Risk envelopes are proposed as a tool for sensitivity analysis of financial investment decisions made under coherent risk measures.

Rockafellar et al. [87] establish one-to-one correspondence among four concepts: expectation bounded coherent risk measures, deviation measures, acceptance sets, and risk envelopes:

$$D(\mathcal{Z}) = E[\mathcal{Z}] - \inf_{Q \in \mathcal{Q}} \{E[\mathcal{Z}Q]\} \tag{2.29}$$

$$\rho(\mathcal{Z}) = \sup_{Q \in \mathcal{Q}} \{-E[\mathcal{Z}Q]\} \tag{2.30}$$

$$\mathcal{A} = \{\mathcal{Z} \mid E[\mathcal{Z}Q] \geq 0; \forall Q \in \mathcal{Q}\} \tag{2.31}$$

The concepts of risk envelopes and deviation measures motivate further research in the risk control theory. The first direction is a generalization leading to a wider class of decomposable risk measures. What are the implications of such a generalization? The decomposition of risk measures and other related questions are researched in Chapter 3 of this thesis.

2.2.6 Exogenous and endogenous risk measures

Rockafellar et al. [87] propose to treat risk measures as a combination of two parts: deviation measure and target. The question of an appropriate target is important in optimization of financial investments because different investors adopt different investment strategies. For instance, certain investors prefer to get profits above a predefined threshold. Thus, they incorporate risk-less financial instruments. Aggressive traders want to get as much profit as they can and they often use risky financial instruments. Due to different preferences, target types for these two categories need to be different. However, the issue of target type is largely unexplored in research and no detailed basic theory has been developed. The common practice is to use portfolio expected value (mean) as a target, see for instance [39], [69], [70], [71], and [87].

Consider an investor who formulates a financial investment problem to minimize risk associated with a decision y. The risk measure ρ for the problem has the following expression:

$$\rho = \rho(\mathcal{X}, y) = t(\mathcal{X}, y) + d(\mathcal{X}, y) \tag{2.32}$$

where $t(\mathcal{X}, y)$ is the target and $d(\mathcal{X}, y)$ is the deviation part of investor's risk measure. A mathematical model for the financial investment problem is then formulated as follows:

$$\min_{y} \quad \{t(\mathcal{X}, y) + d(\mathcal{X}, y)\} \tag{2.33}$$

$$s.t. \qquad y \in \Omega^* \tag{2.34}$$

where Ω^* is the domain of all feasible investment decisions y. According to the standard viewpoint [69], [70], [71], and [87]:

$$t(\mathcal{X}, y) = -E[\langle \mathcal{X}, y \rangle]$$

The fact that the target $t(\mathcal{X}, y)$ depends on the random vector $\mathcal{X}$ of returns of individual financial instruments emphasizes the *endogenous* property of the risk measure ρ. That is *endogenous risk measure* is a risk measure where the target depends on the underlying random vector $\mathcal{X}$. Usually, the target is at the mean. All coherent expectation bounded risk measures are *endogenous* as follows from (2.29). The properties of such risk measures have been studied, see [39], [40], [75], [76], and [77].

An *exogenous risk measure* is a risk measure where target is independent on the random vector $\mathcal{X}$ of individual financial instrument returns. That is, if the target depends only on the decision vector y or some external stochastic parameters, then the risk measure is *exogenous*.

$$t(\mathcal{X}, y) = g(y) \tag{2.35}$$

Exogenous risk measures have not received much attention in research, only limited research has been done, see [13], [50], [81], and [100]. Thus, properties of such measures and possible ways of implementation have not been studied. This thesis explores the properties of *exogenous* and *endogenous* risk measures under specific conditions. The question of minimal conditions for *exogenous* and *endogenous* risk measures to be consistent with FSD is the subject for research in Chapter 3 of this thesis.

2.2.7 Tail risk measures

In the insurance industry, risk control is achieved via the concept of controlling either the probability or the value of maximum loss, see [3] and [8]. This approach motivates tail risk measures. There has been a significant development in this area in the last decade, see [93] and [96].

2.2.7.1 VaR definition, properties and applications

Historically, *value at risk* or *percentile* is an early approach to risk control. Value-at-Risk (VaR), according to [96], is defined as the maximum loss with a specified probability level α. Mathematically, VaR is defined as follows:

$$\mathrm{VaR}_\alpha(-\mathcal{Z}) = F_1^{-\mathcal{Z},-1}(1-\alpha). \tag{2.36}$$

Thus, $\mathrm{VaR}_\alpha(-\mathcal{Z})$ is the left end point of a nonempty interval consisting of the values for which:

$$F_1^{-\mathcal{Z},-1}(1-\alpha) = x. \tag{2.37}$$

From an applications stand-point, this is the minimum value of loss under the condition that α worst-case events will happen. Alternatively, this is the minimum profit if α worst-case events will not happen, see Figure 2.3 for an illustration.

Uryasev [96] assumes that $F_1^{-\mathcal{Z},-1}(\,\cdot\,)$ is a continuous function. Based on this assumption, he concludes that $\mathrm{VaR}_\alpha(-\mathcal{Z})$ is continuous and non-decreasing as a function of α. Uryasev [96] also develops a formula for computing the gradient of

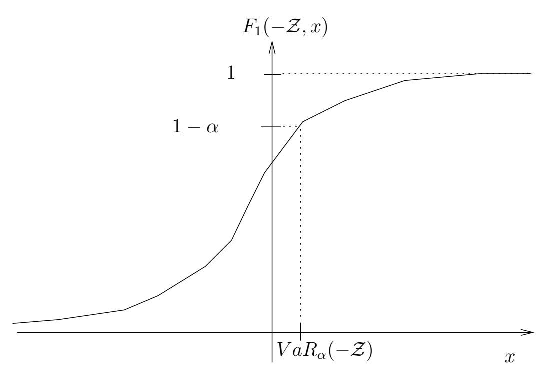

Figure 2.3. $\mathrm{VaR}_\alpha(-\mathcal{Z})$ in graphical representation

$F_1^{l(\mathcal{X},\cdot),-1}(\alpha)$ as a function of decision y and the gradient of $\mathrm{VaR}_\alpha(-\mathcal{Z})$ with respect to α. Where, $l(\mathcal{X},\cdot)$ is denoted as a loss function of random return $\mathcal{X}$ and portfolio y.

$$l(\mathcal{X}, y) = -\mathcal{Z}.$$

VaR has been used in financial investments, see [8], [54], and [97]. However, VaR as a concept of risk control has difficulties. For instance, in order to evaluate VaR, one needs to minimize the inverse cumulative distribution function that is an integral of the probability distribution function. The minimization of integration operation is computationally tedious. Even using decomposition techniques and under assumptions of independence, this approach does not yield significant computational gain because of the difficulty of optimization of an integral over a bounded set. However, VaR approach is widely used in financial applications where an optimization based approach can be avoided, for instance, evaluating a credit score. This is because for a particular decision y the problem of computing VaR is only a problem of evaluating the integral.

Pflug in [78] summarizes the properties of VaR as a function of quantile α and *r.v.* $-\mathcal{Z}$. He describes VaR properties as follows (see Proposition 3 in [78]):

i. $\mathrm{VaR}_\alpha(-\mathcal{Z})$ is translation invariant:

$$\mathrm{VaR}_\alpha(-\mathcal{Z}+c) = \mathrm{VaR}_\alpha(-\mathcal{Z}) + c; \qquad c \in \mathbb{R}^1 \tag{2.38}$$

ii. $\mathrm{VaR}_\alpha(-\mathcal{Z})$ is positively homogeneous:

$$\mathrm{VaR}_\alpha(c \cdot -\mathcal{Z}) = c \cdot \mathrm{VaR}_\alpha(-\mathcal{Z}); \qquad c \in \mathbb{R}^1 \text{ and } c \geqslant 0 \tag{2.39}$$

iii. $\mathrm{VaR}_\alpha(-\mathcal{Z}) = -\mathrm{VaR}_{1-\alpha}(\mathcal{Z})$.

iv. $\mathrm{VaR}_\alpha(-\mathcal{Z})$ is monotonic with respect to FSD:

$$-\mathcal{Z}_1 \preccurlyeq_{\mathrm{FSD}} -\mathcal{Z}_2 \Longrightarrow \mathrm{VaR}_\alpha(-\mathcal{Z}_1) \leq \mathrm{VaR}_\alpha(-\mathcal{Z}_2) \tag{2.40}$$

VaR is not a convex function of either the quantile α or the *r.v.* $-\mathcal{Z}$. Therefore, it is computationally difficult to incorporate VaR as part of a performance measure in a mathematical programming model of financial investment. VaR is also not a deviation measure, see [87]. An advanced extension of VaR is the concept of Conditional value-at-risk (CVaR).

2.2.7.2 CVaR definition, properties and applications

CVaR, which is also called the *Mean Excess Loss*, *Mean Shortfall*, or *Tail VaR,* is yet another approach for risk control. The definition of this term is based on the VaR approach. The purpose of CVaR is to control the expectation of the tail of random variable, see Figure 2.4 for an illustration. CVaR has received extensive attention in the past few years, see [1], [68], [78], [85], [86], and [95]. Mathematically, CVaR is formulated as follows:

$$\mathrm{CVaR}_\alpha(-\mathcal{Z}) = \frac{1}{\alpha} \int\limits_{x \geqslant \mathrm{VaR}_\alpha(-\mathcal{Z})} x \, \mathrm{d}F_1^{-\mathcal{Z}}(x). \tag{2.41}$$

Uryasev and Rockafellar [86] also present an alternative expression for CVaR:

$$T_\alpha(x^*, -\mathcal{Z}) = x^* + \frac{1}{\alpha} \int\limits_{x \geqslant x^*} (x - x^*) \mathrm{d}F_1^{-\mathcal{Z}}(x) \tag{2.42}$$

$$\mathrm{CVaR}_\alpha(-\mathcal{Z}) = \mathrm{VaR}_\alpha(-\mathcal{Z}) + \\ + \frac{1}{\alpha} \int\limits_{x \geqslant \mathrm{VaR}_\alpha(-\mathcal{Z})} (x - \mathrm{VaR}_\alpha(-\mathcal{Z})) \mathrm{d}F_1^{-\mathcal{Z}}(x) = T_\alpha(\mathrm{VaR}_\alpha(-\mathcal{Z}), -\mathcal{Z}) \tag{2.43}$$

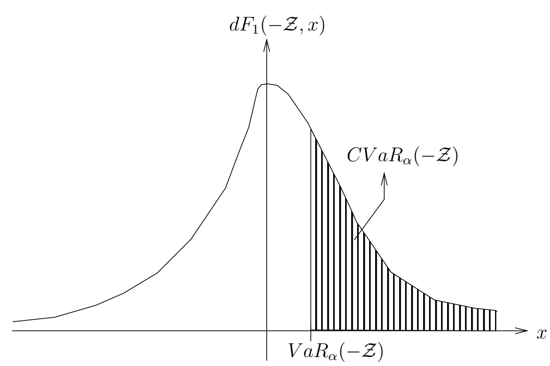

Figure 2.4. $\mathrm{CVaR}_\alpha(-\mathcal{Z})$ in graphical representation

The above expressions underscore the link between VaR and CVaR. CVaR is the expectation of a random variable conditioned on the random variable exceeding the corresponding VaR value.

Uryasev and Rockafellar [85] provide the properties of $T_\alpha(x^*, -\mathcal{Z})$ as a function of x^*. Theorem 1 in [85] establishes the convexity and differentiability of $T_\alpha(x^*, -\mathcal{Z})$. The same theorem also establishes the relationship between CVaR and $T_\alpha(x^*, -\mathcal{Z})$:

$$\mathrm{CVaR}_\alpha(-\mathcal{Z}) = \min_{x^*\in\mathbb{R}^1} \{T_\alpha(x^*, -\mathcal{Z})\} \tag{2.44}$$

$$\mathrm{VaR}_\alpha(-\mathcal{Z}) = \min_{x^*\in\mathbb{R}^1} \mathcal{A}_\alpha(-\mathcal{Z}) \tag{2.45}$$

$$\mathcal{A}_\alpha(-\mathcal{Z}) = \arg\min_{x^*\in\mathbb{R}^1}\{T_\alpha(x^*, -\mathcal{Z})\} \tag{2.46}$$

The proof of (2.44) - (2.46) consists of two major steps. The first one is the proof of convexity and differentiability of $T_\alpha(x^*, -\mathcal{Z})$. Uryasev and Rockafellar prove the first step by using the Shapiro and Wardi [92] results. The second step is to prove the formulas (2.44) – (2.46). Uryasev and Rockafellar use convexity and differentiability of $T_\alpha(x^*, -\mathcal{Z})$ and derive a precise formula for the derivative of $T_\alpha(x^*, -\mathcal{Z})$ with respect to x^*.

The formulas (2.44) - (2.46) state that the minimum value of $T_\alpha(x^*, -\mathcal{Z})$ over all possible $x^*\in\mathbb{R}^1$ is associated with the $\mathrm{CVaR}_\alpha(-\mathcal{Z})$ value, and $\mathrm{VaR}_\alpha(-\mathcal{Z})$ value is the argument that minimizes the functional $T_\alpha(x^*, -\mathcal{Z})$. The main advantage of (2.44) - (2.46) is the unique relationship between VaR and CVaR values. Both $\mathrm{CVaR}_\alpha(-\mathcal{Z})$ and $\mathrm{VaR}_\alpha(-\mathcal{Z})$ values can be found by minimizing the function $T_\alpha(x^*, -\mathcal{Z})$ over x^*. The function $T_\alpha(x^*, -\mathcal{Z})$ has two important mathematical properties, namely, convexity and continuity with respect to x^*, which simplify the minimization process.

CVaR is the expectation of the loss in α worst-case events. Thus, CVaR needs to be minimized in financial applications.Theorem 2 in [85] states that minimizing CVaR over all possible decisions y is equivalent to minimizing $T_\alpha(x^*, l(\mathcal{X}, y))$ over all possible pairs (x^*, y), that is,

$$\min_{y\in\Omega^*}\{\mathrm{CVaR}_\alpha(l(\mathcal{X}, y))\} = \min_{y\in\Omega^*;x^*\in\mathbb{R}^1} \{T_\alpha(x^*, l(\mathcal{X}, y))\} \tag{2.47}$$

The formula (2.47) establishes the equivalence between minimizing *CVaR* and minimizing the function $T_\alpha(x^*, l(\mathcal{X}, y))$ over the pair (x^*, y). Moreover, the pair (x^*, y), that minimizes $T_\alpha(x^*, l(\mathcal{X}, y))$, uniquely defines $\mathrm{VaR}_\alpha(-\mathcal{Z})$ and $\mathrm{CVaR}_\alpha(-\mathcal{Z})$ values.

Uryasev et al. [1] explore the question of incorporating CVaR into mathematical models for financial optimization. They prove that models with CVaR as a deviation measure can be formulated as linear programs if *r.v.* $\mathcal{X}$ has a finite σ-field. In this case, the loss function $l(x, y)$ is piecewise linear and convex with respect to y and convex with respect to x. Several important properties, such as convexity and differentiability of $T_\alpha(x^*, l(\mathcal{X}, y))$ are established in [85].

Pflug [78] summarizes properties of CVaR with respect to the underlying *r.v.* $-\mathcal{Z}$: $\{\mathbb{R}^1, \mathcal{F}(\mathbb{R}^1), \mathcal{P}(\mathcal{F}(\mathbb{R}^1))\}$ (see Proposition 2 in [78]):

i. CVaR is translation invariant:

$$\mathrm{CVaR}_\alpha(-\mathcal{Z} + c) = \mathrm{CVaR}_\alpha(-\mathcal{Z}) + c; \qquad c \in \mathbb{R}^1 \tag{2.48}$$

ii. CVaR is positively homogeneous:

$$\mathrm{CVaR}_\alpha(c \cdot (-\mathcal{Z})) = c \cdot \mathrm{CVaR}_\alpha(-\mathcal{Z}); \qquad c > 0, c \in \mathbb{R}^1 \tag{2.49}$$

iii. If r.v. $\mathcal{L}$ has a finite probability density function and finite first moment then:

$$E[-\mathcal{Z}] = (1 - \alpha)\, \mathrm{CVaR}_\alpha(-\mathcal{Z}) - \alpha\, \mathrm{CVaR}_{1-\alpha}(-\mathcal{Z}) \tag{2.50}$$

iv. CVaR is convex:

$$\mathrm{CVaR}_\alpha(\lambda\, (-\mathcal{Z}_1) + (1 - \lambda)\, (-\mathcal{Z}_2)) = \lambda\, \mathrm{CVaR}_\alpha(-\mathcal{Z}_1) + (1 - \lambda)\, \mathrm{CVaR}(-\mathcal{Z}_2); \quad 0 < \lambda < 1 \tag{2.51}$$

v. CVaR is monotonic with respect to SSD:

$$-\mathcal{Z}_1 \preccurlyeq_{\mathrm{SSD}} -\mathcal{Z}_2 \Longrightarrow \mathrm{CVaR}_\alpha(-\mathcal{Z}_1) \leq \mathrm{CVaR}_\alpha(-\mathcal{Z}_2) \tag{2.52}$$

2.2.7.3 Alternative method of defining CVaR as TVaR

Ogryczak et al. in [67], [68], and [77] develop an alternative approach to measure risk based on tail expectation. They use *first quantile function* as the basis. The desirable property of $F_1^{\mathcal{Z},-1}(p)$, according to Ogryczak et al. [77], is as follows:

$$\mathcal{Z}_1 \succcurlyeq_{\text{FSD}} \mathcal{Z}_2 \qquad \Longleftrightarrow \qquad F_1^{\mathcal{Z}_1,-1}(p) \geqslant F_1^{\mathcal{Z}_2,-1}(p) \qquad \text{for } 0 < p \leqslant 1 \tag{2.53}$$

Ogryczak et al. in [77] claim that $F_1^{\mathcal{Z},-1}(p)$ can be used as a risk measure. Authors also show the consistency of $VaR_\alpha(\mathcal{Z})$ risk measure with FSD using the basic properties of $F_1^{\mathcal{Z},-1}(p)$. $F_1^{\mathcal{Z},-1}(p)$ is not consistent with SSD. However, the second quantile function as a risk measure, defined below, is consistent with SSD.

Definition 2.9. *(Ogryczak and Ruszczynski [77]) Second quantile function is defined by:*

$$F_2^{\mathcal{Z},-1}(p) = \int_0^p F_1^{\mathcal{Z},-1}(\alpha)d\,\alpha \qquad \text{for } 0 < p \leqslant 1.$$

It has been shown in [77] that $F_2^{\mathcal{Z},-1}(p)$ is convex and it is a Fenchel dual for $F_2^{\mathcal{Z}}(x)$ for the pair of arguments (p, x), see Section 31 in [36] for details about convex conjugate (Fenchel dual) functions. This allows authors to directly use results from convex analysis [84]. The convexity of $F_2^{\mathcal{Z},-1}(p)$ simplifies the proof of many properties. Ogryczak and Ruszczynski [77] also establish the following relation:

$$\mathcal{Z}_1 \succcurlyeq_{\text{SSD}} \mathcal{Z}_2 \qquad \Longleftrightarrow \qquad F_2^{\mathcal{Z}_1,-1}(p) \geq F_2^{\mathcal{Z}_2,-1}(p) \qquad \text{for } 0 < p \leqslant 1 \tag{2.54}$$

Ogryczak and Ruszczynski [77] find an alternative way of defining $F_2^{\mathcal{Z},-1}(p)$ based on the convex conjugate connection between $F_2^{\mathcal{Z},-1}(p)$ and $F_2^{\mathcal{Z}}(x)$:

$$\begin{aligned} F_2^{\mathcal{Z},-1}(p) &= p * x - F_2^{\mathcal{Z}}(x) = \\ &= p * x + E[\mathcal{Z} - x | \mathcal{Z} \leqslant x] = p\,E[\mathcal{Z} | \mathcal{Z} \leqslant x] \end{aligned} \tag{2.55}$$

The graph of $F_2^{\mathcal{Z},-1}(p)$ as a function of p is a modification of Lorentz curve and called the Absolute Lorentz Curve (ALC) (see Figure 2.5 for illustration and [76], [77] for details). Ogryczak and Ruszczynski introduce Tail VaR as a risk measure in the following form:

$$\text{TVaR}(\mathcal{Z}, p) = \frac{F_2^{\mathcal{Z},-1}(p)}{p} \tag{2.56}$$

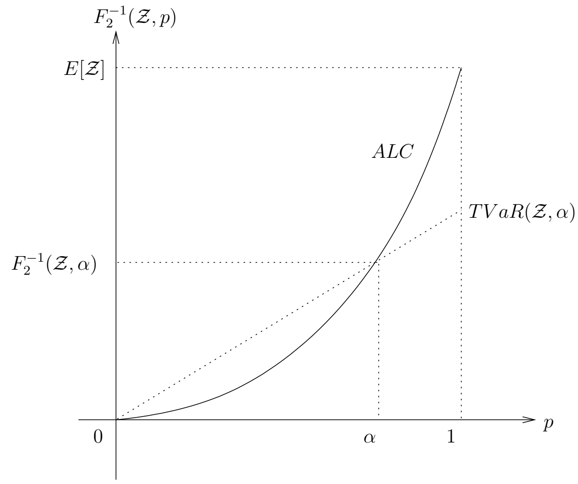

Figure 2.5. Absolute Lorentz Curve (ALC) and TVaR

It follows from the previous results that mean-risk model $(E[\mathcal{Z}], -\,\mathrm{TVaR}(\mathcal{Z}, p))$ is consistent with SSD. Based on the dual representation of $F_2^{\mathcal{Z},-1}(p)$, Ogryczak and Ruszczynski derive two equivalent formulae for computing TVaR:

$$\mathrm{TVaR}(\mathcal{Z}, p) = E[\mathcal{Z}] - \min_{\xi\in\mathbb{R}} E\left[\max\left(\mathcal{Z} - \xi, \frac{1-p}{p}(\xi - \mathcal{Z})\right)\right] \tag{2.57}$$

$$\mathrm{TVaR}(\mathcal{Z}, p) = \max_{\xi\in\mathbb{R}}\left(\xi - \frac{1}{p}\, E[\max(0, \xi - \mathcal{Z})]\right). \tag{2.58}$$

and, equality (2.58) is consistent with Rockafellar et al. [87] results. The difference is that Ogryczak and Ruszczynski use the lower tail of distribution instead of the upper tail in [85], see Figure 2.6.

The above discussion shows the importance of tail risk measures and the attention that they have gained over the years. VaR, CVaR, and TVaR have all been defined in the space of portfolio profit that is appropriate for portfolio optimization problems. However, such definitions of tail characteristics make impossible formulation of scenario generation and approximation problems using tail information, because both problems must be formulated in terms of outcomes of individual securities not

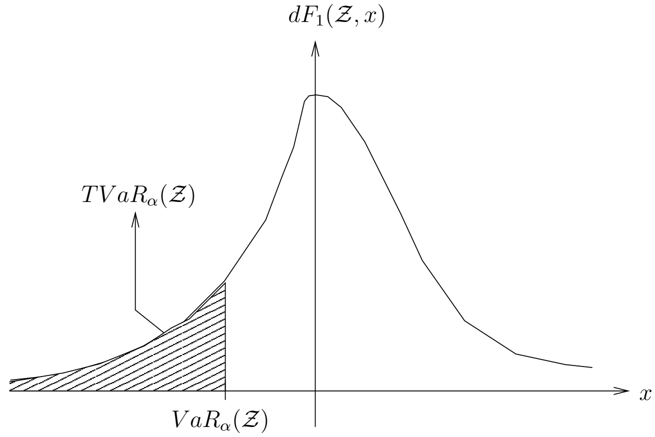

Figure 2.6. TVaR on the portfolio profit distribution function.

composite outcomes of optimal portfolio. An extension of the tail risk measures is proposed in Chapter 4 of this thesis for improving the risk control in financial investment models. The theoretical basis and an algorithm for scenario generation and approximation using tail information is addressed in Chapter 5 of this thesis.

Chapter 3

Rational Level Sets and Rational Risk Measures

3.1 Preliminaries

One of the major problems in financial investment optimization, as indicated in Chapter 2, is the problem of choosing an appropriate objective function for the investment model. There are several methodologies for addressing this problem: utility theory, risk/return trade-off, and coherent risk measures, for example. The latter approach has been developed in the last five years and has gained much attention in the risk management community. The axioms of Artzner et al. [2] for acceptance sets are the cornerstones of the theory of coherent risk measures. Expected bounded coherent risk measures, deviation measures, and risk envelopes are all based on the concept of acceptance sets, see [87]. The coherency property (see page 21 of this thesis) induces linearity of risk measures with respect to the underlying *r.v.* $\mathscr{Z}$ (the future portfolio-profit). However, other useful risk measures that have proven to be efficient in applications are not coherent, for instance, Markowitz's mean-variance and mean-semivariance risk measures, see [70]. The main difference between these measures and coherent risk measures is the non-linearity of the former with respect to the *r.v.* $\mathscr{Z}$.

The linearity of the coherent risk measures arises from the positive homogeneity property in the definition of coherent risk measures, see **Axiom PH** on page 21. The positive homogeneity property is a direct consequence of **Axiom 4** in the definition of acceptance sets. As pointed out in Section 2.2.4, the positive homogeneity property of acceptance sets requires strong assumptions about both the investor's preferences and the financial market. The first assumption is the linearity of investor's utility function. The second assumption is either a frictionless financial market or a market with linear slippage execution (cost of investments), see [26] Section 3.1 for details about slippage modeling. The above two assumptions are almost always violated in reality. Therefore, accepting the assumption about linearity of investor's utility causes underestimation of risk in an investment model. A way to resolve this problem is to develop an alternative concept for describing investor's preferences where utility functions and risk attitudes can accommodate non-linearity.

Using the basic ideas in Artzner et al. [2], a precise description of investor's preferences is developed first in this chapter. Then, this description is generalized to a new class of risk measures. The description of preferences is based on a set of axioms termed "*rational level set (RLS)*". Then, the idea of RLS is utilized to obtain a full characterization of investor's attitude towards risk. The bridge from rational level sets to a new concept termed "*rational risk measures (RRM)*" is built using the results of convex analysis [84]. The resulting concept of rational risk measures is broader than the concept of coherent risk measures. Coherent risk measures with additional restriction will be shown to be a subset of rational risk measures.

3.2 Axioms for rational level set

The problem of investor's choice is reduced to the following form. Let C denote the certainty equivalent of the risky investment $\mathcal{Z}$ to a given investor. Suppose the investor has the knowledge of C given a risky investment $\mathcal{Z}$. This value C may depend on such factors as wealth, risk preferences, market conditions, etc. How C is determined is not the focus here, rather, it is the specific value that is assigned by the investor that is of interest. The value of certainty equivalent C reflects investor's indifference between two alternatives, namely, the random profit $\mathcal{Z}$ from risky investment and the deterministic risk free profit C. For the above interpretation to hold, the amounts of initial investments are assumed to be the same for the two alternatives.

Next, based on the concept of certainty equivalent, the set of all investment opportunities that the investor will accept will be defined. Each investment opportunity is characterized by the future portfolio profit $\mathcal{Z}$ (deterministic or stochastic). Thus, investor's decision about acceptance or rejection is based on the distribution of the profit $\mathcal{Z}$.

Let the value of certainty equivalent to the random profit $\mathcal{Z}$ be denoted by $C(\mathcal{Z})$. A rational investor always has the following implication:

$$\mathcal{Z}_1 \succcurlyeq_{\text{FSD}} \mathcal{Z}_2 \Longrightarrow C(\mathcal{Z}_1) \geqslant C(\mathcal{Z}_2). \tag{3.1}$$

The above is true because rationality is equivalent to the investor's choice always being consistent with first order stochastic dominance (FSD). If one investment always generates higher profit then the other, the rational investor always pick the former one. Now, suppose the investor is faced with two alternatives $\mathcal{Z}_1$ and $\mathcal{Z}_2$. A convex combination of these two alternatives yields the profit $\mathcal{Z}$:

$$\mathcal{Z} = \alpha \cdot \mathcal{Z}_1 + (1-\alpha) \cdot \mathcal{Z}_2, \qquad 0 < \alpha < 1.$$

Clearly the following hold because of linearity of expectation operator:

$$E[\mathcal{Z}] = \alpha \cdot E[\mathcal{Z}_1] + (1-\alpha) \cdot E[\mathcal{Z}_2], \qquad 0 < \alpha < 1.$$

A rational investor will choose $\mathcal{Z}$ if and only if the following hold:

$$C(\mathcal{Z}) \geqslant \alpha\, C(\mathcal{Z}_1) + (1-\alpha)\, C(\mathcal{Z}_2), \qquad 0 < \alpha < 1. \tag{3.2}$$

That is $(\alpha, 1-\alpha)$ lottery between deterministic $C(\mathcal{Z}_1)$ and $C(\mathcal{Z}_2)$ is inferior to the fixed $C(\mathcal{Z})$. This is consistent with inequality (2.9) on page 13 that follows from the definition of *rational* behavior.

A *Rational Level Set (RLS)* is defined on the space of random variables (future portfolio profit) and certainty equivalents. The rational level set $\mathcal{A}^r$ contains all future profits for which the investor accepts an investment decision. The idea for this set is inspired by the definition of acceptance set in [2]. However, axioms for RLS are less restrictive than axioms for acceptance set in [2] and RLS depends on the value of certainty equivalents. The basis for defining rational level sets is the assumption that the investor is *risk averse* and *rational*, see section 2.2.1.

First, axioms for RLS are presented, then detailed explanations for each axiom are given. The rational level set $\mathcal{A}^r(C^L)$ for a particular value C^L of the certainty equivalent is the set of all possible future portfolio values that satisfy the following axioms:

RLS axiom 1. C^L is the lower limit for all certainty equivalents in the rational level set:

$$C^L = \inf_{C \in \mathbb{R}^1} \{ C \,|\, C \in \mathcal{A}^r(C^L) \} \tag{3.3}$$

RLS axiom 2. RLS is closed and convex:

$$\mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{A}^r(C^L) \Longrightarrow \lambda\, \mathcal{Z}_1 + (1-\lambda)\mathcal{Z}_2 \in \mathcal{A}^r(C^L), 0 \leqslant \lambda \leqslant 1 \tag{3.4}$$

The reasoning for the above axioms are the following:

RLS axiom 1. It is assumed that the investor has a lower limit for acceptance among constant returns. The investor can precisely specify this minimum value $C^L \in \mathbb{R}^1$ (which is either positive or negative) and it reflects the outcome generated without risk, for instance an investment in treasury bills. The value of C^L depends on the initial conditions of investments.

RLS axiom 2. The investor accepts diversification among acceptable investments alternatives. In other words, diversification among different alternatives does not increase the riskiness of investment. This is consistent with the concept of *risk aversion*. The investor always prefers a fixed outcome instead of participating in the lottery with expected outcome equal to the fixed one. It is also consistent with Capital Asset Pricing Model (CAPM) where diversification reduces the principal component of risk.

To explain further, suppose the convexity requirement is violated:

$$\mathcal{Z} = \alpha\, \mathcal{Z}_1 + (1-\alpha)\, \mathcal{Z}_2; \qquad \alpha \in (0,1); \qquad \mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{A}^r(C^L)$$

but $\mathcal{Z} \notin \mathcal{A}^r(C^L)$. Then, according to Axiom 1, we must have:

$$C(\mathcal{Z}) < C^L \leqslant \alpha\, C(\mathcal{Z}_1) + (1-\alpha)\, C(\mathcal{Z}_2) \tag{3.5}$$

However, (3.5) violates the inequality (3.2) which must hold for a *rational* investor. Thus, RLS must be a convex set (Axiom 2).

The following RLS properties (in the form of propositions) can be derived based on the definition.

Proposition 3.1. *RLS is consistent with FSD, for any given C^L:*

$$\mathcal{Z}_1 \succcurlyeq_{\text{FSD}} \mathcal{Z}_2 \text{ and } \mathcal{Z}_2 \in \mathcal{A}^r(C^L) \Longrightarrow \mathcal{Z}_1 \in \mathcal{A}^r(C^L) \tag{3.6}$$

Proof. This follows directly from assumption of investor's *rationality*, see (3.1). The investor always prefers higher outcomes over the lower ones. □

Proposition 3.2. *If two given investment alternatives, where one has random profit $\mathcal{Z}$ and the other has a fixed profit C^L, are both accepted, then a convex combination of the two investment alternatives is also accepted:*

$$\mathcal{Z} \in \mathcal{A}^r(C^L) \Longrightarrow (\lambda\, \mathcal{Z} + (1-\lambda)\, C^L) \in \mathcal{A}^r(C^L), 0 \leqslant \lambda \leqslant 1 \tag{3.7}$$

Proof. It follows directly from **RLS axiom 2** that RLS is convex, see (3.4). Because of **RLS axiom 1**, C^L is the minimal acceptable deterministic profit. Thus, both investment alternatives belong to RLS and RLS is a convex set. That validates implication (3.7). □

The major motivation behind RLS is to drop the assumption of positively homogeneity from the definition of acceptance set given by Artzner et al. [2]. Suppose that

$$\mathcal{Z}_1 = \lambda\, \mathcal{Z}_2, \forall \lambda \geqslant 1$$

then $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are both either accepted or rejected if the acceptance sets are used for decision making. RLS is more flexible than the acceptance sets. In the case of RLS, it is possible that $\mathcal{Z}_2$ is accepted but $\mathcal{Z}_1$ is rejected. This reflects the risk aversion of the investor where profits with higher volatility are rejected.

Mathematically, the rationale for dropping the positive homogeneity axiom can be presented as follows. If a random profit scales linearly (with a positive coefficient), then the appropriate initial investments are also supposed to be scaled. As a result, the axiom of positive homogeneous cone is similar to proposition 3.2:

$$\mathcal{Z} \in \mathcal{A}^r(C^L) \Longrightarrow (\lambda\, \mathcal{Z} + (1-\lambda)\, C^L) \in \mathcal{A}^r(C^L), 0 \leqslant \lambda \leqslant 1 \tag{3.8}$$

However, for $\lambda > 1$ the implication in (3.8) translates to:

$$\mathcal{Z} \in \mathcal{A}^r(C^L) \Longrightarrow (\lambda\, \mathcal{Z} - \left|1-\lambda\right| C^L) \in \mathcal{A}^r(C^L), 1 \leqslant \lambda \tag{3.9}$$

The investor borrows money from outside and invests it into a portfolio. First, this assumes the presence of a perfect market where rates for borrowing and lending are equal, which is not true in reality. Second, it implies an arbitrage opportunity if certainty equivalent for portfolio profit $\mathcal{Z}$ is different from C^L, i.e.

$$\mathcal{Z} \sim C(\mathcal{Z}) > C^L$$

Thus, positive homogeneity assumption in the case of coherent risk measures imposes additional implications about financial markets.

The collection of all rational level sets, denoted by $\{\mathcal{A}^r\}$, for all $C^L \in \mathbb{R}^1$, reflects the investor's preferences.

$$\{\mathcal{A}^r\} \equiv \left\{\mathcal{A}^r(C^L) \quad \middle| \quad C^L \in \mathbb{R}^1\right\}$$

Moreover, an optimal investment decision can be defined if the collection $\{\mathcal{A}^r\}$ is specified. In order to find the best investment decision, the C^L level needs to be increased to a level $C^{\max}$ such that RLS for $C^L > C^{\max}$ is empty (no portfolio outcomes in the set). Portfolio outcome corresponding to $C^{\max}$ are the optimal choices.

The advantage of $\{\mathcal{A}^r\}$ is the flexibility in the characterization of preferences. Each investor can define his/her own preferences. Even if some $\mathcal{A}^r(C^L)$ for two different investors coincide, their preferences can be different. The acceptance set does not provide such a flexibility because an acceptance set does not consider certainty equivalents. However, the concept of an RLS collection incorporates acceptance sets that are consistent with FSD.

Proposition 3.3. *A rational level set $\mathcal{A}^r(C^L)$ with $C^L = 0$, along with the additional requirement that $\mathcal{A}^r$ is a positive homogeneous cone, i.e.,*

$$\mathcal{Z} \in \mathcal{A}^r(C^L) \Longrightarrow \lambda\, \mathcal{Z} \in \mathcal{A}^r(C^L), \lambda \geqslant 0 \tag{3.10}$$

is equivalent to $\mathcal{A}^r$ being an acceptance set.

Proof. It is straightforward that **Axiom 1** and **Axiom 2** of acceptance set are satisfied for RLS with $C^L = 0$. **Axiom 3** of acceptance set follows directly from **Axiom 2** of RLS. The **Axiom 4** of acceptance set is valid by the condition (3.10) of the proposition. Thus, all axioms in the definition of acceptance set are satisfied if condition (3.10) is added to the definition of RLS. □

3.3 Rational risk measures (RRM)

The set collection $\{\mathcal{A}^r\}$ of rational level sets completely describes the investor's preferences. It is possible that the number of elements in $\{\mathcal{A}^r\}$ is infinite. Information in this form is very difficult to handle and use. However, the RLS is similar in spirit to the concept of "level sets" in convex analysis, see [84] section 5. In convex analysis, the "epigraph" refers to all function arguments for which the function value is smaller than some target value μ, see Figure 3.1 for example. From an analogous treatment, it is possible to express the investor's full preferences $\{\mathcal{A}^r\}$ through a function. This function is hereby called a *rational risk measure (RRM)*.

Rational risk measures are defined as functions of random profit $\rho(\mathcal{Z})$. It is a mapping

$$\rho\colon \left\{\mathbb{R}^1, \mathcal{F}(\mathbb{R}^1), \mathcal{P}(\mathcal{F}(\mathbb{R}^1))\right\} \times \mathbb{R}^1 \longrightarrow \mathbb{R}^1 \tag{3.11}$$

that satisfies the following axioms:

RRM axiom 1. $\rho(\mathcal{Z})$ is a non-increasing, convex function with respect to $\mathcal{Z}$.

RRM axiom 2. Acceptability property:

$$\mathcal{Z} \geqslant C^L \; a.s. \Longrightarrow \rho(\mathcal{Z}) \leqslant -C^L \tag{3.12}$$

The connection between RRM and FSD is described as following proposition:

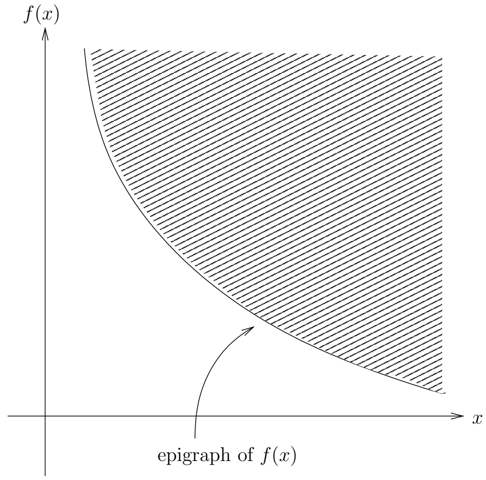

Figure 3.1. Example of epigraph.

Proposition 3.4. *Rational risk measures are consistent with first order stochastic dominance:*

$$\mathcal{Z}_1 \succcurlyeq_{\mathrm{FSD}} \mathcal{Z}_2 \Longrightarrow \rho(\mathcal{Z}_1) \leqslant \rho(\mathcal{Z}_2) \tag{3.13}$$

Proof. The proposition directly follow from **RRM axiom 1** where $\rho(\mathcal{Z})$ is a non-increasing function of its argument. □

From definitions of RLS and RRM, it is clear that RRM is connected to collection of RLS as follows:

$$\rho(\mathcal{Z}) = \inf_{C \in \mathbb{R}^1} \left\{ C \,|\, (C + \mathcal{Z}) \in \mathcal{A}^r(C^L) \right\} \tag{3.14}$$

$$\mathcal{A}^r(C^L) = \left\{ \mathcal{Z} \,|\, \rho(\mathcal{Z}) \leqslant -C^L \right\} \tag{3.15}$$

The next result is the theorem that describes the one-to-one correspondence between rational risk measures and collection $\{\mathcal{A}^r\}$.

Theorem 3.5. *Suppose two collections of rational level sets $\{\mathcal{A}_1^r\}$ and $\{\mathcal{A}_2^r\}$, induce RRM ρ_1 and ρ_2. If $\{\mathcal{A}_1^r\}$ and $\{\mathcal{A}_2^r\}$ coincide, then:*

$$\rho_1(\mathcal{Z}) = \rho_2(\mathcal{Z}), \qquad \forall \mathcal{Z}.$$

Proof. The proof consist of two parts.

Collection of RLS $\Rightarrow$ RRM.

RLS axiom 1 and **RLS axiom 2** imply the convexity of the epigraph of the negative underlying function. The property (3.6) of RLS from Proposition 3.1 implies that the negative underlying function is non-decreasing. Thus, the underlying function is convex and non-increasing. That is the **RRM axiom 1**.

The following implication is straightforward:

RLS axiom 1 $\Rightarrow$ RRM axiom 2

If two epigraphs coincide (for all values $\mu \in \mathbb{R}^1$) then the two functions coincide, see Rockafellar [84].

RRM $\Rightarrow$ Collection of RLS.

RRM axiom 1 implies the convexity of epigraph and the existing of lower limit for each epigraph. That is **RLS axiom 1** and **RLS axiom 2**. At the same time, the following implication is straight forward:

RRM axiom 2 $\Rightarrow$ RLS axiom 1

Based on Rockafellar [84], the convex functions define the level sets uniquely. Thus, if two convex function coincide the level sets for them must also coincide. □

In the foregoing discussion a process for describing an investor's preference is presented via a collection of acceptable future profits and then a method of conversion between this collection and risk measures, (3.14) and (3.15), is developed. This gives the insight for the mechanism of preference description using risk measures. The theorem also gives insight for understanding the difference between coherent and rational risk measures.

If two different investors have the same acceptance set for $C^L = 0$ then their risk measures must coincide if they use coherent risk measures. However, two different rational risk measures can generate the same level set for some values C^L. Decisions made using RRM are consistent with FSD.

$$\mathcal{Z}_1 \succcurlyeq_{\text{FSD}} \mathcal{Z}_2 \Longrightarrow \rho(\mathcal{Z}_1) \leqslant \rho(\mathcal{Z}_2)$$

which follows directly from **RRM axiom 1**. This is a consequence of the investor's rational behavior. For deterministic portfolio profits, the function $-\rho(C)$ is non-decreasing and concave. Additionally, $-\rho(C)$ reflects investor's wealth preferences. That is a utility function, according to section 2.2.1, because the investor is risk averse if the utility function is concave.

$$u(C) = -\rho(C).$$

The relation between RRM and coherent risk measures is presented in the following proposition:

Proposition 3.6. *The coherent risk measures with FSD requirement instead of* ***axiom M*** *in the definition are RRM.*

Proof. There are only two differences between coherent risk measures and rational risk measures:

1. Coherent risk measures have the additional **axiom PH** in the definition.

2. Coherent risk measures satisfy the monotonicity requirement (**axiom M**) almost surely.

The first difference narrows the scope of all available risk measures from non-linear to positive homogeneous functions. Thus, coherent risk measures are a subset of rational risk measures with respect to **axiom PH**. However, the second difference implies that the class of coherent risk measures is broader than the class of rational risk measures because FSD implies almost sure monotonicity but not vise versa. The proposition condition restricts all coherent risk measures to those that satisfy FSD. Thus, coherent risk measures with the additional condition of FSD consistency are rational risk measures. □

The Proposition 3.6 emphasizes the difference between coherent risk measures and rational risk measures. Proposition condition of substituting **axiom M** with FSD consistency is appropriate for application where σ-field of *r.v.* remains fixed. The important fact is that FSD always implies almost sure monotonicity, but the converse is not true, see Figure 3.2 where $\mathcal{Z}_2 \geqslant \mathcal{Z}_1$ *a.s.* but $\mathcal{Z}_2 \not\succeq_{\text{FSD}} \mathcal{Z}_1$.

3.4 Decomposition of risk measures

Let's assume that risk measure $\rho(\mathcal{Z})$ is decomposable and consists of two components: target and deviation. The basis for this assumption is [84] and Markowitz's work [70], [69], [71].

The idea is to separate attitude towards deterministic outcome and attitude towards uncertainty. The target measure reflects utility and the deviation measure reflects attitude toward uncertainty. The combination of these two components produces the risk measure. The assumption about separation of the two measures and additivity of the combination is taken as an axiom, see [87]. Then the risk measure can be presented as follows:

$$\rho(\mathcal{Z}) = -t(\mathcal{Z}) - d(\mathcal{Z})$$

Both $t(\mathcal{Z})$ and $d(\mathcal{Z})$ need to be concave, non-decreasing functions in order for $\rho(\mathcal{Z})$ to be a rational risk measure. Some examples of target and deviation measures are given below.

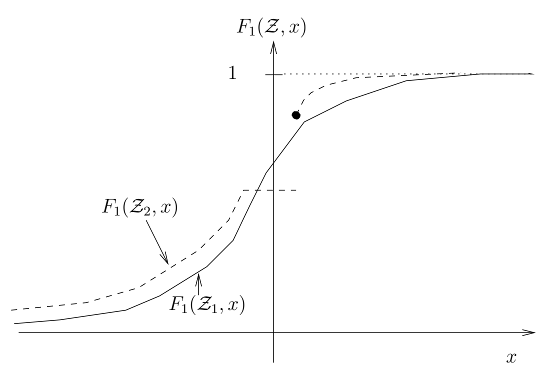

Figure 3.2. Example of $\mathcal{Z}_2 \geqslant \mathcal{Z}_1$ *a.s.* but $\mathcal{Z}_2 \not\succeq_{\text{FSD}} \mathcal{Z}_1$.

3.5 Examples of risk measures

Target measure examples are as follows:

- Expectation.

$$t(\mathcal{Z}) = E[\mathcal{Z}] \tag{3.16}$$

- Value-at-Risk.

$$t(\mathcal{Z}) = \mathrm{VaR}_{\alpha}(\mathcal{Z}) \tag{3.17}$$

- Constant target.

$$t(\mathcal{Z}) = C \tag{3.18}$$

- Non-linear target.

$$t(\mathcal{Z}) = f(E[\mathcal{Z}]) \tag{3.19}$$

  with $f(\cdot)$ concave and non-decreasing function.

Each of the above target measures is consistent with FSD.

Deviation measure examples are as follows:

- Semi-deviation with some power, see [39] for details:

$$d(\mathcal{Z}) = E\left[\left([t(\mathcal{Z}) - \mathcal{Z}]^{+}\right)^{\gamma}\right], \gamma \geqslant 1 \tag{3.20}$$

- CVaR or a weighted combination of different CVaR's:

$$d(\mathcal{Z}) = \frac{1}{\alpha} E\left[\left([\mathrm{VaR}_{\alpha}(\mathcal{Z}) - \mathcal{Z}]^{+}\right)\right] \tag{3.21}$$

$$d(\mathcal{Z}) = \sum_{i} \frac{1}{\alpha_i} E\left[\left([\mathrm{VaR}_{\alpha_i}(\mathcal{Z}) - \mathcal{Z}]^{+}\right)\right] \tag{3.22}$$

- Deviation from a constant target:

$$d(\mathcal{Z}) = E\left[\left([C - \mathcal{Z}]^{+}\right)^{\gamma}\right], \gamma \geqslant 1 \tag{3.23}$$

The investor needs to be cautious because not all combinations of the above target and deviation measures can produce a rational risk measure. For example, the combination of target measure (3.16) and deviation measure (3.20) yields:

$$\rho(\mathcal{Z}) = -E[\mathcal{Z}] - E\left[\left([t(\mathcal{Z}) - \mathcal{Z}]^+\right)^2\right], \tag{3.24}$$

which is also known as mean/semi-variance risk measure and it is not consistent with FSD, see [77].

However, there are risk measures that are guaranteed to be rational. A combination of the deviation measures (3.21), (3.22), and (3.23) with any target measure (3.16)-(3.19) produces RRM because each component separately is FSD consistent and convex. Thus, the resulting risk measure is FSD consistent and convex. The following risk measures belong to this class:

$$\rho(\mathcal{Z}) = -E[\mathcal{Z}] - E\left[\left([C - \mathcal{Z}]^+\right)^\gamma\right], \gamma \geqslant 1 \tag{3.25}$$

$$\rho(\mathcal{Z}) = -E[\mathcal{Z}] - \frac{1}{\alpha}E\left[\left([\mathrm{VaR}_\alpha(\mathcal{Z}) - \mathcal{Z}]^+\right)\right] \tag{3.26}$$

$$\rho(\mathcal{Z}) = -E[\mathcal{Z}] - \sum_i \frac{1}{\alpha_i}E\left[\left([\mathrm{VaR}_{\alpha_i}(\mathcal{Z}) - \mathcal{Z}]^+\right)\right] \tag{3.27}$$

$$\rho(\mathcal{Z}) = -f(E[\mathcal{Z}]) - E\left[\left([C - \mathcal{Z}]^+\right)^\gamma\right], \gamma \geqslant 1 \tag{3.28}$$

$$\rho(\mathcal{Z}) = -f(E[\mathcal{Z}]) - \frac{1}{\alpha}E\left[\left([\mathrm{VaR}_\alpha(\mathcal{Z}) - \mathcal{Z}]^+\right)\right] \tag{3.29}$$

$$\rho(\mathcal{Z}) = -f(E[\mathcal{Z}]) - \sum_i \frac{1}{\alpha_i}E\left[\left([\mathrm{VaR}_{\alpha_i}(\mathcal{Z}) - \mathcal{Z}]^+\right)\right] \tag{3.30}$$

All risk measures (3.25) - (3.30) are rational. However, only (3.26) and (3.27) are coherent risk measures. The risk measures (3.25), (3.28) - (3.30) can violate the coherency requirements because of either non-linearity of $f(\,\cdot\,)$ or $C \neq 0$. Thus, the concept of rational risk measures is more flexible than the concept of coherent risk measures.

If semi-deviation is used with $\gamma > 1$:

$$\rho(\mathcal{Z}) = -E[\mathcal{Z}] - E\left[\left([t(\mathcal{Z}) - \mathcal{Z}]^+\right)^\gamma\right], \tag{3.31}$$

it may not be a RRM because FSD consistency is not guaranteed for (3.31).

A new approach to describe investor's preferences (risk attitude) and to construct appropriate risk measures has been presented in this chapter. The foundation for describing the investor's preferences is the Rational Level Set (RLS) with a specific value C^L for certainty equivalent. A collection of RLS for all possible values of certainty equivalents is a complete description of investor's preferences. The collection of all RLS corresponds uniquely to a Rational Risk Measure (RRM). Such a measure is appropriate for modeling risk in financial optimization problems. Moreover, the proposed RRM contains the coherent risk measures which are FSD consistent, see Proposition 3.6 on page 44.

Chapter 4

New Approach to VaR/CVaR

4.1 Preliminaries

VaR/CVaR concept is an effective way of controlling risks as it was shown in Chapter 2. However, VaR is not appropriate for an optimization framework because VaR lacks of concavity/convexity property. Recently, there was an attempt to solve the financial investment optimization problem using VaR as an objective function, see [80]. Unfortunately, the proposed method gives an approximation solution only. CVaR is more attractive from a computational view point [85], and at the same time, CVaR delivers more information about distribution tail (not just a bound as VaR), see Chapter 2 section 2.2.7. Recently, there have been more studies of *CVaR* application for financial investment problems, see [1], [67], and [68]. These results are promising, but the drawback is the requirement of generating a sample of a random variable in order to formulate the financial optimization problem. A sample must represent tail information that requires large number of points in the sample. If the problem of financial optimization is multiperiod, like in Russell-Yasuda Kasai financial planning model, see [13], then the scenario tree will be enormously large and the formulated problem is computationally tedious. Thus, there is a requirement for an efficient approximation scheme that preserves tail information.

VaR and *CVaR* are not defined through the underlying sample of random vector $\mathcal{X}$. They are defined through the one-dimensional random variable, that is the future portfolio loss (or profit), see section 2.2.7. Therefore, before constructing an approximation scheme that preserves tail information, the theoretical properties of VaR/CVaR with respect to the underlying sample of random vector need to be explored. This chapter describes a new way of defining VaR/CVaR through a sample of random vector. The problem of approximating a given sample using CVaR information is addressed in Chapter 5.

4.2 The concept of SaR/SaG

Assume that the future portfolio profit is generated by returns of the underlying financial instruments as a linear function:

$$\mathcal{Z} = \langle \mathcal{X}, y \rangle \tag{4.1}$$

where $\mathcal{X}$ is the random vector of the future outcomes of the financial instruments and y is a decision that the investor made, i.e. portfolio positions.

Suppose we fix $y \in \Omega^*$ (an investment portfolio). We also define the level of significance α, where $0 < \alpha < 1$. Then the *value at risk (VaR)* and *value at gain (VaG)* can be defined in several steps. The first definition is for *upper part* $D_y^U(x)$ and *lower part* $D_y^L(x)$ of Ω, which is a partition of Ω by $x \in \Omega$ and $y \in \Omega^*$:

$$D_y^L(x) = \{z \in \Omega \mid \langle y, z \rangle < \langle y, x \rangle\} \tag{4.2}$$

$$D_y^U(x) = \{z \in \Omega \mid \langle y, z \rangle > \langle y, x \rangle\} \tag{4.3}$$

The next step in *VaR/VaG* definition is to define *Scenarios at Risk (SaR)* and *Scenarios at Gain (SaG)* for probability level α as additional conditions on *upper part* and *lower part* of (4.2) and (4.3).

$$\mathrm{SaR}_\alpha(\langle \mathcal{X}, y \rangle) = \left\{z \in \Omega \mid \mu\left(D_y^L(z)\right) \leq \alpha\right\} \tag{4.4}$$

$$\mathrm{SaG}_\alpha(\langle \mathcal{X}, y \rangle) = \left\{z \in \Omega \mid \mu\left(D_y^U(z)\right) \leq (1-\alpha)\right\} \tag{4.5}$$

where $\mu(\ )$ is a probability measure of $\mathcal{X}$, see Figure 4.1 for an illustration. SaR contains all outcomes that are located in the lower tail of the random variable. The probability of this lower tail is α. SaG contains all outcomes that are located in the upper tail. The probability of this upper tail is $1-\alpha$.

4.3 VaR/VaG definitions

The original definition of *VaR* is as follows:

$$\mathrm{VaR}_\alpha(\langle \mathcal{X}, y\rangle) = \max_{x\in\mathbb{R}^1}\{F_1^{\langle \mathcal{X}, y\rangle,-1}(\alpha)\} \tag{4.6}$$

VaR can be defined based on the *SaR* concept (4.4) as follows:

$$VaR_\alpha(\langle \mathcal{X}, y\rangle) = \sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X}, y\rangle)}\{\langle y, x\rangle\} \tag{4.7}$$

This definition is consistent with the original definition of *VaR* in [96], but we have defined intermediate sets *SaR/SaG*. The intermediate sets *SaR/SaG* will be useful in the problem of sample approximating. Both *SaR* and *SaG* define the partition of

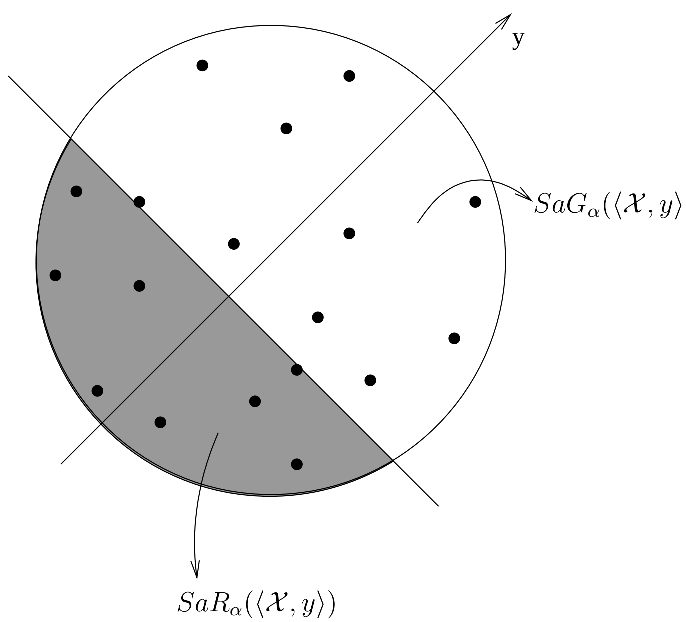

Figure 4.1. SaR and SaG representation.

the random vector domain Ω. Mathematically, this partition is defined through a boundary on Ω that split the *r.v.* domain into two parts with appropriate probabilities α and $1-\alpha$. $\mathrm{SaR}_\alpha(\langle x, y\rangle)$ is the biggest subset of Ω with probability no more than α or the upper limit of the new *r.v* made from the original one such that $1-\alpha$ upper tail values are ignored (or negligible). This is the maximum profit that the investor can gain if all $1-\alpha$ favorable scenarios never happen in the financial investment application. Equation (4.7) defines the upper bound for the following set:

$$D_{VaRG} = \left\{ x \,\middle|\, \int_{D_y^L(x)} d\mu(t) = \alpha \right\} \tag{4.8}$$

where $\mu(t)$ is the probability measure of the r.v. $\mathcal{X}$. The upper bound for the set D_{VaRG} can be defined as follows:

$$VaG_\alpha(\langle \mathcal{X}, y\rangle) = \inf_{x\,\in\,\mathrm{SaG}_\alpha(\langle \mathcal{X}, y\rangle)} \{\langle y, x\rangle\} \tag{4.9}$$

Value at Gain (VaG) is the largest element $\langle y, z\rangle$ for $z \in D_y^U(x)$ with probability of $D_y^U(x)$ no more than $1-\alpha$. In other words, *VaG* is the lower limit of the new *r.v* made from the original one by neglecting the α lower tail values and adapting the probability measure. The interpretation for application of financial investment is similar to *VaR*. *VaG* is the minimum profit that the investor can guarantee if α tail unfavorable events are negligible or non-existent.

As can be seen from the definitions, *VaR* and *VaG* are complementary to each other and adapted to both Lebesgue and countable measure *random variables*, and provide lower and upper bound for the (4.8). Rockafellar and Uryasev in [86] define these values differently, and they are referred to as VaR_α^- and VaR_α^+ . The upper and lower bounds for probability level α can also be defined as following:

$$\alpha_y^L = \int_{\mathrm{SaR}_\alpha(\langle \mathcal{X}, y\rangle)} d\mu(x) \tag{4.10}$$

$$\alpha_y^U = 1 - \int_{\mathrm{SaG}_\alpha(\langle \mathcal{X}, y\rangle)} d\mu(x) \tag{4.11}$$

The usefulness of VaR_α and VaG_α can be seen from the following Lemma.

Lemma 4.1. *For any consistent distribution (dominated by either a Lebesgue or countable measure), for any* $\alpha \in (0,1)$*, and for any* $y \in \mathbf{\Omega}^*$ *at most one of the following inequalities is satisfied:*

$$\mathrm{VaR}_\alpha(\langle \mathcal{X}, y\rangle) < \mathrm{VaG}_\alpha(\langle \mathcal{X}, y\rangle), \tag{4.12}$$

$$\text{or} \qquad \alpha_y^L < \alpha_y^U. \tag{4.13}$$

Inequalities (4.12) and (4.13) follow from the definitions of VaR_α, VaG_α, α_y^L, and α_y^U. The intuition is that it's impossible to have a non-zero probability for an element that does not belong to σ-field $\mathcal{F}(\langle \mathcal{X}, y\rangle)$.

Proof. The proof is based on contradiction. Suppose that both (4.12) and (4.13) hold at the same time. Then from (4.13), there exists an element $\beta \in \mathcal{F}(\langle \mathcal{X}, y\rangle)$ such that:

$$\beta \cap (VaR_\alpha, VaG_\alpha) \neq \varnothing \quad and \quad \mu(\beta) > 0$$

But at the same time from the definitions of *VaR*, *VaG*, and *r.v.* $\langle \mathcal{X}, y\rangle$ we have:

$$\forall \beta \in \mathcal{F}(\mathcal{X}) \quad : \quad \beta \cap (VaR_\alpha, VaG_\alpha) = \varnothing,$$

otherwise, either the **sup** property of VaR or the **inf** property of VaG is violated. Thus, only negligible events can belong to the interval (VaR_α, VaG_α). That is $\mu(\beta) = 0$. This contradiction proves Lemma 4.1. □

4.4 Expressions for CVaR/CVaG

According to section 2.2.7.2, *CVaR* is defined as follows:

$$\mathrm{CVaR}_\alpha(\mathcal{Z}) = \frac{1}{\alpha} \int\limits_{z \geqslant \mathrm{VaR}_\alpha(\mathcal{Z})} z \, \mathrm{d}F_1(\mathcal{Z}, z) \tag{4.14}$$

Conditional Value at Risk (CVaR) and *Conditional Value at Gain (CVaG)* can also be defined as expectation over *Scenarios at Risk (SaR)* and *Scenarios at Gain (SaG)*. *Conditional Value at Risk (CVaR)* has the following expression:

$$CVaR_\alpha(\langle \mathcal{X}, y\rangle) = \frac{1}{\alpha} \int_{\mathrm{SaR}_\alpha(\langle \mathcal{X}, y\rangle)} \langle y, x\rangle d\mu(x) \tag{4.15}$$

and *Conditional Value at Gain (CVaG)* can be defined as follows:

$$CVaG_\alpha(\langle \mathcal{X}, y\rangle) = \frac{1}{1-\alpha} \int_{\mathrm{SaG}_\alpha(\langle \mathcal{X}, y\rangle)} \langle y, x\rangle d\mu(x) \tag{4.16}$$

The interpretation of these values is similar to *VaR* and *VaG* interpretations. *CVaR* is the expectation of the transformed *r.v.* $\langle \mathcal{X}, y\rangle$ if all $1-\alpha$ upper tail events are considered as negligible. Thus, it is an expectation of the new random variable that is produced from the original one by dropping $1-\alpha$ upper tail events and adapting the probability measure μ to the new σ-field. The same interpretation can be done for *CVaG* but instead of dropping upper tail events, the new *r.v.* is produced by dropping the α lower tail events from the original distribution. The only difference between *CVaR/CVaG* and *VaR/VaG* is that *CVaR/CVaG* define expected profits instead of profit boundaries in *VaR/VaG* case.

According to modern probability theory [9], [61], the conditional expectation of a *r.v.* is another *r.v.* with σ-field that belongs to the σ-field of the original *r.v.* The new σ-field $\mathcal{B}_\alpha$ is produced from the original $\mathcal{F}(\Omega)$ if we condition the *r.v.* $\mathcal{X}$ on *SaR/SaG* tails. $\mathcal{B}_\alpha$ has only two elements:

$$\beta_\alpha^1 = \{x \in \mathrm{SaR}_\alpha(\langle \mathcal{X}, y\rangle)\} \tag{4.17}$$

$$\beta_\alpha^2 = \{x \in \mathrm{SaG}_\alpha(\langle \mathcal{X}, y\rangle)\} \tag{4.18}$$

For distributions that are dominated by Lebesgue measure, there is no third component because $\alpha_y^L = \alpha_y^U$. Thus for all such distributions:

$$\mu(\beta_\alpha^1) + \mu(\beta_\alpha^2) = 1$$

For distributions that are dominated by a countable measure or simple *r.v.*, the point split is possible in the case of $\alpha_y^L < \alpha_y^U$. Assume that a boundary point with coordinate x_b and probability p_b exists such that:

$$\langle x_b, y\rangle = \mathrm{VaR}_\alpha(\langle \mathcal{X}, y\rangle) = \mathrm{VaG}_\alpha(\langle \mathcal{X}, y\rangle) \tag{4.19}$$

This assumption is valid according to Lemma 4.1. Then, the point (x_b, p_b) is substituted by two points with coordinates:

$$x_b^1 = x_b^2 = x_b$$

and probabilities:

$$p_b^1 + p_b^2 = p_b$$

such that (4.17) and (4.18) are hold. Based on the fact that $\mathcal{B}_\alpha \in \mathcal{F}(\Omega)$, we can write:

$$E[\langle \mathcal{X}, y\rangle] = E[E[\langle \mathcal{X}, y\rangle \mid \mathcal{B}_\alpha]] = \sum_{i=1}^{2} \int_{\beta_\alpha^i} \langle y, x\rangle d\mu(x) \tag{4.20}$$

By incorporating expressions for $CVaR_\alpha$ (4.15) and $CVaG_\alpha$ (4.16) into (4.20):

$$E[\langle \mathcal{X}, y\rangle] = \alpha \cdot CVaR_\alpha(\langle \mathcal{X}, y\rangle) + (1-\alpha) \cdot CVaG_\alpha(\langle \mathcal{X}, y\rangle) \tag{4.21}$$

That gives relation between *CVaR*, *CVaG* and expected portfolio profit.

4.5 Properties of CVaR/CVaG

The properties of *CVaR/CVaG* with respect to the original scenarios $\mathcal{X}$ and the chosen decision y are explored in this section. These properties will be helpful during formulation and decomposition of an approximation model for the financial investment problem.

The definitions of $CVaR_\alpha$ (4.15) and $CVaG_\alpha$ (4.16) can be rewritten as follows:

$$\begin{aligned} CVaR_\alpha(\langle \mathcal{X}, y\rangle) &= VaR_\alpha\Big(\langle \mathcal{X}, y\rangle\Big) + \\ &+ \frac{1}{\alpha} \cdot \int\limits_{\mathrm{SaR}_\alpha(\langle \mathcal{X}, y\rangle)} \Big(\langle y, x\rangle - VaR_\alpha(\langle \mathcal{X}, y\rangle)\Big) d\mu(x) \end{aligned} \tag{4.22}$$

$$\begin{aligned} CVaG_\alpha(\langle \mathcal{X}, y\rangle) &= VaG_\alpha\Big(\langle \mathcal{X}, y\rangle\Big) + \\ &+ \frac{1}{1-\alpha} \cdot \int\limits_{\mathrm{SaR}_\alpha(\langle \mathcal{X}, y\rangle)} \Big(\langle y, x\rangle - VaG_\alpha(\langle \mathcal{X}, y\rangle)\Big) d\mu(x) \end{aligned} \tag{4.23}$$

We start with the following proposition that asserts positive homogeneity of *CVaR/CVaG*.

Proposition 4.2. *$CVaR_\alpha(\langle \mathcal{X}, y\rangle)$ and $CVaG_\alpha(\langle \mathcal{X}, y\rangle)$ are positively homogeneous with respect to portfolio y, as well as with respect to distribution points $x \in \Omega$, separately.*

Proof. If we rewrite CVaR definition for a fixed values of α and y:

$$CVaR_\alpha(\langle \mathcal{X}, y\rangle) = \sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \{\langle y, x\rangle\} - $$

$$- \frac{1}{\alpha} \int\limits_{\mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \left(\sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \{\langle y, x\rangle\} - \langle y, x\rangle \right) d\mu(x)$$

then for $\lambda > 0$ we can move the multiplier from sup:

$$CVaR_\alpha(\langle \lambda\mathcal{X}, y\rangle)$$

$$= \sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \{\langle y, \lambda x\rangle\} -$$

$$- \frac{1}{\alpha} \int_{\mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \left(\sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \{\langle y, \lambda x\rangle\} - \langle y, \lambda x\rangle \right) d\mu(x) =$$

$$= \sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \{\langle \lambda y, x\rangle\} -$$

$$- \frac{1}{\alpha} \int_{\mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \left(\sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \{\langle \lambda y, x\rangle\} - \langle \lambda y, x\rangle \right) d\mu(x) =$$

$$= \sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \{\lambda \cdot \langle y, x\rangle\} -$$

$$- \frac{\lambda}{\alpha} \int_{\mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \left(\sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \{\langle y, x\rangle\} - \langle y, x\rangle \right) d\mu(x) =$$

$$= \lambda \cdot \left(\sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \{\langle y, x\rangle\} - \right.$$

$$\left. - \frac{1}{\alpha} \int_{\mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \left(\sup_{x\in \mathrm{SaR}_\alpha(\langle \mathcal{X},y\rangle)} \{\langle y, x\rangle\} - \langle y, x\rangle \right) d\mu(x) \right) =$$

$$= \lambda \cdot CVaR_\alpha(\langle \mathcal{X}, y\rangle)$$

An analogous proof follows for *CVaG*. □

Based on (4.22), (4.23), and appealing to Uryasev [96] $CVaR_\alpha$ and $CVaG_\alpha$ can be reformulated as given in the following theorem.

Theorem 4.3. *If* $\mathcal{X}$ *is properly defined r.v. and* α *is a given probability level then* CVaR_α *and* CVaG_α*can be defined by solving the following optimization problems:*

$$\mathrm{CVaR}_\alpha(\langle \mathcal{X}, y\rangle) = \max_{x\in\Omega}\left\{ \langle y, x\rangle - \frac{1}{\alpha}\int_{D_y^L(x)} \langle y, x-t\rangle d\mu(t)\right\} \tag{4.24}$$

$$\mathrm{CVaG}_\alpha(\langle \mathcal{X}, y\rangle) = \min_{x\in\Omega}\left\{ \langle y, x\rangle + \frac{1}{1-\alpha}\int_{D_y^U(x)} \langle y, x-t\rangle d\mu(t)\right\} \tag{4.25}$$

Proof. The proof is fairly tedious and it is moved into Appendix A. □

This theorem is the *SaR/SaG* analog of Theorem 10 in [96] (their main theorem about *CVaR*). If we take Rockafellar and Uryasev [85] definition of *CVaR*:

$$\mathrm{CVaR}_\alpha(\langle \mathcal{X}, y\rangle) = \max_{\xi\in\mathbb{R}}\left\{ \xi - \frac{1}{\alpha}\int_{\Omega} [\xi - \langle y, t\rangle]^+ d\mu(t)\right\} \tag{4.26}$$

$$[f(x)]^+ = \begin{cases} f(x) & | \; f(x) \geqslant 0 \\ 0 & | \; f(x) < 0 \end{cases} \tag{4.27}$$

Similarly, we can write:

$$\mathrm{CVaG}_\alpha(\langle \mathcal{X}, y\rangle) = \min_{\xi\in\mathbb{R}}\left\{ \xi + \frac{1}{1-\alpha}\int_{\Omega} [\langle y, t\rangle - \xi]^+ d\mu(t)\right\} \tag{4.28}$$

These functions are *concave* and *convex* respectively as functions of a decision $y \in \Omega^*$, see [85, Corollary 10].

The Theorem 4.3 allows finding the *CVaR/CVaG* values for a fixed α and a fixed portfolio $y \in \Omega^*$. A comparison with the work in [96] shows that the difference between the formulation in Theorem 4.3 and that in Theorem 10 [96] is in the utility function $g(x,\ y)$. Theorem 10 [96] is more general because $g(x,\ y)$ is a concave function. However, the same utility function can't be applied here because if $g(x,\ y)$ is convex, *CVaR* will not be a concave function with respect to $x \in \Omega$, and if $g(x, y)$ is concave, *CVaG* will not be a convex function with respect to $x \in \Omega$.

Using any non-linear utility function will affect the results in Theorem 4.3 in the following way. If we use a convex utility function $g(x,\ y)$ instead of the inner product (linear function), (4.24) becomes invalid. However, (4.25) will be valid and the proof of Theorem 4.3 will be applicable with minor modifications. The case of convex $g(x,\ y)$ reflects a *risk taking* strategy and we should control portfolio's *Conditional Value at Gain* or conditional expectation of the upper tail. Simply, the investor will make decisions by looking at the favorable events (upper tail) and reject insignificant unfavorable events (lower tail). If we use a concave utility function $g(x, y)$ instead of the inner product, (4.25) becomes invalid. However, (4.24) will be valid and again the proof of Theorem 4.3 will be valid with minor modifications. The case of concave $g(x,\ y)$ reflects a *risk aversion* strategy and the investor should control portfolio's *Conditional Value at Risk* or conditional expectation of the lower tail. Theorem 4.3 also has the following advantages against the result in [96, Theorem 10]:

- *CVaR* in Theorem 4.3 is determined directly as a function of $x \in \mathcal{R}^n$ instead of $f(y,x) \in \mathcal{R}^1$. This allows using formulas (4.24) and (4.25) later during solving approximation problem.
- The direct proof of the convexity of the $F(x,\ y)$ has been incorporated in Theorem 4.3. That gives insight and understanding of *CVaR/CVaG* concept.

Observation: *$CVaR_\alpha$ and $CVaG_\alpha$ as they are defined in Theorem 4.3 look very similar to the convex-conjugate functions used in the Large Deviation Principle (LDP) [20] and convex analysis [84]. The main subject of research in LDP is to evaluate the probabilities of rare events using rate functions. Rate functions bound the probability functionals on open and close sets. That is sufficient to bound probability on the σ-field. In the case of $CVaR_\alpha$ and $CVaG_\alpha$, we impose bounds on the components of*

Conditional Expectations; we also assume some events to be negligible. Thus, the ideas behind two approaches are the same.
One possible advantage of using rate functions instead of integration is the possibility of formulating a financial optimization problem with a tail risk measure. Such a problem will not have integration in the objective function or in the constraints. The main problem is to find the correct rate function for the given distribution and to prove that the rate function will have exact probability bounds.

We pursue further properties of *CVaR/CVaG*. *CVaR/CVaG* as defined as in (4.24) and (4.25) is *concave/convex* functions with respect to $y \in \Omega^*$. Four supporting functions are introduced for determining the values of *CVaR/CVaG*. The first two functions follow directly from Theorem 4.3:

$$F_R^1(\Omega, \mathcal{B}(\Omega), P_\chi, \alpha, y, x) = \langle y, x\rangle - \frac{1}{\alpha}\int\limits_\Omega [\langle y, x - t\rangle]^+ d\mu(t) \tag{4.29}$$

$$F_G^1(\Omega, \mathcal{B}(\Omega), P_\chi, \alpha, y, x) = \langle y, x\rangle + \frac{1}{1-\alpha}\int\limits_\Omega [\langle y, t - x\rangle]^+ d\mu(t) \tag{4.30}$$

The second two functions follow from the definition of *CVaR* in [85] i.e., (4.26) and (4.28) on page 57:

$$F_R^2(\Omega, \mathcal{B}(\Omega), P_\chi, \alpha, y, \xi) = \xi - \frac{1}{\alpha}\int\limits_\Omega [\xi - \langle y, t\rangle]^+ d\mu(t) \tag{4.31}$$

$$F_G^2(\Omega, \mathcal{B}(\Omega), P_\chi, \alpha, y, \xi) = \xi + \frac{1}{1-\alpha}\int\limits_\Omega [\langle y, t\rangle - \xi]^+ d\mu(t) \tag{4.32}$$

Then *CVaR/CVaG* can be defined as follows:

$$\begin{aligned} \mathrm{CVaR}_\alpha(\langle \mathcal{X}, y\rangle) &= \max_x \{F_R^1(\Omega, \mathcal{B}(\Omega), P_\chi, \alpha, y, x)\} = \\ &= \max_\xi \{F_R^2(\Omega, \mathcal{B}(\Omega), P_\chi, \alpha, y, \xi)\} \end{aligned} \tag{4.33}$$

$$\begin{aligned} \mathrm{CVaG}_\alpha(\langle \mathcal{X}, y\rangle) &= \min_x \{F_G^1(\Omega, \mathcal{B}(\Omega), P_\chi, \alpha, y, x)\} = \\ &= \min_\xi \{F_G^2(\Omega, \mathcal{B}(\Omega), P_\chi, \alpha, y, \xi)\} \end{aligned} \tag{4.34}$$

The properties of F_R^1 and F_R^2 are formulated as the following theorem. Analogous properties can be formulated for F_G^1 and F_G^2 by replacing the word *concave* by the word *convex*.

Theorem 4.4. *F_R^1 and F_R^2 as a function of multiple variables have the following properties:*

$$F_R^1(\,\cdot\,,\cdot\,,\cdot\,,x)\,\text{is concave}\,\forall x\in\Omega; \tag{4.35}$$

$$F_R^1(\,\cdot\,,\cdot\,,\cdot\,,y,\cdot\,)\,\text{is concave}\,\forall y\in\Omega^* \tag{4.36}$$

$$F_R^1(\,\cdot\,,\cdot\,,\alpha,\cdot\,,\cdot\,)\,\text{is monotonic, non decreasing}\,\forall\alpha\in(0,1) \tag{4.37}$$

$$F_R^1(\,\cdot\,,P_\chi,\cdot\,,\cdot\,,\cdot\,)\,\text{is linear function} \tag{4.38}$$

$$F_R^1(\Omega,\cdot\,,\cdot\,,\cdot\,,x)\,\text{is jointly concave} \tag{4.39}$$

$$F_R^2(\,\cdot\,,\cdot\,,\cdot\,,\cdot\,,\xi)\,\text{is concave}\,\forall\xi\in\mathbb{R}^1 \tag{4.40}$$

$$F_R^2(\,\cdot\,,\cdot\,,\cdot\,,y,\xi)\,\text{is jointly concave}\,\forall y\in\Omega^*,\forall\xi\in\mathbb{R}^1 \tag{4.41}$$

$$F_R^2(\,\cdot\,,\cdot\,,\alpha,\cdot\,,\cdot\,)\,\text{is monotonic, non decreasing}\,\forall\alpha\in(0,1) \tag{4.42}$$

$$F_R^2(\,\cdot\,,P_\chi,\cdot\,,\cdot\,,\cdot\,)\,\text{is linear function} \tag{4.43}$$

$$F_R^2(\Omega,\cdot\,,\cdot\,,\cdot\,,\xi)\,\text{is jointly concave function}\,\forall\xi\in\mathbb{R}^1 \tag{4.44}$$

Proof.

Property (4.35) follows from the property (4.40) by the following arguments:

$(4.40) \Longrightarrow (4.35)$

$$F_R^2(\,\cdot\,,\cdot\,,\cdot\,,\cdot\,,\gamma\xi_1+(1-\gamma)\xi_2)\geqslant$$
$$\geqslant\gamma F_R^2(\,\cdot\,,\cdot\,,\cdot\,,\cdot\,,\xi_1)+(1-\gamma)F_R^2(\,\cdot\,,\cdot\,,\cdot\,,\cdot\,,\xi_2);\ \forall\gamma\in(0,1)$$

$\Longrightarrow \exists x_1\,,x_2\in\Omega\ \mid\ \xi_1=\langle y,x_1\rangle,\ \ \xi_2=\langle y,x_2\rangle,$

$\gamma\xi_1+(1-\gamma)\xi_2=\langle y,\gamma x_1+(1-\gamma)x_2\rangle$ because of the fixed y.

Property (4.36) follows form the property (4.41) as a direct consequence because one parameter in (4.41) has been fixed.

Property (4.37) follows directly from the function definitions (4.29).

Property (4.38) follows directly from the function definitions (4.29).

Property (4.39) follows from the property (4.44) by applying the same arguments as in (4.40) $\Longrightarrow$ (4.35).

Property (4.40) follows directly from *Theorem 10* and *Corollary 11* [86].

Property (4.41) follows directly from *Theorem 10* and *Corollary 11* [86].

Property (4.42) follows directly from the function definitions (4.31).

Property (4.43) follows directly from the function definitions (4.31).

Property (4.44) follows directly from *Theorem 10* and *Corollary 11* [86]. □

The results of Theorem 4.4 will be used in the process of constructing the approximation algorithm. Theorem 4.4 summarizes all properties of the underlying functions for CVaR definition. The next important properties of CVaR/CVaG are their properties with respect to vector of random outcomes $\mathcal{X}$. Assume that two random vectors $\mathcal{X}^1$ and $\mathcal{X}^2$ defined on the same domain Ω. Let's define the convex combination of these random vectors as follows:

$$\mathcal{X} = \lambda\, \mathcal{X}^1 + (1-\lambda)\, \mathcal{X}^2 =$$
$$= \{\Omega, \lambda\, \mathcal{F}_{\mathcal{X}^1}(\Omega) + (1-\lambda)\, \mathcal{F}_{\mathcal{X}^2}(\Omega), \lambda\, \mathcal{P}(\mathcal{F}_{\mathcal{X}^1}(\Omega)) + (1-\lambda)\, \mathcal{P}(\mathcal{F}_{\mathcal{X}^2}(\Omega))\}$$

where convex combination operator is applied separately to locations and probability measures.

Pflug in [78] derives the properties of CVaR as a function of a univariate random variable, see section 2.2.7.2, that is either the final profit or final loss in financial optimization. Because the approximation problem will work with the domain of original outcomes, we extend Pflug's result in the following proposition.

Proposition 4.5. *For a fixed portfolio y and a fixed probability level α, CVaR is concave and CVaG is convex with respect to the random vector $\mathcal{X}$:*

$$0 \leqslant \lambda \leqslant 1$$

$$\mathrm{CVaR}_\alpha(\langle \lambda\, \mathcal{X}^1 + (1-\lambda)\, \mathcal{X}^2, y \rangle) \geqslant$$
$$\geqslant \lambda\, \mathrm{CVaR}_\alpha(\langle \mathcal{X}^1, y \rangle) + (1-\lambda)\, \mathrm{CVaR}_\alpha(\langle \mathcal{X}^2, y \rangle) \qquad (4.45)$$

$$\mathrm{CVaG}_\alpha(\langle \lambda\, \mathcal{X}^1 + (1-\lambda)\, \mathcal{X}^2, y \rangle) \leqslant$$

$$\leqslant \lambda \,\mathrm{CVaG}_\alpha(\langle \mathcal{X}^1, y\rangle) + (1-\lambda)\,\mathrm{CVaG}_\alpha(\langle \mathcal{X}^2, y\rangle) \tag{4.46}$$

Proof. The following statement:

$$\mathrm{CVaR}_\alpha\big(\lambda \langle y, \mathcal{X}^1\rangle + (1-\lambda)\langle y, \mathcal{X}^2\rangle\big) \geqslant$$
$$\geqslant \lambda\,\mathrm{CVaR}_{\alpha,y}\big(\langle y, \mathcal{X}^1\rangle\big) + (1-\lambda)\,\mathrm{CVaR}_{\alpha,y}\big(\langle y, \mathcal{X}^2\rangle\big)$$

is correct for a fixed y and α based on Pflug [78] result. Thus, $\mathrm{CVaR}_\alpha(\langle \mathcal{X}, y\rangle)$ is a combination of two functions: the first is concave and non-decreasing $\mathrm{CVaR}_\alpha(\tilde{\mathcal{X}})$ and the second is linear $\tilde{\mathcal{X}} = \langle y, \mathcal{X}\rangle$. Combination of a non-decreasing concave function and a linear function is concave due to Rockafellar [84, Section 5]. Inequality for *CVaG* follows from (4.21) and linearity of $E[\langle y, \mathcal{X}\rangle]$ with respect to $\mathcal{X}$. □

4.6 Class of risk measures based on CVaR/CVaG

Suppose the investor expresses his/her preferences in terms of trade-off between favorable and unfavorable events. Then the risk measure for such investor can be presented as follows:

$$-\rho(\mathcal{Z}) = w_1\,\mathrm{CVaR}_\alpha(\mathcal{Z}) + w_2\,\mathrm{CVaG}_\alpha(\mathcal{Z}) \tag{4.47}$$

We assume that investor is rational. Thus, using the rational risk measures is appropriate. Based on (4.21), we can rewrite (4.47) as follows:

$$-\rho(\mathcal{Z}) = w_1\,\mathrm{CVaR}_\alpha(\mathcal{Z}) +$$
$$+\left(\frac{1-\alpha}{\alpha}w_1\right)\mathrm{CVaG}_\alpha(\mathcal{Z}) - \left(\frac{1-\alpha}{\alpha}w_1 - w_2\right)\mathrm{CVaG}_\alpha(\mathcal{Z}) =$$

$$= \frac{w_1}{\alpha}\,E[\mathcal{Z}] - \left(\frac{1-\alpha}{\alpha}w_1 - w_2\right)\mathrm{CVaG}_\alpha(\mathcal{Z}) \tag{4.48}$$

or alternatively:

$$-\rho(\mathcal{Z}) = \left(\frac{\alpha}{1-\alpha}w_2\right)\mathrm{CVaR}_\alpha(\mathcal{Z}) +$$

$$+\left(w_1 - \frac{\alpha}{1-\alpha} w_2\right) \mathrm{CVaR}_\alpha(\mathcal{Z}) + w_2\, \mathrm{CVaG}_\alpha(\mathcal{Z}) =$$

$$= \frac{w_2}{1-\alpha}\, E[\mathcal{Z}] + \left(w_1 - \frac{\alpha}{1-\alpha} w_2\right) \mathrm{CVaR}_\alpha(\mathcal{Z}) \tag{4.49}$$

From the definition of RRM (RRM axiom 1, see page 37) we know that $\rho(\mathcal{Z})$ need to be convex. Thus, $-\rho(\mathcal{Z})$ is concave. The condition for $\rho(\mathcal{Z})$ to be rational is as follows:

$$(1-\alpha)\, w_1 \geqslant \alpha\, w_2 \tag{4.50}$$

because *CVaR* is concave and *CVaG* is convex functions of *r.v.* $\mathcal{Z}$ (Proposition 4.5).

If equality is held in (4.50) then:

$$-\rho(\mathcal{Z}) = w_1\, \mathrm{CVaR}_\alpha(\mathcal{Z}) +$$

$$+\left(\frac{1-\alpha}{\alpha} w_1\right) \mathrm{CVaG}_\alpha(\mathcal{Z}) - \left(\frac{1-\alpha}{\alpha} w_1 - w_2\right) \mathrm{CVaG}_\alpha(\mathcal{Z}) =$$

$$= \frac{w_2}{1-\alpha}\, E[\mathcal{Z}] = \frac{w_1}{\alpha}\, E[\mathcal{Z}] \tag{4.51}$$

Therefore, the risk measure in this case simplifies to the negative value of expectation or alternatively risk measure has only a target component and no deviation measure. This is the case of risk neutrality according to [58].

From a practical standpoint, an investor assigns higher weights to the lower tail in order to be *risk averse*. The representations (4.48)-(4.51) give perfect insight for understanding the nature of risk measures based on *CVaR* and *CVaG*. They also provide the two alternative ways of presenting a risk measure in mathematical formulation of the financial optimization problem. The two alternatives are equivalent and the investor can chose the formula that minimizes computational roundoff error. Now, it is clear that risk aversion can be expressed in the *CVaR* form. Moreover, the degree of risk aversion remains the same for different random variables.

Chapter 5

Scenario Generation and Approximation

5.1 Preliminaries

As discussed in the earlier chapters, a stochastic programming model for financial investment decisions requires an explicit specification of uncertainty in order to compute risk characteristics of future portfolio returns. When multistage investment models are formulated, under a suitable discretization of time, a specific decision tree has to be constructed to depict the underlying uncertainty. In fact, the number of nodes in the decision tree is determined by the degree of discretization of the future uncertainty for individual asset returns coupled with the number of periods in the planning horizon. Therefore, with nodes being nested to form a multiperiod decision tree, and with large sample of outcomes per node, the size of the resulting multistage stochastic program becomes very large. The solution of such a large multistage stochastic program is computationally very tedious. For this reason, it is customary to start with a large enough (discrete) sample of individual asset returns per node, and then use an appropriate approximation technique to summarize the sample with a limited number of outcomes.

In this chapter, it is assumed that the investor solves the financial investment problem using *CVaR* or *CVaG* as a tail risk measure, see Rockafellar and Uryasev [88], [89], and Section 4.6 of this thesis. *CVaR* or *CVaG* are used as a risk measure in the objective function or in the constraints, see the problem formulation in [85], for instance. This implies that probability levels α for which *CVaR*/*CVaG* will be optimized are known. Suppose that a large sample of outcomes for future asset returns is available. This large sample may have been generated randomly under certain distributional assumptions. In the sequel, it is assumed that probability for each sample point is known. However, large number of scenarios can be redundant for portfolio optimization problem and it increases the solution time. The goal is to construct an approximate sample with fewer scenarios than in the original sample. The approximated sample needs to preserve important information about the original sample. The difference between solutions of two investment problems, one with the original sample and another with the approximated sample, needs to be insignificant.

There are many techniques for scenario (sample) approximation in the literature: random (or importance) sampling based techniques that relies on asymptotic properties, see Dantzig and Glynn [19], Ermoliev and Wets [34], Higle and Sen [49], or distributional approximations that strive for preserving the specified moment information of the original large sample, see Birge and Wets [10], Dula [22], Edirisinghe [23], [29], [30], [31], and [33], Frauendorfer [41], Gassman and Ziemba [44], and Pflug [79]. The latter approach typically relies on a solution of the so-called Generalized Moment Problem (GMP). A solution is determined using the underlying properties of the objective function of the stochastic program. In this chapter, a brief description of the GMP-based technique for sample approximation is provided and then conclusions are drawn as to why that method is generally undesirable for the financial optimization problems with tail risk measures. Later in this chapter, a new scenario approximation procedure is developed that will be more appropriate for financial stochastic programs, in particular for those with tail risk measures such as $CVaR/CVaG$.

5.1.1 Generalized moment problem (GMP)

The GMP was first addressed by Kemperman [59] in the context of developing bounds for expectation of a function. Let $f(\,\cdot\,)$ be the function of the *r.v.* $\mathcal{X}$: $\{\Omega, \mathcal{F}(\Omega), \mathcal{P}(\mathcal{F}(\Omega))\}$. Then *generalized moment problem* (GMP) is a problem of evaluating the following integral:

$$f(h(\mathcal{X})) = \int_{\Omega} h(x) d\,F_1(\mathcal{X}, x) \tag{5.1}$$

where $h(x)$ is a following functional:

$$h\colon \Omega \rightarrow \mathbb{R}^1 \tag{5.2}$$

$F_1(\mathcal{X}, \cdot)$ is unknown but moment information about probability measure $\mathcal{P}(\mathcal{F}(\Omega))$ is available. This information is presented via functions g_1, ..., g_n that are Borel measurable functions with respect to $\mathcal{F}(\Omega)$. So, the moments $f(g_i) = v_i, i = 1...n$ are known. Kemperman in [59] claims that this problem is unsolvable in its present form and proposes to evaluate the upper and lower bounds of the integral (5.1):

$$U(f) = U(f|h) = \sup_{\mu} f(h)$$

$$L(f) = L(f|h) = \inf_{\mu} f(h)$$

where μ is varied through all possible probability measures for $\mathcal{F}(\Omega)$ with restriction of $f(g_i) = v_i$ for $j = 1...n$.

Kemperman defines the upper and lower limits as hyperplanes. It is difficult and sometimes impossible to find the lower or upper bound for the formulated problem because of the unbounded domain. The achievement of Kemperman [59] is in defining the minimal requirements for lower and upper bounds to exist and in finding the way to express the bounds through the available moment information. Kemperman [59] uses basic topology, results later confirmed by Karr [57], and the linear structure of the given moment information $\{v_i\}$ to prove that lower and upper bounds lay between two limiting hyperplanes. Thus, the problem of multidimensional integration has been reduced to the problem of finding two limiting hyperplanes. The idea of solving several linear optimization problems instead of a non-linear problem is computationally useful. GMP approach has been applied for solving stochastic linear programs with resources. This area is well developed, see [33], [41], [42], and [43], and it has been adapted to the financial investment problems.

The problem of financial investment over two periods is formulated as follows:

$$\max_{y^1} \; E\left[f^1(x_1, y^1) + \max_{y^2}\{ f^2(x_2, y^2)\}\right] \tag{5.3}$$

$$s.t. \qquad A\,y^1 \qquad = b \tag{5.4}$$

$$Ty^1 + Wy^2 = h \tag{5.5}$$

$$y^1, y^2 \geqslant 0 \tag{5.6}$$

where y^1 is the decision on stage 1 when no future information is available, y^2 is the decision on stage 2 when information about *r.v.* x_1 is available, $f(\,\cdot\,, y)$ is a linear function. This is a problem of linear stochastic programming with resources, see [29] and [30] for details. The application of Kemperman's approach to the financial investment problem by evaluating the upper and lower bound of the specified problem rely on the first and mixed moment information, see [10], [22], [29], [33], and [44].

For instance, Edirisinghe and Ziemba [30] apply GMP result to solve problem (5.3)-(5.6) if function $f^t(x_t, \cdot\,)$ is linear in the decision variables $\{y^1, y^2\}$. The main idea is to find upper and lower bounds on the original problem and improve bounds iteratively until the bounds satisfy a required tolerance. In the course of developing an approximation using this approach, Edirisinghe and Ziemba make several significant improvements:

- Edmudson-Madansky [65] upper bound for expectation of a convex function is improved by deriving Edirisinghe and Ziemba [30] upper bound.
- Mixed moments are incorporated in the procedure of bounds evaluation.
- Approximation problem is reformulated and problem solution gives both optimal investment strategy y and optimal scenarios approximation x.

Other approaches for bounding approximations can be found in [10] and [44]. However, these approaches imply solving the original problem and creating an approximation sample at the same time. The goal of the algorithm is to generate a sample and to embed the sample into the financial investment problem.

5.1.2 Approximation using mixed moment information

Edirisinghe in [23] addresses the problem of separating the sample approximation and bounding procedures, where a technique of approximating scenarios using a simplicial domain is developed. Let Ω be the set of scenarios and Ω^* the available decisions.

The problem is to find the approximating distribution $\mathcal{X}^a = \{x_s, p_s\}_{s=1}^S$ which matches the first moments of the original distribution $\mathcal{X}$ exactly and incorporates the mixed moment information as much as possible.

$$E[\mathcal{X}^a] = E[\mathcal{X}] \tag{5.7}$$

$$E[\mathcal{X}_i^a \mathcal{X}_j^a] \approx E[\mathcal{X}_i \mathcal{X}_j] \tag{5.8}$$

Edirisinghe [23] develops two solution schemes. The first scheme is for random variable with a simplicial bounded domain. The second scheme is for random variable with a bounded rectangular domain. Both approximation schemes have the same theoretical foundations. However, the first scheme is more appropriate because it allows finding a simplicial bound for a given convex, bounded set.

Edirisinghe and You in [33] give the implementation scheme for the first step as finding the simplex Sim such that

$$Sim \supseteq \text{conv}\Big(\{x_i\}_{i=1}^N\Big) \tag{5.9}$$

convex hull of approximation points must include the convex hull of the original distribution $\mathcal{X}$. However, the procedure in the first step is not ideal and simplex Sim can be reduced. Thus, authors propose as the second step to shrink the simplex to $W_a = \{x_s\}_{s=1}^S$. During the second step, Edirisinghe and You use the linear programming technique. The solution of formulated linear program is a new set of approximating points W_a. Unfortunately, the volume of the produced simplex is not minimal because of the trade-off between non-linear and linear formulation.

The proposed scheme of approximation is applicable for financial optimization. Risk measures are highly dependent on the first moment of the underlying scenarios and only slightly on the mixed and second moments. Thus, the approach of preserving first moment information exactly and second moment information with slight variation is justified from practical stand point. This approach according to Edirisinghe [23] generates better upper bound then an approximation using the first moments only. The above scheme is also interesting because it approximates a given convex hull with a simplex and it will be useful later in this thesis for tail approximation problem.

5.1.3 Scenario approximation using tail information

All of the above techniques are applicable if the risk measure in financial optimization problem is based on moment information, for instance Markowitz's mean/variance risk measure. The question of approximating a given sample for financial optimization problem with a tail risk measure (VaR, CVaR, semi-deviation) is an open question. There is no research found in the literature. All of the above methods are inappropriate for such approximation because they do not use tail information. This chapter of my thesis develops the theoretical foundation and an algorithm for approximating the *r.v.* sample for financial optimization problem with a risk measure based on CVaR/CVaG information.

5.2 Formulation of approximation problem

Scenario approximation problem is formulated as follows. The original distribution $\mathcal{X}$ is given. A discrete approximation needs to be defined. CVaR of the approximation needs to be as close as possible to the CVaR of the original distribution for the given probability levels $\{\alpha_k\}$.

Consider the following:

- $\Omega^U \supseteq \Omega$ is a bounded domain on which we will define the values of approximation distribution $\{x_s\}_{s=1}^S$.
- Ω^* is a bounded domain of all possible portfolios y.
- $\mathcal{X} = \{\Omega, \mathcal{B}_\mathcal{X}(\Omega), P_\mathcal{X}\}$ is a random variable representing the original distribution.
- $\mathcal{X}^a = \{x_s,\, p_s\}_{s=1}^S$ is a random variable representing the approximation distribution, where $\text{conv}(\{x_s\}_{s=1}^S) \subset \text{conv}(\Omega^U)$.
- $\{\alpha_k\}_{k=1}^K$ is a set of probability levels. *CVaR* of these levels will be used in the investment model. These levels are specified a priori. K is finite.

Now, we need a criteria for evaluating the precision of the approximating distribution $\mathcal{X}^a$ with respect to the original distribution $\mathcal{X}$. Approximation is good if *CVaR* of approximating distribution will never exceed *CVaR* of original distribution for all portfolios in the feasible domain $y \in \Omega^*$ and *CVaG* of approximated distribution will never be less than *CVaG* of original distribution. In other words, we do not want to overestimate the lower tail expectations (overestimate the profit of unfavorable scenarios). We also do not want to underestimate the upper tail expectations (underestimate the profit of favorable scenarios). These requirements can be formulated as following constraints:

$$\text{For } \forall y \in \Omega^*, \text{and } \alpha \in \{\alpha_k\}_{k=1}^K$$

$$CVaR_\alpha(\langle \mathcal{X}^a,\ y\rangle) \leq CVaR_\alpha(\langle \mathcal{X},\ y\rangle) \tag{5.10}$$

$$CVaG_\alpha(\langle \mathcal{X}^a,\ y\rangle) \geq CVaG_\alpha(\langle \mathcal{X}, y\rangle) \tag{5.11}$$

From (4.21) page 55 in Chapter 4, the relation between upper and lower tail expectations is as follows:

$$E[\langle \mathcal{X}, y\rangle] = \alpha \cdot CVaR_\alpha(\langle \mathcal{X},\ y\rangle) + (1-\alpha) \cdot CVaG_\alpha(\langle \mathcal{X},\ y\rangle). \tag{5.12}$$

Thus, (5.11) can be substituted with the following constraint:

$$E[\mathcal{X}^a] = E[\mathcal{X}] \tag{5.13}$$

According to (5.12), the inequality (5.11) will be automatically satisfied if (5.10) and (5.13) hold.

Also the CVaR's of approximated distribution need to be as close as possible to the CVaR's of original distribution. This is the criteria for evaluating the effectiveness of approximation. We construct the objective function that measures the distance between $\mathcal{X}^a$ and $\mathcal{X}$ in terms of tail expectations for fixed sequence $\{\alpha_k\}$ and all portfolios $y \in \text{bound}(\text{conv}(\Omega^*))$. We propose to use the following function:

$$G(\mathcal{X}^a) \equiv \sum_k \int_{y \in \text{bound}(\text{conv}(\Omega^*))} [CVaR_{\alpha_k}(\langle \mathcal{X},\ y\rangle) - CVaR_{\alpha_k}(\langle \mathcal{X}^a,\ y\rangle)]dy \tag{5.14}$$

Function (5.14) is convex because of convexity of the integrated expression for fixed portfolio y and probability level α. *CVaR* is a concave function with respect to portfolio profit, see [78]. Superposition of linear and concave functions is concave, see [84]. Function (5.14) is also positive for all $\mathcal{X}^a$ satisfying (5.10) and (5.12). $G(\mathcal{X}^a)$ is not a norm because it lacks positive homogeneity and symmetry. However, it can be used as a distance measure because it is non-negative and satisfies the triangular inequality.

5.3 Scenario generation model (SGM)

Now we can formulate the problem of finding the approximation distribution mathematically as follows:

$$G(\mathcal{X}^a) \equiv \min_{\mathcal{X}^a} \quad \sum_k \int_y [CVaR_{\alpha_k}(\langle \mathcal{X},\ y\rangle) - CVaR_{\alpha_k}(\langle \mathcal{X}^a,\ y\rangle)] \tag{5.15}$$

$$s.t. \quad CVaR_{\alpha}(\langle \mathcal{X},\ y\rangle) \geq CVaR_{\alpha}(\langle \mathcal{X}^a,\ y\rangle), \quad \forall \alpha \in \{\alpha_k\}_{k=1}^K \text{ and } \forall y \in \Omega^* \tag{5.16}$$

$$E[\mathcal{X}^a] = E[\mathcal{X}] \tag{5.17}$$

$$\mathcal{X}^a = \{\{x_s, p_s\}_{s=1}^S, x_s \in \Omega^a, \{p_s\} \in P^a\}$$

Using the preceding results (Theorem 4.4) and referring to Rockafellar [84], it follows that CVaR_{α_k} $(\langle \mathcal{X}^a,\ y\rangle)$ is concave as a function of $\mathcal{X}^a$. Therefore, the constraints in (5.16) generate a non-convex domain. Thus, we have a problem with a convex objective function but with a non-convex feasible domain. This problem is extremely difficult to solve. A trivial lower bound for (5.15) is at zero, when approximation coincides with the original distribution (approximation without errors).

We propose the following approach to find a feasible solution for the above problem. First, we simplify the problem to eliminate the integration in (5.15). The simplified problem is then divided into sub-problems which are linear. We design an algorithm to find a quasi-optimum solution of the simplified problem. Finally, using *CVaR* and *CVaG* properties we establish a feasible solution of the scenario approximation problem. As a result, we get $\mathcal{X}^a$ that satisfies (5.16)-(5.17). Computationally, we verify that such an $\mathcal{X}^a$ is a good approximation to (5.16)-(5.17). However, this will not necessarily be true in general.

5.4 Simplified SGM

In order to simplify the problem (5.15)-(5.17) we fix three parameters:

1. S the number of points in the approximation distribution.
2. $\{y_i\}_{i=1}^I$ is set of fixed portfolios with the requirement that $\Omega^* \subseteq \operatorname{conv}\big(\{y_i\}_{i=1}^I\big)$. We assume that $0 \in \operatorname{conv}\big(\{y_i\}_{i=1}^I\big)$.
3. $I_{\mathrm{SaR}} = \big\{I_{SaR_{i,k}}\big\}$ is set of indices that refers to points in Scenarios-at-Risk set. It is an index set (among points in $\mathcal{X}^a$) that specifies for each pair (α_k, y_i) the points of $\mathcal{X}^a$ that belong to $S\,a\,R_{i,k}(\mathcal{X}^a) = S\,a\,R_{\alpha_k}(\langle \mathcal{X}^a, y_i\rangle)$.

The first parameter is easy to fix because the number of points in the approximating distribution is usually defined by decision maker. The difficult problem is to determine $\{y_i\}_{i=1}^I$ because $\operatorname{conv}\{\{y_i\}_{i=1}^I\}$ needs to contain the original domain Ω^* and be as small as possible to avoid approximation errors. Domain Ω^* is bounded. Therefore, we can solve the problem of finding $\{y_i\}_{i=1}^I$ in three ways:

1. Using "simplex approximation", see [33] for details.
2. Using "box approximation" by projecting all points into orthogonal hyperplanes.
3. Assuming $0 \in \Omega^*$. Set:

$$\rho_i^+ \geq \max_{y \in \Omega^*} y_i \qquad \text{and} \qquad \rho_i^- \leq \max_{y \in \Omega^*} y_i$$

   Then $\Omega^* \in \operatorname{conv}\big\{\{\rho_i^+ e_i, -\rho_i^- e_i\}_{i=1}^N\big\}$.

In order to index set I_{SaR}, we need an initially feasible distribution $\mathcal{X}^a$. Once such a distribution is in hand, we can find $I_{SaR_{i,k}}$.

Denote by $D(I_{SaR})$ the set of all approximating distributions $\mathcal{X}^a$ which has the same index sets I_{SaR} for specified α_k and y_i. The set of such distributions is convex because restricting $\mathcal{X}^a$ to $D(I_{SaR})$ induces a finite number of linear constraints on points $\{x_s\}$ and probability measure $\{p_s\}$ of $\mathcal{X}^a$.

Consequently, we can reformulate the simplified SGM as follows:

$$\min_{\mathcal{X}^a} \sum_{\alpha_k, y_i} \Big[CVaR_{\alpha_k}(\langle \mathcal{X},\ y_i\rangle) - CVaR_{\alpha_k}(\langle \mathcal{X}^a,\ y_i\rangle)\Big] \tag{5.18}$$

$$\begin{aligned} s.t. \quad & CVaR_{\alpha}(\langle \mathcal{X}^a,\ y_i\rangle) \leqslant CVaR_{\alpha}(\langle \mathcal{X},\ y_i\rangle), \\ & \forall \alpha \in \{\alpha_k\}_{k=1}^K \text{ and } \forall y \in \Omega^* \end{aligned} \tag{5.19}$$

$$\mathcal{X}^a \in D(I_{SaR}) \tag{5.20}$$

$$E[\mathcal{X}^a] = E[\mathcal{X}] \tag{5.21}$$

This problem has convex objective function but non-convex domain. The simplified problem can have several local optima. It is impossible to check the optimality of a feasible solution because *Constraints Qualification* conditions may not be satisfied (the domain is non-convex, see [5] for details about *constraint qualification*). This problem is non-linear in $\mathcal{X}^a$. We propose a method of finding a quasi-minimum using an iterative procedure based on linear programming techniques.

5.5 Solving the simplified SGM

The main idea of the proposed method is to convert the non-convex area of feasible $\mathcal{X}^a$ into a convex region. We start with any feasible solution $S^a = \{x_s,\ p_s\}_{s=1}^S$ for problem (5.18)-(5.21). Then approximation problem can be formulated in a linear form either by fixing distribution points or by fixing probability measure. Each of the two formulations is linear and can be solved efficiently. The iterative process of solving the two problems is the core of the proposed algorithm. We will prove that this iterative process converges. Using the original distribution $\mathcal{X}$, we can define the following values:

- $CVaR_{i,k}$ the values of $CVaR_{\alpha_k}(\langle \mathcal{X}, y_i\rangle)$ of original distribution.

Then we can formulate the problem of finding an approximating distribution with fixed probability measure $\{p_s\}$ as follows:

$$
\begin{aligned}
f_p^1(x) \equiv \min_{x_s} \quad & \sum_{\alpha_k, y_i} \Big[CVaR_{\alpha_k}(\langle \mathcal{X},\ y_i\rangle) - CVaR_{\alpha_k}(\langle \mathcal{X}^a,\ y_i\rangle) \Big] \\
s.t. \quad & CVaR_{\alpha}(\langle \mathcal{X}^a,\ y_i\rangle) \leqslant CVaR_{\alpha}(\langle \mathcal{X},\ y_i\rangle) \\
& \qquad \forall \alpha \in \{\alpha_k\}_{k=1}^K \text{ and } \forall y \ \in \{y_i\}_{i=1}^I \\
& \mathcal{X}^a \in D(I_{\mathrm{SaR}}) \\
& E[\mathcal{X}^a] = E[\mathcal{X}]
\end{aligned}
\tag{5.22}
$$

We can also formulate the problem of finding an approximating distribution with fixed points $\{x_s\}$ as follows:

$$
\begin{aligned}
f_x^2(p) \equiv \min_{p_s} \quad & \sum_{\alpha_k, y_i} \Big[CVaR_{\alpha_k}(\langle \mathcal{X},\ y_i\rangle) - CVaR_{\alpha_k}(\langle \mathcal{X}^a,\ y_i\rangle) \Big] \\
s.t. \quad & CVaR_{\alpha}(\langle \mathcal{X}^a,\ y_i\rangle) \leqslant CVaR_{\alpha}(\langle \mathcal{X}, y_i\rangle) \\
& \qquad \forall \alpha \in \{\alpha_k\}_{k=1}^K \text{ and } \ \forall y \in \{y_i\}_{i=1}^I \\
& \mathcal{X}^a \in D(I_{\mathrm{SaR}}) \\
& E[\mathcal{X}^a] = E[\mathcal{X}]
\end{aligned}
\tag{5.23}
$$

Both problems have convex domains because $\mathrm{CVaR}_{\alpha_k}(\langle \mathcal{X}^a,\ \ y_i\rangle)$ for fixed probability measure and $S\,a\,R$ can be written as a linear function. The same is true for $\mathrm{CVaR}_{\alpha_k}(\langle \mathcal{X}^a,\ \ y_i\rangle)$ for fixed points of distribution $\{x_s\}$ and I_{SaR}. We solve these two problems iteratively. The optimal solution of one problem is the input data for another and vise versa. The cycle is terminated when either objective function stops to improve or growth rate of objective function becomes below numerical round-off error. Convergence of the cycle in a finite number of steps is claimed in the following proposition.

Proposition 5.1. *For two consecutive steps (5.22) and (5.23) of the cycle, the following inequalities are satisfied:*

$$f_{x^i}^2(p^{i+1}) \leqslant f_{p^i}^1(x^i)$$

or:

$$f^2_{x^i}(p^i) \geqslant f^1_{p^i}(x^{i+1})$$

Proof. Let's mark:

$$\triangle CVaR_{k,i} = CVaR_{\alpha_k}(\langle \mathcal{X},\ y_i \rangle) - CVaR_{\alpha_k}(\langle \mathcal{X}^a,\ y_i \rangle)$$

Also, suppose that $f^1_p(x)$ is optimized first. Then:

$$f^1_p(\hat{x}) \equiv \min_{x_s} \sum_{k,i} \triangle CVaR_{k,i}$$

which deliver the optimal solution $\hat{x}$ and

$$f^2_{\hat{x}}(\hat{p}) \equiv \min_{p_s} \sum_{k,i} \triangle CVaR_{k,i}$$

$$p \in \boldsymbol{P}$$

where $\boldsymbol{P}$ is convex area defined by constraints in (5.22). Then:

$$p, \hat{p} \in \boldsymbol{P}$$

Thus, it follows that:

$$f^1_p(\hat{x}) \geqslant f^2_{\hat{x}}(\hat{p})$$

because $D(I_{\text{SaR}})$ remains the same from iteration to iteration. The second inequality of **proposition** can be proved analogously using the fact that

$$x^i, x^{i+1} \in \boldsymbol{X}$$

where $\boldsymbol{X}$ is a convex set of feasible points defined by the problem (5.23). □

Using the results of Proposition 5.1 we conclude that objective functions $f^1_p(x)$ and $f^2_x(p)$ in the iteration cycle are non-increasing. The objective functions in the iterative cycle are also bounded from below by 0. This guarantees that the cycle stops in a finite number of step by either reaching 0 or stopping objective functions improvement. Thus, we find a quasi-optimum solution for problem (5.18) - (5.21). It is not an optimal solution because additional restrictions on $\mathcal{X}^a$ have been imposed. We term it quasi-optimal because solution is optimal for two sub-problems. The solution is optimal for formulation with fixed $\{x\}$ when $\{p\}$ are parameters and for formulation with fixed $\{p\}$ when $\{x\}$ are parameters.

5.6 Feasible solution for SGM

The next step is to convert $\mathcal{X}^a$ into a feasible solution for (5.15)-(5.17). The approximating distribution may violate the requirements (5.16):

$$CVaR_\alpha(\langle \mathcal{X}^a,\ y_i\rangle) \leqslant CVaR_\alpha(\langle \mathcal{X}, y_i\rangle)$$

$$\forall \alpha \in \{\alpha_k\}_{k=1}^K \quad \text{and} \quad \forall y \in \operatorname{conv}(\{y_i\}_{i=1}^I)$$

This requirements can be checked using the following proposition.

Proposition 5.2. *Conditions*

$$CVaR_\alpha(\langle \mathcal{X}^a, y\rangle) \leqslant CVaR_\alpha(\langle \mathcal{X}, y\rangle); \qquad \forall y \in \operatorname{conv}\big(\{y_i\}_{i=1}^I\big)$$

are satisfied if and only if:

$$CVaR_\alpha(\langle \mathcal{X}^a, y\rangle) \leqslant CVaR_\alpha(\langle \mathcal{X}, y\rangle); \qquad \forall y \in \operatorname{bound}\big(\operatorname{conv}\big(\{y_i\}_{i=1}^I\big)\big)$$

Proof. According to the first condition:

$$\forall y \in \operatorname{conv}(\{y_i\}_{i=1}^I) : \exists\, \tilde{y} \in \operatorname{bound}(\operatorname{conv}(\{y_i\}_{i=1}^I)) \rightarrow y = \lambda\, \tilde{y};\ \lambda \in [0,1]$$

Due to the result in Proposition 4.2, *CVaR* is *positively homogeneous* with respect to portfolio y. Therefore, the following hold:

$$\begin{aligned} & CVaR_\alpha(\langle \mathcal{X}^a, \tilde{y}\rangle) \leqslant CVaR_\alpha(\langle \mathcal{X}, \tilde{y}\rangle) \\ \Longrightarrow\ & \lambda CVaR_\alpha(\langle \mathcal{X}^a, y\rangle) \leqslant \lambda CVaR_\alpha(\langle \mathcal{X}, y\rangle) \\ \Longrightarrow\ & CVaR_\alpha(\langle \mathcal{X}^a, y\rangle) \leqslant CVaR_\alpha(\langle \mathcal{X}, y\rangle). \end{aligned}$$

Conversion of the above implications is trivial, that is:

$$\begin{aligned} & CVaR_\alpha(\langle \mathcal{X}^a, y \rangle) \leqslant CVaR_\alpha(\langle \mathcal{X}, y \rangle) \\ \Longrightarrow & \lambda CVaR_\alpha(\langle \mathcal{X}^a, y \rangle) \leqslant \lambda CVaR_\alpha(\langle \mathcal{X}, y \rangle) \\ \Longrightarrow & CVaR_\alpha(\langle \mathcal{X}^a, \tilde{y} \rangle) \leqslant CVaR_\alpha(\langle \mathcal{X}, \tilde{y} \rangle). \end{aligned}$$

The Figure 5.1 below is a graphical illustration of the above results. □

Thus, $\mathcal{X}^a$ can be corrected early to satisfy the first condition of the Proposition 5.2 by using the positively homogeneous property of *CVaR*. With all theoretical background, we can specify the algorithm for finding the approximating distribution.

5.7 Algorithm for finding an approximation

Suppose we have the following values:

1. S - number of points in the approximation distribution.

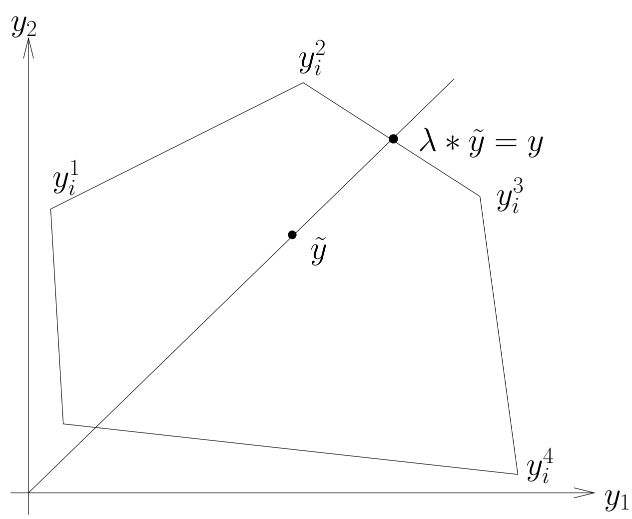

Figure 5.1. Graphical illustration of Proposition 5.2 proof

2. $\{\alpha_k\}_{k=1}^{K}$ - set of probability levels. Approximation will use only finite set of probability levels.

3. $\{y_i\}_{i=1}^{I}$ - set of fixed portfolios with the requirement $\Omega^* \subseteq \mathrm{conv}\{\{y_i\}_{i=1}^{I}\}$.

Let $S^a = \{x_s\}_{s=1}^{S}$. Using the original distribution $\mathcal{X}$ we can define the following values:

1. $CVaR_{i,k}$ the values of $CVaR_{\alpha_k}(\langle \mathcal{X}, y_i \rangle)$ of original distribution $\mathcal{X}$.

2. m is the mean of original distribution $\mathcal{X}$.

3. $l_i = \min_h \{\langle y_i, x_h \rangle \,\big|\, x_h \in \Omega\}$. The minimal return for each basic portfolio y_i.

5.7.1 Find an initially feasible solution

We propose to fix the directions from the mean. Each point of approximation distribution will be in the form $m + d_s$ where $\{d_s\}_{s=1}^{S}$ is a set of directions to be determined. Define the set of directions $\{d_s\}_{s=1}^{S}$ such that $\mathrm{conv}\big\{\{m + d_s\}_{s=1}^{S}\big\}$ covers the domain of original distribution Ω. Let $\{I_s\}_{s=1}^{S}$ be a family of partitions of I which may be overlapped. We assume that for each $s = \overline{1, S}$ set $\{y_{i_s}\}_{i_s=1}^{I_s}$ has been assigned such that $\{y_{i_s}\}_{i_s=1}^{I_s} \subset \{y_i\}_{i=1}^{I}$.

$$
\begin{array}{lll}
\min\limits_{d_s} & \sum\limits_{s} \lambda_s & \\
s.t. & d_s & = \sum\limits_{j} \lambda_{j,s}(x_j - m) \\
 & \langle y_{i_s}, m + d_s \rangle & \leqslant l_{i_s} \\
 & \lambda_{j,s} & \geqslant \emptyset \\
 & \lambda_s - \lambda_{j,s} & \geqslant \emptyset
\end{array}
$$

The problem above is separable in each $s \in S$. Therefore, it can be effectively parallelized. Each direction is associated with one point of approximation distribution

$$x_s = m + d_s$$

The next step is to define the initial probability measure to satisfy the constraint (5.21).

$$\begin{aligned} \max_{p_s} \quad & p & \\ s.t. \quad & p & \leqslant p_s \\ & \sum_s p_s & = 1 \\ & \sum_s d_s p_s & = \emptyset \\ & p_s & \geqslant \emptyset \end{aligned}$$

This step guarantees existence of solution which satisfy (5.21). Probability measure and directions need to be defined because of requirement of fixing SaR and either probability measure or point coordinates for solving procedure.

5.7.2 Define optimization parameters

Compute the values of $CVaR_{i,k}^a = CVaR_{\alpha_k}(\mathcal{X}^a, y_i)$. Also define $SaR_{i,k}$ sets. The boundary points for each combination (α_k, y_i) (the index of split atoms, [86]) is defined as $SaR_{i,k}^b$. If we have:

$$\exists \alpha_k, y_i: CVaR_{\alpha_k}(\langle \mathcal{X}^a, y_i \rangle) \geqslant E[\mathcal{X}^a] \tag{5.24}$$

then problem has no feasible solution in this SaR partition and number of approximated points S need to be increased. Otherwise we can construct feasible solution using positive homogeneity property of CVaR from Proposition 4.5.

Procedure can be like this. First, define the new random variable:

$$\tilde{\mathcal{X}}^a = \mathcal{X}^a - m = \left(\{\tilde{x}_s = x_s - m\}_{s=1}^S, \{\tilde{p}_s = p_s\}_{s=1}^S\right)$$

Because of the (5.24) and transformation which put mean $E[\mathcal{X}^a] = m$ into the origin all CVaR for $\tilde{\mathcal{X}}^a$ are negative. Thus, CVaR of $\tilde{\mathcal{X}}^a$ will be decreased by multiplication $\tilde{\mathcal{X}}^a$ on $\lambda > 1$. The question of picking λ is trivial question of defining the maximum over the finite set of values. The reverse transformation:

$$\mathcal{X}^a = \tilde{\mathcal{X}}^a + m = \left(\{x_s = \tilde{x}_s - m\}_{s=1}^S, \{p_s = \tilde{p}_s\}_{s=1}^S\right)$$

will produce new $\mathcal{X}^a$ which still satisfy (5.21) but with lower CVaR values.

5.7.3 Adjust points in the approximation domain

The following problem is solved using linear programming technique:

$$
\begin{aligned}
f_x^1(p) \equiv \min_{x_s} \quad & \sum_{k,i} \triangle CVaR_{k,i} & \\
s.t. \quad & \langle y_i, x_s \rangle \leqslant \langle y_i, x_{s^*} \rangle & s \in I_{SaR_{k,i}}; \\
& & s^* \in I_{SaR^b_{k,i}} \\
& \langle y_i, x_s \rangle \geqslant \langle y_i, x_{s^*} \rangle & s \notin I_{SaR_{k,i}}; \\
& & s^* \in I_{SaR^b_{k,i}} \\
& \sum_s x_s p_s = m & \\
\frac{1}{\alpha_k}\left(\sum_{s \in I_{SaR_{k,i}}} \langle y_i, x_s \rangle p_s + \triangle p_{k,i} \langle y_i, x_{s^*} \rangle \right) & + \triangle CVaR_{k,i} & = CVaR_{k,i} \\
& \triangle CVaR_{k,i} \geqslant \emptyset &
\end{aligned}
$$

5.7.4 Adjust probability measure

The following problem is solved using linear programming technique:

$$
\begin{aligned}
f_p^2(x) \equiv \min_{p_s} \quad & \sum_{k,i} \triangle CVaR_{k,i} & \\
s.t. \quad & \sum_{s \in I_{SaR_{k,i}}} p_s & \geqslant \alpha_k \\
& \sum_{s \in I_{SaR_{k,i}}} p_s + \triangle p_{k,i} & = \alpha_k \\
\frac{1}{\alpha_k}\left(\sum_{s \in I_{SaR_{k,i}}} \langle y_i, x_s \rangle p_s + \triangle p_{k,i} \langle y_i, x_{s^*} \rangle \right) & + \triangle CVaR_{k,i} & = CVaR_{k,i} \\
& \sum_s x_s p_s = m & \\
& \sum_s p_s = 1 &
\end{aligned}
$$

This problem will always have solution because of convex domain and existence of initially feasible solution. Steps 3 (section 5.7.3) and 4 (section 5.7.4) are repeated iteratively until objective function stop to improve.

Using the results of Proposition 5.1 we conclude that objective function in the iteration cycle is non-increasing. Algorithm will stop when objective function either becomes $\emptyset$ or objective function improvement is less than the pre-specified tolerance. Thus, we find a local optimum for problem (5.18) - (5.21).

5.7.5 Find feasible solution for SGM

The next step is to convert the $\mathcal{X}^a$ into feasible solution for (5.15)-(5.17). The boundary of conv($\{y_i\}_{i=1}^{I}$) is formed by the set of hyperplaines $\{H_j\}_{j=1}^{J}$. In order to check the conditions (5.10), we need to solve the following problem for each $H \in \{H_j\}_{j=1}^{J}$.

$$Er_{k,j} = \min_{y \in H_j} (CVaR_{\alpha_k}(\langle \mathcal{X}, y \rangle) - CVaR_{\alpha_k}(\langle \mathcal{X}^a, y \rangle)) \quad (5.25)$$

If at least one of $Er_{k,j}$ is negative the **approximation distribution** $\mathcal{X}^a$ need to be corrected.

From now we assume that $\{H_j\}_{j=1}^{J}$ has been formed and the approximate distribution $\mathcal{X}^a$ has been found as a solution of (5.18)-(5.21).

Thus, we know the values of *CVaR* for basic portfolios. Now the problem is to find the portfolios $\{y^*_{j,k}\}$; $y^\star_{j,k} \in H_j$ for which the inequality (5.10) is violates with maximum value. We will change the inequality formulation based on the concavity property of *CVaR* (Theorem 4.3).

$$y^*_{j,k} = \sum_{m=1}^{M} \rho_m \cdot y_m; \quad \forall y_m \in \{y_i\}_{i=1}^{I} \cap H_j \quad (5.26)$$

$$CVaR_{\alpha_k}(\langle \mathcal{X}^a, y^*_{j,k} \rangle) \leq \sum_{m=1}^{M} \rho_m \cdot CVaR_{\alpha_k}(\langle \mathcal{X}, y_m \rangle) \leq CVaR_{\alpha_k}(\langle \mathcal{X}, y^*_{j,k} \rangle) \quad (5.27)$$

For each hyperplaine H_j and each α_k we can formulate the problem as the following LP problem.

$$\begin{aligned}
\max_{\rho_j,\xi} \quad & \left(CVaR^a - CVaR^d\right) \\
s.t. \quad CVaR^d - \sum_{m=1}^{M} \rho_m \cdot CVaR^d_{k,m} &= \emptyset \\
y - \sum_{m=1}^{M} \rho_m \cdot y_m &= \emptyset \\
\xi - \langle y, x_s \rangle - P^a_s + N^a_s &= \emptyset \\
\xi - \frac{1}{\alpha_k} \sum_{s=1}^{S} P^a_s \cdot p_s - CVaR^a &= \emptyset \\
\sum_{m=1}^{M} \rho_m &= 1 \\
\rho_m, P^a_s, N^a_s &\geq \emptyset
\end{aligned}$$

It is easy to evaluate the size of the above LP:

- Number of variables $M + 2S + 4$.
- Number of constraints $S + 4$.

According to [86], [85] the optimal solution for the problem above always has

$$CVaR^a_{j,k} = \max_{\xi \in \mathbb{R}} \{F^2_R(\mathcal{X}^a, \alpha_k, y^\star_{j,k}, \xi)\}$$

After getting portfolios $\{y^\star_{j,k}\}$ we can define the subset $\{y^\star_{j,k}\}^\sim$ for which:

$$CVaR^a_{j,k} - CVaR^d_{j,k} = ErA_{j,k} > \emptyset$$

If the set is not empty then the approximation is incorrect. We use positive homogeneity property of $C\ V\ a\ R$ to make correction. First we define the maximum necessary multiplier:

$$\lambda = \max_{y^\star_{j,k}} \left\{ \frac{\langle y^\star_{j,k}, E[\mathcal{X}^a] \rangle - CVaR^d_{j,k}}{\langle y^\star_{j,k}, E[\mathcal{X}^a] \rangle - CVaR^a_{j,k}} \right\}$$

Then each point of approximation distribution is presented in the form:

$$x_s = E[\mathcal{X}^a] + d_s$$

and adjusted as follows:

$$x_s^{\text{new}} = E[\mathcal{X}^a] + \lambda \cdot d_s$$

The new approximation distribution will satisfy all necessary constraints by construction.

5.8 Numerical example

In this section we present a computational example to illustrate the algorithmic progress. The LP problems in the approximation algorithm are solved using *GLPK* package, see [66] for details about *GLPK*. For details about the algorithmic implementation, see Appendix B.

The example demonstrates how the approximation algorithm works. The original distribution consists of 16 points (see Figure 5.2). Financial optimization model will use *CVaR* for three probability levels (see Table 5.1).

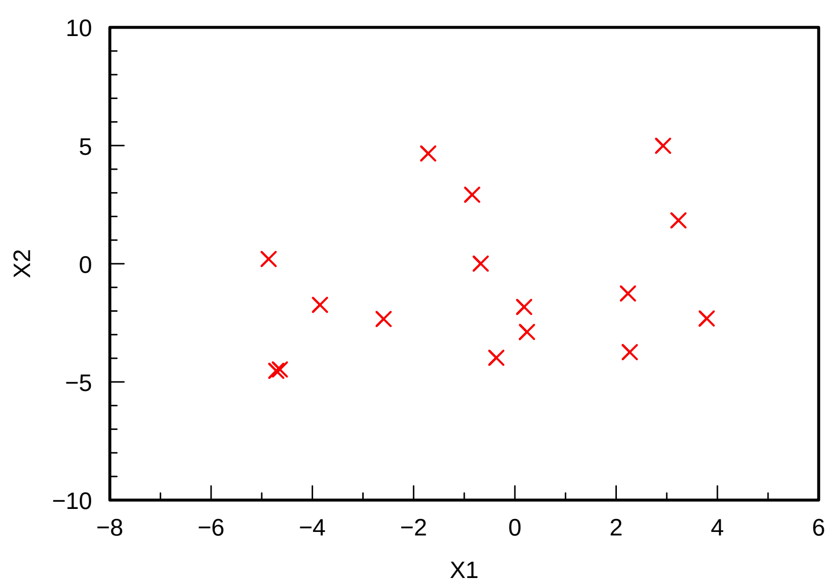

Figure 5.2. Original distribution

Table 5.1. The set of fixed probability levels

α_1	10%
α_2	20%
α_3	30%

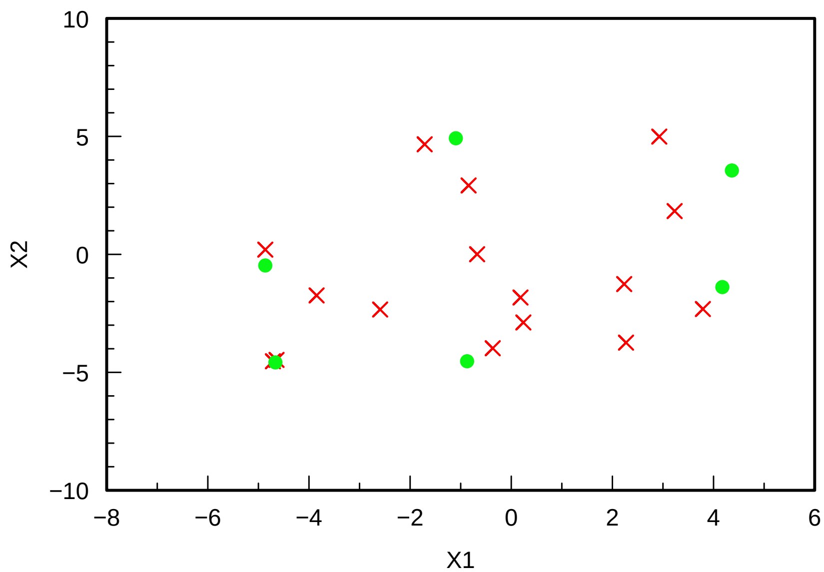

Figure 5.3. Original distribution and Initial approximation

We approximate the original distribution with a 6-point distribution. After Step 1 of the proposed algorithm the initial approximation distribution is found (see Figure 5.3).

At the end of Steps 3 and Step 4 of the proposed algorithm, the approximation is found, see Figure 5.4. This distribution is optimal for the "simplified" problem. In Figure 5.5, we present the approximating distribution that is feasible for the "ideal" formulation. This is the approximation $\mathcal{X}^a$ used in the financial optimization model.

The next step in research is to explore the question of approximation precision, in Chapter 6.

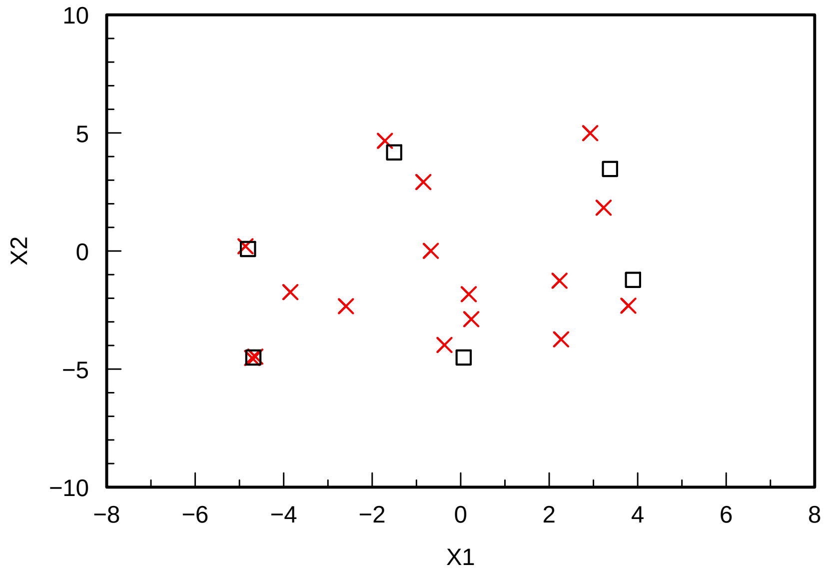

Figure 5.4. Original distribution and quasi-optimal approximation

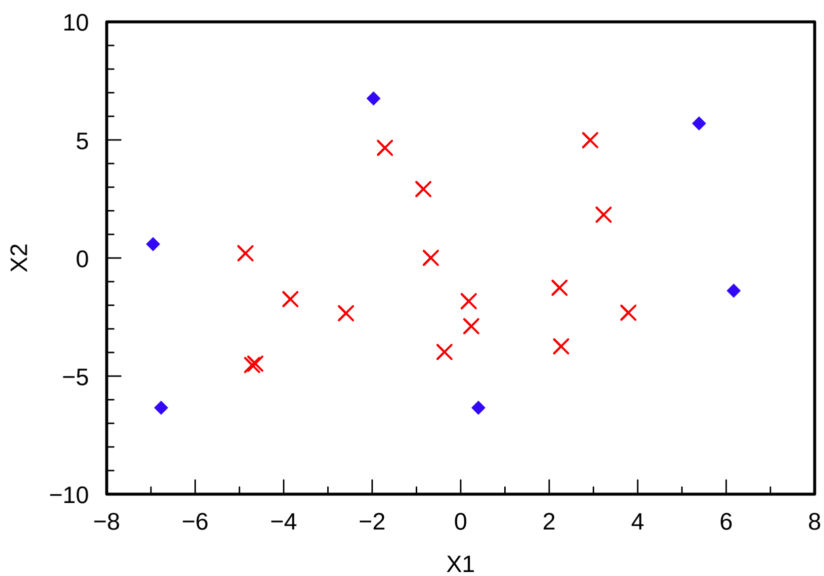

Figure 5.5. Original distribution and final approximation

Chapter 6

Computational Testing of The Scenario Approximation Scheme

6.1 Preliminaries

The objective of this chapter is to provide empirical evidence for efficiency of the scenario approximation algorithm developed in chapters 4 and 5. The portfolio optimization model used for empirical validation of the scenario approximation scheme is first presented. Then, the scope of the experimentation is sketched, and finally, the results of the experiment are analyzed.

6.2 Problem formulation of portfolio optimization

The notations from section 2.1 page 10 are used for model description. However, previous notations are not specific enough for precise model description. Thus, the extended notations are introduced below before the formulation of financial optimization model.

6.2.1 Model notations

Let's denote the following parameters:

- N is the number of securities in the financial optimization model.
- y^I is the initial positions in number of shares.

- ρ is the vector of market prices for a share of each stock.
- $\mathcal{R}$ is the random variable of future returns.

  $$\mathcal{R} \equiv \{x_j, p_j\}$$

  where x_j is the return vector of scenario j and p_j is the probability of scenario j.
- m is the vector of expectations of the random variable $\mathcal{R}$.
- $\{\alpha_k, w_k\}$ is the set of probability levels and weights for computing weighted CVaR deviation measure.
- u and l are vectors of upper and lower limits of share positions.
- B is the investment budget in dollars.
- y is the decision to be made. This is the vector of number of shares that investor needs to create.

The wealth is calculated as inner product of dollar positions and stock returns:

$$W = \langle x \cdot \rho, y \rangle$$

where $x \cdot \rho$ is a vector of dollar profit for each security computed as Rademaher product of two vectors. With all of the above notations a one period financial optimization model can be formulated, see formulation (2.3)-(2.5) on page 11, if the following components are specified: the scenario generation method, the risk function $f(\cdot)$, and the set of constraints for financial optimization model.

6.2.2 Scenario generation and approximation

The uncertainty of future returns is modeled using Normal distribution. The recent studies, see [33], [45], [51], [62], and [73], show that normal distribution can be a good initial approximation for the real distribution in financial optimization models. Thus, normal distribution has been picked as the commonly accepted method of scenario generation. The precision of such procedure and scenario generation methods that give better results are beyond the scope of this thesis.

For a computational procedure to generate a sample of normal distribution with a given mean, standard deviation, and correlation see [82, section 7.2]. The main idea of the procedure is to generate a sample of two independent uniformly distributed random variables. This sample could be converted into the sample of two independent normally distributed random variables with 0 mean and identity matrix as variance/covariance matrix. Adjusting mean and variance/covariance matrix involves only trivial multiplications on variance/covariance matrix and the addition of the desired mean. This is one of the fastest known methods for sample generation.

The approximation technique was developed in chapters 4 and 5. This technique will be used for scenario approximation in financial optimization model.

The next important component of the financial optimization model is risk measure $f(\cdot)$ for the objective function, see (2.3) on page 11.

6.2.3 Objective function

According to the theory, the rational risk measure is decomposable into two additive components, see section 3.4 on page 45, target measure $t(\cdot)$ and deviation measure $d(\cdot)$:

$$f(W) = t(W) + d(W)$$

The target measure $t(\cdot)$ for experimentation is the expected wealth:

$$t(W) = -E[W]$$

The deviation measure $d(\cdot)$ for experimentation is of two types:

- Weighted CVaR deviation measure:

$$d(W) = -\sum_{k=1}^{K} w_k \, \mathrm{CVaR}_{\alpha_k}(W)$$

- Expected mean semi-deviation measure:

$$d(W) = E\left[\left([t(W) - W]^+\right)\right]$$

Thus, the financial optimization model can use two different rational risk measures: mean/weighted CVaR and mean/semi-deviation. These two measures are coherent risk measures and there are the most frequently used measures in the literature, see [1], [26], [28], [39], [67], and [68].

6.2.4 Constraints

There are two constraints for the model:

- Budget:

$$\sum_{i=1}^{N} \left(y_i - y_i^I\right) \cdot \rho_i \leqslant B$$

  Assume a fixed budget is available for investments.

- Position limits:

$$l_i \leqslant y_i \leqslant u_i$$

  These constraints are usually initiated by investor to induce diversification, so as to reduce the risk of over-exposure in a single or a very few securities.

6.2.5 Models for experiments

Experiments are designed to investigate, for example, the effect of the levels of uncertainty, number of securities, and the sample size on the quality of the approximation as well as computational time. For this purpose a basic investment optimization model is utilized in two different variants, namely, the deviation component of the risk measure is varied in the search of optimal investment choices. The experimentation will be done using two different models:

1. Model with weighted CVaR deviation measure.

$$\max_{y} \quad \sum_{i=1}^{N} m_i \cdot y_i \cdot \rho_i + \quad w_0 \cdot \sum_{k=1}^{K} w_k \cdot \mathrm{CVaR}(\mathcal{R}, y, \alpha_k) \tag{6.1}$$

$$\sum_{i=1}^{N} \left(y_i - y_i^I\right) \cdot \rho_i \leqslant B \tag{6.2}$$

$$l_i \leqslant y_i \leqslant u_i \quad (6.3)$$

2. Model with downside mean semi-deviation measure.

$$\max_{y^*} \quad \sum_{i=1}^{N} m_i \cdot y_i \cdot \rho_i + \lambda \sum_{s=1}^{S} \left[\sum_{i=1}^{N} (m_i - x_{i,s}) \cdot y_i \cdot \rho_i \right]^{+} \cdot p_s \tag{6.4}$$

$$\sum_{i=1}^{N} \left(y_i - y_i^I \right) \cdot \rho_i \leqslant B \tag{6.5}$$

$$l_i \leqslant y_i \leqslant u_i \qquad (6.6)$$

6.2.6 Decision parameters

The solution of the portfolio optimization model either (6.1)-(6.3) or (6.4)-(6.6) is the vector of optimal investments y^*. This vector represents the optimal positions in number of shares that the investor creates. The presented financial optimization models are over simplified models which disregard practical features such as transaction costs, margin restrictions, slippage execution costs, multi-period planning horizon. All of these features have been disregarded because the objective of the experiment is to validate efficiency of the scenario approximation scheme.

6.3 Computational experiments

6.3.1 Purpose of experimentation

The validation of the scenario approximation scheme efficiency is performed in three steps. The first step is the comparison of the computational time consumed by the scenario approximation scheme and optimization versus the time required for optimization using the original scenarios. Such comparison addresses the question: when is the scenario approximation scheme more efficient than solving the original problem? The second step is to explore how the number of securities in the model, number of approximation points and number of probability levels in the weighted CVaR deviation measure affect the time consumed by the scenario approximation scheme. These results are interesting from an application standpoint because they give preliminary ideas about optimal settings for the scenario approximation scheme. The last step is to compare efficient frontiers generated by the original problems with efficient frontiers generated by the approximate problems (the financial optimization models with approximate scenarios). These comparisons should confirm the theoretical result that CVaR of the approximation problem will never exceed CVaR of the original problem, see Proposition 5.2 on page 76. Then the efficient frontiers of the original problem will be built using optimal solution of approximation problems. The distance between original and simulated efficient frontiers will show the error related to the scenario approximation scheme.

6.3.2 Efficiency metrics

Performance metrics need to be constructed as validation criteria for efficiency of the scenario approximation scheme. Because evaluation will be done based on the time consumed by the financial optimization model and distance between efficient frontiers there are three performance metrics to measure the quality of approximation:

1. CPU time consumed by program to solve the approximation and optimization problem together or only optimization problem if approximation procedure is skipped.

2. Improvement of approximation procedure. This is done by building the efficient frontiers for the approximation problems.

3. Quality of the investment in comparison to the original sample. This is done by constructing the efficient frontiers for original scenarios using optimal solutions of the original and approximation problems.

6.3.3 Method of experiment

A computer code was developed that uses history of equity prices and performs forecast, scenario generation, scenario approximation (if necessary), and financial investments optimization. The optimization results are the output. The input data uses a history of 250 days of equity prices. Twenty companies from Standard & Poor's 100 are used as the basic data from which four different data sets have been formed:

1. Data set with 5 stocks (SP5).
2. Data set with 10 stocks (SP10).
3. Data set with 15 stocks (SP15).
4. Data set with 20 stocks (SP20).

The experiments are divided into three phases:

Phase 1. This tests the efficiency of the proposed approximation scheme for different settings. The experiment is organized using the financial optimization model with weighted CVaR deviation measure (6.1)-(6.3). There are three different probability levels for the financial optimization model, see Table 6.1. The model is run for each level with different number of original scenarios. The number of scenarios in the scenario approximation scheme is fixed. The experiment is performed for data set: SP5. Results are summarized in the form of tables and graphs.

Phase 2. Dependency of computation time of the scenario approximation procedure under different approximation parameters is explored. The financial optimization models with weighted CVaR deviation measure (6.1)-(6.3) and three different probability levels, see Table 6.1, are used for experimentation. The number of scenarios in the original problem is fixed. The number of points

Table 6.1. Three probability levels for CVaR optimization model

Title	Probabilities
Level 1	0.1
Level 2	0.1 0.25
Level 3	0.05 0.1 0.25

in the scenario approximation scheme is varied from 6 to 102. The experiment is run for all four data sets: SP5, SP10, SP15, SP20. Results are summarized in the form of tables and figures.

Phase 3. The efficiency of the scenario approximation scheme is analyzed using the graphs of efficiency frontiers. The first experiment is to test the improvements of efficient frontiers of the approximation problems when the number of points in the scenario approximation scheme is increased. This experiment is done for the financial optimization model with weighted CVaR deviation measure with three different probability levels, see Table 6.1, and one data set SP5. The next experiment is to check the approximation precision when number of probability levels in the scenario approximation scheme is increased, see Table 6.2 for list of probability levels. This experiment is performed with the same financial optimization model and data set.

The same type of experimentation will be done for the financial optimization model with mean semi-deviation measure, see (6.4)-(6.6). The first experiment is to check the precision of the scenario approximation scheme when number of points in the scheme is increased. Then the scenario approximation scheme is run with fixed number of points in approximation (36 points) but different probability levels, see Table 6.3, and the resulting efficient frontiers are analyzed.

6.3.4 Computer code implementation

The implemented computer code consists of independent modules. Each module has specific functionality and connected to a model manager module. The model manager processes all results and manages data flows. The method for data transfer among modules is pipe mechanism in GLIBC, see [14] chapter 15. The advantages of this approach are simplicity in implementation and ease of application debugging. Each module is debugged in stand-alone mode. This form of program organization will allow adding new functionality to the program during further development. The scheme of program is presented on Figure D.1 page 135.

Table 6.2. List of probability levels for the scenario approximation scheme and the financial optimization model with weighted CVaR deviation measure

Levels	Probabilities				
1 level	0.100				
2 levels	0.050	0.100			
3 levels	0.050	0.100	0.250		
4 levels	0.025	0.075	0.100	0.250	
5 levels	0.025	0.075	0.100	0.150	0.250
6 levels	0.010	0.050	0.075	0.100	0.150
					0.250
7 levels	0.010	0.050	0.075	0.100	0.150
				0.200	0.250
8 levels	0.010	0.035	0.065	0.080	0.100
			0.150	0.200	0.250
9 levels	0.010	0.035	0.065	0.080	0.100
		0.125	0.175	0.225	0.250
10 levels	0.010	0.025	0.050	0.075	0.090
	0.100	0.125	0.175	0.225	0.250

Table 6.3. List of probability levels for the scenario approximation scheme and the financial optimization model with mean semi-deviation measure.

Levels	Probabilities				
1 level	0.50				
2 levels	0.25	0.50			
3 levels	0.25	0.50	0.75		
4 levels	0.10	0.35	0.50	0.75	
5 levels	0.10	0.35	0.50	0.65	0.90
6 levels	0.10	0.30	0.40	0.50	0.65
					0.90
7 levels	0.10	0.30	0.40	0.50	0.60
				0.70	0.90
8 levels	0.10	0.20	0.30	0.40	0.50
			0.60	0.70	0.90
9 levels	0.10	0.20	0.30	0.40	0.50
		0.60	0.70	0.80	0.90
10 levels	0.10	0.20	0.30	0.40	0.45
	0.50	0.60	0.70	0.80	0.90

List of modules with brief explanations:

Data Reading. This module reads a file with historical prices and return the array of prices for a specified data range.

Forecast. This module makes the forecast of first and second moments using history of rate of returns. The forecast method is simple moving average.

Scenarios Generation. This module generates a sample with the specified number of points. The Normal distribution is used for sample generation, see section 6.2.2 for details.

Scenarios Approximation. This module approximates a given sample with a smaller one using CVaR/CVaG information, see chapters 4 and 5 for details.

Financial Optimization Model. This module solves a specified financial optimization problem.

Program Manager. This module provides interconnection between all program components and forms the output of the program.

The described program package is used for running the experiment. The hardware and software specifications are given in Table D.1 on page 134.

6.4 Experimentation results

6.4.1 Efficiency of approximation scheme

The methodology for experiment has been described in section 6.3.3. Here the results of **phase 1** of the experiment are presented and discussed. The results of running the financial optimization model with weighted CVaR deviation measure, one probability level (Level 1, see Table 6.1), and data set SP5 are presented in Table E.1. The minimum number of points required in the original scenarios when the scenario approximation scheme uses 33 points to be time efficient is around 4000. However, this number heavily depends on the number of points in the approximation distribution, see Figure E.1. The results of running the financial optimization model with weighted CVaR deviation measure, two probability levels (Level 2, see Table 6.1), and data set SP5 are presented in Table E.2. The minimum number of points required in the original scenarios when the scenario approximation scheme uses 33 scenarios will be time efficient is around 3000 scenarios. However, this number depends on the number of points in approximation distribution, see Figure E.2 for illustration. The results of running the financial optimization model with weighted CVaR deviation measure, three probability levels (Level 3, see Table 6.1), and data set SP5 are presented in Table E.3. The minimum number of points required in the original scenarios when the scenario approximation scheme uses 33 scenarios will be time efficient is around 4000 scenarios. The graphical representation of this result is given on Figure E.3.

Thus, I conclude that number of points in the original scenarios need to exceed 3000 in order for the scenario approximation scheme to be efficient.

6.4.2 Effect on CPU time

This section presents experimentation results of **phase 2**, see section 6.3.3. The results of running the financial optimization model with weighted CVaR deviation measure, three different probability levels (Level 1, Level2, and Level 3), and data set SP5 are presented in the Table E.4 and Figure E.4. The analogous results for data set SP10 presented on Figure E.5, for data set SP15 presented on Figure E.6, and for data set SP20 presented on Figure E.7. The dependency of CPU time on the number of points in the scenario approximation scheme is non-linear. Thus, one needs to be very careful when specifying the number of points for the scenario approximation scheme, otherwise the CPU time for the approximation scheme and optimization will exceed CPU time of optimization of the original problem. It is also clear from Figures E.4, E.5, E.6, and E.7 that the CPU time consumed by the scenario approximation scheme heavily depends on the number of probability levels in the scheme and number of securities in the financial optimization model.

6.4.3 Precision of scenario approximation procedure

The results of **phase 3** of the experiment, see section 6.3.3, are presented in this section. The first experiment is to test the improvements in approximation quality when the number of points in the scenario approximation scheme is increased. The results of running the financial optimization model with weighted CVaR deviation measure, three different probability levels (Level 1, Level2, and Level 3), and data set SP5 are presented on Figures E.8, E.18, and E.28. The analogous results are obtained for data set SP10, see Figures E.10, E.20, and E.30, data set SP15, see Figures E.12, E.22, and E.32, and data set SP20, see Figures E.14, E.24, and E.34. As evident from the graphs, after some time the approximation quality fails to improve. That gives additional insight in to the question of optimal number of points in the scenario approximation scheme. However, this area is still open for further research. The second part of this experiment is to compare the quality of the approximation solution with the efficient frontier of the original problem. This is done by simulating the optimal solutions of the approximation problem against the original one. The results of simulating the financial optimization model with weighted CVaR deviation measure, three different probability levels (Level 1, Level2, and Level 3), and data set SP5 are presented on Figures E.9, E.19, and E.29. The analogous results are obtained for data set SP10, see Figures E.11, E.21, and E.31, data set SP15, see Figures E.13, E.23, and E.33, and data set SP20, see Figures E.15, E.25, and E.35. The general conclusion is that the efficient frontier simulated using the optimal solutions from the approximation problem becomes very close to the original efficient frontier when number of points in the scenario approximation scheme exceeds 30.

The next part of this phase is to check the precision of the scenario approximation scheme when number of probability levels in the scheme is increased, see Table 6.2 for the list of probability levels. The results of running the financial optimization model with weighted CVaR deviation measure, three different probability levels (Level 1, Level2, and Level 3) in optimization, 36 points in the scenario approximation scheme, and data set SP5 are presented on Figures E.16, E.26, and E.36. The efficient frontiers do not improve at all. Thus, increasing the number of probability levels in the scenario approximation scheme does not improve the quality of approximation. The second part of this experiment is to compare the quality of approximation solution with the quality of optimal solution for the original problem. This is done by simulating the optimal solutions of approximation problem against the original distribution. The results of simulating the financial optimization model with weighted CVaR deviation measure, three different probability levels (Level 1, Level2, and Level 3), 36 points in the scenario approximation scheme, and data set SP5 are presented on Figures E.17, E.27, and E.37. The approximation efficient frontier does not improve when the number of probability levels in the scenario approximation scheme is increased. Thus, the best approximation is achieved when probability levels in approximation scheme match the probability levels in the financial optimization problem.

The last experiment is to check the precision of approximation scheme applied to model with semi-deviation measure. The results of running the financial optimization model with mean semi-deviation measure, one probability level in the scenario approximation scheme (0.5), and data set SP5 are presented on Figure E.38. The analogous results are obtained for data set SP10, see Figure E.40, data set SP15, see Figure E.42, and data set SP20, see Figure E.44. Conclusion here is the same as it is for the financial optimization model with weighted CVaR deviation measure. The quality of approximation is increased with larger number of points in the scenario approximation scheme. The results of simulating the optimal solutions of the financial optimization model with mean semi-deviation measure, one probability level in the scenario approximation scheme (0.5), and data set SP5 are presented on Figures E.39. The analogous results are obtained for data set SP10, see Figure E.41, data set SP15, see Figure E.43, and data set SP20, see Figure E.45. The simulated efficient frontiers of the approximation problem becomes very close to the original efficient frontier.

Thus, the theoretical result that CVaR of the approximation problem will never exceed CVaR of the original problem, see Proposition 5.2 on page 76 is confirmed in each experiment. The efficiency of the proposed scenario approximation scheme is proven in both CPU time and solution quality.

Additionally, the empirical testing shows that the proposed scenario approximation scheme can be applied efficiently not only for the financial optimization models with weighted CVaR deviation measure but also for models with mean semi-deviation measure.

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Appendixes

Appendix A

Proof of Theorem 4.3

The proof heavily uses the fact that α increases with $\langle y, x\rangle$ and it decreases with $\langle y, x\rangle$, as well as it uses the upper and lower bounds on D_{VaRG}.

Proof. (4.24) and (4.25) are reformulations of (4.22) and (4.23) using upper $D_y^U(x)$ and lower $D_y^L(x)$ tails. Thus, we need to prove that (4.24) and (4.25) deliver the correct value of x. The proof for (4.24) will be given here, a slight modification of which can be applied for proving (4.25). Let's define the functional $F\colon \mathbb{R}^n \longrightarrow \mathbb{R}^1$:

$$F(x) = \langle y, x\rangle - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(x)} \langle y, x - t\rangle d\mu(t) \tag{A.1}$$

Function $F(\,\cdot\,)$ is concave because: $\langle y, x\rangle$ is a linear function of x by definition of inner product, $\frac{1}{\alpha} \cdot \int_{D_y^L(x)} \langle y, x - t\rangle d\mu(t)$ is an integration over a variable domain and convexity of this function needs to be proved directly. That is, for function:

$$f(x) \equiv \frac{1}{\alpha} \int\limits_{D_y^L(x)} \langle y, x - t\rangle d\mu(t)$$

we need to prove:

$$f(\gamma x + (1-\gamma) z) \leq \gamma f(x) + (1-\gamma) f(z). \tag{A.2}$$

Without loss of generality we can assume that:

$$\forall x, z \in \Omega$$

and

$$\langle y, x\rangle \geq \langle y, z\rangle$$

$$D_y^L(\gamma x + (1-\gamma) z) = D_y^L(z) \cup \left(D_y^L(z)^c \cap D_y^L(\gamma x + (1-\gamma) z)\right) \subseteq D_y^L(x)$$

Now we can make the following transformation:

$$f(\gamma x+(1-\gamma)z)=\frac{1}{\alpha}\int\limits_{D_y^L(\gamma x+(1-\gamma)z)}\langle y,(\gamma x+(1-\gamma)z)-t\rangle d\mu(t)=$$

$$=\frac{\gamma}{\alpha}\int\limits_{D_y^L(\gamma x+(1-\gamma)z)}\langle y,x-t\rangle d\mu(t)+(1-\gamma)\frac{1}{\alpha}\int\limits_{D_y^L(\gamma x+(1-\gamma)z)}\langle y,z-t\rangle d\mu(t)\leq$$

$$\leq\frac{\gamma}{\alpha}\int\limits_{D_y^L(x)}\langle y,x-t\rangle d\mu(t)+(1-\gamma)\frac{1}{\alpha}\int\limits_{D_y^L(\gamma x+(1-\gamma)z)}\langle y,z-t\rangle d\mu(t)=$$

$$=\frac{\gamma}{\alpha}\int\limits_{D_y^L(x)}\langle y,x-t\rangle d\mu(t)+$$

$$+(1-\gamma)\frac{1}{\alpha}\int\limits_{D_y^L(z)}\langle y,z-t\rangle d\mu(t)-$$

$$-(1-\gamma)\frac{1}{\alpha}\int\limits_{D_y^L(z)^c\cap D_y^L(\gamma x+(1-\gamma)z)}|\langle y,z-t\rangle|\,d\mu(t)\leq$$

$$\leq\frac{\gamma}{\alpha}\int\limits_{D_y^L(x)}\langle y,x-t\rangle d\mu(t)+(1-\gamma)\frac{1}{\alpha}\int\limits_{D_y^L(z)}\langle y,z-t\rangle d\mu(t)=$$

$$=\gamma f(x)+(1-\gamma)f(z)$$

Thus, it follows that (A.2) holds. Based on the convexity of $f(x)$, the concavity of $F(x)$ is straightforward.

Based on the concavity of $F(x)$ we can derive that $F(x)$ has a unique global maximum on Ω and every local maximum is a global maximum. The most difficult step in the proof is to show that VaR_α is a local maximum for $F(x)$. This will be proved in two steps:

Step 1. Prove that $F(\bar{x})<F(x^*)$ for

$$VaR_{\alpha\,y}=\langle y,x^*\rangle;\quad \langle y,x^*\rangle=\langle y,\bar{x}\rangle+\epsilon;\quad \epsilon\longrightarrow^+ 0.$$

Step 2. Prove that $F(\bar{x}) < F(x^*)$ for

$$Va\,R_{\alpha\,y} = \langle y, x^* \rangle; \quad \langle y, x^* \rangle = \langle y, \bar{x} \rangle - \epsilon; \quad \epsilon \longrightarrow^{+} 0.$$

Let start with step 1 assuming that:

$$Va\,R_{\alpha\,y} = \langle y, x^* \rangle; \quad x^* = \bar{x} + \nu; \quad \langle y, x^* \rangle = \langle y, \bar{x} \rangle + \epsilon; \quad \langle y, \nu \rangle = \epsilon; \quad \epsilon \longrightarrow^{+} 0$$

Then

$$F(\bar{x}) = \langle y, \bar{x} \rangle - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})} \langle y, \bar{x} - x \rangle d\mu(x) =$$

$$= \langle y, x^* \rangle - \epsilon - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})} \langle y, x^* - x - \nu \rangle d\mu(x) =$$

$$= \langle y, x^* \rangle - \epsilon - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})} \langle y, x^* - x \rangle d\mu(x) + \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})} \langle y, \nu \rangle d\mu(x).$$

Since $\epsilon \longrightarrow^{+} 0$ is true by assumption, $D_y^L(\bar{x}) \subset D_y^L(x^*)$ is correct by construction, and the following implication is true:

$$\int\limits_{D_y^L(\bar{x})} d\mu(x) < \alpha \quad \Rightarrow \quad \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})} \langle y, \nu \rangle d\mu(x) \quad - \quad \epsilon < 0$$

transformation can be continued as follows:

$$F(\bar{x}) <$$

$$< \langle y, x^* \rangle - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(x^*)} \langle y, x^* - x \rangle d\mu(x) - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})^C \cap D_y^L(x^*)} \langle y, x^* - x \rangle d\mu(x) =$$

$$= F(x^*) - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})^C \cap D_y^L(x^*)} \langle y, \nu \rangle d\mu(x) \quad \leq \quad F(x^*)$$

Because $\frac{1}{\alpha} \cdot \int_{D_y^L(\bar{x})^C \cap D_y^L(x^*)} \langle y, \nu \rangle d\mu(x) \geq 0$ we derive that $F(\bar{x}) < F(x^*)$.

The step 2 of the proof is as follows:

$$Va\,G_{\alpha} = \langle y, x^* \rangle; \quad x^* = \bar{x} - \nu; \quad \langle y, x^* \rangle = \langle y, \bar{x} \rangle - \epsilon; \quad \langle y, \nu \rangle = \epsilon; \quad \epsilon \longrightarrow^{+} 0$$

then

$$F(\bar{x}) = \langle y, \bar{x} \rangle - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})} \langle y, \bar{x} - x \rangle d\mu(x) \quad =$$

$$= \quad \langle y, x^* \rangle + \epsilon - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})} \langle y, x^* - x + \nu \rangle d\mu(x) =$$

$$= \langle y, x^* \rangle + \epsilon - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})} \langle y, x^* - x \rangle d\mu(x) - \frac{1}{\alpha} \cdot \int\limits_{D_y^L(\bar{x})} \langle y, \nu \rangle d\mu(x).$$

Observing, that $D_y^L(x^*) \subset D_y^L(\bar{x})$ is correct by construction. The following equality:

$$\int\limits_{D_y^L(x^*)} d\mu(x) + \int\limits_{\langle y, x^* \rangle}^{\langle y, x^* \rangle} d\mu(x) \quad = \quad \alpha_U \geq \alpha$$

is also hold by construction and it implies the following:

$$\epsilon \leq \frac{1}{\alpha} \cdot \int\limits_{D_y^L(x^*)} \langle y, \nu \rangle d\mu(x) + \frac{1}{\alpha} \cdot \int\limits_{\langle y, x^* \rangle}^{\langle y, x^* \rangle} \langle y, \nu \rangle d\mu(x)$$

Then, the transformations can be continued as follows:

$$F(\bar{x}) \leq$$

$$\leq \langle y, x^* \rangle + \frac{1}{\alpha} \cdot \int\limits_{D_y^L(x^*)} \langle y, \nu \rangle d\mu(x) + \frac{1}{\alpha} \cdot \int\limits_{\langle y, x^* \rangle}^{\langle y, x^* \rangle} \langle y, \nu \rangle d\mu(x) -$$

$$- \frac{1}{\alpha} \cdot \int\limits_{D_y^L(x^*)} \langle y, x^* - x \rangle d\mu(x) + \frac{1}{\alpha} \cdot \int\limits_{D_y^L(x^*)^C \cap D_y^L(\bar{x})} \langle y, x - x^* \rangle d\mu(x) -$$

$$- \frac{1}{\alpha} \cdot \int\limits_{D_y^L(x^*)} d\mu(x).$$

the following equality is true by definition:

$$\int_{\langle y,x^*\rangle}^{\langle y,x^*\rangle} \langle y, x^* - x\rangle d\mu(x) = 0$$

and the transformation can be continued as follows:

$$\begin{aligned}
&F(\bar{x}) < \\
&< F(x^*) + \frac{1}{\alpha}\cdot \int_{D_y^L(x^*)} \langle y, \nu\rangle d\mu(x) + \\
&\qquad + \frac{1}{\alpha}\cdot \int_{D_y^L(x^*)^C \cap D_y^L(\bar{x})} \langle y, \nu\rangle d\mu(x) - \frac{1}{\alpha}\cdot \int_{D_y^L(\bar{x})} \langle y, \nu\rangle d\mu(x) < \\
&< F(x^*) + \frac{1}{\alpha}\cdot \int_{D_y^L(\bar{x})} \langle y, \nu\rangle d\mu(x) - \frac{1}{\alpha}\cdot \int_{D_y^L(\bar{x})} \langle y, \nu\rangle d\mu(x) = F(x^*)
\end{aligned}$$

which yields $F(\bar{x}) < F(x^*)$. That completes the step 2 of the proof. Because there are only two possible direction of variating for function $F(x)$, we proved that $F(x^*)$ is a local maximum. Because of the concavity of $F(\,\cdot\,)$ the local maximum is a global one. Thus (4.24) is proved. The equality (4.25) can be proved analogously. □

Appendix B

Approximation Algorithm Implementation

B.1 Input data and notation

Let's mark:

$m = E[\mathcal{X}]$

x_j are points of original distribution $\mathcal{X}$.

The following values are consider as given:

szDim. This is the number of financial investment instruments in the model.

szOrgPnt. This is the number of sample points in the original distribution $\mathcal{X}$.

szAprxPnt. This is the number of sample points in the approximating distribution $\mathcal{X}^a$.

szPrbLvl. This is the number of probability levels specified for SGM.

szBscPrt. This is the number of basic portfolios for SGM.

szHyprPt. This is the number of supporting hyperplanes for simplicial bounded area.

Index for finite sets:

$$\begin{aligned} i &= \overline{1, \text{szDim}} \\ j &= \overline{1, \text{szOrgPnt}} \\ s &= \overline{1, \text{szAprxPnt}} \\ k &= \overline{1, \text{szPrbLvl}} \\ l &= \overline{1, \text{szBscPrt}} \\ m &= \overline{1, \text{szHyprPt}} \\ t &- \text{for any other purpose} \end{aligned}$$

B.2 Define directions

Original problem formulation for one direction:

$$\begin{array}{lcl} \min\limits_{d_s} & \sum \lambda & \\ s.t. & d_s & = \sum \lambda_j (x_j - m) \\ & \langle y_{i_s}, m + d_s \rangle & \leqslant l_{i_s} \\ & \lambda_j & \geqslant \emptyset \\ & \lambda - \lambda_j & \geqslant \emptyset \end{array}$$

Conversion in GLPK format:

column identifiers:

$$\begin{array}{lll} \text{CV1}_i & = d_{s,i} & i = \overline{1, \text{szDim}} \\ \text{CV2}_j & = \lambda_j & j = \overline{1, \text{szOrgPnt}} \\ \text{CV3} & = \lambda & 1 \end{array}$$

column limits:

$$
\begin{aligned}
-\infty \leq \; & \mathrm{CV1}_i \; \leq +\infty \\
\emptyset \leq \; & \mathrm{CV2}_j \; \leq +\infty \\
\emptyset \leq \; & \mathrm{CV3} \; \leq +\infty
\end{aligned}
$$

row identifiers with limits:

$$
\begin{aligned}
\emptyset = \; & \mathrm{RV1}_i \; = \emptyset \\
\emptyset \leq \; & \mathrm{RV2}_j \; \leq +\infty \\
-\infty \leq \; & \mathrm{RV3}_{i_s} \; \leq l_{i_s} - \sum y_{i_s,i} \cdot m_i
\end{aligned}
$$

problem formulation:

$$
\begin{aligned}
\min \quad & \sum \emptyset \cdot \mathrm{CV1}_i + \sum \emptyset \cdot \mathrm{CV2}_j + 1 \cdot \mathrm{CV3} \\
s.t. \quad & 1 \cdot \mathrm{CV1}_i + \sum_j [m_i - x_{j,i}] \cdot \mathrm{CV2}_j && = \mathrm{RV1}_i \\
& 1 \cdot \mathrm{CV3} + [-1] \cdot \mathrm{CV2}_j && = \mathrm{RV2}_j \\
& \sum_i y_{i_s,i} \cdot \mathrm{CV1}_i && = \mathrm{RV3}_{i_s}
\end{aligned}
$$

B.3 Define probabilities

Original problem formulation for one direction:

$$
\begin{aligned}
\max_{p_s} \quad & p \\
s.t. \quad & p \leqslant p_s
\end{aligned}
$$

$$
\begin{aligned}
\sum p_s &= 1 \\
\sum d_s p_s &= \emptyset \\
p_s &\geqslant 0
\end{aligned}
$$

Conversion in GLPK format:

column identifiers:

$$
\begin{aligned}
\mathrm{CV1}_s &= p_s \quad s = \overline{1, \mathrm{szAprxPnt}} \\
\mathrm{CV2} &= p \quad 1
\end{aligned}
$$

column limits:

$$
\begin{aligned}
\emptyset \leq \ & \mathrm{CV1}_s \leq 1 \\
\emptyset \leq \ & \mathrm{CV2} \leq 1
\end{aligned}
$$

row identifiers with limits:

$$
\begin{aligned}
-\infty \leq \ & \mathrm{RV1}_s \leq \emptyset \\
1 = \ & \mathrm{RV2} = 1 \\
0 = \ & \mathrm{RV3}_i = 0
\end{aligned}
$$

problem formulation:

$$
\begin{aligned}
\min \quad & \sum \emptyset \cdot \mathrm{CV1}_s + 1 \cdot \mathrm{CV2} \\
s.t. \quad & [-1] \cdot \mathrm{CV1}_s + 1 \cdot \mathrm{CV2} = \mathrm{RV1}_s \\
& \sum 1 \cdot \mathrm{CV1}_s = \mathrm{RV2} \\
& \sum_s d_{s,i} \cdot \mathrm{CV1}_s = \mathrm{RV3}_i
\end{aligned}
$$

B.4 Adjust points

Original problem formulation for one direction:

$$
\begin{array}{lll}
\min\limits_{x_s} & \sum \triangle CVaR_{k,i} & \\
s.t. & \langle y_i, x_s\rangle \leqslant \langle y_i, x_{s^*}\rangle & s \in I_{SaR_{k,i}}; \\
 & & s^* \in I_{SaR^b_{k,i}} \\
 & \langle y_i, x_s\rangle \geqslant \langle y_i, x_{s^*}\rangle & s \notin I_{SaR_{k,i}}; \\
 & & s^* \in I_{SaR^b_{k,i}} \\
 & \sum x_s p_s = m & \\
\end{array}
$$

$$
\frac{1}{\alpha_k}\left(\sum_{s\in SaR_{k,i}} \langle y_i, x_s\rangle p_s + \triangle p_{k,i}\langle y_i, x_{s^*}\rangle\right) + \triangle CVaR_{k,i} \ = CVaR_{k,i}
$$

$$
\triangle CVaR_{k,i} \geqslant \emptyset
$$

Conversion in GLPK format:

column identifiers:

$$
\begin{array}{lll}
\mathrm{CV1}_{s,i} & = x_{s,i} & s = \overline{1, \mathrm{szAprxPnt}}, i = \overline{1, \mathrm{szDim}} \\
\mathrm{CV2}_{l,k} & = \triangle CVaR_{l,k} & l = \overline{1, \mathrm{szBscPrt}}, k = \overline{1, \mathrm{PrbLvl}}
\end{array}
$$

column limits

$$
\begin{array}{rcl}
-\infty \leq & \mathrm{CV1}_{s,i} & \leq +\infty \\
\emptyset \leq & \mathrm{CV2}_{l.k} & \leq +\infty
\end{array}
$$

row identifiers with limits:

$$
\begin{array}{rcl}
-\infty \leq & \mathrm{RV1}_{\# I_{SaR_{k,i}}} & \leq \emptyset \\
-\infty \leq & \mathrm{RV2}_{\#(\Omega^a / I_{SaR_{k,i}})} & \leq \emptyset
\end{array}
$$

$$CVaR_{k,l} = \quad \mathrm{RV3}_{k,l} \quad = CVaR_{k,l}$$

$$m_i = \quad \mathrm{RV4}_i \quad = m_i$$

problem formulation:

$$\begin{aligned}
\min \quad & \sum \emptyset \cdot \mathrm{CV1}_{s,i} + \sum 1 \cdot \mathrm{CV2}_{l,k} \\
s.t. \quad & \sum_{s \in I_{SaR_{k,i}}} \sum_i y_{l,i} \cdot \mathrm{CV1}_{s,i} + \sum_{s \in I_{SaR^b_{k,i}}} \sum_i [-y_{l,i}] \cdot \mathrm{CV1}_{s,i} && = \mathrm{RV1}_{\# I_{SaR_{k,i}}} \\
& \sum_{s \in I_{SaR^b_{k,i}}} \sum_i y_{l,i} \cdot \mathrm{CV1}_{s,i} + \sum_{s \notin I_{SaR_{k,i}}} \sum_i [-y_{l,i}] \cdot \mathrm{CV1}_{s,i} && = \mathrm{RV2}_{\#(\Omega^a / I_{SaR_{k,i}})} \\
& \sum_{s \in I_{SaR_{k,i}}} \sum_i \frac{y_{l,i} \cdot p_s}{\alpha_k} \cdot \mathrm{CV1}_{s,i} + \sum_{s \in I_{SaR^b_{k,i}}} \sum_i \frac{y_{l,i} \cdot \triangle p_{k,l}}{\alpha_k} \cdot \mathrm{CV1}_{s,i} + \\
& \qquad + \mathrm{CV2}_{l,k} && = \mathrm{RV3}_{k,l} \\
& \sum_s p_s \cdot \mathrm{CV1}_{s,i} && = \mathrm{RV4}_i
\end{aligned}$$

B.5 Adjust probabilities

Original problem formulation for one direction:

$$\begin{aligned}
\min_{p_s} \quad & \sum \triangle CVaR_{k,i} \\
s.t. \quad & \sum_{s \in I_{SaR_{k,i}} \bigcup I_{SaR^b_{k,i}}} p_s && \geqslant \alpha_k \\
& \sum_{s \in I_{SaR_{k,i}}} p_s + \triangle p_{k,i} && = \alpha_k \\
& \frac{1}{\alpha_k} \left(\sum_{s \in I_{SaR_{k,i}}} \langle y_i, x_s \rangle p_s + \triangle p_{k,i} \langle y_i, x_{s^*} \rangle \right) + \triangle CVaR_{k,i} && = CVaR_{k,i}
\end{aligned}$$

$$\sum x_s p_s = m$$

$$\sum p_s = 1$$

Conversion in GLPK format:

column identifiers:

$$\begin{aligned}
\text{CV1}_s &= p_s & s &= \overline{1, \text{szAprxPnt}} \\
\text{CV2}_{l,k} &= \triangle p_{l,k} & l &= \overline{1, \text{szBscPrt}}, k = \overline{1, \text{PrbLvl}} \\
\text{CV3}_{l,k} &= \triangle CVaR_{l,k} & l &= \overline{1, \text{szBscPrt}}, k = \overline{1, \text{PrbLvl}}
\end{aligned}$$

column limits:

$$\begin{aligned}
\emptyset \leq & \ \text{CV1}_s & \leq 1 \\
\emptyset \leq & \ \text{CV2}_{l,k} & \leq \alpha_k \\
\emptyset \leq & \ \text{CV3}_{l,k} & \leq +\infty
\end{aligned}$$

row identifiers with limits:

$$\begin{aligned}
\alpha_k \leq & \ \text{RV1}_{k,l} & \leq +\infty \\
\alpha_k = & \ \text{RV2}_{k,l} & = \alpha_k \\
CVaR_{k,l} = & \ \text{RV3}_{k,l} & = CVaR_{k,l} \\
m_i = & \ \text{RV4}_i & = m_i \\
1 = & \ \text{RV5} & = 1
\end{aligned}$$

problem formulation:

$$\begin{aligned}
\min \quad & \sum \emptyset \cdot \text{CV1}_s + \sum \emptyset \cdot \text{CV2}_{l,k} + \sum 1 \cdot \text{CV3}_{l,k} \\
s.t. \quad & \sum_{s \in I_{SaR_{k,i}}} 1 \cdot \text{CV1}_s + \sum_{s \in I_{SaR^b_{k,i}}} 1 \cdot \text{CV1}_s & = \text{RV1}_{k,l} \\
& \sum_{s \in SaR_{k,l}} 1 \cdot \text{CV1}_{s,i} + 1 \cdot \text{CV2}_{k,l} & = \text{RV2}_{k,l}
\end{aligned}$$

$$\sum_{s \in I_{SaR_{k,i}}} \left[\sum_i \frac{y_{l,i} \cdot x_{s,i}}{\alpha_k} \right] \cdot \mathrm{CV1}_s + \sum_{s \in SaR^b_{k,l}} \left[\sum_i \frac{y_{l,i} \cdot x_{s,i}}{\alpha_k} \right] \cdot \mathrm{CV2}_{k,l} + \mathrm{CV3}_{l,k} = \mathrm{RV3}_{k,l}$$

$$\sum_s x_{s,i} \cdot \mathrm{CV1}_s = \mathrm{RV4}_i$$

$$\sum 1 \cdot \mathrm{CV1}_s = \mathrm{RV5}$$

B.6 Finding a feasible solution

Original problem formulation for one direction:

$$\begin{aligned}
\max_{\rho_m, \xi} \quad & \left(CVaR^a - CVaR^d \right) \\
s.t. \quad & -CVaR^d + \sum_{t=1}^{T} \rho_t \cdot CVaR^d_{k,t} = \emptyset \\
& -y + \sum_{t=1}^{T} \rho_t \cdot y_t = \emptyset \\
& \xi - \langle y, x_s \rangle - P^a_s \leq \emptyset \\
& \xi - \frac{1}{\alpha_k} \sum_{s=1}^{S} P^a_s \cdot p_s - CVaR^a = \emptyset \\
& \sum_{t=1}^{T} \rho_t = 1 \\
& \rho_t, P^a_s \geq \emptyset
\end{aligned}$$

Conversion in GLPK format:

column identifiers:

$$\begin{aligned}
\mathrm{CV1}_s &= P^a_s & s &= \overline{1, \mathrm{szAprxPnt}} \\
\mathrm{CV2}_t &= \rho_t & t &= \overline{1, \mathrm{szHyprPt}} \\
\mathrm{CV3}_i &= y & i &= \overline{1, \mathrm{szDim}}
\end{aligned}$$

$$\begin{aligned} \text{CV4} &= \xi & 1 \\ \text{CV5} &= CVaR^a & 1 \\ \text{CV6} &= CVaR^d & 1 \end{aligned}$$

column limits:

$$\begin{aligned} \emptyset \leq & \quad \text{CV1}_s & \leq +\infty \\ \emptyset \leq & \quad \text{CV2}_t & \leq 1 \\ -\infty \leq & \quad \text{CV3}_i & \leq +\infty \\ -\infty \leq & \quad \text{CV4} & \leq +\infty \\ -\infty \leq & \quad \text{CV5} & \leq +\infty \\ -\infty \leq & \quad \text{CV6} & \leq +\infty \end{aligned}$$

row identifiers with dimensions:

$$\begin{aligned} \text{RV1} &: 1 \\ \text{RV2}_i &: i = \overline{1, \text{szDim}} \\ \text{RV3}_s &: s = \overline{1, \text{szAprxPnt}} \\ \text{RV4} &: 1 \\ \text{RV5} &: 1 \end{aligned}$$

row limits:

$$\begin{aligned} \emptyset = & \quad \text{RV1} & = \emptyset \\ \emptyset = & \quad \text{RV2}_i & = \emptyset \\ -\infty \leq & \quad \text{RV3}_s & \leq \emptyset \\ \emptyset = & \quad \text{RV4} & = \emptyset \\ 1 = & \quad \text{RV5} & = 1 \end{aligned}$$

problem formulation:

$$
\begin{aligned}
\max \quad & \sum \emptyset \cdot \mathrm{CV1}_s + \sum \emptyset \cdot \mathrm{CV2}_t + \sum \emptyset \cdot \mathrm{CV3}_i \\
& + \sum \emptyset \cdot \mathrm{CV4} + \sum 1 \cdot \mathrm{CV5} + \sum [-1] \cdot \mathrm{CV6} \\
s.t. \quad & [-1] \cdot \mathrm{CV6} + \sum \mathrm{CVaR}^d_{k,t} \cdot \mathrm{CV2}_t && = \mathrm{RV1} \\
& [-1] \cdot \mathrm{CV3}_i + \sum_t y_{t,i} \cdot \mathrm{CV2}_t && = \mathrm{RV2}_i \\
& [-1] \cdot \mathrm{CV1}_s + \sum_i [-x_{s,i}] \cdot \mathrm{CV3}_i + 1 \cdot \mathrm{CV4} && = \mathrm{RV3}_s \\
& \sum_s \left[-\frac{p_s}{\alpha_k} \right] \cdot \mathrm{CV1}_s + 1 \cdot \mathrm{CV4} + [-1] \cdot \mathrm{CV5} && = \mathrm{RV4} \\
& \sum_t 1 \cdot \mathrm{CV2}_t && = \mathrm{RV5}
\end{aligned}
$$

Appendix C

Financial Optimization Model Implementation

C.1 Parameters for financial optimization model

Let's mark:

$m = E[\mathcal{X}]$

x_j are points in the distribution $\mathcal{X}$.

The following values are consider as given:

szDim. This is the number of financial investment instruments in the model.

szDistrPnt. This is the number of sample points in the original distribution $\mathcal{X}$.

szPrbLvl. This is the number of probability levels for CVaR weighted summation.

Index for finite sets:

$$
\begin{aligned}
i &= \overline{1, \text{szDim}} \\
s &= \overline{1, \text{szDistrPnt}} \\
k &= \overline{1, \text{szPrbLvl}}
\end{aligned}
$$

Additional parameters:

- ξ_k is VaR value for α_k probability level.
- $P_{s,k}$ is the value of $[\mathrm{VaR} - \langle x_s \cdot \rho, y^* \rangle]^+$ for particular scenario x_s and investment portfolio y^*.
- CVaR_k is $\mathrm{CVaR}(\mathcal{R}, y^*, \alpha_k)$

C.2 Model with weighted CVaR deviation measure

$$
\begin{aligned}
\max_{y^*, \xi_k, P_{s,k}} \quad & \sum_i m_i \cdot y_i^* \cdot \rho_i + \sum_k w_k \cdot CVaR_k \\
s.t. \quad & \xi_k - \sum_i y_i^* \cdot \rho_i \cdot x_{s,i} - P_{s,k} \leqslant \emptyset \\
& \xi_k - \frac{1}{\alpha_k} \sum_s P_{s,k} \cdot p_s - CVaR_k = \emptyset \\
& \sum_i \left(y_i^* - y_i^I \right) \cdot \rho_i \leqslant B \\
& l_i \leqslant y_i^* \leqslant u_i \\
& P_{s,k} \geqslant \emptyset
\end{aligned}
$$

Conversion in GLPK format:

column identifiers:

$$
\begin{aligned}
\mathrm{CV\_Opt\_Prt}_i &= y_i^* & i &= \overline{1, \mathrm{szDim}} \\
\mathrm{CV\_VaR}_k &= \xi_k & k &= \overline{1, \mathrm{szPrbLvl}} \\
\mathrm{CV\_Pos\_Dif}_{s,k} &= P_{s,k} & s &= \overline{1, \mathrm{szDistrPnt}}, k = \overline{1, \mathrm{szPrbLvl}} \\
\mathrm{CV\_CVaR}_k &= \mathrm{CVaR}_k & k &= \overline{1, \mathrm{szPrbLvl}}
\end{aligned}
$$

column limits:

$$
\begin{aligned}
l_i \leq \quad & \text{CV\_Opt\_Prt}_i && \leq u_i \\
-\infty \leq \quad & \text{CV\_VaR}_k && \leq +\infty \\
\emptyset \leq \quad & \text{CV\_Pos\_Dif}_{s,k} && \leq +\infty \\
-\infty \leq \quad & \text{CV\_CVaR}_k && \leq +\infty
\end{aligned}
$$

row identifiers with dimensions:

$$
\begin{aligned}
\text{RV\_Pos\_Def}_{s,k} &: \; s = \overline{1, \text{szDistrPnt}}, k = \overline{1, \text{szPrbLvl}} \\
\text{RV\_CVaR\_Def}_k &: \; k = \overline{1, \text{szPrbLvl}} \\
\text{RV\_Budget\_Rst} &: \; 1
\end{aligned}
$$

row limits:

$$
\begin{aligned}
-\infty \leqslant \quad & \text{RV\_Pos\_Def}_{s,k} && \leqslant \emptyset \\
\emptyset = \quad & \text{RV\_CVaR\_Def}_k && = \emptyset \\
-\infty \leqslant \quad & \text{RV\_Budget\_Rst} && \leqslant B + \sum \rho_i \cdot y_i^I
\end{aligned}
$$

problem formulation:

$$
\begin{aligned}
\max \quad & \sum_i m_i \cdot \text{CV\_Opt\_Prt}_i + \sum_k \emptyset \cdot \text{CV\_VaR}_k + \\
& + \sum_{s,k} \emptyset \cdot \text{CV\_Pos\_Dif}_{s,k} + \sum_k \omega_k \cdot \text{CV\_CVaR}_k \\
s.t. \quad & \text{CV\_VaR}_k + [-1] \cdot \text{CV\_Pos\_Dif}_{s,k} + \sum_i [-x_{s,i}] \cdot \text{CV\_Opt\_Prt}_i = \text{RV\_Pos\_Def}_{s,k} \\
& \text{CV\_VaR}_k + \sum_s \left[-\frac{p_s}{\alpha_k} \right] \cdot \text{CV\_Pos\_Dif}_{s,k} + [-1] \cdot \text{CV\_CVaR}_k = \text{RV\_CVaR\_Def}_k \\
& \sum_i \rho_i \cdot \text{CV\_Opt\_Prt}_i = \text{RV\_Budget\_Rst}
\end{aligned}
$$

C.3 Model with downside mean semi-deviation deviation measure

$$\max_{y^*,\mathrm{Pos}_s} \quad \sum_i m_i \cdot y_i^* \cdot \rho_i + \lambda \cdot \sum_s \mathrm{Pos}_s \cdot p_s$$

$$s.t. \quad \sum_i y_i^* \cdot \rho_i \cdot (m_i - x_{s,i}) - \mathrm{Pos}_s \quad \leqslant \emptyset$$

$$\sum_i \left(y_i^* - y_i^I\right) \cdot \rho_i \leqslant B$$

$$l_i \leqslant y_i^* \leqslant u_i$$

$$\mathrm{Pos}_s \geqslant \emptyset$$

Conversion in GLPK format:

column identifiers:

$$\mathrm{CV\_Opt\_Prt}_i \;= y_i^* \qquad i = \overline{1, \mathrm{szDim}}$$

$$\mathrm{CV\_Pos\_Dev}_s \;= \mathrm{Pos}_s \quad s = \overline{1, \mathrm{szDistrPnt}}$$

column limits:

$$l_i \leq \; \mathrm{CV\_Opt\_Prt}_i \; \leq u_i$$

$$\emptyset \leq \; \mathrm{CV\_Pos\_Dev}_s \; \leq +\infty$$

row identifiers with dimensions:

$$\mathrm{RV\_Pos\_Def}_s \;:\; s = \overline{1, \mathrm{szDistrPnt}}$$

$$\mathrm{RV\_Budget\_Rst} \;:\; 1$$

row limits:

$$-\infty \leqslant \quad \mathrm{RV\_Pos\_Def}_s \quad \leqslant \emptyset$$

$$-\infty \leqslant \; \mathrm{RV\_Budget\_Rst} \; \leqslant B + \sum \rho_i \cdot y_i^I$$

problem formulation:

$$\begin{aligned}
\max \quad & \sum m_i \cdot \text{CV_Opt_Prt}_i + \sum \,(-\lambda \cdot p_s) \cdot \text{CV_Pos_Dev}_s \\
s.t. \quad & [-1] \cdot \text{CV_Pos_Dev}_s + \sum_i [(m_i - x_{s,i}) \cdot \rho_i] \cdot \text{CV_Opt_Prt}_i && = \text{RV_Pos_Def}_s \\
& \sum \rho_i \cdot \text{CV_Opt_Prt}_i && = \text{RV_Budget_Rst}
\end{aligned}$$

Appendix D

Specifications and Schematic

Table D.1. Hardware and software specification for experiment

Component	Specification
Processor	AMD Athlon XP 2100+
Memory type	DDR 333
Memory capacity	256 MB
Operating system	GNU/Linux (Slackware 10.0)
Compiler	GCC 3.3.4
Libraries	STL 3.3.2
LP solver	GLPK 4.8

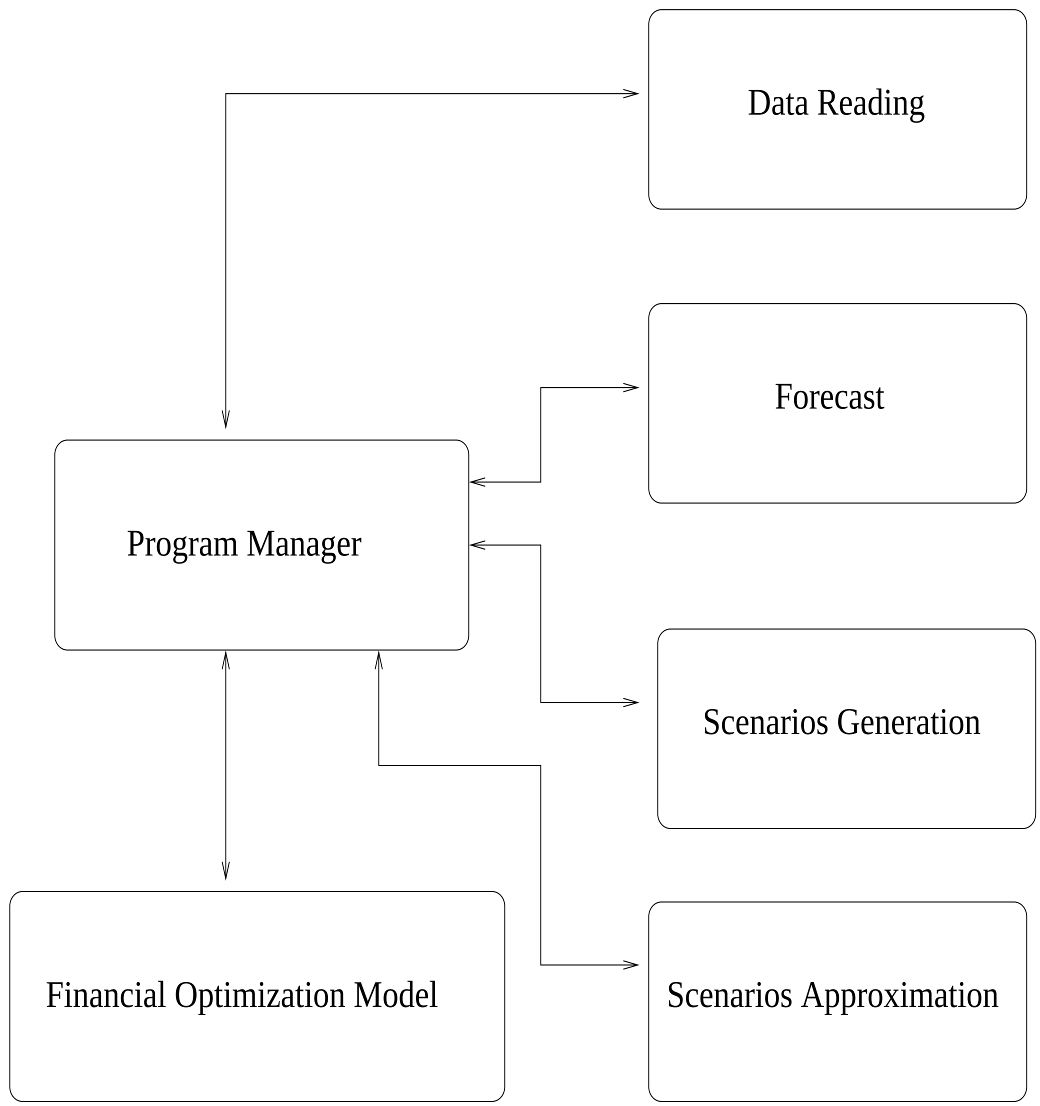

Figure D.1. The scheme of program for financial optimization

Appendix E

Experiment Results

Table E.1. CPU time in seconds for solving model with weighted CVaR deviation measure (Level 1) and data set SP5

Scenarios	Original	Number of points in approximation						
		6	12	24	33	42	53	64
100	0.0174	0.0326	0.1551	0.827	2.472	6.04	15.84	43.95
500	0.1350	0.0548	0.2089	1.046	1.828	6.47	13.51	48.13
1000	0.4656	0.0816	0.2024	0.889	2.037	3.79	18.21	36.26
3000	1.2249	0.2131	0.3188	1.211	2.699	5.93	16.71	35.17
5000	19.1421	0.3108	0.4396	1.173	2.896	6.58	18.75	25.45
10000	15.1211	0.5884	0.7495	1.692	3.801	7.64	13.95	25.36

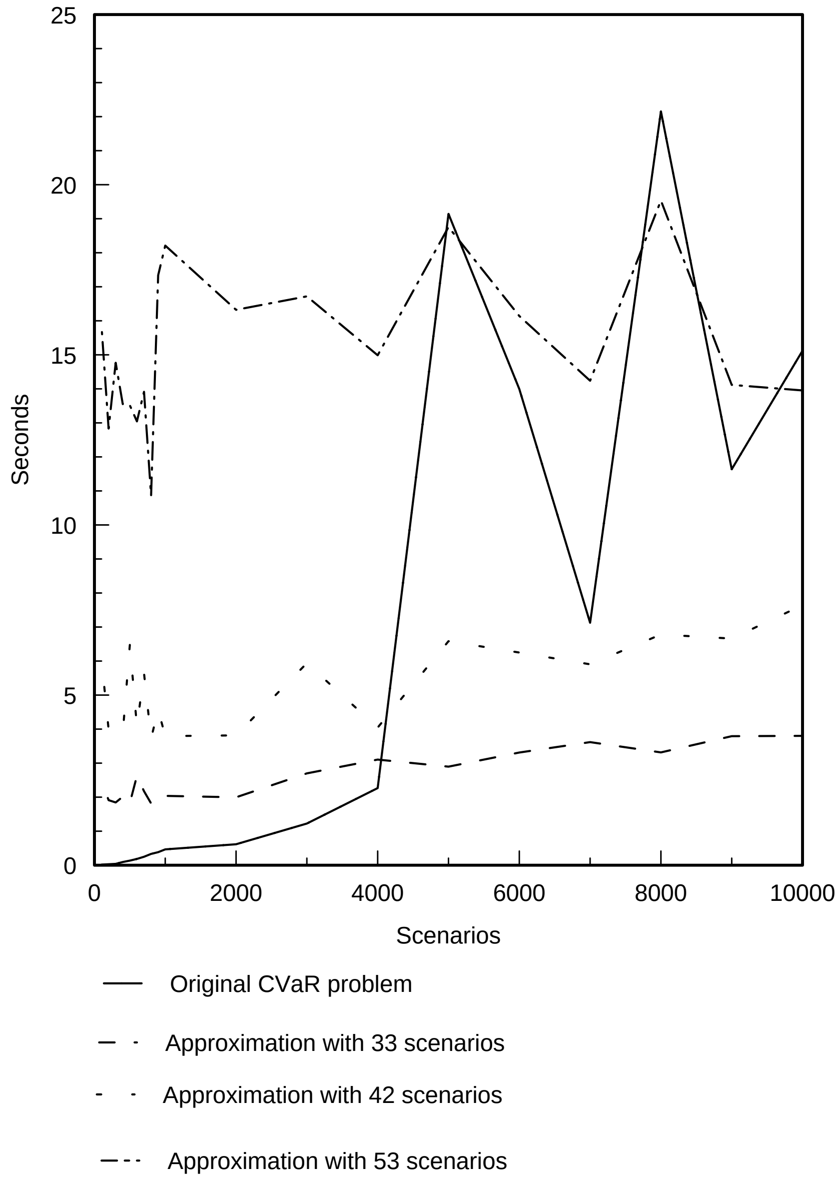

Figure E.1. CPU time comparison for solving model with weighted CVaR deviation measure (Level 1) and data set SP5

Table E.2. CPU time in seconds for solving model with weighted CVaR deviation measure (Level 2) and data set SP5

Scenarios	Original	Number of points in approximation						
		6	12	24	33	42	53	64
100	0.0414	0.1325	0.2603	2.34	13.77	39.99	90.36	240.37
500	0.2297	0.1312	0.3867	2.04	15.19	39.25	125.73	276.20
1000	0.7739	0.2215	0.4159	2.77	21.72	34.18	123.59	382.43
3000	6.7675	0.3088	0.5264	3.10	11.24	33.87	120.12	301.32
5000	19.7204	0.3994	0.6505	2.09	11.46	39.87	111.56	313.71
10000	98.7505	0.6949	0.9581	3.23	15.67	46.88	109.82	320.28

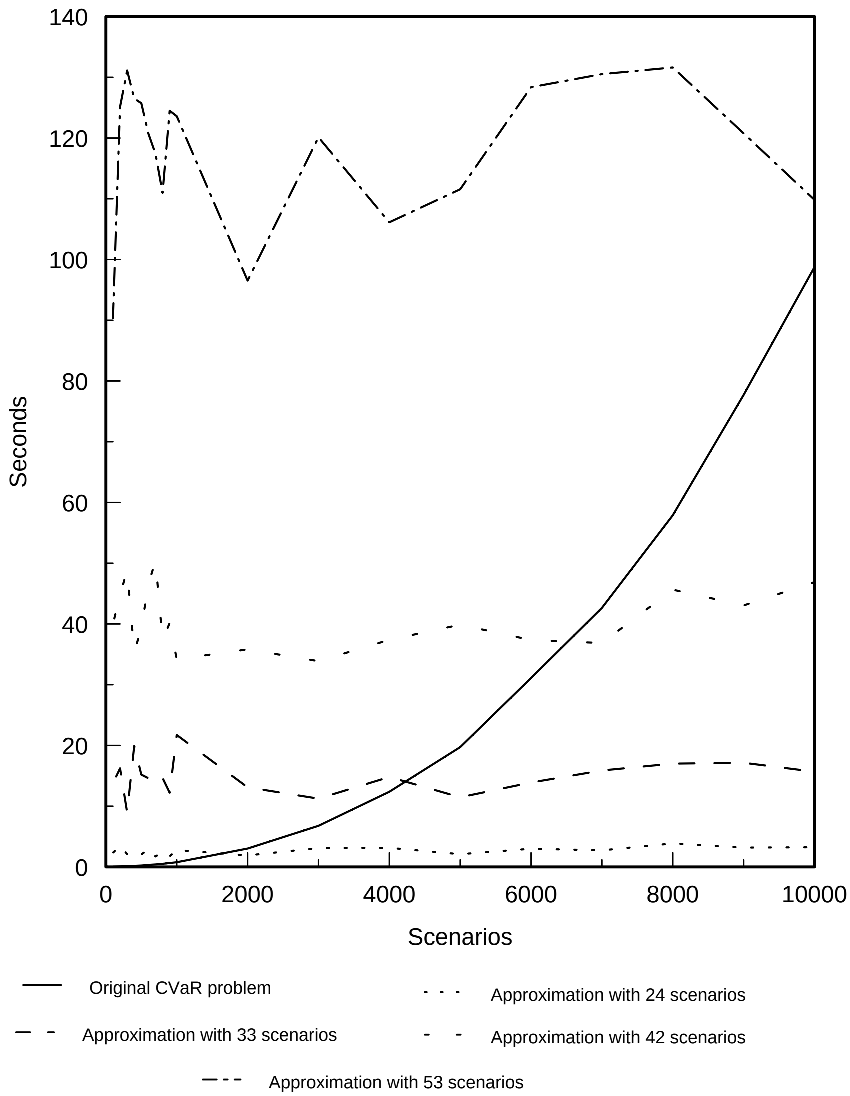

Figure E.2. CPU time comparison for solving model with weighted CVaR deviation measure (Level 2) and data set SP5

Table E.3. CPU time in seconds for solving model with weighted CVaR deviation measure (Level 3) and data set SP5

Scenarios	Original	Number of points in approximation						
		6	12	24	33	42	53	64
100	0.04	0.148	0.478	2.86	17.85	41.03	266.66	567.44
500	0.36	0.160	0.559	4.05	20.74	60.95	243.42	663.14
1000	1.32	0.162	0.532	5.116	22.83	71.40	230.93	652.16
3000	12.11	0.393	0.786	4.59	23.55	65.18	215.05	697.52
5000	38.03	0.414	1.369	3.79	17.91	70.13	246.03	839.04
10000	181.39	0.717	1.174	4.81	27.35	80.15	304.10	784.52

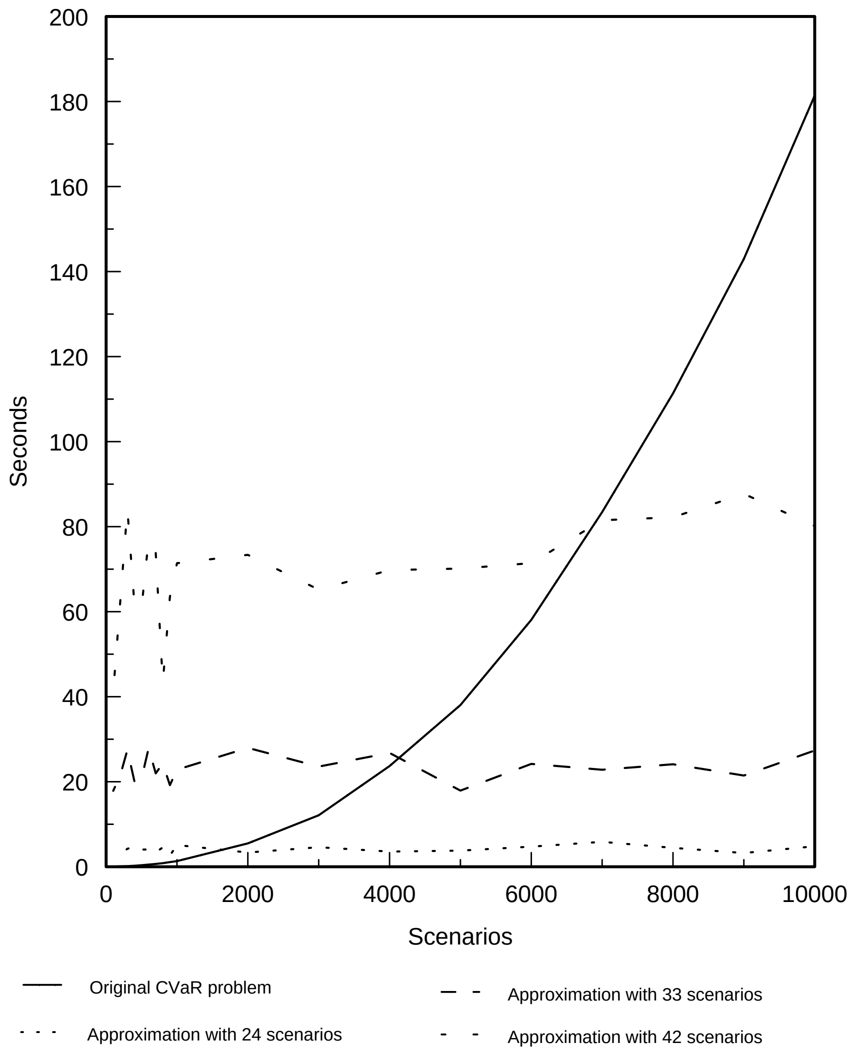

Figure E.3. CPU time comparison for solving model with weighted CVaR deviation measure (Level 3) and data set SP5

Table E.4. CPU time in seconds for solving model with weighted CVaR deviation measure, 3000 original scenarios and data set SP5

Number of approximation points	CVaR model with 1 Level	2 Levels	3 Levels
6	0.1923	0.2094	0.2258
12	0.3177	0.4373	0.6281
18	0.5366	1.0801	1.2760
24	1.0940	3.0756	4.4677
30	1.6054	7.1292	7.0359
36	3.5111	18.4708	44.2272
42	5.8888	34.0953	64.9417
48	8.4266	62.7366	144.3783
54	17.4137	127.4400	299.4389
60	29.5426	185.7776	647.3501
66	36.6286	307.8309	871.6054
72	35.8680	582.4237	1513.5684
78	110.1980	705.9582	2238.1759
84	117.3843	1094.4741	2700.4273
90	202.7877	1722.4632	3629.0156
96	202.9797	2324.3468	4311.0073
102	394.4830	2802.8796	5742.7305

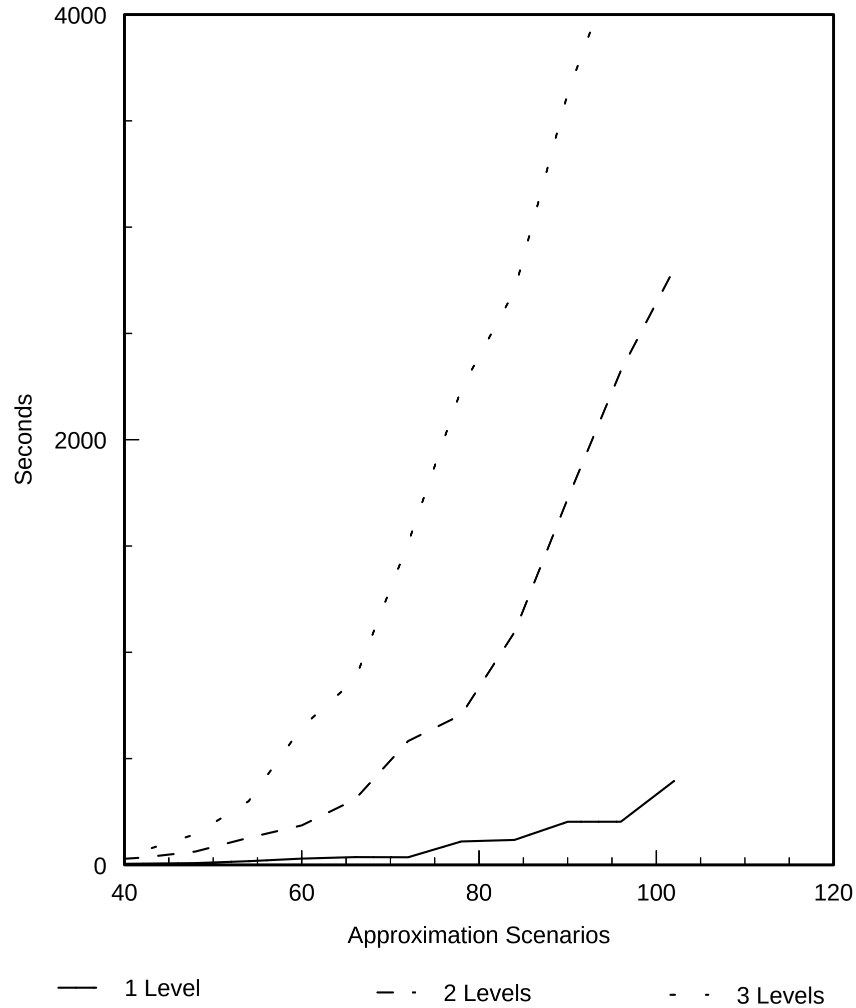

Figure E.4. CPU time comparison for solving model with weighted CVaR deviation measure, 3000 original scenarios and data set SP5

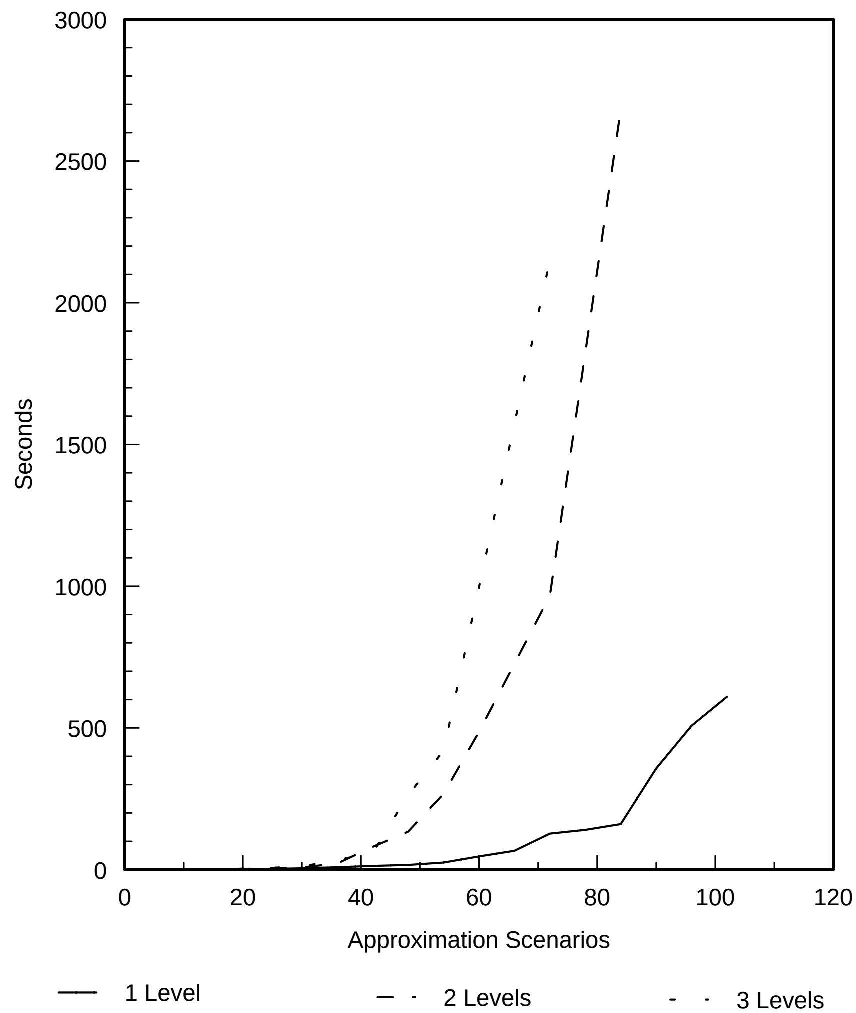

Figure E.5. CPU time comparison for solving model with weighted CVaR deviation measure, 3000 original scenarios and data set SP10

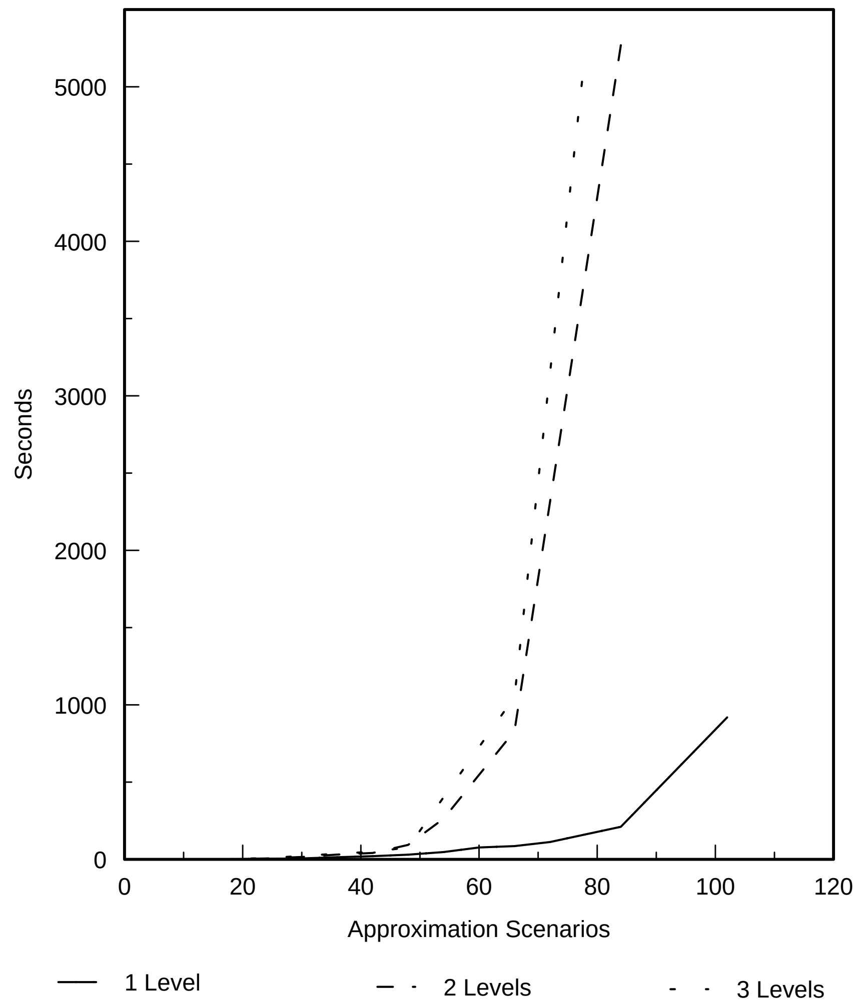

Figure E.6. CPU time comparison for solving model with weighted CVaR deviation measure, 3000 original scenarios and data set SP15

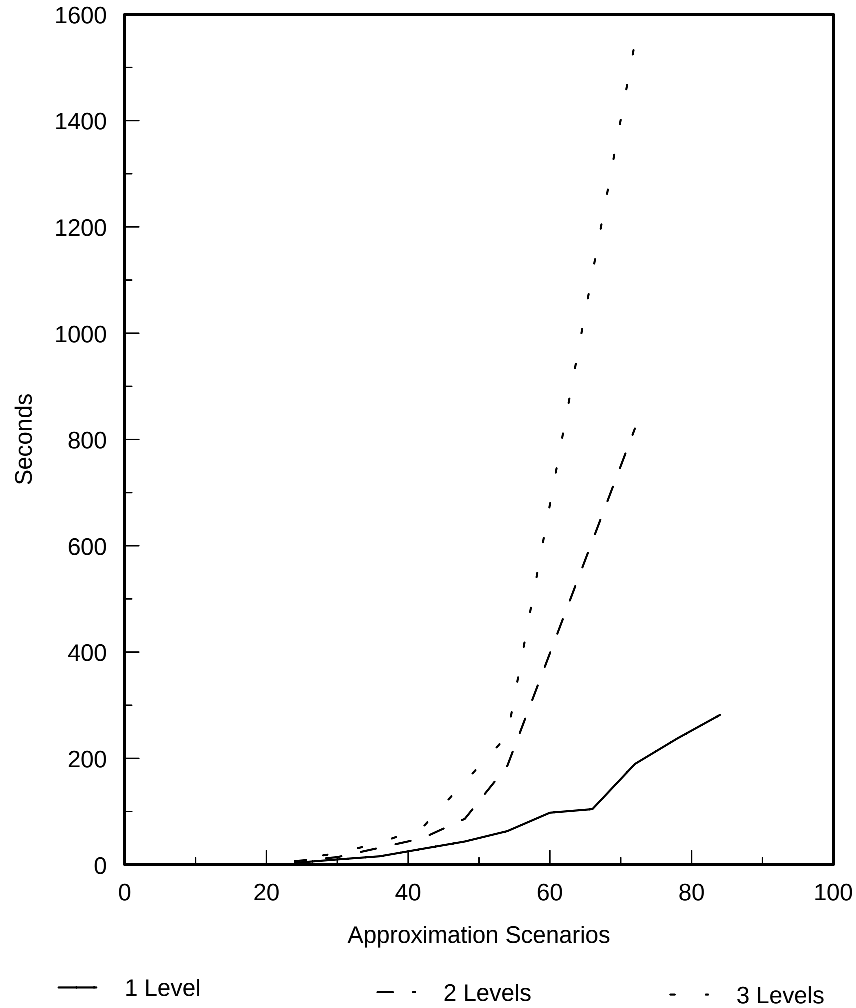

Figure E.7. CPU time comparison for solving model with weighted CVaR deviation measure, 3000 original scenarios and data set SP20

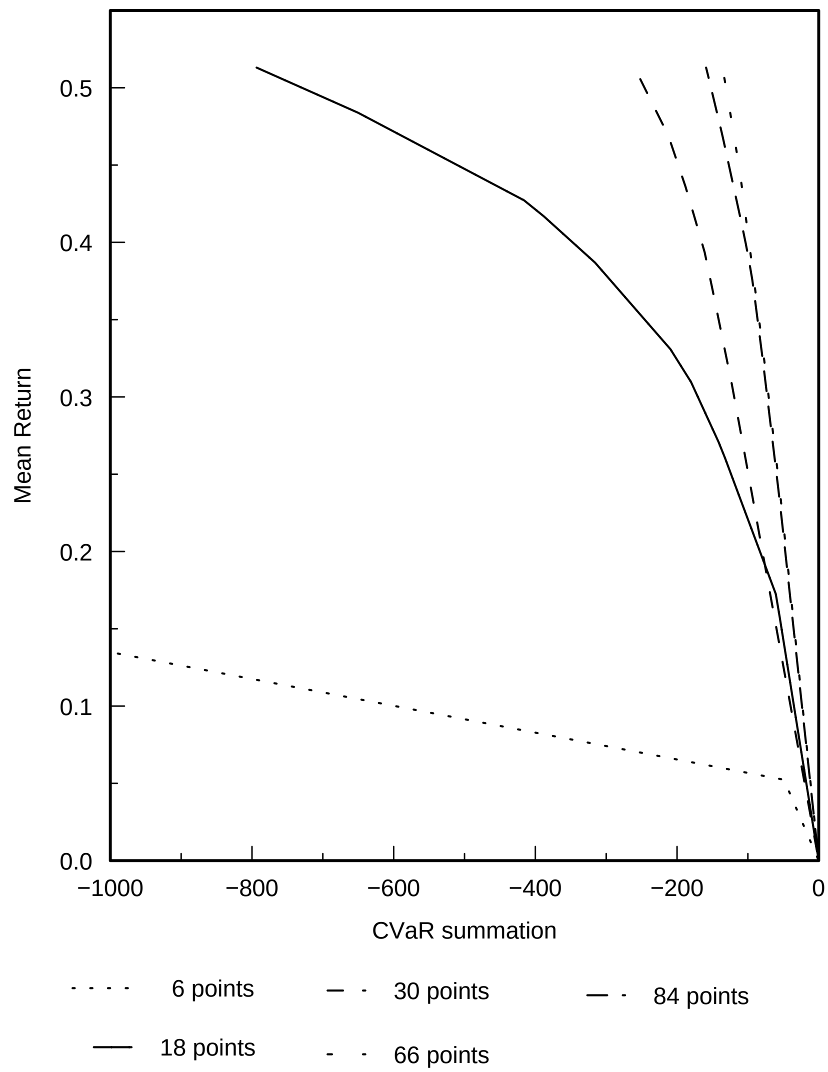

Figure E.8. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 1) and data set SP5 with increasing number of points in approximation

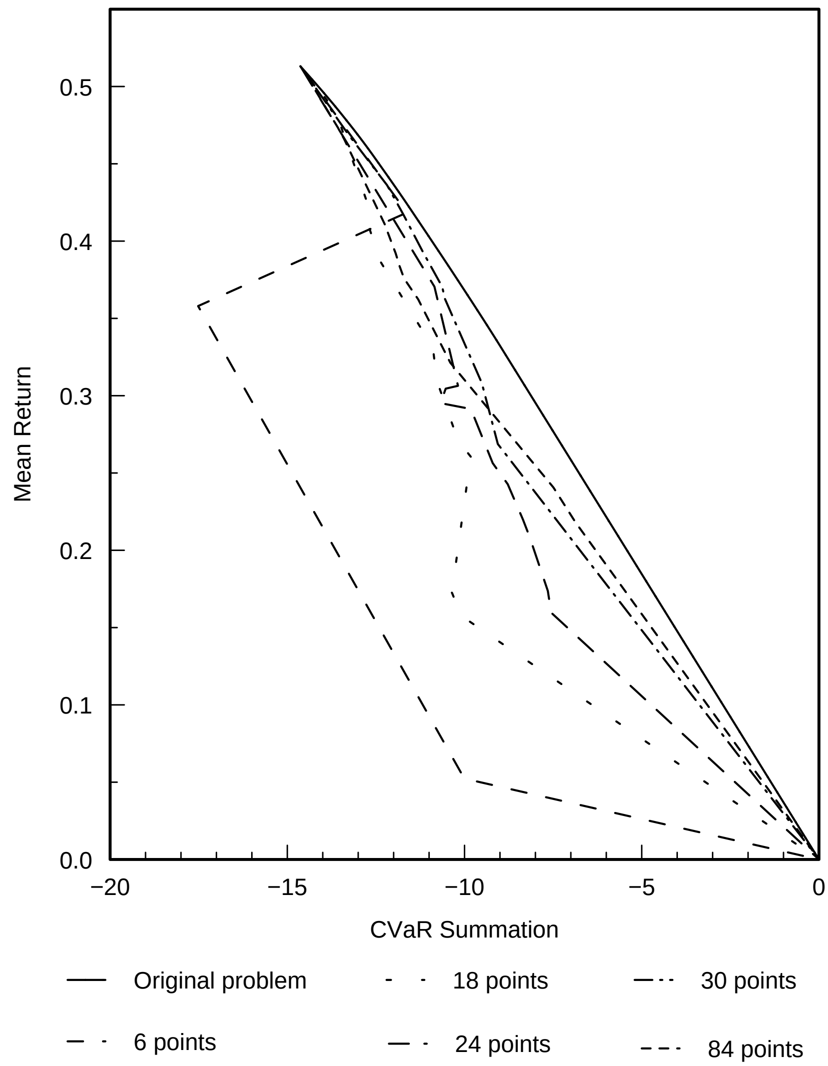

Figure E.9. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 1) and data set SP5 with increasing number of points in approximation

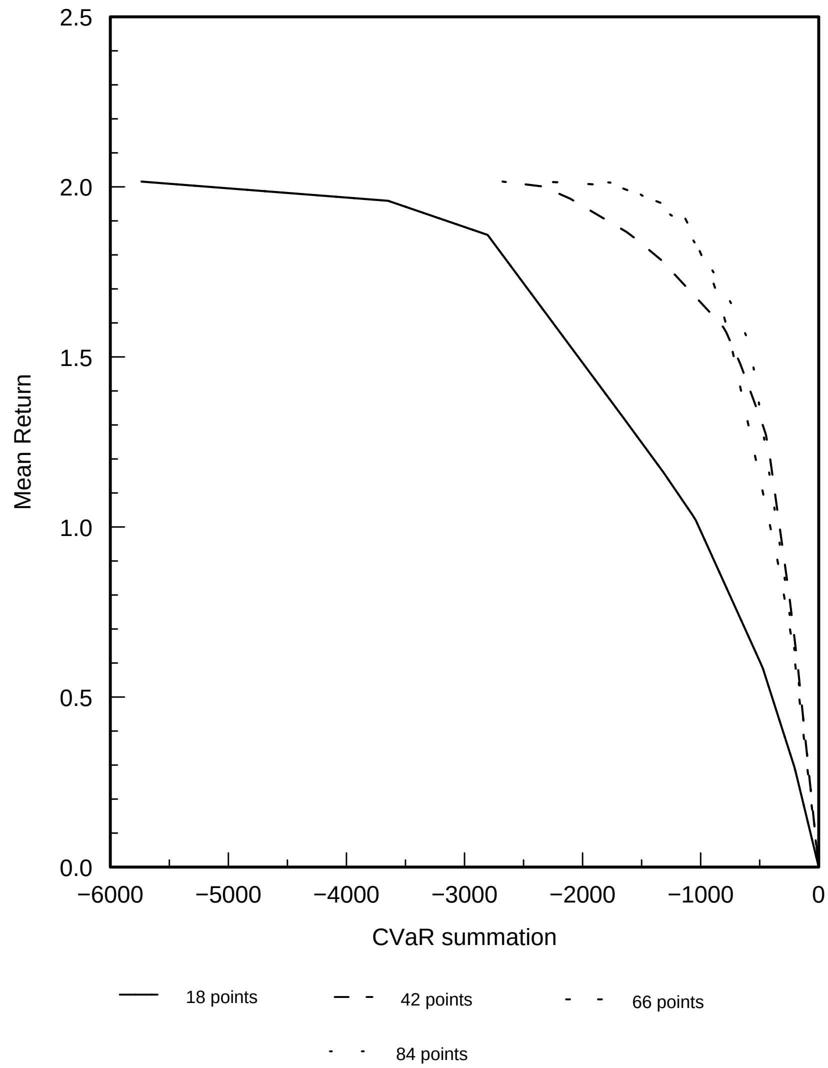

Figure E.10. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 1) and data set SP10 with increasing number of points in approximation

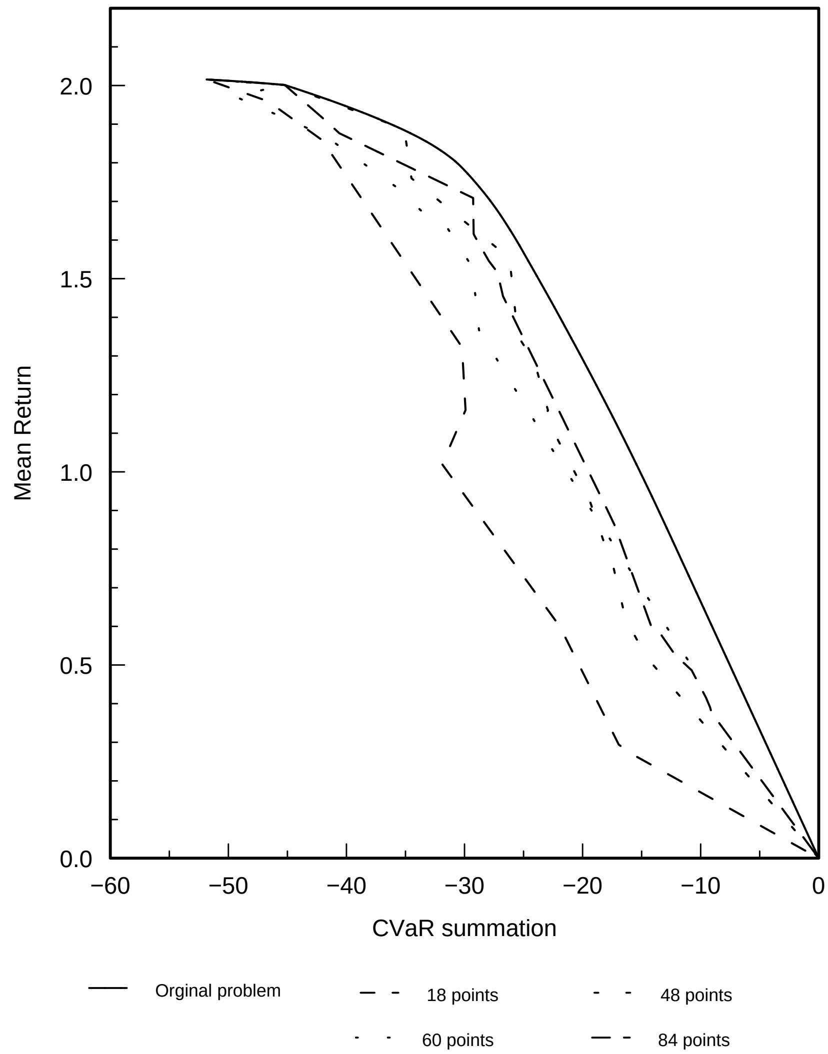

Figure E.11. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 1) and data set SP10 with increasing number of points in approximation

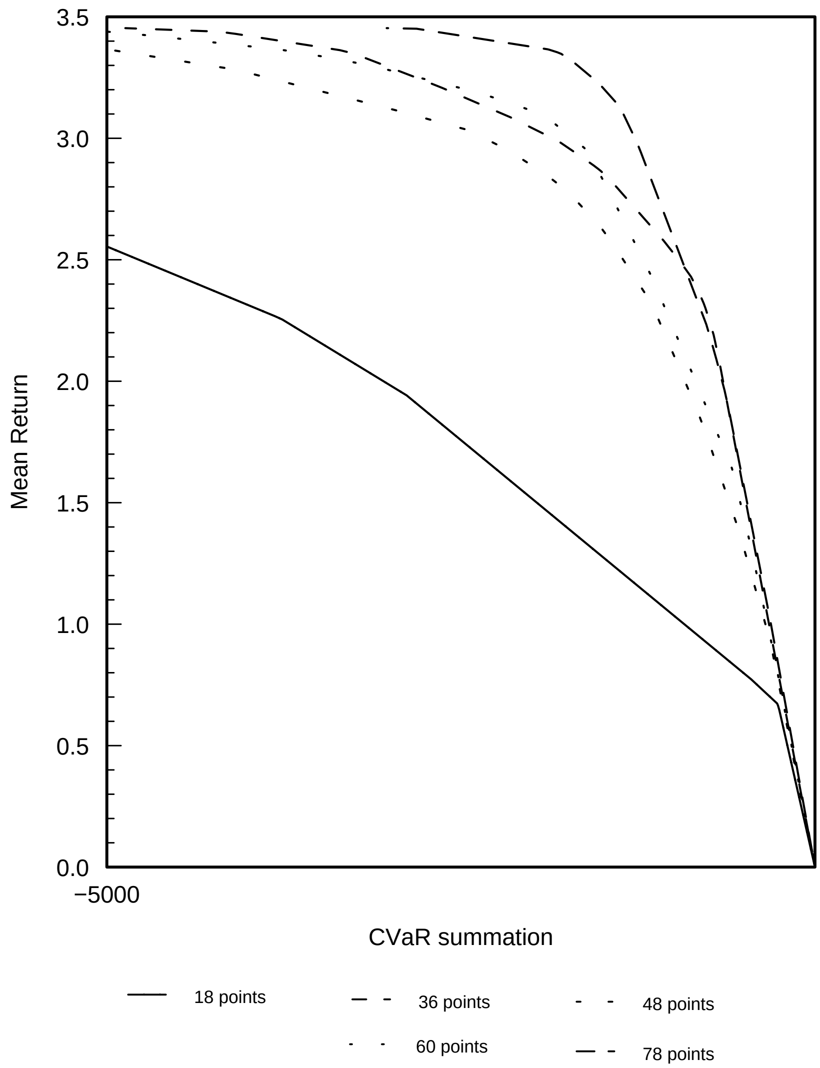

Figure E.12. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 1) and data set SP15 with increasing number of points in approximation

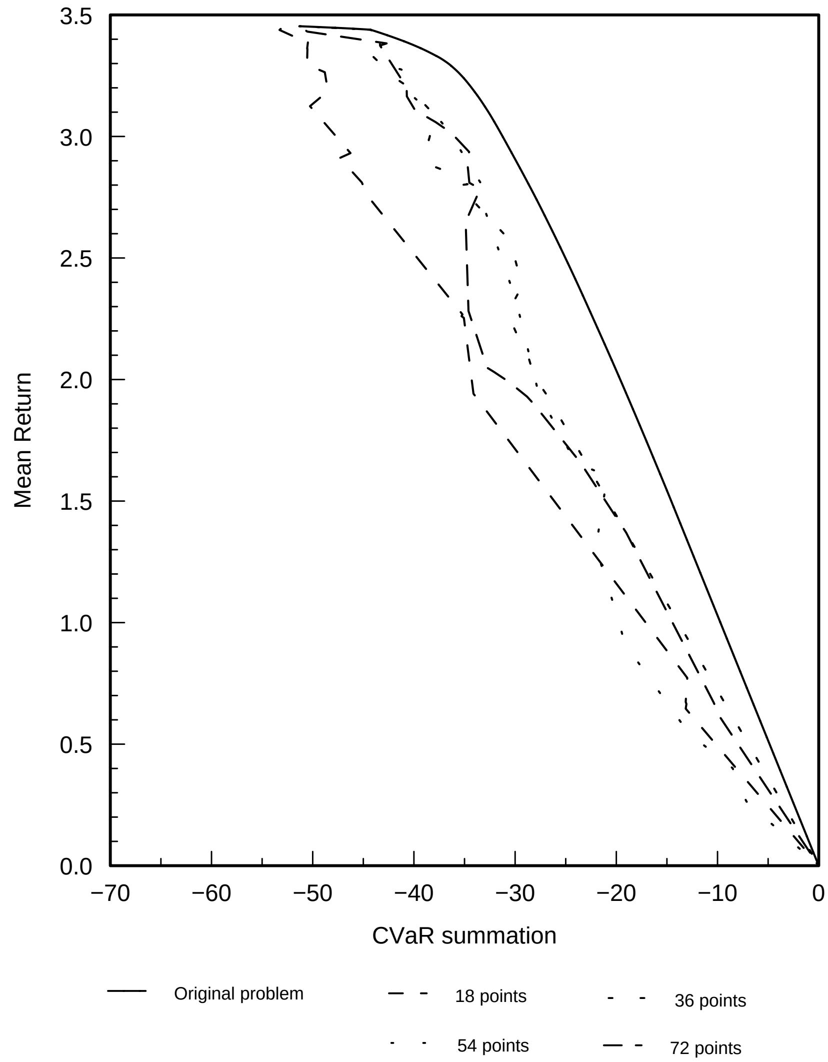

Figure E.13. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 1) and data set SP15 with increasing number of points in approximation

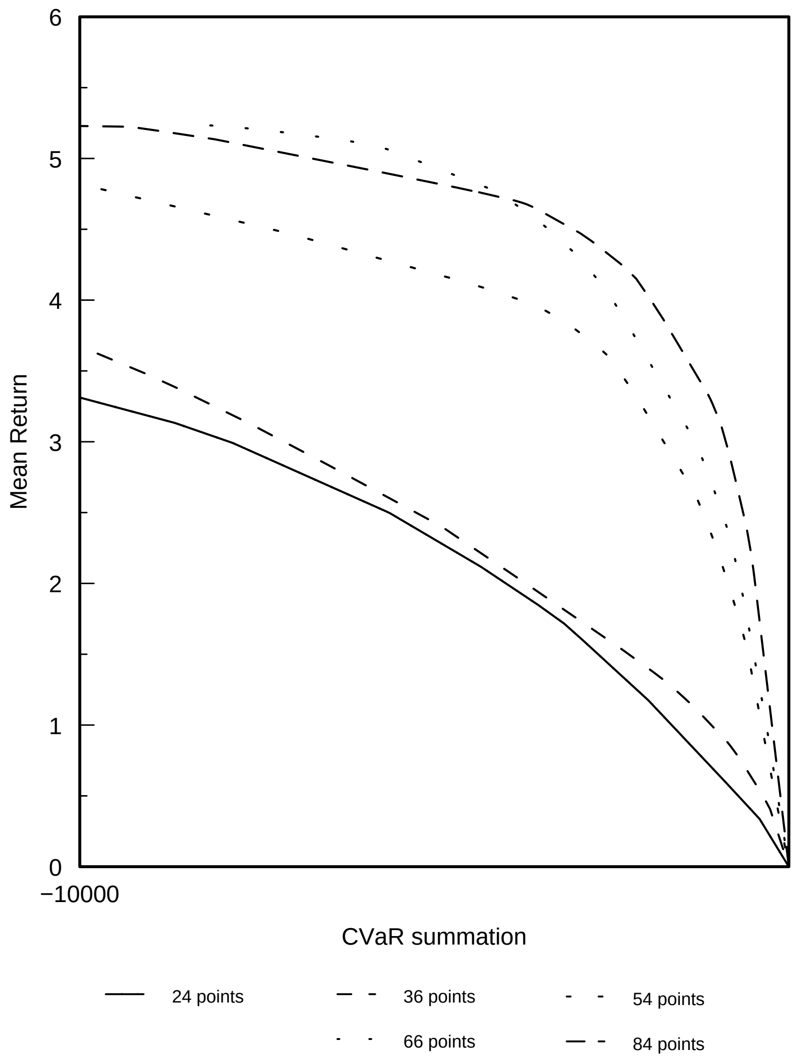

Figure E.14. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 1) and data set SP20 with increasing number of points in approximation

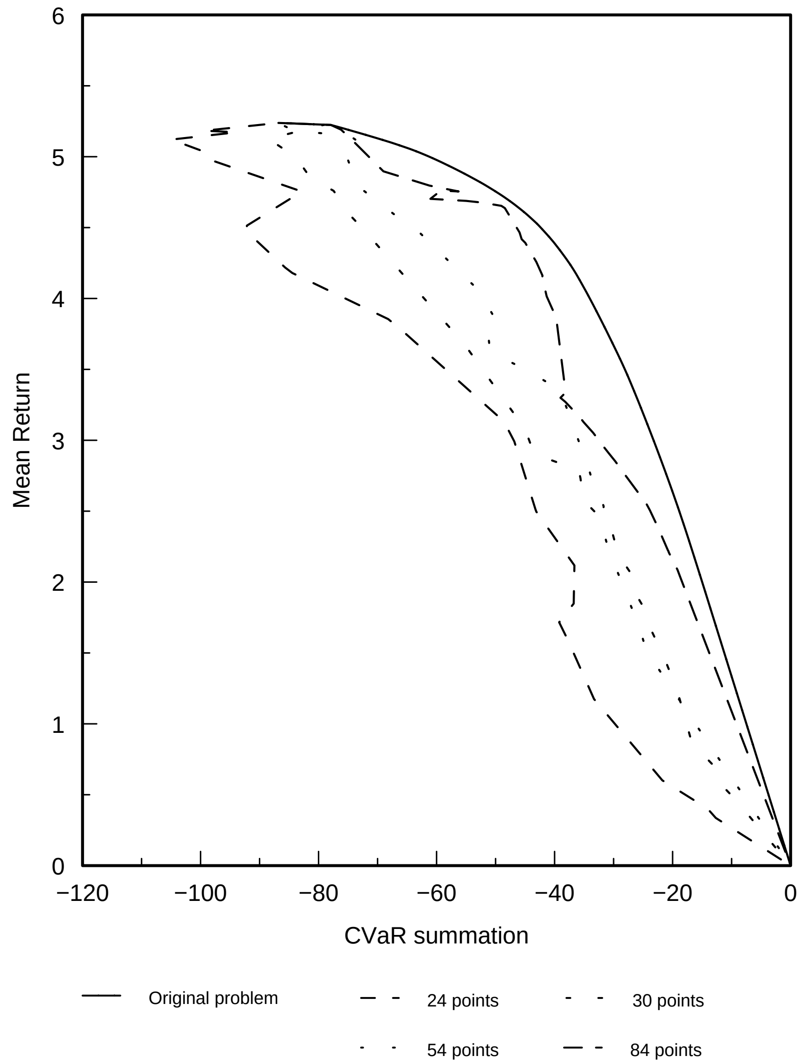

Figure E.15. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 1) and data set SP20 with increasing number of points in approximation

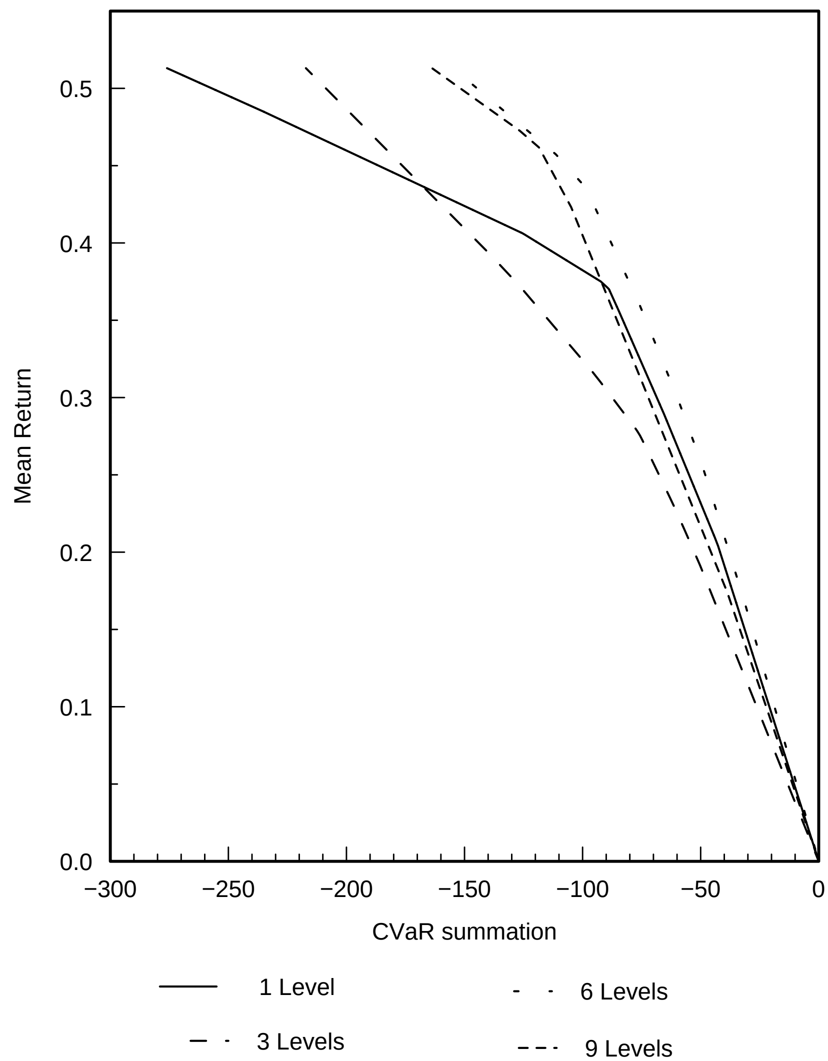

Figure E.16. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 1) and data set SP5 with increasing number of probability levels in approximation

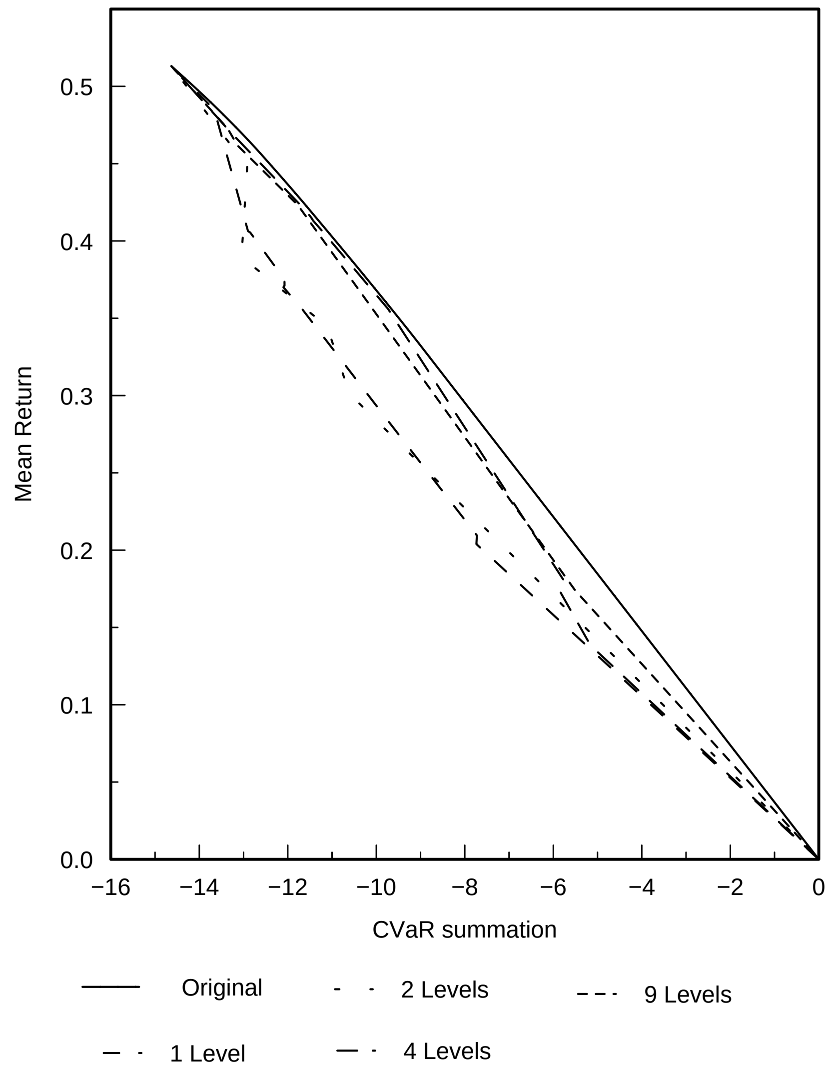

Figure E.17. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 1) and data set SP5 with increasing number of probability levels in approximation

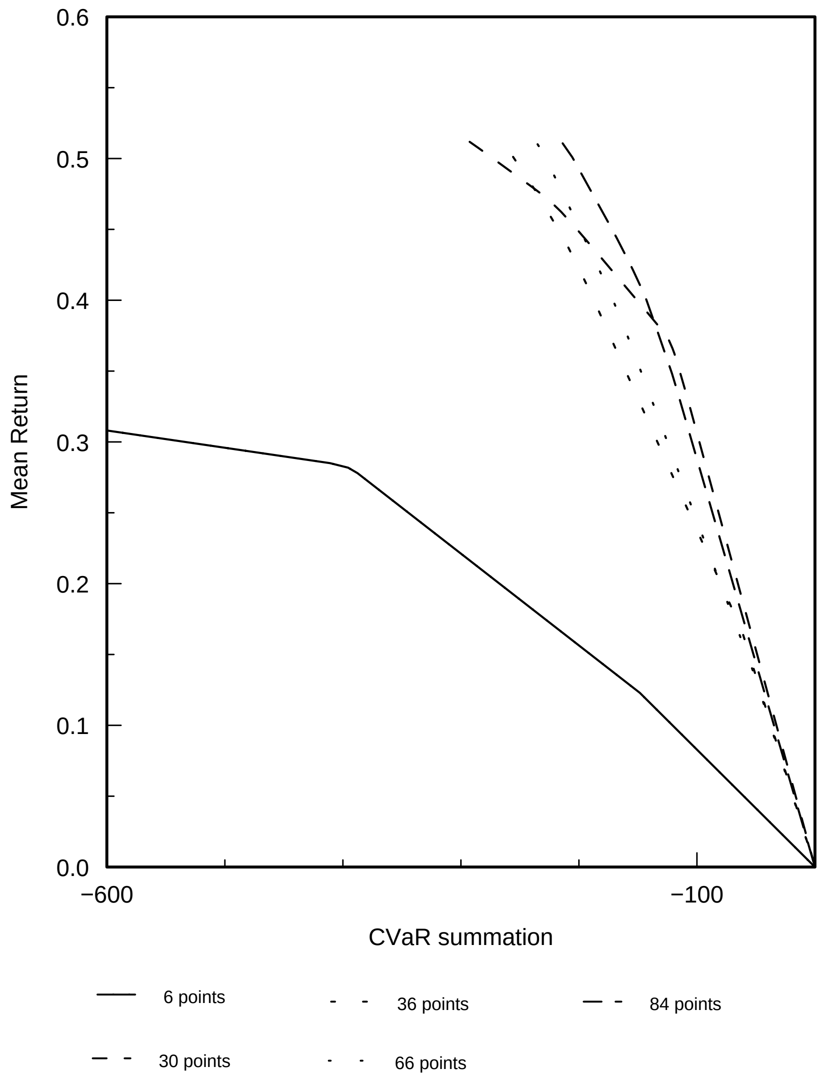

Figure E.18. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 2) and data set SP5 with increasing number of points in approximation

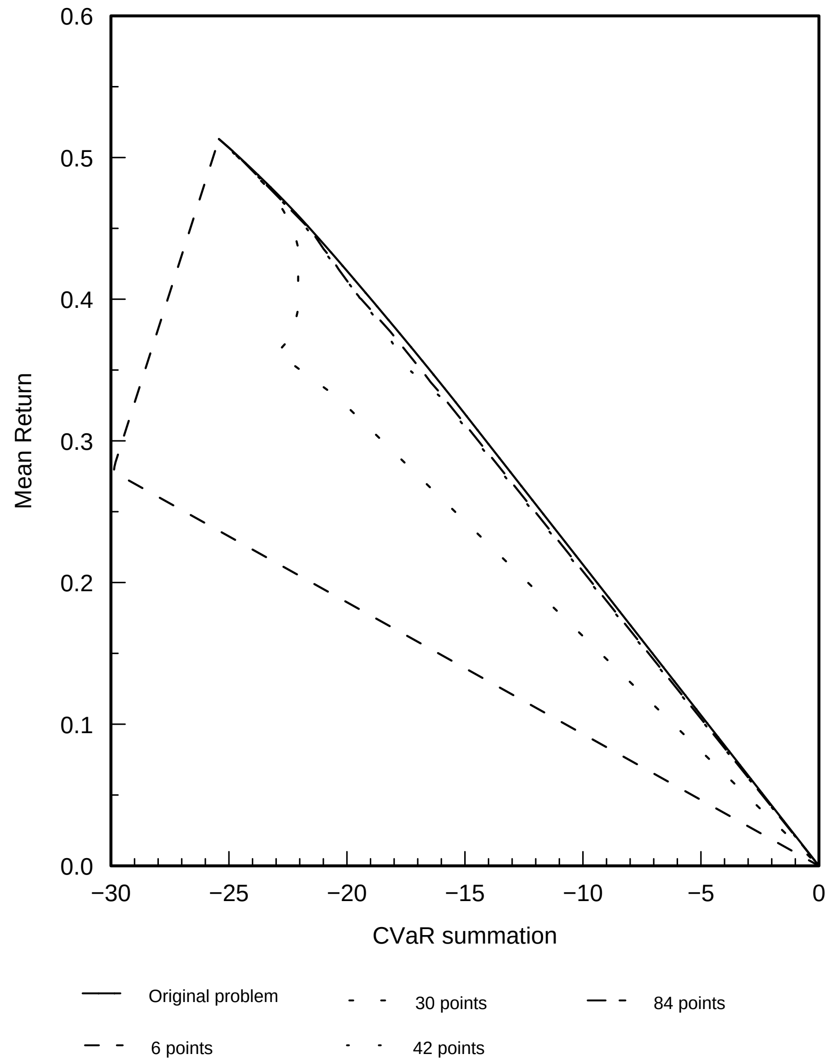

Figure E.19. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 2) and data set SP5 with increasing number of points in approximation

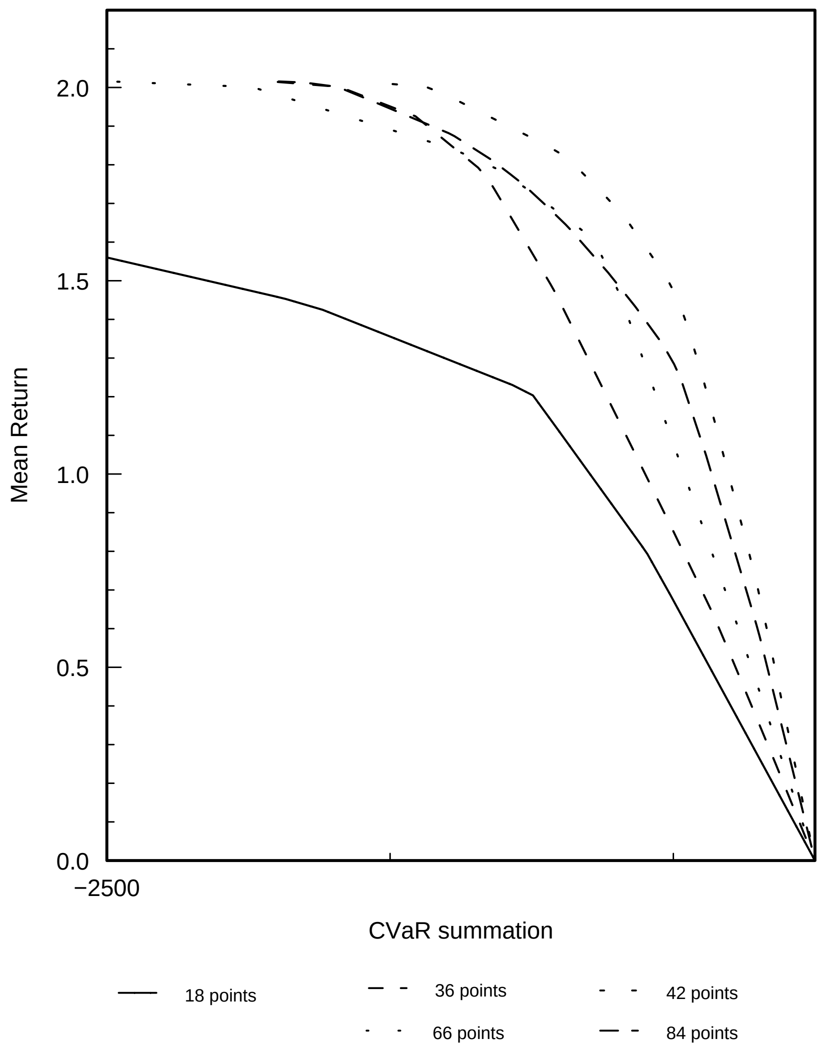

Figure E.20. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 2) and data set SP10 with increasing number of points in approximation

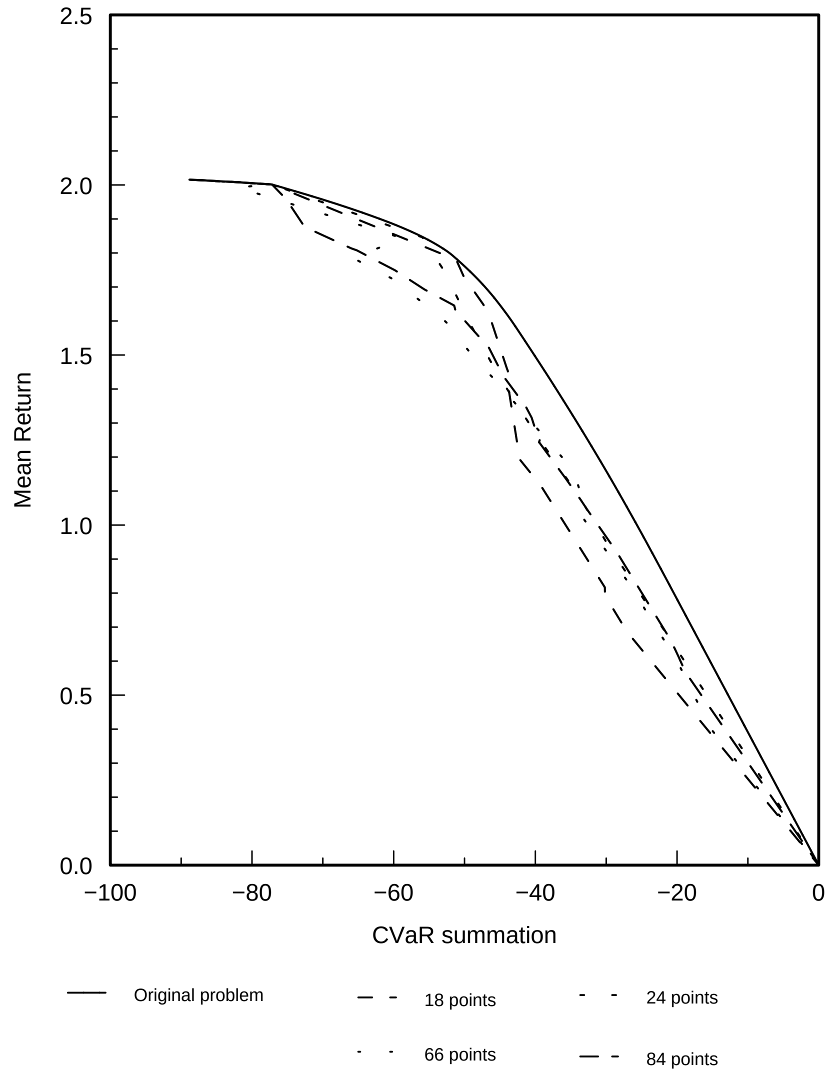

Figure E.21. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 2) and data set SP10 with increasing number of points in approximation

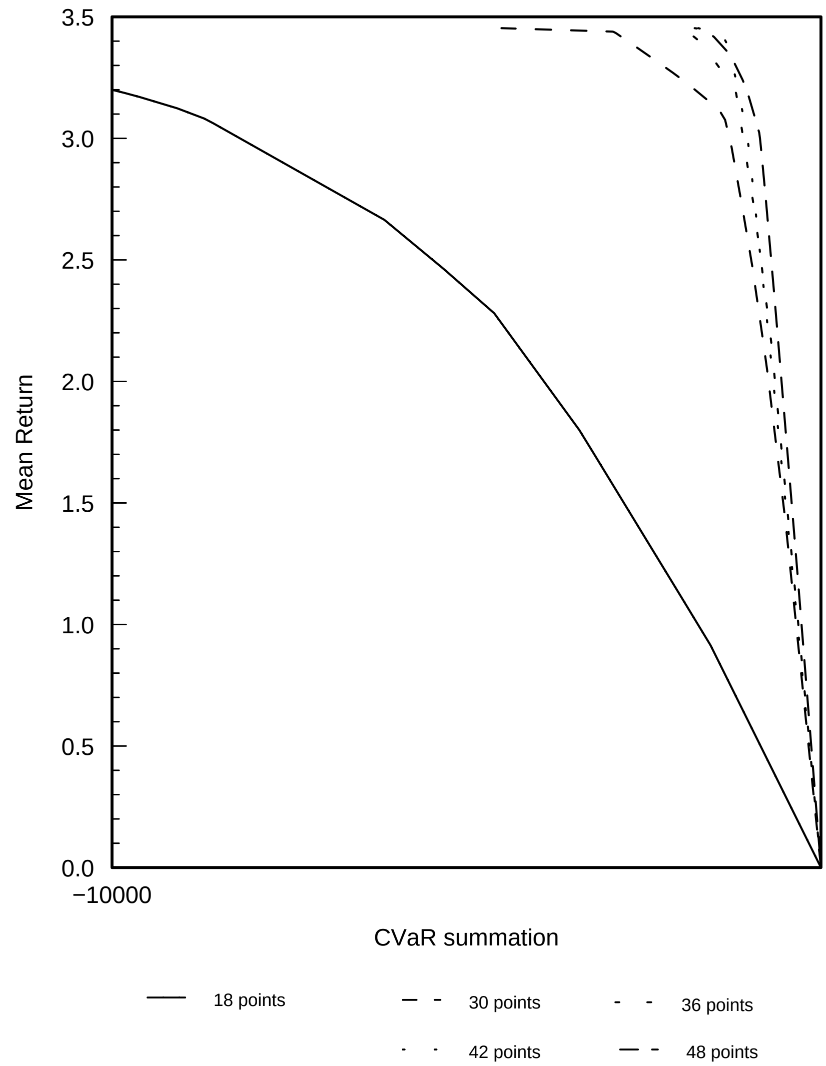

Figure E.22. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 2) and data set SP15 with increasing number of points in approximation

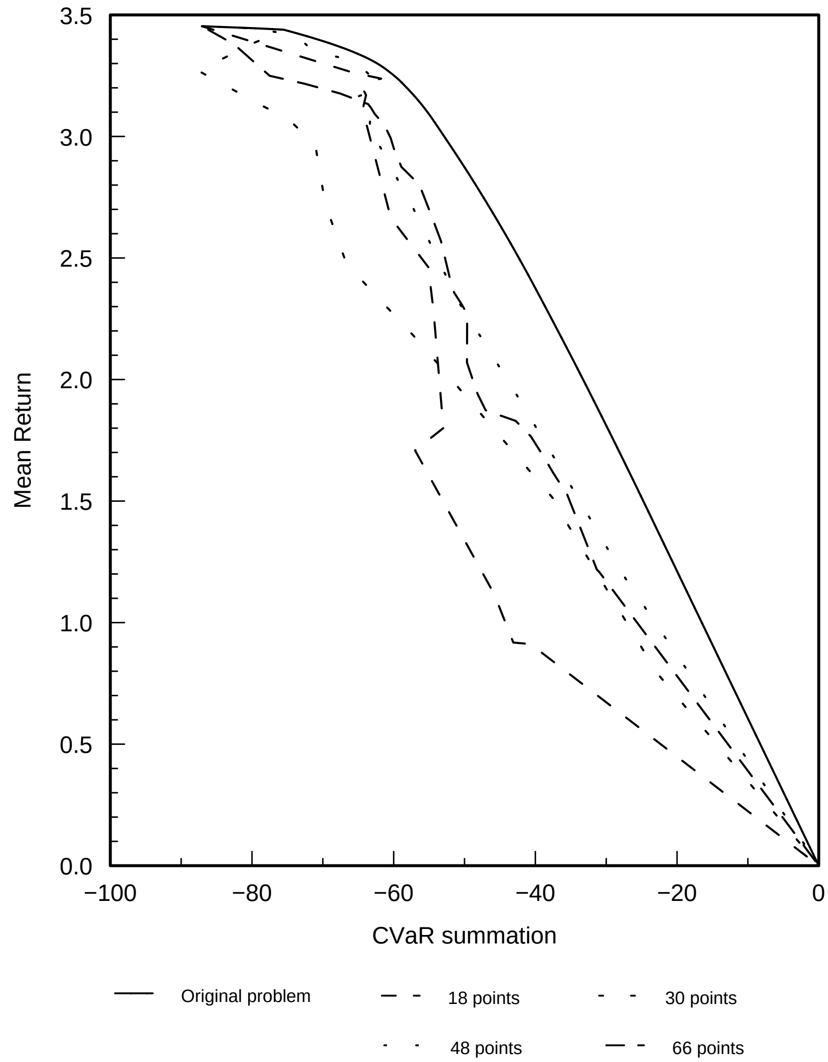

Figure E.23. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 2) and data set SP15 with increasing number of points in approximation

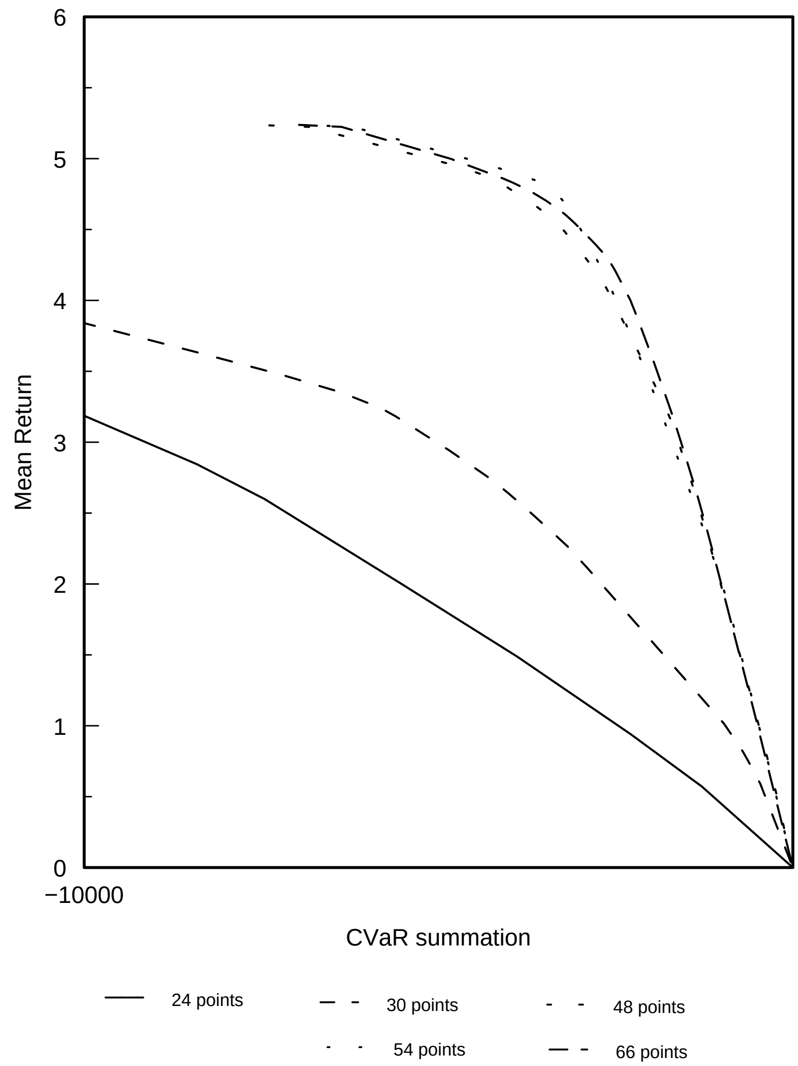

Figure E.24. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 2) and data set SP20 with increasing number of points in approximation

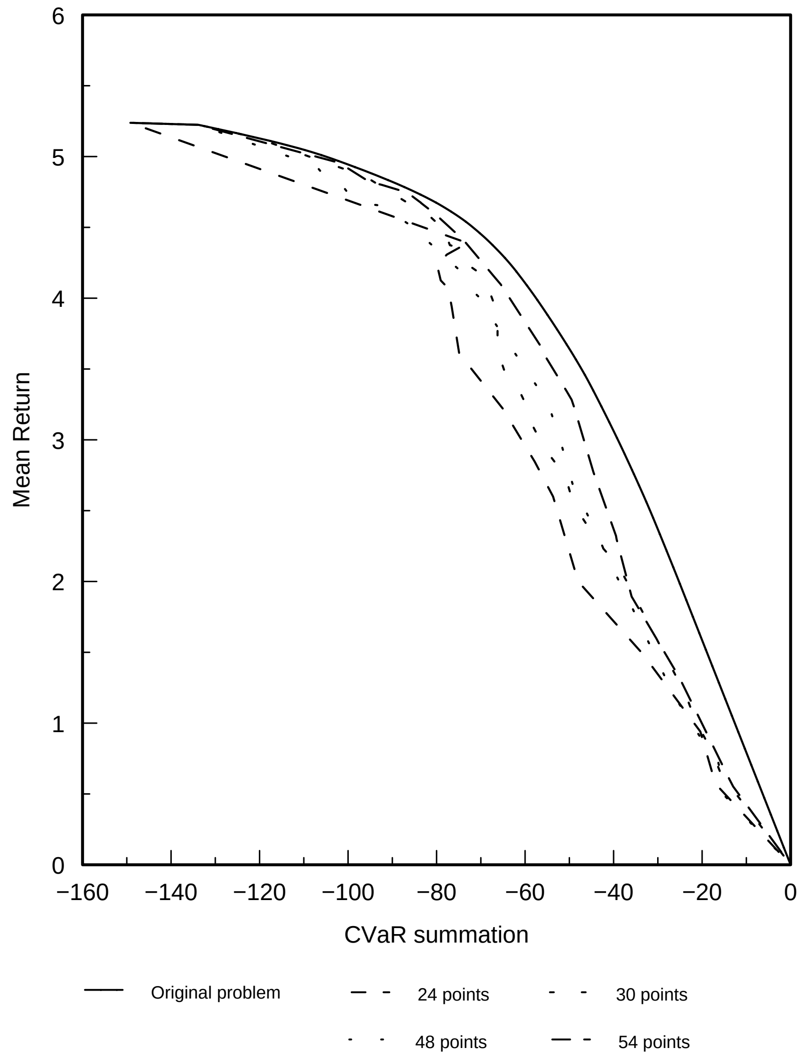

Figure E.25. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 2) and data set SP20 with increasing number of points in approximation

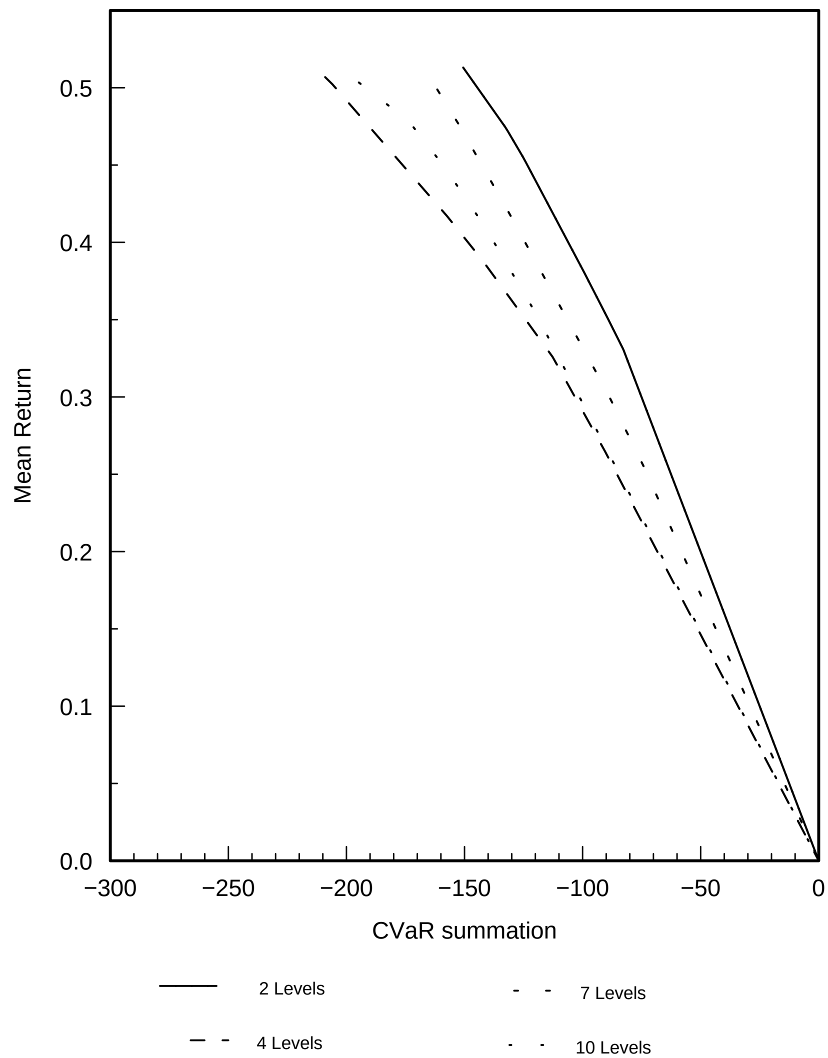

Figure E.26. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 2) and data set SP5 with increasing number of probability levels in approximation

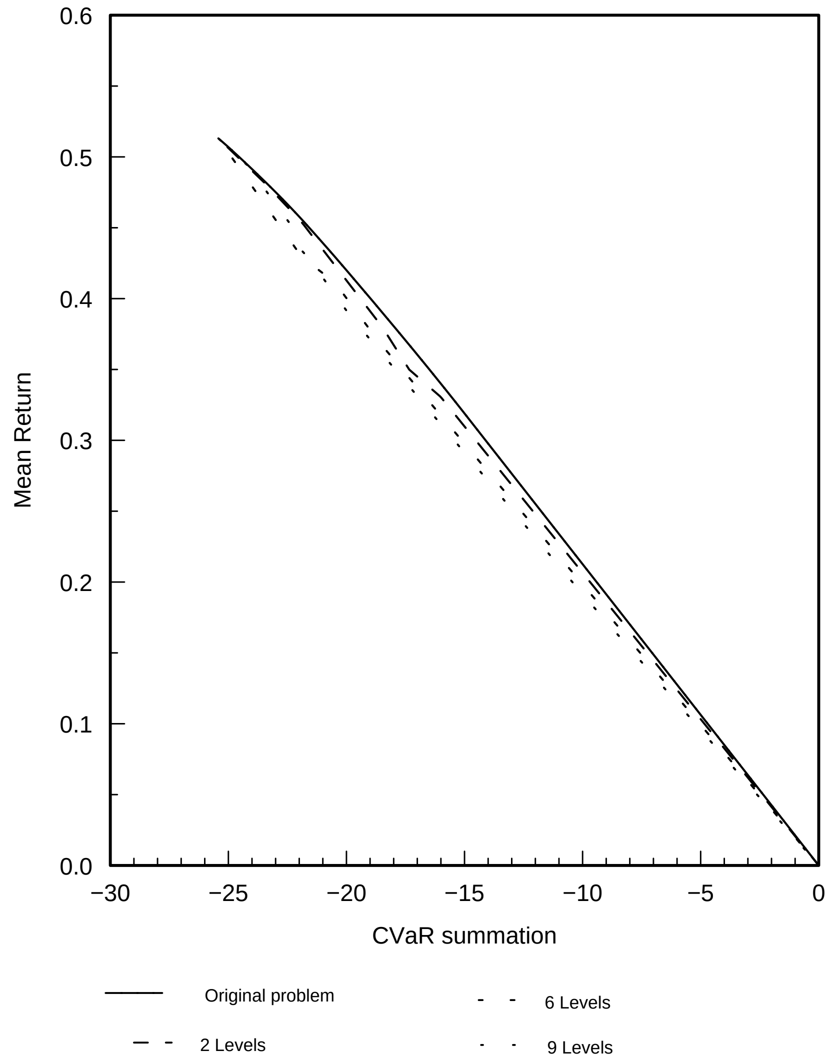

Figure E.27. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 2) and data set SP5 with increasing number of probability levels in approximation

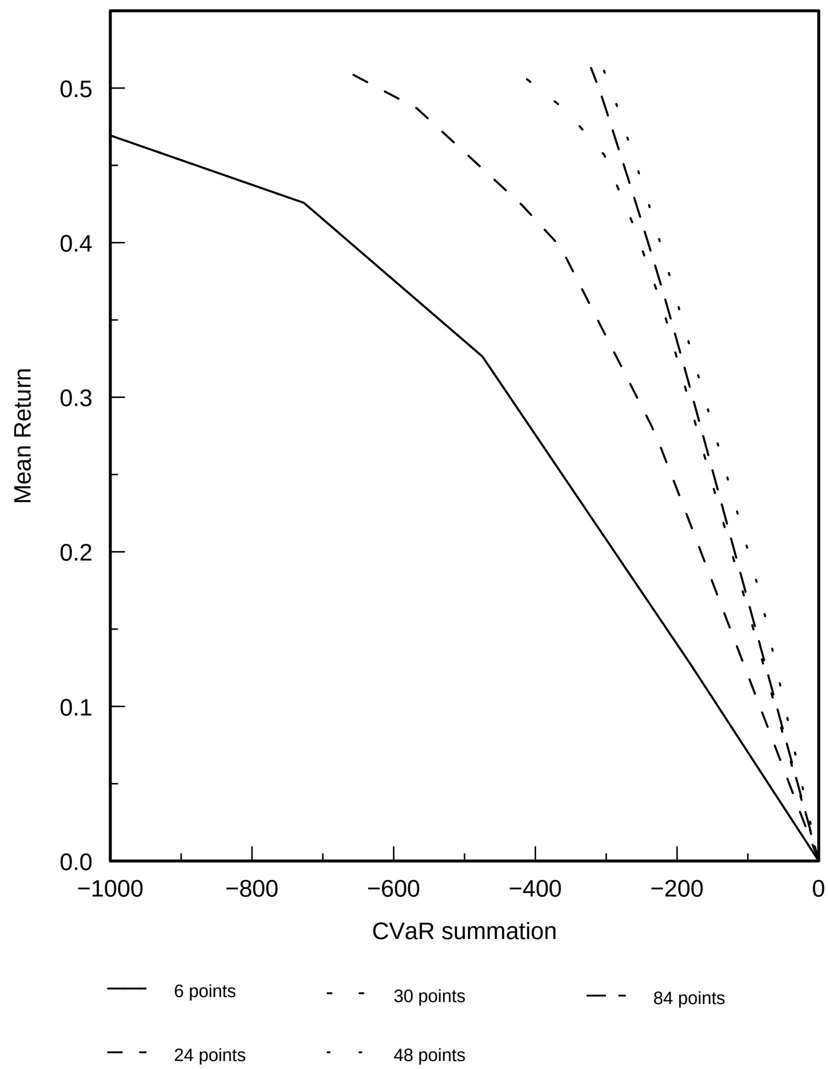

Figure E.28. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 3) and data set SP5 with increasing number of points in approximation

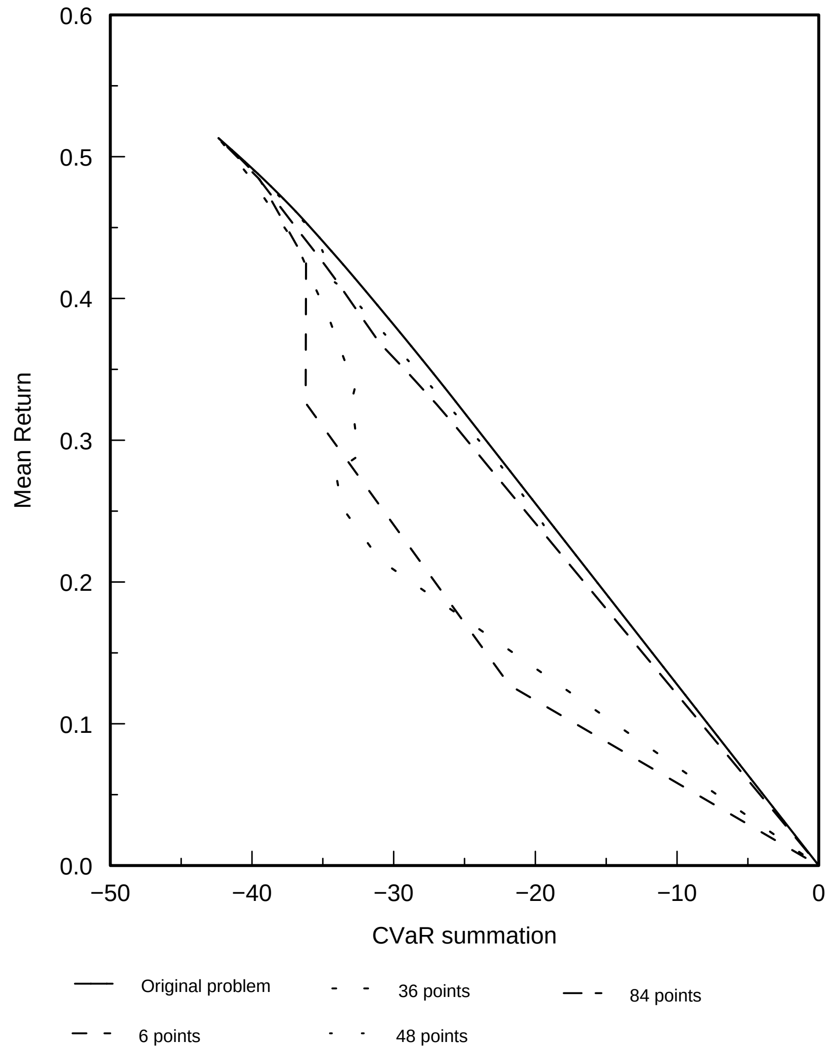

Figure E.29. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 3) and data set SP5 with increasing number of points in approximation

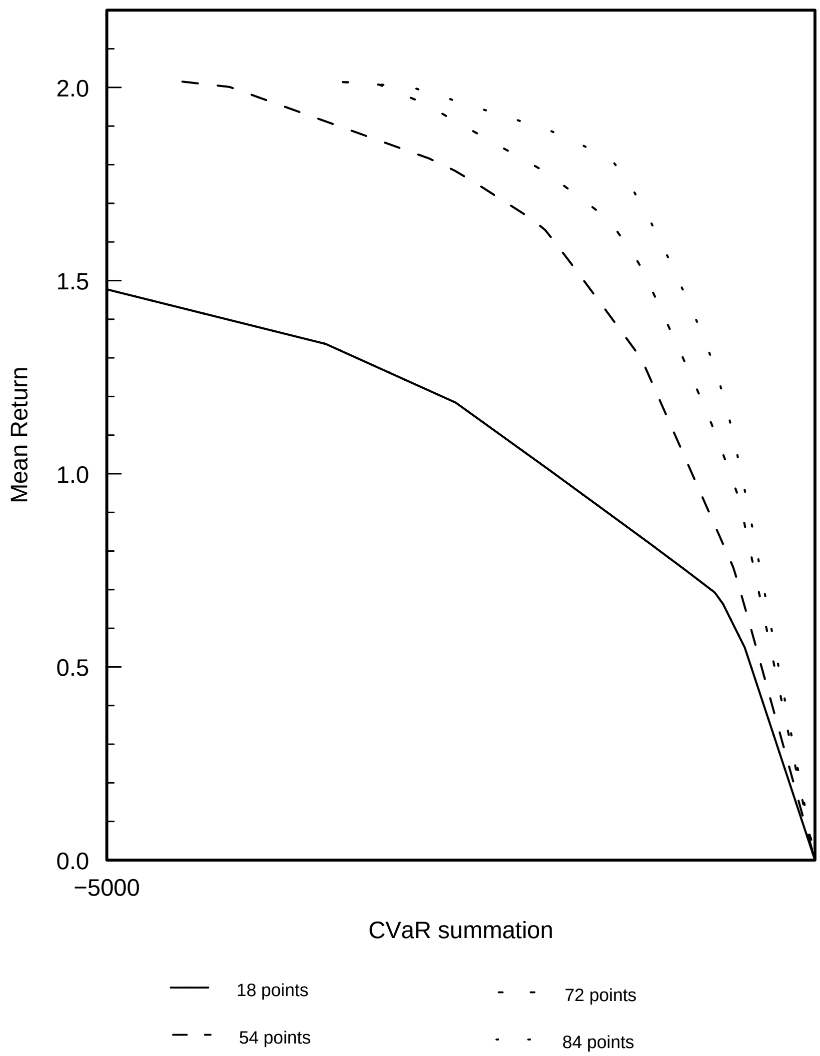

Figure E.30. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 3) and data set SP10 with increasing number of points in approximation

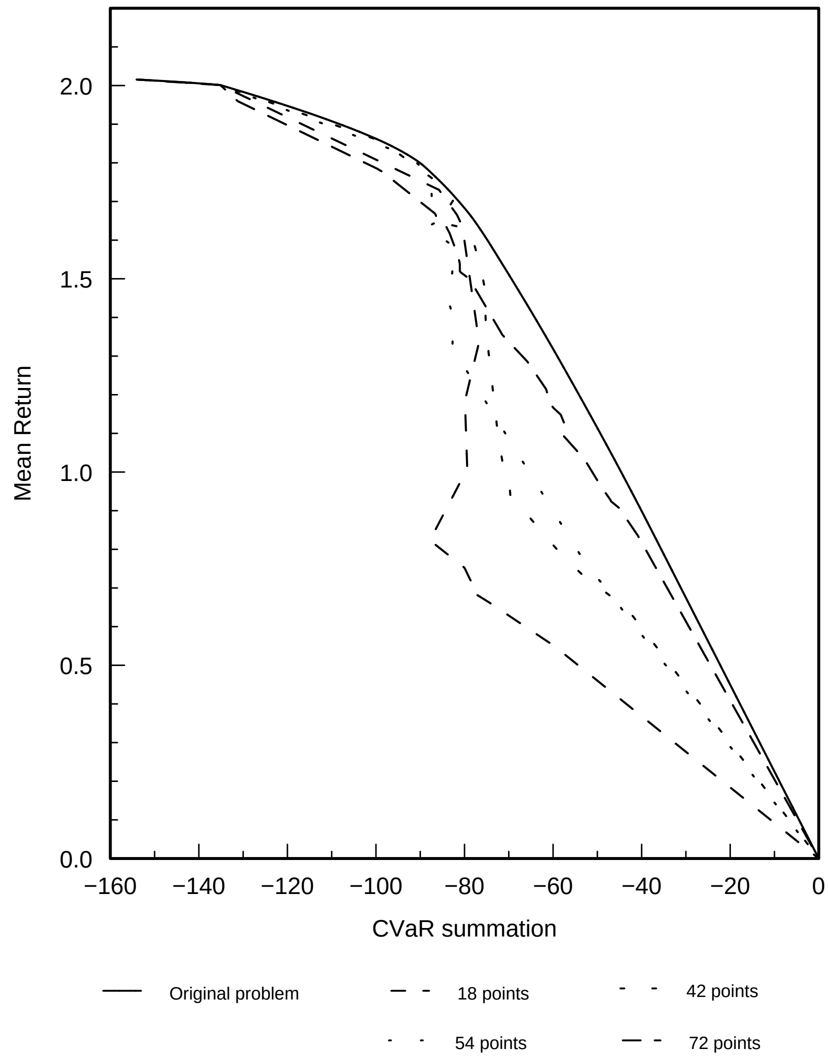

Figure E.31. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 3) and data set SP10 with increasing number of points in approximation

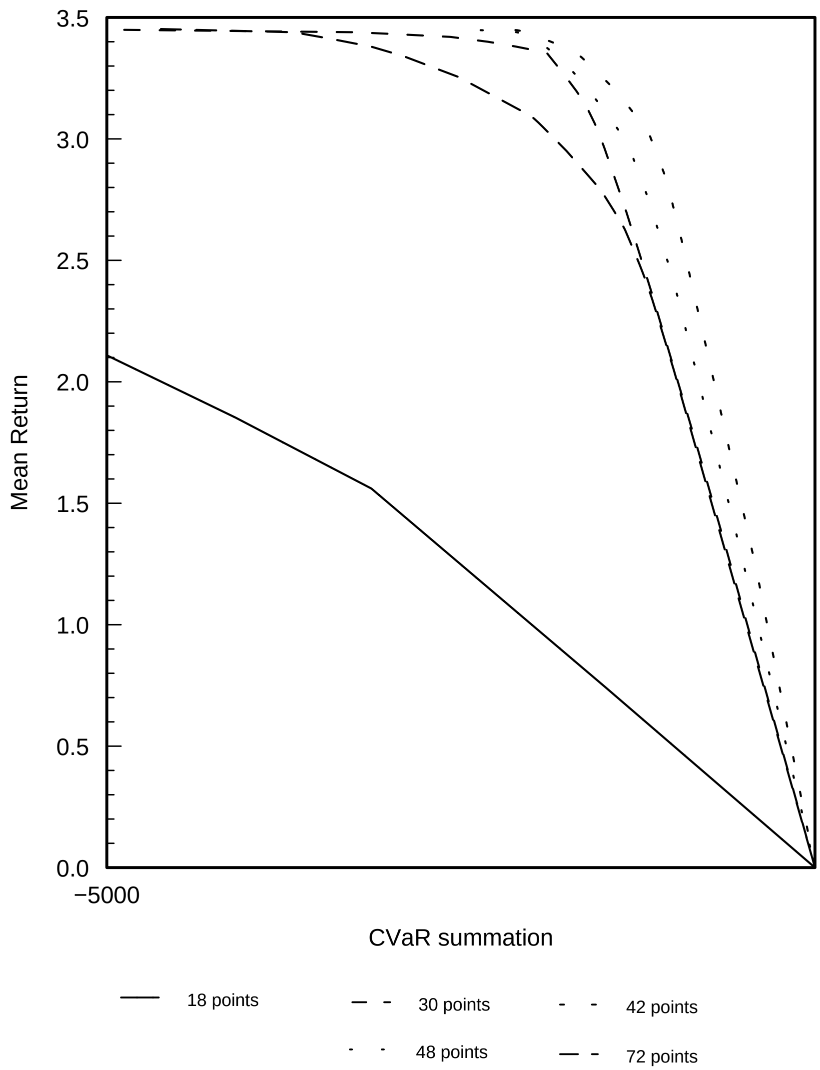

Figure E.32. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 3) and data set SP15 with increasing number of points in approximation

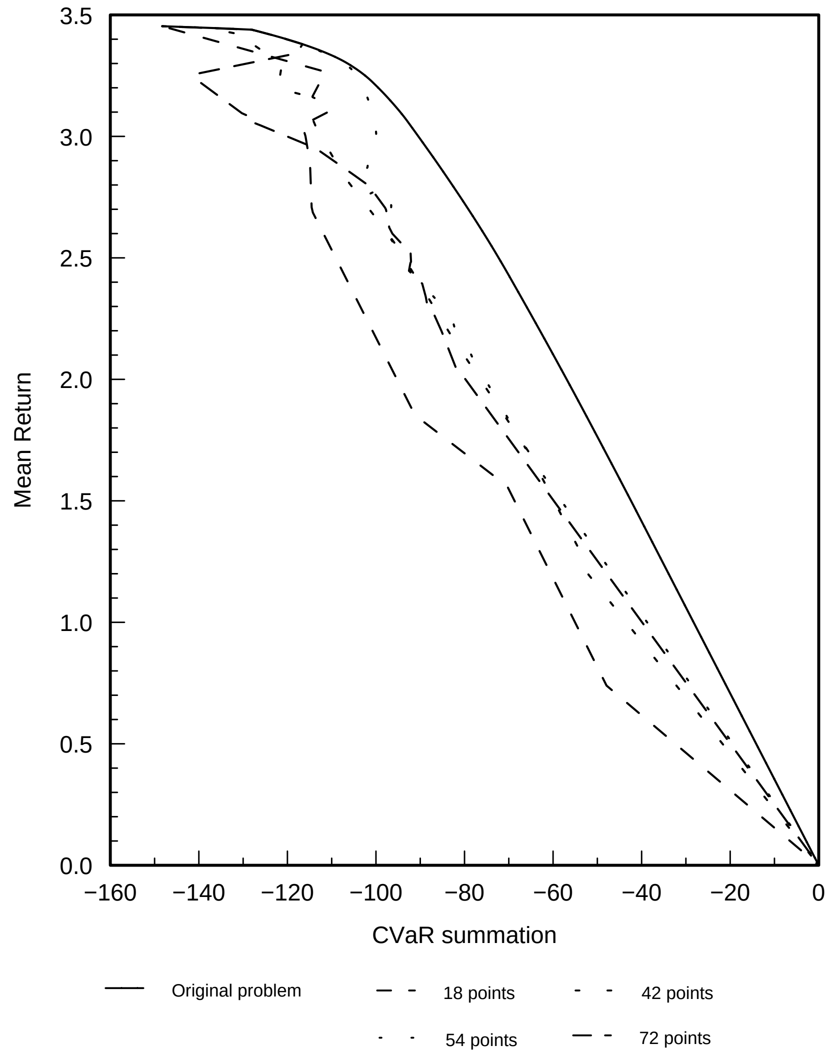

Figure E.33. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 3) and data set SP15 with increasing number of points in approximation

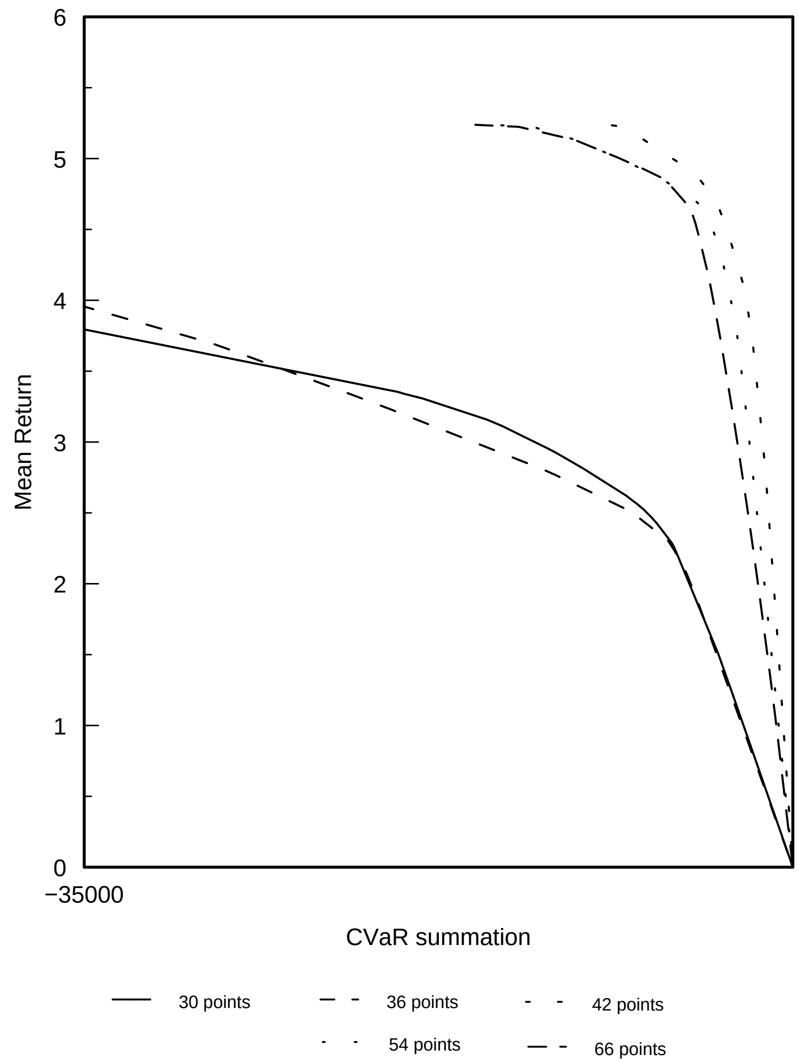

Figure E.34. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 3) and data set SP20 with increasing number of points in approximation

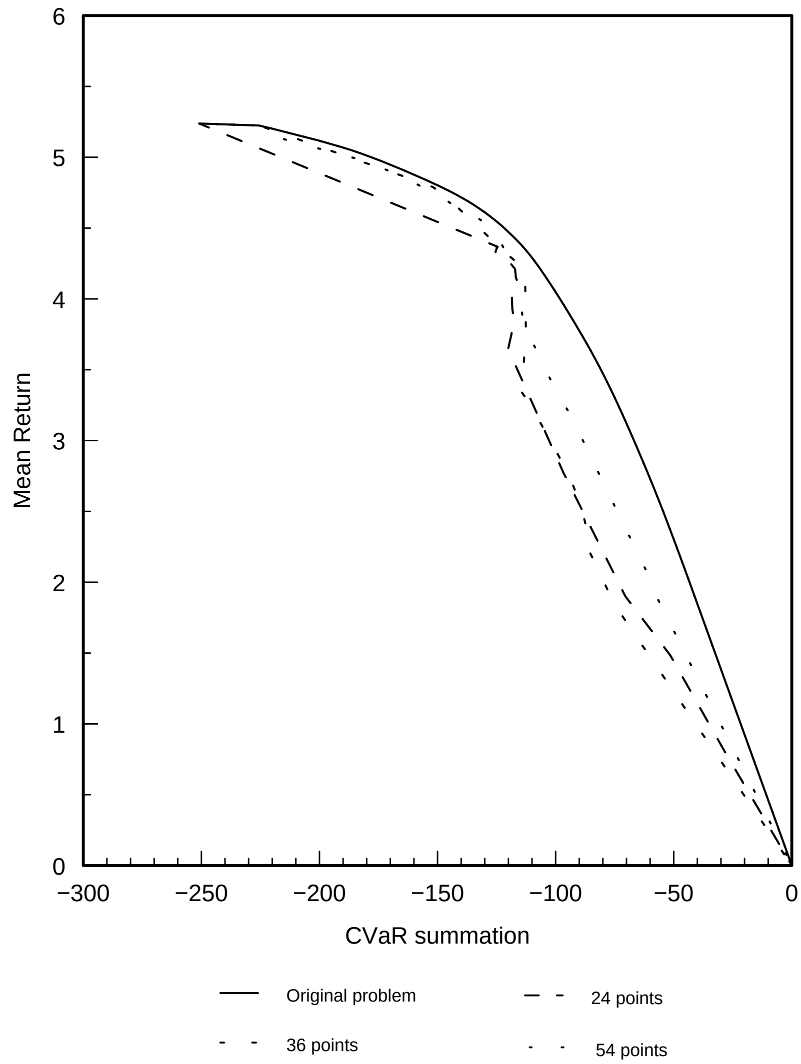

Figure E.35. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 3) and data set SP20 with increasing number of points in approximation

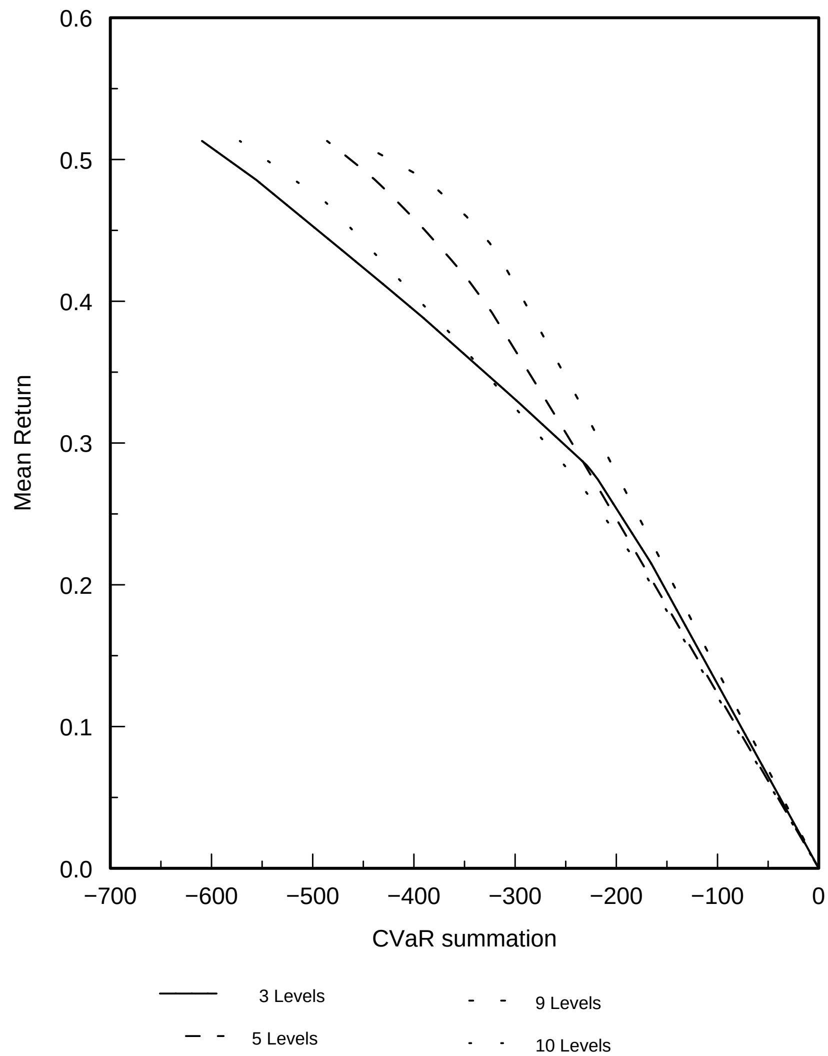

Figure E.36. Progress of approximation efficient frontiers for model with weighted CVaR deviation measure (Level 3) and data set SP5 with increasing number of probability levels in approximation

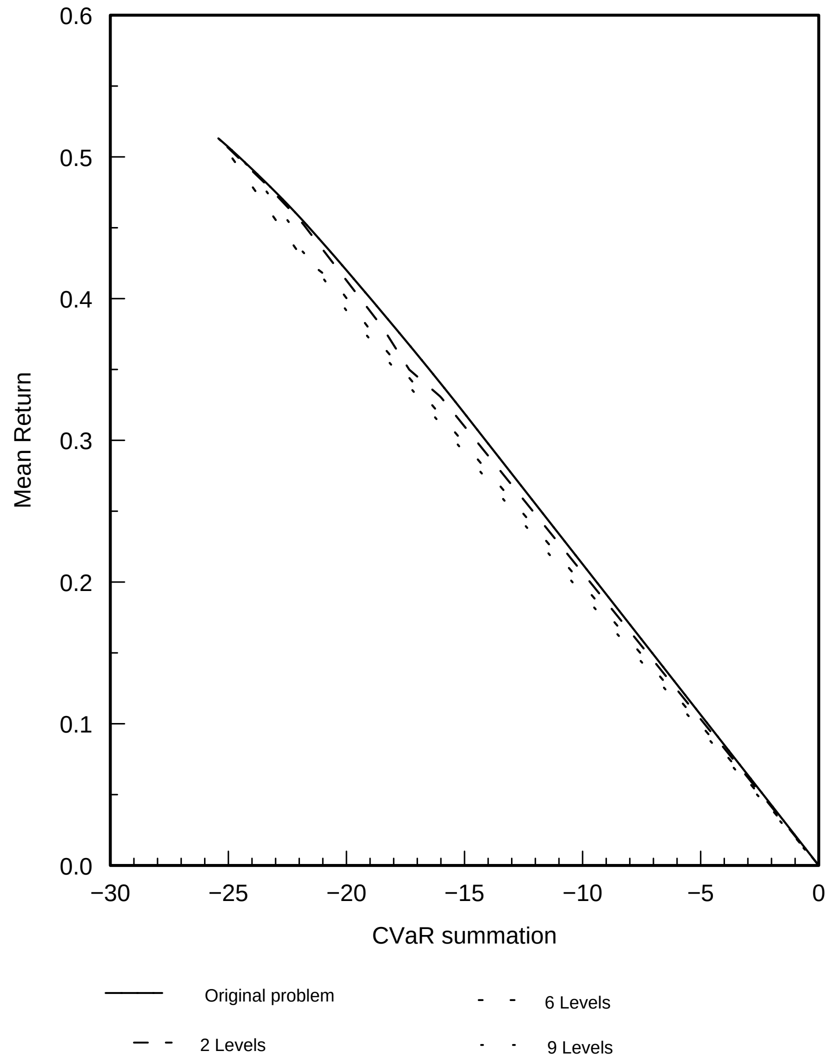

Figure E.37. Simulation of approximation optimal solutions against the original formulation for model with weighted CVaR deviation measure (Level 3) and data set SP5 with increasing number of probability levels in approximation

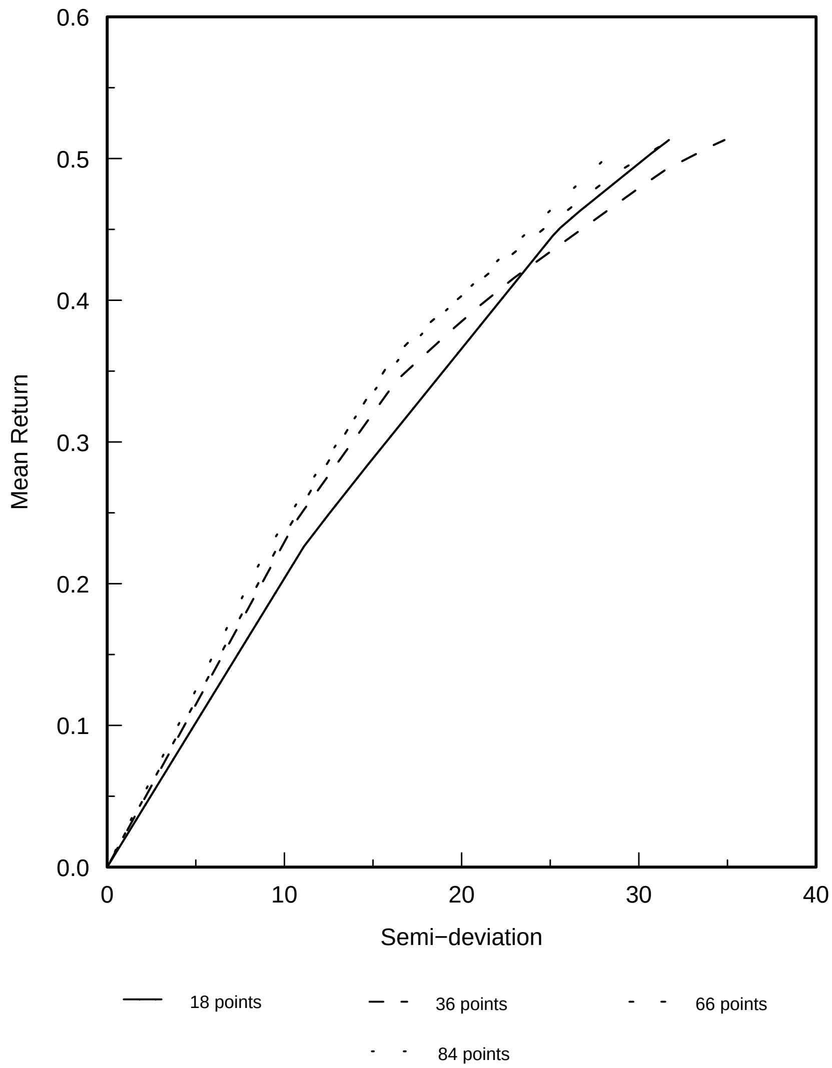

Figure E.38. Progress of approximation efficient frontiers for model with mean semi-deviation measure and data set SP5 with increasing number of points in approximation

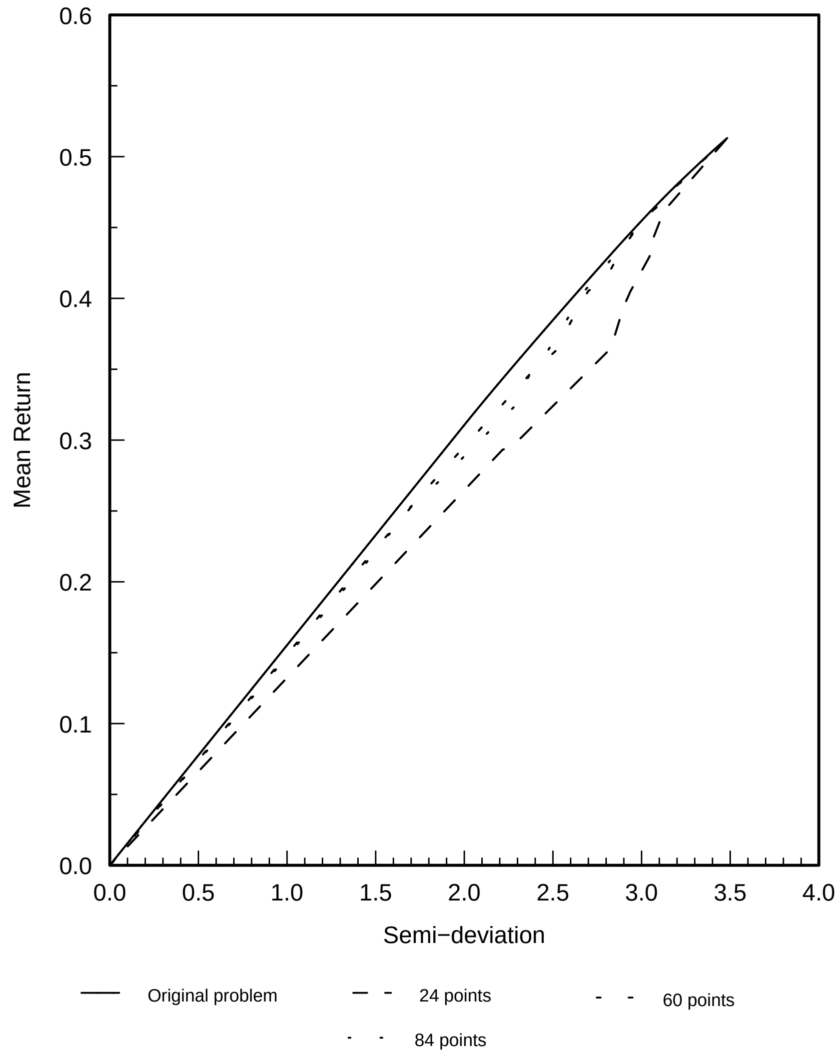

Figure E.39. Simulation of approximation optimal solutions against the original formulation for model with mean semi-deviation measure and data set SP5 with increasing number of points in approximation

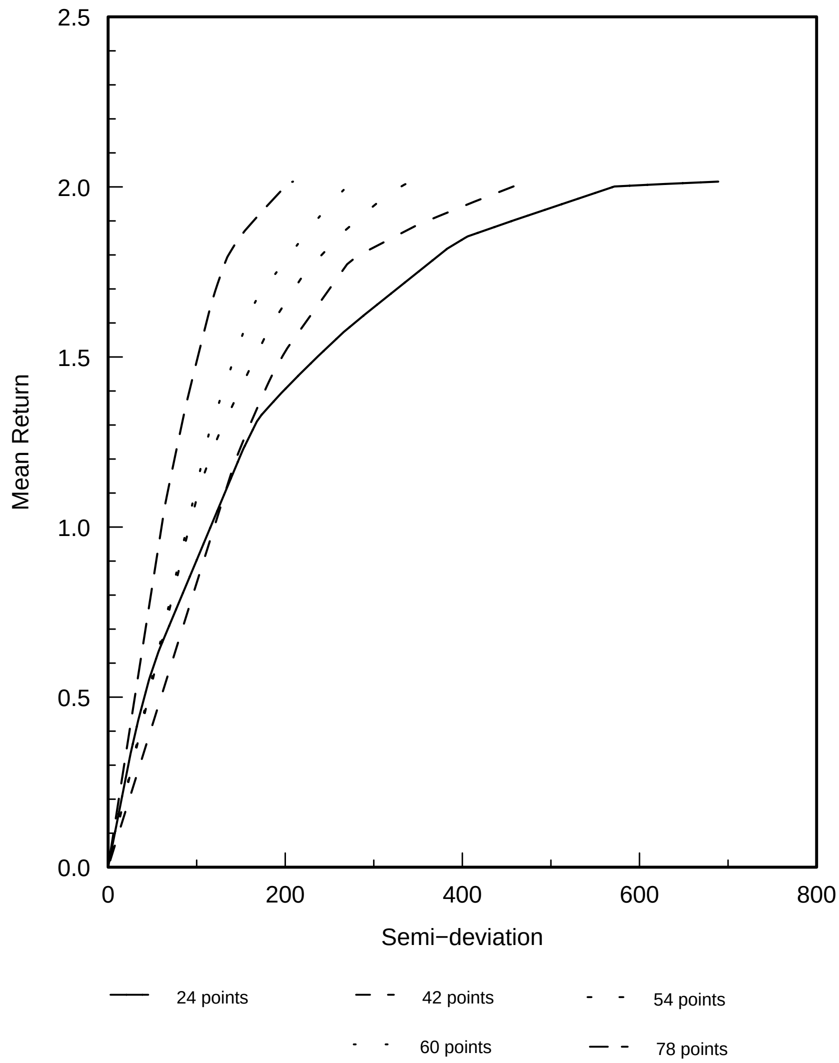

Figure E.40. Progress of approximation efficient frontiers for model with mean semi-deviation measure and data set SP10 with increasing number of points in approximation

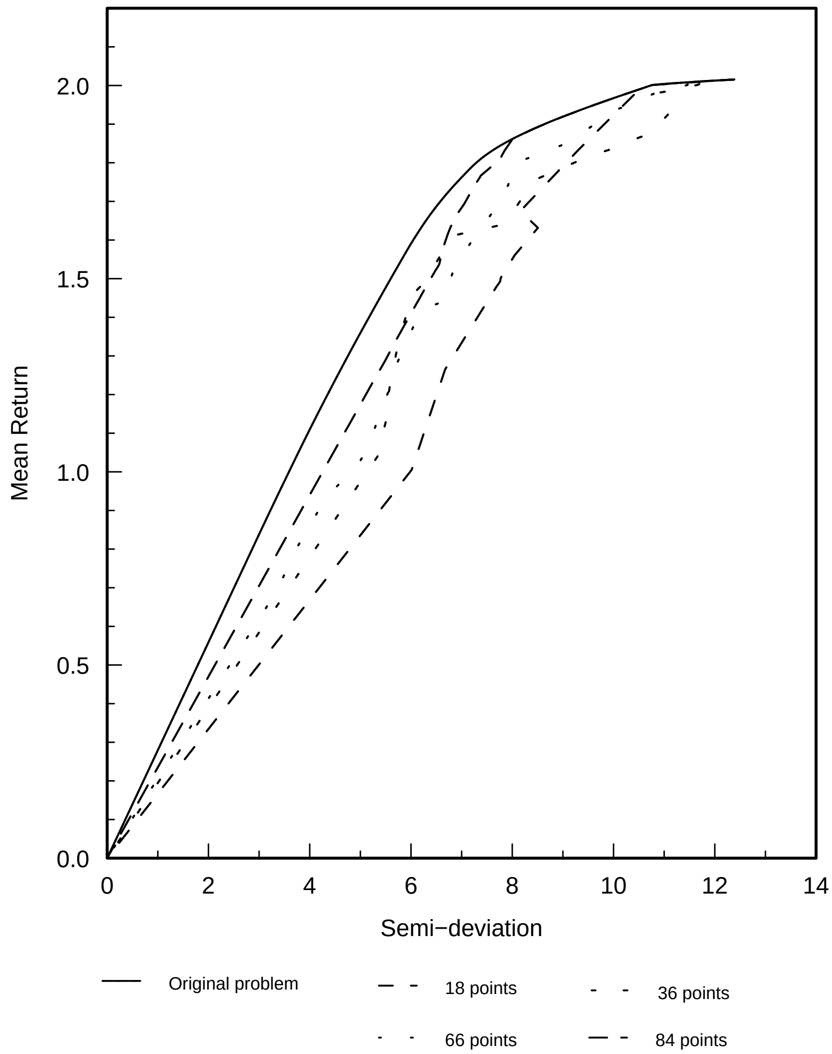

Figure E.41. Simulation of approximation optimal solutions against the original formulation for model with mean semi-deviation measure and data set SP10 with increasing number of points in approximation

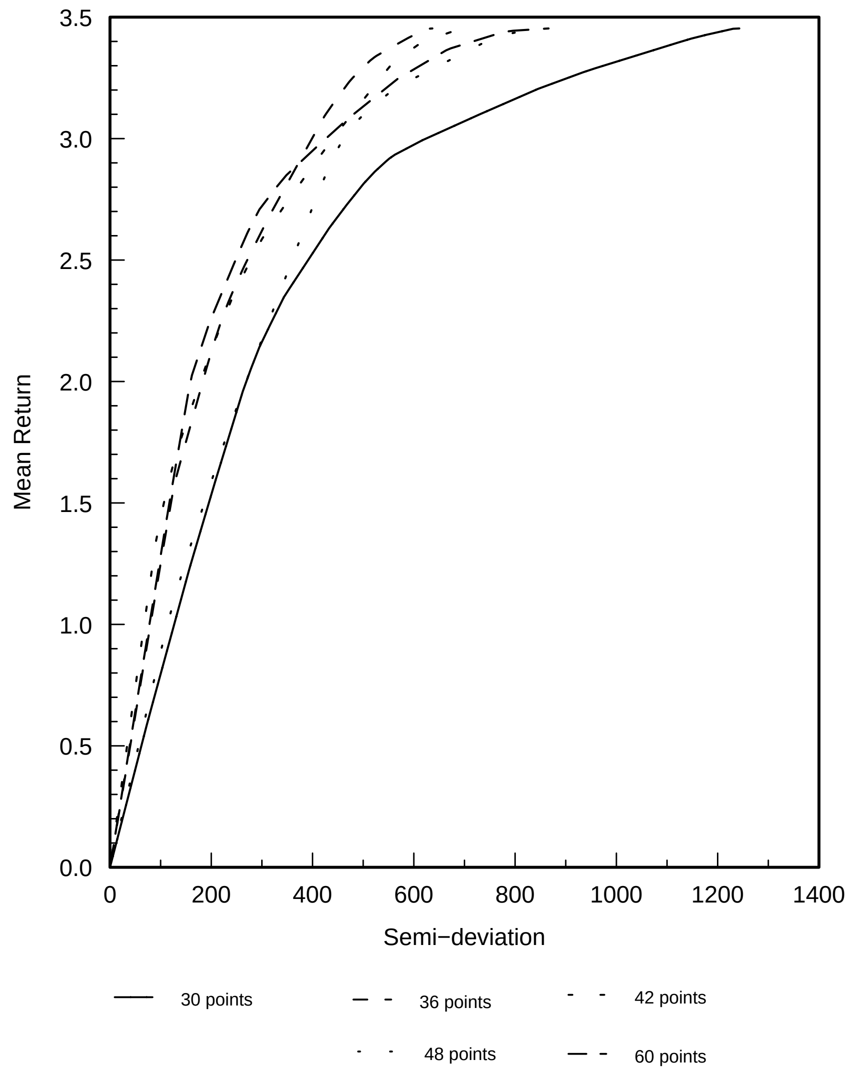

Figure E.42. Progress of approximation efficient frontiers for model with mean semi-deviation measure and data set SP15 with increasing number of points in approximation

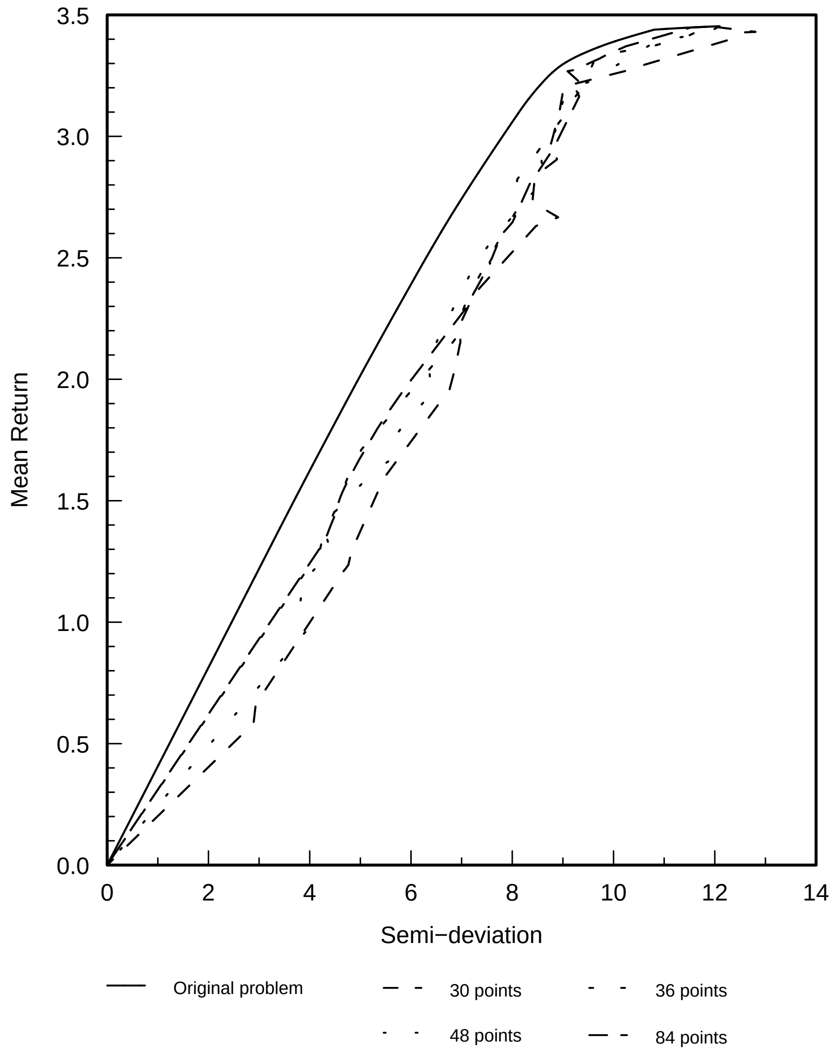

Figure E.43. Simulation of approximation optimal solutions against the original formulation for model with mean semi-deviation measure and data set SP15 with increasing number of points in approximation

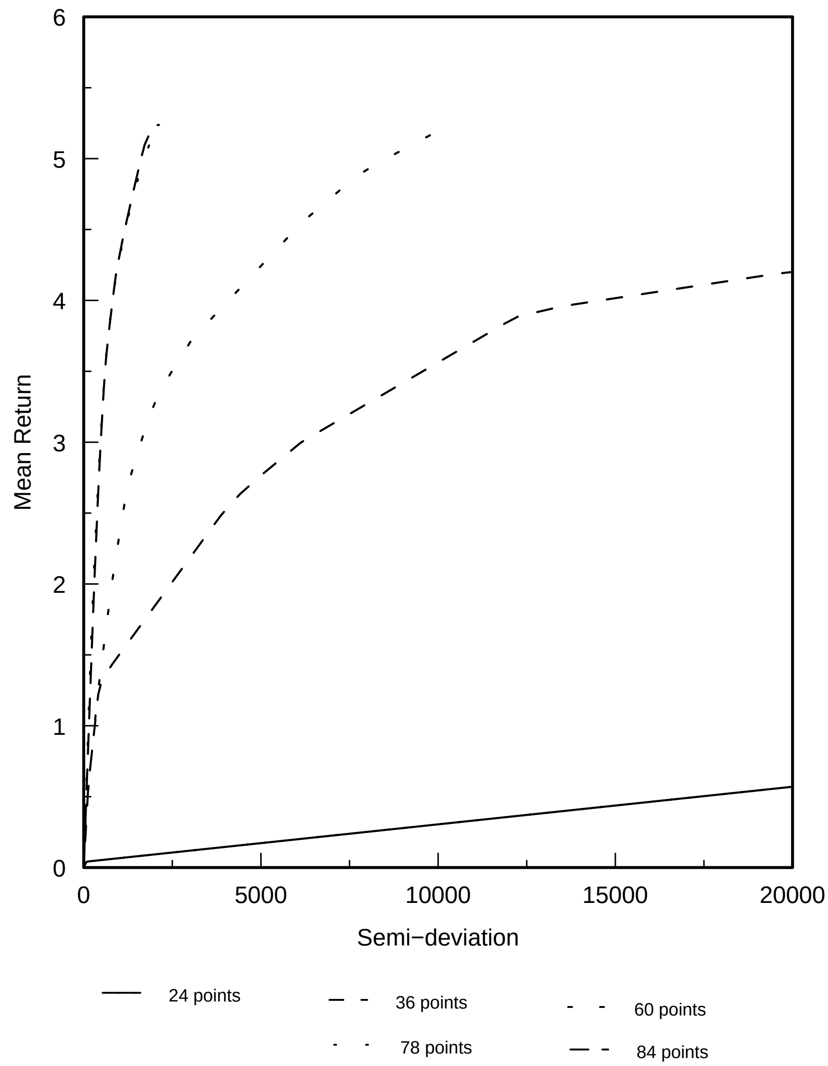

Figure E.44. Progress of approximation efficient frontiers for model with mean semi-deviation measure and data set SP20 with increasing number of points in approximation

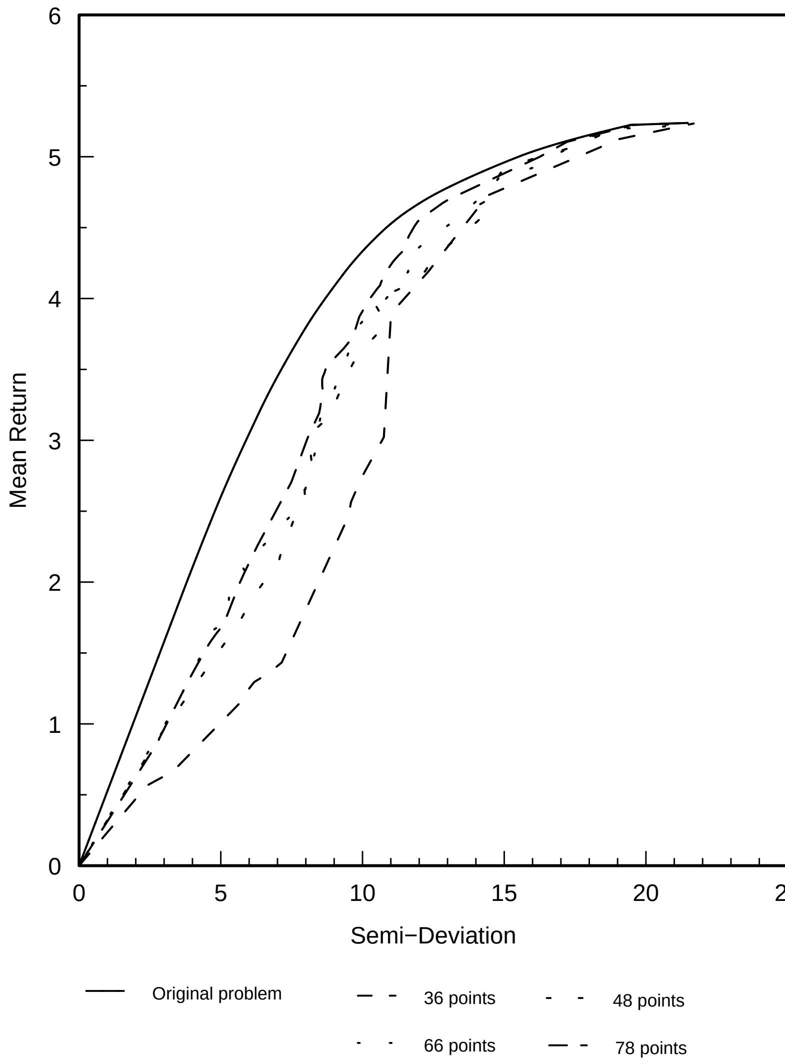

Figure E.45. Simulation of approximation optimal solutions against the original formulation for model with mean semi-deviation measure and data set SP20 with increasing number of points in approximation

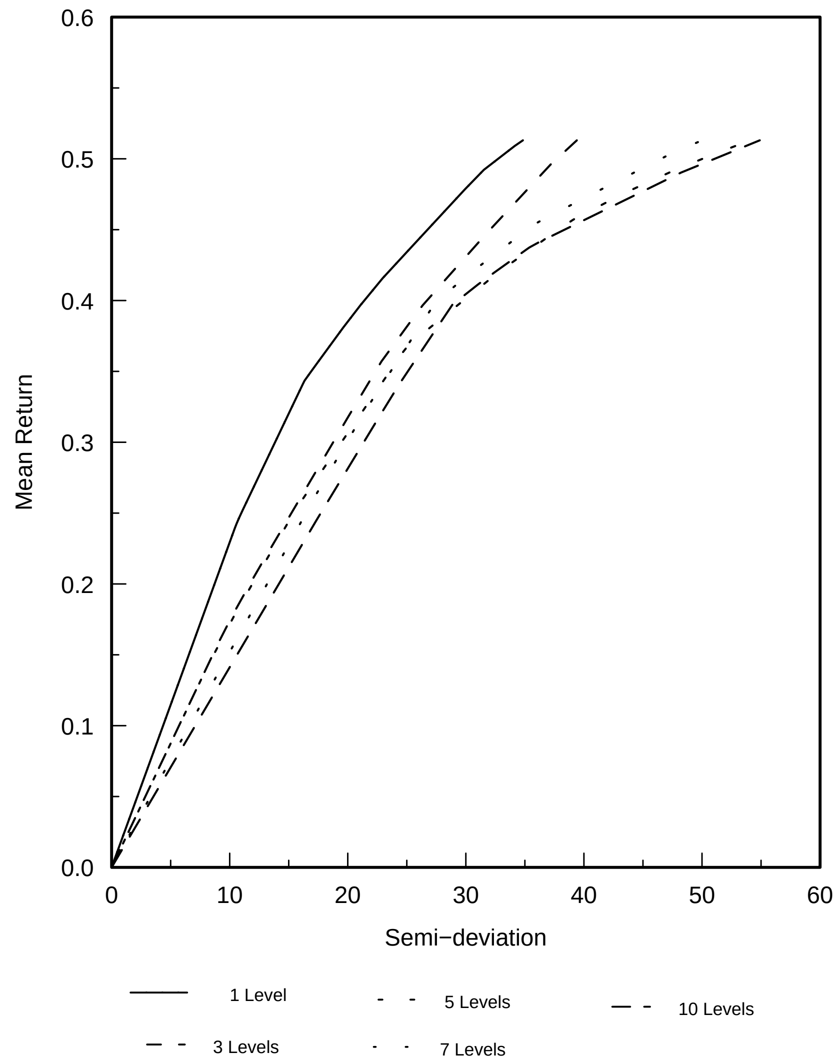

Figure E.46. Progress of approximation efficient frontiers for model with mean semi-deviation measure and data set SP5 with increasing number of probability levels in approximation

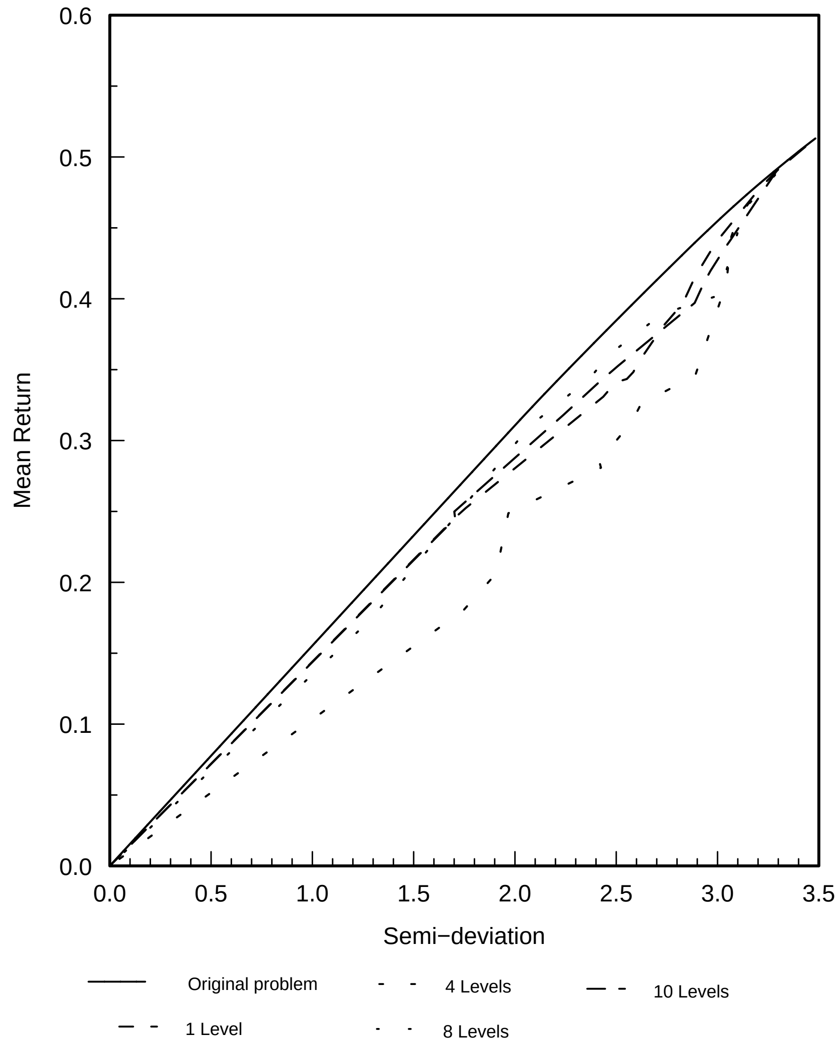

Figure E.47. Simulation of approximation optimal solutions against the original formulation for model with mean semi-deviation measure and data set SP5 with increasing number of probability levels in approximation

Vita

Maksym Bychkov was born in Kharkov, Ukraine on October 8, 1973. He was raised in Kharkov and went to special physico-mathematical school $\mathcal{N}$27. He graduated from Kharkov State Polytechnic University in 1996 with a M.S. in computer science with minor in Artificial Intelligence (A.I.). He has worked in Ukrainian Scientific Research Institute of Environmental Problems (USRIEP), Kharkov from 1995 till 2000. Maksym Bychkov was admitted for doctoral program in University of Tennessee, Knoxville in 2000.

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