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# Long Time Asymptotics of Ornstein-Uhlenbeck Processes in Poisson Random Media

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To the Graduate Council:

I am submitting herewith a dissertation written by Fei Xing entitled "Long Time Asymptotics of Ornstein-Uhlenbeck Processes in Poisson Random Media." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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# Long Time Asymptotics of Ornstein-Uhlenbeck Processes in Poisson Random Media

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*To my beloved parents Guangyou Xing and Yanfei Ding, for their endless understanding and encouragement.*

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*Learning without thought is labor lost; thought without learning is perilous. — Confucius*

# Abstract

The Models of Random Motions in Random Media (RMRM) have been shown to have fruitful applications in various scientific areas such as polymer physics, statistical mechanics, oceanography, etc. In this dissertation, we consider a special model of RMRM: the Ornstein-Uhlenbeck process in a Poisson random medium and investigate the long time evolution of its random energy. We give complete answers to the long time asymptotics of the exponential moments of the random energy with both positive and negative coefficients, under both quenched and annealed regimes. Through these results, we find out a dramatic difference between the long time behavior of the Brownian motion dynamics and the Ornstein-Uhlenbeck dynamics in the Poisson random medium.

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# Chapter 1

# Introduction

Concerns of Random Motions in Random Media (RMRM) arise when researchers try to understand the interaction between the evolution of a random particle movement and the random environment where it stays in. RMRM is one of the most active research fields in probability theory in the past few decades, having many applications to areas such as astrophysics, oceanography, chemical reactions, statistical mechanics and partial differential equations (PDE). We refer the readers to [8, 19, 21] for background, motivation, applications and fundamental results.

The general model of RMRM is formulated as following. Let $\{X(t,\varpi)\}_{t\in\mathbb{R}^+}$ be a stochastic process representing the evolution of some random movement or curve growth over time. For instance, one can treat $X(t,\varpi)$ as the location of a particle with random movement realization $\varpi$ at time $t$, or view $\{X(s,\varpi)\}_{0\le s\le t}$ as the shape of a random polymer chain up to time $t$, in the $d-$dimensional Euclidean space $\mathbb{R}^d$. On the other hand, *independent* of the law of $\{X(t,\varpi)\}_{t\in\mathbb{R}^+}$, the $\mathbb{R}^d$ space is filled with a random medium $\{V(x,\omega)\}_{x\in\mathbb{R}^d}$, where the value of $V(x,\omega)$ can be interpreted by different meanings ranging from the reward function to the potential function * at each position $x$ for every random realization $\omega$. With this set up, $\int_0^t V(X(s,\varpi),\omega)\,\mathrm{d}s$ quantifies the total energy accumulated by the particle from starting time 0 up to

*Due to the broad applications of RMRM in polymer physics, sometimes *random media* are also called *random potentials* in literature.

time $t$. Notice that due to the two systems of randomness in the construction of a RMRM (i.e. the particle movements and the media), there are two different regimes for these types of models that can be studied. Those studies of the random energy given the media $\omega$ are called the *quenched* regime. The *annealed* regime, on the other hand, is obtained by averaging the quenched objects over all possible random media. Throughout this dissertation, denote $\mathbf{P}$ and $\mathbf{E}$ as the law and expectation of a random medium, respectively. Similarly, denote $\mathbb{P}_x$ and $\mathbb{E}_x$ as the law and expectation of $X(t,\varpi)$ starting at position $x$, respectively. Without causing confusion, let us take $X(t) := X(t,\varpi)$ for simplicity in the rest of the dissertation. The following exponential moments are of great interest due to their strong connections with various fields, such as PDE in mathematics, survival probability of polymer chains in polymer physics, and random Gibbs measure in statistical physics:

$$u_{\pm}^{\omega}(t,x) \stackrel{\text{def}}{=} \mathbb{E}_x \exp\left\{\pm \int_0^t V(X(s),\omega)\,\mathrm{d}s\right\} \quad \text{(Quenched)}$$

$$U_{\pm}(t,x) \stackrel{\text{def}}{=} \mathbf{E}\otimes\mathbb{E}_x \exp\left\{\pm \int_0^t V(X(s),\omega)\,\mathrm{d}s\right\} \quad \text{(Annealed)}$$

Throughout this paper, we call $u_{+}^{\omega}$, $u_{-}^{\omega}$, $U_{+}$, $U_{-}$ the quenched exponential moment with positive coefficient, the quenched exponential moment with negative coefficient, the annealed exponential moment with positive coefficient, and the annealed exponential moment with negative coefficient, respectively. Research on long time asymptotics of those exponential moments have become very active in the past few decades. To make the idea of long time asymptotics clearer, let us take the quenched exponential moment with positive coefficient as an example, and the problem can be formulated as follows: we look for suitable long time growth *rate* $a(t)$ and the corresponding almost surely constant $\lambda \notin \{0,\pm\infty\}$ such that

$$\lim_{t\to\infty}\frac{1}{a(t)}\log\mathbb{E}_x\exp\left\{\int_0^t V(X(s),\omega)\,\mathrm{d}s\right\} = \lambda.$$

As an example of RMRM, the models of Brownian motion (BM) in homogeneous Poisson random media have been studied extensively in literature due to their applications in a wide range of scientific areas such as random polymer model in chemistry [8], parabolic Anderson model in physics [15], and so on. The research on the long time asympotics of Brownian motion in Poisson random media can be traced back to 1970s. In their seminal paper in 1975, Donsker and Varadhan [9] discovered that the long time asymptotic of the annealed exponential moment $U_-$ for BM in Poisson random media exhibits a decay rate $a(t) = t^{d/(d+2)}$, using their ground-breaking large deviation theory. Whereas the development of the quenched regime for BM in Poisson random media appeared much later. There were big breakthroughs for the quenched regime in the 1990s. Carmona and Molchanov [3] studied the long time asymptotic of $u_+^\omega$ for BM in Poisson random media and they obtained the suitable growth rate $a(t) = \frac{t \log t}{\log \log t}$. Around the same time, Sznitman [27] proved that $a(t) = \frac{t}{(\log t)^{2/d}}$ is the correct rate for $u_-^\omega$ for BM in Poisson potential using his powerful method of enlargement of obstacles. Since then, there have been many advances in this area. To mention a few here: Gärtner et al. [16] obtained the almost surely second order long time asymptotic of exponential moment with positive coefficient for BM in certain Poisson media. Most recently, Chen [7] investigated the BM in a Poisson media of the gravitational field type and obtained long time asymptotics of renormalized exponential moments for both positive and negative coefficients cases.

Brownian motions, as the continuous analogues of simple random walks, have very strong diffusive behaviors. However, various real world random dynamics which are influenced by certain known factors, such as friction and mean-reverting effect, are often presenting non-diffusive or even stationary phenomenons One such dynamic is the Ornstein-Uhlenbeck (O-U) process, which was first introduced by Leonard Ornstein and George Eugene Uhlenbeck in the 1930s to describe the velocity of a massive Brownian particle under the influence of friction [29]. Since then the O-U processes have been discovered to have many applications in a wide range of areas such as noisy relaxation process and Langevin equations in physics, interest rates, currency exchange rates, and commodity prices in financial mathematics, and model for peptide bond angle of molecules in biochemistry, etc. See for instance [4, 18, 26] for introductions and applications. One way to formulate the O-U dynamic is through the following stochastic differential equation:

$$\mathrm{d}X(t) = -X(t)\,\mathrm{d}t + \mathrm{d}W(t). \tag{1.1}$$

It is classical results [25] that the O-U dynamic is an ergodic Markov process with a Normal distributed invariant distribution. Notice that, due to the pull-back effect of the $-X(t)\,\mathrm{d}t$ term in (1.1), the O-U process tends to stay near its equilibrium position 0, which is quite different from the behavior of BM dynamic.

Motivated by O-U processes' crucial roles in various areas of real-world applications as well as their different dynamical behavior from the BM, we are interested in investigating a type of new RMRM model: the O-U process in homogeneous Poisson potential. In particular, we ask the following question:

> "Are there differences between the long time asymptotic behaviors of O-U processes and BM, in a Poisson random medium?"

The goal of the presented work is to give a complete answer to the long time asymptotics of exponential moments with both positive and negative coefficients for O-U processes in homogeneous Poisson random media, under both quenched and annealed regimes. The results in this work provide a better understanding of the interaction between the O-U dynamics and the Poisson random media, which is potentially fruitful in statistical physics, finance and biochemistry.

**Organization of the Dissertation**

The rest of the paper is organized as follows. In Chapter 2 we describe the O-U process in Poisson potential model in details, present the main results of the dissertation, and compare the results with the counterpart of Brown motion studied in [3, 9, 27]. Chapter 3 characterizes the spectral structure of the O-U semigroup, which will be used extensively in later chapters for the proof of the asymptotic results. In Chapter 4 we consider the quenched regime and give the proof of the corresponding long time asymptotics. In Chapter 5 we provide the proof of long time asymptotics for our model under the annealed regime. Chapter 6 discusses several possible avenues for future work of this topic. Some mathematical backgrounds as well as proofs of several technical lemmas are included in Appendix A.

# Chapter 2

# Model Set Up and the Main Results

In this chapter we set up the model and then present the main theorems of the dissertation. We first list the notations and basic definitions which will be used through the paper in Section 2.1. Section 2.2 introduces the Ornstein-Uhlenbeck process and its related properties that will be used in later proofs. We define the Poisson potential which serves as the random media in our model. Furthermore, we give a path description of the random potential in Section 2.3. Equipped with these, we state the main results of the dissertation and compare them with the cases of Brownian motion in Section 2.4.

In the whole dissertation, we consider the model on $\mathbb{R}^d$ with $d \geq 1$.

## 2.1 Notation and Basic Definitions

Throughout the dissertation, $p(x, y, t)$ denotes the transition probability of a Markov process $X$ from position $x$ at time 0 to position $y$ at time $t > 0$.

$\mathbb{Z}_+$ is the set of all positive integers.

**P** and **E** stand for the probability law and the corresponding expectation of the random media, respectively. Similar, $\mathbb{P}_x$ and $\mathbb{E}_x$ stand for the probability law and the corresponding expectation of the random motion starting at position $x$, respectively.

We use $\omega_d$ to denote the volume of the unit $d-$dimensional ball.

*i.o.* is short for infinitely often and *a.s.* is short for almost surely.

Denote the domain of an operator $A$ by $\mathcal{D}(A)$.

$B(x, R)$ is the ball centering at $x$ of radius $R$.

$\mathcal{B}(\mathbb{R}^d)$ is the collection of all Borel sets on $\mathbb{R}^d$.

$\text{supp}(K) \overset{\text{def.}}{=}$ the closure of $\{x : K(x) \neq 0\}$ is called the support of a function $K$.

We denote the first exit time of a stochastic process $X(t)$ from the inside of a $R-$ball by $\tau_R$, that is, $\tau_R = \inf\{t \geq 0 : X(t) \notin B(0, R)\}$.

$\mathbb{R}^+$ denotes the set of all non-negative real numbers.

## 2.2 Random Motions: Ornstein-Uhlenbeck Processes

We model the random motion by a $d-$dimensional Ornstein-Uhlenbeck processes $\{X(t)\}_{t\in\mathbb{R}^+} = \{X_1(t), \cdots, X_d(t)\}$ which satisfies the following stochastic differential equation:

$$\mathrm{d}X(t) = -X(t)\,\mathrm{d}t + \mathrm{d}W(t) \tag{2.1}$$

with $X(0) = x$, where $W(t)$ is a $d-$dimensional Brownian motion of which margin is a one dimensional standard Brownian motion. Under this setting, it is well known that (see [25])

- $X$ is a homogeneous ergodic *Markov Process*. Hence, given the present state of $X$, the future and past behaviors of $X$ are independent. Indeed, $X$ has the transition density

$$p(x,y,t) = \frac{1}{\left(\pi(1-e^{-2t})\right)^{d/2}} \exp\left\{-\frac{|y-xe^{-t}|^2}{1-e^{-2t}}\right\} \quad x,y \in \mathbb{R}^d, t>0, \tag{2.2}$$

and the invariant distribution $\mu(x) \sim N(0, I_d/2)$, where $I_d$ is the $d$ by $d$ identity matrix. In the following dissertation, we denote the density function of $\mu$ by $\phi$, that is

$$\mu(\mathrm{d}x) = \phi(x)\,\mathrm{d}x = (2\pi)^{-d/2} \exp\{-|x|^2\}\,\mathrm{d}x. \tag{2.3}$$

- $X$ is a *Gaussian process*. That is, any finite linear combination of samples of $X$ is Normal (also known as Gaussian) distributed: for all $c_1, \cdots, c_N \in \mathbb{R}, t_1, \ldots, t_N \in \mathbb{R}^+$, $\sum_{k=1}^N c_k X(t_k)$ is Normal distributed. In fact, $X$ as a stochastic process has the same distribution as a time changed Brownian motion $B(\cdot)$:

$$X(t) \overset{d}{=} xe^{-t} + \frac{e^{-t}}{\sqrt{2}} B(e^{2t}-1). \tag{2.4}$$

According to (2.4), it is also straight forward to see that $X$ has $N(0, I_d/2)$ distributed invariant distribution.

**Remark 1.** *The following equation (2.5) can be derived from (2.2), (2.3) and the time reversal property of $X$:*

$$\phi(y)^{-1} p(x,y,t) = \frac{1}{(1-e^{-2t})^{d/2}} \exp\left\{|x|^2 - \frac{|x-ye^{-t}|^2}{1-e^{-2t}}\right\} = \phi(x)^{-1} p(y,x,t). \tag{2.5}$$

*We will revisit this equation several times in the later proofs.*

## 2.3 Random Media: Poisson Random Media

The positions of random obstacles is modeled by a Poisson point process $\omega(\cdot)$ with intensity measure $\nu(\mathrm{d}x) = \lambda\,\mathrm{d}x$ $(\lambda > 0)$. A Poisson point process $\{\omega(A)\}_{A \in \mathcal{B}(\mathbb{R}^d)}$ is a measure-valued random variable such that for any Borel set $A$ in $\mathbb{R}^d$:

1. $\omega(\emptyset) = 0$ almost surely.

2. For any disjoint sets $A_1$ and $A_2$, $\omega(A_1)$ and $\omega(A_2)$ are independent random variables.

3. $\omega(A)$ is a Poisson distributed random variable of parameter $\lambda \cdot \text{volume}(A)$.

Furthermore, we assume the influence of each Poisson point to the environment is local, captured by a deterministic *shape function* $K(\cdot)$. More precisely, we make the following assumptions:

**Assumption** The shape function $K$ is nonnegative, continuous and compactly supported. Without loss of generality, assume $\text{supp}(K) \subset B(0, L)$ and $\max_{x \in \mathbb{R}^d} K(x) > 0$.

Therefore, the *Poisson media* $V(\cdot)$ defined by

$$V(x) = \int_{\mathbb{R}^d} K(x - y)\, \omega(\mathrm{d}y)$$

measures the accumulated impact of all the Poisson points at position $x$. Figure 2.1 illustrates simulations of a 2-dimensional Poisson point process on the $[-40, 40]^2$ square with $\lambda = 0.05$ and the corresponding Poisson media $V$ with a compactly supported shape function $K$.

We include some long range estimates of the Poisson random media $V$ in Appendix A.3.

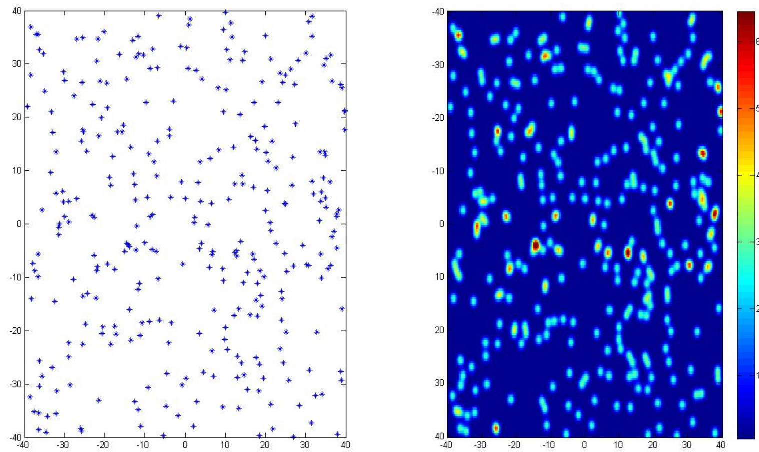

**Figure 2.1:** Simulation of Poisson point process (left) and the corresponding Poisson media $V$ (right).

## 2.4 Main Theorems: Long Time Asymptotics

As introduced in Chapter 1, we are interested in the long time asymptotics of the following quantities:

$$u^{\omega}_{\pm}(x,t) = \mathbb{E}_x \exp\left\{\pm \int_0^t V(X(s))\,\mathrm{d}s\right\}, \quad \text{(Quenched Regime)} \tag{2.6}$$

and

$$U_{\pm}(x,t) = \mathbf{E} \otimes \mathbb{E}_x \exp\left\{\pm \int_0^t V(X(s))\,\mathrm{d}s\right\}. \quad \text{(Annealed Regime)} \tag{2.7}$$

For the quenched exponential moments $u^{\omega}_{\pm}$, incorporating the Feynman-Kac formula as well as the infinitesimal operator structure of $X$, we know that $u^{\omega}_{\pm}$ solve the following parabolic PDE with random potentials $V$ and $-V$, respectively:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u - x \cdot \nabla u \pm V(x,\omega)\cdot u, && (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ u(0,x) &= 1, && x \in \mathbb{R}^d. \end{aligned} \tag{2.8}$$

Therefore, understanding the long time asymptotics of (2.6) provides information on the the long time behavior of the solution of PDE (2.8).

As to $U_-$, here we provide two ways to visualize the quantity: For the first viewpoint, take the Poisson integral (see Appendix A.1 for more details on Poisson integral) to

$$\mathbb{E}_x \exp\left\{-\int_0^t V(X(s))\,\mathrm{d}s\right\},$$

then we have

$$\begin{aligned}&\mathbf{E}\otimes\mathbb{E}_x \exp\left\{-\int_0^t V(X(s))\,\mathrm{d}s\right\}\\&=\mathbb{E}_x \exp\left\{-\lambda\int_{\mathbb{R}^d}\left(1-\exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right)\,\mathrm{d}y\right\}.\end{aligned} \tag{2.9}$$

If we take the Poisson points as "hard obstacles", i.e. the shape function $K$ satisfies

$$K(x)=\begin{cases}+\infty & \text{if } |x|<\delta\\ 0 & \text{otherwise.}\end{cases}$$

We see that $\int_{\mathbb{R}^d}(1-\exp\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\})\,\mathrm{d}y$ equals to the total volume of the region swept by the $\delta-$neighborhood of the path of $X$ from 0 to $t$, denoted by $|C_t^\delta(X(\cdot))|$. In literature, $C_t^\delta$ is called the $\delta-$sausage of the process $X$ [2, 9]. Hence, $U_-$ measures the exponential moment of the $\delta-$sausage of O-U process.

Another perspective to understand $U_-(x,t)$ is to view is as the survival probability of an O-U process in the $\delta-$Poisson traps until time $t$. Indeed, if we take each Poisson point as a trap and assume the O-U process being killed when it first runs into a $\delta$ neighborhood of those Poisson points, then the survival probability of the O-U process up to time $t$ could be expressed as:

$$\mathbf{P}\otimes\mathbb{P}_x(\tau>t)=\mathbf{E}\otimes\mathbb{E}_x \exp\left\{-\int_0^t V(X(s))\,\mathrm{d}s\right\},$$

where $\tau = \inf\{t \geq 0, X(t) \in \delta \text{ neighborhood of a Poisson point}\}$ is the survival time of the O-U particle and $V$ is the hard obstacles modeled as above. For details, see Section 2.5 in [2].

Under the above settings for the RMRM model, we obtain the following long time asymptotics of the exponential moments for O-U processes $X$ in homogeneous Poisson random media $V$, for both the quenched regime as well as the annealed regime. For each regime, we consider the exponential moments in both positive and negative coefficients situations.

**Theorem 1** (Quenched Regime)**.** $\mathbf{P}-$*almost surely,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\} = \lambda_1,$$

*and*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \exp\left\{-\int_0^t V(X(s))\,\mathrm{d}s\right\} = -\lambda_2,$$

*where* $\lambda_1, \lambda_2 \in (0, \infty)$ *are non-degenerate random variables with the following variational representations*

$$\lambda_1 = \sup_{g\in\mathcal{F}_\infty} \left\{\int_{\mathbb{R}^d} \left(-\frac{1}{2}|\nabla g|^2 + V(x)g^2(x)\right)\phi(x)\,\mathrm{d}x\right\}$$
$$\lambda_2 = \inf_{g\in\mathcal{F}_\infty} \left\{\int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla g|^2 + V(x)g^2(x)\right)\phi(x)\,\mathrm{d}x\right\}$$

*where* $\mathcal{F}_\infty = \left\{g \in \mathcal{C}_0^\infty(\mathbb{R}^d) : \int_{\mathbb{R}^d} g^2(x)\phi(x)\,\mathrm{d}x = 1\right\}$ *and* $\mathcal{C}_0^\infty(\mathbb{R}^d)$ *is the set of all smooth functions on* $\mathbb{R}^d$ *with compact support.*

According to the Feynman-Kac formula, we have the following corollary straight forward from Theorem 1.

**Corollary 2.** *The solutions of the PDEs in (2.8) have exponential growth/decay speed almost surely. More precisely,*

$$\lim_{t\to\infty} \frac{1}{t} \log u_+^\omega(x,t) = \lambda_1,$$

*and*

$$\lim_{t\to\infty} \frac{1}{t} \log u_-^\omega(x,t) = -\lambda_2,$$

*where* $\lambda_1, \lambda_2 \in (0,\infty)$ *are non-degenerate random variables.*

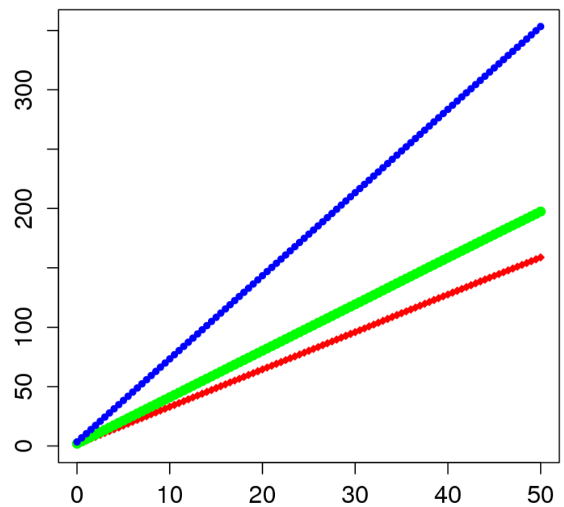


**Figure 2.2:** $x = t$, $y = \log u_+^\omega(0,t)$. Averaging 500 O-U samples in 3 realizations of the Poisson media.

**Remark 2.** *For the BM case, Carmona and Molchanov's result [3] shows that*

$$\lim_{t\to\infty} \frac{\log\log t}{t\log t} \log \mathbb{E}_x \exp\left\{\int_0^t V(B(s))\,\mathrm{d}s\right\} = d \max_{x\in\mathbb{R}^d} K(x), \quad \mathbf{P} - \textit{almost surely}$$

*whereas Sznitman's result [27] shows that*

$$\lim_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log \mathbb{E}_x \exp\left\{-\int_0^t V(B(s))\,\mathrm{d}s\right\} = c \quad \mathbf{P} - \textit{almost surely.}$$

*Comparing their results with ours, we have the following observations. First, both rates are different from the O-U dynamics: the* $u_+^\omega$ *for BM has faster growth rate*

$\exp\{c\frac{t\log\log t}{\log t}\}$ *comparing with O-U dynamics' $e^{ct}$, while the $u_-^\omega$ for BM yields a slower decay rate* $\exp\{-c\frac{t}{(\log t)^{2/d}}\}$ *than O-U dynamics' $e^{-ct}$. Second, even though $u_+^\omega$ and $u_-^\omega$ are random variables of which values depend on each realization $\omega$ of the random media $V$, the constants $\lambda$ in both cases of the quenched exponential moments under the BM dynamics are almost surely not affected by the randomness of the Poisson potential. However, the constants we obtained for O-U dynamics are taking random values that are highly influenced by the random media. So these phenomenons reveal that BM has a relatively stabilized interaction with the Poisson random media.*

**Remark 3.** *Due to the dramatic path behavior differences between O-U processes (non-diffusive) and BMs (diffusive), the strategies executed well for BM case do not work here for the O-U dynamics anymore. For instance, the approach proposed by Carmona and Molchanov for the quenched exponential moment of BM in Poisson media needs to quickly send the random motion to a small ball which is far away from the origin point and let it stay inside for the rest time (see also [7] for an excellent summary in details). The effectiveness of this strategy for BM counts on its diffusive nature. However, to require the same behavior for O-U processes is extremely hard since there is a strong intention for an O-U process to come back to the equilibrium position when the O-U particle moves far apart from the equilibrium position. Indeed, it turns out that the cost of such procedure is not affordable for us to achieve the correct long time asymptotic. Therefore, we need to find alternative method to handle the O-U model.*

*Our proof of the long time exponential moment asymptotic for the quenched regime proceeds by analyzing spectral structure of the following semigroup* $\{T_t^f\}_{t\in\mathbb{R}^+}$

$$T_t^f g(x) = \mathbb{E}_x \exp\left\{\int_0^t f(X(s))\,\mathrm{d}s\right\} g(X(t)).$$

*For the case of potential function $f$ being bounded and deterministic, the classical potential theory and large deviation theory for Markov processes ensures that the long time limit of* $\frac{1}{t}\log\mathbb{E}_x\exp\left\{\int_0^t f(X(s))\,\mathrm{d}s\right\}$ *is closely related to the principle eigenvalue* *of the infinitesimal operator of $T_t$. Furthermore, the principle eigenvalue has a variational representation [10, 28]. Inspired by this idea, we aim to derive similar variational representation in our quenched model, in which case the potential function $V$ is random and blows up in infinity. We achieve this by using local approximation techniques to the semigroup in Chapter 4. By analyzing the variational representation formula, we manage to get the desired long time asymptotic.*

**Theorem 3** (Annealed Positive Regime)**.** *Let Poisson potential $V(\cdot)$ be defined as before. For all $d \in \mathbb{Z}_+$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log\log \mathbf{E} \otimes \mathbb{E}_x \exp\left\{ \int_0^t V(X(s))\, \mathrm{d}s \right\} = \max_{x\in\mathbb{R}^d} K(x).$$

**Remark 4.** *Following the similar argument in Section 5.1 with some mild adjustments, a careful reader will find out that the same asymptotic result holds if we replace the O-U process $X(\cdot)$ by Brownian motion $B(\cdot)$. This phenomenon indicates that, for the positive exponential moment case, it is the the overall impact of the Poisson potential, rather than the random motions, plays the dominant role to the long time asymptotic of the annealed exponential moment.*

**Theorem 4** (Annealed negative regime)**.** *Let Poisson potential $V(\cdot)$ be defined as in Section 2.3. For all $d \in \mathbb{Z}_+$, we have*

$$\lim_{t\to\infty} \frac{1}{(\log t)^{d/2}} \mathbf{E} \otimes \mathbb{E}_x \exp\left\{ -\int_0^t V(X(s))\, \mathrm{d}s \right\} = -\lambda\, \omega_d,$$

*where $\omega_d$ is the volume of the unit $d-$dimensional ball and $\lambda > 0$ is the intensity of the intensity measure $\nu(\mathrm{d}x) = \lambda\, \mathrm{d}x$ for the Poisson point process $\{\omega(A)\}_{A\in\mathcal{B}(\mathbb{R}^d)}$.*

**Remark 5.** *Applying a very similar approach covered in Section 5.2, the same asymptotic result also holds for the hard obstacle situation, as introduced early in (2.4).*

**Remark 6.** *In their seminal paper [9], Donsker and Varadhan showed that both the negative exponential moments of the soft obstacle and hard obstacle (also known as*

*"Wiener sausage") have an exponential decay with rate* $t^{d/(d+2)}$*, i.e.*

$$\lim_{t\to\infty} \frac{1}{t^{d/(d+2)}} \mathbf{E} \otimes \mathbb{E}_x \exp\left\{-\int_0^t V(B(s))\,\mathrm{d}s\right\} = -c \quad c > 0.$$

*The results for the O-U process and the BM are consistent because the O-U process generates smaller sausage than Brownian motion in general due to the pull-back force to the equilibrium position, which is the origin in our model.*

# Chapter 3

# Spectral Structures of the Ornstein-Uhlenbeck Semigroup

In this Chapter, we characterize the spectral structure of certain global as well as killed O-U semigroups. In particular, we derive the variational representations (see (3.15) and (3.16)) of the principle eigenvalues of the infinitesimal operators of these O-U semi-groups. These variational representations will play a crucial role in proving the quenched exponential regime in Chapter 4.

Section 3.1 lists the function spaces which will be applied extensively in the current and the follow up Chapters. Section 3.2 concentrates on presenting the spectral structure of certain O-U semigroups. A summary of background knowledge for self-adjoint operators and their spectral structures can be found at Section A.6 in the Appendix.

## 3.1 Function Space Notations

In the following, we are going to present some analytic results of functional of $\{X(t)\}_{t\geq 0}$. First we list some notations for functional spaces of which will be applied extensively in this section:

- $\mathcal{L}^2(\mathbb{R}^d, \mu)$ – $L^2$ space on $\mathbb{R}^d$ with reference measure $\mu$;
- $\mathcal{L}^2(B(0,R), \mu)$ – $L^2$ space on $B(0,R)$ with reference measure $\mu$;
- $Poly(\mathbb{R}^d)$ – space of all polynomials on $\mathbb{R}^d$;
- $\mathcal{C}_0^\infty(\mathbb{R}^d)$ – smooth function on $\mathbb{R}^d$ with compact support;
- $W^{1,2}(\mathbb{R}^d, \mu) = \{g \in \mathcal{L}^2(\mathbb{R}^d, \mu) : |\nabla g| \in \mathcal{L}^2(\mathbb{R}^d, \mu)\}$, where $\nabla g$ is defined in the weak derivative sense;
- $\mathcal{F}_\infty = \{g \in \mathcal{C}_0^\infty(\mathbb{R}^d) : ||g||_\mu = 1\}$ where $||\cdot||_\mu$ is the $L^2$ norm;
- $\mathcal{F}_R = \{g \in \mathcal{C}_0^\infty(\mathbb{R}^d) : \text{supp}(g) \in B(0,R), ||g||_\mu = 1\}$;
- $\mathcal{P} = \{g \in Poly(\mathbb{R}^d) : ||g||_\mu = 1\}$.

**Remark 7.**

- *$W^{1,2}(\mathbb{R}^d, \mu)$ is a Hilbert space under the Sobolev norm $\sqrt{||g||_\mu^2 + ||\nabla g||_\mu^2}$ (see Section A.4 in Appendix for the proof).*
- *$\mathcal{C}_0^\infty(\mathbb{R}^d)$ and $Poly(\mathbb{R}^d)$ are both dense in $W^{1,2}(\mathbb{R}^d, \mu)$ under the Sobolev norm. Hence, any function $g \in Poly(\mathbb{R}^d)$ can be approximated by functions in $\mathcal{C}_0^\infty(\mathbb{R}^d)$ in the Sobolev norm sense.*

## 3.2 Spectral Structures of the Ornstein-Uhlenbeck Semigroup

Let $f(x)$ be a bounded continuous function on $\mathbb{R}^d$. We define the following family of linear operators $\{T_t\}_{t\geq 0}$ on $\mathcal{L}^2(\mathbb{R}^d, \mu)$: for each $g \in \mathcal{L}^2(\mathbb{R}^d, \mu)$,

$$T_t^f g(x) := \mathbb{E}_x \left( \exp\left\{ \int_0^t f(X(s))\, \mathrm{d}s \right\} g(X(t)) \right). \tag{3.1}$$

Similarly, for $g \in \mathcal{L}^2(B(0,R),\mu)$, we define

$$T_t^{f,R} g(x) = \mathbb{E}_x \left( \exp\left\{ \int_0^t f(X(s))\, \mathrm{d}s \right\} g(X(t)) \mathbf{1}_{\{\tau_R > t\}} \right), \tag{3.2}$$

where $\tau_R \stackrel{\text{def}}{=} \inf\{t \geq 0 : X(t) \notin B(0,R)\}$ is the first exit time of $X$ from the ball $B(0,R)$.

Since O-U process $X$ is time-reversal Markov process, $\{T_t^f\}_{t\geq 0}$ and $\{T_t^{f,R}\}_{t\geq 0}$ are semigroups where each operator is bounded and self-adjoint. In particular, for the case of $f \equiv 0$, $T_t^0$ and $T_t^{0,R}$ correspond to the semigroup of Markov processes $X$ and the semigroup of the killed Markov processes $X$ on the boundary $\partial B(0,R)$, respectively.

Let $L^f$ and $L^{f,R}$ be the infinitesimal operators for $\{T_t^f\}_{t\geq 0}$ and $\{T_t^{f,R}\}_{t\geq 0}$, respectively. In particular, when $f \equiv 0$, $L^0$ and $L^R$ are the infinitesimal operators for Markov process $X$ and the Markov process $X$ being killed at boundary $\partial B(0,R)$, respectively. The following Feynman-Kac formula for $\{T_t\}_{t\geq 0}$ on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ holds (see, e.g., Chapter VII & Chapter VIII, [25]):

**Proposition 5.** *For all $g(x) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,*

$$L^f g(x) = \lim_{t\to 0^+} \frac{T_t^f g(x) - g(x)}{t} = -x \cdot \nabla g(x) + \frac{1}{2}\Delta g(x) + f(x) g(x).$$

*Proof.* For the case of $f \equiv 0$, applying Itō's formula to $g(X(t))$ gives us

$$\mathrm{d}g(X(t)) = \left( -X(s) \cdot \nabla g(X(t)) + \frac{1}{2}\Delta g(X(t)) \right) \mathrm{d}t + \nabla g(X(t)) \cdot \mathrm{d}W(t).$$

Hence the infinitesimal operator $L^0$ can be written as

$$L^0 g(x) = -x \cdot \nabla g(x) + \frac{1}{2}\Delta g(x),$$

for all $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. See for instance [22].

As to general $f$, first notice that

$$\exp\left\{\int_0^t f(X(s))\,\mathrm{d}s\right\} = 1 + \int_0^t f(X(s))\exp\left\{\int_s^t f(X(r))\,\mathrm{d}r\right\}\mathrm{d}s.$$

Multiplying both sides by $g(X(t))$, taking expectation and then applying Markov property on the right side, we get

$$\begin{aligned} T_t^f g(x) &= T_t^0 g(x) + \int_0^t \mathbb{E}_x\left(f(X(s))\mathbb{E}_{X(s)}\left(\exp\left\{\int_0^{t-s} f(X(r))\,\mathrm{d}r\right\} g(X(t-s))\right)\right)\mathrm{d}s \\ &= T_t^0 g(x) + \int_0^t \mathbb{E}_x\big(f(X(s))T_{t-s}g(X(t-s))\big)\,\mathrm{d}s, \end{aligned}$$

which yields to

$$\begin{aligned} L^f g(x) = \lim_{t\to 0^+}\frac{T_t^f g(x) - g(x)}{t} &= L^0 g(x) + f(x)g(x) \\ &= -x\cdot\nabla g(x) + \frac{1}{2}\Delta g(x) + f(x)g(x). \end{aligned}$$

□

From Proposition 5, we observe that $L^f$ have the following symmetric quadratic form on $C_0^\infty(\mathbb{R}^d)$:

**Proposition 6.** *For $g, h \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,*

$$\langle L^f g, h\rangle_\mu = \int_{\mathbb{R}^d} f(x)g(x)h(x)\phi(x)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d}(\nabla g\cdot\nabla h)\,\phi(x)\,\mathrm{d}x, \tag{3.3}$$

*which admits that $L^f$ is a symmetric operator on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ with respect to $\mu$, i.e. $\langle L^f g, h\rangle_\mu = \langle g, L^f h\rangle_\mu$.*

*Proof.* From Proposition 5,

$$\langle L^f g, h\rangle_\mu = \int_{\mathbb{R}^d}(-x\cdot\nabla g)\;h\phi\,\mathrm{d}x + \int_{\mathbb{R}^d}\frac{1}{2}\Delta g\,h\,\phi\,\mathrm{d}x + \int_{\mathbb{R}^d} f\,g\,h\,\phi\,\mathrm{d}x. \tag{3.4}$$

Recall $\phi(x) = \pi^{-d/2}\exp\{-|x|^2\}$. By divergence theorem, the second integral on the right side of (3.4) becomes

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{2}\left(\Delta g(x)\right) h(x)\phi(x)\,\mathrm{d}x \\
=& -\int_{\mathbb{R}^d} \frac{1}{2}\nabla g \cdot \nabla\left(h(x)\phi(x)\right)\,\mathrm{d}x \\
=& -\int_{\mathbb{R}^d} \frac{1}{2}\left(\nabla g \cdot \nabla h\right)\phi(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} \left(x \cdot \nabla g\right) h(x)\phi(x)\,\mathrm{d}x.
\end{aligned}$$

Therefore,

$$\langle L^f g, h\rangle_\mu = \int_{\mathbb{R}^d} f(x)g(x)h(x)\phi(x)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} \left(\nabla g \cdot \nabla h\right)\phi(x)\,\mathrm{d}x.$$

□

Notice that $\mathcal{C}_0^\infty(\mathbb{R}^d)$ and $Poly(\mathbb{R}^d)$ are both dense in $W^{1,2}(\mathbb{R}^d, \mu)$ and the quadratic form on the right side of (3.3) is continuous (both in $g$ and $h$) under the Sobolev norm, we know that the same quadratic form in (3.3) also holds on $Poly(\mathbb{R}^d)$:

**Corollary 7.** *For all $g, h \in Poly(\mathbb{R}^d)$, we have*

$$\langle L^f g, h\rangle_\mu = \langle g, L^f h\rangle_\mu = \int_{\mathbb{R}^d} f(x)g(x)h(x)\phi(x)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} \left(\nabla g \cdot \nabla h\right)\phi(x)\,\mathrm{d}x. \quad (3.5)$$

In order to apply the powerful spectral representation toolbox for self-adjoint operators to $L^f$, we need to extend the description of $L^f$ to a larger function space than $Poly(\mathbb{R}^d)$ and $\mathcal{C}_0^\infty(\mathbb{R}^d)$. In fact, from (3.5) we have $\langle Lg, g\rangle_\mu \leq \sup_{x\in\mathbb{R}^d}|f(x)| \cdot ||g||_\mu^2$ for all $g \in Poly(\mathbb{R}^d) \cup \mathcal{C}_0^\infty(\mathbb{R}^d)$, which implies that $L$ is upper semi-bounded due to the boundedness of $f$. According to the Friedrichs' extension theorem in Section A.6, $L^f$ admits a self-adjoint extension. For simplicity, we still use the same notation for the Friedrichs's extension of $L^f$ and still call it the infinitesimal generator of the semigroup $T_t^f$. Denote $\mathcal{D}(L^f)$ as the domain of the self-adjoint operator $L^f$, that is, $\mathcal{D}(L^f)$ is the collection of all the $\mathcal{L}^2(\mathbb{R}^d, \mu)$ functions $g$ such that $L^f g \in \mathcal{L}^2(\mathbb{R}^d, \mu)$.

From Proposition 6 and Corollary 7, it is clear that $\mathcal{C}_0^\infty(\mathbb{R}^d) \cup Poly(\mathbb{R}^d) \subset \mathcal{D}(L^f) \subset \mathcal{L}^2(\mathbb{R}^d, \mu)$.

Next, we aim to describe $L^f$ on the domain $\mathcal{D}(L^f)$ by the same quadratic form formulated in (3.5). This could be achieved by approximation using Hermite polynomials.

For $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, define

$$\hat{H}_{\mathbf{n}}(x) = \prod_{i=1}^{d} H_{n_i}(x_i),$$

where $\{H_n\}_{n\in\mathbb{N}}$ is the family of one dimensional Hermite polynomials, that is,

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}.$$

We know that each $\hat{H}_{\mathbf{n}}$ is an eigenfunction of $L^0$ with eigenvalue $-|\mathbf{n}|$, where $|\mathbf{n}| = \sum_{i=1}^d n_i$. That is

$$L^0 \hat{H}_{\mathbf{n}} = -x \cdot \nabla \hat{H}_{\mathbf{n}} + \frac{1}{2} \Delta \hat{H}_{\mathbf{n}} = -|\mathbf{n}| \hat{H}_{\mathbf{n}}.$$

Furthermore, normalize these eigenvalues by $e_{\mathbf{n}} = \hat{H}_{\mathbf{n}} / ||\hat{H}_{\mathbf{n}}||_\mu$, $\mathbf{n} \in \mathbb{N}^d$. Then $\{e_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}}$ becomes an orthonormal basis of $\mathcal{L}^2(\mathbb{R}^d, \mu)$. See section 2.3.4, Dunkl and Xu [11] for details.

Using standard approximation techniques in $\mathcal{L}^2(\mathbb{R}^d, \mu)$, we have the following isometry result:

**Proposition 8.** *Given $g \in \mathcal{L}^2(\mathbb{R}^d, \mu)$, then $g \in W^{1,2}(\mathbb{R}^d, \mu)$ if and only if*

$$\sum_{\mathbf{n}\in\mathbb{N}^d} (2|\mathbf{n}| + 1) \langle g, e_{\mathbf{n}} \rangle_\mu^2 < \infty. \tag{3.6}$$

*Furthermore, for any $g, h \in W^{1,2}(\mathbb{R}^d, \mu)$, we have*

$$\begin{aligned} \int_{\mathbb{R}^d} g(x)h(x)\phi(x)\,\mathrm{d}x &= \sum_{\mathbf{n}\in\mathbb{N}^d} \langle g, e_{\mathbf{n}}\rangle_\mu \langle h, e_{\mathbf{n}}\rangle_\mu, \\ \int_{\mathbb{R}^d} (\nabla g \cdot \nabla h)\phi(x)\,\mathrm{d}x &= \sum_{\mathbf{n}\in\mathbb{N}^d} 2|\mathbf{n}|\langle g, e_{\mathbf{n}}\rangle_\mu \langle h, e_{\mathbf{n}}\rangle_\mu. \end{aligned} \tag{3.7}$$

With the above isometry identity, we can prove that the quadratic form $\langle g, L^f g\rangle_\mu$ has the following representation on $\mathcal{D}(L^f)$:

**Lemma 3.0.1.** *We have $\mathcal{D}(L^f) \subset W^{1,2}(\mathbb{R}^d, \mu)$. Furthermore,*

$$\langle g, L^f g\rangle_\mu = \int_{\mathbb{R}^d} f(x)g^2(x)\phi(x)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g|^2\phi(x)\,\mathrm{d}x \quad \textit{for } g \in \mathcal{D}(L). \tag{3.8}$$

*Proof.* Let $g \in \mathcal{D}(L^f)$. For any $n \in \mathbb{N}$, write $g_n(x) = \sum_{|\mathbf{k}|\le n} \langle g, e_{\mathbf{k}}\rangle_\mu e_{\mathbf{k}}(x) \in Poly(\mathbb{R}^d)$. Then

$$\langle L^f g_n, g\rangle_\mu = \int_{\mathbb{R}^d} f(x)g_n(x)g(x)\phi(x)\,\mathrm{d}x - \sum_{|\mathbf{k}|\le n} |\mathbf{k}|\langle g, e_{\mathbf{k}}\rangle_\mu^2. \tag{3.9}$$

Since $\{e_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$ is an orthonormal basis of $\mathcal{L}^2(\mathbb{R}^d, \mu)$, we know

$$g_n \to g \quad \text{as } n \to \infty$$

in $\mathcal{L}^2(\mathbb{R}^d, \mu)$. Consequently,

$$\lim_{n\to\infty} \langle L^f g_n, g\rangle_\mu = \lim_{n\to\infty} \langle g_n, L^f g\rangle_\mu = \langle g, L^f g\rangle_\mu < \infty. \tag{3.10}$$

On the other hand, due to the boundedness of $f(x)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)g_n(x)g(x)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)g^2(x)\phi(x)\,\mathrm{d}x < \infty. \tag{3.11}$$

Let $n$ tend to infinity in (3.9). By (3.10) and (3.11), we have $\sum_{\mathbf{k}\in\mathbb{N}^d} |\mathbf{k}|\langle g, e_{\mathbf{k}}\rangle_\mu^2 < \infty$. This implies that $g \in W^{1,2}(\mathbb{R}^d, \mu)$ by (3.6) from Proposition 8. Furthermore, from

(3.7) and (3.9),

$$\begin{aligned}\langle g, L^f g\rangle_\mu &= \int_{\mathbb{R}^d} f(x)g^2(x)\phi(x)\,\mathrm{d}x - \sum_{\mathbf{k}\in\mathbb{N}^d}^{\infty} |\mathbf{k}|\langle g, e_{\mathbf{k}}\rangle_\mu^2 \\ &= \int_{\mathbb{R}^d} f(x)g^2(x)\phi(x)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2\phi(x)\,\mathrm{d}x.\end{aligned}$$

□

From the classical result of relations between a semigroup and its infinitesimal operator (for instance, see [23]),

$$T_t^f = \exp\left\{tL^f\right\} \tag{3.12}$$

on $\mathcal{L}^2(\mathbb{R}^d, \mu)$. From (3.12) and the spectral representation of self-adjoint operator

$$T_t^f = \int_{-\infty}^{\infty} \exp\{t\lambda\}\, E^f(\mathrm{d}\lambda),$$

where $\{E^f(\lambda); -\infty < \lambda < \infty\}$ is the corresponding resolution of identity for self-adjoint operator $L^f$. In addition, for any $g \in \mathcal{L}^2(\mathbb{R}^d, \mu)$,

$$\langle g, T_t^f g\rangle_\mu = \int_{-\infty}^{\infty} \exp\{t\lambda\} m_g^f(\mathrm{d}\lambda), \tag{3.13}$$

where $m_g^f$ is the spectral measure on $\mathbb{R}$ induced by the distribution function $F^f(\lambda) \equiv \langle g, E^f(\lambda)g\rangle_\mu$ with

$$m_g^f(\mathbb{R}) = ||g||_\mu^2.$$

Moreover, the measure $m_g^f$ is bounded above by

$$\lambda^f \equiv \sup_{g\in\mathcal{D}(L^f), ||g||_\mu = 1} \langle g, L^f g\rangle_\mu. \tag{3.14}$$

Recall that $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{D}(L^f)$ under the Soblev norm and

$$\mathcal{F}_\infty = \left\{g \in \mathcal{C}_0^\infty(\mathbb{R}^d) : ||g||_\mu = 1\right\},$$

then we have

$$\lambda^f = \sup_{g\in\mathcal{F}_\infty} \left\{ \int_{\mathbb{R}^d} f(x)g^2(x)\phi(x)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2\phi(x)\,\mathrm{d}x \right\}. \tag{3.15}$$

Next, we would like to transfer similar spectral properties from $T_t^f$ and $L^f$ to $T_t^{f,R}$ and $L^{f,R}$. Indeed, $\mathcal{L}^2(B(0,R),\mu)$ can be imbedded in $\mathcal{L}^2(\mathbb{R}^d,\mu)$ by the mapping $U : \mathcal{L}^2(B(0,R),\mu) \to \mathcal{L}^2(\mathbb{R}^d,\mu)$, where

$$(Ug)(x) = \begin{cases} g(x) & \text{if } x \in B(0,R) \\ 0 & \text{if } x \notin B(0,R) \end{cases}$$

Thus $\mathcal{L}^2(B(0,R),\mu)$ and $W^{1,2}(B(0,R),\mu)$ can be regarded as a closed subspace of $\mathcal{L}^2(\mathbb{R}^d,\mu)$ and $W^{1,2}(\mathbb{R}^d,\mu)$, respectively.

The following definition of local operator can be found in different literatures, for instance, Getoor [17]:

**Definition 1.** *An operator $Q$ in $\mathcal{L}^2(\mathbb{R}^d,\mu)$ is called a local operator if for any $h \in \mathcal{D}(Q)$ and any open set $G$ with Lebesgue measure 0 on the boundary, one has $hI_G \in \mathcal{D}(Q)$ and $I_GQh = Q(I_Gh)$ as elements of $\mathcal{L}^2(\mathbb{R}^d,\mu)$.*

We know from lemma 3.0.1 that $L^f$ is a local operator. Therefore, Theorem 4.2 and Theorem 4.3 in Getoor [17] yields to the fact that $\mathcal{D}(L^{f,R}) = \mathcal{D}(L^f) \cap \mathcal{L}^2(B(0,R),\mu)$ and $L^f g(x) = L^{f,R}g(x)$ for all $g \in \mathcal{D}(L^{f,R})$. Combine with (3.8), we have

$$\langle g, L^{f,R}g\rangle_\mu = \int_{B(0,R)} f(x)g^2(x)\phi(x)\,\mathrm{d}x - \frac{1}{2}\int_{B(0,R)} |\nabla g(x)|^2\phi(x)\,\mathrm{d}x$$

for any $g \in \mathcal{D}(L^{f,R})$.

Now we turn to $T_t^{f,R}$. Repeat the similar argument carried out for $T_t^f$, we know that $T_t^{f,R}$ has the spectral representation:

$$T_t^{f,R} = \int_{-\infty}^{\infty} \exp\{t\lambda\}\, E^{f,R}(\mathrm{d}\lambda),$$

where $\{E^{f,R}(\lambda); -\infty < \lambda < \infty\}$ is the corresponding resolution of identity for self-adjoint operator $L^{f,R}$. In addition, for any $g \in \mathcal{L}^2(B(0,R),\mu)$,

$$\langle g, T_t^{f,R} g\rangle_\mu = \int_{-\infty}^{\infty} \exp\{t\lambda\} m_g^{f,R}(\mathrm{d}\lambda),$$

where $m_g^{f,R}$ is known as spectral measure on $\mathbb{R}$ induced by the distribution function $F^{f,R}(\lambda) \equiv \langle g, E^{f,R}(\lambda) g\rangle_\mu$ with

$$m_g^{f,R}(\mathbb{R}) = ||g||_\mu^2.$$

Furthermore, $m_g^{f,R}$ is bounded above by

$$\begin{aligned} \lambda^{f,R} &\equiv \sup_{g\in\mathcal{D}(L^{f,R}),||g||_\mu=1} \langle g, L^{f,R} g\rangle_\mu \\ &= \sup_{g\in\mathcal{F}_R} \left\{ \int_{\mathbb{R}^d} f(x) g^2(x)\phi(x)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 \phi(x)\,\mathrm{d}x \right\}. \end{aligned} \tag{3.16}$$

# Chapter 4

# Long Time Asymptotics: Quenched Regime

In this Chapter, we give the proof of Theorem 1. The proof is followed by two steps. First, we derive variational formulas for $\lambda_1$ and $\lambda_2$. Second, we investigate the variational formulas and obtain that $\lambda_1, \lambda_2 \in (0, \infty)$.

## 4.1 Variational Formulas for the Rates

**Proposition 9.** *The following large deviation result holds* $\mathbf{P}$*-a.s.:*

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \exp\left\{\pm \int_0^t V(X(s))\,\mathrm{d}s\right\} \\ &= \sup_{g\in\mathcal{F}_\infty} \left\{\int_{\mathbb{R}^d} \left(-\frac{1}{2}|\nabla g|^2 \pm V(x)g^2(x)\right)\phi(x)\,\mathrm{d}x\right\}, \end{aligned} \tag{4.1}$$

*where* $\mathcal{F}_\infty = \left\{g \in \mathcal{C}_0^\infty(\mathbb{R}^d) : \int_{\mathbb{R}^d} g^2(x)\phi(x)\,\mathrm{d}x = 1\right\}$.

In the following subsections, we will discuss the proof of Proposition 9 under the position exponential as well as negative exponential situations, respectively. As we mentioned earlier, the main challenge here is to deal with *unbounded* potential function $V$. This challenge is highlighted more for the positive exponential situation and the solution we provide is to use local approximation. With this regard, we will give full proof with details for the positive exponential situation. As to the negative exponential case, since the argument is very similar, we will sketch the proof and highlight those parts which need different attention from the positive exponential situation.

### 4.1.1 Exponential Moments with Positive Coefficients

*Proof.* For $n \in \mathbb{Z}^+$, define $V_n = V \wedge n$. Since $V_n$ is a bounded function, the spectral representation techniques discussed in Chapter 3 can be applied here. In fact, choose $g \in \mathcal{F}_\infty$ and notice that $V \geq 0$. Then we have

$$\begin{aligned}
&\mathbb{E}_x \exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\} \\
\geq &\mathbb{E}_x \exp\left\{\int_1^t V_n(X(s))\,\mathrm{d}s\right\} \\
\geq &||g||_\infty^{-2}\mathbb{E}_x\left(g(X(1))\exp\left\{\int_1^t V_n(X(s))\,\mathrm{d}s\right\}g(X(t))\right) \\
= &||g||_\infty^{-2}\mathbb{E}_x\left(g(X(1))\mathbb{E}_{X(1)}\exp\left\{\int_0^{t-1} V_n(X(s))\,\mathrm{d}s\right\}g(X(t-1))\right) \\
= &||g||_\infty^{-2}\int_{\mathbb{R}^d} p(x,y,1)g(y)T_{t-1}^{V_n}g(y)\,\mathrm{d}y,
\end{aligned} \tag{4.2}$$

where recall that $p(x, y, 1)$ is the transition density of $X$ from $x$ at time 0 to $y$ at time 1 and $T_{t-1}^{V_n}$ is the semigroup defined in (3.1).

Recall from (2.5) that

$$p(x,y,1)\phi^{-1}(y) = c\exp\left\{|x|^2 - \frac{|x-ye^{-1}|^2}{1-e^{-2}}\right\}.$$

So $p(x, y, 1)\phi^{-1}(y)$, as a function of $y$, is bounded below by a positive number on the compact support of $g$. Therefore, combine with (4.2), we get

$$\mathbb{E}_x \exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\} \geq c\int_{\mathbb{R}^d} g(y)T_{t-1}^{V_n}g(y)\phi(y)\,\mathrm{d}y = c\langle g, T_{t-1}^{V_n}g\rangle_\mu. \tag{4.3}$$

By applying spectral representation and Jensen's inequality, we have

$$
\begin{aligned}
\langle g, T_{t-1}^{V_n} g\rangle_\mu =& \int_{-\infty}^{\infty} e^{(t-1)\lambda} m_g(\mathrm{d}\lambda) \\
\geq & e^{(t-1)\int_{-\infty}^{\infty}\lambda\, m_g(\mathrm{d}\lambda)} \\
=& \exp\left\{(t-1)\langle g, L^{V_n} g\rangle_\mu\right\} \\
=& \exp\left\{(t-1)\int_{\mathbb{R}^d}\left(-\frac{1}{2}|\nabla g|^2 + V_n(x) g^2(x)\right)\phi(x)\,\mathrm{d}x\right\}.
\end{aligned}
\tag{4.4}
$$

From (4.3) and (4.4), we get

$$
\liminf_{t\to\infty}\frac{1}{t}\log \mathbb{E}_x \exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\} \geq -\frac{1}{2}\int_{\mathbb{R}^d}\left(|\nabla g|^2 - 2V_n(x)g^2(x)\right)\phi(x)\,\mathrm{d}x.
$$

Let $n \to \infty$ and then take supreme over all $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, we obtain the lower bound.

Next, we turn to the upper bound. To prove the upper bound, we need the following localization estimate, of which proof is given in the Appendix, Section A.5.

**Lemma 4.0.2.** *Put $\gamma_t = \alpha t^{1/2}\log t$ for some constant $\alpha > 0$. Then $\mathbf{P}$-a.s.,*

$$
\lim_{t\to\infty}\frac{\mathbb{E}_x\left(\exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\}\mathbf{1}_{\{\tau_{\gamma_t}>t\}}\right)}{\mathbb{E}_x\exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\}} = 1,
$$

*where $\tau_R = \inf\{t > 0 : X(t) \notin B(0,R)\}$.*

By Lemma 4.0.2 and Lemma A.0.8, $\mathbf{P}-$a.s. there exist $c_1, c_2 > 0$ (the choice of $c_1, c_2$ depends on the realization of random media $\omega(\cdot)$) such that for all large $t$

$$
\mathbb{E}_x \exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\} \leq c_1 \mathbb{E}_x\left(\exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\}\mathbf{1}_{\{\tau_{\gamma_t}>t\}}\right),
$$

and

$$
\sup_{x\in B(0,\gamma_t)} V(x) \leq c_2 \log t.
$$

Therefore, for all $t$ sufficiently large, we have

$$\begin{aligned}
&\mathbb{E}_x \exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\}\\
\le& c_1\mathbb{E}_x\left(\exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\}\mathbf{1}_{\{\tau_{\gamma_t}>t\}}\right)\\
\le& c_1 t^{c_2}\mathbb{E}_x\left(\exp\left\{\int_1^t V(X(s))\,\mathrm{d}s\right\}\mathbf{1}_{\left\{\sup\limits_{1\le s\le t}|X(t)|\le\gamma_t\right\}}\right)\\
=& c_1 t^{c_2}\mathbb{E}_x\left(\mathbf{1}_{\{|X(1)|\le\gamma_t\}}\exp\left\{\int_1^t V(X(s))\,\mathrm{d}s\right\}\mathbf{1}_{\left\{\sup\limits_{1\le s\le t}|X(t)|\le\gamma_t\right\}}\mathbf{1}_{\{|X(t)|\le\gamma_t\}}\right).
\end{aligned}\tag{4.5}$$

Let $|g_t|\le 1$ be a smooth function such that, $g_t(y)\equiv 1$ on $B(0,\gamma_t)$ and $g_t(y)\equiv 0$ outside $B(0,\gamma_t+2)$. Denote $h_t=c_t^{-1}g_t$, such that $||h_t||_\mu=1$. Clearly, the normalizing constant

$$c_t=\left(\int_{\mathbb{R}^d} g_t^2(x)\phi(x)\,\mathrm{d}x\right)^{1/2}\le\left(\int_{B(0,\gamma_t+2)}\phi(x)\,\mathrm{d}x\right)^{1/2}<1.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}_x\left(\mathbf{1}_{\{|X(1)|\le\gamma_t\}}\exp\left\{\int_1^t V(X(s))\,\mathrm{d}s\right\}\mathbf{1}_{\left\{\sup\limits_{1\le s\le t}|X(t)|\le\gamma_t\right\}}\mathbf{1}_{\{|X(t)|\le\gamma_t\}}\right)\\
\le&\mathbb{E}_x\left(g_t(X(1))\exp\left\{\int_1^t V(X(s))\,\mathrm{d}s\right\}\mathbf{1}_{\left\{\sup\limits_{1\le s\le t}|X(t)|\le\gamma_t+2\right\}}g_t(X(t))\right)\\
\le&\mathbb{E}_x\left(h_t(X(1))\exp\left\{\int_1^t V(X(s))\,\mathrm{d}s\right\}\mathbf{1}_{\left\{\sup\limits_{1\le s\le t}|X(t)|\le\gamma_t+2\right\}}h_t(X(t))\right)\\
=&\int_{\mathbb{R}^d}p(x,y,1)\phi(y)^{-1}h_t(y)T_{t-1}^{V,\gamma_t+2}h_t(y)\phi(y)\,\mathrm{d}y\\
\le& c\exp\left\{|x|^2\right\}\langle h_t,T_{t-1}^{V,\gamma_t+2}h_t\rangle_\mu,
\end{aligned}\tag{4.6}$$

where the notation $T_t^{V,\gamma_t+2}$ is defined as in (3.2):

$$T_t^{V,R}g(x)=\mathbb{E}_x\left(\exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\}g(X(t))\mathbf{1}_{\{\tau_R>t\}}\right),$$

and the last inequality in (4.6) holds since $p(x,y,1)\phi(y)^{-1} \le c\exp\{|x|^2\}$ by (2.5).

Recall from (**??**) and (3.16) that the semigroup $T_t^{V,\gamma_t+2}$ has the spectral representation

$$\langle h_t, T_{t-1}^{V,\gamma_t+2} h_t\rangle_\mu = \int_{-\infty}^{\infty} e^{(t-1)\lambda} m_{h_t}^{\gamma_t+2}(\mathrm{d}\lambda)$$

and the smallest supporting set of probability measure $m_{h_t}^{\gamma_t+2}$ is bounded above by

$$\sup_{h\in\mathcal{D}(L^{V,\gamma_t+2})} \langle h, L^{\gamma_t+2} h\rangle_\mu = \sup_{g\in\mathcal{F}_{\gamma_t+2}} \left\{ -\frac{1}{2}\int_{\mathbb{R}^d} \left(|\nabla g(x)|^2 - 2V(x)g(x)^2\right)\phi(x)\,\mathrm{d}x \right\},$$

where

$$\mathcal{F}_{\gamma_t+2} = \left\{ g\in\mathcal{C}_0^\infty(B(0,\gamma_t+2)) : \int_{B(0,\gamma_t+2)} g^2(x)\phi(x)\,\mathrm{d}x = 1 \right\}.$$

Hence,

$$\begin{aligned} \langle h_t, T_{t-1}^{V,\gamma_t+2} h_t\rangle_\mu &\le \exp\left\{ (t-1) \sup_{g\in\mathcal{F}_{\gamma_t+2}} \left\{ -\frac{1}{2}\int_{\mathbb{R}^d} \left(|\nabla g|^2 - 2Vg^2\right)\phi(x)\,\mathrm{d}x \right\} \right\} \\ &\le \exp\left\{ (t-1) \sup_{g\in\mathcal{F}_\infty} \left\{ -\frac{1}{2}\int_{\mathbb{R}^d} \left(|\nabla g|^2 - 2Vg^2\right)\phi(x)\,\mathrm{d}x \right\} \right\}. \end{aligned} \tag{4.7}$$

Combine (4.5), (4.6) and (4.7), we obtain the upper bound

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t}\log\mathbb{E}_x \exp\left\{ \int_0^t V(X(s))\,\mathrm{d}s \right\} \\ &\le \sup_{g\in\mathcal{F}_\infty} \left\{ -\frac{1}{2}\int_{\mathbb{R}^d} \left(|\nabla g(x)|^2 - 2V(x)g(x)^2\right)\phi(x)\,\mathrm{d}x \right\}. \end{aligned}$$

□

### 4.1.2 Exponential Moments with Negative Coefficients

Here we sketch the proof for the negative exponential moment situation.

*Proof.* First, we consider the lower bound. Keep the same notation $V_n = V \wedge n$ as before. Notice that

$$\mathbb{E}_x \exp\left\{-\int_0^t V(X(s))\,\mathrm{d}s\right\} \geq e^{-n}\mathbb{E}_x \exp\left\{-\int_1^t V_n(X(s))\,\mathrm{d}s\right\}.$$

Repeat the similar procedures in (4.2), (4.3) and (4.4), we get

$$\mathbb{E}_x \exp\left\{-\int_0^t V(X(s))\,\mathrm{d}s\right\} \geq C\exp\left\{-(t-1)\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla g|^2 + V_n(x)g^2(x)\right)\,\mathrm{d}x\right\},$$

where $C$ is constant determined by $g$ and $n$. Hence, we have

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x \exp\left\{-\int_0^t V(X(s))\,\mathrm{d}s\right\} \geq -\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla g|^2 + V_n(x)g^2(x)\right)\,\mathrm{d}x.$$

Let $n$ go to infinity and take supreme over $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, then we obtain the lower bound:

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x \exp\left\{-\int_0^t V(X(s))\,\mathrm{d}s\right\} \geq \sup_{g\in\mathcal{F}_\infty}\left\{-\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla g|^2 + V(x)g^2(x)\right)\,\mathrm{d}x\right\}. \tag{4.8}$$

Next, we turn to the upper bound. Since $-V$ is bounded above by 0, the proof of the upper bound is straight forward and do not need localization treatment as before. Indeed, we have

$$\begin{aligned}\mathbb{E}_x \exp\left\{-\int_0^t V(X(s))\,\mathrm{d}s\right\} \leq& \mathbb{E}_x \exp\left\{-\int_1^t V(X(s))\,\mathrm{d}s\right\}\\ =&\int_{\mathbb{R}^d} p(x,y,1)\phi(y)^{-1}T_{t-1}^{-V}\mathbf{1}\,\phi(y)\,\mathrm{d}y \\ \leq& c\exp\left\{|x|^2\right\}\langle \mathbf{1}, T_{t-1}^{-V}\mathbf{1}\rangle_\mu,\end{aligned} \tag{4.9}$$

where the last inequality holds once again due to the fact that $p(x,y,1)\phi(y)^{-1} \leq c\exp\{|x|^2\}$. Apply the spectral representation (3.13) and the spectral measure estimate (3.15) to $\langle \mathbf{1}, T_{t-1}^{-V}\mathbf{1}\rangle_\mu$, we have

$$\langle \mathbf{1}, T_{t-1}^{-V}\mathbf{1}\rangle_\mu \leq \exp\left\{(t-1)\sup_{g\in\mathcal{F}_\infty}\left\{-\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla g|^2 + V(x)g^2(x)\right)\,\mathrm{d}x\right\}\right\}. \tag{4.10}$$

Combine (4.9) and (4.10), we obtain the upper bound

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x\exp\left\{-\int_0^t V(X(s))\,\mathrm{d}s\right\} \leq \sup_{g\in\mathcal{F}_\infty}\left\{-\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla g|^2 + V(x)g^2(x)\right)\,\mathrm{d}x\right\}. \tag{4.11}$$

Put (4.8) and (4.11) together, we get the desired result. □

Using standard approximation treatment, we have,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x\exp\left\{\pm\int_0^t V(X(s))\,\mathrm{d}s\right\}\\
&= -\inf_{f\in\mathcal{F}_\infty}\left\{\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla f|^2 \mp V(x)f^2(x)\right)\phi(x)\,\mathrm{d}x\right\},\\
&= -\inf_{f\in\mathcal{P}}\left\{\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla f|^2 \mp V(x)f^2(x)\right)\phi(x)\,\mathrm{d}x\right\},
\end{aligned} \tag{4.12}$$

where

$$\mathcal{P} = \left\{g \in \mathrm{Poly}(\mathbb{R}^d), ||g||_\mu = 1\right\}.$$

For the convenience of the analysis in Section 4.2, we rewrite (4.12) with respect to Lebesgue measure. Let $\mathcal{E} = \left\{\tilde{f}(x) \stackrel{\text{def}}{=} f(x)e^{-\frac{|x|^2}{2}} : f \in \mathcal{P}\right\}$, then $||\tilde{f}||_2 = \pi^{d/2}$, where $||\cdot||_2$ is the classic $L^2$-norm. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^d}|\nabla f|^2\phi(x)\,\mathrm{d}x = \pi^{-d/2}\int_{\mathbb{R}^d}\left|\nabla\tilde{f} + x\tilde{f}(x)\right|^2\,\mathrm{d}x\\
&= \pi^{-d/2}\int_{\mathbb{R}^d}\left(\left|\nabla\tilde{f}\right|^2 + |x|^2\tilde{f}^{\,2}\right)\,\mathrm{d}x + 2\pi^{-d/2}\int_{\mathbb{R}^d}\left(x\cdot\nabla\tilde{f}\right)\tilde{f}\,\mathrm{d}x.
\end{aligned} \tag{4.13}$$

Applying divergence theorem to the second integral in (4.13),

$$\int_{\mathbb{R}^d}\left(x\cdot\nabla\tilde{f}\right)\tilde{f}\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^d}x\cdot\nabla\tilde{f}^{\,2}(x)\,\mathrm{d}x = -\frac{d}{2}\int_{\mathbb{R}^d}\tilde{f}^2(x)\,\mathrm{d}x = -\frac{d}{2}\pi^{d/2}. \tag{4.14}$$

Hence, by (4.13) and (4.12), the quenched long time asymptotic results become

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x\exp\left\{\pm\int_0^t V(X(s))\,\mathrm{d}s\right\} \\
&= -\frac{1}{2}\pi^{-d/2}\inf_{g\in\mathcal{E}}\left\{\int_{\mathbb{R}^d}|\nabla g|^2+\left(|x|^2\mp 2V(x)\right)g^2\,\mathrm{d}x\right\}+\frac{d}{2}.
\end{aligned}
\tag{4.15}
$$

## 4.2 Analysis of $\lambda_1$ and $\lambda_2$

In this section, we analyze (4.15) and prove theorem 1.

**Lemma 4.0.3.**

$$
\inf_{g\in\mathcal{E}}\left\{\int_{\mathbb{R}^d}|\nabla g|^2+|x|^2g^2\,\mathrm{d}x\right\}=d\pi^{d/2},
\tag{4.16}
$$

*where the minimizers are* $g_0(x)=\pm e^{-|x|^2/2}$.

*Proof.* By (4.14),

$$
\begin{aligned}
\frac{1}{2}d\pi^{d/2} &= -\int_{\mathbb{R}^d}(x\cdot\nabla g)g\,\mathrm{d}x \le \left(\int_{\mathbb{R}^d}|x|^2g^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^d}|\nabla g|^2\,\mathrm{d}x\right)^{1/2} \\
&\le \frac{1}{2}\left(\int_{\mathbb{R}^d}|x|^2g^2(x)\,\mathrm{d}x+\int_{\mathbb{R}^d}|\nabla g(x)|^2\,\mathrm{d}x\right).
\end{aligned}
$$

To make both inequalities equal, we need $g_0(x)\cdot x=-\nabla g_0(x)$. Under the condition that $||g_0||_2=1$, we have $g_0(x)=\pm e^{-|x|^2/2}$. Clearly, $g_0\in\mathcal{E}$. ☐

To get what stated in theorem 1 we need to show the following Proposition. Throughout the proof, use the same notation as in lemma 4.0.3: $g_0(x)=e^{-|x|^2/2}$.

**Proposition 10.** *Let*

$$
\begin{aligned}
\lambda_1 &= -\frac{1}{2}\pi^{-d/2}\inf_{g\in\mathcal{E}}\left\{\int_{\mathbb{R}^d}|\nabla g|^2+\left(|x|^2-2V(x)\right)g^2\,\mathrm{d}x\right\}+\frac{d}{2}, \\
\lambda_2 &= \frac{1}{2}\pi^{-d/2}\inf_{g\in\mathcal{E}}\left\{\int_{\mathbb{R}^d}|\nabla g|^2+\left(|x|^2+2V(x)\right)g^2\,\mathrm{d}x\right\}-\frac{d}{2}.
\end{aligned}
$$

*Then* $\mathbf{P}-$*a.s.,* $\lambda_1,\lambda_2\in(0,\infty)$ *are non-degenerate random variables.*

*Proof.* First, consider $\lambda_1$. By Lemma A.0.8, $|x|^2 - 2V(x)$ has a (random) lower bound $C(\omega)$ on $\mathbb{R}^d$. Then, we have

$$\inf_{g\in\mathcal{E}}\left\{\int_{\mathbb{R}^d}|\nabla g|^2+\left(|x|^2-2V(x)\right)g^2\,\mathrm{d}x\right\}\geq\inf_{g\in\mathcal{E}}\left\{\int_{\mathbb{R}^d}\left(|x|^2-2V(x)\right)g^2\,\mathrm{d}x\right\}$$
$$\geq\inf_{g\in\mathcal{E}}\left\{C(\omega)\int_{\mathbb{R}^d}g(x)^2\,\mathrm{d}x\right\}=C(\omega)\pi^{d/2}.$$

Therefore, $\mathbf{P}$−a.s., $\lambda_1\leq\frac{1}{2}(d-C(\omega))<\infty$. On the other hand,

$$\begin{aligned}&\inf_{g\in\mathcal{E}}\left\{\int_{\mathbb{R}^d}\left(|\nabla g|^2+\left(|x|^2-2V(x)\right)g^2(x)\right)\,\mathrm{d}x\right\}\\&\leq\int_{\mathbb{R}^d}|\nabla g_0|^2+\left(|x|^2-2V(x)\right)g_0^2(x)\,\mathrm{d}x\\&=d\pi^{d/2}-2\int_{\mathbb{R}^d}V(x)e^{-|x|^2/2}\,\mathrm{d}x<d\pi^{d/2}\quad\mathbf{P}-a.s.\end{aligned}$$

The last inequality holds since $\mathbf{P}(V\equiv 0\text{ on }\mathbb{R}^d)=0$. Therefore, $\mathbf{P}$−a.s.

$$\lambda_1=-\frac{1}{2}\pi^{-d/2}\inf_{g\in\mathcal{E}}\left\{\int_{\mathbb{R}^d}\left(|\nabla g|^2+\left(|x|^2-2V(x)\right)g^2\right)\,\mathrm{d}x\right\}+\frac{d}{2}>0.$$

To prove the non-degeneracy of $\lambda_1$, it suffices to show $\mathbf{P}(\lambda_1>\alpha)>0$ for any $\alpha>0$.

By continuity of $K$, there exists $r>0$ such that $K(x)>K(0)/2$ for all $x\in B(0,r)$. Then for any $x\in B(0,r/2)$

$$\begin{aligned}V(x)&=\int_{\mathbb{R}^d}K(x-y)\,\omega(\mathrm{d}y)\geq\int_{B(0,r/2)}K(x-y)\,\omega(\mathrm{d}y)\\&\geq\int_{B(0,r/2)}\frac{K(0)}{2}\,\omega(\mathrm{d}y)=\frac{K(0)}{2}\omega(B(0,r/2)).\end{aligned}$$

Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^d} V(x) g_0^2(x)\,\mathrm{d}x \geq & \int_{B(0,r/2)} V(x) g_0^2(x)\,\mathrm{d}x \\
\geq & \frac{K(0)}{2}\omega(B(0,r/2)) \int_{B(0,r/2)} e^{-|x|^2}\,\mathrm{d}x \\
\geq & c\,\omega(B(0,r/2)).
\end{aligned}
\tag{4.17}
$$

From (4.17) we get

$$
\begin{aligned}
\lambda_1 \geq & -\frac{1}{2}\pi^{-d/2}\int_{\mathbb{R}^d} \left(|\nabla e^{-|x|^2}|^2 + |x|^2 e^{-|x|^2} - V(x) e^{-|x|^2}\right)\,\mathrm{d}x + \frac{d}{2} \\
\geq & \; c\omega(B(0,r/2)),
\end{aligned}
$$

which implies that

$$
\mathbf{P}(\lambda_1 \geq cn) \geq \mathbf{P}(\omega(B(0,r/2)) = n) > 0.
$$

As to $\lambda_2$, the upper bound holds since

$$
\begin{aligned}
\lambda_2 \leq & \frac{1}{2}\pi^{-d/2}\int_{\mathbb{R}^d} |\nabla g_0|^2 + \left(|x|^2 + 2V(x)\right) g_0^2\,\mathrm{d}x - \frac{d}{2} \\
= & \pi^{-d/2}\int_{\mathbb{R}^d} V(x) e^{-|x|^2}\,\mathrm{d}x < \infty,
\end{aligned}
$$

where the last inequality holds by Lemma A.0.8.

For the lower bound, denote $F(g) := \int_{\mathbb{R}^d} |\nabla g|^2 + (|x|^2 + 2V(x))\, g^2(x)\,\mathrm{d}x$. Notice that for $g = g_0$ a.s.,

$$
\begin{aligned}
F(g_0) = & \int_{\mathbb{R}^d} \left(|\nabla g_0|^2 + |x|^2 + 2V(x)\right) g_0^2(x)\,\mathrm{d}x \\
= & d\pi^{d/2} + 2\int_{\mathbb{R}^d} V(x) e^{-|x|^2/2}\,\mathrm{d}x > d\pi^{d/2} + \delta_1 \quad \mathbf{P} - a.s.,
\end{aligned}
\tag{4.18}
$$

for some $\delta_1 > 0$.

For $g \neq g_0$, by lemma 4.0.3,

$$\begin{aligned} F(g) &= \int_{\mathbb{R}^d} \left( |\nabla g|^2 + \left( |x|^2 + 2V(x) \right) g^2(x) \right) \, \mathrm{d}x \\ &\geq \int_{\mathbb{R}^d} \left( |\nabla g_0|^2 + |x|^2 g_0^2(x) \right) \, \mathrm{d}x + \delta_2 = d\pi^{d/2} + \delta_2 \quad \mathbf{P} - a.s. \end{aligned} \tag{4.19}$$

for some $\delta_2 > 0$.

Therefore, from (4.18), (4.19) and the continuity of $F$ on $\mathcal{E}$ under Sobolev norm,

$$\lambda_2 = \frac{1}{2}\pi^{-d/2} \inf_{g \in \mathcal{E}} \left\{ \int_{\mathbb{R}^d} \left( |\nabla g|^2 + \left( |x|^2 + 2V(x) \right) g^2 \right) \, \mathrm{d}x \right\} - \frac{d}{2} > 0.$$

As to the non-degeneracy of $\lambda_2$, by continuity of $K$ and the construction of $V$, we know that $V$ has a positive probability of greater than any large value in a compact set. Therefore,

$$\lambda_2 \geq \pi^{-d/2} \inf_{g \in \mathcal{E}} \left\{ \int_{\mathbb{R}^d} \left( |x|^2 + 2V(x) \right) g^2(x) \, \mathrm{d}x \right\} - \frac{d}{2} \geq \inf_{x \in \mathbb{R}^d} (|x|^2 + 2V(x)) - \frac{d}{2} := c$$

happens with a positive probability. □

# Chapter 5

# Long Time Asymptotics: Annealed Regime

In this Chapter, we give the detailed proofs of Theorem 3 and Theorem 4.

## 5.1 Exponential Moments with Positive Coefficients

*Proof of Theorem 3.* Notice that

$$\int_0^t V(X(s))\,\mathrm{d}s = \int_{\mathbb{R}^d}\int_0^t K(X(s)-y)\,\mathrm{d}s\,\omega(\mathrm{d}y).$$

Using Fubini Theorem and Poisson integrals (see Appendix A.1), we have

$$\mathbf{E}\otimes\mathbb{E}_x \exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\} = \mathbb{E}_x \exp\left\{\lambda\int_{\mathbb{R}^d}\left(\exp\left\{\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}-1\right)\mathrm{d}y\right\}. \tag{5.1}$$

First we establish the upper bound of (5.1). Jensen's inequality yields to

$$\exp\left\{\int_0^t K(X(s)-y)\,\mathrm{d}s\right\} \le \frac{1}{t}\int_0^t \exp\left\{tK(X(s)-y)\right\}\mathrm{d}s.$$

Hence, we have

$$
\begin{aligned}
&\mathbb{E}_x \exp\left\{\lambda \int_{\mathbb{R}^d} \left(\exp\left\{\int_0^t K(X(s)-y)\,\mathrm{d}s\right\} - 1\right)\,\mathrm{d}y\right\} \\
\leq &\mathbb{E}_x \exp\left\{\frac{\lambda}{t}\int_0^t \int_{\mathbb{R}^d} \left(e^{tK(X(s)-y)} - 1\right)\mathrm{d}y\,\mathrm{d}s\right\} \\
= &\exp\left\{\lambda \int_{\mathbb{R}^d} \left(e^{tK(y)} - 1\right)\,\mathrm{d}y\right\},
\end{aligned}
\tag{5.2}
$$

where the last equality holds due to the shift invariance of the Lebesgue measure. Since $K(\cdot)$ is compactly supported and $\mathrm{supp}(K) \subset B(0, L)$, the integral in the last line of (5.2) equals to the restriction of its domain on $B(0, L)$. Therefore, combining (5.1) and (5.2) we obtain the upper bound:

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \log\log \mathbf{E} \otimes \mathbb{E}_x \exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\} \\
\leq &\limsup_{t\to\infty} \frac{1}{t} \log\left(\lambda\,|B(0,L)|\exp\left\{t \max_{x\in\mathbb{R}^d} K(x)\right\}\right) = \max_{x\in\mathbb{R}^d} K(x).
\end{aligned}
\tag{5.3}
$$

Next, we consider the lower bound. For any $\epsilon > 0$, by the continuity of $K$ there exists a ball $B(x_0, \delta)$ such that

$$
K(y) > \max_{x\in\mathbb{R}^d} K(x) - \epsilon, \quad \text{for all } y \in B(x_0, \delta). \tag{5.4}
$$

Hence, our strategy for the lower bound is to restrict the O-U process $X(\cdot)$ inside a small ball up to time $t$ so that the exponentials will get main contribution from the maximum of the shape function $K$. More precisely,

$$
\begin{aligned}
&\mathbb{E}_x \exp\left\{\lambda \int_{\mathbb{R}^d} \left(\exp\left\{\int_0^t K(X(s)-y)\,\mathrm{d}s\right\} - 1\right)\,\mathrm{d}y\right\} \\
&\geq \mathbb{E}_x \left(\exp\left\{\lambda \int_{B(x-x_0,\delta/2)} \left(e^{\int_0^t K(X(s)-y)\,\mathrm{d}s} - 1\right)\,\mathrm{d}y\right\} \mathbf{1}_{\left\{\sup\limits_{0\leq s\leq t} |X(s)-x|<\delta/2\right\}}\right) \\
&\geq \exp\left\{\lambda\,|B(x-x_0,\delta/2)| \left(e^{t\cdot(\max\limits_{x\in\mathbb{R}^d} K(x)-\epsilon)} - 1\right)\right\} \cdot \mathbb{P}_x\left(\sup_{0\leq s\leq t} |X(s)-x| < \delta/2\right),
\end{aligned}
\tag{5.5}
$$

where the last inequality holds due to (5.4) and the fact that $X(s) - y \in B(x_0, \delta)$ given the condition that $X(s) \in B(x, \delta/2)$ and $y \in B(x - x_0)$ for all $0 \leq s \leq t$.

Using the classical small ball estimate for Gaussian processes, for instance [20], the cost of restricting Gaussian process $X$ in a small ball up to $t$ is exponentially small:

$$
\mathbb{P}_x\left(\sup_{0\leq s\leq t} |X(s)-x| < \delta/2\right) \asymp e^{-ct}, \quad \text{for some } c > 0. \tag{5.6}
$$

Therefore, combine with (5.1), (5.5) and (5.6) we have

$$
\liminf_{t\to\infty} \frac{1}{t} \log\log \mathbf{E} \otimes \mathbb{E}_x \exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\} \geq \max_{x\in\mathbb{R}^d} K(x) - \epsilon.
$$

The lower bound is obtained by letting $\epsilon$ go to $0^+$.

Together with (5.3) we get the full result of Theorem 3. □

## 5.2 Exponential Moments with Negative Coefficients

In this section, we shall prove the Theorem 4. Notice that by using Poisson integral again, to prove Theorem 4 is equivalent to prove the following Proposition:

**Proposition 11.** *For any bounded, compactly supported shape function $K(\cdot) \geq 0$, we have*

$$\lim_{t\to\infty} \frac{1}{(\log t)^{d/2}} \log \mathbb{E}_x \exp\left\{-\lambda \int_{\mathbb{R}^d} \left(1 - \exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right) \mathrm{d}y\right\} = -\lambda\,\omega_d,$$

*where $\omega_d$ denotes the volume of $d-$dimensional unit ball and $\lambda > 0$ is the intensity of the Poisson point process $\omega(\cdot)$, the same notations as we described before in Theorem 4.*

### 5.2.1 Lower Bound

For any given $t$, denote $R_{\beta,t} = \sqrt{\beta \log t}$ with $\beta > 1$. By restricting $X(\cdot)$ in the ball $B(0, R_{\beta,t})$ up to time $t$, we have

$$\begin{aligned}
&\mathbb{E}_x \exp\left\{-\lambda \int_{\mathbb{R}^d} \left(1 - \exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right) \mathrm{d}y\right\} \\
\geq &\mathbb{E}_x \exp\left\{\left\{-\lambda \int_{\mathbb{R}^d} \left(1 - \exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right) \mathrm{d}y\right\} \mathbf{1}_{\left\{\sup\limits_{0\leq s\leq t} |X(s)| < R_{\beta,t}\right\}}\right\} \\
= &\mathbb{E}_x \exp\left\{\left\{-\lambda \int_{B(0,R_{\beta,t}+L)} \left(1 - \exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right) \mathrm{d}y\right\} \mathbf{1}_{\left\{\sup\limits_{0\leq s\leq t} |X(s)| < R_{\beta,t}\right\}}\right\},
\end{aligned}$$

where the last equality holds simply due to the fact that the support of $K(\cdot)$ is inside the ball $B(0, L)$ hence the function inside the spacial integral vanishes outside $B(0, R_{\beta,t} + L)$. Using the simple fact that $1 - e^x < 1$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{E}_x \exp\left\{-\lambda \int_{\mathbb{R}^d} \left(1 - \exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right) \mathrm{d}y\right\} \\
\geq &e^{-\lambda\,\omega_d (R_{\beta,t}+L)^d} \mathbb{P}_x \left(\sup_{0\leq s\leq t} |X(s)| < R_{\beta,t}\right).
\end{aligned}$$

Applying the Lemma 5.0.4 below and noticing that $\beta > 1$, we thus obtain

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{(\log t)^{d/2}} \log \mathbb{E}_x \exp\left\{-\lambda \int_{\mathbb{R}^d} \left(1 - \exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right) \mathrm{d}y\right\}\\
\geq &\liminf_{t\to\infty} \frac{1}{(\log t)^{d/2}} \left(-\lambda\,\omega_d(\sqrt{\beta\log t}+L)^d - ct^{-\frac{\beta-1}{2}}(\log t)^{\frac{d-1}{2}}\right)\\
= &-\beta^{d/2}\,\lambda\,\omega_d.
\end{aligned}$$

Therefore, we get the lower bound of Proposition 11 by letting $\beta$ go to $1^+$.

Now we turn to prove the technical Lemma used early in the proof of the lower bound. This Lemma tells us that the probability of restricting O-U process $X$ up to time $t$ is close to 1 if we select the radius of the ball carefully.

**Lemma 5.0.4.** *Take $R_{\beta,t} = \sqrt{\beta\log t}$ $(\beta > 1)$, then for all $t$ large enough,*

$$\mathbb{P}_x\left(\sup_{0\leq s\leq t}|X(s)| < R_{\beta,t}\right) \overset{\log}{\succ\succ} \exp\left\{-t^{-\frac{\beta-1}{2}}(\log t)^{\frac{d-1}{2}}\right\}.^{*} \tag{5.7}$$

*Proof.* Let $\gamma(t) \nearrow \infty$ be an increasing function of $t$ of which growth speed is slow enough. For instance, choose $\gamma_t \prec\prec \log t$. Observe that:

$$\begin{aligned}
&\mathbb{P}_x\left(\sup_{\gamma(t)\leq s\leq t}|X(s)| \leq R_{\beta,t}\right)\\
=&\mathbb{P}_x\left(\sup_{0\leq s<\gamma(t)}|X(s)| > R_{\beta,t} \text{ and } \sup_{\gamma(t)\leq s\leq t}|X(s)| \leq R_{\beta,t}\right) + \mathbb{P}_x\left(\sup_{0\leq s\leq t}|X(s)| \leq R_{\beta,t}\right).
\end{aligned} \tag{5.8}$$

Therefore, in order to prove (5.7), it suffices to check

$$\mathbb{P}_x\left(\sup_{0\leq s<\gamma(t)}|X(s)| > R_{\beta,t}\right) \overset{\log}{\prec\prec} \exp\left\{-t^{-\frac{\beta-1}{2}}(\log t)^{\frac{d-1}{2}}\right\} \tag{5.9}$$

and

$$\mathbb{P}_x\left(\sup_{\gamma(t)\leq s\leq t}|X(s)| \leq R_{\beta,t}\right) \overset{\log}{\succ\succ} \exp\left\{-t^{-\frac{\beta-1}{2}}(\log t)^{\frac{d-1}{2}}\right\}. \tag{5.10}$$

---

* $f(t) \overset{\log}{\succ\succ} g(t)$ means $\log f(t)/\log g(t) \to +\infty$ as $t \to +\infty$.

Indeed, by Lemma A.0.9 in Section A.5, the following inequality holds for all $t$ large enough

$$\mathbb{P}_x\left(\sup_{0\le s<\gamma(t)}|X(s)|>R_{\beta,t}\right)\le c_1\frac{\gamma(t)}{t^{c_2}},\quad \text{where } c_1,c_2>0. \tag{5.11}$$

Hence, since $\gamma(t)\prec\prec\log t$, (5.9) holds. Next, we show (5.10). For any $g\in\mathcal{C}_0^\infty(B(0,R_{\beta,t}))$ with $||g||_{2,\mu}=1$, we have

$$\begin{aligned}&\mathbb{P}_x\Big(\sup_{\gamma(t)\le s\le t}|X(s)|<R_{\beta,t}\Big)=\mathbb{E}_x\left\{\mathbf{1}_{\left\{\sup_{\gamma(t)\le s\le t}|X(s)|<R_{\beta,t}\right\}}\right\}\\ \ge&||g||_\infty^{-2}\mathbb{E}_x\left\{g(X(\gamma(t)))g(X(t))\mathbf{1}_{\left\{\sup_{\gamma(t)\le s\le t}|X(s)|<R_{\beta,t}\right\}}\right\}.\end{aligned}\tag{5.12}$$

Using Markov property of $X$,

$$\begin{aligned}&\mathbb{E}_x\left\{g(X(\gamma(t)))g(X(t))\mathbf{1}_{\left\{\sup_{\gamma(t)\le s\le t}|X(s)|<R_{\beta,t}\right\}}\right\}\\ &=\mathbb{E}_x\left\{g(X(\gamma(t)))\mathbb{E}_{X(\gamma(t))}\left\{g(X(t-\gamma(t)))\mathbf{1}_{\left\{\sup_{0\le s\le t-\gamma(t)}|X(s)|<R_{\beta,t}\right\}}\right\}\right\}.\end{aligned}\tag{5.13}$$

Hence,

$$\mathbb{P}_x\Big(\sup_{\gamma(t)\le s\le t}|X(s)|<R_{\beta,t}\Big)\ge||g||_\infty^{-2}\int_{B(0,R_{\beta,t})}p(x,y,\gamma(t))g(y)T^{0,R_{\beta,t}}_{t-\gamma(t)}g(y)\,\mathrm{d}y,\tag{5.14}$$

where $p(x,y,t)$ is the probability density of $X$ staring from $x$ and ending at $y$ at time $t$. The semigroup $T_t^{0,R}$ on $\mathcal{L}^2(B(0,R),\mu)$ is defined as in Chapter 3:

$$T_t^{0,R}g(x)\overset{\text{def.}}{=}\mathbb{E}_x\left\{g(X(t))\mathbf{1}_{\left\{\sup_{0\le s\le t}|X(s)|\le R\right\}}\right\}.$$

Notice from (2.5) and the fact that $\gamma(t)\prec\prec\log t$, we know $p(x,y,\gamma(t))\phi(y)^{-1}$ is uniformly bounded below on $B(0,R_{\beta,t})$ for large $t$. That is,

$$\liminf_{t\to\infty}\inf_{y\in B(0,R_{\beta,t})}p(x,y,\gamma(t))\phi(y)^{-1}>C.$$

Hence, combining with (5.14), $\mathbb{P}_x\left(\sup_{\gamma(t)\le s\le t}|X(s)|<R_{\beta,t}\right)$ has the following Dirichlet form lower bound:

$$\begin{aligned}\mathbb{P}_x\left(\sup_{\gamma(t)\le s\le t}|X(s)|<R_{\beta,t}\right)&\ge C||g||_\infty^{-2}\int_{B(0,R_{\beta,t})}g(y)T^{R_{\beta,t}}_{t-\gamma t}g(y)\phi(y)\,\mathrm{d}y\\&=C||g||_\infty^{-2}\langle g,T^{0,R_{\beta,t}}_{t-\gamma(t)}g\rangle_\mu.\end{aligned}\tag{5.15}$$

By using spectral representation for $T_t^{0,R_{\beta,t}}$ in (5.14), we have

$$\begin{aligned}&\langle g,T^{R_{\beta,t}}_{t-\gamma(t)}g\rangle_\mu=\int_{-\infty}^{\infty}e^{(t-\gamma(t))\lambda}\,m_g^{0,R_{\beta,t}}(\mathrm{d}\lambda)\ge\exp\left\{(t-\gamma(t))\int_{-\infty}^{\infty}\lambda\,m_g^{0,R_{\beta,t}}(\mathrm{d}\lambda)\right\}\\&=\exp\left\{(t-\gamma(t))L^{0,R_{\beta,t}}g\right\}=\exp\left\{-\frac{t-\gamma(t)}{2}\int_{B(0,R_{\beta,t})}|\nabla g(y)|^2\,\phi(y)\,\mathrm{d}y\right\},\end{aligned}\tag{5.16}$$

where the inequality holds due to Jensen's inequality.

Choose $h_t$: $\mathbb{R}\to[0,1]$ as a smooth function such that $h_t(x)\equiv1$ for $|x|<R_{\beta,t}-2$, $h_t(x)\equiv0$ for $|x|>R_{\beta,t}$, and $|h_t'|<1$ for all $x\in\mathbb{R}$. Define $g_t:\mathbb{R}^d\to\mathbb{R}$ as $g_t(x)=c_th_t(|x|)$, where $c_t>0$ is the normalizing constant such that $||g_t||_{2,\mu}=1$. Use $g_t$ in (5.15), (5.16) and notice $c_t=||g_t||_\infty$, we have

$$\mathbb{P}_x\left(\sup_{\gamma(t)\le s\le t}|X(s)|<R_{\beta,t}\right)\ge C\exp\left\{-\frac{t-\gamma(t)}{2}\int_{B(0,R_{\beta,t})}h_t'(|y|)^2\,\phi(y)\,\mathrm{d}y\right\}.\tag{5.17}$$

To achieve the desired lower bound, we need to estimate $\int_{B(0,R_{\beta,t})}h_t'(|y|)^2\,\phi(y)\,\mathrm{d}y$. In fact, using the sphere integral, we have for $t$ sufficiently large

$$\int_{B(0,R_{\beta,t})}h_t(|y|)^2\phi(y)\,\mathrm{d}y\le c_1\int_{\sqrt{\beta\log t}-2}^{\sqrt{\beta\log t}}r^{d-1}e^{-r^2}\,\mathrm{d}r\le c_2(\log t)^{\frac{d-1}{2}}e^{-\frac{\beta+1}{2}\log t}.$$

Therefore, we have

$$\mathbb{P}\left\{\sup_{\gamma(t)\le s\le t}|X(s)|\le R_{\beta,t}\right\}\overset{\log}{\succ\succ}\exp\left\{-t^{-\frac{\beta-1}{2}}(\log t)^{\frac{d-1}{2}}\right\}.\tag{5.18}$$

From (5.8), (5.11) and (5.18), we have

$$\mathbb{P}_x\left(\sup_{0\le s\le t}|X(s)|<R_{\beta,t}\right)\overset{\log}{\succeq}\exp\left\{-t^{-\frac{\beta-1}{2}}(\log t)^{\frac{d-1}{2}}\right\}.$$

□

### 5.2.2 Upper Bound

Now we turn to the upper bound. Take $R_{\beta,t}=\sqrt{\beta\log t}$ with $\beta<1$. Since $K(\cdot)\ge 0$, we have

$$\begin{aligned}
&\mathbb{E}_x\exp\left\{-\lambda\int_{\mathbb{R}^d}\left(1-\exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right)\mathrm{d}y\right\}\\
\le&\mathbb{E}_x\exp\left\{-\lambda\int_{B(0,R_{\beta,t})}\left(1-\exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right)\mathrm{d}y\right\}\\
=&I_1(t,\beta,\delta)+I_2(t,\beta,\delta),
\end{aligned}$$

where

$$I_1(t,\beta,\delta)=\mathbb{E}_x\left(\exp\left\{-\lambda\int_{B(0,R_{\beta,t})}\left(1-\exp\left\{\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right)\mathrm{d}y\right\}\mathbf{1}_{B_{\beta,\delta}}\right),$$

$$I_2(t,\beta,\delta)=\mathbb{E}_x\left(\exp\left\{-\lambda\int_{B(0,R_{\beta,t})}\left(1-\exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\right)\mathrm{d}y\right\}\mathbf{1}_{B^c_{\beta,\delta}}\right),$$

and $B_{\beta,\delta}$ is defined as

$$B_{\beta,\delta}=\left\{\frac{1}{(\log t)^{d/2}}\int_{B(0,R_{\beta,t})}\exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\mathrm{d}y\le\delta\right\}.$$

In the following, we will show that $I_1(t,\beta,\delta)$ makes the main contribution to the upper bound while $I_2(t,\beta,\delta)$ is negligible comparing with $I_1(t,\beta,\delta)$. Indeed, notice that

$$I_1(t,\beta,\delta) \le \exp\left\{-\lambda\,|B(0,R_{\beta,t})| + \lambda\,\delta(\log t)^{\frac{d}{2}}\right\} = \exp\left\{-\lambda\,\omega_d(\beta\log t)^{\frac{d}{2}} + \lambda\,\delta(\log t)^{\frac{d}{2}}\right\}.$$

Hence,

$$\lim_{t\to\infty}\frac{1}{(\log t)^{d/2}}\log I_1(t,\beta,\delta) \le -\lambda\,\omega_d\beta^{d/2} + \lambda\,\delta. \tag{5.19}$$

Next we prove the smallness of $I_2(t,\beta,\delta)$. By Chebyshev inequality,

$$I_2(t,\beta,\delta) \le \mathbb{P}_x(B^c_{\beta,\delta}) \le \frac{1}{\delta(\log t)^{d/2}}\int_{B(0,R_{\beta,t})}\mathbb{E}_x\exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\mathrm{d}y. \tag{5.20}$$

Recall $K \ge 0$, then the expectation on the right side of (5.20) has the following upper bound estimate

$$\begin{aligned}
&\mathbb{E}_x\exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\\
\le&\mathbb{E}_x\exp\left\{-\int_1^t K(X(s)-y)\,\mathrm{d}s\right\}\\
=&\int_{\mathbb{R}^d} p(x,y,1)\mathbb{E}_y\exp\left\{-\int_0^{t-1} K(X(s)-y)\,\mathrm{d}s\right\}\mathrm{d}y\\
=&\int_{\mathbb{R}^d}\frac{p(x,y,1)}{\phi(y)}\mathbb{E}_y\exp\left\{-\int_0^{t-1} K(X(s)-y)\,\mathrm{d}s\right\}\phi(y)\,\mathrm{d}y
\end{aligned}$$

From (2.5),

$$p(x,y,1)\phi^{-1}(y) = c\exp\left\{|x|^2 - \frac{|x-ye^{-1}|^2}{1-e^{-2}}\right\} \le ce^{|x|^2},$$

for some $c>0$. This implies that $p(x,y,1)\phi^{-1}(y)$, as a function of $y$, is uniformly upper bounded. Therefore,

$$\mathbb{E}_x\exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\} \le c_1\langle 1, T^K_{t-1}1\rangle_\mu = c_1\int_{-\infty}^{\infty} e^{(t-1)\lambda}m_1(\mathrm{d}\lambda), \tag{5.21}$$

where spectral measure $m_1$ is a probability measure, see again in Section A.6 for details. From (3.14) and (3.15), The support of $m_1$ is bounded from above by

$$\sup_{\substack{g\in\mathcal{D}(L^K)\\ ||g||_{2,\mu}=1}} \langle g, L^K g\rangle_\mu = -\inf_{g\in\mathcal{F}_\infty}\left\{\int_{\mathbb{R}^d}\left(K(x-y)g^2(x)+\frac{1}{2}|\nabla g(x)|^2\right)\phi(x)\,\mathrm{d}x\right\}. \quad (5.22)$$

Summarizing (5.21) and (5.22), we get

$$\begin{aligned}&\mathbb{E}_x\exp\left\{-\int_0^t K(X(s)-y)\,\mathrm{d}s\right\}\\ &\le c\exp\left\{-(t-1)\inf_{g\in\mathcal{F}_\infty}\left\{\int_{\mathbb{R}^d}\left(K(x-y)g^2(x)+\frac{1}{2}|\nabla g|^2\right)\phi(x)\,\mathrm{d}x\right\}\right\},\end{aligned}$$

for some $c>0$. Define $\Lambda(\beta,t)$ as

$$\Lambda(\beta,t)=\inf_{g\in\mathcal{F}_\infty}\inf_{y\in B(0,R_{\beta,t})}\left\{\int_{\mathbb{R}^d}\left(K(x-y)g^2(x)+\frac{1}{2}|\nabla g|^2\right)\phi(x)\,\mathrm{d}x\right\}. \quad (5.23)$$

Then from (5.20), we observe that

$$I_2(t,\beta,\delta)\le C\exp\big\{-(t-1)\Lambda(\beta,t)\big\},$$

for some $C>0$. According to the Lemma 5.0.5 below and the fact that $\beta<1$, we have

$$\lim_{t\to\infty}\frac{1}{(\log t)^{d/2}}\log I_2(t,\beta,\delta)=-\infty. \quad (5.24)$$

Therefore, combine (5.19), (5.24) and let $\beta\to 1^-$, $\delta\to 0^+$, we obtain the upper bound. □

**Lemma 5.0.5.** *For $\Lambda(\beta,t)$ as defined in (5.23) and $0<\beta<1$, the following inequality holds for $t$ large enough*

$$\Lambda(\beta,t)\ge ct^{-\beta}\quad \textit{for some } c>0.$$

*Proof.* Given $t > 0$, denote

$$\mathcal{F}_1(t) = \left\{ g \in \mathcal{F}_\infty : \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 \phi(x) \, \mathrm{d}x > \lambda t^{-\beta} \right\}$$

and

$$\mathcal{F}_2(t) = \left\{ g \in \mathcal{F}_\infty : \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 \phi(x) \, \mathrm{d}x \le \lambda t^{-\beta} \right\},$$

where $\lambda > 0$ is a constant, of which value will be determined later. We have the following inequality

$$\Lambda(\beta, t) \ge \min \left\{ \inf_{g \in \mathcal{F}_2(t)} \inf_{y \in B(0, R_{\beta,t})} \int_{\mathbb{R}^d} K(x-y) g^2(x) \phi(x) \, \mathrm{d}x, \lambda t^{-\beta} \right\}. \tag{5.25}$$

For any $g \in \mathcal{F}_2(t)$, let $\bar{g} = \int_{\mathbb{R}^d} g(x)\phi(x) \, \mathrm{d}x$ be the expectation of $g$ with respect to the Normal distribution $\mu$. By triangular inequality,

$$\begin{aligned} &\left( \int_{\mathbb{R}^d} K(x-y) g^2(x) \phi(x) \, \mathrm{d}x \right)^{1/2} \\ \ge &\left( \int_{\mathbb{R}^d} K(x-y) \bar{g}^2 \phi(x) \, \mathrm{d}x \right)^{1/2} - \left( \int_{\mathbb{R}^d} K(x-y) (g(x) - \bar{g})^2 \phi(x) \, \mathrm{d}x \right)^{1/2}. \end{aligned}$$

Apply the Poincaré inequality for Normal distribution to $g$ (See Section A.2 in the Appendix), we get

$$1 - \bar{g}^2 = \int_{\mathbb{R}^d} \left( g(x) - \bar{g} \right)^2 \phi(x) \, \mathrm{d}x \le C \int_{\mathbb{R}^d} |\nabla g|^2 \phi(x) \, \mathrm{d}x < 2C\lambda t^{-\beta},$$

which leads to

$$\sqrt{1 - 2C\lambda t^{-\beta}} \le \bar{g} \le 1, \tag{5.26}$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^d} K(x-y)\left(g(x)-\bar{g}\right)^2 \phi(x)\,\mathrm{d}x \\
\leq &\max_{x\in\mathbb{R}^d} K(x) \int_{\mathbb{R}^d} \left(g(x)-\bar{g}\right)^2 \phi(x)\,\mathrm{d}x \\
\leq &2\max_{x\in\mathbb{R}^d} K(x) C\lambda t^{-\beta}.
\end{aligned}
\tag{5.27}
$$

From (5.26) and (5.27), we have

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^d} K(x-y) g^2(x)\phi(x)\,\mathrm{d}x\right)^{1/2} \\
\geq &\sqrt{1-2C\lambda t^{-\beta}} \left(\int_{\mathbb{R}^d} K(x-y)\phi(x)\,\mathrm{d}x\right)^{1/2} - \left(2\max_{x\in\mathbb{R}^d} K(x) C\lambda t^{-\beta}\right)^{1/2}.
\end{aligned}
\tag{5.28}
$$

Notice that for large $t$

$$
\begin{aligned}
&\inf_{y\in B(0,R_{\beta,t})} \int_{\mathbb{R}^d} K(x-y)\phi(x)\,\mathrm{d}x \\
&\geq c_1 \inf_{y\in B(0,R_{\beta,t})} \int_{B(0,\delta)} \phi(x-y)\,\mathrm{d}x \\
&\geq c_2 \exp\left\{-R_{\beta,t}^2\right\} \\
&= c_2 t^{-\beta},
\end{aligned}
$$

for for some suitable constants $c_1, c_2 > 0$. Choose $\lambda > 0$ sufficiently small but fixed, we obtain from (5.28) that

$$
\inf_{g\in\mathcal{F}_2(t)} \inf_{y\in B(0,R_{\beta,t})} \int_{\mathbb{R}^d} K(x-y) g^2(x)\phi(x)\,\mathrm{d}x \geq c_3 t^{-\beta}.
$$

Therefore, combine with (5.25) we have

$$
\Lambda(\beta,t) \geq c_4 t^{-\beta} \qquad \text{for some } c_4 > 0.
$$

$\square$

# Chapter 6

# Future research

Some of my plans for future research include the followings:

- There are a number of applications rising from other science fields that address the importance of the *time dependent* random media, such as moving catalysts or traps in chemistry reaction, chiral medium in electromagnetic fields. Thus, I am going to investigate the behavior of O-U processes in certain time dependent random media. For the case of BM, it has been studied in [13, 14]. However, the methodology will most likely be different for OU processes due to its friction effect, as one have seen in my dissertation.

- The macroscopic systems of OU dynamics have ubiquitous applications in physics, chemistry, biology and engineering. Unlike the single OU particle, the output processes of large composite OU systems appear a long time memory (non-Markov) behavior with a universal scaling limit: fractional Brownian Motion(fBM)[12]. Motivated by this, I am going to investigate the model of fBM in random media, which in return answers the question on long time macroscopic behavior of OU dynamics in random media. One possible approach is to rewrite the fBM by a integral of BM and then to apply Gaussian techniques.

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# Appendix

# Appendix A

# Appendix

## A.1 Poisson Integrals

Recall $\{\omega(\mathrm{d}x)\}_{x\in\mathbb{R}^d}$ is a Poisson random measure (also known as Poisson point process) with intensity measure $\nu(\mathrm{d}x) = \lambda\,\mathrm{d}x$, $\lambda > 0$. Integrals of the form

$$\mathbf{E}\exp\left\{\int_{\mathbb{R}^d} g(x)\,\omega(\mathrm{d}x)\right\},$$

for some measurable functions $g$, if well-defined, are essentially the moment generating functions of stochastic integrals over Poisson random measure $\omega(\mathrm{d}x)$. The definitions of stochastic integrals over infinitely divisible random measures and their characteristic functions as well as moment generating functions have been discussed thoroughly in [24]. Hence, as an example of infinitely divisible random measure, we have the following Poisson integral characterization: (see also [6] for Poisson random measure case)

**Proposition 12.** *Let $\{\omega(A)\}_{A\in\mathbb{R}^d}$ be a Poisson random measure with intensity measure $\nu(\mathrm{d}x) = \lambda\,\mathrm{d}x$ $(\lambda > 0)$. A Borel measurable function $g(x)$ is integrable* *on $\mathbb{R}^d$ with respect to $\omega(\mathrm{d}x)$ if and only if*

$$\int_{\mathbb{R}^d} 1 - e^{-|g(x)|}\,\mathrm{d}x < \infty.$$

*Furthermore, if $g$ is integrable with respect to $\omega$, the following Poisson integral formula holds*

$$\mathbf{E}\exp\left\{\int_{\mathbb{R}^d} g(x)\,\omega(\mathrm{d}x)\right\} = \exp\left\{\lambda\int_{\mathbb{R}^d}\left(e^{g(x)}-1\right)\,\mathrm{d}x\right\}.$$

## A.2 Poincaré Inequality for Normal Distribution

The Poincaré inequality allows one to obtain bounds on a function using bounds on its derivatives and the geometry of its domain of definition. The following Poincaré inequality for Normal distribution $\mu(\cdot)$ can be found in [1], therein Theorem 1.6.4.

**Lemma A.0.6.** *Let $f \in W^{1,2}(\mu)$. It holds that for some $C > 0$,*

$$\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu - \left(\int_{\mathbb{R}^d} f\,\mathrm{d}\mu\right)^2 \le C\int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu.$$

## A.3 Basic Properties of the Poisson potential

In this section, we first give a large deviation type upper tail estimate of Poisson distributed random variables:

**Lemma A.0.7.** *Let $Y$ be a Poisson distributed random variable with parameter $\lambda > 0$. For any $\sigma > 0$, the following large deviation result holds*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbf{P}\left(Y \ge \sigma\frac{t}{\log t}\right) = -\sigma.$$

*Hence, we have $\mathbf{P}\left(Y \ge \sigma\frac{\log t}{\log\log t}\right) = t^{-\sigma(1+o(1))}$ as $t$ goes to $\infty$.*

*Proof.* By Stirling formula, we have

$$\mathbf{P}(Y = k) = \frac{1}{\sqrt{2\pi k}} \left(\frac{\lambda e}{k}\right)^k e^{-\lambda}(1 + o(1)). \tag{A.1}$$

Notice that

$$\mathbf{P}(Y = k) < \mathbf{P}(Y \geq k) = \frac{\lambda^k}{k!} e^{-\lambda} \sum_{j=0}^{\infty} \frac{k!}{(j+k)!} \lambda^j \leq e^{\lambda} \mathbf{P}(Y = k). \tag{A.2}$$

Take $k = \left[\sigma \frac{t}{\log t}\right]$, by (A.1) and (A.2), we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{P}\left(Y \geq \sigma \frac{t}{\log t}\right) = \lim_{t\to\infty} \frac{1}{t} \left(-\sigma \frac{t}{\log t}\right) \log\left(\sigma \frac{t}{\log t}\right) = -\sigma.$$

□

**Remark 8.** *Lemma A.0.7 implies that this tail estimate does not rely on the Poisson parameter* $\lambda$.

Using the above upper tail estimate and a standard Borel-Cantelli argument, we have the following Lemma.

**Lemma A.0.8.** *With probability one,*

$$\lim_{R\to\infty} \frac{\log\log R}{\log R} \max_{|x|<R} V(x) = d \max_{x\in\mathbb{R}^d} K(x).$$

*Proof.* First, we prove the upper bound. Notice that $B(0, R)$ can be covered by $cR^d$ evenly spaced unit ball, i.e.

$$B(0, R) \subset \cup_{x\in\Lambda_R} B(x, 1) \quad \text{and } |\Lambda_R| \sim cR^d,$$

where $\Lambda_R$ is the collection of centers of the unit covers. Hence,

$$\max_{|x|<R} V(x) \leq \max_{z\in\Lambda_R} \max_{x\in B(0,1)} V(x + z). \tag{A.3}$$

Since the support of $K(\cdot)$ is contained in $B(0,L)$, we have

$$\begin{aligned}\sup_{x\in B(0,1)} V(x+z) &= \sup_{x\in B(0,1)} \int_{\mathbb{R}^d} K(x+z-y)\,\omega(\mathrm{d}y) \\ &\le \max_{x\mathbb{R}^d} K(x)\omega\left(B\left(z,L+1\right)\right)\end{aligned} \tag{A.4}$$

By lemma A.0.7 and the fact that $\{\omega(B(z,L+1)\}_{z\in\Lambda_R}$ are identically distributed,

$$\begin{aligned}&\mathbf{P}\left(\max_{z\in\Lambda_R} \omega\left(B\left(z,L+1\right)\right) \ge \sigma \frac{\log R}{\log\log R}\right) \\ &\le \sum_{z\in\Lambda_R} \mathbf{P}\left(\omega\left(B\left(0,L+1\right)\right) \ge \sigma \frac{\log R}{\log\log R}\right) \le cR^{d-\sigma}.\end{aligned} \tag{A.5}$$

Choose $r_n = 2^n$ and $\sigma = d+\epsilon$. From (A.5), the following infinite series converge:

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\max_{z\in\Lambda_{r_n}} \omega\left(B\left(z,L+1\right)\right) \ge (d+\epsilon)\frac{\log r_n}{\log\log r_n}\right) < \infty.$$

Borel-Cantelli lemma tells us

$$\mathbf{P}\left(\left\{\frac{\log\log r_n}{\log r_n} \max_{z\in\Lambda_{r_n}} \omega\left(B\left(z,L+1\right)\right) \ge d+\epsilon\right\} \quad i.o.\right) = 0,$$

which implies that

$$\limsup_{n\to\infty} \frac{\log\log r_n}{\log r_n} \max_{z\in\Lambda_{r_n}} \omega(B(z,r_2)) \le d \quad a.s.$$

For arbitrary $R \in \mathbb{R}^+$, there exists an $n \in \mathbb{N}$ such that $r_n \le R < r_{n+1}$. Therefore, we have

$$\begin{aligned}&\limsup_{R\to\infty} \frac{\log\log R}{\log R} \max_{z\in\Lambda_R} \omega(B(z,L+1)) \\ \le &\limsup_{n\to\infty} \frac{\log\log r_{n+1}}{\log r_n} \max_{z\in\Lambda_{r_{n+1}}} \omega(B(z,L+1)) \\ = &\limsup_{n\to\infty} \frac{\log\log r_{n+1}}{\log r_{n+1}} \max_{z\in\Lambda_{r_{n+1}}} \omega(B(z,L+1)) = d.\end{aligned}$$

Combine this with (A.3) and (A.4), we obtain the upper bound.

Next, we prove the lower bound. For any $\epsilon > 0$, by continuity of $K$, we can find a $\delta > 0$ such that $K(x) > \max_{x\in\mathbb{R}^d} K(x) - \epsilon$ for $x \in B(x_0, \delta)$ and those balls $\{B(x_0 + x, \delta) : x \in \mathbb{Z}^d\}$ are mutually disjoint. Hence, the elements in $\{\omega(B(x_0 + x, \delta))\}_{x\in\mathbb{Z}^d}$ are independent, identically distributed. From lemma A.0.7,

$$\mathbf{P}\left(\max_{x\in\mathbb{Z}^d\cap B(0,n)} \omega\left(B\left(x_0 + x, \delta\right)\right) < \sigma\frac{\log n}{\log\log n}\right) = \mathbf{P}\left(\omega\left(B\left(0, \delta\right)\right) < \sigma\frac{\log n}{\log\log n}\right)^{cn^d}$$
$$\sim \left(1 - n^{-\sigma}\right)^{cn^d} \sim \exp\left\{-cn^{d-\sigma}\right\}.$$

Take $\sigma = d - \epsilon$. We have

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\max_{x\in\mathbb{Z}^d\cap B(0,n)} \omega\left(B\left(x_0 + x, \delta\right)\right) < \sigma\frac{\log n}{\log\log n}\right) < \infty.$$

Using Borel-Cantelli lemma again,

$$\liminf_{n\to\infty} \frac{\log\log n}{\log n} \max_{x\in\mathbb{Z}^d\cap B(0,n)} \omega(B(x_0 + x, \delta)) \geq d - \epsilon \quad \mathbf{P} - a.s. \tag{A.6}$$

Notice that

$$\max_{x\in B(0,R)} V(x) \geq \max_{x\in\mathbb{Z}^d\cap B(0,R)} V(x) \geq \left(\max_{x\in\mathbb{R}^d} K(x) - \epsilon\right) \max_{x\in\mathbb{Z}^d\cap B(0,n)} \omega(B(x_0 + x, \delta)).$$

Combine with (A.6) and let $\epsilon$ go to 0, then we obtain the lower bound. □

## A.4 $W^{1,2}(\mathbb{R}^d, \mu)$

In this part, we define the weak derivative under measure $\mu$ and then prove several basic properties of $W^{1,2}(\mathbb{R}^d, \mu)$.

**Definition 2.** *Suppose $u, v \in \mathcal{L}^2(\mathbb{R}^d, \mu)$ where $\mu(\mathrm{d}x) = \phi(x)\,\mathrm{d}x$. We say that $v$ is the weak partial derivative of $u$ with respect to variable $x_i$, $1 \leq i \leq d$, written as*

$$\frac{\partial}{\partial x_i} u = v,$$

*provided*

$$\int_{\mathbb{R}^d} v(x)w(x)\phi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^d} u(x)\frac{\partial w}{\partial x_i}\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} u(x)w(x)\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x$$

*for all test function $w \in C_0^\infty(\mathbb{R}^d)$.*

It is a standard argument to show that a weak derivative, if it exists, is uniquely defined to a set of measure zero.

**Proposition 13.** *The Sobolev space $W^{1,2}(\mathbb{R}^d, \mu)$ is a Hilbert space.*

*Proof.* Assume $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in $W^{1,2}(\mathbb{R}^d, \mu)$. Then $\{u_n\}_{n=1}^\infty$ and $\left\{\frac{\partial}{\partial x_k} u_n\right\}_{n=1}^\infty$ $(1 \leq k \leq d)$ are all Cauchy sequences in $\mathcal{L}^2(\mathbb{R}^d, \mu)$. Since $\mathcal{L}^2(\mathbb{R}^d, \mu)$ is complete, there exists functions $\hat{u}, \hat{u}_1, \cdots, \hat{u}_d \in \mathcal{L}^2(\mathbb{R}^d, \mu)$ such that

$$\frac{\partial}{\partial x_k} u_n \to \hat{u}_k, \text{ for } 1 \leq k \leq d \text{ and } u_n \to \hat{u} \quad \text{in } \mathcal{L}^2(\mathbb{R}^d, \mu).$$

We now claim that

$$\frac{\partial}{\partial x_k} \hat{u} = \hat{u}_k \quad \text{for } 1 \leq k \leq d. \quad \text{Hence, } \hat{u} \in \widehat{W}^{1,2}(\mathbb{R}^d, \mu), . \tag{A.7}$$

To verify this assertion, fix $w \in C_c^\infty(\mathbb{R}^d)$. Then for $1 \leq k \leq d$

$$\begin{aligned}
&\int_{\mathbb{R}^d} \hat{u} \frac{\partial}{\partial x_k}(w\phi)\,\mathrm{d}x \\
=&\pi^{-d/2}\left(\int_{\mathbb{R}^d} \hat{u}(x)\frac{\partial}{\partial x_k} w(x) e^{-|x|^2}\,\mathrm{d}x - 2\int_{\mathbb{R}^d} \hat{u}(x) w(x) x_k e^{-|x|^2}\,\mathrm{d}x\right) \\
=&\pi^{-d/2}\lim_{n\to\infty}\left(\int_{\mathbb{R}^d} u_n(x)\frac{\partial}{\partial x_k} w(x) e^{-|x|^2}\,\mathrm{d}x - 2\int_{\mathbb{R}^d} u_n(x) w(x) x_k e^{-|x|^2}\,\mathrm{d}x\right) \\
=&\pi^{-d/2}\lim_{n\to\infty}\int_{\mathbb{R}^d} \frac{\partial}{\partial x_k} u_n(x) w(x) e^{-|x|^2}\,\mathrm{d}x \\
=&\int_{\mathbb{R}^d} \hat{u}_k(x) w(x)\phi(x)\,\mathrm{d}x.
\end{aligned}$$

Thus (A.7) is valid. Since therefore $\frac{\partial}{\partial x_k} u_n \to \frac{\partial}{\partial x_k}\hat{u}$ in $\mathcal{L}^2(\mathbb{R}^d, \mu)$ for all $1 \leq k \leq d$, we see that $u_n \to \hat{u}$ in $W^{1,2}(\mathbb{R}^d, \mu)$, as required. □

## A.5 Proof of Lemma 4.0.2

First, we need the following Gaussian tail type upper bound estimate for the proof of Lemma 4.0.2:

**Lemma A.0.9.** *Let $\{X(t)\}_{t\geq 0}$ be the O-U process defined in (2.2). There exists $c > 0$ and $a_x > 0$ such that for all $a > a_x$, the following inequality holds for all $t > 0$:*

$$\mathbb{P}_x\left(\sup_{0\leq s\leq t} |X(s)| > a\right) \leq 2t \exp\left\{-ca^2\right\}.$$

*Proof.* First, let $x = 0$. Since $X(t)$ is an asymptotically stationary Gaussian process, by classical Gaussian tail estimate (for reference, see [20]) there exists $c_1 > 0$ and $a_0 > 0$ such that

$$\max\left\{\mathbb{P}_0\left(|X(k)| > a\right), \mathbb{P}_0\left(\sup_{0\leq s\leq 1} |X(s)| > a\right)\right\} \leq \exp\left\{-c_1 a^2\right\},$$

for $a > a_0$ and all $k \in \mathbb{N}$.

For $a > a_0$, by Markov property and (2.4), we have

$$\begin{aligned}
&\mathbb{P}_0\left(\sup_{k\le s\le k+1}|X(s)|>a\right) = \int_{-\infty}^{\infty}\mathbb{E}_x\left(\mathbf{1}_{\left\{\sup_{0\le s\le 1}|X(s)|>a\right\}}\right) p(0,x,k)\,\mathrm{d}x \\
\le &\int_{|x|\le\frac{a}{2}}\mathbb{P}_x\left(\sup_{0\le s\le 1}|X(s)|>a\right)p(0,x,k)\,\mathrm{d}x+\mathbb{P}_0\left(|X(k)|>\frac{a}{2}\right) \\
= &\int_{|x|\le\frac{a}{2}}\mathbb{P}_0\left(\sup_{0<s\le 1}|X(s)+xe^{-s}|>a\right)p(0,x,k)\,\mathrm{d}x+\mathbb{P}_0\left(|X(k)|>\frac{a}{2}\right) \\
\le &\mathbb{P}_0\left(\sup_{0\le s\le 1}|X(s)|>\frac{a}{2}\right)+\mathbb{P}_0\left(|X(k)|>\frac{a}{2}\right)\le 2\exp\left\{-\frac{c_1}{4}a^2\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{P}_0\left(\sup_{0\le s\le k+1}|X(s)|>a\right) &\le \mathbb{P}_0\left(\sup_{0\le s\le k}|X(s)|>a\right)+\mathbb{P}_0\left(\sup_{k\le s\le k+1}|X(s)|>a\right) \\
&\le \mathbb{P}_0\left(\sup_{0\le s\le k}|X(s)|>a\right)+2\exp\left\{-\frac{c_1}{4}a^2\right\}.
\end{aligned}$$

Repeating this procedure and let $c := c_1/4$, we have

$$\mathbb{P}_0\left(\sup_{0\le s\le t}|X(s)|>a\right)\le 2t\exp\left\{-ca^2\right\}.$$

For general $x$, notice that

$$\begin{aligned}
&\mathbb{P}_x\left(\sup_{0\le s\le t}|X(s)|>a\right)=\mathbb{P}_0\left(\sup_{0\le s\le t}|xe^{-s}+X(s)|>a\right) \\
&\le \mathbb{P}_0\left(\sup_{0\le s\le t}|X(s)|>a-|x|\right).
\end{aligned}$$

Let $a_x := a_0 + x$, then for all $a > a_x$ we have

$$\mathbb{P}_x\left(\sup_{0\le s\le t}|X(s)|>a\right)\le 2t\exp\left\{-ca^2\right\},$$

□

Using lemma A.0.9, we can prove Lemma 4.0.2:

*Proof of Lemma 4.0.2.* Notice that

$$\begin{aligned}
0 \le \mathbb{E}_x \exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\} - \mathbb{E}_x\left(\exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\}\mathbf{1}_{\left\{\tau_{Q_{\gamma(t)}}>t\right\}}\right) \\
= \sum_{n=1}^{\infty} \mathbb{E}_x\left(\exp\left\{\int_0^t V(X(s))\,\mathrm{d}s\right\}\mathbf{1}\left\{\tau_{Q_{n\gamma(t)}} \le t < \tau_{Q_{(n+1)\gamma(t)}}\right\}\right) \\
\le \sum_{n=1}^{\infty} \exp\left\{t \max_{x\in Q_{(n+1)\gamma(t)}} V(x)\right\}\mathbb{P}_x\left(\tau_{Q_{n\gamma(t)}} \le t\right).
\end{aligned}$$

We know from Lemma A.0.8 that there exists a constant $c_1 > 0$ such that, with probability one,

$$\max_{x\in Q_R} V(x) \le c_1 \log R$$

for all sufficiently large $R$. Moreover, from lemma A.0.9 we have

$$\mathbb{P}_x\left(\tau_{Q_a} \le t\right) = \mathbb{P}_x\left(\sup_{0\le s\le t}|X(s)| > a\right) \le 2t\exp\left\{-ca^2\right\},$$

for $a > a_x$. Therefore, with probability one, for sufficiently large $t$ and all $n$, we have

$$\begin{aligned}
&\exp\left\{t \max_{x\in Q_{(n+1)\gamma(t)}} V(x)\right\}\mathbb{P}_x\left(\tau_{Q_{n\gamma(t)}} \le t\right) \\
\le& 2t\exp\left\{c_1 t\left(\frac{1}{2}\log t + \log\log t + \log(\alpha(n+1))\right) - c\alpha^2 n^2 t(\log t)^2\right\} \\
\le& 2t\exp\left\{-\frac{c}{2}\alpha^2 nt\left(\log t\right)^2\right\} \\
=& 2t\cdot t^{-\frac{c_2}{2}\alpha^2 nt\log t}.
\end{aligned}$$

Notice that the second inequality holds for $t$ sufficiently large and uniformly in $n$.

Therefore,

$$\sum_{n=1}^{\infty} \exp\left\{t \max_{x\in Q_{(n+1)\gamma(t)}} V(x)\right\}\mathbb{P}_x\left(\tau_{Q_{n\gamma(t)}} \le t\right) \le \frac{2t^{-\frac{c_2}{2}\alpha^2 t\log t+1}}{1 - t^{-\frac{c_2}{2}\alpha^2 t\log t}} \quad \text{for large } t. \tag{A.8}$$

Let $t \to \infty$, the left hand side of (A.8) goes to 0, which completes the proof. □

## A.6 Spectral Representation of Self-Adjoint Operators

In this section, we briefly introduce the spectral theory for self-adjoint operators which appears several time in the dissertation. For more details on this topic, see [5] and [30].

In this section, we assume $H$ is a separable real Hilbert space.

**Definition 3.** *A linear operator $A$ is said to be* densely defined *if its domain $\mathcal{D}(A)$ is dense in $H$.*

Let $A$ be a densely defined operator on $H$.

**Definition 4.** *The* adjoint *operator of $A$ is the operator $A^* : y \to y^*$ defined by*

$$\langle Ax, y\rangle = \langle x, y^*\rangle \quad \textit{for } x \in \mathcal{D}(A).$$

**Definition 5.** *$A$ is said to be* symmetric *if*

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \textit{for } x, y \in \mathcal{D}(A).$$

If $A$ is a symmetric operator, then clearly $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $Ay = Ay^*$ for all $y \in \mathcal{D}(A)$.

**Definition 6.** *A symmetric operator $A$ is said to be* self-adjoint *if $\mathcal{D}(A) = \mathcal{D}(A^*)$.*

If $A$ is a bounded operator on $H$, symmetric and self-adjoint are the same. However, for unbounded operators, there is an example that an operator is symmetric but not self-adjoint. Since there are nice and powerful spectral representations for self-adjoint operators, we hope to extend a symmetric operator $A$ to a larger domain such that it becomes self-adjoint. Indeed, for a special class of operators this is possible:

**Definition 7.** *A linear operator $A$ is said to be* upper semi-bounded (or lower semi-bounded), *if*

$$\sup_{x\in\mathcal{D}(A),||x||=1}\langle x, Ax\rangle < \infty \quad (or \inf_{x\in\mathcal{D}(A),||x||=1}\langle x, Ax\rangle > -\infty).$$

**Theorem 14** (Friedrich's Extension Theorem)**.** *A semi-bounded symmetric operator $A$ can be extended into a self-adjoint operator. More precisely, there is a self-adjoint operator $\tilde{A}$ on $H$ such that $\mathcal{D}(A) \subset \mathcal{D}(\tilde{A})$ for every $x \in \mathcal{D}(A)$.*

In the following, we focus on self-adjoint operators and introduce their spectral structures.

**Definition 8.** *A* projection operator *$P$ is a bounded self-adjoint operator on Hilbert space $H$ such that $P^2 = P$.*

It is clear that $||P|| \leq 1$ and $P$ is symmetric, hence $P$ is self-adjoint.

**Definition 9.** *A family $\{E(\lambda) : -\infty < \lambda < \infty\}$ of projection operators on $H$ is called a $resolution of identity$ if*

1. *$E(\lambda) \circ E(\mu) = E(\lambda \wedge \mu)$ for any $-\infty < \lambda, \mu < \infty$;*

2. *$E(-\infty)$ is zero operator and $E(\infty)$ is identity operator. Also, $E(\lambda+) = E(\lambda)$ for all $\lambda \in \mathbb{R}$, where $E(-\infty), E(\infty)$ and $E(\lambda+)$ are the linear operators defined as following:*

$$E(\pm\infty)(x) = \lim_{\lambda\to\pm\infty} E(\lambda)(x), \quad E(\lambda+)(x) = \lim_{\mu\to\lambda^+} E(\mu)(x) \quad \text{for all } x \in H.$$

Notice from Definition 9 that resolution of identity looks like an operator version of the distribution function in probability theory. Indeed, there is a strong bond between these two. For any $x \in H$ with $||x|| = 1$, the function

$$F_x(\lambda) = \langle E(\lambda)(x), x\rangle = ||E(\lambda)x||^2$$

is a probability distribution function on $\mathbb{R}$. We write $\mu_x$ as the probability measure generated by $F_x(\cdot)$ and call $\mu_x$ *spectral measure*.

For any Borel-measurable function $\xi(\lambda)$ on $\mathbb{R}$, the linear operator

$$\int_{-\infty}^{\infty} \xi(\lambda) E(\mathrm{d}\lambda) \tag{A.9}$$

is called *spectral integral* on domain $\mathcal{D}_\xi \subset H$, where

$$\mathcal{D}_\xi = \left\{ x \in H : \int_{-\infty}^{\infty} |\xi(\lambda)|^2 \, \mu_x(\mathrm{d}\lambda) < \infty \right\}.$$

Moreover, we have

$$\left\langle \left( \int_{-\infty}^{\infty} \xi(\lambda) E(\mathrm{d}\lambda) \right)(x), x \right\rangle = \int_{-\infty}^{\infty} \xi(\lambda) \mu_x(\mathrm{d}\lambda), \tag{A.10}$$

$$\left\| \left( \int_{-\infty}^{\infty} \xi(\lambda) E(\mathrm{d}\lambda) \right)(x) \right\|^2 = \int_{-\infty}^{\infty} |\xi(\lambda)|^2 \mu_x(\mathrm{d}\lambda). \tag{A.11}$$

It turns out that the linear operator in (A.9) is self-adjoint. On the other hand, all the self adjoint operator have the spectral integral representation (see [30]):

**Theorem 15** (Spectral Integral Representation)**.** *For any self-adjoint operator $A$, there is a unique resolution of identity $\{E(\lambda) : -\infty < \lambda < \infty\}$ such that*

$$A = \int_{-\infty}^{\infty} \lambda E(\mathrm{d}\lambda), \tag{A.12}$$

*where the domain of $A$ is*

$$\mathcal{D}(A) = \left\{ x \in H : \int_{-\infty}^{\infty} |\lambda|^2 \, \mu_x(\mathrm{d}\lambda) < \infty \right\}.$$

Also, we use the following result in the dissertation.

**Proposition 16.** *Let the $A$ be the self-adjoint operator given in (A.12). For any $x \in H$, the spectral measure $\mu_x$ is supported by the interval $[c_0, c_1]$, where*

$$c_0 = \inf_{x \in \mathcal{D}(A), \|x\|=1} \langle x, Ax \rangle, \text{ and } c_1 = \sup_{x \in \mathcal{D}(A), \|x\|=1} \langle x, Ax \rangle.$$

# Vita

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