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Big Homotopy Theory

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To the Graduate Council:

I am submitting herewith a dissertation written by Keith Gordon Penrod entitled "Big Homotopy Theory." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Big Homotopy Theory

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This dissertation is dedicated to Archimedes, who was my inspiration for becoming a mathematician.

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Abstract

Cannon and Conner [1] developed the theory of "big fundamental groups." This is meant to expand on the notion of fundamental group and is a powerful tool that can be used for distinguishing spaces that are not distinguishable using the fundamental group. Turner [6] proved several classical results, such as covering theory and Seifert-VanKampen for big fundamental groups. The purpose of this paper is to expand on the the theory, to refine the definitions, and to give more examples. Also, in this paper, we define big higher homotopy groups analogous to the way classical higher homotopy groups are defined.

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Chapter 1

Preliminaries

Cannon and Conner, in a three-paper series ([1], [2], and [3]) to investigate the Hawaiian earring and other one-dimensional spaces, developed the notion of "big fundamental group", whose construction is motivated by and similar to that of the (classical) fundamental group. In fact, it coincides with the fundamental group in the case of second-countable spaces. Turner did wrote a thesis [6] expanding on this theory, including covering space theory and Seifert-VanKampen for the big fundamental group. This paper will explore the extension of this theory to higher dimensions. That is, in [1], the big fundamental group, denoted as Π_1, was defined. This paper will define generalizations of these "big" groups and also big groups for higher dimensional homotopy.

The motivation for studying these groups is to expand upon the power of homotopy theory. Classical homotopy theory has been used in many ways to distinguish topological spaces and slight modifications to the theory may help us to have better tools for such study. For example, there are spaces whose fundamental group is trivial but whose big fundamental group is non-trivial. Indeed, there are many spaces which are not path connected, but are "big path connected".

This paper heavily uses concepts from three distinct categories, and therefore three different fields of study. Since there is much overlap of notation in these fields, to help minimize confusion, we will hold to the following conventions in notation.

1. The letters X, Y, and Z will be reserved for arbitrary topological spaces.

2. The letters S and T will be reserved to denote generic totally-ordered spaces and I and J will be used for connected, compact totally-ordered spaces.

3. Any product of totally-ordered sets will be assumed to be endowed with the lexicographical order, unless explicitly stated otherwise.

4. The letters α, β, γ, and δ will be reserved for cardinals and ordinals. A cardinal will be identified with the least ordinal of its cardinality.

5. The letters λ, μ, ν will be reserved to denote paths (or big paths) in a topological space.

6. The symbols $<$ and $\leq$ will be used for any total order. If clarity is needed, notation such as $<_T$ or $\leq_T$ will mean the order defined on the set T.

7. The overline notation, such as $\bar{T}$, will always mean Dedekind completion of a totally-ordered set. Topological closure and interior will be denoted by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ respectively.

Chapter 2

Order Theory

Due to the heavy use of order theory in this paper, we take a moment here to describe the theory in brief. Many of the results here are common in the literature, but are included here with brief proofs for convenience.

2.1 Preliminaries

We start off with the very basics of order theory. This is mostly for the reader who is unfamiliar with total orders and basic results in order theory.

Definition 2.1.1. A *totally-ordered set* also called a *toset* is a set T with a binary operator $\leq$ that satisfies the following.

1. (anti-symmetry) $a \leq b$ and $b \leq a$ implies $a = b$,
2. (transitivity) $a \leq b$ and $b \leq c$ implies $a \leq c$
3. (trichotomy) for any $a, b \in T$, $a \leq b$ or $b \leq a$ (in the case where both are true, we conclude $a = b$, by anti-symmetry)

In this case, we call $\leq$ a *total order*. Also, $a < b$ will be understood to mean $a \leq b$ and $a \neq b$. An *initial point* or *minimal element*, if it exists, is a point $0_T \in T$ such that $0_T \leq t$ for all $t \in T$. Similarly, a *terminal point* or *maximal element* in T is a point 1_T such that $t \leq 1_T$ for all $t \in T$. We may also write 0_T as $\min T$ and 1_T as $\max T$.

We note here that although the symbols 0 and 1, when used with subscripts, will denote the minimal and maximal elements of any toset, when used without subscripts they will always mean the real numbers they typically represent, and also identified with the first and second ordinals respectively. Also, given a toset T and subsets $A, B \subset X$, the notation $A < B$ will be understood to mean for all $a \in A$ and all $b \in B$ we have $a < b$.

Definition 2.1.2. Given a toset T, the *reverse* of T or the *opposite* of T is the set T^{op} with the same elements as T (but denoted $x^{\text{op}} \in T^{\text{op}}$ when $x \in T$ to indicate the opposite order is to be assumed) and order $\leq^{\text{op}}$ such that $x^{\text{op}} \leq^{\text{op}} y^{\text{op}}$ if $y \leq x$ (in T).

As an example, take $T = \omega_0$, the set of natural numbers. Then, essentially, T^{op} can be thought of as the set of negative integers, since that is the order type that it has.

Definition 2.1.3. An *order-preserving map* is a function $f : (T, \leq_T) \to (S, \leq_S)$ such that $f(a) \leq_S f(b)$ whenever $a \leq_T b$. It is *strictly order preserving* if $f(a) <_S f(b)$ whenever $a <_T b$. Since this extra condition necessitates that f is injective, this kind of map is also called an *embedding* or, more specifically, an *order embedding*.

The function $f : T \to S$ is *order-reversing* if the function $f : T \to S^{\text{op}}$ (equivalently, the function $f : T^{\text{op}} \to S$) is order-preserving. That is, whenever $a \leq_T b$, we have $f(b) \leq_S f(a)$.

Note that if $(T, \leq)$ is a toset and $A \subset T$ then $(A, \leq)$, where $\leq$ is restricted to only elements of A, is also a toset and the inclusion map $i : A \hookrightarrow T$ is an embedding.

Definition 2.1.4. A *well-ordering* on a set T is a total order such that every non-empty subset has a minimal element. A set which is well-ordered is called an *ordinal*.

The well-ordering theorem, which is equivalent to the axiom of choice in first-order logic, states that every set can be given an order that is a well-order. The theory of transfinite induction is built upon well-ordered sets and will be used in this paper. This is analogous to what some call "strong induction" (on the set of natural numbers), but is applicable to ordinals other than ω_0.

Theorem 2.1.5 (Principle of transfinite induction)**.** *Let alpha be a well-ordered set and $A \subset alpha$. Let 0 denote the least element of α. Suppose that $0 \in A$ and that for any $\beta \in \alpha$ such that $\gamma \in A$ for all $\gamma < \beta$ we have $\beta \in A$. Then $A = \alpha$.*

This principle is used in precisely the same way as the principle of induction on the set of natural numbers. It can be used in a proof, showing that a certain property holds for all ordinals less than a given ordinal, or it can be used to define a function inductively.

Definition 2.1.6. Let $\{T_\beta \mid \beta \in \alpha\}$ be a collection of tosets indexed by the ordinal α. Define the *lexicographical order* on the product space $T = \prod_{\beta\in\alpha} T_\beta$ as follows. Given two points $a = (a_\beta), b = (b_\beta) \in T$, we say $a < b$ if there is an index $\beta \in \alpha$ such that $a_\gamma = b_\gamma$ for all $\gamma < \beta$ and $a_\beta < b_\beta$.

We see that this does indeed define a total order on T because if a and b are distinct, then they differ in some coordinate. Since the coordinates are well-ordered, there is a least coordinate where they differ, which becomes the β required by the definition. Unfortunately, it is not possible for this object to be the product in the category of totally-ordered sets, since the projection functions $\pi_\beta : T \to T_\alpha$ are not even order-preserving. However, we do have the following result which will be useful.

Proposition 2.1.7. *Let $\{f_\beta : S_\beta \to T_\beta\}$ be a collection of order-preserving maps indexed by the ordinal α. Then the product function $f : \prod_{\beta\in\alpha} S_\beta \to \prod_{\beta\in\alpha} T_\beta$ given by $f(a) = (f_\beta(a_\beta))_{\beta\in\alpha}$ is order-preserving. If f_β is an embedding for all β then so is f.*

Proof. Suppose that $a = (a_\gamma) < b = (b_\gamma)$ in $S = \prod S_\beta$. Let $\beta \in \alpha$ be such that $a_\gamma = b_\gamma$ for $\gamma < \beta$ and $a_\beta < b_\beta$. Then for $\gamma < \beta$, we see that $(f(a))_\gamma = f_\gamma(a_\gamma) = f_\gamma(b_\gamma) = (f(b))_\gamma$ and $(f(a))_\beta = f_\beta(a_\beta) \le f_\beta(b_\beta) = (f(b))_\beta$. Hence, $f(a) \le f(b)$. If f_β is strictly order-preserving, then we see that $f(a) < f(b)$. □

Definition 2.1.8. A toset T can be endowed with a natural topology called the *order topology*, wherein a basis of open sets is formed by the collection of open intervals

$$(a, b) = \{x \in T \mid a < x < b\} \tag{2.1}$$

and open rays

$$(-\infty, a) = \{x \in T \mid x < a\} \tag{2.2}$$

$$(a, \infty) = \{x \in T \mid a < x\} \tag{2.3}$$

Unless otherwise specified, all tosets in this paper will be assumed to be endowed with the order topology.

2.2 Dedekind Completion

Here we introduce the concept of "complete" for a totally-ordered set. While these concepts are still basic in order theory, they may be less known to someone who has not specifically studied the field.

Definition 2.2.1. Given a point $x \in T$, the point x' is called the *successor* of x if $x < x'$ and $x < y$ implies $x' \leq y$ for all $y \in T$. In this case, x is called the *predecessor* of x'. For clarity, the terms *immediate successor* or *immediate predecessor* may be used.

In a well-ordered set, every element (excepting the maximal element, if it exists) has a successor. However, not every element has a predecessor. Such an element is sometimes called a *limit* (usually in the context of ordinals, in which case it is called a *limit ordinal*). In the set $\mathbb{Z}$ of integers, each point has a successor and a predecessor. In the set $\mathbb{Q}$ of rational numbers, no point has a successor nor a predecessor.

Definition 2.2.2. Given a toset T and a subset $A \subset T$, we say that A is *bounded above* if there is $x \in T$ such that $a \leq x$ for all $a \in A$, in which case x is called an *upper bound* on A. An element $x \in T$ is the *least upper bound* or *supremum* of A if it is an upper bound and for any upper bound y of A we have $x \leq y$ (such an element, if it exists, is clearly unique). In this case, we write $x = \sup A$. A toset T is said to be *complete* (more specifically *order-complete* or *Dedekind complete*) if every non-empty subset A which is bounded above has a supremum.

Definition 2.2.3. Given a toset T and subset $A \subset T$, we say that A is *dense in* T (or *order-dense*, to distinguish from the topological use of the word) if for any $x, y \in T$ with $x < y$ there is $z \in A$ with $x < z < y$. We say that T is (order-)*dense* if any such A exists (in which case, $A = T$ will also work). If T is dense and order complete, then it is called a *linear continuum.*

It may seem confusing or ambiguous to use the word "dense" with its topological definition and with the order-theory definition just given. However, we see that at least for the purposes of this paper (since all tosets will be endowed with the order topology), this is never an issue since the two definitions coincide. More precisely, we have the following result.

Proposition 2.2.4. *Let T be a toset and $A \subset T$. If A is order-dense in T then A is a topologically-dense subset of T.*

Proof. Let $t \in T$ and (a, b) a basic neighborhood of t. Since A is order-dense in T, there is $s \in A$ such that $a < s < t$. Thus, $s \in (a, b) \cap A$, hence A is topologically dense in T. □

We note that the converse is false. For example, a two-point set $T = \{1, 2\}$ is not order-dense (and has no order-dense subset), but for any topological space X, X is topologically-dense in itself. However, if T is order-dense and $A \subset T$ is topologically-dense in T then A is order-dense in T.

Proposition 2.2.5. *Let T be a toset with the order topology. Then T is connected if and only if it is a linear continuum.*

Proof. Suppose that T is not complete. Let $A \subset T$ be nonempty with an upper bound $b \in T$ but no least upper bound. Let B be the set of all upper bounds of A in T. Since there is no least upper bound, we see that $B = \bigcup_{b \in B} (b, \infty)$, hence B is open. Now define $C = \bigcup_{a \in A} (-\infty, a)$. Since A has no least upper bound (in particular, no maximal element), we see that $A \subset C$. In fact, it is seen that C and B separate T, hence T is not connected.

Now suppose that T is not dense. That is, some point $t \in T$ has a successor t'. Then it is easily seen that $(-\infty, t')$ and (t, ∞) form a separation, hence T is not connected.

Finally, suppose that T is a linear continuum. Suppose that $A \subsetneq T$ is open. Let $x \in T \setminus A$. Define $A' = (-\infty, x) \cap A$. We see that A' is bounded above by x, hence it has a least upper bound, x_0. Given any points $a < x_0 < b$ in T, we see that there is $c \in A'$ such that $a \leq c < x_0$, since a is not an upper bound on A'. Hence, x_0 cannot be in the interior of A^c, thus T is connected. □

Define the terms "lower bound" and "greatest lower bound" (also refered to as "infimum") to be the dual of "upper bound" and "least upper bound" respectively. That is, if T is a toset and $A \subset T$ then $a \in T$ is a *lower bound on* A if a^{op} is an upper bound on A^{op} in T^{op} and is the *greatest lower bound* if a^{op} is the least upper bound on A^{op}. In this case, we write $a = \inf A$ If every nonempty set $A \subset T$ that is bounded below has a least upper bound, we say that T has the *greatest lower bound property.* If T is linearly complete then it also has the greatest lower bound property.

Theorem 2.2.6. *Suppose that T and S are linear continua, that $A \subset T$ is dense in T, and that there is an order-preserving map $f : A \to S$. Also suppose that A and $f(A)$ are either both bounded above or neither is, and that the same holds for bounded below. Then f extends to an order-preserving map $\tilde{f} : T \to S$. If $f(A)$ is dense in S then the extension is unique and surjective. If f is an embedding, then $\tilde{f}$ is as well.*

Proof. Suppose that A is bounded below. By hypothesis, we then have that $f(A)$ is as well. Let B be the set of lower bounds on A, let $s_0 \in S$ be a lower bound of $f(A)$, and define $\tilde{f}(b) = s_0$ for all $b \in B$. For $t \in T \setminus B$, define $\tilde{f}(t) = \sup \{f(a) \mid a \in A,\ a \leq t\}$. We see that $\tilde{f}$ is well-defined since supremums always exist in S. We also note that it extends f since $\sup \{f(a) \mid a \in A,\ a \leq t\} = f(t)$ if $t \in A$, since $t = \sup \{a \in A,\ a \leq t\}$. Now we show that $\tilde{f}$ is order-preserving. Suppose that $a \leq b$ in T. Then we have that $(-\infty, a) \subset (-\infty, b)$ and therefore $f((-\infty, a) \cap A) \subset f((-\infty, b) \cap A)$, hence $\tilde{f}(a) = \sup f((-\infty, a) \cap A) \leq \sup f((-\infty, b) \cap A) = \tilde{f}(b)$.

Next we show that if $f(A)$ is dense in S then $\tilde{f}$ is unique. Suppose that $g : T \to S$ is order-preserving and $g|_A = f$. Given $t \in T$, we see that $t = \sup \{a \in A,\ a \leq t\}$, since A is dense in T. Also, since g is order-preserving, we see that $g(t) \geq \sup \{f(a) \mid a \in A,\ a \leq t\} = \tilde{f}(t)$.

Suppose that $g(t) > \tilde{f}(t)$. Since $f(A)$ is dense in S, we see that there is $y \in f(A)$ such that $\tilde{f}(t) < y < g(t)$, and since f is order-preserving, we see that $y = f(s)$ for some $s < t$. This contradicts that $g(s) = f(s)$. Hence, we must have $g(t) = \tilde{f}(t)$, so the extension is unique. A similar argument shows that $\tilde{f}$ is surjective. Given $s \in S$, we see that $s = \sup \{b \in f(A) \mid b \leq s\}$, so we define $t = \sup \{a \in A \mid f(a) \leq s\}$ and see that $\tilde{f}(t) = s$.

Finally, all that is left is to show that $\tilde{f}$ is strictly order-preserving if f is. In particular, we need to show that if $a < b$ then $\sup f((-\infty, a) \cap A) < \sup f((-\infty, b) \cap A)$. Since A is dense, there are $c, d \in A$ such that $a < c < d < b$. Since f is strictly order-preserving, we see that $\tilde{f}(a) = \sup f((-\infty, a) \cap A) \leq f(c) < f(d) \leq \sup f((-\infty, b) \cap A) = \tilde{f}(b)$. □

We have shown some of the benefits of a toset being complete, and we will see many more such advantages later in the paper. Because this is such a useful property, we introduce here one way to "enrich" a toset so that it becomes order complete. The method given here is well-known and well-understood. A different method will be explored in a later section.

Definition 2.2.7. Let T be a toset and $A \subset T$. Then A is *downward closed* in T if whenever $a \in A$ and $x < a$ we have $x \in A$. Let $\bar{T}$ denote the set of all downward closed sets in T that have no maximal point. Then $\bar{T}$ is called the *Dedekind completion* of T. Order $\bar{T}$ by set inclusion (this is only a partial order on the whole power set of a set, but when restricting to downward-closed sets, it is seen to be a total order).

The Dedekind completion of a toset is not always an extension of the set, in the way that a compactification or metric completion is. Rather, there are times when the Dedekind completion actually has fewer elements. For example, the Dedekind completion of any finite set is a single point, namely the empty set, since any non-empty finite toset has a maximal element. Next, consider the set of integers. Its completion consists of two points, namely $\bar{Z} = \{\emptyset, \mathbb{Z}\}$. Thus, when extending a function to its Dedekind completion, we cannot ask for an extension in the normal sense of the word, since the original set may not embed in its completion. However, we do have this result, which is as strong as can be hoped for.

Proposition 2.2.8. *If $f : S \to T$ is an order-embedding, then there is an order-embedding $\tilde{f} : \bar{S} \to \bar{T}$ that extends f in the sense that $\tilde{f}((-\infty, s)) = (-\infty, f(s))$ for all $s \in S$ such that $(-\infty, s)$ has no maximal element.*

Proof. Given $A \in \bar{S}$, define $\tilde{f}(A)$ to be the downward closed hull of $f(A)$. That is $\tilde{f}(A) = \{t \in T \mid \exists a \in A \text{ with } t \leq f(a)\}$. By definition, $\tilde{f}(A)$ is downward closed, so we show that it has no maximal element. Assume that $x \in \tilde{f}(A)$ is a maximal element. Then, it must follow that $x \in f(A)$. Also, given any $t \in f(A)$, we see that $t \leq x$ since f is order-preserving. Therefore, x is a maximal element of $f(A)$. Since f is injective, we see that this means that $f^{-1}(x)$ is the maximal element of A, which contradicts the choice of A. Hence, $\tilde{f}(A)$ has no maximal element, and thus $\tilde{f}$ is well-defined. It is clearly order-preserving.

We show that $\tilde{f}$ is injective. Suppose that $A < B \in \bar{S}$. Let $x \in B \setminus A$. Then we see that for any $t \in \tilde{f}(A)$, $t < f(x)$, since $A < x$ and $t \leq f(a)$ for some $a \in A$, hence $x \notin \tilde{f}(A)$. But $x \in B$, so $f(x) \in \tilde{f}(B)$. Thus, $\tilde{f}(A) \neq \tilde{f}(B)$. □

Theorem 2.2.9. *Given the toset T, its Dedekind completion $\bar{T}$ is order complete.*

Proof. Let $\mathcal{A} \subset \bar{T}$ be non-empty and bounded above. If $\mathcal{A}$ has a maximal element, we are done, so we assume $\mathcal{A}$ has no maximal element. Define the set $B \subset T$ to be the union of all $A \in \bar{T}$ such that A is not an upper bound of $\mathcal{A}$ (note that elements of $\bar{T}$ are subsets of T). If there are no such elements, it follows that $\mathcal{A}$ is a single point and therefore its only element is its supremum. Otherwise, B is non-empty. In fact, given $A \in \mathcal{A}$, we see that A is not an upper bound on $\mathcal{A}$ and hence $A \subset B$. Thus, $A \leq B$ (in $\bar{T}$) for all $A \in \mathcal{A}$, hence B is an upper bound on $\mathcal{A}$. Given any other upper bound C, it is easily seen that C must contain every point that is not an upper bound on $\mathcal{A}$ and hence must contain B. □

The Dedekind completion $\bar{T}$ of any toset has $\emptyset$ as its minimal element. It has a maximal element, namely the union of all sets which have no maximal element, which may be T itself if it has no maximal element. The Dedekind completions of the set of rationals, the real open interval $(0, 1)$, and the closed interval $[0, 1]$ are all $[0, 1]$.

Proposition 2.2.10. *If T is a linear continuum, then $\bar{T}$ is isomorphic (and homeomorphic) to T, with first and last points attached if they were not present in T*

Corollary 2.2.11. *The Dedekind completion operator is idempotent. That is, for any toset T, we have $\bar{\bar{T}} \cong \bar{T}$.*

Theorem 2.2.12. *If T is a dense toset, then there is an injective map $i : T \to \bar{T}$ that is continuous and order-preserving. In this case, $i(T)$ is dense in $\bar{T}$.*

Proof. For each $t \in T$, it is clear that the open ray $(-\infty, t)$ is a downward closed set and has no maximal point (since t, by hypothesis, has no immediate predecessor). Therefore, it follows that $(-\infty, t) \in \bar{T}$, so define the map $i : T \to \bar{T}$ so that $i(t) = (-\infty, t)$. Clearly i is strictly order preserving. Now, let (A, B) be an open interval in $\bar{T}$. Let $x \in i^{-1}((A, B))$. It suffices to show that there is an open interval (a, b) containing x that is also in the preimage of (A, B). Since $A < i(x)$ in $\bar{T}$, it follows that x is an upper bound for A. Also, x cannot be the supremum of A because that would imply that $A = (-\infty, x) = i(x)$, which is not true. Therefore, let a be an upper bound of A with $a < x$.

Now, since $i(x) < B$ in $\bar{T}$, it follows that x is not an upper bound on the set B in T, so there is $b \in B$ with $x < b$. It follows that $(a, b) \subset i^{-1}((A, B))$. Hence i is also continuous.

Finally, we show that $i(T)$ is dense in $\bar{T}$. It suffices to show that given any two points $A, B \in \bar{T}$ (with $A < B$), there is a point in the image of i that lies between A and B. Since A and B are distinct subsets of T, there is a point $x \in B \setminus A$. Since B has no maximal element, there is $y \in B$ with $x < y$. Also, since A and B are downward-closed sets, it follows that $A \subset (-\infty, x) \subsetneq (-\infty, y) \subsetneq B$, as desired. □

Corollary 2.2.13. *If $f : T \to S$ is an order-preserving map with S a linear continuum, T dense, and $f(T)$ bounded above and below in S, then there is an order-preserving map $\tilde{f} : \bar{T} \to S$ that extends f (in the sense that if $i : T \to \bar{T}$ is the embedding guaranteed by Theorem 2.2.12 then $\tilde{f} \circ i = f$). If $f(T)$ is dense in S then $\tilde{f}$ is surjective. If f is injective, then so is $\tilde{f}$.*

This follows from Theorem 2.2.6.

Corollary 2.2.14. *Suppose that T is order-dense in the toset S. Then we have that $|S| \leq |2^T|$.*

Proof. Since T is dense in S, T is a dense toset. Therefore it embeds in its Dedekind completion $\bar{T}$. Since each element of $\bar{T}$ is a subset of T, we see that $\bar{T} \subset 2^T$, hence $|\bar{T}| \leq |2^T|$. Let $i : T \to \bar{T}$ be the canonical embedding. Then by the previous result, this extends to an embedding $\tilde{i} : S \to \bar{T}$. Hence $|S| \leq |\bar{T}| \leq |2^T|$, as desired. □

2.3 Big Intervals

Here we define what a "big interval" is and introduce some concepts that will be useful when discussing big homotopy theory.

Theorem 2.3.1. *Let T be a toset. The following are equivalent.*

1. *T is compact and connected.*
2. *T is connected and has a first and last point.*
3. *T is a linear continuum and has a first and last point.*

Proof. It follows immediately from Proposition 2.2.5 that (2) $\Leftrightarrow$ (3). So we show (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

Assume that T is compact and connected. Suppose that it has no first point. Then the collection $\{(x, \infty) \mid x \in T\}$ is an open cover with no finite subcover. This is a contradiction, therefore T has a first point. The same method is used to show that T has a last point.

Now assume that T is a linear continuum with first and last point. It follows from Proposition 2.2.5 that T is connected. Let $\mathcal{U}$ be an open cover of T. We may assume that $\mathcal{U}$ consists of open intervals (and rays). Let $U_0 \in \mathcal{U}$ be the set containing the minimal element of T. Define x_1 as the right endpoint of U_0 (that is, $U_0 = (-\infty, x_1)$). Then let $U_1 \in \mathcal{U}$ be such that $x_1 \in U_1$. Continue this process as necessary, so that $x_i \in U_i \in \mathcal{U}$ and x_i is the right endpoint of U_{i-1}. We need to choose the sequence $\{x_i\}$ so that x_i is the maximal element of T for some i.

Let $A \subset T$ be the set of all points which can be connected to x_0 through a finite number of elements of U_0. We wish to show that $A = T$. Let $x = \sup A$ and assume, for a contradiction, that $x \neq 1_T$. Suppose that $x_1 < x_2 < \cdots$ is a such that $x = \sup\{x_i\}$ and each x_i is defined as above. Let $U \in \mathcal{U}$ be a neighborhood of x. It follows that all but finitely many x_i are in U. Let n be minimal such that $x_n \in U$. Then replace x_n with x and U_n with U. Then we see that U contains a point greater than x, which contradicts that $x = \sup A$. Hence, we must have that $\sup A = 1_T$ and therefore T is compact. □

Definition 2.3.2. A *big interval* is a toset X that satisfies any of the equivalent conditions in Theorem 2.3.1.

Any closed interval in $\mathbb{R}$ is a big interval. Indeed, even a degenerate closed interval—a single point—is a big interval. Given any dense toset X, its Dedekind completion $\bar{X}$ is a big interval. This is a direct corollary of Theorem 2.2.9, together with the observation that the Dedekind completion always has a maximal and minimal element.

Just assuming these few things—that a space is compact, connected, and totally-ordered—is very helpful in many ways. In fact, many proofs concerning homotopy theory only relies on these three properties of the real interval $[0, 1]$, and therefore they carry over into big homotopy theory. One of the most useful facts is as follows. In general, if S and T are tosets and $f : S \to T$ it is possible for f to be continuous but not order-preserving and also to be order-preserving but not continuous. However, with big intervals, we have the following result relating continuity with order preservation.

Theorem 2.3.3. *Let T be a toset and S a big interval. Then the function $f : T \to S$ is continuous if it is surjective and order-preserving.*

Proof. Suppose that $f : T \to S$ is surjective and order-preserving. It suffices to show that the preimage of an open interval is open. Let $a < b$ in S and define $A = f^{-1}((a, b))$, so we show that A is open. We know that (a, b) is non-empty because S is dense and that A is non-empty because f is surjective. So, let $t \in A$. Since S is dense, there are $c, d \in S$ with $a < c < f(t) < d < b$. Since f is surjective, we have $s, u \in T$ so that $f(s) = c$ and $f(u) = d$. Since f is order-preserving, we see that $t \in (s, u) \subset A$. Hence, A is open. □

When defining the fundamental group, the operation necessary to define the group law is that of path concatenation. In the case of big intervals, since each big path may have a distinct domain, we define concatenation for big intervals themselves first.

Definition 2.3.4. Let S and T be tosets. Define the *concatenation* $S * T$ as follows. As a set, $S * T = S \sqcup T$, with the order $\leq$ such that

1. $a \leq b$ if $a, b \in S$ and $a \leq_S b$
2. $a \leq b$ if $a, b \in T$ and $a \leq_T b$
3. $a \leq b$ if $a \in S$ and $b \in T$.

In the case that S has maximal element 1_S and T has minimal element 0_T, we also define $S \vee T = S * T/(1_S \sim 0_T)$.

Note that $\vee$ as defined here is the same as the topological notion of wedge, since it is the disjoint union of two spaces where one point of each is identified. We need not specify in each case which point of each space is the wedge point, since it is always taken to be the maximal element of the first term and the minimal element of the second.

Lemma 2.3.5. *Let S, T be tosets. Then there is a canonical isomorphism $\overline{S * T} \cong \bar{S} \vee \bar{T}$.*

Proof. By definition, $\overline{S * T}$ is the set of all downward closed sets in $S * T$ which have no maximal element. Since $S < T$ in $S * T$, we see that a downward closed set $A \subset S * T$ satisfies either $A \subset S$ or $A \supset S$. In the former case, we see that A is a downward closed set in S with no maximal element, and thus is an element of $\bar{S}$. In the latter case, we see that $A \setminus S$ is a downward closed set in T with no maximal element and is therefore an element of $\bar{T}$. This defines an injection $\overline{S * T} \leftrightarrow \bar{S} * \bar{T}$ which is surjective with the single exception that it misses the maximal element of $\bar{S}$ in the case that S has no maximal element or that it misses the minimal element of $\bar{T}$ in the case that S does have a maximal element. But, we see that in either case, exactly one of $1_{\bar{S}}$ and $0_{\bar{T}}$ fails to correspond to an element of $\overline{S * T}$ under the correspondence given. Since these two points are identified in $\bar{S} \vee \bar{T}$, we see that this yields a bijection $\phi : \overline{S * T} \to \bar{S} \vee \bar{T}$. This is easily seen to be order-preserving. $\square$

Definition 2.3.6. Let I and J be big intervals and X a set. Suppose that $f : I \to X$ and $g : J \to X$ are such that $f(1_I) = g(0_J)$. Then we can define the *concatenation of f and g*, denoted $f * g : I \vee J \to X$, by

$$f * g(t) = \begin{cases} f(t) & t \in I \\ g(t) & t \in J \end{cases}$$

and we can define the *reverse of f*, denoted $f^{\mathrm{op}} : I^{\mathrm{op}} \to X$, by $f^{\mathrm{op}}(t^{\mathrm{op}}) = f(t)$.

Since I and I^{op} are equal as sets, f and f^{op} are the same function, the key difference being if f is thought of as a path then f^{op} is the same path in reverse direction.

This is a good start to the domain we want for "big paths". And in [1], all possible big intervals were used. However, the nice thing about the definition of the classical fundamental group is that all maps are from the same space, namely the unit real interval $[0,1]$. The reason this is possible is because this interval (or any real interval, for that matter) has two useful properties. The first is that $[0,1] \vee [0,1] \cong [0,1]$ and that $[0,1]^{\mathrm{op}} \cong [0,1]$ (the interval is order-isomorphic to itself in the reverse order). Thus we will enrich the above definition as follows.

Definition 2.3.7. A toset T is *self-similar* if it satisfies the properties

1. $T \vee T \cong T$
2. $T^{\mathrm{op}} \cong T$

where $\cong$ denotes order isomorphism, which (in the case of big intervals) implies topological homeomorphism as well.

Given any toset T, we may construct a self-similar toset $\hat{T}$, which one might call the *self-similar set generated by T*. Indeed, let $\hat{T} = \mathbb{Q} \times (T \vee T^{\mathrm{op}})$. Intuitively, this is constructed by attaching T^{op} to the end of T and then gluing together countably many of these in the order type of the rational numbers. Note that if T is infinite, then $\left|\hat{T}\right| = |T|$, since as a set $\hat{T}$ is essentially the disjoint union of countably many copies of T. Here we point out that if T is self-similar, then its Dedekind completion $\bar{T}$ is as well.

Definition 2.3.8. Given a cardinal α, the toset T is *α-dense* if for all $s, t \in T$ with $s < t$, we have $|(s,t)| = \alpha$. T is *strongly dense* if it is $|T|$-dense. Note that with this definition, the term "order dense" is equivalent to ω_0-dense.

Definition 2.3.9. Given an ordinal α and a toset S, let S^α be the set of all α-sequences in S, endowed with the lexicographical order. For a point $s_0 \in S$, let $(S, s_0)^\alpha$ be the subset of S^α consisting of all points x where there is an index $\beta \in \alpha$ such that $x_\gamma = s_0$ for all $\gamma \geq \beta$.

We see that given S, s_0 as above and any two ordinals $\alpha < \beta$, there is a canonical inclusion $i : S^\alpha \to S^\beta$, given by adding on s_0 in each coordinate in $\beta \setminus \alpha$. Then we see that $(S, s_0)^\alpha = \varinjlim_{\beta \in \alpha} S^\beta$. If we suppose that $|S| < |\alpha|$, then $|(S, s_0)^\alpha| = \sup_{\beta<\alpha} \left|S^\beta\right| = \sup_{\beta<\alpha} \left|2^\beta\right|$. So, if we assume GCH, then $|(S, s_0)^\alpha| = |\alpha|$. Indeed, for any countable set S with at least two points, the statement that $|(S, s_0)^\alpha| = \alpha$ for every infinite cardinal α is equivalent to GCH.

Proposition 2.3.10. *Given a cardinal α and pointed set (S, s_0) with S dense, we have that $(S, s_0)^\alpha$ is dense in S^α.*

Proof. Let $a, b \in S^\alpha$ with $a < b$. Let β be the least coordinate in α where $a_\beta \neq b_\beta$. Let $s \in S$ be such that $a_\beta < s < b_\beta$. Then define the point $c \in (S, s_0)^\alpha$ as follows.

$$c_\gamma = \begin{cases} a_\gamma & \gamma < \beta \\ s & \gamma = \beta \\ s_0 & \gamma > \beta \end{cases}. \tag{2.4}$$

Then we see that $a < c < b$, so $(S, s_0)^\alpha$ is dense in S^α. □

Proposition 2.3.11. *Let α be an ordinal and S a toset. If S has a first (respectively, last) point 0_S (resp., 1_S), then so does S^α. If S is dense then S^α is strongly dense. If S is complete with first and last point then S^α is complete.*

Proof. The first point 0_{S^α} is given by the constant sequence $x_\gamma = 0_S$ and the last point 1_{S^α} is given by the constant $x_\gamma = 1_S$.

Suppose that S is dense. Given $a < b \in S^\alpha$, let β be the first coordinate where a and b differ, thus $a_\beta < b_\beta$. Since S is dense, there is $s \in S$ with $a_\beta < s < b_\beta$. Define $x \in S^\alpha$ so that $x_\gamma = a_\gamma$ for $\gamma < \beta$ and $x_\beta = s$. Then, for any assignment of x_γ where $\gamma > \beta$, we see that $a < x < b$. Hence $|(a,b)| \geq \left|S^{\alpha\setminus\beta}\right| = |S^\alpha|$. Thus, in fact, S^α is strongly dense.

Now suppose that S is complete. Let $A \subset S^\alpha$ be bounded above. We define the point x inductively and then show that it is the supremum of A. For this discussion, the following notation will be useful. Given $t \in S^\alpha$ and $\beta \in \alpha$, let $t[\beta] = \{s \in S^\alpha \mid s_\gamma = t_\gamma \; \forall \gamma < \beta\}$. Let $x_0 = \sup\{a_0 \mid a \in A\}$. Then for $\beta \in \alpha$ where x_γ is defined for all $\gamma < \beta$, consider the set $A \cap x[\beta]$. Intuitively, this is the set of all points in A which agree with x in all coordinates before β. If this set is empty, define $x_\beta = 0_S$ (and it is seen that from this point on, all coordinates of x will be 0_S). Otherwise, define $x_\beta = \sup\{a_\beta \mid a \in A \cap x[\beta]\}$. Now we verify that x is indeed the least upper bound of A.

We first show that x is an upper bound of A. If there is no β for which the set $A \cap x[\beta]$ is empty, then we see that x is at least as big as any element of A in each coordinate up to the first coordinate where it differs from x, and therefore it is an upper bound. So, assume there is β such that $A \cap x[\beta]$ is empty. Let β_0 be the least such coordinate. This means there is no $a \in A$ such that $a_\gamma = \sup\{a_\gamma \mid a \in A \cap x[\gamma]\}$ for all $\gamma < \beta_0$. It follows then that given any $a \in A$, there is $\gamma < \beta_0$ such that $a_\gamma < \sup\{a_\gamma \mid a \in A \cap x[\gamma]\} = x_\gamma$, hence $a < x$. So, x is an upper bound.

Now suppose that $y < x$. Let β be the minimal coordinate where x and y differ. Thus, $y_\beta < x_\beta$. Thus we see that $A \cap x[\beta] = \emptyset$ is impossible, since that would mean $x_\beta = 0_S$. Therefore, $x_\beta = \sup\{a_\beta \mid a \in A \cap x[\beta]\}$. Since $y_\beta < x_\beta$, there is $a \in A \cap x[\beta]$ such that $y_\beta < a_\beta$, thus y is not an upper bound on A. Hence, x is the least upper bound. □

Corollary 2.3.12. *If I is a big interval, then so is I^α.*

Proposition 2.3.13. *Let S be a toset. If S is self-similar then for any $s_0 \in S$ and any ordinal α, we have that $(S, s_0)^\alpha$ and S^α are self-similar.*

Proof. Suppose that S is self-similar. Let $r : S \to S$ be an order-reversing isomorphism. Then we see the map $\tilde{r} : S^\alpha \to S^\alpha$ given by $\tilde{r}(x_\gamma) = (r(x_\gamma))$ is an order-reversing isomorphism and it carries $(S, s_0)^\alpha$ to itself. Let $S_1 = S_2 = S$ and $T_1 = T_2 = S^\alpha$. The different indices are to distinguish points in $S_1 * S_2$ (or $S_1 \vee S_2$) as coming from either S_1 or S_2. Let $\varphi : S_1 \vee S_2 \to S$ be an order-preserving isomorphism. Also for clarity of discussion, we define $\varphi_1 = \varphi|_{S_1}$ and $\varphi_2 = \varphi|_{S_2}$ and for a point $a \in S_1 \vee S_2$, we use the notation a^1 to indicate $a \in S_1$ and a^2 to indicate $a \in S_2$. Thus, we will use 1^1 to mean the maximal element of S_1, 0^2 to mean the maximal element of S_2 and so forth.

First, we consider the case where S has no maximal point or no minimal point. In this case, $S_1 \vee S_2 = S_1 * S_2$. Also, it is seen that T will also lack a maximal or minimal point (coinciding with that lacking in S), thus $T_1 \vee T_2 = T_1 * T_2$. Then construct a map $\Phi : T_1 * T_2 \to S^\alpha$ by

$$\Phi(x)_0 = \begin{cases} \varphi_1(x_0) \in S_1 & \text{if } x \in T_1 \\ \varphi_2(x_0) \in S_2 & \text{if } x \in T_2 \end{cases} \tag{2.5}$$

and $\Phi(x)_\gamma = x_\gamma$ for $\gamma > 0$. Note that this is an isomorphism. So we get that $T_1 * T_2 \cong S^\alpha = T$, hence T is self-similar. Also, we see that this isomorphism preserves $(S, s_0)^\alpha$, so $(S, s_0)^\alpha$ is self-similar as well.

Now we suppose that S has a first and last point, 0_S and 1_S. Thus, T does as well, which we denote 0_T and 1_T. The idea for constructing this map is similar to that as before, but before compacting two copies of S in one coordinate, it is necessary to first compact copies of S in subsequent coordinates, as necessary. The formula for the map $\Phi : T_1 \vee T_2 \to T$ is as follows.

$$\Phi(x)_\gamma = \begin{cases} \varphi_i(x_\gamma) & \text{if } x \in T_i \text{ and } x_\delta = 1^1, 0^2 \text{ for all } \delta < \gamma \\ x_\gamma & \text{otherwise} \end{cases}. \tag{2.6}$$

To see that this is an isomorphism, we construct a 2-sided inverse $\Psi : T \to T_1 \vee T_2$ using the obvious definition

$$\Psi(x)_\gamma = \begin{cases} \varphi^{-1}(x_\gamma) & \text{if } x_\delta = \varphi_1(1_T) \text{ for all } \delta < \gamma \\ x_\gamma & \text{otherwise} \end{cases}. \tag{2.7}$$

Now, given $x \in T_1 \vee T_2$, we calculate $[\Psi(\Phi(x))]_\gamma$. Without loss of generality, we may assume $x \in T_1$. Then, if $x_\delta = 1^1$ for all $\delta < \gamma$ then we have that $\Phi(x)_\delta = \varphi_1(1)$ for all $\delta < \gamma$ and $\Phi(x)_\gamma = \varphi_1(x_\gamma)$. Thus $[\Psi(\Phi(x))]_\gamma = \varphi^{-1}(\varphi_1(x_\gamma)) = x_\gamma$. Otherwise, we see that $\Phi(x)_\gamma = x_\gamma$ and $[\Psi(\Phi(x))] = x_\gamma$. Hence, $\Psi\Phi = \mathrm{id}_{T_1 \vee T_2}$.

The calculation for the reverse composition is similar. Given $x \in T$, we calculate $[\Phi(\Psi(x))]_\gamma$. If $x_\delta = \varphi_1(1_T)$ for all $\delta < \gamma$ then $\Psi(x)_\delta = 1^1$ for all $\delta < \gamma$ and $\Psi(x)_\gamma = \varphi^{-1}(x_\gamma)$, thus $[\Phi(\Psi(x))]_\gamma = \varphi(\varphi^{-1}(x_\gamma)) = x_\gamma$. Otherwise, $\Psi(x)_\gamma = x_\gamma$ and $[\Phi(\Psi(x))]_\gamma = x_\gamma$, hence $\Phi\Psi = \mathrm{id}_T$. Thus $T = S^\alpha$ is self-similar. □

2.4 Cauchy Completion

In this section, we give a different approach to the concept of completing a given toset. This approach will help with the understanding of the Dedekind completion, and will be seen to be equivalent to the Dedekind completion in the case where the original toset is order dense. This approach is a generalization of the construction of the set of real numbers from the set of rational numbers using Cauchy sequences. We define a "weak Cauchy sequence", which is a generalization of a Cauchy sequence, and then define the Cauchy completion to be the set of Cauchy sequences in the given toset (modulo a certain equivalence relation), just as is done with the standard construction of the real numbers.

Here we introduce an abuse of notation. In a toset T, given elements $a \leq b$ in T, the notation $[a, b]$ is interpreted to mean the closed interval between a and b. That is $[a, b] = \{t \in T \mid a \leq t \leq b\}$. However, in the current discussion it will be useful to slightly broaden the usage of this notation. For example, if we do not wish to bother about which of a and b is larger, we may write $[a, b]$ without knowing. In this case, it is understood that we mean $[\min\{a, b\}, \max\{a, b\}]$. More generally, given any finite set of points $\{t_1, \ldots, t_n\} \subset T$, we write $[t_1, \ldots, t_n]$ to mean the *convex hull* of the given set. That is $[t_1, \ldots, t_n] = [\min t_1, \ldots, t_n, \max t_1, \ldots, t_n] = \{t \in T \mid t_i \leq t \leq t_j \text{ for some } i, j\}$.

Definition 2.4.1. Given an ordinal α and a topological space X, an *α-sequence in X* is a function $a : \alpha \to X$. Given $\beta \in \alpha$, we may write a_β to mean $a(\beta)$. We say that the sequence *converges to* $x_0 \in X$ if for every open neighborhood U of x_0 there is an index $\beta \in \alpha$ such that $a_\gamma \in U$ for all $\gamma > \beta$. In this case, we write $a_\beta \to x_0$ and we say that x_0 is the *limit of* a.

Definition 2.4.2. Let α be an ordinal and $\mathcal{A} = \{A_\beta \mid \beta \in \alpha\}$ an α-sequence of sets. Define the *limit superior of* $\mathcal{A}$ to be $\limsup A_\alpha = \bigcap_{\beta \in \alpha} \bigcup_{\gamma \geq \beta} A_\gamma$. Then, given a space X and an α-sequence $a : \alpha \to X$, define the *limit superior of* a to be the limit superior of the collection $\{A_\beta\}$ given by $A_\beta = [a_\beta, a_{\beta+1}]$. That is, $\limsup a = \bigcap_{\beta \in \alpha} \bigcup_{\gamma \geq \beta} [a_\gamma, a_{\gamma+1}]$.

Definition 2.4.3. Let T be a toset and $a : \alpha \to T$ an α-sequence. Then we say that f is *weakly Cauchy* or is a *weak Cauchy sequence* if $\limsup a$ is at most one point.

Note that in the case $T = \mathbb{R}$, all Cauchy sequences are weakly Cauchy, but not all weakly Cauchy sequences are Cauchy. For example, the sequence given by $a_n = n$ is weakly Cauchy but not Cauchy. In fact, every monotone sequence in any toset is weakly Cauchy.

Definition 2.4.4. An α-sequence $a : \alpha \to T$ is *monotone* (or *weakly monotone*) if it is either order-preserving (in which case it is called *non-decreasing*) or order-reversing (in which case it is called *non-increasing*). If it is strictly order-preserving (or strictly order-reversing), it is called *strictly monotone*, in which case it may be either *increasing* or *decreasing*.

Proposition 2.4.5. *Suppose that* $a : \alpha \to T$ *is an* α*-sequence in* T *such that* $\limsup a = \{t_0\}$*. Then* $a_\beta \to t_0$*.*

Proof. Let (s, t) be an open interval containing t_0. Since $\limsup a = \{t_0\}$, we see that there is $\beta \in \alpha$ such that $\left(\bigcup_{\gamma \geq \beta} [a_\gamma, a_{\gamma+1}] \right) \subset (s, t)$, since the intersection of all such sets is $\{t_0\}$ hence they cannot all intersect the complement of (s, t). Therefore $a_\gamma \in (s, t)$ for all $\gamma \geq \beta$. $\square$

Definition 2.4.6. Let T be a toset and α a cardinal. Let $\mathcal{C}(T, \alpha)$ be the set of all weak Cauchy β-sequences in T, with $\beta \leq \alpha$. Define a relation $\sim$ on $\mathcal{C}(T, \alpha)$ as follows. Given an α_1-sequence a and an α_2-sequence b, we say $a \sim b$ if $\bigcap_{\substack{\beta_1 \in \alpha_1 \\ \beta_2 \in \alpha_2}} \bigcup_{\substack{\gamma_1 \geq \beta_1 \\ \gamma_2 \geq \beta_2}} [a_{\gamma_1}, a_{\gamma_1+1}, b_{\gamma_2}, b_{\gamma_2+1}]$ is at most one point.

Lemma 2.4.7. *Suppose that a is an α_1-sequence and b is an α_2-sequence in the toset T. Suppose also that* $\bigcap_{\substack{\beta_1\in\alpha_1\\ \beta_2\in\alpha_2}} \bigcup_{\substack{\gamma_1\geq\beta_1\\ \gamma_2\geq\beta_2}} [a_{\gamma_1}, a_{\gamma_1+1}, b_{\gamma_2}, b_{\gamma_2+1}] = \{t_0\}$. *Then $a_\gamma \to t_0$ and $b_\gamma \to t_0$.*

This follows from noticing that for every open interval (s,t) containing t_0, there are indices $\beta_1 \in \alpha_1$ and $\beta_2 \in \alpha_2$ such that $\left(\bigcup_{\substack{\gamma_1\geq\beta_1\\ \gamma_2\geq\beta_2}} [a_{\gamma_1}, b_{\gamma_2}] \right) \subset (s,t)$.

Lemma 2.4.8. *The relation $\sim$ defined on $\mathcal{C}(T,\alpha)$ in Definition 2.4.6 above is an equivalence relation.*

Proof. It is clearly symmetric and reflexive. We show that it is transitive. So, assume that $a, b, c \in \mathcal{C}(T,\alpha)$, that $a \sim b$, and $b \sim c$. Let α_1, α_2, and α_3 denote the domains on which a, b, and c (respectively) are defined.

We see that for all $\gamma_1 \in \alpha_1$, $\gamma_2 \in \alpha_2$, and $\gamma_3 \in \alpha_3$ $[a_{\gamma_1}, a_{\gamma_1+1}, c_{\gamma_3}, c_{\gamma_3+1}] \subset ([a_{\gamma_1}, a_{\gamma_1+1}, b_{\gamma_2}, b_{\gamma_2+1}] \cup [b_{\gamma_2}, b_{\gamma_2+1}, c_{\gamma_3}, c_{\gamma_3+1}])$ and therefore if we define

$$A = \bigcap_{\substack{\beta_1\in\alpha_1\\ \beta_3\in\alpha_3}} \bigcup_{\substack{\gamma_1\geq\beta_1\\ \gamma_3\geq\beta_3}} [a_{\gamma_1}, a_{\gamma_1+1}, c_{\gamma_3}, c_{\gamma_3+1}] \tag{2.8}$$

$$B = \bigcap_{\substack{\beta_1\in\alpha_1\\ \beta_2\in\alpha_2}} \bigcup_{\substack{\gamma_1\geq\beta_1\\ \gamma_2\geq\beta_2}} [a_{\gamma_1}, a_{\gamma_1+1}, b_{\gamma_2}, b_{\gamma_2+1}] \tag{2.9}$$

$$C = \bigcap_{\substack{\beta_2\in\alpha_2\\ \beta_3\in\alpha_3}} \bigcup_{\substack{\gamma_2\geq\beta_2\\ \gamma_3\geq\beta_3}} [b_{\gamma_2}, b_{\gamma_2+1}, c_{\gamma_3}, c_{\gamma_3+1}] \tag{2.10}$$

then we have $A \subset B \cup C$. If at least one of B, C is empty, then we are done since each can have at most one point. So, we may assume that $B = \{t_1\}$ and $C = \{t_2\}$. By Lemma 2.4.7, we see that $a_\gamma \to t_1$, $b_\gamma \to t_1$, $b_\gamma \to t_2$, and $c_\gamma \to t_2$. Since T is Hausdorff, sequences cannot converge to two different points, thus we must have that $t_1 = t_2$ since $b_\gamma \to t_1$ and $b_\gamma \to t_2$. Therefore, we see that $A = B = C = \{t_1\}$. And thus $a \sim c$. □

Definition 2.4.9. Let T be a toset, α a cardinal, and $\mathcal{C}(T,\alpha)$ as in Definition 2.4.6. Define the set $\hat{T}(\alpha) = \mathcal{C}(T,\alpha)/\sim$. Endow $\hat{T}(\alpha)$ with the order $\prec$ so that $[a] \prec [b]$ if there are $s, t \in T$ and $\beta_1 \in \alpha_1,\ \beta_2 \in \alpha_2$ such that $a_{\gamma_1} \leq s < t \leq b_{\gamma_2}$ for all $\gamma_1 \geq \beta_1,\ \gamma_2 \geq \beta_2$.

Lemma 2.4.10. *Suppose that a is a weak Cauchy α-sequence in the toset T. Further suppose there is $t \in T$ and that there is a cofinal set $A \subset \alpha$ for which $a_\gamma \le t$ for all $\gamma \in A$. Then for any weakly Cauchy α'-sequence a' with $a \sim a'$ and any $t' > t$, there is an index $\beta \in \alpha'$ such that $a'_\gamma < t'$ for all $\gamma \ge \beta$.*

Proof. The conclusion may be restated as follows: For any $t' > t$, there is no cofinal set $B \subset \alpha'$ for which $a'_\gamma \ge t'$ for all $\gamma \in B$. This is what we will show.

Suppose for contradiction that there is such a set. Then we see that for any $\beta \in \alpha, \beta' \in \alpha'$ we have $[t, t'] \subset \bigcup_{\substack{\gamma \ge \beta \\ \gamma' \ge \beta'}} [a_\gamma, a_{\gamma+1}, a'_{\gamma'}, a'_{\gamma'+1}]$, and hence $t, t' \in \bigcap_{\substack{\beta \in \alpha \\ \beta' \in \alpha'}} \bigcup_{\substack{\gamma \ge \beta \\ \gamma' \ge \beta'}} [a_\gamma, a_{\gamma+1}, a'_{\gamma'+1}]$, which contradicts that this set has at most one point. Thus no such cofinal set exists. □

Theorem 2.4.11. *Let T be a toset and α a cardinal. Let $\hat{T}(\alpha)$ be as in Definition 2.4.9. The relation $\prec$ defined on $\hat{T}(\alpha)$ is a total order.*

Proof. The fact that $\prec$ is well-defined (that is, that it is not dependent on which representative of each equivalence class is chosen) follows immediately from Lemma 2.4.10.

It is easily seen that $\prec$ is anti-symmetric and transitive. So we show trichotomy. It is clear that at most one of $[a] \prec [b]$ and $[b] \prec [a]$ is true. Suppose that $[a] = [b]$. In other words, $a \sim b$. Then, we see that given any two points $s < t$ there are indices $\beta_1 \in \alpha_1, \beta_3 \in \alpha_3$ (using the same convention as above) such that $\bigcup_{\substack{\gamma_1 \ge \beta_1 \\ \gamma_3 \ge \beta_3}} [a_{\gamma_1}, b_{\gamma_3}]$ contains at most one of $\{s, t\}$. Hence, $[a] \prec [b]$ and $[b] \prec [a]$ are each impossible.

Now suppose that $[a] \ne [b]$. This means that there are at least two points in the set $\bigcap_{\substack{\beta_1 \in \alpha_1 \\ \beta_3 \in \alpha_3}} \bigcup_{\substack{\gamma_1 \ge \beta_1 \\ \gamma_3 \ge \beta_3}} [a_{\gamma_1}, a_{\gamma_1+1}, b_{\gamma_1}, b_{\gamma_1+1}]$, call them s and t, with $s < t$. This means that for some cofinal sets $X \subset \alpha_1$, $Y \subset \alpha_3$ of indices, either $a_\gamma \le s$ for all $\gamma \in X$ or $b_\gamma \le s$ for all $\gamma \in Y$. Without loss of generality, assume the former. This means that there is another cofinal set $\{\gamma'_\delta\}$ such that $t \le b_{\gamma'_\delta}$ for all δ. It then follows from Lemma 2.4.10 that $[a] \prec [b]$. □

Definition 2.4.12. Let X be a topological space. Given a cardinal α, we say that X is *α-separable* if it has a dense subset A with $|A| \le \alpha$.

Theorem 2.4.13. *Let T be a toset. Given any cardinal α, there is a canonical order embedding $i : T \to \hat{T}(\alpha)$, with $i(T)$ topologically dense in $\hat{T}(\alpha)$. If T is α-separable, then $\hat{T}(\alpha)$ is order complete.*

Proof. Define $i(t)$ to be the equivalence class represented by the constant α-sequence (t). This is clearly an order-embedding, since given points $s < t$ in T, we have $[(s)] \prec [(t)]$.

To see that $i(T)$ is topologically dense in $\hat{T}(\alpha)$, let a and b be α_1- and α_2-sequences in T such that $[(a)] \prec [(b)]$. Thus, there are $s < t$ in T and indices $\beta_1 \in \alpha_1$, $\beta_2 \in \alpha_2$ such that $a_{\gamma_1} \leq s < t \leq b_{\gamma_2}$ for all $\gamma_1 \geq \beta_1$, $\gamma_2 \geq \beta_2$. It follows that $[(a)] \preceq [(s)] \prec [(t)] \preceq [(b)]$. Then the result follows from the observation that $[(s)] = i(s)$ and $[(t)] = i(t)$.

Now suppose that T is α-separable. Let $A \subset \hat{T}(\alpha)$ be bounded above. We show that A has a supremum. Let b be an upper bound of A. If there is no point $t \in i(T)$ such that $A \leq t \leq b$, then it follows that b is the supremum of A, since $i(T)$ is dense in $\hat{T}(\alpha)$. So, we may assume that such t exists. Let $t_0 = i^{-1}(t)$.

Let $A^+ = \{t \in T \mid A \leq i(t)\}$. Since T is α-separable, there is $S \subset T$ which is dense with $|S| \leq \alpha$. Index the elements of S with α. We define the sequence a as follows. Let $a_0 = t_0$. Then, suppose that β is such that a_γ is defined for all $\gamma < \beta$. we define a_β to be the element of the set $S \cap A^+ \cap \bigcap_{\gamma<\beta} (-\infty, a_\gamma]$ which has minimal index. If this intersection is empty, then terminate the sequence. Then a is a weak Cauchy sequence since it is monotone.

We show that $[a]$ is an upper bound on A. Suppose that there is $[a'] \in A$ with $[a] \prec [a']$. Then, since $i(T)$ is dense in $\hat{T}(\alpha)$, there are $s, t \in T$ such that $[a] \preceq i(s) \prec i(t) \preceq [a']$. But it is seen in the definition of a that given any γ and any $x \notin A^+$, $x \leq a_\gamma$. Since $i(s), i(t) \leq [a'] \in A$, we see that $s, t \notin A^+$, hence $[a] \preceq i(s), i(t)$ is impossible. Therefore, no such a' exists. Hence, $[a]$ is an upper bound on A. It is seen to be the least upper bound by noting that it is dominated by every upper bound on A from the set $i(T)$, which is dense in $\hat{T}(\alpha)$. □

Corollary 2.4.14. *Let T be a toset that is order dense and α-separable. Then $\hat{T}(\alpha)$ is order-isomorphic to the Dedekind completion $\bar{T}$.*

Proof. By Theorem 2.4.13, we see that $i(T)$ is topologically dense in $\hat{T}(\alpha)$. Since T is order dense, $i(T)$ is also order dense. Then we conclude that $i(T)$ is order dense in $\hat{T}(\alpha)$ and therefore that $\hat{T}(\alpha)$ is order dense. □

It then follows immediately that if T is a dense toset then $\hat{T}(|T|) \cong \bar{T}$, since T is clearly $|T|$-dense.

Chapter 3

Big Fundamental Group

3.1 Cannon and Conner's Π_1

In this section, we give the definition of the "big fundamental group" (denoted by Π_1) that was given by Cannon and Conner in [1]. Then we will give an alternate definition and discuss how the two are related.

Definition 3.1.1. A *big path* in a space X is a function $f : I \to X$, where I is a big interval. A *big loop* is a big path whose endpoints coincide. These terms will also be used to mean their images, respectively, in X.

Definition 3.1.2. Given continuous maps $f, g : X \to Y$, a *big homotopy from f to g* is a function $F : X \times J \to Y$, where J is a big interval, $F_{0_J} = f$, and $F_{1_J} = g$. In this case, we say that f and g are *big homotopic*. If there is $A \subset X$ such that $F(a, t) = F(a, 0_J)$ for all $a \in A$ and all $t \in J$ then we say that F is a *big homotopy rel A* and that f and g are *big homotopic rel A*.

Definition 3.1.3. A *big rectangle* is a product $I \times J$ where I and J are both big intervals. A *big path homotopy* between big paths $f, g : I \to X$ is a big homotopy $F : I \times J \to X$ for some big interval J rel $\{0_I, 1_I\}$.

Definition 3.1.4. Given a toset T, a *monotone decomposition* is an equivalence relation $\sim$ on T such that each equivalence class is either a single point or a closed interval.

It will be noted here that a monotone decomposition is equivalent to an order-preserving map in the following sense.

Proposition 3.1.5. *If I and J are big intervals and $f : I \to J$ is a surjective order-preserving map, then the equivalence relation $x \sim y$ on I given by $x \sim y$ if $f(x) = f(y)$ is a monotone decomposition.*

In other words, for every point $a \in J$, we have $f^{-1}(a)$ is either a single point or a closed interval. The proof is relatively simple, and we give a brief sketch here. In general, if $f : I \to J$ is an order-preserving map (for just arbitrary tosets I, J), point-inverses are intervals—possibly open, closed, or half-open (or degenerate, in the case of a single point). Also, order-preserving maps also preserve supremums, whenever they exist. Since they always exist in big intervals, the result follows.

Corollary 3.1.6. *Given a big interval I and a monotone decomposition $\sim$, the order topology on $I/\sim$ coincides with the quotient topology.*

Suppose that $f : I \to X$ and $g : J \to X$ are big paths. Suppose further that there is a monotone decomposition $\sim$ on I such that f is constant on each equivalence class. Further suppose that $J \cong I/\sim$ and that the induced map $\tilde{f}$ equals g. (Equivalently, suppose there is a surjective order-map $h : I \to J$ so that $f = g \circ h$.) Then we will identify f and g as if they were the same path.

Along the same line, if $f : I \to X$ and $g : J \to X$ are big paths, we say they are equivalent if there is a big homotopy $H : I' \times J' \to X$ such that there are monotone decompositions $p_1 : I' \to I$ and $p_2 : I' \to J$ such that $p_1 \circ (H|_{I' \times 0_{J'}}) = f$ and $p_2 \circ (H|_{I' \times 1_{J'}}) = g$. It is seen in [1] that this is an equivalence relation. In this paper, we abuse terminology by calling the equivalence classes "big homotopy classes". This terminology is not strictly correct since by the definition of big homotopy, two maps with different domains cannot be big homotopic.

Theorem 3.1.7. *Let I and J be big intervals. The following are equivalent.*

1. *There is an order embedding $f : I \to J$.*

2. *There is A dense in I and an order embedding $g : A \to J$.*

3. There is an order surjection $h : J \to I$.

Proof. Condition 2 follows automatically from condition 1. We suppose that 2 holds and show that 3 also holds. Define h as follows. For any point $j \in J$ such that $g^{-1}([0_J, j]) = \emptyset$, define $h(j) = 0_I$. Otherwise, define $h(j) = \sup g^{-1}([0_J, j])$. This is well-defined since any nonempty set in I is bounded above (by 1_I) and therefore has a supremum. It is clearly order-preserving. To see that it is surjective, let $i \in I$ be given. Then we see that since A is dense in I, we have that $i = \sup\{a \in A \mid a \leq i\}$. Therefore, for $j = \sup\{g(a) \mid a \in A,\ a \leq i\}$ we see that $h(j) = i$.

Finally, suppose that 3 holds. Define the map $f : I \to J$ by $f(i) = \sup h^{-1}(i)$. Since h is surjective, $h^{-1}(i)$ is always nonempty so f is well-defined. Now we show that f is strictly order-preserving. Let $a < b$ in I. Since h is order-preserving, we see that the pre-image of a point is always a closed interval (possibly degenerate). Thus $h^{-1}(a) = [s, t]$ and $h^{-1}(b) = [s', t']$ for some $s, s', t, t' \in J$. Since $a \neq b$, we see that $[s, t] \cap [s', t'] = \emptyset$. Thus, we must have that $t' < s' \leq t'$, and by definition $f(a) = t$ and $f(b) = t'$. Thus f is strictly order preserving. □

Definition 3.1.8. Given a pointed topological space (X, x_0), define $\Pi_1(X, x_0)$ to be the set of big path homotopy classes of big loops based at x_0. We call this set the *big fundamental group of* X. Endowed with the multiplication $[\lambda] * [\mu] = [\lambda * \mu]$ this set becomes a group, with inverse operation $[\lambda]^{-1} = [\lambda^{\mathrm{op}}]$ and trivial element represented by the inclusion map $\{x_0\} \hookrightarrow X$.

It is seen in [1, Theorem 4.20] that if X is Hausdorff then $\Pi_1(X, x_0)$ is a set and therefore also a group. In fact, examining the proof of that result, we see that what was proven is the following statement. Given a space X, there is a cardinal α (depending on X) such that for any continuous map $f : I \times C \to X$ (with I a big interval and C compact Hausdorff) there is a map $f' : I' \times C \to X$ with $|I'| \leq \alpha$ and an order-preserving surjection $p : I \to I'$ such that if $f(t, u) \neq f'(p(t), u)$ for some $t \in I$, $u \in C$, then $f(t, u)$ and $f'(p(t), u)$ are related with the relation $\sim$ as defined below. The following theory will then conclude that $\Pi_1(X, x_0)$ is always a set (rather than a proper class) and therefore always a group.

Definition 3.1.9. Let X be a topological space. Given $x \in X$, define N_x to be the intersection of every open neighborhood of x. That is, $N_x = \bigcap_{\substack{U \ni x \\ U \text{ open}}}$. Then given $y \in X$, we say that $x \sim y$ if $y \in N_x$.

Note that the statement that X is a T_1 space is equivalent to saying that $N_x = \{x\}$ for all $x \in X$. Thus, using the same notation as in the discussion above, $f(t, u) \neq f'(p(t), u)$ is impossible, meaning that $f(\cdot, u)$ is constant on the pre-image of any point under p, hence f factors through the quotient p.

The following theorem is adapted from [1, 4.20]. The proof is omitted since it is the same as found in that paper.

Theorem 3.1.10. *Let X be a non-empty topological space. Then there is a cardinal $\alpha(X)$ with the following property. Suppose I is a big interval, C is a compact Hausdorff space, and $f : I \times C \to X$ is continuous. Then there is an order epimorphism $p : I \to I'$ with $|I'| \leq \alpha(X)$ and a function $f' : I' \times C \to X$ such that $f(t, u) \sim f'(p(t), u)$ for all $(t, u) \in I \times C$.*

Proposition 3.1.11. *Suppose that $f, g : X \to Y$ are continuous and that for all $x \in X$ such that $f(x) \neq g(x)$, we have that $f(x) \sim g(x)$ in Y. Then f and g are (classically) homotopic.*

Proof. We define the homotopy $F : X \times I \to Y$ in the seemingly naive way by

$$F(x, t) = \begin{cases} f(x) & t \leq \frac{1}{2} \\ g(x) & t > \frac{1}{2} \end{cases}. \tag{3.1}$$

Now we show that F is indeed continuous (every other requirement for it to be a homotopy from f to g is trivial). We proceed by showing that it is continuous at each point of the domain, so assume that $(x, t) \in X \times I$ and $U \subset Y$ an open neighborhood of $F(x, t)$ are given. If $t \neq \frac{1}{2}$, then F is clearly continuous at (x, t), since it is (on some open neighborhood of (x, t)) either $f \times \mathrm{id}_I$ or $g \times \mathrm{id}_I$. So, we need only consider the case $t = \frac{1}{2}$.

We see that $f(x) \in U$ by hypothesis. Also by hypothesis, we know that $g(x) \in N_{f(x)}$. Therefore, it follows that $g(x) \in U$ since it is contained in every neighborhood of $f(x)$. Since f and g are continuous, there are V_1 and V_2 open neighborhoods of x in X such that $f(V_1) \subset$

U and $g(V_2) \subset U$. Then define $V = (V_1 \cap V_2) \times I$. We see that $F(V \times I) \subset f(V) \cup g(V) \subset U$, as desired. □

This isn't quite sufficient yet. In fact, we need to extend $\sim$ to an equivalence relation. However, since homotopy is an equivalence relation, it is seen that we can do so without violating the result stated in the above proposition. With this information, and the proof found in [1], we now have the following result.

Theorem 3.1.12. *For any pointed space (X, x_0), the big fundamental group $\Pi_1(X, x_0)$ is a set (not a proper class) and therefore also a group.*

3.2 General Big Fundamental Groups

In this section, we start to discuss alternate definitions to the big fundamental group defined by Cannon and Conner. One such way is by specifying which big interval(s) will be allowed to be the domain of a "big path" and which big rectangle(s) will be allowed to be the domain of a "big path homotopy". The advantage here is being able to specify precisely which kinds of big paths and which kinds of big homotopies will be allowed. The disadvantage is that it may miss some things that could be picked up by Π_1 as defined by Cannon-Conner. This method requires much discussion about collections of big intervals, since this terminology is cumbersome, we affectionately call such a collection a "tapestry", since it can be thought of as a bunch of strings of various lengths.

Definition 3.2.1. A collection $\mathcal{I}$ of big intervals is called a *tapestry*. Let **Tap** denote the category whose objects are tapestries. Given tapestries $\mathcal{I}, \mathcal{J}$, a morphism between them, called a *tapestry map*, is a function $\mathcal{F} : \mathcal{I} \to \mathcal{J}$ together with order-preserving functions $f_I : I \to \mathcal{F}(I)$ for all $I \in \mathcal{I}$.

Note that in a tapestry $\mathcal{I}$ there is no assumed order on $\mathcal{I}$ but each element $I \in \mathcal{I}$ is a big interval and therefore has an assumed total order, which may be denoted $\leq_I$. It is seen that in **Tap**, an epimorphism is a surjective function $\mathcal{F} : \mathcal{I} \to \mathcal{J}$ such that f_I is surjective for all $I \in \mathcal{I}$. Similarly, $\mathcal{F}$ is a monomorphism if it is injective and f_I is injective for all I. A morphism $\mathcal{F}:\mathcal{I}\to\mathcal{J}$ will be called *weakly monomorphic* or a *weak monomorphism* if each f_I is injective, but not necessarily $\mathcal{F}$ itself. Also note that by Theorem 3.1.7, existence of a weak monomorphism $\mathcal{F}:\mathcal{I}\to\mathcal{J}$ is equivalent to a collection of order-preserving epimorphisms $f_I:\mathcal{F}(I)\to I$ for all $I\in\mathcal{I}$.

So, suppose $\mathcal{I}$ is a collection of big intervals. It would be insufficient to say that this collection would then become the set of legal domains for what we will consider to be a "big path", since it may or may not be closed under the two operations of concatenation and reversal, which we require them to be for turning the set of homotopy classes into a monoid.

Definition 3.2.2. Given a tapestry $\mathcal{I}$, define $\bar{\mathcal{I}}$ as the collection of all sets of the form $J_1 * J_2 * \cdots * J_n$ where $n\geq 1$ is any natural number, $I_1,\dots,I_n\in\mathcal{I}$, and J_i is either I_i or I_i^{op}. We call this set the *weave of* $\mathcal{I}$.

Proposition 3.2.3. *Given a tapestry morphism $\mathcal{F}:\mathcal{I}\to\mathcal{J}$, there is a canonical morphism $\bar{\mathcal{F}}:\bar{\mathcal{I}}\to\bar{\mathcal{J}}$.*

This is merely the observation that $\mathcal{I}$ and $\mathcal{J}$ serve as a (monoid) generating set for $\bar{\mathcal{I}}$ and $\bar{\mathcal{J}}$ respectively, and that the morphism $\mathcal{I}\to\mathcal{J}$ can be extended to be a monoid homomorphism on their weaves $\bar{\mathcal{I}}\to\bar{\mathcal{J}}$.

Definition 3.2.4. Given two tapestries $\mathcal{I}$ and $\mathcal{J}$ and a pointed space (X,x_0), define $\pi_1^{(\mathcal{I},\mathcal{J})}(X,x_0)$ to be the set of $(\mathcal{I},\mathcal{J})$-homotopy classes of $\mathcal{I}$-loops where an $\mathcal{I}$*-loop* is a big loop whose domain is a member of $\bar{\mathcal{I}}$ and an $(\mathcal{I},\mathcal{J})$*-homotopy* is a big path homotopy whose domain is of the form $I\times J$ where $I\in\bar{\mathcal{I}}$ and $J\in\bar{\mathcal{J}}$. Just as before, endow $\pi_1^{(\mathcal{I},\mathcal{J})}(X,x_0)$ with multiplication $*$ given by $[f]*[g]=[f*g]$ and $[f]^{\mathrm{op}}=[f^{\mathrm{op}}]$ to make it a monoid.

This definition is handy because it allows one to specify precisely which types of big paths will be allowed and also (independently) which types of big homotopies will be allowed. For example, we could let $\mathcal{I}$ consist of just the closed long line and $\mathcal{J}$ just the real interval $[0,1]$. Recall that the closed long line is the one-point compactification of the set $\omega_1\times[0,1)$ with the lexicographical order, where ω_1 is the first uncountable ordinal. Then, in this monoid, paths would be maps from a finite number of concatenations of the long line, together with its reverse, and two paths would represent the same homotopy class if they are classically homotopic maps.

Unfortunately, this is not always a group. As an example, consider the space X to be the long line with its endpoints identified, which we will call the *long circle*, and let x_0 be the point where the identification was made. Consider the path $f : I \to X$, where I denotes the long line, to wrap one time around the long circle (in the one direction in which this is possible, and without backtracking). Then we see that the reverse $f^{\mathrm{op}} : I^{\mathrm{op}} \to X$, where it wraps in the opposite direction around X, is not an inverse to f since neither concatenation $f * f^{\mathrm{op}}$ nor $f^{\mathrm{op}} * f$ is homotopic to the constant map. Interestingly enough, the concatenation $f^{\mathrm{op}} * f$ can be homotoped "arbitrarily close" to the constant map, but not all the way. The interval $[0,1]$ is simply too small to perform the full homotopy. However, $\pi_1^{(\mathcal{I},\mathcal{I})}(X, x_0) \cong \mathbb{Z}$. Also, it is noted that $\pi_1(X, x_0)$—the classical fundamental group—is trivial, since it is not possible for a classical path to wrap all the way around the long circle.

Theorem 3.2.5. *Let $\mathcal{I}$ and $\mathcal{J}$ be tapestries with a weak monomorphism $\mathcal{F} : \mathcal{I} \to \mathcal{J}$. Then for any space (X, x_0), we have that $\pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$ is a group.*

Proof. We never actually justified that $\pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$ is a monoid, so first we will prove that it satisfies all of the group axioms.

First, we show that the operation $*$ is well-defined. Suppose that $\lambda \sim \lambda'$ and $\mu \sim \mu'$. Let $F : I \times J \to X$ be a homotopy from λ to λ' and $G : K \times L \to X$ be a homotopy from μ to μ'. Then we see that $F * G : (I \vee K) \times (J \vee L) \to X$ given by

$$F * G(s,t) = \begin{cases} F(s,t) & s \in I, t \in J \\ G(s,t) & s \in K, t \in L \\ x_0 & \text{otherwise} \end{cases} \tag{3.2}$$

is a homotopy from $\lambda * \mu$ to $\lambda' * \mu'$. Thus $[\lambda] * [\mu] = [\lambda * \mu] = [\lambda' * \mu'] = [\lambda'] * [\mu']$.

Next we show associativity. In fact, concatenation for tosets is associative. That is, for tosets S, T, U, we have $S * (T * U) = (S * T) * U$. Therefore for $\lambda : S \to X$, $\mu : T \to X$,

$\nu : U \to X$, with $\lambda(1_S) = \mu(0_T)$ and $\mu(1_T) = \nu(0_U)$, we have that $\lambda * (\mu * \nu) = (\lambda * \mu) * \nu$, hence $[\lambda] * ([\mu] * [\nu]) = ([\lambda] * [\mu]) * [\nu]$.

It is easily seen that the identity is represented by a constant function. Let $\lambda : \{x_0\} \to X$ be inclusion. Then for any other big path $\mu : J \to X$, we see that $\mu * \lambda = \lambda * \mu = \mu$, hence $[\lambda] * [\mu] = [\lambda * \mu] = [\mu]$, and similarly for $[\mu] * [\lambda]$, hence $[\lambda]$ is the identity.

Finally, we show that given $\lambda : I \to X$, the reverse λ^{op} represents the inverse of $[\lambda]$ in $\pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$. This is the part where we require the additional hypothesis, that there is a weak monomorphism $\mathcal{F} : \mathcal{I} \to \mathcal{J}$. Since there is a monomorphism $f_I : I \to \mathcal{F}(I)$, we see that there is an order-preserving surjection $p : J \to I$, where $J = \mathcal{F}(I)$. We define the homotopy $H : (I * I^{\text{op}}) \times J \to X$ by

$$H(s,t) = \begin{cases} \lambda(s) & s \in I, s < p(t) \\ \lambda^{\text{op}}(s) & s = x^{\text{op}} \in I^{\text{op}}, x < p(t) \\ \lambda(t) & \text{otherwise} \end{cases} \tag{3.3}$$

.

In other words, what this homotopy does is at time $t = 0_J$, this is the constant map $\lambda(0_I) = x_0$. Then at time $t \in J$, it is the path that goes along λ until $s = p(t)$, then stays at $\lambda(p(t))$ until $s = p(t)^{\text{op}}$ and then reverses back along the part of λ that was traversed earlier. Hence, at time $t = 1_I$, this is the concatenation $\lambda * \lambda^{\text{op}}$.

We now need to show that H is continuous. We do this using the pasting lemma. Define the sets A, B, and C by

$$A = \{(a,b) \in I \times J \mid a \leq p(b)\} \tag{3.4}$$

$$B = \{(a^{\text{op}},b) \in I^{\text{op}} \times J \mid a \leq p(b)\} \tag{3.5}$$

$$C = \{(a,b) \in I \times J \mid p(b) \leq a\} \cup \{(a^{\text{op}},b) \in I^{\text{op}} \times J \mid p(b) \leq a\}. \tag{3.6}$$

Then we see that $(I * I^{\text{op}}) \times J = A \cup B \cup C$, also each of A, B, C is closed in $(I * I^{\text{op}}) \times I$. Now we define $F : I \times J \to X$ by $F(s,t) = \lambda(s)$, $F^{\text{op}} : I^{\text{op}} \times J \to X$ by $F^{\text{op}}(s,t) = \lambda^{\text{op}}(s)$, and $G : (I * I^{\text{op}}) \times J \to X$ by $G(s,t) = \lambda(p(t))$. It is easily seen that each of F, F^{op}, and G

is continuous. Also, we note that $F|_A = H|_A$, $F^{\mathrm{op}}|_B = H|_B$, and $G|_C = H|_C$. Hence, H is continuous by the pasting lemma. A similar proof shows that $\lambda^{\mathrm{op}} * \lambda$ is also nul-homotopic, and thus $[\lambda]^{-1} = [\lambda^{\mathrm{op}}]$, so $\pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$ is indeed a group. □

Another useful theorem will answer the question of when we can compare two of these big fundamental groups. To give some intuition, imagine that there's a "big path" between two points in a space. It would make sense that any "bigger path" would also be able to connect those two points, since there's merely "more stuff" in the bigger path. In fact, this is the case. Also, if there is a "big path homotopy" between two big paths, then a "bigger homotopy" can also connect those two big paths. Now we formalize this.

Remark 3.2.6. Suppose that I and J are big intervals and that there is a surjection $p : I \to J$. Also suppose there is a big path $\lambda : J \to X$. Then the composition $\lambda \circ p : I \to X$ is a big path and is homotopic to λ.

Corollary 3.2.7. *Suppose that I and J are big intervals, that J embeds (order-preserving, not necessarily continuously) into I, and that $\lambda : J \to X$ is a big path. Then there is a big path $\tilde{\lambda} : I \to X$.*

This follows immediately from Theorem 3.1.7, and the remark above.

Theorem 3.2.8. *Suppose that $\mathcal{F} : \mathcal{I} \to \mathcal{I}'$ and $\mathcal{G} : \mathcal{J} \to \mathcal{J}'$ are weak monomoprhisms. Then there is a canonical homomorphism $\varphi : \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0) \to \pi_1^{(\mathcal{I}',\mathcal{J}')}(X, x_0)$. If there is a weak monomorphism $\mathcal{F}' : \mathcal{I}' \to \mathcal{I}$ then φ is surjective. If there is a weak monomorphism $\mathcal{G}' : \mathcal{J}' \to \mathcal{J}$ then φ is injective.*

Proof. By Proposition 3.2.3, it suffices to show that there is a well-defined map taking any big path whose domain is an element of $\mathcal{I}$ to one whose domain is an element of $\mathcal{I}'$ that respects homotopy classes.

Let $\lambda : I \to X$ be a big path. Then let $f_I : I \to \mathcal{F}(I)$ be an embedding and $p_I : \mathcal{F}(I) \to I$ be an epimorphism guaranteed by Theorem 3.1.7. Define $\tilde{\lambda} = \lambda \circ p_I$. Let $[\lambda]$ denote the equivalence class of λ in $\pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$ and $[\tilde{\lambda}]'$ the class of $\tilde{\lambda}$ in $\pi_1^{(\mathcal{I}',\mathcal{J}')}(X, x_0)$. Then we define $\varphi([\lambda]) = [\tilde{\lambda}]'$. To see that this map is well-defined, suppose that $\lambda : I \to X$ and

$\mu : I \to X$ are $(\mathcal{I}, \mathcal{J})$-homotopic. Let $F : I \times J \to X$ be a big path homotopy from λ to μ. Let $g_J : J \to \mathcal{G}(J)$ be an embedding and $q_J : \mathcal{G}(J) \to J$ an epimorphism. Then we see that the homotopy $H : \mathcal{F}(I) \times \mathcal{G}(J) \to X$ given by $H(s,t) = F(p_I(s), q_J(t))$ is a homotopy between $\tilde{\lambda}$ and $\tilde{\mu}$. Thus $[\tilde{\lambda}]' = [\tilde{\mu}]'$, hence φ is well-defined.

Now suppose there is a weak monomorphism $\mathcal{F}' : \mathcal{I}' \to \mathcal{I}$. Let $\lambda : I' \to X$ be a big path. Since there is a monomorphism $f_{I'} : I' \to I$, there is an epimorphism $p_{I'} : I \to I'$. Thus $\lambda \circ p : I \to X$ is a big path, and it is seen that $[\varphi(\lambda \circ p)]' = [\lambda]'$. Hence φ is surjective.

Finally, suppose that there is a weak monomorphism $\mathcal{G}' : \mathcal{J}' \to \mathcal{J}$. By the argument above, we see that this implies that $\lambda : I' \to X$ and $\mu : I' \to X$ are $(\mathcal{I}, \mathcal{J})$-homotopic if they are $(\mathcal{I}', \mathcal{J}')$-homotopic. Thus, φ is injective. □

Corollary 3.2.9. *Suppose that there are weak monomorphisms $\mathcal{F} : \mathcal{I} \to \mathcal{I}'$, $\mathcal{F}' : \mathcal{I}' \to \mathcal{I}$, $\mathcal{G} : \mathcal{J} \to \mathcal{J}'$, and $\mathcal{G}' : \mathcal{J}' \to \mathcal{J}$. Then for any pointed space (X, x_0), we have that $\pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0) \cong \pi_1^{(\mathcal{I}',\mathcal{J}')}(X, x_0)$.*

Now we have the following result comparing these generalized big fundamental groups with Cannon-Conner's big fundamental group Π_1.

Theorem 3.2.10. *Given any pointed topological space (X, x_0), there is a tapestry $\mathcal{I} = \mathcal{I}(X)$ such that $\pi_1^{(\mathcal{I},\mathcal{I})}(X, x_0) \cong \Pi_1(X, x_0)$.*

Proof. From Theorem 3.1.10, we see that there is a cardinal α such that any big path $\lambda : J \to X$ is big homotopic to a big path $f' : I' \to X$ with $|I'| \leq \alpha$. Let $\mathcal{I}$ be a collection of big intervals consisting of exactly one big interval of each isomorphism type where each element $I \in \mathcal{I}$ has cardinality at most α. Since there are (up to order isomorphism) at most 2^α distinct big intervals of cardinality α, we see that $\mathcal{I}$ is a set. Then we see that every big path has a representative in $\pi_1^{(\mathcal{I},\mathcal{I})}(X, x_0)$. Also, two big paths are big homotopic exactly when they are $\mathcal{I}$-homotopic. □

Corollary 3.2.11. *If X and $\mathcal{I} = \mathcal{I}(X)$ are as above and there is a weak monomorphism $\mathcal{F} : \mathcal{I} \to \mathcal{J}$, then $\Pi_1(X, x_0) \cong \pi_1^{(\mathcal{J},\mathcal{J})}(X, x_0)$.*

3.3 Classical Results

In this section, we discuss many of the results from classical homotopy theory and how they carry over into the theory of big homotopy groups. Emma Turner [6] wrote about some results such as covering theory and the Seifert-VanKampen theorem. Other results such as those concerning retractions and deformation retractions extend very well without any major alteration to the original proofs. For each result in this section, we assume that the tapestries $\mathcal{I}$ and $\mathcal{J}$ are fixed and that there is a weak monomorphism $\mathcal{I} \to \mathcal{J}$, so that $\pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$ is guaranteed to be a group.

Definition 3.3.1. Given a continuous function between pointed spaces $f : (X, x_0) \to (Y, y_0)$, the *induced map on* $\pi_1^{(\mathcal{I},\mathcal{J})}$, denoted $f_* : \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0) \to \pi_1^{(\mathcal{I},\mathcal{J})}(Y, y_0)$, is given by $f_*([\lambda]) = [f \circ \lambda]$.

Theorem 3.3.2. *The operator* $\pi_1^{(\mathcal{I},\mathcal{J})} : \mathbf{Top}^* \to \mathbf{Grp}$ *is a covariant functor.*

Proof. We saw in Theorem 3.2.5 that for any pointed space (X, x_0), the set $\pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$ is a group. (Recall that for this section, we are always assuming that there is a weak monomorphism $\mathcal{I} \to \mathcal{J}$.)

Now, we show that for any continuous map $f : (X, x_0) \to (Y, y_0)$, the induced map f_* is a homomorphism. That is, for $[\lambda], [\mu] \in \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$, we have $f_*([\lambda] * [\mu]) = f_*([\lambda * \mu]) = [f \circ (\lambda * \mu)] = [(f \circ \lambda) * (f \circ \mu)] = [f \circ \lambda] * [f \circ \mu] = f_*([\lambda]) * f_*([\mu])$.

It is easily seen that if $i : X \to X$ is the identity, then i_* is as well. Now, given $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$, we see that $(g \circ f)_*([\lambda]) = [g \circ f \circ \lambda] = g_*([f \circ \lambda]) = g_*(f_*([\lambda])) = (g_* \circ f_*)([\lambda])$, hence $(g \circ f)_* = g_* \circ f_*$. Thus, $\pi_1^{(\mathcal{I},\mathcal{J})}$ is a functor. □

Theorem 3.3.3. *Suppose that there are weak monomorphisms* $\mathcal{F} : \mathcal{I} \to \mathcal{I}'$ *and* $\mathcal{G} : \mathcal{J} \to \mathcal{J}'$*. Then the homomorphisms* $\varphi : \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0) \to \pi_1^{(\mathcal{I}',\mathcal{J}')}(X, x_0)$ *guaranteed by Theorem 3.2.8 give a natural transformation from* $\pi_1^{(\mathcal{I},\mathcal{J})}$ *to* $\pi_1^{(\mathcal{I}',\mathcal{J}')}$*.*

Proof. We seek to show that the following diagram commutes.

$$\begin{array}{ccc} \pi_1^{(\mathcal{I},\mathcal{J})}(X,x_0) & \xrightarrow{f_*} & \pi_1^{(\mathcal{I},\mathcal{J})}(Y,y_0) \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \varphi_Y} \\ \pi_1^{(\mathcal{I}',\mathcal{J}')}(X,x_0) & \xrightarrow{f_*} & \pi_1^{(\mathcal{I}',\mathcal{J}')}(Y,y_0) \end{array} \tag{3.7}$$

So, given a big path $\lambda : I \to X$ with $I \in \mathcal{I}$, we calculate $f_* \circ \varphi_X([\lambda]) = [f \circ \tilde{\lambda}]$ where there is an order-preserving surjection $p : I' \to I$ and $\tilde{\lambda} : I' \to X$ is such that $\lambda \circ p = \tilde{\lambda}$. That is, $f_* \circ \varphi_X([\lambda]) = [f \circ \lambda \circ p]$. Now we calculate $\varphi_Y \circ f_*([\lambda]) = \varphi_Y([f \circ \lambda]) = [f \circ \lambda \circ p]$, as desired. □

Consider the class of all pairs of tapestries $(\mathcal{I},\mathcal{J})$ such that there is a weak monomorphism $\mathcal{I} \to \mathcal{J}$. Given any two pairs $(\mathcal{I},\mathcal{J})$ and $(\mathcal{I}',\mathcal{J}')$, let $\varphi_{(\mathcal{I},\mathcal{J})}^{(\mathcal{I}',\mathcal{J}')} : \pi_1^{(\mathcal{I},\mathcal{J})} \to \pi_1^{(\mathcal{I}',\mathcal{J}')}$ be the natural transformation as above. Then this class is directed in the sense that given any two pairs $(\mathcal{I},\mathcal{J})$ and $(\mathcal{I}',\mathcal{J}')$, there is a pair, namely $(\mathcal{I} \cup \mathcal{I}', \mathcal{J} \cup \mathcal{J}')$ such that the natural transformations $\varphi_{(\mathcal{I},\mathcal{J})}^{(\mathcal{I}\cup\mathcal{I}',\mathcal{J}\cup\mathcal{J}')}$ and $\varphi_{(\mathcal{I}',\mathcal{J}')}^{(\mathcal{I}\cup\mathcal{I}',\mathcal{J}\cup\mathcal{J}')}$ both exist.

$$\begin{array}{ccc} \pi_1^{(\mathcal{I},\mathcal{J})} & & \\ & \searrow^{\varphi_{(\mathcal{I},\mathcal{J})}^{(\mathcal{I}\cup\mathcal{I}',\mathcal{J}\cup\mathcal{J}')}} & \\ & & \pi_1^{(\mathcal{I}\cup\mathcal{I}',\mathcal{J}\cup\mathcal{J}')} \\ & \nearrow_{\varphi_{(\mathcal{I}',\mathcal{J}')}^{(\mathcal{I}\cup\mathcal{I}',\mathcal{J}\cup\mathcal{J}')}} & \\ \pi_1^{(\mathcal{I}',\mathcal{J}')} & & \end{array} \tag{3.8}$$

We will call this system the *system of generalized fundamental groups.*

Definition 3.3.4. Suppose that **C** and **D** are categories. The *functor category* denoted $\mathbf{D}^{\mathbf{C}}$ is the category whose objects are functors $\mathbf{C} \to \mathbf{D}$ and whose morphisms are natural transformations.

Note that the proof of Theorem 3.2.8 can be slightly modified to show that for any tapestries $(\mathcal{I},\mathcal{J})$, there is a homomorphism $\varphi_{(\mathcal{I},\mathcal{J})} : \pi_1^{(\mathcal{I},\mathcal{J})}(X,x_0) \to \Pi_1(X,x_0)$, and also the proof above may be slightly modified to show that this defines a natural transformation from $\pi_1^{(\mathcal{I},\mathcal{J})}$ to Π_1. Then we have the following result.

Theorem 3.3.5. *The functor Π_1 is the direct limit of the system of generalized fundamental groups described above.*

Proof. First we show that whenever there is a weak monomorphism $(\mathcal{I},\mathcal{J}) \to (\mathcal{I}',\mathcal{J}')$, and thus the homomorphism $\varphi_{(\mathcal{I},\mathcal{J})}^{(\mathcal{I}',\mathcal{J}')}$ exists, the following diagram commutes.

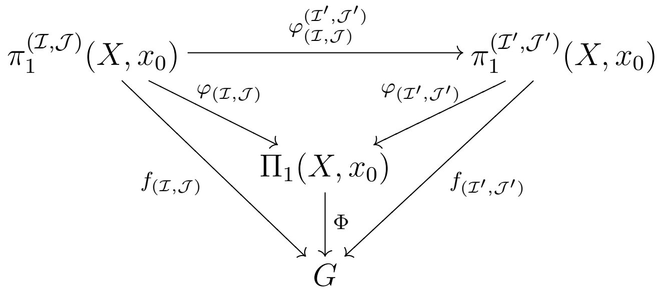
$$\begin{array}{ccc} \pi_1^{(\mathcal{I},\mathcal{J})} & \xrightarrow{\varphi_{(\mathcal{I},\mathcal{J})}^{(\mathcal{I}',\mathcal{J}')}} & \pi_1^{(\mathcal{I}',\mathcal{J}')} \\ & {\scriptstyle \varphi_{(\mathcal{I},\mathcal{J})}} \searrow \quad \swarrow {\scriptstyle \varphi_{(\mathcal{I}',\mathcal{J}')}} & \\ & \Pi_1 & \end{array} \tag{3.9}$$

Fix a pointed space (X, x_0). Let λ be an $\mathcal{I}$-loop in X. Let $[\lambda]_{\mathcal{J}}$ denote its $\mathcal{J}$-homotopy class, and similarly for $\mathcal{J}'$ and let $[\lambda]$ denote its big homotopy class (that is, its equivalence class in $\Pi_1(X, x_0)$). Then we see that $\varphi_{(\mathcal{I},\mathcal{J})}([\lambda]_{\mathcal{J}}) = [\lambda]$ and that $\varphi_{(\mathcal{I},\mathcal{J})}^{(\mathcal{I}',\mathcal{J}')}([\lambda]_{\mathcal{J}}) = [\lambda]_{\mathcal{J}'}$. Finally, we see that $\varphi_{(\mathcal{I}',\mathcal{J}')}([\lambda]_{\mathcal{J}'}) = [\lambda]$, as desired.

Now suppose that G is a group such that for any pair $(\mathcal{I},\mathcal{J})$ there is a homomorphism $f_{(\mathcal{I},\mathcal{J})} : \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0) \to G$ such that anytime there is a monomorphism $(\mathcal{I},\mathcal{J}) \to (\mathcal{I}',\mathcal{J}')$, the following diagram commutes.

$$\begin{array}{ccc} \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0) & \xrightarrow{\varphi_{(\mathcal{I},\mathcal{J})}^{(\mathcal{I}',\mathcal{J}')}} & \pi_1^{(\mathcal{I}',\mathcal{J}')}(X, x_0) \\ & {\scriptstyle f_{(\mathcal{I},\mathcal{J})}} \searrow \quad \swarrow {\scriptstyle f_{(\mathcal{I}',\mathcal{J}')}} & \\ & G & \end{array} \tag{3.10}$$

We define a map $\Phi : \Pi_1(X, x_0) \to G$ as follows. Let $\lambda : I \to X$ be a big loop in X. By Theorem 4.20 of [1], we see that there is a cardinal α such that any big homotopy $F : I \times J \to X$ of λ factors through a homotopy $F : I \times J' \to X$ with $|J'| \leq \alpha$. So, if we let $\mathcal{J}$ be a tapestry consisting of one big interval of each order type of cardinality at most α, then we see that $[\lambda]_{\mathcal{J}} = [\lambda]$. So we define $\mathcal{I} = \{I\}$ and $\Phi([\lambda]) = f_{(\mathcal{I},\mathcal{J})}([\lambda]_{\mathcal{J}})$. It follows that for any $(\mathcal{I}',\mathcal{J}')$ with weak monomorphisms $\mathcal{I} \to \mathcal{I}'$ and $\mathcal{J} \to \mathcal{J}'$, we have that $[\lambda]_{\mathcal{J}'} = [\lambda]_{\mathcal{J}} = [\lambda]$. Which means that $f_{(\mathcal{I},\mathcal{J})}([\lambda]_{\mathcal{J}}) = f_{(\mathcal{I}',\mathcal{J}')}([\lambda]_{\mathcal{J}'}) = \Phi([\lambda])$, hence the following diagram commutes, as desired.



(3.11)

□

Lemma 3.3.6. *If $r : (X, x_0) \to (A, x_0)$ is a retraction, then the induced map r_* is surjective and the map i_* induced by inclusion $i : A \to X$ is injective.*

Proof. We have that $r \circ i : A \to A$ is the identity, hence the induced map $(r \circ i)_* = r_* \circ i_*$ is as well. Thus, we must have r_* is surjective and i_* is injective. □

Definition 3.3.7. Given two functions $f : X \to Y$ and $g : X \to Y$, we say that f and g are *$\mathcal{J}$-homotopic* if there are $J \in \bar{\mathcal{J}}$ and a homotopy $F : X \times J \to Y$ such that $F|_{X\times\{0_J\}} = f$ and $F|_{X\times\{1_J\}} = g$. In this case, we write $f \sim g$ or if $\mathcal{J}$ is unclear from context, we may write $f \sim_{\mathcal{J}} g$.

Lemma 3.3.8. *If $f, g : (X, x_0) \to (Y, y_0)$ are $\mathcal{J}$-homotopic then the induced maps $f_*, g_* : \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0) \to \pi_1^{(\mathcal{I},\mathcal{J})}(Y, y_0)$ are equal.*

Proof. Let $F : X \times J \to Y$ be a homotopy from f to g. Given $[\lambda] \in \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$, we see that $G(x, t) = F(\lambda(x), t)$ is a homotopy from $f \circ \lambda$ to $g \circ \lambda$, hence $f_*([\lambda]) = [f \circ \lambda] = [g \circ \lambda] = g_*([\lambda])$. □

Definition 3.3.9. Let X be a space and $A \subset X$. A *$\mathcal{J}$-deformation retraction* from X to A is a map $r : X \times J \to X$, for some $J \in \bar{\mathcal{J}}$ if $r(a, t) \in A$ for all $a \in A$ and all $t \in J$ and $r(x, 1_J) \in A$ for all $x \in X$. In this case A is called a *$\mathcal{J}$-deformation retract* of X. If, in addition, $r(a, t) = a$ for all $a \in A$ and all $t \in J$ then r is called a *strong $\mathcal{J}$-deformation retraction*, and A is called a *strong $\mathcal{J}$-deformation retract*.

Lemma 3.3.10. *If $r : X \times J \to A$ is a strong $\mathcal{J}$-deformation retraction with $x_0 \in A$ then the map $i_* : \pi_1^{(\mathcal{I},\mathcal{J})}(A, x_0) \to \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$ induced by inclusion $i : A \to X$ is an isomorphism.*

Proof. We see from Lemma 3.3.6 that i_* is injective, so we show that it is surjective. Given $[\lambda] \in \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$, define the loop $\mu : I \to A$, I being the domain of λ, by $\mu(t) = r(\lambda(t), 1_J)$. Then we see that $f(\lambda(t), t)$ is a homotopy from λ to μ, so we have that $\lambda \sim \mu$ in X, and $i_*([\mu]) = [\mu] = [\lambda]$. □

Definition 3.3.11. Two spaces X and Y are said to be $\mathcal{J}$*-homotopy equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. In this case, we say that each of f and g is a $\mathcal{J}$*-homotopy equivalence*, that f and g are $\mathcal{J}$*-homotopy inverses*, and we write $X \simeq Y$ or $X \simeq_{\mathcal{J}} Y$.

Proposition 3.3.12. *If $f : X \to Y$ is a homotopy equivalence then $f_* : \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0) \to \pi_1^{(\mathcal{I},\mathcal{J})}(Y, f(x_0))$ is an isomorphism for all $x_0 \in X$.*

Proof. We see from Lemma 3.3.8 that $(f \circ g)_* = f_* \circ g_*$ is the identity on $\pi_1^{(\mathcal{I},\mathcal{J})}(Y, y_0)$ and $(g \circ f)_* = g_* \circ f_*$ is the identity on $\pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$. Since $f_* \circ g_*$ is surjective, f_* is surjective. Since $g_* \circ f_*$ is injective, f_* is injective. Thus f_* is bijective. □

Definition 3.3.13. A space X is $\mathcal{J}$*-contractible* if it is $\mathcal{J}$-homotopy equivalent to a point. Equivalently, if there is $x_0 \in X$ that is a deformation retract of X.

Corollary 3.3.14. *If X is $\mathcal{J}$-contractible, then $\pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0) = 1$.*

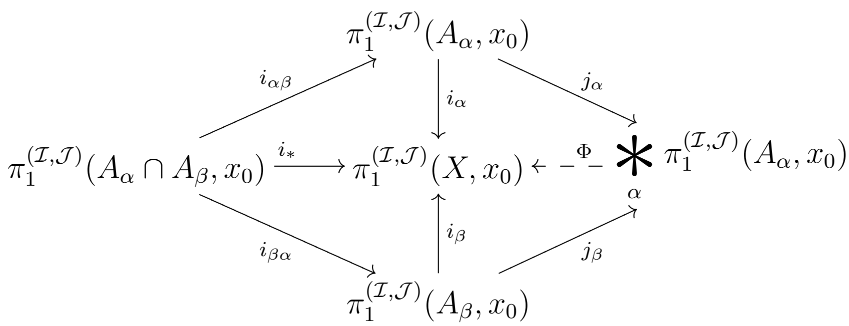
Turner [6] outlined the sketch of a proof of the Seifert-VanKampen theorem for Π_1 in the case where X is covered by two open sets. Here we give the more general result for an arbitrary cover and for any generalized $\pi_1^{(\mathcal{I},\mathcal{J})}$. Note that by Theorem 3.2.10, this gives the result for Π_1 as well. The proof given here is a rough sketch of that given by Allen Hatcher in [5]. It is easily seen that the exact same proof as given by Hatcher can be used in the case of big fundamental groups.

Definition 3.3.15. Suppose that A_α and A_β are subsets of X each containing x_0. Then define the maps $i_\alpha : \pi_1^{(\mathcal{I},\mathcal{J})}(A_\alpha, x_0) \to \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$ and $i_{\alpha\beta} : \pi_1^{(\mathcal{I},\mathcal{J})}(A_\alpha \cap A_\beta, x_0) \to \pi_1^{(\mathcal{I},\mathcal{J})}(A_\alpha, x_0)$ to be those induced by inclusion.

Perhaps one of the most interesting and useful results in classical homotopy theory is the Seifert-VanKampen Theorem. It is noted in [**?**] that this holds for the big fundamental group Π_1 as well. A sketch of the proof will be given here, but it will be noted that using any classical proof will work, simply replacing any occurrence of the real interval $[0, 1]$ with an arbitrary big interval. The statement and sketch of the proof given below follow that given in [5].

Theorem 3.3.16 (Seifert-VanKampen)**.** *Let (X, x_0) be a pointed space and $\{A_\alpha\}$ an open covering of X each containing x_0 and such that every intersection of the form $A_\alpha \cap A_\beta$ is $\mathcal{I}$-path-connected. Then there is a unique surjective homomorphism $\Phi : \mathop{\ast}_{\alpha} \pi_1^{(\mathcal{I},\mathcal{J})}(A_\alpha, x_0) \to \pi_1^{(\mathcal{I},\mathcal{J})}(X, x_0)$ such that the diagram commutes.*

(3.12)



If, in addition, every intersection of the form $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then the kernel of Φ is the normal subgroup generated by elements of the form $j_\alpha i_{\alpha\beta}([\lambda]) j_\beta i_{\beta\alpha}([\lambda]^{-1})$, where $[\lambda] \in \pi_1^{(\mathcal{I},\mathcal{J})}(A_\alpha \cap A_\beta, x_0)$.

Proof. To avoid ambiguity, given a loop λ in A_α, let $[\lambda]_\alpha$ denote its $\mathcal{J}$-homotopy class in A_α and $[\lambda]_X$ its $\mathcal{J}$-homotopy class in X. The requirement that the diagram commutes guarantees that Φ is unique, since it must be defined by $\Phi([\lambda]_\alpha) = [\lambda]_X$ whenever $[\lambda]_\alpha \in \pi_1^{(\mathcal{I},\mathcal{J})}(A_\alpha, x_0)$ for any α, and it must extend to be a homomorphism by $\Phi([\lambda]_\alpha \ast [\mu]_\beta) = \Phi([\lambda]_\alpha) \ast \Phi([\mu]_\beta) = [\lambda]_X \ast [\mu]_X = [\lambda \ast \mu]_X$.

To see that Φ is surjective, let $\lambda : I \to X$ be a loop in X. Since $\{A_\alpha\}$ is an open covering of X, $\{\lambda^{-1}(A_\alpha)\}$ is an open covering of I. Since I is compact, there is a finite subcovering.

In fact, there is a finite partition $0_I = s_0 < s_1 < \cdots < s_{n+1} = 1_I$ and a finite sequence $B_1, \ldots B_n$ with $B_i \in \{A_\alpha\}$ such that $\lambda|_{[s_i, s_{i+1}]}$ lies in B_i and $\lambda(s_i) \in B_i$ for $1 \leq i \leq n$.

For each i, note that $B_i \cap B_{i+1}$ is path connected, so let μ_i be a path in $B_i \cap B_{i+1}$ from x_0 to $\lambda(s_i)$. Also, define $\lambda_i : I \to X$ so that $\lambda_i|_{[s_i,s_{i+1}]} = \lambda|_{[s_i,s_{i+1}]}$ and $\lambda_i(t) = x_0$ for $t \in I \setminus [s_i, s_{i+1}]$. So we see that

$$\lambda \sim \lambda_0 * \lambda_1 * \cdots * \lambda_n \tag{3.13}$$

$$\sim \lambda_0 * \mu_1^{\text{op}} * \mu_1 * \lambda_1 * \cdots * \mu_n^{\text{op}} * \mu_n * \lambda_n \tag{3.14}$$

where $\sim$ denotes $\mathcal{J}$-homotopic in X. Then, finally, since

$$\lambda_0 * \mu_1^{\text{op}} * \mu_1 * \lambda_1 * \cdots * \mu_n^{\text{op}} * \mu_n * \lambda_n = (\lambda_0 * \mu_1^{\text{op}}) * (\mu_1 * \lambda_1 * \mu_2^{\text{op}}) * \cdots * (\mu_n * \lambda_n) \tag{3.15}$$

we see that λ is in the image of Φ, since the right-hand side of equation 3.15 represents an element of $\mathop{\ast}_\alpha \pi_1^{(\mathcal{I},\mathcal{J})}(A_\alpha, x_0)$.

The proof of the final assertion is omitted due to its length and complexity, but it directly parallels the proof given in [5] for the classical case. □

3.4 Ideal Intervals

In this section, we discuss certain properties that will be nice for big intervals to satisfy. In particular, we would like to make big intervals which have as many properties in common with the real interval $[0, 1]$ as possible. The properties we will be discussing are as follows. An "ideal interval" will be a big interval which satisfies all of the following properties.

1. Self similarity: The concatenation of the interval with itself yields the original interval, $I \vee I \cong I$.

2. Richness: That the interval essentially contains all the information of all intervals up to a certain cardinality. That is, given any toset T with $|T| \leq \alpha$ there is an embedding $T \hookrightarrow I$.

3. Separability: That the interval has a dense set of a given cardinality. In particular, it is desired that for some α, I is α-rich and α-separable.

4. Homogeneity: The interval, without its endpoints, is homogeneous.

Recall that for big intervals I and J, there is an injective (order) map $J \hookrightarrow I$ precisely when there is a surjective (order) map $J \to I$. Thus, a big interval I is α-rich among big intervals if it can map (order-preserving) onto any big interval which is α-separable. Thus, in essence, I can replace many different big intervals in calculating big fundamental groups.

One of the convenient things about working with the classical fundamental group is that there is always just one domain. As mentioned earlier, this is made possible by the fact that the real unit interval is self-similar. That is, when concatenated with itself or reversed, it remains unchanged. Since these are the only two properties necessary to guarantee that the one interval is always sufficient, we give the following definition.

Definition 3.4.1. Suppose that I is a self-similar big interval. Then we define $\pi_1^I(X, x_0)$ to be the set of all I-homotopy classes of I-loops in (X, x_0).

Proposition 3.4.2. *Given a self-similar big interval I, define $\mathcal{I} = \{I\}$. Then we have that $\pi_1^I(X, x_0) \cong \pi_1^{(\mathcal{I},\mathcal{I})}(X, x_0)$.*

Proof. Fix (order-preserving) homeomorphisms $f : I \vee I \to I$ and $g : I^{\text{op}} \to I$. Given a big loop $\lambda : I \vee I \to X$, we see that $\lambda f^{-1} : I \to X$ is $\mathcal{I}$-homotopic to λ. Similarly, given a loop $\mu : I^{\text{op}} \to X$, we see that μg^{-1} is $\mathcal{I}$-homotopic to μ. Since every element of $\bar{\mathcal{I}}$ is of the form $I_1 \vee I_2 \vee \cdots \vee I_n$ where I_i is either or I^{op}, we see that pre-composing appropriately with f^{-1} and g^{-1} gives the desired isomorphism. □

Another property that we may want the ideal big interval to satisfy is that it can achieve any path that any other big interval of lesser cardinality can achieve. So, we give the following definition.

Definition 3.4.3. A toset T is *α-rich* if for any toset S with $|S| \leq \alpha$, there is an embedding $S \hookrightarrow T$.

Constructing an α-rich toset for any α is not that difficult. For finite α, there is only one order type of cardinality α, so we assume α is infinite. Then if T is a toset of cardinality α, we may assume that as sets, $T = \alpha$. Then the order on T is simply a subset of the Cartesian product $T \times T = \alpha \times \alpha$. Thus, there are at most $|2^\alpha|$ distinct order types of cardinality α. Hence, the toset consisting of the concatenation of all of them (in any order) has cardinality at most $|2^\alpha|$ and is clearly α-rich.

Proposition 3.4.4. *Given a pointed space (X, x_0), let $\mathcal{I}$ be a tapestry guaranteed by Theorem 3.2.10 and let $\alpha = \sup\{|I| \mid I \in \mathcal{I}\}$. Let J be a big interval which is self-similar and α-rich. Then $\Pi_1(X, x_0) \cong \pi_1^{(\mathcal{I},\mathcal{I})}(X, x_0) \cong \pi_1^J(X, x_0)$.*

This result is easily seen when noting that since J is α-rich and every element $I \in \mathcal{I}$ has cardinality no greater than α, there is a weak monomorphism $\mathcal{F} : \mathcal{I} \to \{J\}$. Then this result follows from Corollary 3.2.11

Lemma 3.4.5. *Suppose that I and J are both self-similar and that there is an epimorphism $f : I \to J$ and a monomorphism $g : I \to J$. Then we have that $\pi_1^I(X, x_0) \cong \pi_1^J(X, x_0)$ for any pointed space (X, x_0).*

Proof. This follows from Corollary 3.2.9, when noting that a monomorphism $g : I \to J$ is a weak monomorphism from $\{I\}$ to $\{J\}$ (in fact, it is a tapestry monomorphism) and that by Theorem 3.1.7, an epimorphism $f : I \to J$ yields a monomorphism $f' : J \to I$ and hence a weak monomorphism from $\{J\}$ to $\{I\}$. $\square$

Proposition 3.4.6. *Assume that S is a toset with $|S| \geq 3$ and $s_0 \in S$ is neither minimal nor maximal. Then $(S, s_0)^\alpha$ is α-rich (and hence S^α is as well).*

Proof. Assume that there are at least three points $a < s_0 < b$ in S and T is a toset with $|T| \leq |\alpha|$. Index the elements of T with $|\alpha|$. Define the map $f : T \to (S, s_0)^\alpha$ as follows. Given $\beta \in |\alpha|$, define

$$f(t_\beta)_\gamma = \begin{cases} a & \text{if } \gamma < \beta,\ t_\beta < t_\gamma \\ b & \text{if } \gamma < \beta,\ t_\beta > t_\gamma \\ s_0 & \text{otherwise} \end{cases}. \tag{3.16}$$

Note that in the case $\beta = 0$, the condition $\gamma < \beta$ is never satisfied, so $f(t_0)_\gamma = s_0$ for all γ.

We verify that this is a strictly order-preserving map. Suppose that $t_\beta < t_\gamma$ are given. For any $\delta < \min\{\beta, \gamma\}$, we have three cases. The first case is $t_\delta < t_\beta$, in which case $f(t_\beta)_\delta = f(t_\gamma)_\delta = b$. The second is $t_\beta < t_\delta < t_\gamma$, and so $f(t_\beta)_\delta = a < b = f(t_\gamma)_\delta$. The last case is $t_\gamma < t_\delta$, thus $f(t_\beta)_\delta = f(t_\gamma)_\delta = a$. In any of these cases, we see that $f(t_\beta)_\delta \leq f(t_\gamma)_\delta$ for all $\delta < \min\{\beta, \gamma\}$. Now for $\delta = \min\{\beta, \gamma\}$, we have two cases. If $\delta = \beta$, we see that $f(t_\beta)_\beta = s_0 < b = f(t_\gamma)_\beta$. If $\delta = \gamma$, then we see that $f(t_\beta)_\gamma = a < s_0 = f(t_\gamma)_\gamma$ hence in either case $f(t_\beta) < f(t_\gamma)$. Thus f is strictly order preserving. Finally, since $(S, s_0)^\alpha$ embeds in S^α, we see that T embeds in S^α, so S^α is also $|\alpha|$-rich. □

Proposition 3.4.7. *If I and J are both self-similar, α-rich, and each has a dense subset of cardinality α, then $\pi_1^I(X, x_0) \cong \pi_1^J(X, x_0)$ for any pointed space (X, x_0).*

Proof. Let $A \subset I$ be dense of cardinality no more than α. Then we see that there is an embedding $f : A \to J$. By Theorem 3.1.7, we see that this implies that there is an order embedding $g : I \to J$ and an order epimorphism $h : I \to J$, so we obtain the result by applying Lemma 3.4.5. □

This result allows us to define $\pi_1^\alpha(X, x_0)$ for any α such that there is an interval I that is self-similar, α-rich, and has a dense subset of cardinality α. We saw in the previous chapter that if α is a strong limit cardinal, then $[0,1]^\alpha$ is such a big interval. Since any two such intervals of the same cardinality yield the same big fundamental group, this definition is well-defined.

Definition 3.4.8. Let α be a cardinal such that there is a big interval I that is self-similar, α-rich, and α-separable. Then define $\pi_1^\alpha = \pi_1^I$.

Now we must ask the question, do such big intervals actually exist? To answer this, we construct them as follows. But first we note that the real interval $[0,1]$ is such an interval, for $\alpha = \omega_0$. That is, it has a countable dense set (namely $\mathbb{Q} \cap [0,1]$), $\mathbb{Q}$ is ω_0-rich, and therefore $[0,1]$ is as well, and $[0,1]$ is clearly self-similar. Thus $\pi_1^{\omega_0} = \pi_1$ is just the classical fundamental group. But now we show that such intervals exist for arbitrarily large cardinality.

Definition 3.4.9. A cardinal α is called a *strong limit cardinal* if $2^\beta < \alpha$ for any $\beta < \alpha$.

We note here that strong limit cardinals exist for arbitrarily large cardinalities. That is, given a cardinal α, there is a strong limit cardinal β with $\alpha \leq \beta$. In fact, we do this using the following construction. Note that the least such cardinal is $\aleph_0$, since no finite cardinal is a limit cardinal but given any finite cardinal α, 2^α is also finite. So, we define $\beth_0 = \aleph_0$. Then, for any ordinal α where $\beth_\alpha$ is defined, define $\beth_{\alpha+1} = 2^{\beth_\alpha}$. Finally, for any limit ordinal α, define $\beth_\alpha = \bigcup_{\beta<\alpha} \beth_\beta$.

Lemma 3.4.10. *For any limit ordinal α, we have that $\beth_\alpha$ as defined above is a strong limit cardinal.*

Proof. Suppose that $\beta < \beth_\alpha$ is given. Then we see that there is $\gamma < \alpha$ such that $\beta \leq \beth_\gamma$. It follows that $2^\beta \leq 2^{\beth_\gamma} = \beth_{\gamma+1} < \beth_\alpha$, as desired. □

Proposition 3.4.11. *Given a strong limit cardinal α and a pointed set (S, s_0) with $|S| \leq \alpha$, we have that $|(S, s_0)^\alpha| = \alpha$.*

Proof. We saw above that $|(S, s_0)^\alpha| = \sup_{\beta<\alpha} \left|S^\beta\right| = \sup_{\beta<\alpha} 2^\beta$. Since α is a strong limit cardinal, we see that $2^\beta < \alpha$ for all $\beta < \alpha$, hence $\sup_{\beta<\alpha} 2^\beta = \sup_{\beta<\alpha} \beta = \alpha$. □

It may be the case that for any cardinal α, there is a big interval which is self-similar, α-rich, and has a dense subset of cardinality α. However, all we have shown is that such big intervals exist when α is a strong limit cardinal, and that such cardinals exist of arbitrarily large cardinality, yielding the following result.

Proposition 3.4.12. *Given any pointed space (X, x_0), there is a cardinal α for which $\Pi_1(X, x_0) \cong \pi_1^\alpha(X, x_0)$.*

Proof. Theorem 4.20 from [1] guarantees a cardinal $\alpha(X)$ such that if I is any big interval, C any compact Hausdorff space, and $f : I \times C \to X$ continuous, then there is a big interval I' with $|I'| \leq \alpha(X)$ such that f factors through $I' \times C$. Then by Lemma 3.4.10 we see that there is a strong limit cardinal α such that $\alpha \geq \alpha(X)$. Then $[0,1]^\alpha$ is a big interval which is self-similar, α-rich, and contains a dense subset of cardinality α. Hence $\Pi_1(X, x_0) \cong \pi_1^\alpha(X, x_0)$. □

Chapter 4

Big π_n

4.1 Construction

Just as we did with the big fundamental group, we define big homotopy groups of higher dimensions using maps from big cubes. The boundary of the cube will be required to map to a single point, as is required with classical homotopy theory. For this chapter, we will only be using self-similar big intervals because that is the easiest way to ensure that the theory is well-defined. Thus, every big interval will be assumed to be self-similar, unless stated otherwise.

Definition 4.1.1. A *big n-cube* is the Cartesian product I^n, where I is a big interval. The *boundary* of I^n, denoted ∂I^n is defined to be $\bigcup_{i=1}^{n}\prod_{j=1}^{n} A_{ij}$ where A_{ij} is $\{0_I, 1_I\}$ if $i = j$ and I otherwise.

Here we note that we are breaking from the earlier convention that I^n be endowed with the order topology. Instead, we wish for I^n to be given the product topology, so that this object is analogous to the classical n-cube $[0,1]^n$.

Definition 4.1.2. Let I be a big interval. Given a pointed space (X, x_0), define the set $\pi_n^I(X, x_0)$ to be homotopy classes (using I as the parameter space, and rel ∂I^n) of maps from the pair $(I^n, \partial I^n)$ to the pair (X, x_0). Let $\varphi : I \to I_1 \vee I_2$ (where $I_1 = I_2 = I$) and $r : I \to I^{\text{op}}$ be isomorphisms. Given two elements $[\lambda], [\mu] \in \pi_n^I(X, x_0)$, define the *concatenation* $\lambda * \mu : (I^n, \partial I^n) \to (X, x_0)$ by

$$\lambda * \mu(t_1, t_2, \ldots, t_n) = \begin{cases} \lambda(\varphi(t_1), t_2, \ldots, t_n) & \text{if } \varphi(t_1) \in I_1 \\ \mu(\varphi(t_1), t_2, \ldots, t_n) & \text{if } \varphi(t_1) \in I_2 \end{cases}. \tag{4.1}$$

And define the *reverse* of λ, by $\bar{\lambda} = \lambda \circ r$.

We could try for more generality by allowing different big intervals to be mapped into the space when calculating this homotopy group, but as we saw in the one-dimensional case of the matter, no information is lost by restricting to just one interval.

Theorem 4.1.3. *Given a self-similar big interval I and a pointed space (X, x_0), we have that $\pi_n^I(X, x_0)$ is a group, with multiplication given by $[\lambda] * [\mu] = [\lambda * \mu]$ and inverse $[\lambda]^{-1} = [\bar{\lambda}]$.*

Since exactly the same proof for the one-dimensional case applies here as well, it is omitted.

Theorem 4.1.4. *For any self-similar big interval, the operator $\pi_n^I : \mathbf{Top}^* \to \mathbf{Gp}$ is a functor. Given an order epimorphism $J \to I$, the induced map $\pi_n^I \to \pi_n^J$ is a natural transformation.*

It is not automatic that this group is Abelian. The classical trick of "sliding" the maps around in the extra space may or may not be possible. For that we give the following definition.

Definition 4.1.5. Let I be a big interval. We say that I is *strongly self-similar* if the following hold

1. Given any $a, b \in I$ with $a < b$ there is an isomorphism $f : I \to [a, b]$ and an I-isotopy $F : I \times I \to I$ from id_I to f.

2. Given $c, d \in I$ with $c < d$ there is an I-isotopy $F : I \times I \to I$ rel $\{0_I, 1_I\}$ such that $F_{0_I} = \mathrm{id}_I$, $F_{1_I}(a) = c$, and $F_{1_I}(b) = d$.

Note that the condition given in this definition, which is merely that a big interval can be "shrunk" to any (non-degenerate) closed interval is sufficient for our purposes. That is, to obtain the ability to "slide", merely concatenate two of these shrinking isotopies, with the order reversed on one. That is, if F is a shrink from I to $[a,b]$ and G is a shrink from I to $[c,d]$ then $F^{\mathrm{op}} * G$ is a slide from $[a,b]$ to $[c,d]$.

Proposition 4.1.6. *If I is strongly self-similar, then $\pi_n^I(X,x_0)$ is Abelian for any pointed space (X,x_0) and any $n \geq 2$.*

Proof. Let $\lambda, \mu : I^n \to X$ be n-loops. Let $0_I < a < b < c < d < 1_I$ and let $f_1 : I \to [a,b]$ and $f_2 : I \to [c,d]$ be isomorphisms and $\phi_1, \phi_2 : I \times I \to I$ be isotopies from id_I to f_1 and f_2 respectively. Let $\phi_3 : I \times I \to I$ be an isotopy sliding $[a,b]$ to $[c,d]$. Let $\varphi : I \vee I \to I$ be an isomorphism. For higher dimensions, we simply apply the identity in each coordinate past the second, so we may assume $n = 2$.

Then the classical trick of sliding the two maps λ and μ past each other is performed in the following moves. First, to shrink each one, we use $(\phi_1 \times \phi_1) * (\phi_2 \times \phi_2)$. That is, in the first coordinate, we shrink the left half of $I \vee I$ to $[a,b]$ and the right half to $[c,d]$ and in the second coordinate we shrink I to $[a,b]$ on the left and to $[c,d]$ on the right. Then we apply $(\phi_3 \circ \varphi) \times \mathrm{id}_{[a,b]}$ on the left portion and $(\phi_3^{\mathrm{op}} \circ \varphi) \times \mathrm{id}_{[c,d]}$ on the right portion. Then, we expand each piece by applying $(\phi_1^{\mathrm{op}} \times \phi_2^{\mathrm{op}}) * (\phi_2^{\mathrm{op}} \times \phi_1^{\mathrm{op}})$. This provides a homotopy from $\lambda * \mu$ to $\mu * \lambda$. Hence $\pi_n^I(X,x_0)$ is Abelian. $\square$

Bibliography

Bibliography

[1] James Cannon and Greg Conner, *The big fundamental group, big Hawaiian earrings, and the big free groups*, Topology Appl. **106** (2000), no. 3, 273–291. MR 1775710 (2002k:55029) v, 2, 16, 26, 27, 28, 29, 30, 38, 46

[2] ———, *The combinatorial structure of the Hawaiian earring group*, Topology Appl. **106** (2000), no. 3, 225–271. MR 1775709 (2001g:20020) 2

[3] ———, *On the fundamental groups of one-dimensional spaces*, Topology Appl. **153** (2006), no. 14, 2648–2672. MR 2243740 2

[4] Katsuya Eda, *The fundamental groups of one-dimensional spaces and spatial homomorphisms*, Topology Appl. **123** (2002), no. 3, 479–505. MR 1924047 (2003e:55015)

[5] Allen Hatcher, *Algebraic Topology*, 1 ed., Cambridge University Press, December 2001. 40, 41, 42

[6] Emma Turner, *Big fundamental groups and covering spaces*, Master's thesis, Brigham Young University, 1999. v, 2, 36, 40

Vita

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He obtained his Bachelor of Science degree from Brigham Young University in April 2007 with a major in mathematics and a minor in Japanese. In August of the same year, he obtained his Master of Science, also from Brigham Young. With this dissertation, he will graduate with a Doctor of Philosophy in May 2013 and start a career as a math professor.