University of Tennessee, Knoxville
Trace: Tennessee Research and Creative Exchange

Doctoral Dissertations   Graduate School

12-2011

Baryon Spectrum Analysis using Dirac's Covariant Constraint Dynamics

Joshua Franklin Whitney
jwhitne2@utk.edu

Recommended Citation

To the Graduate Council:

I am submitting herewith a dissertation written by Joshua Franklin Whitney entitled "Baryon Spectrum Analysis using Dirac's Covariant Constraint Dynamics." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

Horace W. Crater, Major Professor

We have read this dissertation and recommend its acceptance:

Lloyd M. Davis, Christian G. Parigger, John, S. Steinhoff

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

Baryon Spectrum Analysis using Dirac's Covariant Constraint Dynamics

A Dissertation Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Joshua Franklin Whitney

December 2011

Dedication

This work is dedicated to my cat Azalea, who passed away on January 6th, 2010 after being my closest friend and companion for 17 years. She will always be missed.

Acknowledgements

First and foremost, I am grateful for my thesis advisor, Dr. Horace Crater, for the long hours, dedication and assistance he has given in order to bring me to this point. I am extremely blessed for the training he has given me in the areas of quantum field theory, particle physics and constraint dynamics. He has truly devoted himself to turning me into a better physicist. In addition, Dr. Cheuk-yin Wong and Dr. Christian Parigger have both been instrumental in filling in the holes in my knowledge of coupling coefficients and numerical methods, respectively, and Dr. Parigger has been a constant source of encouragement during the difficult times when it felt as if this would never be done. I am also thankful to Dr. Lloyd Davis for being willing to be a part of my thesis committee on relatively short notice and for his helpful comments that arise from him being in a much different field from my own.

I am grateful to my friends and family for the unending support given over the course of this work. They have been nothing but encouraging and proud even as it dragged on much longer than it should have. My friends have remained close and loyal even as I had to repeatedly isolate myself in order to continue working as quickly as possible and I thank them for their dedication.

I thank Dr. Dan Cross, Dr. Ray Bloomer and Dr. J.C. McDonald at King College for their early training and support. They gave me both the tools and the confidence to begin down this road in graduate school in the first place. I also thank Dr. Ted Barnes for his excellent classes in quantum mechanics and quantum field theory that cemented my decision to go into the field of theoretical particle physics.

I am fortunate to have done my thesis work at the University of Tennessee Space Institute with so many helpful staff members, most notably Charlene Hane. I am very grateful for their assistance will all things administrative.

Finally, my wife Heather, for being the only woman in the world who could truly understand what it is I'm going through, having just defended her own Ph. D. She has supported me financially after my research funds ran out and has been a constant driving force to continue to push through to the end and make her as proud of me as I am of her.

Abstract

The energy spectrum of the baryons is determined by treating each of them as a three-body system with the interacting forces coming from a set of two-body potentials that depend on both the distance between the quarks and the spin and orbital angular momentum coupling terms. Constraint dynamics is first reviewed for a relativistic two-body system in order to assemble the necessary two body framework for the three-body problem and then we review the different types of covariant two-body interactions involved in constraint dynamics, including vector and scalar, and solve the problem of energy eigenstates using constraint dynamics. The Two Body Dirac equations of constraint dynamics derived by Crater and Van Alstine, matched with the quasipotential formalism of Todorov as the underlying two-body formalism are used, as well as the three-body constraint formalism of Sazdjian to integrate the three two-body equations into a single relativistically covariant three body equation for the bound state energies. The results are analyzed and compared to experiment using a best fit method and several different algorithms, including a gradient approach, and Monte Carlo method.

Contents

List of Tables

List of Figures

1 Introduction

Recent quark model calculations done by Crater et al. [1, 2, 3] using covariant two-body Dirac equations in a relativistic constraint dynamics formalism have given a good description of the meson masses for both light and heavy quarks, using world scalar and vector potentials that depend on merely one or two parameters. The good quality of the fit has been attributed to the exact two-body kinematics merged with a QCD interaction potential based on vector and scalar potentials that uses a minimal number of variable parameters. These non-perturbative, i.e. numerical results hold up well when compared to other methods for meson spectroscopy. The vector potentials, in turn, have a structure originally derived from the classical electrodynamics of Wheeler and Feynman.[4] This structure can also be obtained from quantum electrodynamics by using a covariant three dimensional truncation of the Bethe-Salpeter equation based on the Todorov quasipotential, which is then compared to the Two-Body Dirac equations.[4] The comparison is done in order to identify the appropriate invariant potential functions that will be used in the potential model.

A usual common ground for a two-body bound state problem (almost regardless of which approach is taken) is the Bethe-Salpeter equation, which is generally not used in its full (in other words, four dimensional) form due to difficulties with the relative time coordinate. The Two-Body Dirac equations of constraint dynamics give a covariant, three-dimensional version of the Bethe-Salpeter equation that solves the problem of relative time and relative coordinates. Originally, the Two-Body Dirac equations of constraint dynamics arose from a supersymmetric treatment of two pseudoclassical constraints (with Grassmann variables in place of gamma matrices) for scalar interactions, which were then quantized. Later work led to the development of the form of the Two-Body Dirac equations that were not limited to scalar interactions. Covariant constraint dynamics, at its core, is a method which allows connections between non-independent operators to be built in a system that has more variables than it has observables, such as occurs with the mass shell constraint $p^2 + m^2 = 0$. It is inherently four-dimensional, but only three of the momentum variables are independent. [4]

In this work, we review the Hamiltonian constraint dynamics formalism for the two-body system and apply it to the three-body quark problem for baryon spectroscopy. The derivations of the Two-Body Dirac equations are reviewed along with their potentials for a relativistic, spin-dependent formalism. In taking the two-body equations to a three-body system we still regard the system as the naive quark model in that each interaction is between each pair of quarks and there is no over-arching three-body interaction to be considered, but the system now has three sets of interactions instead of just one. Thus, all of our interactions are still two-body interactions, but for three sets of quarks. This works out relatively simply for

some of the interacting potential but has required extensive reworking for others, such as the spin-spin and spin-orbit interactions. Not surprisingly then, a large portion of this work is devoted to the development of the two-body theory as this provides both the tools for working the three-body case along with the possible drawbacks to consider as we the formalism is extended to larger systems (e.g. compatibility of the constraint equations). The end result of this, as will be seen later, is that the theoretical considerations that must be applied in order to go to the three-body system are relatively simple, but the actual mechanics of doing so are not and multiple methods that are concise and rigorous (such as separation of variables and relative coordinates) in the two-body case cannot be applied to the three-body one, at least not without a substantial adaptation of the two-body formalism.

The Hamiltonian Constraint Dynamics formalism [5] allows for a relativistic method of accounting for two-body effects. In addition, Constraint Dynamics as developed by Crater and Van Alstine [4] provides not only the usual spin interaction dependence seen in the Dirac equation but also additional terms needed to make the approach mathematically consistent. It is correct and useful in both a classical and a quantum mechanical formalism and as such we show how it is derived and applied for a two-body system. Furthermore, we show how the world scalar and vector interactions come about in this formalism and how they are used. All of the interacting potentials that are used in this work (and the two-body system as well) are dependent on these vector and scalar potentials.

The variational principle is used with a Gaussian basis wavefunction of total JM to solve our eigenvalue equation. We show how the Gaussian basis can be expanded in a variational theory in order to achieve an eigenvalue closer to the true ground state. The matrix must be truncated after a reasonable limit is reached, but in theory an infinitely large variational matrix would give the exact ground state. Since the potentials are dependent on the energy eigenvalue, the standard approach to using the variational principle does not readily apply. A recursion algorithm is therefore used so that the effective Hamiltonian H has an embedded E dependence, but this E will change as we approach convergence. The program then is designed to iteratively solve these equations until a desired level of convergence is reached. As in the two-body case, the solution to our Hamiltonian can be done exactly for the kinematics but including the interacting potentials requires a numerical treatment. Also as in the two-body case, these interacting potentials depend only on one or two parameters. The numerical fitting routine uses a chi-squared minimization Monte Carlo routine combined with a simplified gradient approach to acquire a best fit for the spectrum of known baryons. We have compared our numerical results to both experimental data and to other theories, most notably the approach of Capstick and Isgur([6]), while also comparing the quark masses and potential parameters we obtained in our fit with those found in two-body results by Crater et al. [1] [2] [3]. Ideally, since this model uses the same basic model for the three-body system as has been used for the two-body system, one would

expect the parameters to be similar. The actual baryon masses are, of course, not compared to the meson results, but the masses of the quarks and values for other shared parameters are.

1.1 Quarks and the Quark Model for the Baryons

This section contains a brief review of relevant definitions of quarks and baryons as well as references to recent work on the baryon three-body problem. Quarks were introduced in the beginning of 1964 by Gell-Mann and Zweig[7] as an idea evolving from considerations of unitary symmetry. Quarks are considered to be the fundamental building blocks of hadronic matter and each hadron is a bound state of quarks and/or antiquarks. Quarks are considered to be fermions in that they have odd half-integer spins.

The baryons are three-quark bound states having an odd half-integer total angular momentum j. They experience the strong nuclear force, and are described by Fermi-Dirac statistics. The lowest-lying baryon states are three-quark states with orbital angular momentum $l = 0$, of which there are 8 states with total $j = \frac{1}{2}$ (the octet). This work begins with a focus on the octet, but quickly proceeds to a much larger number of baryons, such as the decuplet, which consists of $l = 0$ three-quark states with total j of $\frac{3}{2}$ [7], as well as most of the other baryon radial and orbital excitations (see Appendix B for the complete listing).

Quantum field theory, from which the Bethe-Salpeter equation is directly derived, originated in the 1920s from the problem of combining a quantum mechanical theory with an electromagnetic field[8]. The theory was constructed by expressing the field's internal degrees of freedom as an infinite set of harmonic oscillators and applying a canonical quantization to them. The theory, of course, needed to combine relativity and quantum mechanics for the electrons as well, which is the reason for the importance of the Dirac equation.

Quantum chromodynamics (QCD), which plays an important role in this work, is the theory of the strong interaction. The strong interaction is the force responsible for describing the interactions of quarks and gluons inside of a hadron and QCD itself possesses the properties of both asymptotic freedom and linear confinement.[8] Asymptotic freedom, which in our model comes from the vector potential term, allows for quarks and gluons to interact very weakly in high energy interactions where the force is approximately independent of r. However, quarks also exhibit linear confinement and it would take an infinite amount of energy to separate two quarks, which is modeled in our scalar potential term.

As far as work done with 3-body systems similar to this work, Capstick and Isgur have studied the three-quark system in a relativized quark potential model with chromodynamics. Toward that end, they have developed a model that supports the phenomenology of nonrelativistic calculations. It excludes the spin-orbit interactions in baryons and leads to spectra and internal compositions similar to those of the nonrelativistic model [6]. In a similar fashion to our work, they use a set of two-body potentials that have

been previously derived and tested for the meson spectrum, as we do here with the meson model created by Crater et al.([1, 2, 3]). It still does follow similar qualitative (for spectra and internal compositions) properties as the nonrelativistic quark model, since the goal of that work was to overcome the shortfalls of the nonrelativistic approach, such as the nonrelativistic treatment of both the quark motion and quark dynamics. Isgur and Karl have also studied the ground state baryon octet in a quark model with flavor-independent confinement and color hyperfine interactions in an even earlier study using a similar model to the later model of Capstick and Isgur, with good results for the octet [9]. Neither of these methods, however, use the relativistic constraint dynamics formalism used here, which allows us to create a completely covariant formalism. Capstick and Isgur also use basis functions that are always antisymmetric under exchange of quarks 1 and 2, which is a simplifying approximation we do not use, though it is worth noting it is useful for the vast majority of the spectrum (most have 2 quarks that are the same flavor).

Löring et. al. have written a series of three papers treating light baryon resonances within a relativistically covariant quark model based on the Bethe-Salpeter equation and demonstrate how to solve the Bethe-Salpeter by reduction to the Salpeter equation [10]. Again, this is a significantly different formalism from our approach, but as we stated above, the Bethe-Salpeter equation tends to be a common point in most approaches to this problem.

By contrast to those approaches, Sazdjian uses the manifestly covariant formalism with constraints for the construction of relativistic wave equations which describe the dynamics of N interacting spin 0 or spin $\frac{1}{2}$ particles with $N \geq 2$. The system is governed by a single dynamical wave equation that determines the eigenvalue of the total mass squared of the system. For $N = 2$ his equations are essentially the same as presented below and his notation has been adapted accordingly. The interactions in the general N case are introduced through two-body potentials, but many body potentials can also be incorporated [11]. Sazdjian's N body eigenvalue equations provide a useful starting point for work on the manifestly covariant relativistic three-body problem.

In this work, some of the simplifying substitutions derived by Isgur have been used [9] and the interaction potentials have been treated as separate two-body interactions. The constraint dynamics formalism that has been successful for the two-body problem applications to meson spectroscopy is reviewed and used, [12] with a focus on how to extend this to the relativistic three-body formalism for the baryons. In addition to not assuming that two quarks are the same mass as Isgur does, we also do not consider full three-body interactions, choosing instead to split our problem into just three two-body interactions.

1.2 History of Dirac's Equation and its Two-Body Counterpart

As with any good discussion of things quantum, we begin with the non-relativistic Schrödinger equation for a single particle

$$\frac{\mathbf{p}^2}{2m}\Psi(\mathbf{r}, t) + V(\mathbf{r})\Psi(\mathbf{r}, t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}, t). \tag{1}$$

This equation, developed in 1926, forms the foundation for modern quantum mechanics, where $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$. Natural units will be used, setting $\hbar = c = 1$. This equation is not relativistically covariant (meaning that it does not treat $\mathbf{r}$ and t equally), but there are very well-established methods to solve this equation. Thus, if whichever wave equation is used can be modified to fit this form, then the solution is essentially known. The ability to put an equation into a Schrödinger-like form will prove to be invaluable in solving far more complicated systems.

Since the Schrödinger equation is non-relativistic, the next obvious step comes in relativizing it. By taking the standard relativistic equation for total energy, $E^2 = \mathbf{p}^2c^2 + m^2c^4 = \mathbf{p}^2 + m^2$ (in natural units), and substituting the quantum analogs of $\mathbf{p}$ and E, one arrives at the Klein-Gordon equation for particles without spin[14]

$$(p^2 + m^2)\Psi = 0, \tag{2}$$

[1]where $p^2 = -(\boldsymbol{\nabla}^2 - \frac{\partial^2}{\partial t^2})$(the d'Alembert operator). This equation looks promising in that it treats $\mathbf{x}$ and t symmetrically, however, it has difficulties with negative energy states and the interaction form did not include a spin term, which is a necessity for this work.

After seeing the issues with the Klein-Gordon equation, Dirac then set himself upon fixing the problems it had, namely the negative probability densities due to the lack of first-order derivatives in time, not to mention negative energies. This lead to the development of his equation in 1928, which is

$$(\gamma^\mu p_\mu + m)\Psi = 0, \tag{3}$$

where γ^μ are the Dirac matrices. This equation is fully relativistic, consistent with the principles of quantum mechanics (including positive probability densities), and designed for spin-$\frac{1}{2}$[15] particles. It does not, however, remove the issue of negative energy solutions, but it does provide an explanation for them in

---

[1]Our metric $\eta^{\mu\nu}$ is defined as $\begin{Bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$.

quantum field theory, in which the solution for a negative energy electron is interpreted as a solution for a positive energy positron. Thus, it is a first-order equation in time (like the Schrödinger equation) and accounts for the existence of antiparticles. In the presence of world scalar (S) and four-vector interactions (A), these equations become

$$\underbrace{\left((p - A)^2 + (m + S)^2\right)\ \psi = 0}_{\text{Klein-Gordon}} \qquad \underbrace{\left(\gamma^\mu \left(p - A\right)_\mu + (m + S)\right)\ \psi = 0}_{\text{Dirac}} . \tag{4}$$

The next logical step is then to create a two-body equation for spin-$\frac{1}{2}$ particles since there now existed a one-body equation. Breit was the first to develop a 2-body Dirac equation in 1929, formed by summing two free-particle Dirac Hamiltonians with a spin dependent interaction obtained by substituting Dirac $\boldsymbol{\alpha}$ matrices for velocities in the semi-relativistic electrodynamic interactions of Darwin[16, 17, 18], given by

$$E\boldsymbol{\Psi} = \{\boldsymbol{\alpha}_1 \cdot \mathbf{p}_1 + \beta_1 m_1 + \boldsymbol{\alpha}_2 \cdot \mathbf{p}_2 + \beta_2 m_2 - \frac{\alpha}{r}[1 - \frac{1}{2}(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_1 \cdot \hat{\mathbf{r}}\ \boldsymbol{\alpha}_2 \cdot \hat{\mathbf{r}})]\}\boldsymbol{\Psi} . \tag{5}$$

While this equation was useful for the perturbative calculation of the electromagnetic bound-state spectrum of multi-electron atoms, which is what Breit designed it for, it was not a well-defined wave equation, meaning (for terms beyond Coulomb) it could only be used perturbatively, and it was also not relativistically covariant. In an attempt to reproduce results from the Breit equation in weak-coupling perturbation theory, Bethe and Salpeter derived a two-body wave equation[19] from quantum field theory. While their equation is in agreement with quantum electrodynamics, its use as a relativistic bound-state equation has difficulties. Nakanishi[20] discovered that the Bethe-Salpeter equation would give negative-norm solutions as a result of relative time, a relativistic degree of freedom.

Todorov created an effective potential method for the two-body system[21, 22] by generalizing the Einstein condition $E^2 = \mathbf{p}^2 + m^2$ to an effective particle of relative motion. He attempted to solve the problem that the Bethe-Salpeter equation had with relative energy and relative time by creating an equation for a particle of relative motion relativistically, but three-dimensional. He introduced the relative momentum, an energy, and a mass of the particle of relative motion that satisfied the relation of the Einstein condition between mass, energy, and momentum.

In order to address the problem of relative time and energy, Todorov, Kalb, Van Alstine, and others[23] derived covariant, canonical mechanics for the two-body system from Dirac's Hamiltonian constraint dynamics. In such a system the dynamics are given by a set of constraints that act both to constrain the motion in phase-space and to generate the Hamiltonian. The development of Dirac's constrained Hamiltonian formalism for the two-body problem to include spin with scalar, vector, tensor, pseudoscalar, and pseudovector

interactions for spin - $\frac{1}{2}$ particles has been established by Crater and Van Alstine[24, 25, 4, 26, 27, 28] and used extensively in this work.

2 Hamiltonian Constraint Dynamics

This section begins our review of the Hamiltonian constraint dynamics formalism. We begin with the two-body system in order to fully develop the set of tools that are used in the three-body system, but it will become apparent that the two-body methods require extensive reworking in order to use the same basic formalism for the three-body problem. This formalism is useful because it provides a method of accounting for two-body relativistic effects that incorporate the automatic spin interaction dependence that the Dirac methodology provides and also provides a covariant, three dimensional truncation of the Bethe-Salpeter equation. We review constraint dynamics first for a classical system and then for a quantum mechanical system. We derive the constraints that arise from mass shell conditions and provide a review of the interacting potentials. We first derive the entire system of constraints and interacting potentials for a classical system and then we show how it is quantized. In this section we describe how the constraints arise in the context of relativistic classical mechanics. The Lagrangian for a free particle

$$L = -m\sqrt{-\dot{x}^2},$$
(6)

has canonical four-momentum

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{-\dot{x}^2}},$$
(7)

and Legendre Hamiltonian

$$\mathcal{H}_L = p_\mu \dot{x}^\mu - L = \frac{m\dot{x}^2}{\sqrt{-\dot{x}^2}} + m\sqrt{-\dot{x}^2} = 0.$$
(8)

From $p^2 = \frac{m^2\dot{x}^2}{-\dot{x}^2} = -m^2$, one arrives at the mass shell constraint with $(p = (\varepsilon, \mathbf{p}))$

$$\mathcal{H}_0 = p^2 + m^2 = -\varepsilon^2 + \mathbf{p}^2 \approx 0,$$
(9)

with the weak equality sign $\approx$ indicating that the equality can only be imposed once equations of motion have been evaluated. When one has constraints, the Hamiltonian, called the Dirac Hamiltonian, is $\mathcal{H}_D = \mathcal{H}_L + \lambda \times$constraints. Note that when the Legendre Hamiltonian vanishes, the constraint itself serves as the

Hamiltonian, $\mathcal{H} = p^2 + m^2$. In this context we can introduce interactions by using a generalized mass shell constraint

$$p^2 + m^2 + \Phi \approx 0, \tag{10}$$

where Φ is some interacting potential to be determined later. These equations provide the necessary definitions for discussing the two-body Hamiltonians which form the basis for most of our work.

2.1 Two-Body Hamiltonians for Relativistic Constraint Dynamics

In order to understand the relativistic three-body problem one must first be very familiar with the relativistic two-body one, since the three-body formalism we use is based on two-body interactions (our three-body problem is essentially written as three two-body problems). This section therefore describes the methods used to set up two-body Hamiltonians. In the two-body case, in analogy with what occurs in the 1-body case, we start with a 2-body Lagrangian that would lead not to one but to two mass shell constraints:

$$
\begin{aligned}
\mathcal{H}_{10} &= p_1^2 + m_1^2 \approx 0, \\
\mathcal{H}_{20} &= p_2^2 + m_2^2 \approx 0,
\end{aligned} \tag{11}
$$

and as in the one-body case interactions would be introduced by mass shell constraints of the form

$$
\begin{aligned}
\mathcal{H}_1 &= p_1^2 + m_1^2 + \Phi_1(x, P) \approx 0, \\
\mathcal{H}_2 &= p_2^2 + m_2^2 + \Phi_2(x, P) \approx 0,
\end{aligned} \tag{12}
$$

where x is the relative coordinate and P is the total momentum, defined as

$$
\begin{aligned}
x &= x_1 - x_2, \\
P &= p_1 + p_2.
\end{aligned} \tag{13}
$$

The invariants $\Phi_i(x, P)$ are called quasipotentials due to having center of momentum (c.m.) energy dependent potentials that describe deviations from the free mass shell constraint. Here again, the weak equality sign means that these constraints are only to be applied after the Poisson brackets (see e.g. Eq.(15) below) have been evaluated. The Hamiltonian in terms of the constraints only is[29]

8

$$\mathcal{H} = \lambda_1 \mathcal{H}_1 + \lambda_2 \mathcal{H}_2, \tag{14}$$

where λ_i are unknown Lagrange multipliers. The constraints are constant in time, meaning that $\{\mathcal{H}_i, \mathcal{H}\} = 0$, provided that

$$\{\mathcal{H}_1, \mathcal{H}_2\} \approx 0, \tag{15}$$

which is known as the compatibility condition. If we explicitly work out these Poisson brackets, we arrive at the equation

$$2p_1 \cdot \{p_1, \Phi_2\} + 2p_1 \cdot \{\Phi_1, p_2\} + \{\Phi_1, \Phi_2\} \approx 0. \tag{16}$$

One assumes that the invariant functions are dependent on two coordinates and the total energy of the system, as given by

$$\Phi_i = \Phi_i(\frac{x_\perp^2}{2}, \frac{x_{||}^2}{2}, w), \tag{17}$$

in which w is the total c.m. energy,

$$
\begin{aligned}
w^2 &= -P^2, \\
x_{||} &= -\frac{x \cdot P}{w^2} P = x \cdot \hat{P} \hat{P}, \\
\hat{P} &= \frac{P}{w}, \\
\hat{P}^2 &= -1, \\
x_\perp &= x - x_{||}, \\
P \cdot x_\perp &= 0, \\
\{x_\perp^\mu, p^\nu\} &= (\eta^{\mu\nu} - \hat{P}^\mu \hat{P}^\nu) \, .
\end{aligned}
\tag{18}
$$

Here, $x_{||}$ and $x_\perp$ are the components parallel and perpendicular to the total momentum, respectively. Thus, the compatibility condition becomes

$$-4p_1 \cdot x_\perp \frac{\partial \Phi_2}{\partial x_\perp^2} - 4p_1 \cdot x_{||} \frac{\partial \Phi_2}{\partial x_{||}^2} - 4p_2 \cdot x_\perp \frac{\partial \Phi_1}{\partial x_\perp^2} - 4p_2 \cdot x_{||} \frac{\partial \Phi_1}{\partial x_{||}^2} + \{\Phi_1, \Phi_2\} \approx 0, \tag{19}$$

for which the solution is (in the simplest case)

$$\Phi_1 = \Phi_2 = \Phi(\frac{x_\perp^2}{2}, w), \tag{20}$$

which gives

$$\{\mathcal{H}_1, \mathcal{H}_2\} = -4P \cdot x_\perp \frac{\partial \Phi(\frac{x_\perp^2}{2}, w)}{\partial x_\perp^2} = 0. \tag{21}$$

Since the non-relativistic relative momentum is defined by

$$\mathbf{p} = \frac{m_2 \mathbf{p}_1 - m_1 \mathbf{p}_2}{m_1 + m_2} = \frac{m_1 m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2) = \mu \mathbf{v} , \tag{22}$$

it is natural to take as its relativistic counterpart

$$p \equiv \frac{1}{w} \left(\varepsilon_2 p_1 - \varepsilon_1 p_2 \right), \tag{23}$$

where ε_1 and ε_2 are the center of momentum energies of particles 1 and 2, respectively, defined so that

$$\varepsilon_1 + \varepsilon_2 = w. \tag{24}$$

Then Eq.(13) and Eq.(23) imply

$$p_1 = \frac{\varepsilon_1 P}{w} + p \equiv \varepsilon_1 \hat{P} + p \quad \text{and} \quad p_2 = \frac{\varepsilon_2 P}{w} - p \equiv \varepsilon_2 \hat{P} - p, \tag{25}$$

which leads to

$$\mathcal{H}_1 - \mathcal{H}_2 = p_1^2 - p_2^2 + m_1^2 - m_2^2 = \left(\varepsilon_2^2 - \varepsilon_1^2 \right) + \frac{2}{w} \left(\varepsilon_1 + \varepsilon_2 \right) P \cdot p + m_1^2 - m_2^2 \approx 0 . \tag{26}$$

Since the ε_i are the c.m. energies, this implies

$$
\begin{aligned}
\varepsilon_1 &= \frac{-p_1 \cdot P}{w} = \frac{-p_1^2 - (p_1 \cdot p_2)}{w} = \frac{-P^2 - p_1^2 + p_2^2}{2w} \approx \frac{w^2 + m_1^2 - m_2^2}{2w}, \\
\varepsilon_2 &= \frac{-p_2 \cdot P}{w} = \frac{-p_2^2 - (p_2 \cdot p_1)}{w} = \frac{-P^2 - p_2^2 + p_1^2}{2w} \approx \frac{w^2 + m_2^2 - m_1^2}{2w},
\end{aligned}
\tag{27}
$$

which in turn gives

$$\varepsilon_1 - \varepsilon_2 \approx \frac{m_1^2 - m_2^2}{w}. \tag{28}$$

If instead one defines

$$\varepsilon_1 - \varepsilon_2 \equiv \frac{m_1^2 - m_2^2}{w}, \tag{29}$$

then

$$\varepsilon_1^2 - \varepsilon_2^2 = m_1^2 - m_2^2, \tag{30}$$

and (26) is a derived constraint[2] with the simple form

$$P \cdot p \approx 0. \tag{31}$$

For the remaining constraint one may use any combination

$$\mathcal{H} = \tilde{\lambda}_1 \mathcal{H}_1 + \tilde{\lambda}_2 \mathcal{H}_2 , \tag{32}$$

of the constraints. The same constraint is obtained provided that $\tilde{\lambda}_1 + \tilde{\lambda}_2 = 1$. The reason is that the constraint $\mathcal{H}_1 - \mathcal{H}_2 = 2P \cdot p \approx 0$. Thus,

$$\begin{aligned}
\mathcal{H}_1 &= p_1^2 + m_1^2 + \Phi = 2\varepsilon_1 p \cdot \hat{P} + p^2 - \varepsilon_1^2 + m_1^2 + \Phi \\
&\approx p^2 - b^2 + \Phi \\
&\approx \mathcal{H}_2 = -2\varepsilon_2 p \cdot \hat{P} + p^2 - \varepsilon_2^2 + m_2^2 + \Phi \\
&\approx p^2 - b^2 + \Phi,
\end{aligned} \tag{33}$$

where

$$b^2 = \varepsilon_1^2 - m_1^2 = \varepsilon_2^2 - m_2^2 = \lambda(w^2, m_1^2, m_2^2)/4w^2, \tag{34}$$

with the triangle function λ is defined as

$$\lambda(w^2, m_1^2, m_2^2) = w^4 + m_1^4 + m_2^4 - 2w^2 m_1^2 - 2w^2 m_2^2 - 2m_1^2 m_2^2. \tag{35}$$

Thus we have

$$\mathcal{H}_1 \approx \mathcal{H}_2 \approx \mathcal{H} \approx p^2 - b^2 + \Phi, \tag{36}$$

as (weakly) equivalent constraint forms. Without interactions, the solution to Eq.(33) becomes

---

[2] The condition Eq.(20) is known as the relativistic version of Newton's third law. As stated there it is a strong equality. It can be relaxed to the weak form $\Phi_1 - \Phi_2 = p \cdot P\chi(x_\perp) \approx 0$ and still lead to compatible constraints.

$$w = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}, \tag{37}$$

indicating exact relativistic two-body kinematics in the center of momentum system (in the c.m.. frame, $p = (0, \mathbf{p})$). Thus, we now have a classical two-body Hamiltonian with exact relativistic two-body kinematics and interacting potentials, though the potentials are still in a general form. Therefore, we must now determine the specific form of the interacting potentials Φ, first for world scalar and vector interactions.

2.2 World Scalar Interactions

This section discusses the world scalar interaction, beginning with a single, spinless particle[25]. We first derive the scalar interaction for a one-body Hamiltonian and then use that result to derive the two-body analog of said Hamiltonian. In addition, we use knowledge of the Klein-Gordon equation (and thus what the equations must reduce down to given certain conditions) to derive the exact form in which the scalar potential appears. The substitution $m \to m + S$, defined as M, is used for a particle in an external scalar field. The Lagrangian, the momentum, and Legendre Hamiltonian are then

$$L = -(m + S)\sqrt{-\dot{x}^2}, \tag{38}$$

$$p = \frac{M\dot{x}}{\sqrt{-\dot{x}^2}}, \tag{39}$$

$$\mathcal{H}_L = \frac{M\dot{x}^2}{\sqrt{-\dot{x}^2}} + M\sqrt{-\dot{x}^2} = 0. \tag{40}$$

The momentum equation then directly leads to the constraint for the one-body case

$$p^2 + M^2 = p^2 + (m + S)^2 \approx 0. \tag{41}$$

For the two-body system, one begins with the free constraints of Eq.(11). The fact that these constraints need to reduce down to a Klein-Gordon form in the case of one mass being much greater than the other mass implies that scalar interactions should be introduced by $m_i \to M_i$,

$$\mathcal{H}_{10} \quad \rightarrow \quad \mathcal{H}_1 = p_1^2 + M_1^2 = p_1^2 + m_1^2 + \Phi_1(x_\perp, P) \approx 0,$$

$$\mathcal{H}_{20} \quad \rightarrow \quad \mathcal{H}_2 = p_2^2 + M_2^2 = p_2^2 + m_2^2 + \Phi_2(x_\perp, P) \approx 0. \tag{42}$$

where

$$M_i = m_i + S_i. \tag{43}$$

From the compatibility condition Eq.(15) and Eq.(20)

$$\Phi_1 \quad = \quad \Phi_2 \equiv \Phi_S,$$

$$\Phi_1 \quad = \quad M_1^2 - m_1^2 \; = 2m_1 S_1 + S_1^2,$$

$$\Phi_2 \quad = \quad M_2^2 - m_2^2 = 2m_2 S_2 + S_2^2, \tag{44}$$

resulting in

$$M_1^2 - M_2^2 = m_1^2 - m_2^2 \; , \tag{45}$$

the solution of which is

$$M_1 \quad = \quad m_1 \cosh \mathcal{L} + m_2 \sinh \mathcal{L} \; , \tag{46}$$

$$M_2 \quad = \quad m_2 \cosh \mathcal{L} + m_1 \sinh \mathcal{L} \; ,$$

where $\mathcal{L}$, S_1,and S_2 are functions of $x_\perp$ in terms of a single underlying invariant function governing the scalar interaction which will be called $S(x_\perp)$.

In order to determine $\mathcal{L}$, S_1, S_2, and thus Φ_S one constructs a mass shell condition for a single particle of relative motion analogous to

$$p^2 + (m + S)^2 \approx 0. \tag{47}$$

This single-particle Klein-Gordon equation has the form

$$\mathbf{p}^2 + \Phi \approx \varepsilon^2 - m^2, \tag{48}$$

13

where

$$\Phi = 2mS + S^2. \tag{49}$$

To continue further, there need to be analogous equations for the two-body case. For a two-body system one has (from Eq.(33))

$$p^2 + \Phi \approx b^2. \tag{50}$$

To parallel what is found in the one-body Klein-Gordon case, one uses the energy and mass for the effective particle of relative motion introduced by Todorov. These are defined by

$$
\begin{aligned}
\varepsilon_w &= \frac{w^2 - m_1^2 - m_2^2}{2w}, \\
m_w &= \frac{m_1 m_2}{w},
\end{aligned}
\tag{51}
$$

which satisfy the Einstein condition

$$b^2 = \varepsilon_w^2 - m_w^2. \tag{52}$$

Thus

$$p^2 + \Phi_S \approx \varepsilon_w^2 - m_w^2, \tag{53}$$

which is directly analogous to the one-body case Eq.(48)

Let us define the four vector momentum analogous to the single particle p^μ

$$\mathfrak{p}^\mu \equiv p^\mu + \frac{\varepsilon_w}{w} P^\mu . \tag{54}$$

In the c.m. system this is

$$\mathfrak{p}^\mu \approx (0, \mathbf{p}) + \frac{\varepsilon_w}{w}(w, 0) = (\varepsilon_w, \mathbf{p}) . \tag{55}$$

14

Since

$$
\begin{aligned}
\mathfrak{p}^2 &= p^2 - \varepsilon_w^2 + 2\frac{\varepsilon_w}{w}p \cdot P, \\
&\approx p^2 - \varepsilon_w^2, \\
b^2 &= \varepsilon_w^2 - m_w^2,
\end{aligned}
\tag{56}
$$

the Hamiltonian Eq.(33)

$$
\mathcal{H} = p^2 - b^2 + \Phi_S \approx 0,
\tag{57}
$$

then assumes the Klein-Gordon form

$$
\mathcal{H} = \mathfrak{p}^2 + m_w^2 + \Phi_S.
\tag{58}
$$

By looking at the single particle Klein-Gordon equations Eq.(48), Eq.(47), and Eq.(49) one can see that the proper choice of Φ_S to bring it to the form required is

$$
\Phi_S = 2m_w S + S^2,
\tag{59}
$$

which finally gives us a Hamiltonian of

$$
\mathcal{H} = \mathfrak{p}^2 + (m_w + S)^2 = p^2 - \varepsilon_w^2 + (m_w + S)^2 \approx 0.
\tag{60}
$$

Note that this constraint can be obtained from its free form $\mathcal{H}_0 = \mathfrak{p}^2 + m_w{}^2$ by replacing m_w with $m_w + S$. This structure for the interaction is supported by two independent field-theoretic arguments, one classical and one quantum-mechanical[30, 31, 32]. In addition, the forms given in Eq.(51) are confirmed in both classical and quantum field theory[3].[48]This formalism is still classical, but we will derive the quantum analog of it and the other potentials.

2.3 World Vector Interactions

This section describes the methods of treating the world vector (A^μ) interaction, which will include time-like and electromagnetic-like vector interactions as well. In a similar manner to our treatment of the scalar potential, we derive a modified mass-shell constraint with a vanishing Legendre Hamiltonian. Unlike

---

[3]This [11] structure and these kinematical forms for the effective mass and energy arise from eikonal approximate sums of ladder and cross ladder diagrams arising from a scalar field theory. [28]

the scalar potential, there are multiple types of vector interactions and we will review the time-like and electromagnetic-like vector interactions in addition to deriving the basic form of the potential. Lastly, we will combine the electromagnetic-like vector interaction with the scalar interaction.

With the choice [4, 26]

$$L = -m\sqrt{-\dot{x}^2} + A{\cdot}\dot{x} \ , \tag{61}$$

one defines the interaction for a single particle with a vector field. The conjugate momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{-\dot{x}^2}} + A, \tag{62}$$

which is rewritten as

$$\pi = p - A = \frac{-m\dot{x}}{\sqrt{-\dot{x}^2}}. \tag{63}$$

This equation leads us to the mass shell constraint of

$$\mathcal{H} = (p - A)^2 + m^2 \approx 0 \ , \tag{64}$$

which from Eq.(63) becomes

$$\mathcal{H} = \left(\pi^2 + m^2\right) \approx 0, \tag{65}$$

and now, as before, the Legendre Hamiltonian is zero.

$$\mathcal{H}_L = \sum p\dot{x} - L = \frac{-m\dot{x}^2}{\sqrt{-\dot{x}^2}} - A\dot{x} - m\sqrt{-\dot{x}^2} + A\dot{x} = 0 \ . \tag{66}$$

Similarly to what was done for the two-body scalar case, one writes in four vector notation the classical minimal substitution

$$p_i^\mu \rightarrow p_i^\mu - A_i^\mu = \pi_i^\mu, \tag{67}$$

which causes the free particle constraints to become the interacting ones below

$$\mathcal{H}_1 = \pi_1^2 + m_1^2 \approx 0, \tag{68}$$

$$\mathcal{H}_2 = \pi_2^2 + m_2^2 \approx 0.$$

Assuming translational invariance so that the potentials depend on coordinates only through the relative coordinate x, these have the form

$$\mathcal{H}_1 = p_1^2 + m_1^2 + \Phi_1(x, p_1, p_2) \approx 0 , \tag{69}$$

$$\mathcal{H}_2 = p_2^2 + m_2^2 + \Phi_2(x, p_1, p_2) \approx 0 . \tag{70}$$

Now that the basic form has been derived, we continue to the specific time-like and electromagnetic-like vector interactions.

2.3.1 Time-like Vector Interactions

For time-like vector interactions one sets[4, 26]

$$A_1^\mu = \hat{P}^\mu \mathcal{V}_1,$$

$$A_2^\mu = \hat{P}^\mu \mathcal{V}_2,$$

$$\pi_1^\mu = E_1 \hat{P}^\mu + p^\mu,$$

$$\pi_2^\mu = E_2 \hat{P}^\mu - p^\mu, \tag{71}$$

which, using the above definition (Eq.(69)) for the Hamiltonian with a vector potential, leads to

$$\mathcal{H}_1 = \pi_1^2 + m_1^2 = p_1^2 + m_1^2 + \Phi_1 \approx 0.$$

$$\mathcal{H}_2 = \pi_2^2 + m_2^2 = p_2^2 + m_2^2 + \Phi_2 \approx 0. \tag{72}$$

where

$$\Phi_1 = \pi_1^2 - p_1^2 = -E_1^2 + \varepsilon_1^2 + 2E_1 p \cdot \hat{P} = -2p_1 \cdot A_1 + A_1^2 = 2\varepsilon_1 \mathcal{V}_1 - \mathcal{V}_1^2,$$

$$\Phi_2 = \pi_2^2 - p_2^2 = -E_2^2 + \varepsilon_2^2 - 2E_2 p \cdot \hat{P} = -2p_2 \cdot A_2 + A_2^2 = 2\varepsilon_2 \mathcal{V}_2 - \mathcal{V}_2^2. \tag{73}$$

Note that these are the vector equivalents to Eq.(42), Eq.(43), and Eq.(44), respectively.

The compatibility condition, here in weak form (see above footnote), is

$$\Phi_1 \approx -E_1^2 + \varepsilon_1^2 \approx \Phi_2 \approx -E_2^2 + \varepsilon_2^2 \equiv \Phi_{\mathcal{V}}(x_\perp), \tag{74}$$

where $\Phi_{\mathcal{V}}(x_\perp)$ is defined by

$$\Phi_{\mathcal{V}}(x_\perp) = -E_1^2 + \varepsilon_1^2 = -E_2^2 + \varepsilon_2^2, \tag{75}$$

or

$$E_1^2 - E_2^2 = \varepsilon_1^2 - \varepsilon_2^2. \tag{76}$$

The solution for Eq.(76) is

$$E_1 = \varepsilon_1 \cosh \mathcal{J} + \varepsilon_2 \sinh \mathcal{J} , \tag{77}$$

$$E_2 = \varepsilon_2 \cosh \mathcal{J} + \varepsilon_1 \sinh \mathcal{J} ,$$

where $\mathcal{J}$, $\mathcal{V}_1$, and $\mathcal{V}_2$ are functions of $x_\perp$ in terms of a single underlying invariant function governing the scalar interaction which will be called $\mathcal{V}(x_\perp)$.

Now we want to find the analogy of Eq.(60) for time-like vector interactions. The one-body Klein-Gordon equation for this is

$$\mathbf{p}^2 - (E - \mathcal{V})^2 + m^2 \approx 0, \tag{78}$$

which can also be written as

$$\mathbf{p}^2 + \Phi \approx E^2 - m^2, \tag{79}$$

18

with

$$\Phi = 2E\mathcal{V} - \mathcal{V}^2. \tag{80}$$

The two-body constraint of Eq.(33)

$$\mathcal{H}_1 \approx \mathcal{H}_2 \approx \mathcal{H} = p^2 - b^2 + \Phi_{\mathcal{V}}(x_\perp), \tag{81}$$

assumes a Klein-Gordon form

$$\mathcal{H} = \mathfrak{p}^2 + m_w^2 + \Phi_{\mathcal{V}} = p^2 - \varepsilon_w^2 + m_w^2 + \Phi_{\mathcal{V}} \approx 0. \tag{82}$$

By looking at Eq.(78), Eq.(79), and Eq.(80) one can see that

$$\begin{aligned}
\Phi_{\mathcal{V}} &= 2\varepsilon_w\mathcal{V} - \mathcal{V}^2, \\
\mathcal{H} &= \mathfrak{p}^2 + m_w^2 + 2\varepsilon_w\mathcal{V} - \mathcal{V}^2 = p^2 + m_w^2 - (\varepsilon_w - \mathcal{V})^2 \approx 0.
\end{aligned} \tag{83}$$

Note that this constraint can be obtained from its free form $\mathcal{H}_0 = \mathfrak{p}^2 + m_w{}^2$ by replacing $\mathfrak{p}$ with $\mathfrak{p} - \mathcal{V}\hat{P}$ or ε_w with $\varepsilon_w - \mathcal{V}$.

2.3.2 Electromagnetic-like Interactions

For spinless particles, there are four independent vectors : x_1 , x_2 , p_1 , and p_2 . This means the A_i^μ can be written as[26]

$$\begin{aligned}
A_1^\mu &= \alpha_1(x_\perp)p_1^\mu + \beta_1(x_\perp)p_2^\mu , \\
A_2^\mu &= \alpha_2(x_\perp)p_2^\mu + \beta_2(x_\perp)p_1^\mu.
\end{aligned} \tag{84}$$

Four vector terms that would produce unobservable gauge changes proportional to x^μ or require the absolute position x_i^μ of particles have been omitted. Such interactions include both space- and time-like four vector interactions, and under special relations between the $\alpha's$ and $\beta's$ are called electromagnetic-like interactions due to how the potentials share similarities with the related time and space components of vector potentials in classical electromagnetism. Using the variables $\hat{P}^\mu$ and p^μ, these can be incorporated into $\pi_i = p_i - A_i$

as

$$\pi_1^\mu = E_1 \hat{P}^\mu + G_1 p^\mu \ ,$$
$$\pi_2^\mu = E_2 \hat{P}^\mu - G_2 p^\mu \ , \tag{85}$$

Thus

$$\pi_1 - p_1 = -A_1 = (E_1 - \varepsilon_1)\hat{P} + (G_1 - 1)p \ ,$$
$$A_1 = (\varepsilon_1 - E_1)\hat{P} - (G_1 - 1)p \ ,$$
$$\pi_2 - p_2 = -A_2 = (E_2 - \varepsilon_2)\hat{P} - (G_2 - 1)p \ ,$$
$$A_2 = (\varepsilon_2 - E_2)\hat{P} + (G_2 - 1)p \ . \tag{86}$$

When A_i is time-like (parallel to $\hat{P}$), $G_1 - 1 = G_2 - 1 = 0$ and

$$G_1 = G_2 = G_{TL} = 1 \ . \tag{87}$$

One assumes that

$$G_1 = G_2 = G \equiv \exp(\mathcal{G}) \tag{88}$$

for general vector interaction. This implies that E_1, E_2, and G depend on two invariant functions, which will be called A and $\mathcal{V}$. A is redefined instead of G as a basic unit in the potential by choosing $G = G(A)$. The most general choice for E_i is then $E_i = E_i(A, \mathcal{V})$, so

$$\pi_1^\mu = E_1(A, \mathcal{V})\hat{P}^\mu + G(A)p^\mu \ ,$$
$$\pi_2^\mu = E_2(A, \mathcal{V})\hat{P}^\mu - G(A)p^\mu \ , \tag{89}$$

which modifies the free expressions

$$p_1^\mu = \varepsilon_1 \hat{P}^\mu + p^\mu,$$
$$p_2^\mu = \varepsilon_2 \hat{P}^\mu - p^\mu, \tag{90}$$

the case of pure time like invariant $\mathcal{V} = 0$ and

$$E_i(A) \equiv G(A)(\varepsilon_i - A), \tag{91}$$

with

$$G^2 = \frac{1}{1 - 2A/w}, \tag{92}$$

corresponds to what will be called the electromagnetic-like vector interaction. With this choice the Hamiltonian is then

$$
\begin{aligned}
\mathcal{H}_1 &= \pi_1^2 + m_1^2 = -E_1^2 + m_1^2 + G^2 p^2 + 2E_1 G p \cdot \hat{P} \approx -E_1^2 + m_1^2 + G^2 p^2 \\
&\approx \mathcal{H}_2 = \pi_2^2 + m_2^2 = -E_2^2 + m_2^2 + G^2 p^2 - 2E_2 G p \cdot \hat{P} \approx -E_2^2 + m_2^2 + G^2 p^2 \\
&\approx \mathcal{H} \equiv p^2 - b^2 + \Phi_A \\
&= G^2(p^2 - (\varepsilon_1 - A)^2 + m_1^2/G^2) = G^2(p^2 - (\varepsilon_2 - A)^2 + m_2^2/G^2) \\
&= G^2(p^2 + 2\varepsilon_w A - A^2 - b^2) \tag{93}
\end{aligned}
$$

where Eqs.(91) and (92) have been used to obtain the result in the last line above. Thus one arrives at

$$\Phi_A = (G^2 - 1)(p^2 - b^2) + G^2(2\varepsilon_w A - A^2) \tag{94}$$

so that similar to Eq.(83)

$$\mathcal{H} = G^2(p^2 + m_w^2 - (\varepsilon_w - A)^2). \tag{95}$$

Note that since $\varepsilon_1^2 - E_1^2 = \varepsilon_2^2 - E_2^2$ one still has the hyperbolic Eq.(77), but now Eq.(91) implies that

$$\mathcal{J} = -\mathcal{G}. \tag{96}$$

21

2.3.3 Scalar and Electromagnetic-like Combination

If one combines both scalar- and electromagnetic-like interactions, one has

$$
\begin{aligned}
\mathcal{H}_1 &= \pi_1^2 + M_1^2 = -E_1^2 + M_1^2 + G^2 p^2 + 2E_1 G p \cdot \hat{P} \approx -E_1^2 + M_1^2 + G^2 p^2 \\
&\approx \mathcal{H}_2 = \pi_2^2 + M_2^2 = -E_2^2 + M_2^2 + G^2 p^2 - 2E_2 G p \cdot \hat{P} \approx -E_2^2 + M_2^2 + G^2 p^2 \\
&\approx \mathcal{H} = p^2 - b^2 + \Phi_A + \Phi_S. \\
&= G^2(p^2 - (\varepsilon_1 - A)^2 + M_1^2/G^2) = G^2(p^2 - (\varepsilon_2 - A)^2 + M_2^2/G^2).
\end{aligned}
\tag{97}
$$

Following the analogy between Eqs.(83) and (95) it is plausible, in light of Eq.(60), to take

$$
\begin{aligned}
\mathcal{H} &= G^2(p^2 + (m_w + S)^2 - (\varepsilon_w - A)^2) \\
&= G^2(p^2 - b^2 + 2m_w S + S^2 + 2\varepsilon_w A - A^2).
\end{aligned}
\tag{98}
$$

If one takes Φ_A as given in Eq.(94) then

$$
\begin{aligned}
\Phi_S &= (G^2 - 1)(p^2 - b^2) + G^2(p^2 - b^2 + 2m_w S + S^2 + 2\varepsilon_w A - A^2) \\
&= G^2 \left(2m_w S + S^2\right),
\end{aligned}
\tag{99}
$$

which, unlike Eq.(59) depends on the invariant A responsible for vector interactions. This equation and the last line of Eq.(97) imply

$$
\begin{aligned}
M_1^2 &= m_1^2 + G^2 \left(2m_w S + S^2\right), \\
M_2^2 &= m_1^2 + G^2 \left(2m_w S + S^2\right),
\end{aligned}
\tag{100}
$$

and so

$$
M_1^2 - M_2^2 = m_1^2 - m_2^2.
\tag{101}
$$

This again gives the hyperbolic solution of Eq.(46). We have derived the general form of the vector potential and then shown how it specializes to the time-like and electromagnetic-like forms for a one-body Hamiltonian and then continued to the derivation of the two-body Hamiltonian equations. In addition, we have then shown the combination of the scalar and electromagnetic-like vector interactions.

What we have shown so far is entirely classical, so now that the potentials are completely defined in a classical formalism, we will quantize this system.

2.3.4 Quantum Constraint Dynamics for the Two-Body Relativistic System

Now that we have described all of the relevant interactions relating to the two-body Hamiltonians classically, it becomes pertinent to review Dirac's method of conversion to quantum mechanics from classical mechanics. First, the constraints derived above now become conditions on the wave function, i.e. $\mathcal{H} \approx 0 \rightarrow \mathcal{H}\psi \approx 0$. Secondly, Poisson brackets are now commutators. In the quantum case one must use

$$[x_\perp^\mu, p^\nu] = i(\eta^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}) , \tag{102}$$

as a check to verify that the commutator $[\mathcal{H}_1, \mathcal{H}_2]$ (the quantum compatibility condition, equivalent to the classical $\{\mathcal{H}_1, \mathcal{H}_2\}$) on the wave function vanishes. This ensures that the quantum versions of the constraints

$$\mathcal{H}_1\psi = 0 , \tag{103}$$
$$\mathcal{H}_2\psi = 0,$$

are compatible wave function conditions, and

$$P \cdot p \approx 0 \rightarrow P \cdot p\psi = 0 . \tag{104}$$

Now we see how this applies to each of the interacting potentials.

2.3.5 Quantum Constraint Dynamics for World Scalar Interaction

This is a brief review of the methods for adapting the above section of the scalar interaction into the quantum realm. By setting $\Phi_S = 2m_w S + S^2$ (the scalar potential), the condition

$$P \cdot p \approx 0, \tag{105}$$

gives

$$P \cdot p \approx 0 \rightarrow P \cdot p\psi = 0 , \tag{106}$$

and the quantized Hamiltonian is

$$\mathcal{H} = \mathfrak{p}^2 + m_w^2 + \Phi_S \approx 0 \rightarrow (\mathfrak{p}^2 + (m_w + S)^2)\psi = 0. \tag{107}$$

The eigenvalue form of the Hamiltonian above is, in the center of momentum system,

$$(\mathbf{p}^2 + 2m_wS + S^2)\psi = b^2(w)\psi. \tag{108}$$

Comparing this to the nonrelativistic energy expression

$$\mathbf{p}^2\psi(\mathbf{r}) + 2mU(\mathbf{r})\psi(\mathbf{r}) = 2mE\psi(\mathbf{r}), \tag{109}$$

one can see that $2m_wS + S^2$ and $b^2(w)$ are analogous to $2mU(\mathbf{r})$ and $2mE$, respectively. The scalar interaction has then been properly defined in the quantum realm, so we continue to a description of the vector interaction in a quantum formalism.

2.3.6 Quantum Constraint Dynamics for World Vector Interactions

To construct the quantum versions of the $\mathcal{H}_i$'s we must maintain the minimal substitution form of π_i^μ. Classically, the underlying scalars $(S, \ \mathcal{V}, \ A)$ may depend on $x_\perp^2, x_\perp \cdot p, p^2$,and w^2 or $x_\perp^2, l^2, p^2$, and w^2 where

$$\begin{aligned} l^2 &= x_\perp^2 p^2 - (x_\perp \cdot p)^2, \\ &= (\mathbf{r} \times \mathbf{p})^2, \text{(in center of momentum system)} \end{aligned} \tag{110}$$

is the invariant square of the relative angular momentum. A scale transformation on the wavefunction can eliminate linear dependence on $x_\perp \cdot p$. The system variables p and $\hat{P}$ are well defined if the space is restricted so that P^2 has only time-like eigenvalues. Assuming that the E_i $[= G(\varepsilon_i - A_i)]$ are Hermitian, then starting with the classical expression equation

$$\pi_1^\mu = E_1(A)\hat{P}^\mu + G(A)p^\mu, \tag{111}$$

the quantum Hermitian version of which is

$$\begin{aligned} (\pi_1^\mu)_{\mathrm{H}} &= \frac{\pi_1^\mu + \pi_1^{\mu\dagger}}{2} = E_1(A)\hat{P}^\mu + \frac{G(A)p^\mu + p^\mu G(A)}{2} \\ &= E_1(A)\hat{P}^\mu + \frac{G(A)p^\mu + G(A)p^\mu + [p^\mu, G(A)]}{2} \\ &= E_1(A)\hat{P}^\mu + G(A)p^\mu + \frac{1}{2i}\nabla^\mu G(A) \ . \end{aligned} \tag{112}$$

These equations combined with

$$
\begin{aligned}
E_1(A) &= G(\varepsilon_1 - A_1), \\
E_2(A) &= G(\varepsilon_2 - A_2) \, ,
\end{aligned}
\tag{113}
$$

give us the Hermitian forms

$$
\begin{aligned}
\pi_1^\mu &= G\left[\hat{P}^\mu(\varepsilon_1 - A) + p^\mu + \frac{1}{2i}\nabla^\mu\mathcal{G}\right] \, , \\
\pi_2^\mu &= G\left[\hat{P}^\mu(\varepsilon_2 - A) - p^\mu - \frac{1}{2i}\nabla^\mu\mathcal{G}\right] \, ,
\end{aligned}
\tag{114}
$$

where $\mathcal{G} = \ln G$. Using the quantum brackets

$$
[x_\perp^\mu, p^\nu] = i\left(\eta^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right) = i(\eta^{\mu\nu} + \hat{P}^\mu\hat{P}^\nu) \, ,
\tag{115}
$$

One verifies that the commutator $[\mathcal{H}_1, \mathcal{H}_2]$ on the wavefunction vanishes and that the quantum analogs of the classical constraints $(\mathcal{H}_i = 0)$ become conditions on the wave function

$$
\begin{aligned}
\mathcal{H}_1\psi &= (\pi_1^2 + m_1^2)\psi = 0 \, , \\
\mathcal{H}_2\psi &= (\pi_2^2 + m_2^2)\psi = 0 \, .
\end{aligned}
\tag{116}
$$

Due to the two-body system being an isolated one and ψ being an eigenfunction of the total momentum, P^μ becomes a constant of the motion. The difference constraint

$$
(\mathcal{H}_1 - \mathcal{H}_2)\psi = 0,
\tag{117}
$$

then gives

$$
P \cdot p\psi = 0 \, ,
\tag{118}
$$

which is now a differential equation as $p^\mu = -i\partial/\partial x_\mu$ in the coordinate representation. The quantum counterpart to the remaining independent constraint $(\mathcal{H} \approx 0)$ is $\mathcal{H}\psi = 0$ for the system wave function. In the c.m. system this yields

$$\mathcal{H}\psi = G^2[\mathbf{p}^2 - (\varepsilon_w - A)^2 + m_w^2 + \frac{2}{i}\nabla\mathcal{G}\cdot\mathbf{p} - \frac{1}{2}\nabla^2\mathcal{G} - \frac{3}{4}(\nabla\mathcal{G})^2]\psi = 0 \ . \tag{119}$$

When separate time and electromagnetic-like vectors and the world scalar interaction are used then our quantum constraint equation can be written as [4]

$$G^2[\mathbf{p}^2 - (\varepsilon_w - A)^2 + 2\varepsilon_w\mathcal{V} - \mathcal{V}^2 + (m_w + S)^2 + \frac{2}{i}\boldsymbol{\nabla}\mathcal{G}\cdot\mathbf{p} - \frac{1}{2}\boldsymbol{\nabla}^2\mathcal{G} - \frac{3}{4}(\nabla\mathcal{G})^2]\psi = 0 \ . \tag{120}$$

If one lets $\phi = G\psi$, the above equations become

$$\left[\mathbf{p}^2 - (\varepsilon_w - A)^2 + \frac{1}{2}\boldsymbol{\nabla}^2\mathcal{G} + \frac{1}{4}(\boldsymbol{\nabla}\mathcal{G})^2\right]\phi = 0, \tag{121}$$

and[4]

$$\left[\mathbf{p}^2 - (\varepsilon_w - A)^2 + 2\varepsilon_w\mathcal{V} - \mathcal{V}^2 + (m_w + S)^2 + \frac{1}{2}\boldsymbol{\nabla}^2\mathcal{G} + \frac{1}{4}(\boldsymbol{\nabla}\mathcal{G})^2\right]\phi = 0 \ , \tag{122}$$

respectively[25], which is the general Klein-Gordon form for two spinless particles. The terms with $\mathcal{G}$ can be regarded as quantum recoil corrections. That is, they would vanish if one of the particles becomes very massive. This completes the review of the relativistic quantum mechanical fundamentals of constraint dynamics for spinless particles, so we will now continue on to the relativistic forms for spin-one-half particles (i.e. the Dirac equation and how we arrive at the two-body Dirac equations).

2.4 One Body Dirac Equation

Since the potential framework that is used to extend the above wave equation for the two-body Klein-Gordon equations to the two-body Dirac equations in the constraint formalism has been assembled, it is helpful to briefly review the one-body Dirac equation in order to see how spin dependence comes into play. Defining the operator $\mathcal{S}$ as[4]

$$\mathcal{S} = \gamma_5(\boldsymbol{\gamma}\cdot\mathbf{p} - E\beta + M), \tag{123}$$

the Dirac equation can be written in the form

$$\begin{aligned}
\mathcal{S}\psi &= \gamma_5(\boldsymbol{\gamma}\cdot\mathbf{p} - E\beta + M)\psi \tag{124}\\
&= (-\beta\boldsymbol{\Sigma}\cdot\mathbf{p} + E\beta\gamma_5 + M\gamma_5)\psi = 0,
\end{aligned}$$

---
[4]The γ_5 here does not affect the physics but simplifies the compatibility conditions in the two-body case.

where $E = \varepsilon - \mathcal{V}$, and $M = m + S$, accounting for the scalar and vector interactions to be defined later. The 4×4 matrices are[14]

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{125}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{126}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{127}$$

With the γ_5 matrix

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{128}$$

and the four component spinor written in terms of two component subspinors

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{129}$$

Then the Dirac equation becomes

$$(-\boldsymbol{\sigma} \cdot \mathbf{p})\psi_1 + (E + M)\psi_2 = 0,$$

$$(-\boldsymbol{\sigma} \cdot \mathbf{p})\psi_2 + (E + M)\psi_1 = 0, \tag{130}$$

by "splitting" the matrix equation into two linear ones. If one solves the second equation for ψ_2 and substitutes into the first, one obtains

$$[\mathbf{p}^2 - E^2 + M^2 + i(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \ln(E + M))\boldsymbol{\sigma} \cdot \mathbf{p}]\psi_1 = 0, \tag{131}$$

and similarly solving for ψ_2 from the first and substituting into the second

$$[\mathbf{p}^2 - E^2 + M^2 - i(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \ln(E - M))\boldsymbol{\sigma} \cdot \mathbf{p}]\psi_2 = 0. \tag{132}$$

If one expands using an identity of Pauli $\boldsymbol{\sigma}$ matrices $(\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k)$ and take $\boldsymbol{\nabla} \ln(X) = \ln'(X)\hat{\mathbf{r}}$, one arrives at

$$[\mathbf{p}^2 - E^2 + M^2 + i \ln'(E+M)\hat{\mathbf{r}} \cdot \mathbf{p} - \frac{\ln'(E+M)}{r}\mathbf{L} \cdot \boldsymbol{\sigma}]\psi_1 \quad = \quad 0,$$

$$[\mathbf{p}^2 - E^2 + M^2 - i \ln'(E-M)\hat{\mathbf{r}} \cdot \mathbf{p} + \frac{\ln'(E-M)}{r}\mathbf{L} \cdot \boldsymbol{\sigma}]\psi_2 \quad = \quad 0, \tag{133}$$

and thus we can see how the reduced Dirac formalism reveals the familiar spin dependence with the $\frac{\mathbf{L}}{r}$ term and Darwin interactions, which together give the correct spectral results for all orbital angular momentum for Hydrogen[33]. This well-known result is explicitly displayed here to demonstrate in a simple form what the more complex two-body Dirac formalism does later.

Now that the familiar formalism[14] for one-body Dirac equations has been reviewed, we can discuss two-body Dirac equations.

2.5 Two-Body Dirac Equations

This section presents a review of two-body Dirac equations[25, 26]. The Dirac equations for two free particles can be written in the form

$$\mathcal{S}_{10}\psi \quad = \quad (\theta_1 \cdot p_1 + m_1\theta_{51})\psi = \gamma_{51}i\sqrt{\frac{1}{2}}(\gamma_1^\mu p_1^\mu + m_1)\psi = 0, \tag{134}$$

$$\mathcal{S}_{20}\psi \quad = \quad (\theta_2 \cdot p_2 + m_2\theta_{52})\psi = \gamma_{52}i\sqrt{\frac{1}{2}}(\gamma_2^\mu p_2^\mu + m_2)\psi = 0.$$

The theta matrices are defined as[35]

$$\theta_i^\mu \quad \equiv \quad i\sqrt{\frac{1}{2}}\gamma_{5i}\gamma_i^\mu, \ \mu = 0, 1, 2, 3 \ , i = 1, 2,$$

$$\theta_{5i} \quad \equiv \quad i\sqrt{\frac{1}{2}}\gamma_{5i}, \tag{135}$$

and these satisfy the anti-commutation and commutation relations

$$
\begin{aligned}
\left[\theta_i^\mu, \theta_i^\nu\right]_+ &= -\eta^{\mu\nu}, \\
\left[\theta_{5i}, \theta_i^\mu\right]_+ &= 0, \\
\left[\theta_{5i}, \theta_{5i}\right]_+ &= -1, \\
\left[\theta_i^\mu, \theta_j^\nu\right]_- &= 0, \ i \neq j, \\
\left[\theta_{5i}, \theta_{5j}\right]_- &= 0, \ i \neq j, \\
\left[\theta_i^\mu, \theta_{5j}\right]_- &= 0, \ i \neq j,
\end{aligned}
\tag{136}
$$

In terms of variables defined earlier

$$
p_1 = \varepsilon_1 \hat{P} + p, \quad p_2 = \varepsilon_2 \hat{P} - p,
\tag{137}
$$

where $\hat{P} = P/w$, Eq.(134) can now be written as

$$
\begin{aligned}
\mathcal{S}_{10}\psi &= (\theta_1 \cdot p + \varepsilon_1 \theta_1 \cdot \hat{P} + m_1 \theta_{51})\psi = 0, \\
S_{20}\psi &= (-\theta_2 \cdot p + \varepsilon_2 \theta_2 \cdot \hat{P} + m_2 \theta_{52})\psi = 0.
\end{aligned}
\tag{138}
$$

Using Eq.(136) one can show that

$$
[\mathcal{S}_{10}, S_{20}]\psi = 0.
\tag{139}
$$

The projections of these theta matrices satisfy

$$
\begin{aligned}
\left[\theta_i \cdot \hat{P}, \theta_i \cdot \hat{P}\right]_+ &= 1, \\
\left[\theta_i \cdot \hat{P}, \theta_{i\perp}^\mu\right]_+ &= 0,
\end{aligned}
\tag{140}
$$

with $\theta_{i\perp}^\mu = \theta_{i\nu}(\eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu)$. These theta matrices are Dirac matrices modified in such a way as to ensure that the operators $\mathcal{S}_{10}$ and $\mathcal{S}_{20}$ are the square root operators of the corresponding mass-shell operators, that is $\mathcal{S}_{10}^2 = -\frac{1}{2}(p_1^2 + m_1^2)$ and $S_{20}^2 = -\frac{1}{2}(p_2^2 + m_2^2)$[5].

The difference

---

[5]In the ordinary Dirac matrix formalism, the Klein-Gordon operator $(p_1^2 + m_1^2)$ is obtained not from the direct square of Dirac operator $\gamma_1 \cdot p_1 + m_1$ but from multiplications by the related operator $-\gamma_1 \cdot p_1 + m_1$.

$$(\mathcal{S}_{10}^2 - \mathcal{S}_{20}^2)\psi = 0 = \frac{1}{2}(p_1^2 + m_1^2 - p_2^2 - m_2^2)\psi, \tag{141}$$

leads to the equation

$$P \cdot p\psi = \frac{1}{2}[w(\varepsilon_1 - \varepsilon_2) - (m_1^2 - m_2^2)]\psi = 0, \tag{142}$$

which is the same constraint on the relative momentum that eliminates the relative energy in the center of momentum frame that appeared in the spinless case.

Covariant Dirac α and β matrices are defined as

$$\begin{aligned}
\beta_i &= -\gamma_i \cdot \hat{P} = 2\theta_{5i}\theta_i \cdot \hat{P}, \\
\alpha_i^\mu &= 2\theta_{i\perp}^\mu \theta_i \cdot \hat{P},
\end{aligned} \tag{143}$$

along with the Σ matrix

$$\Sigma_i^\mu = \gamma_{5i}\alpha_i^\mu = 2\sqrt{2}i\theta_{5i}\,\theta_i \cdot \hat{P}\,\theta_{\perp i}, \quad i = 1, 2. \tag{144}$$

These take on the simpler forms $\alpha_i^\mu = (0, \boldsymbol{\alpha}_i)$ and $\Sigma_i^\mu = (0, \boldsymbol{\Sigma}_i)$ in the center of momentum system for which $\hat{P} = (1, \mathbf{0})$. In that case the free two-body Dirac equations take the more familiar forms of

$$\theta_{5i}(\boldsymbol{\gamma}_i \cdot \mathbf{p} - \beta_i\varepsilon_i + m_i)\psi = 0 \; ; \; i = 1, 2, \tag{145}$$

analogous to Eq.(124), or

$$(\boldsymbol{\gamma}_i \cdot \mathbf{p} - \beta_i\varepsilon_i + m_i)\psi = 0 \; ; \; i = 1, 2. \tag{146}$$

If one makes the naive substitutions for scalar interactions defined by Eq.(43), the Dirac equations would become

$$\begin{aligned}
\mathcal{S}_1\psi &= (\theta_1 \cdot p + \varepsilon_1\theta_1 \cdot \hat{P} + M_1\theta_{51})\psi = 0, \\
\mathcal{S}_2\psi &= (-\theta_2 \cdot p + \varepsilon_2\theta_2 \cdot \hat{P} + M_2\theta_{52})\psi = 0,
\end{aligned} \tag{147}$$

which are then not compatible, that is

$$[\mathcal{S}_1, \mathcal{S}_2]_- \psi = [\theta_1 \cdot p, M_2 \theta_{52}]_- + [M_1 \theta_{51}, -\theta_2 \cdot p]_-$$

$$= -i(\partial M_1 \cdot \theta_1 \theta_{52} + \partial M_2 \cdot \theta_2 \theta_{51})\psi \neq 0. \tag{148}$$

Using supersymmetry arguments and the third law condition of Eq.(45) and Eq.(46), one can obtain the compatible spin-dependent constraints $\mathcal{S}_1$ and $\mathcal{S}_2$ below[35],

$$\mathcal{S}_1 \psi = (\theta_1 \cdot p + \varepsilon_1 \theta_1 \cdot \hat{P} + M_1 \theta_{51} - i\partial\mathcal{L} \cdot \theta_2 \theta_{52}\theta_{51})\psi = 0, \tag{149}$$
$$\mathcal{S}_2 \psi = (-\theta_2 \cdot p + \varepsilon_2 \theta_2 \cdot \hat{P} + M_2 \theta_{52} + i\partial\mathcal{L} \cdot \theta_1 \theta_{52}\theta_{51})\psi = 0,$$

which commute with each other. These equations are known as the external potential forms, due to how each particle's potential is created by the other particle. They satisfy

$$[\mathcal{S}_1, \mathcal{S}_2] = 0, \tag{150}$$

provided that

$$\partial\mathcal{L} = \frac{\partial M_1}{M_2} = \frac{\partial M_2}{M_1}. \tag{151}$$

The strong compatibility due to a supersymmetry is responsible for producing the spin-dependent recoil terms depending on $\partial\mathcal{L}$. It can be shown that these terms vanish when one particle is infinitely massive relative to the other ($M_1 >> M_2$ or vice versa, giving $\partial\mathcal{L} = \frac{\partial M_1}{M_2} = \frac{\partial M_1}{\infty} = 0$ or $\partial\mathcal{L} = \frac{\partial M_2}{M_1} = \frac{\partial M_2}{\infty} = 0$), thereby recovering the one-body Dirac equation. The operators S_1 and S_2 satisfy the relation

$$(\mathcal{S}_1^2 - \mathcal{S}_2^2)\psi = -\frac{1}{2}(p_1^2 + m_1^2 - p_2^2 - m_2^2)\psi = -P \cdot p\psi = 0, \tag{152}$$

thus showing that the relative momentum remains orthogonal to the total momentum.

The above difficulty with compatibility from Eq.(148)can also be overcome if the scalar interaction is replaced by vector, pseudoscalar, pseudovector, or tensor potentials as described below. We first rewrite the Dirac equations Eq.(149) for scalar interactions in the hyperbolic form of[28]

$$\mathcal{S}_1\psi \quad = \quad (\cosh(\Delta) \, \mathbf{S}_1 + \sinh(\Delta) \, \mathbf{S}_2)\psi = 0,$$

$$S_2\psi \quad = \quad (\cosh(\Delta) \, \mathbf{S}_2 + \sinh(\Delta) \, \mathbf{S}_1)\psi = 0, \tag{153}$$

where

$$\mathbf{S}_1\psi \quad \equiv \quad (S_{10}\cosh(\Delta) \, + S_{20}\sinh(\Delta) \,)\psi = 0, \tag{154}$$

$$\mathbf{S}_2\psi \quad \equiv \quad (S_{20}\cosh(\Delta) \, + S_{10}\sinh(\Delta) \,)\psi = 0,$$

and

$$\Delta = -\theta_{51}\theta_{52}\mathcal{L}(x_\perp). \tag{155}$$

Here, $\mathbf{S}_1$ and $\mathbf{S}_2$ are auxiliary constraint operators.

It turns out that for all arbitrary Δ satisfying

$$[P \cdot p, \Delta]\psi \quad = \quad 0,$$

$$\Delta(x) \quad = \quad \Delta(x_\perp),$$

$$P \cdot p\psi \quad = \quad 0, \tag{156}$$

this leads to weak compatibility on both sets of constraints

$$[\mathcal{S}_1, \mathcal{S}_2]\psi \quad = \quad 0,$$

$$[\mathbf{S}_1, \mathbf{S}_2]\psi \quad = \quad 0. \tag{157}$$

The function Δ is defined as

$$\Delta_\mathcal{L} = -\mathcal{L}\theta_{51}\theta_{52} \equiv -1\frac{\mathcal{L}}{2}\mathcal{O}_1, \ \mathcal{O}_1 = -\gamma_{51}\gamma_{52}, \tag{158}$$

for scalar interactions,

$$\Delta_\mathcal{J} = \mathcal{J}\hat{P} \cdot \theta_1 \hat{P} \cdot \theta_2 \equiv \mathcal{O}_2\frac{\mathcal{J}}{2} = \beta_1\beta_2\frac{\mathcal{J}}{2}\mathcal{O}_1, \tag{159}$$

for time-like vector interactions, and

$$\Delta_{\mathcal{G}} = \mathcal{G}\theta_{1\perp} \cdot \theta_{2\perp} \equiv \mathcal{O}_3 \frac{\mathcal{G}}{2} = \gamma_{1\perp} \cdot \gamma_{2\perp} \frac{\mathcal{G}}{2} \mathcal{O}_1, \tag{160}$$

for space-like vector interactions. Here, $\mathcal{L}, \mathcal{J}$, and $\mathcal{G}$ are all invariant arbitrary functions of $x_\perp$ related to each interaction. For the axial interactions, see [28],[35].

2.5.1 Quantum Analogy and Reduction to Schrödinger-like Form

Similar to the one-body case, this section discusses an analogy of the two-body for converting to Schrödinger-like equation. This reveals one of the advantages to our methods in that methods for solving Schrödinger-like equations are very well known and so if we can convert any formalism into this form, a final solution becomes much more feasible. For the combined scalar, time-like vector, and space-like vector, the equations Eq.(154,153) take the form below after bringing the free Dirac operators through to the right [28],[35]

$$\mathcal{S}_1 \psi = (G\theta_1 \cdot p + E_1 \theta_1 \cdot \hat{P} + M_1 \theta_{51} + i\frac{G}{2}(\theta_2 \cdot \partial \mathcal{G}\mathcal{O}_3 + \theta_2 \cdot \partial \mathcal{J}\mathcal{O}_2 - \theta_2 \cdot \partial \mathcal{L}\mathcal{O}_1))\psi = 0, \tag{161}$$

$$\mathcal{S}_2 \psi = (-G\theta_2 \cdot p + E_2 \theta_2 \cdot \hat{P} + M_2 \theta_{52} - i\frac{G}{2}(\theta_1 \cdot \partial \mathcal{G}\mathcal{O}_3 + \theta_1 \cdot \partial \mathcal{J}\mathcal{O}_2 - \theta_1 \cdot \partial \mathcal{L}\mathcal{O}_1))\psi = 0. \tag{162}$$

In the two-body case, the wavefunction is now a 16 component spinor, with each individual ψ_i having four components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{163}$$

The end result of the matrix multiplications of $\mathcal{S}_i \psi = 0$ is a set of eight simultaneous equations for the Dirac spinors $\psi_1, \psi_2, \psi_3, \psi_4$. There is a set of four simultaneous equations from the constraint of the two-body Dirac equations that $\mathcal{S}_1 \psi = 0$ and another set of four simultaneous equations from the second of the constraint two-body Dirac equations $\mathcal{S}_2 \psi = 0$. In order to fully solve these equations, we define the terms

$$\phi_\pm \equiv \psi_1 \pm \psi_4, \tag{164}$$

$$\chi_\pm \equiv \psi_2 \pm \psi_3, \tag{165}$$

$$D_1^{++} = e^{\mathcal{G}}(\sigma_1 \cdot p - \frac{i}{2}\sigma_2 \cdot \partial(\mathcal{J} - \mathcal{L} + \mathcal{G}\sigma_1 \cdot \sigma_2)), \tag{166}$$

$$D_2^{++} = e^{\mathcal{G}}(\sigma_2 \cdot p - \frac{i}{2}\sigma_1 \cdot \partial(\mathcal{J} - \mathcal{L} + \mathcal{G}\sigma_1 \cdot \sigma_2)), \tag{167}$$

$$D_1^{-+} = e^{\mathcal{G}}(\sigma_1 \cdot p - \frac{i}{2}\sigma_2 \cdot \partial(\mathcal{J} + \mathcal{L} + \mathcal{G}\sigma_1 \cdot \sigma_2)), \tag{168}$$

$$D_1^{--} = e^{\mathcal{G}}(\sigma_1 \cdot p + \frac{i}{2}\sigma_2 \cdot \partial(\mathcal{J} + \mathcal{L} - \mathcal{G}\sigma_1 \cdot \sigma_2)), \tag{169}$$

$$E_1 = \varepsilon_1 \cosh(\mathcal{J}) + \varepsilon_2 \sinh(\mathcal{J}), \tag{170}$$

$$E_2 = \varepsilon_2 \cosh(\mathcal{J}) + \varepsilon_1 \sinh(\mathcal{J}), \tag{171}$$

$$M_1 = m_1 \cosh(\mathcal{L}) + m_2 \sinh(\mathcal{L}), \tag{172}$$

$$M_2 = m_2 \cosh(\mathcal{L}) + m_1 \sinh(\mathcal{L}), \tag{173}$$

and work out the matrices to obtain

$$D_1^{++}\phi_+ = E_1\chi_+ - M_1\chi_-, \tag{174}$$

$$D_2^{+-}\phi_+ = E_2\chi_+ + M_2\chi_-, \tag{175}$$

$$D_1^{-+}\chi_+ = E_1\phi_+ - M_1\phi_-, \tag{176}$$

$$D_1^{--}\chi_- = -E_1\phi_- + M_1\phi_+. \tag{177}$$

These equations can be solved through a system of substitution and elimination similar to what was done for the one-body Dirac equation (Eq.(134)), since there are now four equations and four unknowns. The result is[35]

$$\begin{aligned}
[E_1 D_1^{-+}&\frac{1}{E_1 M_2 + E_2 M_1}(M_2 D_1^{++} - M_1 D_2^{++}) + \\
M_1 D_1^{--}&\frac{1}{E_1 M_2 + E_2 M_1}(E_2 D_1^{++} + E_1 D_2^{++})]\phi_+ \\
= \quad (E_1^2 &- M_1^2)\phi_+.
\end{aligned} \tag{178}$$

Further expansion of terms reveals [35]

$$[p^2 + 2m_w S + S^2 + 2\varepsilon_w A - A^2$$

$$-[2\mathcal{G}' - \frac{E_2 M_2 + E_1 M_1}{E_2 M_1 + E_1 M_2}(\mathcal{L} - \mathcal{G})']i\hat{r} \cdot p - \frac{1}{2}\nabla^2 \mathcal{G} - \frac{1}{4}(\mathcal{G})'^2 - (\mathcal{G}' + \mathcal{L}')^2 + \frac{E_2 M_2 + E_1 M_1}{E_2 M_1 + E_1 M_2}\frac{1}{2}\mathcal{G}'(\mathcal{L} - \mathcal{G}')$$

$$+\frac{L \cdot (\sigma_1 + \sigma_2)}{r}[\mathcal{G}' - \frac{1}{2}\frac{E_2 M_2 + E_1 M_1}{E_2 M_1 + E_1 M_2}(\mathcal{L} - \mathcal{G})'] - \frac{L \cdot (\sigma_1 - \sigma_2)}{2r}\frac{E_2 M_2 - E_1 M_1}{E_2 M_1 + E_1 M_2}(\mathcal{L} - \mathcal{G})'$$

$$+\sigma_1 \cdot \sigma_2(\frac{1}{2}\nabla^2 \mathcal{G} + \frac{1}{2r}\mathcal{L}' + \frac{1}{2}(\mathcal{G}')^2 - \frac{1}{2}\mathcal{G}'(\mathcal{L} - \mathcal{G})'\frac{E_2 M_2 + E_1 M_1}{E_2 M_1 + E_1 M_2})$$

$$+\sigma_1 \cdot \hat{r}\sigma_2 \cdot \hat{r}(\frac{1}{2}\nabla^2 \mathcal{L} - \frac{3}{2r}\mathcal{L}' + \mathcal{G}'\mathcal{L}' - \frac{1}{2}\mathcal{L}'(\mathcal{L} - \mathcal{G})'\frac{E_2 M_2 + E_1 M_1}{E_2 M_1 + E_1 M_2})$$

$$+\frac{i}{2}(\mathcal{L} + \mathcal{G})'(\sigma_1 \cdot \hat{r}\sigma_2 \cdot p + \sigma_2 \cdot \hat{r}\sigma_1 \cdot p) + \frac{i}{2}(\mathcal{L} - \mathcal{G})\frac{E_1 M_2 - E_2 M_1}{E_2 M_1 + E_1 M_2}\frac{L \cdot (\sigma_1 \times \sigma_2)}{r}]\phi_+$$

$$\equiv (p^2 + \Phi)\phi_+$$

$$= b^2(w)\phi_+, \tag{179}$$

We work in the c.m. frame in which $\hat{P} = (1, \mathbf{0})$ and $\hat{r} = (0, \hat{\mathbf{r}})$. The final four component wave functions $\psi_\pm, \eta_\pm$ that appear in Eq. (179) are defined by [2]

$$\phi_\pm = \exp(\mathcal{F} + \mathcal{K}\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})\psi_\pm = (\exp \mathcal{F})(\cosh \mathcal{K} + \sinh \mathcal{K}\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})\psi_\pm,$$

$$\chi_\pm = \exp(\mathcal{F} + \mathcal{K}\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})\eta_\pm = (\exp \mathcal{F})(\cosh \mathcal{K} + \sinh \mathcal{K}\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})\eta_\pm, \tag{180}$$

in which

$$\mathcal{F} = \frac{1}{2}\log \frac{\mathcal{D}}{\varepsilon_2 m_1 + \varepsilon_1 m_2} - \mathcal{G},$$

$$\mathcal{D} = E_2 M_1 + E_1 M_2,$$

$$\mathcal{K} = \frac{(\mathcal{L} - \mathcal{J})}{2}. \tag{181}$$

In analogy to what occurs in the decoupled form of the Schrödinger equation for the individual single particle wave function, this substitution has the convenient property that in the resultant bound state equation, the coefficients of the first order relative momentum terms vanish and we are left with a more concise form

$$(\mathbf{p}^2 + \Phi_{12} - b^2)\psi = \{\mathbf{p}^2 + (m_w + S)^2 - (\varepsilon_w - A)^2 + \Phi_D + \mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\Phi_{SO} + \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\Phi_{SOT}$$

$$+ \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\Phi_{SS} + (3\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\Phi_T + \mathbf{L} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\boldsymbol{\Phi}_{SOD} + i\mathbf{L} \cdot \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2\Phi_{SOX}\}\psi_+ = 0. \tag{182}$$

Here, $\mathcal{J}$, $\mathcal{G}$ and $\mathcal{L}$ are arbitrary functions of $x_\perp$ related to each interaction, as defined in Eq.(158), Eq.(159), and Eq.(160) and $\mathcal{G} = -\mathcal{J}$. Thus is derived a two-body Schrödinger-like equation for scalar, time-like and space-like vector interactions. The $2m_w S + S^2 + 2\varepsilon_w A - A^2$ term is the classical interaction potential term (which also appears in the spinless Klein-Gordon equations), the $L \cdot (\sigma_1 \pm \sigma_2)$ terms represent effective magnetic field dipole moments and Thomas precession, and $\sigma_1 \cdot \hat{r}\sigma_2 \cdot \hat{r}$, $\sigma_1 \cdot \sigma_2$ and $\sigma_i \cdot \hat{r}\sigma_j \cdot p$ terms arise from dipole-dipole interactions and their relativistic corrections. The main focus of this work has been to derive a similar equation for the three-body baryon system.

This is the framework for the two-body system in a fully relativistic formalism, so from here we go to larger systems. Sazdjian[11] has done considerable work on the N-body system, which will be reviewed shortly. Although he does not deal with spin dependence with as much detail as done here, he does provide a very useful framework for the N-body problem in a constraint formalism.

2.6 Two Body Dirac Equations: Explicit Forms of the Potentials

Since the forms of the potentials in the three-body case are similar to those in the two-body case, it is of use to (very) briefly describe the two-body interacting potentials and how they affect the wavefunction. This section then contains a review of how the operators of the tensor, spin-spin, spin-orbit, spin-orbit difference and spin-orbit exchange work on a $\langle jlsn|$ state coupling, where n is a radial quantum number. In the three-body case there will be two $n's$, one for each relative coordinate, but here there is just one.

First, our quasipotential (energy dependent effective potential) is defined by

$$\Phi = \Phi_{SI} + \Phi_D + \mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\Phi_{SO} + \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\Phi_{SOT} \tag{183}$$
$$+ \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \Phi_{SS} + (3\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\Phi_T + \mathbf{L} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\boldsymbol{\Phi}_{SOD} + i\mathbf{L} \cdot \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2 \Phi_{SOX},$$
$$\Phi_{SI} = 2m_w S + S^2 - 2\varepsilon_w A - A^2$$

where the potential terms[6]

$$\Phi_D, \Phi_{SO}, \Phi_{SOT}, \Phi_{SS}, \Phi_T, \boldsymbol{\Phi}_{SOD}, \Phi_{SOX}$$

are all collections of two-body terms depending on the masses, distances between the two particles, energies of the two particles and the energy of the system. The explicit forms of these are therefore not important

---

[6]The subscript on quasipotential Φ_D refers to Darwin. It consist of what are called Darwin terms, those that are the two-body analogue of terms that accompany the spin-orbit term in the one-body Pauli reduction of the ordinary one-body Dirac equation, and ones related by canonical transformations to Darwin interactions [2], momentum dependent terms arising from retardation effects. The subscripts on the other quasipotentials refer respectively to SO (spin-orbit), SOD (spin-orbit difference), SOX (spin-orbit cross terms), SS (spin-spin), T (tensor), SOT (spin-orbit-tensor)

to the current discussion of the operators and so will be left in this form for simplicity's sake. The spin independent and Darwin terms have no spin operators and so when used on a $\langle jlsn|$ state they just give

$$\langle jlsn|\Phi_{SI}|jl's'n'\rangle = \delta_{ll'}\delta_{ss'}\langle n|\Phi_{SI}|n'\rangle,$$

$$\langle jlsn|\Phi_D|jl's'n'\rangle = \delta_{ll'}\delta_{ss'}\langle n|\Phi_{SI}|n'\rangle.$$

These are the only simple terms though, as the spin-orbit gives

$$\langle jlsn|\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\Phi_{SO}|jl's'n'\rangle = [j(j+1)-l(l+1)-2]\delta_{ll'}\delta_{ss'}\delta_{s1}\langle n|\Phi_{SO}|n'\rangle.$$

As we will show later, while in the two-body case this $l,j,$ and s are for the entire system, in the three-body problem it is just for each pair of particles and so this spin-orbit function requires additional (and extensive) manipulation in order to reach a completely coupled $|JLS\rangle$ state *for each set of particles*. The emphasis here is important as this is the main difficulty in going from the two-body formalism to a three-body one, in this work as well as others.

The tensor and spin-orbit tensor terms allow for coupling of different l states as well as identical l states, as shown ([49])

Tensor:

$$\langle jj+11n|(3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Phi_T|jj+11n'\rangle = -\frac{2j+4}{2j+1}\langle n|\Phi_T|n'\rangle,$$

$$\langle jj+11n|(3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Phi_T|jj-11n'\rangle = 6\frac{\sqrt{j(j+1)}}{2j+1}\langle n|\Phi_T|n'\rangle,$$

$$\langle jj-11n|(3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Phi_T|jj+11n'\rangle = 6\frac{\sqrt{j(j+1)}}{2j+1}\langle n|\Phi_T|n'\rangle,$$

$$\langle jj-11n|(3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Phi_T|jj-11n'\rangle = -\frac{2j-4}{2j+1}\langle n|\Phi_T|n'\rangle. \tag{184}$$

Spin-orbit Tensor:

$$\langle jj+11n|\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\Phi_{SOT}|jj+11n'\rangle = -\frac{1}{2j+1}[-2j-4]\langle n|\Phi_{SOT}|n'\rangle,$$

$$\langle jj+11n|\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\Phi_{SOT}|jj-11n'\rangle = [-2j-4]2\frac{\sqrt{j(j+1)}}{2j+1}\langle n|\Phi_{SOT}|n'\rangle,$$

$$\langle jj-11n|\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\Phi_{SOT}|jj+11n'\rangle = [2j-2]2\frac{\sqrt{j(j+1)}}{2j+1}\langle n|\Phi_{SOT}|n'\rangle,$$

$$\langle jj-11n|\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\Phi_{SOT}|jj-11n'\rangle = \frac{1}{2j+1}[2j-2]\langle n|\Phi_{SOT}|n'\rangle. \tag{185}$$

The spin-orbit difference and spin-orbit exchange *only* allow couplings between different spin states

$$\langle jl1n|\mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\boldsymbol{\Phi}_{SOD}|jl'1n'\rangle = 0$$

$$\langle jl1n|\mathbf{L}\cdot(\boldsymbol{\sigma_1}-\boldsymbol{\sigma_2})\boldsymbol{\Phi}_{SOD}|jl'0n'\rangle = [\{1-(j-l)^2\}\{(j+l)(j+l+2)\}]\langle n|\boldsymbol{\Phi}_{SOD}|n'\rangle$$

$$\langle jl0n|\mathbf{L}\cdot(\boldsymbol{\sigma_1}-\boldsymbol{\sigma_2})\boldsymbol{\Phi}_{SOD}|jl'1n'\rangle = [\{1-(j-l)^2\}\{(j+l)(j+l+2)\}]\langle n|\boldsymbol{\Phi}_{SOD}|n'\rangle$$

$$\langle jl0n|\mathbf{L}\cdot(\boldsymbol{\sigma_1}-\boldsymbol{\sigma_2})\boldsymbol{\Phi}_{SOD}|jl'0n'\rangle = 0 \tag{186}$$

$$\langle jl1n|i\mathbf{L}\cdot\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2\Phi_{SOX}|jl'1n'\rangle = 0$$

$$\langle jl1n|i\mathbf{L}\cdot\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2\Phi_{SOX}|jl'0n'\rangle = [\{1-(j-l)^2\}\{(j+l)(j+l+2)\}]\langle n|\boldsymbol{\Phi}_{SOD}|n'\rangle$$

$$\langle jl0n|i\mathbf{L}\cdot\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2\Phi_{SOX}|jl'1n'\rangle = -[\{1-(j-l)^2\}\{(j+l)(j+l+2)\}]\langle n|\boldsymbol{\Phi}_{SOD}|n'\rangle$$

$$\langle jl0n|i\mathbf{L}\cdot\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2\Phi_{SOX}|jl'0n'\rangle = 0 \tag{187}$$

The exact derivations of these potential terms are done in ([1, 2, 3]). Since one of the goals of this work is to compare essentially the same methods that worked well for the meson spectrum to the baryon spectrum, we use these same potential terms in mostly the same form as they appear in the pure two-body case. The two-body operators are therefore loosely defined and described in preparation for their adaptation to the three-body potential. Now we will derive definitions for the scalar and vector potentials used in our model and from that define the two-body potentials $\Phi_D, \Phi_{SO}, \Phi_{SOT}, \Phi_{SS}, \Phi_T, \boldsymbol{\Phi}_{SOD}$, and Φ_{SOX}.

2.7 Explicit Forms of the QCD Model Potentials

The authors of [39] have used a sophisticated form of the static quark potential developed by Adler and Piran [40], one that has ties at all length scales to field theoretic data and from this obtained good agreement with the quarkonium spectrum from experimental data. However, it is much more common in nonrelativistic treatments to use the static quark Cornell potential[41] for potential model studies,

$$V(r) = -\frac{\alpha_c}{r} + br, \tag{188}$$

as in [42, 43]. Although not displaying asymptotic freedom, it does give the dominant Coulomb-like behavior as well as the linear quark confinement. Early on a model was proposed by Richardson for a static potential which both depends only a single scale size Λ and interpolates in a simple way between asymptotic freedom

and linear confinement [44]. Richardson's model for the static interquark potential in momentum space is

$$\tilde{V}(\mathbf{q}) = -\frac{16\pi}{27}\frac{1}{\mathbf{q}^2 \ln(1 + \mathbf{q}^2/\Lambda^2)}, \tag{189}$$

arising from the assumption that

$$\tilde{V}(\mathbf{q}) = -\frac{4\alpha_s(\mathbf{q}^2)}{3\mathbf{q}^2}, \tag{190}$$

(including the color factor $-4/3$). It is important to note that this is for a $\mathbf{q}\bar{\mathbf{q}}$ singlet state for the meson spectrum. In order to properly account for asymptotic freedom, we must have $\mathbf{q}^2/\Lambda^2 >> 1$, which gives

$$\alpha_s(\mathbf{q}^2) \to \frac{8\pi}{27}\frac{1}{\ln(\mathbf{q}^2/\Lambda^2)}. \tag{191}$$

On the other hand, the property of linear confinement requires that for $\Lambda r >> 1$, $V(r) \propto r$ or equivalently that for $\mathbf{q}^2/\Lambda^2 << 1$ one must impose $\alpha_s(\mathbf{q}^2) \sim \mathbf{q}^{-2}$. The interpolation of Eq. (189) is not tied at all in the intermediate region and only roughly tied in the large r region to any field theoretic data. Nevertheless it provides a convenient one-parameter form for the static quark potential. In coordinate space it has the form

$$V(r) = \frac{8\pi\Lambda^2 r}{27} - \frac{8\pi f(\Lambda r)}{27r}, \tag{192}$$

where $f(\Lambda r)$ is given by a complicated integral transform[7] that displays the asymptotic freedom behavior for $r \to 0$ of

$$f(\Lambda r) \to -\frac{1}{\ln \Lambda r}, \tag{193}$$

while for $r \to \infty$,

$$f(\Lambda r) \to 1. \tag{194}$$

A simpler model for the potential function $f(r)$, which we apply in this model and one which displays the same large and small r behavior is[8]

$$V(r) = \frac{8\pi\Lambda^2 r}{27} - \frac{16\pi}{27r \ln(e^2 + 1/(\Lambda r)^2)}. \tag{195}$$

It amounts to replacing Richardson's $f(\Lambda r)$ by $2/\ln(e^2 + 1/(\Lambda r)^2)$, having the same limits. The forms of

[7]In addition to the spin independent nonrelativistic model presented in [44] see also a relativistic extension of it given in [45].

[8]An earlier coordinate space form that displays asymptotic freedom as well as linear quark confinement proposed in [46] is $V = (8\pi/27)(1 - \lambda r)^2/(r \ln \lambda r)$.

the scalar and vector invariant potentials

$$
\begin{aligned}
S &= \frac{8\pi\Lambda^2 r}{27} \\
A &= -\frac{16\pi}{27r\log(Ke^2 + \frac{B}{(\Lambda r)^2})} + \frac{e_1 e_2}{4\pi r}
\end{aligned}
\tag{196}
$$

are used to construct all of the individual Φ terms. The explicit forms of these potentials that are derived from the vector and scalar potentials are given in Appendix A. In the case of the baryons these are slightly changed due to a different color factor ($-\frac{4\alpha_s}{3}$ becomes $-\frac{2\alpha_s}{3}$ due to this being quark-quark and not quark-antiquark) to

$$
\begin{aligned}
S &= \frac{4\pi\Lambda^2 r}{27} \\
A &= -\frac{8\pi}{27r\log(Ke^2 + \frac{B}{(\Lambda r)^2})} + \frac{e_1 e_2}{4\pi r}
\end{aligned}
\tag{197}
$$

and also of course there is no longer just one interaction but three, so r becomes $r_{12}, r_{13},$ or r_{23}, depending on which potential we are currently discussing.

The technique that Crater et al. used in the two body problem for finding the eigenvalues is called the Inverse Power Method and its application depends on the variables being separable. Unlike the two-body problem, the variables are not separable in the three-body problem. This requires the use of the variational principle, which in turn requires a basis. Therefore, now that the potentials are defined, we turn to defining a basis for the two-body system and from there extending it to the three-body.

2.8 Gaussian Basis Functions

Here we will briefly detail how our wavefunctions are used to construct the basis for use with the variational theorem. In the case of a nonrelativistic, energy independent Hamiltonian the standard way of using the variational principle is having the wavefunction vary by some alpha, i.e.

$$
\begin{aligned}
\langle \Psi(\alpha)|H|\Psi(\alpha)\rangle &= E(\alpha), \\
\frac{\partial E}{\partial \alpha} &= 0, \\
E(\alpha_0) &\geq E_{actual}.
\end{aligned}
\tag{198}
$$

In order to expand the basis into a more general matrix eigenvalue equation, we define.

$$
\begin{aligned}
|\Psi\rangle &= \sum_n c_n |\Psi_n\rangle, \\
\langle\Psi|H|\Psi\rangle &= \sum_{n,m} c_n c_m^* \langle\Psi_m|H|\Psi_n\rangle, \\
\langle\Psi|\Psi\rangle &= \sum_{n,m} c_n c_m^* \langle\Psi|\Psi\rangle = \sum_{n,m} c_n c_m^* B_{mn}.
\end{aligned}
\tag{199}
$$

Since the basis we will choose for the baryons is not orthogonal, B is not the usual Kronecker delta. Using the method of Lagrange multipliers (subject to the constraint $\langle\Psi|\Psi\rangle = 1$) gives that

$$
\frac{\partial}{\partial c_m^*}(\langle\Psi|H|\Psi\rangle - E\langle\Psi|\Psi\rangle) = \sum_{n,m} c_n \langle\Psi_m|H|\Psi_n\rangle - E\sum_{n,m} c_n B_{mn} = 0,
\tag{200}
$$

we arrive at the eigenvalue equation

$$
H_{mn}c_n = EB_{nm}c_n,
\tag{201}
$$

or in matrix form (where $\mathbb{H}$ and $\mathbb{B}$ are matrices, $\underline{c}$ is a vector and E is a scalar eigenvalue)

$$
\mathbb{H}\underline{c} = E\mathbb{B}\underline{c}.
\tag{202}
$$

In the non-relativistic two-body problem with tensor coupling, the s and d states ($l = j + 1$ and $l = j - 1$) are mixed, so we need a mixed wavefunction

$$
\begin{aligned}
|\Psi\rangle &= \sum_n C_{n+}|\Psi_{n+}\rangle + \sum_n C_{n-}|\Psi_{n-}\rangle, \\
- &\rightarrow l = j - 1, \\
+ &\rightarrow l = j + 1.
\end{aligned}
\tag{203}
$$

Using this Ψ in Eq.(199) gives

$$
\begin{aligned}
\langle\Psi|H|\Psi\rangle &= \sum_{n,m}(c_m^{*+}c_n^+ H_{mn}^{++} + c_m^{*-}c_n^+ H_{mn}^{-+} + c_m^{*+}c_n^- H_{mn}^{+-} + c_m^{*-}c_n^- H_{mn}^{--}), \\
\langle\Psi|\Psi\rangle &= \sum_{n,m}(c_m^{*+}c_n^+ + c_m^{*-}c_n^+ + c_m^{*+}c_n^- + c_m^{*-}c_n^-),
\end{aligned}
\tag{204}
$$

and as before, we take the partial derivative with respect to the complex conjugate of one of the c parameters

$$\frac{\partial}{\partial C_m^{*+}} \langle \Psi | H | \Psi \rangle - E \langle \Psi | \Psi \rangle = H_{mn}^{++} c_n^+ + H_{mn}^{+-} c_n^- = E B_{mn}^{++} c_n^+$$

$$H_{mn}^{--} c_n^- + H_{mn}^{-+} c_n^+ = E B_{mn}^{--} c_n^- \tag{205}$$

giving us

$$\begin{pmatrix} \mathbb{H}^{++} & \mathbb{H}^{+-} \\ \mathbb{H}^{-+} & \mathbb{H}^{--} \end{pmatrix} \begin{pmatrix} \underline{c}^+ \\ \underline{c}^- \end{pmatrix} = E \begin{pmatrix} \mathbb{B}^{++} & 0 \\ 0 & \mathbb{B}^{--} \end{pmatrix} \begin{pmatrix} \underline{c}^+ \\ \underline{c}^- \end{pmatrix} \tag{206}$$

Now we must define a wavefunction for our basis.

2.8.1   Gaussian wavefunctions and Infinite Interval Discretization

This section describes how our wavefunctions comes about for our basis. The potentials in our Hamiltonian have both short distance and long distance effects, so we need a basis wavefunction that can accurately account for that. We define a wavefunction in terms of some parameter α that determines the effect of the Gaussian wavefunction for short and long interactions. We then split the wavefunction into those two parts (short and long) and discretize it to a certain N value, from where we get our basis. The wavefunction is originally defined in an infinite vector space, so we must truncate it in order to work with it.

Boris Kupershmidt, a mathematician,[50] has suggested a Laplace transform/Gaussian basis

$$\psi(\mathbf{x}) = \int_0^\infty d\alpha q(\alpha) \sqrt{\frac{1}{a^3} \sqrt{\frac{\alpha^3}{\pi^3}}} \exp(-\frac{\alpha \mathbf{x}^2}{2a^2}) \tag{207}$$

where ψ is essentially the Fourier transform of some function d. In order to work with this function, we split the integral into two pieces, one with boundaries from zero to one and and the other with boundaries from one to infinity, so that

$$\psi(\mathbf{x}) = \int_0^1 d\alpha q(\alpha) \sqrt{\frac{1}{a^3} \sqrt{\frac{\alpha^3}{\pi^3}}} \exp(-\frac{\alpha \mathbf{x}^2}{2a^2})$$

$$+ \int_1^\infty d\alpha q(\alpha) \sqrt{\frac{1}{a^3} \sqrt{\frac{\alpha^3}{\pi^3}}} \exp(-\frac{\alpha \mathbf{x}^2}{2a^2}). \tag{208}$$

By replacing α with $1/\beta$ in the first half of the equation (so that the integral from 0 to 1 now becomes 1 to

infinity) we get

$$\psi(\mathbf{x}) = \int_1^\infty d\beta q(1/\beta) \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{\beta^3 \pi^3}}} \exp(-\frac{\mathbf{x}^2}{2\beta a^2})$$

$$+ \int_1^\infty d\alpha q(\alpha) \sqrt{\frac{1}{a^3} \sqrt{\frac{\alpha^3}{\pi^3}}} \exp(-\frac{\alpha \mathbf{x}^2}{2a^2}), \tag{209}$$

and from there, replacing integrals with sums over arbitrarily large N, this discretizes to

$$\psi(\mathbf{x}) = \sum_{n=1}^N c_n \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{n^3 \pi^3}}} \exp(-\frac{\mathbf{x}^2}{2na^2})$$

$$+ \sum_{n=1}^N d_n \sqrt{\frac{1}{a^3} \sqrt{\frac{n^3}{\pi^3}}} \exp(-\frac{n\mathbf{x}^2}{2a^2})$$

$$= (c_1 + d_1) \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{\pi^3}}} \exp(-\frac{\mathbf{x}^2}{2a^2}) + c_2 \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{8\pi^3}}} \exp(-\frac{\mathbf{x}^2}{4a^2}) + d_2 \sqrt{\frac{1}{a^3} \sqrt{\frac{8}{\pi^3}}} \exp(-\frac{\mathbf{x}^2}{a^2}) + ...$$

$$= e_1 \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{\pi^3}}} \exp(-\frac{\mathbf{x}^2}{2a^2})$$

$$+ \sum_{n=2}^N c_n \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{n^3 \pi^3}}} \exp(-\frac{\mathbf{x}^2}{2na^2}) + \sum_{n=2}^N d_n \sqrt{\frac{1}{a^3} \sqrt{\frac{n^3}{\pi^3}}} \exp(-\frac{n\mathbf{x}^2}{2a^2}). \tag{210}$$

So, for $N = 1$ we have

$$\psi(\mathbf{x}) = e_1 \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{\pi^3}}} \exp(-\frac{\mathbf{x}^2}{2a^2}), \tag{211}$$

For $N = 2$ we have

$$\psi(\mathbf{x}) = e_1 \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{\pi^3}}} \exp(-\frac{\mathbf{x}^2}{2a^2})$$

$$+ c_2 \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{8\pi^3}}} \exp(-\frac{\mathbf{x}^2}{4a^2}) + d_2 \sqrt{\frac{1}{a^3} \sqrt{\frac{8}{\pi^3}}} \exp(-\frac{\mathbf{x}^2}{a^2}) \tag{212}$$

Note that the original wavefunction from $N = 1$ remains as the first term. This is true for all N.

For $N \geq 3$

$$\psi(\mathbf{x}) = e_1 \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{\pi^3}}} \exp(-\frac{\mathbf{x}^2}{2a^2})$$

$$+ \sum_{n=2}^N c_n \sqrt{\frac{1}{a^3} \sqrt{\frac{1}{n^3 \pi^3}}} \exp(-\frac{\mathbf{x}^2}{2na^2}) + \sum_{n=2}^N d_n \sqrt{\frac{1}{a^3} \sqrt{\frac{n^3}{\pi^3}}} \exp(-\frac{n\mathbf{x}^2}{2a^2}) \tag{213}$$

or more symmetrically

$$
\begin{aligned}
\psi(\mathbf{x}) &= \sum_{n=1}^{2N-1} e_n \sqrt{\frac{1}{a^3}} \sqrt{\frac{f_n^3}{\pi^3}} \exp(-\frac{f_n \mathbf{x}^2}{2a^2}), \\
f_n &= \frac{1}{n}; \ 1 \le n \le N; \\
f_n &= n + 1 - N; \ N + 1 \le n \le 2N - 1
\end{aligned}
\tag{214}
$$

As we can see from the $N = 2$ case, the order of the matrix increases as $2N - 1$. Each matrix element of the Hamiltonian matrix is constructed from the expectation value of the Hamiltonian with two of these wavefunctions. For example, for the $N = 2$ case, our general wavefunction $|\psi_n(f_n)\rangle$ is

$$
\begin{aligned}
n &= 1 \rightarrow |\psi_1(1)\rangle, \\
n &= 2 \rightarrow |\psi_2(\tfrac{1}{2})\rangle, \\
n &= 3 \rightarrow |\psi_3(2)\rangle,
\end{aligned}
\tag{215}
$$

and thus we have the 3x3 matrix

$$
\begin{pmatrix}
\langle \psi_1|H|\psi_1\rangle & \langle \psi_1|H|\psi_2\rangle & \langle \psi_1|H|\psi_3\rangle \\
\langle \psi_2|H|\psi_1\rangle & \langle \psi_2|H|\psi_2\rangle & \langle \psi_2|H|\psi_3\rangle \\
\langle \psi_3|H|\psi_1\rangle & \langle \psi_3|H|\psi_2\rangle & \langle \psi_3|H|\psi_3\rangle
\end{pmatrix} .
\tag{216}
$$

As can be inferred from the values of f_n for $n > 1$, this basis allows the wavefunction to account for both the long-rage and short-range interactions of the Hamiltonian. Smaller f_n values-such as for $n = 2$ in the above example-allow for long-range interactions while larger f_n values (like the $n = 3$ wavefunction) account for the short-range interactions.

Graphically, this can be seen in the plots for the N=1 through 5 (not normalized) Gaussians. These are all given with unit width for simplicity, as they are merely intended to show how the sum of Gaussians spreads the possible wavefunctions. Figure 6 shows how these combine and allow some to account for long-range interactions while others are more short-range.
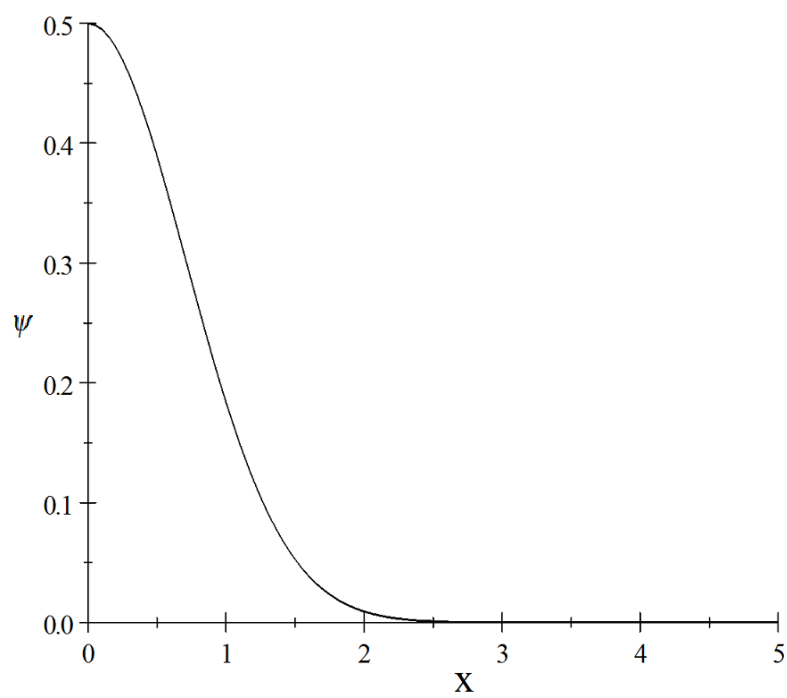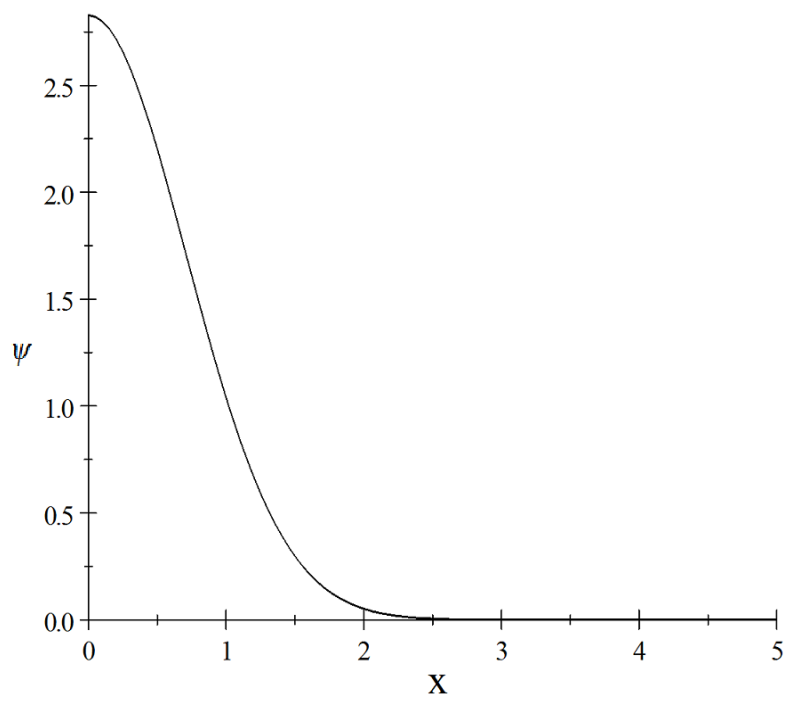
Figure 1: N=1 Gaussian

Figure 2: N=2 Gaussian
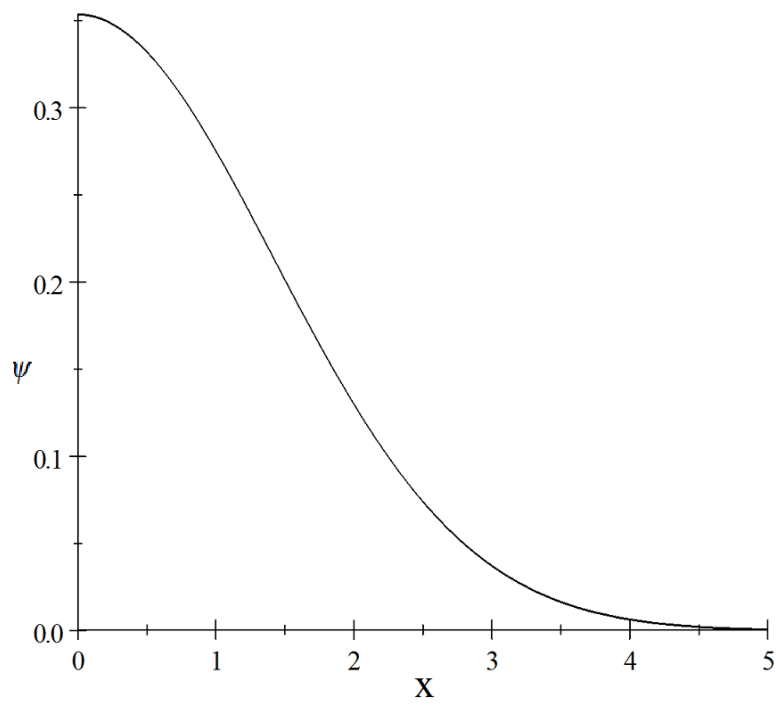
Figure 3: N=3 Gaussian
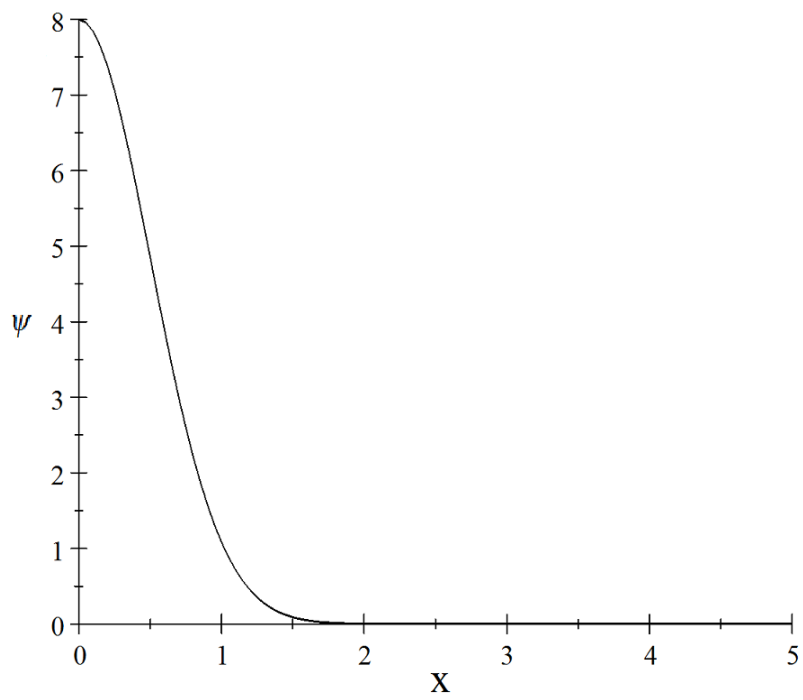
Figure 4: N=4 Gaussian
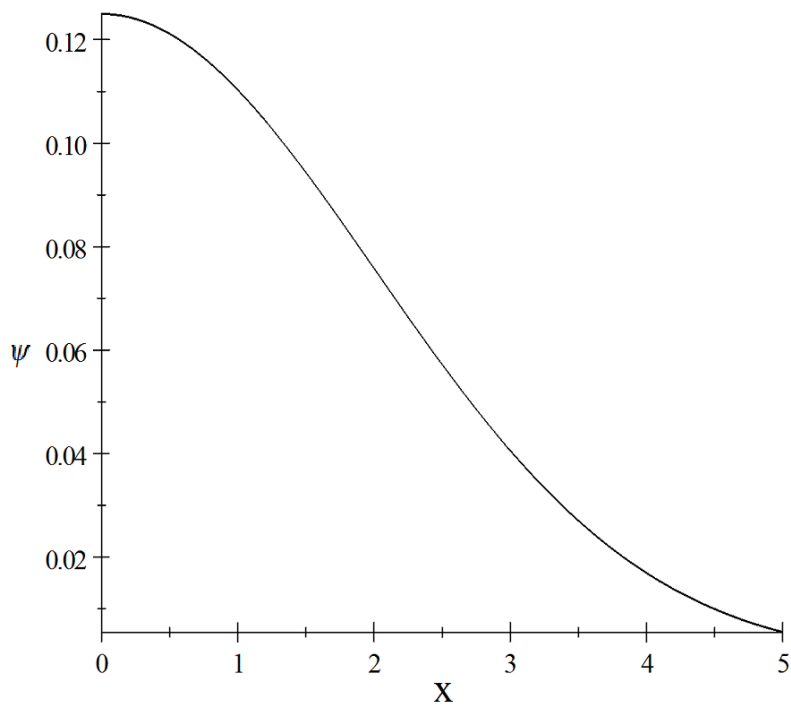
Figure 5: N=5 Gaussian
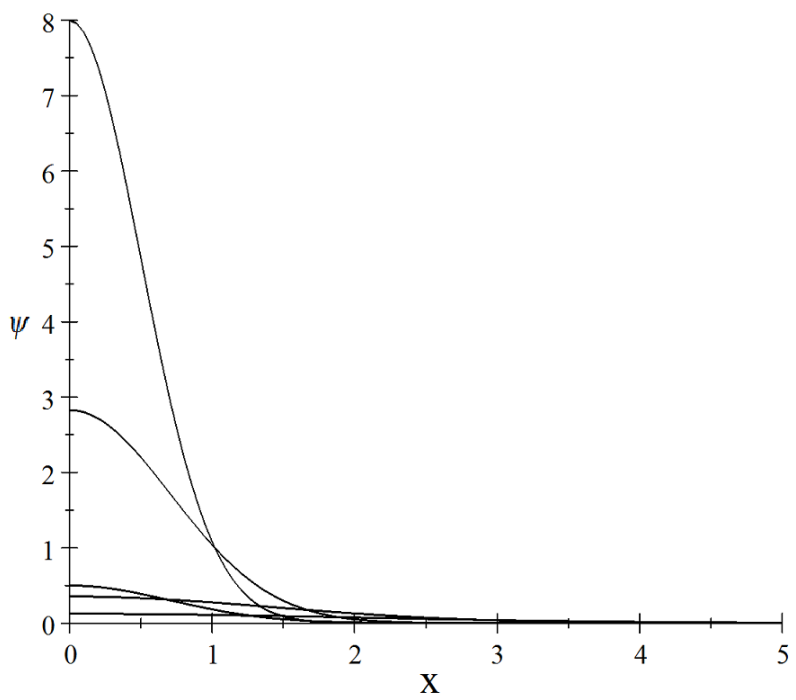
Figure 6: N=1 through 5 Gaussians

In a similar manner, we can now also write our B matrix from Eq.(202) as

$$
\begin{aligned}
\psi(\mathbf{x}) &= \sum_{n=1}^{2N-1} e_n \psi_n(\mathbf{x}); \\
B_{nm} &= \int d^3 x \psi_n^*(\mathbf{x}) \psi_m(\mathbf{x}) \\
&= \sqrt{\frac{1}{a^6} \sqrt{\frac{f_n^3 f_m^3}{\pi^6}}} \int d^3 x \exp(-\frac{(f_n + f_m)\mathbf{x}^2}{2a^2}) \\
&= \sqrt{f_n^{3/2} f_m^{3/2}} \frac{\sqrt{8}}{(f_n + f_m)^{3/2}} = \sqrt{\frac{8 f_n^{3/2} f_m^{3/2}}{(f_n + f_m)^3}}
\end{aligned}
\tag{217}
$$

Thus we get an analytical form for the B matrix that remains the same regardless of coordinate transformations. Note also that this becomes one in the case of $f_n = f_m = 1$, which is expected. This completes our review of the two-body formalism. Since we are attempting to reach a convergence point with as few Gaussians as possible, we do not necessarily include as many wavefunctions as is possible. So for $N = 2$, we only begin with two wavefunctions for each coordinate (giving a 4x4 matrix) and then go to three wavefunctions (going to 9x9). Similarly, $N = 3$ can have up to 5 wavefunctions per coordinate, but we only add one at a time in order to more quickly converge the energy eigenvalues.

3   The Three Body Problem

Now that we have completed our review of constraint dynamics for the two-body problem, we move on to the three-body one. Doing so is not a trivial task, as many pieces of our Hamiltonian become considerably more complicated, or at the very least, considerably different, from those used in the two-body problem. We will begin our review of the three body problem with a simple non-relativistic Hamiltonian with a harmonic oscillator potential. As is common for three body approaches, we set up a relative coordinate system from the coordinates of the quarks themselves. From there, we change the potential to a power law and add in an ad-hoc spin dependent term to acquire crude spectral results for our potential, the purpose of which being to familiarize ourselves and the reader with the larger problem at hand, as while this is a vastly simpler problem, it still has the same form as the fully relativistic, spin-dependent Hamiltonian. The actual spectral results are found using a χ^2 minimization routine.

After the non-relativistic results, we move on to Sazdjian's relativistic N body formalism that we then adapt to a three-body formalism and from there further modify to a Schrödinger-like form, as we did in the two-body. We then once again set up our relative coordinate system, but use a more direct and advanced method to do so in order to avoid mathematical difficulties for the intrinsically more complicated system

(i.e. dealing with an error function as we do below is not difficult, but it would become a much bigger issue with a more complicated Hamiltonian). Finally, we modify the two-body potentials as needed in order to fit a three-body problem. Some of these potentials are simple and require little other than changing an interaction between two particles to three interactions between three sets of two particles, while others require a nearly complete overhaul of the original potentials from the two-body system.

3.1 Nonrelativistic Three Body Problem

Before going to Sazdjian's work on the N-body relativistic approach (and from there to our three-body case), we begin with a simple non-relativistic three-body approach. The purpose of this is to provide a much simpler analogue to the work done in the following sections in order to more readily facilitate an understanding of the workings of the system. Therefore, we first solve the non-relativistic problem for a three-body system using relative coordinates for a simple harmonic oscillator. Then, we replace the oscillator potential with a power law potential and add in an ad-hoc spin-dependent term and numerically solve the equations. We limit our work here to the octet and use a chi-squared minimization routine to acquire the best fit for the baryon octet. The non-relativistic Hamiltonian is

$$H = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{p}_3^2}{2m_3} + V(\mathbf{r}_1, \mathbf{r}_2) + V(\mathbf{r}_2, \mathbf{r}_3) + V(\mathbf{r}_3, \mathbf{r}_1), \tag{218}$$

which in the simplified case of three coupled harmonic oscillators becomes

$$H = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{p}_3^2}{2m_3} + \frac{1}{2}\Lambda^3(\mathbf{r}_1 - \mathbf{r}_2)^2 + \frac{1}{2}\Lambda^3(\mathbf{r}_1 - \mathbf{r}_3)^2 + \frac{1}{2}\Lambda^3(\mathbf{r}_2 - \mathbf{r}_3)^2 , \tag{219}$$

where Λ^3 is the usual spring constant with Λ having the dimensions of energy. Using the canonical transformation [9]

$$
\begin{aligned}
\boldsymbol{\rho} &= \mathbf{r}_2 - \mathbf{r}_3, \ \ \mathbf{p}_\rho = \mathbf{p}_2 - \mathbf{p}_3 \\
\boldsymbol{\lambda} &= \frac{Mm_2}{(m_2 + m_3)m_1}\mathbf{r}_2 + \frac{Mm_3}{(m_2 + m_3)m_1}\mathbf{r}_3, \ \mathbf{p}_\lambda = \frac{m_3\mathbf{p}_2 - m_2\mathbf{p}_3}{m_2 + m_3} \\
\mathbf{R} &= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3}, \ \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3
\end{aligned}
\tag{220}
$$

where

$$M = m_1 + m_2 + m_3, \ m_\rho = \frac{m_1(m_2 + m_3)}{(m_1 + m_2 + m_3)}, \ m_\lambda = \frac{m_2 m_3}{m_2 + m_3}, \tag{221}$$

we arrive at

$$H = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}_\lambda^2}{2m_\lambda} + \frac{\mathbf{p}_\rho^2}{2m_\rho} + \frac{1}{2}\Lambda^3\{\frac{m_3^2 + m_2^2 + (m_2 + m_3)^2}{(m_2 + m_3)^2}\boldsymbol{\rho}^2 - \frac{2}{m_2 + m_3}(m_2 - m_3)\boldsymbol{\rho}\cdot\boldsymbol{\lambda}+2\boldsymbol{\lambda}^2\}. \qquad (222)$$

As a side note, similar to in Isgur's work, in the simplified case of $m_2 = m_3$ (most of the octet has two masses equal, so this is reasonable for this case), the Hamiltonian becomes

$$H = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}_\lambda^2}{2m_\lambda} + \frac{\mathbf{p}_\rho^2}{2m_\rho} + \frac{3}{4}\Lambda^3(\boldsymbol{\rho}^2+2\boldsymbol{\lambda}^2). \qquad (223)$$

 It is interesting to note that this substitution bears a resemblance to a quark-diquark system in that in setting up our coordinate systems two quarks are favored over the third, but the similarities end there as everything past that point treats the system as symmetrically as possible. Also, the oscillator case is similar to a typical phonon problem; however, since this case is just a simple example of what will be a general power law potential, it is not particularly relevant to the discussion at hand.

   This Hamiltonian Eq.(223) is fairly simple to solve (for the ground state) using the Gaussian wave function

$$\Psi(\boldsymbol{\rho}, \boldsymbol{\lambda}) = \frac{\alpha_\rho^{3/2}\alpha_\lambda^{3/2}}{\pi^{3/2}}e^{-\alpha_\rho^2\rho^2/2}e^{-\alpha_\lambda^2\lambda^2/2}, \qquad (224)$$

where

$$\alpha_\rho = (3\Lambda^3 m_\rho)^{1/4}, \qquad (225)$$

$$\alpha_\lambda = (3\Lambda^3 m_\lambda)^{1/4}.$$

   Now we will begin to build upon the simplest state by adding in spin dependent terms. The spin terms typically have the form

$$\sum_{i<,j}\frac{c}{m_i m_j}\mathbf{s}_i\cdot\mathbf{s}_j\boldsymbol{\nabla}^2\mathcal{V}(\mathbf{r}_i, \mathbf{r}_j), \qquad (226)$$

(with c an adjustable parameter) that modifies the spin independent two-body potential and will be treated as three two-particle interactions. This spin term should not be taken as a derived term, but it is important to note that it bears some similarities to one that is derived later for Two-Body Dirac equations. The purpose of adding this spin term is simply to see how it might be treated in this much simpler case than the one with which we will actually be working. The results of working out the quark model expectation value $\mathbf{s}_i\cdot\mathbf{s}_j$ terms are shown in Appendix C, along with the quark state vector structure for the baryons (Appendix B). It should be noted that though this addition does technically make the problem relativistic,

we are still using the non-relativistic kinetic and potential terms from earlier, so for all intents and purposes it is a "non-relativistic problem with spin." Moving from the oscillator to the general case, the Hamiltonian now is

$$
\begin{aligned}
H \;=\; & \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{p}_3^2}{2m_3} + \mathcal{V}(\mathbf{r}_1, \mathbf{r}_2) + \mathcal{V}(\mathbf{r}_1, \mathbf{r}_3) + \mathcal{V}(\mathbf{r}_2, \mathbf{r}_3) \\
& + \sum_{i<,j} \frac{c}{m_i m_j} \mathbf{s}_i \cdot \mathbf{s}_j \boldsymbol{\nabla}^2 \mathcal{V}(\mathbf{r}_i, \mathbf{r}_j).
\end{aligned}
\tag{227}
$$

For the purpose of further exemplifying the techniques we will use for the relativistic case, we choose $V = \frac{1}{2}\Lambda(\Lambda|\mathbf{r}_i - \mathbf{r}_j|)^p$, which with the substitution from Eq.(220) makes the Hamiltonian

$$
\begin{aligned}
H \;=\; & \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}_\rho^2}{2m_\rho} + \frac{\mathbf{p}_\lambda^2}{2m_\lambda} + \frac{1}{2}\Lambda^{p+1}\{|\boldsymbol{\rho}|^p + |-\frac{m_3}{m_2+m_3}\boldsymbol{\rho} - \boldsymbol{\lambda}|^p + |\boldsymbol{\lambda} - \frac{m_3}{m_2+m_3}\boldsymbol{\rho}|^p\} \\
& + \sum_{i,j} \frac{c}{m_i m_j} \mathbf{s}_i \cdot \mathbf{s}_j \boldsymbol{\nabla}^2 \mathcal{V}(\mathbf{r}_i, \mathbf{r}_j),
\end{aligned}
\tag{228}
$$

where p is an arbitrary general power. The kinetic term can always be worked out exactly and is, working in the center of momentum system,

$$
\begin{aligned}
\langle \Psi | T | \Psi \rangle \;=\; & \frac{\alpha_\rho^3 \alpha_\lambda^3}{\pi^3} \int e^{-\alpha_\rho^2 \boldsymbol{\rho}^2} e^{-\alpha_\lambda^2 \boldsymbol{\lambda}^2} \frac{\mathbf{P}_\rho^2}{2m_\rho} e^{-\alpha_\rho^2 \boldsymbol{\rho}^2} \\
& + e^{-\alpha_\rho^2 \boldsymbol{\rho}^2} e^{-\alpha_\lambda^2 \boldsymbol{\lambda}^2} \frac{\mathbf{P}_\lambda^2}{2m_\lambda} e^{-\alpha_\lambda^2 \boldsymbol{\lambda}^2} d^3\lambda d^3\rho \\
\;=\; & \frac{\alpha_\rho^2}{64\pi m_\rho} + \frac{\alpha_\lambda^2}{64\pi m_\lambda},
\end{aligned}
\tag{229}
$$

as can the first potential term of $|\sqrt{2}\boldsymbol{\rho}|^p$, yielding, for integer values of p,

$$
\begin{aligned}
\langle \Psi | |\sqrt{2}\boldsymbol{\rho}|^p | \Psi \rangle \;=\; & \frac{\alpha_\rho^3 \alpha_\lambda^3}{\pi^3} \int |\boldsymbol{\rho}|^p e^{-\alpha_\rho^2 \boldsymbol{\rho}^2} e^{-\alpha_\lambda^2 \boldsymbol{\lambda}^2} d^3\lambda d^3\rho = \frac{1}{2} \frac{\alpha_\rho^3 \alpha_\lambda^2}{\pi^2} \frac{1}{\alpha_\rho^{p+1}} \mathcal{C} \\
\;=\; & \frac{1}{2} \frac{\alpha_\lambda^2}{\pi^2} \frac{1}{\alpha_\rho^p} \mathcal{C}, \\
\mathcal{C} \;=\; & \frac{1}{2} * \frac{3}{2} * \frac{5}{2} * ...(\frac{p+1}{2}).
\end{aligned}
\tag{230}
$$

The other two potential terms do not work out analytically in the general case though and require some numerical work. Through the substitution

$$
\boldsymbol{\lambda}' = -\frac{m_3}{m_2+m_3}\boldsymbol{\rho} - \boldsymbol{\lambda},
\tag{231}
$$

Table 1: Nonrelativistic model parameter values and baryon masses

Particle	Mass(MeV)	Mass(MeV)(experimental)	Constants	value
u	317			
d	322			
s	495			
p	971	938	Constants	value
n	973	939	Λ	175 MeV
Σ^+	1146	1189	c	1.22 (dimensionless)
Σ^0	1148	1192	p	2.05
Σ^-	1150	1197		
Ξ^0	1323	1314		
Ξ^-	1325	1321		
Λ	1125	1115		

one can move some of the more difficult terms (the $|-\frac{m_3}{m_2+m_3}\boldsymbol{\rho}-\boldsymbol{\lambda}|^p$ and $|\boldsymbol{\lambda}-\frac{m_3}{m_2+m_3}\boldsymbol{\rho}|^p$ terms) into the Gaussian wavefunction, which gives us

$$\langle\Psi||-\frac{m_3}{m_2+m_3}\boldsymbol{\rho}-\boldsymbol{\lambda}|^p|\Psi\rangle\frac{1}{2}\Lambda^{p+1} = \frac{1}{2}\Lambda^{p+1}\int\exp(-\alpha_\rho^2\rho^2)\exp(-\alpha_\lambda^2(-\frac{m_3}{m_2+m_3}\boldsymbol{\rho}-\boldsymbol{\lambda}')^2)\lambda'^p d^3\rho d^3\lambda'. \quad (232)$$

The term in the second exponential above results in

$$-\alpha_\lambda^2(-\frac{m_3}{m_2+m_3}\rho-\lambda')^2 = -\alpha_\lambda^2(\lambda'^2 + \frac{m_3^2}{(m_2+m_3)^2}\rho^2 + \frac{m_3}{m_2+m_3}\lambda'\rho\cos\theta), \quad (233)$$

which will cause the Hamiltonian to integrate to an error function in λ'. This can be approximated and then the remaining integral over ρ can be done numerically. The other term $(|\boldsymbol{\lambda}-\frac{m_3}{m_2+m_3}\boldsymbol{\rho}|^p)$ is done in a similar fashion and also results in an error function. We use a Monte Carlo modeling approach in a chi-squared minimization routine. The numerical results are as follows for the baryon octet

So, we have now completed a simplified version of the three-body problem and obtained some crude spectral results. We have used a relative coordinate system for an oscillator basis at first and then changed our potential to a power law with spin-dependent terms. From there, we numerically solved the problem and used a χ^2 minimization routine to acquire a best fit for the parameters (the quark masses and the constant Λ).

3.2 Sazdjian's N body formalism and the Three Body Problem

This section will focus on reviewing Sazdjian's work on the two-body and N-body systems[11]. We describe his derivations for the N-body problem and distill them down to a three-body formalism that we can then use for our specific problem. By using similar constraints to those defined in the two-body case

for compatibility, he derives the same eigenvalue equation as in Eq.(33)

$$[p_{a\perp}^2 + \Phi_a - b^2]\psi(x_{a\perp}) = [\mathcal{H}_a - b^2 \psi](x_{a\perp}) = 0, \tag{234}$$

where $p_\perp = p - \widehat{P}\widehat{P} \cdot p$ is the transverse momentum.

Sazdjian applies the covariant formalism with constraints to the N-particle case. The compatibility condition is then

$$[\mathcal{H}_a, \mathcal{H}_b]\Psi = 0 \quad (a, b = 1, ..., N), \tag{235}$$

which are $N(N-1)/2$ in number and give conditions on the interaction potentials (Φ_a). Later it will be necessary to specialize Sazdjian's work to the three-body system so that it more directly applies to the three-body problem at hand.

Sazdjian now includes all two-body interactions into each Φ_a in order to construct separable interactions, i.e.,

$$\Phi_a = \sum_b \Phi_{ab} + W_a, \tag{236}$$

where W_a are remainders (in other words, three-body potentials) to be specified later. These potentials are symmetric ($\Phi_{ab} = \Phi_{ba}$) and are solutions of the compatibility equation

$$[p_a^2 + m_a^2 + \Phi_{ab}, p_b^2 + m_b^2 + \Phi_{ba}] = 0, \tag{237}$$

which since $\Phi_{ab} = \Phi_{ba}$ can be written as

$$[p_b^2 - p_a^2, \Phi_{ab}] = 0. \tag{238}$$

The two-body potentials Φ_{ab} cannot alone satisfy the global compatibility condition. Therefore Sazdjian adds the W_a terms. The equations that must be satisfied by the W_a potentials are

$$[p_2^2, W_a] - [p_1^2, W_b] = -[\sum_{a,b} \Phi_{ac} + W_a, \sum_{a,b} \Phi_{bc} + W_b] \quad (a, b = 1, ..., N). \tag{239}$$

Sazdjian makes a few remarks at this point. First, the total potentials would not have a closed form like in the two-body case (where Φ is a function of $x_\perp$). Second, in the quantized two-body case, the potentials would become non-local operators, due to the two-body momentum operators $P_{ab} = p_a + p_b$ no longer representing the total momentum of the system and therefore not possessing corresponding eigenvalues. On the other hand the total momentum in two-body case is a number (an eigenvalue). He then abandons this

approach and instead takes a simpler one that works only for confined systems where questions of correct cluster decompositions do not have to be addressed.

Sazdjian finds that in working with the free N-body system, one can be guided by the simplifying features of the two-body system. Since the system must reduce to that of the free case in the absence of interactions, he found it useful to begin with the N-body system without any interacting potentials. In the free case the wave equations are

$$(p_a^2 + m_a^2)\Psi(x_{1,...}x_N) = 0 \quad (a = 1, ..., N). \tag{240}$$

Let us introduce longitudinal and transverse momenta, which are defined by

$$
\begin{aligned}
p_a &= p_{||a} + p_{\perp a}, \\
p_{\perp a} &= p_a + p_a \cdot \hat{P}\hat{P},
\end{aligned}
\tag{241}
$$

where

$$
\begin{aligned}
\hat{P} &= \frac{P}{\sqrt{-P^2}} \\
P &= p_1 + p_2 + ...p_n
\end{aligned}
$$

and the c.m energies of the individual particles which are

$$
\begin{aligned}
p_{||a} &= -p_a \cdot \hat{P}\hat{P} = \varepsilon_a \hat{P}, \\
p_{||a}^2 &= -\varepsilon_a^2.
\end{aligned}
\tag{242}
$$

By subtracting the two wave equations from each other we obtain

$$(p_b^2 - p_a^2)\Psi = (m_a^2 - m_b^2)\Psi, \tag{243}$$

or

$$(p_{||b}^2 - p_{||a}^2)\Psi = (\varepsilon_a^2 - \varepsilon_b^2)\Psi = [(m_a^2 - m_b^2) - (p_{b\perp}^2 - p_{a\perp}^2)]\Psi \quad (a, b = 1, 2, 3). \tag{244}$$

Even in the free case, however, the longitudinal momenta $(p_{||})$ cannot be taken as the analogs of the longitudinal momenta for the two-body case since $p_{b\perp}^2 - p_{a\perp}^2 \neq 0$. Therefore, one must define generalized longitudinal momenta.

Sazdjian defines U as the operator of the canonical transformation

$$U = \exp[i \sum_a x_{aL} k_{aL}], \tag{245}$$

where k_{aL} are functions of the transverse momenta, total mass squared, and free mass, satisfying the condition

$$\sum_a k_{aL} = 0, \tag{246}$$

and x_{aL} is a function of the particle position

$$x_{aL} = x_a \cdot \hat{P}. \tag{247}$$

The operator U transforms the ε_i into ε_i', where

$$\varepsilon_a' = \varepsilon_a + k_{aL}, \tag{248}$$

The operators k_i are chosen such that the new (transformed) wavefunction is an eigenfunction of $p_{||a}'$ with eigenvalues ε_a' that satisfy (dropping the primes)

$$\varepsilon_b^2 - \varepsilon_a^2 = m_b^2 - m_a^2 \quad (a, b = 1, ..., N). \tag{249}$$

By dividing the above relation by $(\varepsilon_a + \varepsilon_b)$ and summing over all relations with respect to b for fixed a, Sazdjian arrives at N equations:

$$N\varepsilon_a - \sum_b \frac{(m_a^2 - m_b^2)}{(\varepsilon_a + \varepsilon_b)} = w, (a = 1, ..., N), \tag{250}$$

where w is the total c.m. energy

$$w = \sum_b \varepsilon_b. \tag{251}$$

These equations cannot be solved simply except in the two-body case, which gives

$$
\begin{aligned}
2\varepsilon_1 - \frac{(m_1^2 - m_2^2)}{(\varepsilon_1 + \varepsilon_2)} &= w, \\
2\varepsilon_2 - \frac{(m_2^2 - m_1^2)}{(\varepsilon_2 + \varepsilon_1)} &= w,
\end{aligned} \tag{252}
$$

which reduce down to Eq.(29)

$$\varepsilon_1 - \varepsilon_2 = \frac{m_1^2 - m_2^2}{w}, \qquad (253)$$

as expected. He shows it is possible, however, to find an approximate solution by using successive iterations in the general case.

In order that Eq. (249) be satisfied, the operators k_a must satisfy the relations

$$(\varepsilon_a + k_{aL})^2 - (\varepsilon_b + k_{bL})^2 = (m_a^2 - m_b^2) + (p_{\perp a}^2 - p_{\perp b}^2) \qquad (a, b = 1...N). \qquad (254)$$

By using a similar procedure to that for Eq.(250), he derives the relation

$$N k_{aL} + \sum_b \frac{(p_{b\perp} - p_{a\perp}) + (\varepsilon_a - \varepsilon_b)(k_{aL} + k_{bL})}{(\varepsilon_a + \varepsilon_b + k_{aL} + k_{bL})} = 0 \qquad (a = 1...N). \qquad (255)$$

An approximate solution for the $k's$ can be obtained by keeping those terms which contribute in the non-relativistic limit. In that limit, ε_a and $p_{a\perp}$ behave as c^2, which means that k_a behaves as c^0. So, by using those approximations one comes up with the end result

$$\varepsilon_a = \frac{w}{N} + \frac{1}{N} \sum_{b=1}^{N} \frac{(m_a - m_b)}{[1 + (w - M)/(2m_a m_b \sum_{c=1}^{N} \frac{1}{2m_c})]} (a = 1, ..., N). \qquad (256)$$

This equation gives the constituent c.m. energies of each particle in terms of the total c.m. energy and the masses. In the $N = 2$ case, we recover the exact two-body equations for the epsilons of

$$\begin{aligned}
\varepsilon_1 &= \frac{w}{2} + \frac{1}{2} \frac{(m_1 - m_2)}{[1 + (w - m_1 - m_2)/(m_1 + m_2)]} = \frac{m_1^2 - m_2^2 + w^2}{2w}, \\
\varepsilon_2 &= \frac{w}{2} + \frac{1}{2} \frac{(m_2 - m_1)}{[1 + (w - m_1 - m_2)/(m_1 + m_2)]} = \frac{-m_1^2 + m_2^2 + w^2}{2w}, \\
\varepsilon_1 - \varepsilon_2 &= \frac{m_1^2 - m_2^2}{w},
\end{aligned} \qquad (257)$$

and so our full N-body equation can now read as

$$\sum_{a=1}^{N} [-\varepsilon_a^2 + N \frac{p_{a\perp}^2/(2\varepsilon_a)}{\sum_{b=1}^{N} 1/2\varepsilon_b} + m_a^2] \psi = 0. \qquad (258)$$

The equation (258) determines the total c.m. energy w in terms of the masses. Again, for $N = 2$, this becomes

$$[-\varepsilon_1^2 + 2 \frac{p_{1\perp}^2 \varepsilon_2}{\varepsilon_1 + \varepsilon_2} + m_1^2] + [-\varepsilon_2^2 + 2 \frac{p_{2\perp}^2 \varepsilon_1}{\varepsilon_1 + \varepsilon_2} + m_2^2] \psi = 0, \qquad (259)$$

56

which, since

$$\varepsilon_1^2 - m_1^2 = \varepsilon_2^2 - m_2^2 = b^2,$$

$$\varepsilon_1 + \varepsilon_2 = w, \tag{260}$$

is the same as the equation derived in the two-body system earlier

$$[\frac{p_{1\perp}^2 \varepsilon_2}{w}] + [\frac{p_{2\perp}^2 \varepsilon_1}{w}]\psi = (\varepsilon_2 + \varepsilon_1)\frac{p_\perp^2}{w}\psi = (p_\perp^2 - b^2)\psi = 0. \tag{261}$$

Finally, he also derives the system of N Klein-Gordon equations

$$\{-\varepsilon_a^2 + [\sum_{b=1}^{N} \frac{p_{b\perp}^2}{(2\varepsilon_b)}]/(\sum_{c=1} \frac{1}{2\varepsilon_c}) + m_a^2\}\Psi = 0 \qquad (a = 1, ..., N). \tag{262}$$

Sazdjian finds that in the interacting case, the N non-independent wave equations are kinematic in nature and should not be modified by the interactions. For example, particle a "feels" an interaction potential Φ_a that enters additively into its kinetic energy term by the relation

$$p_{\perp a}^2 \rightarrow p_{\perp a}^2 + \Phi_a. \tag{263}$$

Eq.(258) then becomes

$$\sum_a [-\varepsilon_a^2 + N\frac{(p_{a\perp}^2 + \Phi_a)/(2\varepsilon_a)}{\sum_b 1/2\varepsilon_b} + m_a^2]\psi = 0, \tag{264}$$

which in the two-body case is

$$[\frac{p_{1\perp}^2 + \Phi_1}{w}\varepsilon_2] + [\frac{p_{2\perp}^2 + \Phi_2}{w}\varepsilon_1]\psi = b^2\psi \tag{265}$$

which since $p_{1\perp}^2 = p_{2\perp}^2$ and $\Phi_1 = \Phi_2$ reduces to Eq. (234). In the general N-body case, the individual wave equations become

$$\{-\varepsilon_a^2 + [\sum_a \frac{(p_{a\perp}^2 + \Phi_a)}{(2\varepsilon_a)}]/(\sum_c \frac{1}{2\varepsilon_c}) + m_a^2\}\Psi = 0 \qquad (a = 1, ..., N). \tag{266}$$

For compatibility of these wave equations, we must have

$$\Phi_a = \sum_{b \neq a}^{N} \Phi_{ab}(x_{ab\perp}),$$

$$x_{ab\perp}^\mu = (x_a^\mu - x_b^\mu) - P^\mu \hat{P} \cdot (x_a^\mu - x_b^\mu), \tag{267}$$

where again, P is the total momentum[9]

$$P = \sum_{a=1}^{N} p_a. \tag{268}$$

The Φ in this equation is chosen to have the same functional dependence on S and A as in Eq.(179). Sazdjian does not deal directly with the potential (hence his free case here with potential added at the end), so part of this work has been to use the potential from the Two-Body Dirac equations in the Sazdjian three-body equations.

3.2.1 Our Adaptation of Sazdjian's Three-Body Generalization

In order to apply the work done by Sazdjian to our problem, we have to specialize his N-body equations to the three-body system and derive the appropriate Hamiltonian, eventually ending up with an equation that looks very much like a nonrelativistic Schrödinger equation and is thus much easier with which to work. He obtains the following approximation for the three-body eigenvalue equation, which is the expanded three body version of Eq.(264). Specializing this equation to the three-body case for the baryons and expanding terms, we get

$$
\begin{aligned}
0 = {} & [\varepsilon_1^2 - m_1^2 - 3 \frac{p_{1\perp}^2 + \Phi_1}{\varepsilon_1(1/\varepsilon_1 + 1/\varepsilon_2 + 1/\varepsilon_3)} \\
& + \varepsilon_2^2 - m_2^2 - 3 \frac{p_{2\perp}^2 + \Phi_2}{\varepsilon_2(1/\varepsilon_1 + 1/\varepsilon_2 + 1/\varepsilon_3)} \\
& + \varepsilon_3^2 - m_3^2 - 3 \frac{p_{3\perp}^2 + \Phi_3}{\varepsilon_3(1/\varepsilon_1 + 1/\varepsilon_2 + 1/\varepsilon_3)}] \psi(x_{12\perp}, x_{23\perp}, x_{31\perp}),
\end{aligned}
\tag{269}
$$

in which the epsilons, representing the total energy of each quark, are

$$
\begin{aligned}
\varepsilon_1 = {} & \frac{w}{3} + \frac{1}{3} \frac{m_1 - m_2}{1 + (w - M)/(m_1 m_2(1/m_1 + 1/m_2 + 1/m_3)} \\
& + \frac{1}{3} \frac{m_1 - m_3}{1 + (w - M)/(m_1 m_3(1/m_1 + 1/m_2 + 1/m_3)} \\
\varepsilon_2 = {} & \frac{w}{3} + \frac{1}{3} \frac{m_2 - m_1}{1 + (w - M)/(m_1 m_2(1/m_1 + 1/m_2 + 1/m_3)} \\
& + \frac{1}{3} \frac{m_2 - m_3}{1 + (w - M)/(m_2 m_3(1/m_1 + 1/m_2 + 1/m_3)} \\
\varepsilon_3 = {} & \frac{w}{3} + \frac{1}{3} \frac{m_3 - m_1}{1 + (w - M)/(m_1 m_3(1/m_1 + 1/m_2 + 1/m_3)} \\
& + \frac{1}{3} \frac{m_3 - m_2}{1 + (w - M)/(m_2 m_3(1/m_1 + 1/m_2 + 1/m_3)}
\end{aligned}
\tag{270}
$$

---

[9]Note that this dependence of the potential on the part of the potential that depends on component of the relative coordinates perpendicular to the momentum of the total system, is allowed as long as the issue of cluster decomposition need not be addressed. In that case, where one may separate out a part of the system from the remaining part, it is not meaningful to require the potentials to depend on the total momemtum of the total system instead of that of its subconstituents.

and the potentials Φ_i are linear combinations of the two-body interacting potentials

$$
\begin{aligned}
\Phi_1 &= \Phi_{12}(x_{12\perp}, \varepsilon_1, \varepsilon_2) + \Phi_{23}(x_{23\perp}, \varepsilon_2, \varepsilon_3) \\
\Phi_2 &= \Phi_{23}(x_{23\perp}, \varepsilon_2, \varepsilon_3) + \Phi_{31}(x_{31\perp}, \varepsilon_3, \varepsilon_1) \\
\Phi_3 &= \Phi_{31}(x_{31\perp}, \varepsilon_3, \varepsilon_1) + \Phi_{12}(x_{12\perp}, \varepsilon_1, \varepsilon_2).
\end{aligned}
\tag{271}
$$

This equation is essentially the three-body version of the two-body equation:

$$
\mathcal{H} = \frac{(p_1^2 + m_1^2 + \Phi)}{2\varepsilon_1} + \frac{(p_2^2 + m_2^2 + \Phi)}{2\varepsilon_2},
\tag{272}
$$

as long as one restricts oneself to confining interactions.

In the above equations for the epsilons (Eq.(270)), we define

$$
\begin{aligned}
p_{i\perp} &= p_i + p_i \cdot \hat{P}\hat{P} \\
x_{ij\perp} &= x_{ij} + x_{ij} \cdot \hat{P}\hat{P} \\
\hat{P} &= \frac{P}{\sqrt{-P^2}} \\
P &= p_1 + p_2 + p_3 \\
E &= w - M \equiv \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - m_1 - m_2 - m_3
\end{aligned}
\tag{273}
$$

in order to make them into a more usable form. Through some extensive algebraic manipulations, we obtain the much simplified equations of

$$
\begin{aligned}
\varepsilon_1 - m_1 &= E f_1(E, m_1, m_2, m_3) \\
\varepsilon_2 - m_2 &= E f_2(E, m_1, m_2, m_3) \\
\varepsilon_3 - m_3 &= E f_3(E, m_1, m_2, m_3)
\end{aligned}
\tag{274}
$$

in which the $f_i's$ are collections of functions, given by [51]

$$
\begin{aligned}
&f_1(E, m_1, m_2, m_3) \\
&= \frac{3m_2^2 m_3^2 + 3m_2^2 m_3 m_1 + 3m_2 m_3^2 m_1 + m_1 m_2^2 E + 2m_2^2 m_3 E + 2m_3^2 E m_2 + m_3^2 E m_1 + m_3 E^2 m_2}{3\left(m_1 m_2 + m_2 m_3 + m_3 m_1 + m_3 E\right)\left(m_1 m_2 + m_2 m_3 + m_3 m_1 + m_2 E\right)}
\end{aligned}
$$

59

$$f_2(E, m_1, m_2, m_3)$$

$$= \frac{3m_1^2 m_2 m_3 + 3m_2 m_3^2 m_1 + m_3^2 E m_2 + 2m_3^2 E m_1 + 3m_1^2 m_3^2 + 2m_3 m_1^2 E + m_3 E^2 m_1 + m_1^2 m_2 E}{3\left(m_1 m_2 + m_2 m_3 + m_3 m_1 + m_3 E\right)\left(m_1 m_2 + m_2 m_3 + m_3 m_1 + m_1 E\right)}$$

$$f_3(E, m_1, m_2, m_3)$$

$$= \frac{3m_2^2 m_3 m_1 + 3m_1^2 m_2 m_3 + 2m_1 m_2^2 E + m_2^2 m_3 E + 3m_1^2 m_2^2 + m_3 m_1^2 E + 2m_1^2 m_2 E + m_2 E^2 m_1}{3\left(m_1 m_2 + m_2 m_3 + m_3 m_1 + m_2 E\right)\left(m_1 m_2 + m_2 m_3 + m_3 m_1 + m_1 E\right)}, (275)$$

so that

$$\varepsilon_1(E, m_1, m_2, m_3) = m_1 + E f_1(E, m_1, m_2, m_3),$$

$$\varepsilon_2(E, m_1, m_2, m_3) = m_2 + E f_2(E, m_1, m_2, m_3),$$

$$\varepsilon_3(E, m_1, m_2, m_3) = m_3 + E f_3(E, m_1, m_2, m_3). \tag{276}$$

Now we want to go back to the full three-body equation (Eq.(269)) in order to create a Schrödinger-like form. Our bound state equation is then rewritten as

$$\begin{aligned}
&[\left(\varepsilon_1^2 - m_1^2\right)\left(1/\varepsilon_1 + 1/\varepsilon_2 + 1/\varepsilon_3\right) - 3\frac{p_1^2 + \varepsilon_1^2 + \Phi_1}{\varepsilon_1} \\
&+ \left(\varepsilon_2^2 - m_2^2\right)\left(1/\varepsilon_1 + 1/\varepsilon_2 + 1/\varepsilon_3\right) - 3\frac{p_2^2 + \varepsilon_2^2 + \Phi_2}{\varepsilon_2} \\
&+ \left(\varepsilon_3^2 - m_3^2\right)\left(1/\varepsilon_1 + 1/\varepsilon_2 + 1/\varepsilon_3\right) - 3\frac{p_3^2 + \varepsilon_3^2 + \Phi_3}{\varepsilon_3}]\psi(x_{12\perp}, x_{23\perp}, x_{31\perp}) \\
&= 0. \tag{277}
\end{aligned}$$

And once again, in order to simplify down to a Schrödinger-like form, we collect functions into the definitions

$$\begin{aligned}
b_1(E, m_1, m_2, m_3) &= \left(\varepsilon_1(E, m_1, m_2, m_3) + m_1\right) \\
&\times (1/\varepsilon_1(E, m_1, m_2, m_3) + 1/\varepsilon_2(E, m_1, m_2, m_3) + 1/\varepsilon_3(E, m_1, m_2, m_3))/6,
\end{aligned}$$

$$\begin{aligned}
b_2(E, m_1, m_2, m_3) &= \left(\varepsilon_2(E, m_1, m_2, m_3) + m_2\right) \\
&\times (1/\varepsilon_1(E, m_1, m_2, m_3) + 1/\varepsilon_2(E, m_1, m_2, m_3) + 1/\varepsilon_3(E, m_1, m_2, m_3))/6,
\end{aligned}$$

$$\begin{aligned}
b_3(E, m_1, m_2, m_3) &= \left(\varepsilon_3(E, m_1, m_2, m_3) + m_3\right) \\
&\times (1/\varepsilon_1(E, m_1, m_2, m_3) + 1/\varepsilon_2(E, m_1, m_2, m_3) + 1/\varepsilon_3(E, m_1, m_2, m_3))/6. (278)
\end{aligned}$$

So, our eigenvalue equation is now

$$
\begin{aligned}
\{E[&f_1(E, m_1, m_2, m_3)b_1(E, m_1, m_2, m_3) \\
&+f_2(E, m_1, m_2, m_3)b_2(E, m_1, m_2, m_3) \\
&+f_3(E, m_1, m_2, m_3)b_3(E, m_1, m_2, m_3)] \\
&-\frac{p_{1\perp}^2 + \Phi_{12} + \Phi_{13}}{2\varepsilon_1(E, m_1, m_2, m_3)} \\
&-\frac{p_{2\perp}^2 + \Phi_{23} + \Phi_{12}}{2\varepsilon_2(E, m_1, m_2, m_3)} \\
&-\frac{p_{3\perp}^2 + \Phi_{31} + \Phi_{23}}{2\varepsilon_3(E, m_1, m_2, m_3)}]\}\psi(x_{12\perp}, x_{23\perp}, x_{31\perp}) \\
= \quad & 0,
\end{aligned}
\tag{279}
$$

and finally, defining

$$
\begin{aligned}
&f_1(E, m_1, m_2, m_3)b_1(E, m_1, m_2, m_3) \\
&+f_2(E, m_1, m_2, m_3)b_2(E, m_1, m_2, m_3) \\
&+f_3(E, m_1, m_2, m_3)b_3(E, m_1, m_2, m_3) = F
\end{aligned}
\tag{280}
$$

we acquire a (relatively) simple Schrödinger-like form of the Hamiltonian:

$$
H = \frac{1}{F}\left(\frac{p_{1\perp}^2 + \Phi_{12} + \Phi_{13}}{2\varepsilon_1(E, m_1, m_2, m_3)} + \frac{p_{2\perp}^2 + \Phi_{23} + \Phi_{12}}{2\varepsilon_2(E, m_1, m_2, m_3)} + \frac{p_{3\perp}^2 + \Phi_{31} + \Phi_{23}}{2\varepsilon_3(E, m_1, m_2, m_3)}\right).
\tag{281}
$$

As a check, note that for $E = 0$

$$
\begin{aligned}
F(0) = &f_1(0, m_1, m_2, m_3)b_1(0, m_1, m_2, m_3) \\
&+f_2(0, m_1, m_2, m_3)b_2(0, m_1, m_2, m_3) \\
&+f_3(0, m_1, m_2, m_3)b_3(0, m_1, m_2, m_3) \\
= \quad & 1
\end{aligned}
\tag{282}
$$

so that when $E = 0$, this becomes an ordinary non-relativistic Hamiltonian like Eq.(218). So now, we have gone from Sazdjian's N-body formalism to a condensed three-body one that is much easier to work with in the constraint dynamics approach, as the Hamiltonian is now in a familiar form. This equation has the distinct advantage that it is like, for the purposes of solving it anyway, a non-relativistic Schrödinger equation. It is, of course, still relativistic, but it is now in a form that is much more easily recognizable and

usable than the one which Sazdjian left us, Eq.(266). It is important to note the recursive nature of this equation as this becomes highly relevant in the numerical studies. The Φ's are dependent on the $\varepsilon's$ and the $w's$ and so we must begin with an initial guess and solve the equation iteratively until an acceptable level of convergence is met.

4 The relativistic three body problem

It should be noted that this model and our relativistic generalizations below are often referred to as the "naive quark model" in that it does not account for the swarm of gluons and quark-anti-quark pair interactions directly. Rather, the model has all of the interacting forces existing between each quark-quark pair.

The process of going from a two-body system to a three-body one is not as straightforward as one might expect. Since the interacting potentials are limited to each quark-quark pair, there are now three times as many terms. Additionally, there are three sets of coordinates $(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_2 - \mathbf{r}_3, \mathbf{r}_1 - \mathbf{r}_3)$ instead of just one distance between the quarks as in the two-body case. This is best treated with a relative coordinate substitution to reduce the number of coordinates from three to two. The coordinate transform used in this work is similar to and uses the same notation as Isgur's ([6]) work, but is not an identical transformation due to treating the more general case of all three quarks as different masses, rather than just two.

The sections that follow describe the methods used in going from a two-body system to a three-body one, mostly dealing with the potentials and coordinate transforms. We also describe our Gaussian basis functions and the reasoning behind them. Since we are still merely using two quark interaction potentials (see Eq.(183)), the scalar (S), vector (A) and Darwin (Φ_D) terms, all simply expand from one term to three, to account for all three interactions. However, the spin-spin (Φ_{SS}), spin-orbit ($\Phi_{SO}, \Phi_{SOT}, \Phi_{SOX}$) and tensor ($\Phi_T$) terms require manipulations of total J, total L and total S, that the two-body system neither accounts for nor has to deal with and thus have been extensively reworked for the three-body system.

4.1 Coordinate system transforms and tensor substitutions

One of the more difficult parts of the three-body problem is also possibly the most fundamental: a proper coordinate system. Therefore, this section provides a description of how the coordinate system is set up for the relativistic three-body problem. Defining a coordinate system that is both correct and useful for a three-body problem is non-trivial and requires extensive manipulations and even some of what would be considered mathematical tricks. One of the simplest and most common ways to begin handling a three-body system is to redefine the coordinate system so that there are only two relative coordinates instead of three.

In the following section we describe the way the coordinate system is defined for the three-body system and then reduced to two relative coordinates, plus a "center of mass". We then address how those are further simplified with additional coordinate transforms in order to analytically solve as much of the problem as possible before going to numerical methods.

Let us return to Eq. (281). Our goal is to create a coordinate system in which the kinetic terms are analytic and the variational principle will be used to solve for the energy eigenvalues. The general form of each Φ_{ij} is

$$\Phi_{ij} = 2m_{wij}S_{ij} + S_{ij}^2 + 2\varepsilon_{wij}A_{ij} - A_{ij}^2 + \Phi_{Dij} + \mathbf{L}\cdot(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)\Phi_{SOij} + \boldsymbol{\sigma}_i\cdot\hat{\mathbf{r}}_{ij}\boldsymbol{\sigma}_j\cdot\hat{\mathbf{r}}_{ij}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\Phi_{SOTij}$$

$$+ \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\Phi_{S\,\mathrm{Si}\,j} + (3\boldsymbol{\sigma}_i\cdot\hat{\mathbf{r}}_{ij}\boldsymbol{\sigma}_j\cdot\hat{\mathbf{r}}_{ij}-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\Phi_{Tij} + \mathbf{L}\cdot(\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_j)\boldsymbol{\Phi}_{SODij}, \tag{283}$$

where

$$\begin{aligned}
S_{ij} &= \frac{4\pi\Lambda^2 r_{ij}}{27}, \\
A_{ij} &= -\frac{8\pi}{27 r_{ij}\ln(Ke^2 + \frac{B}{(\Lambda r_{ij})^2})} + \frac{e_1 e_2}{4\pi r_{ij}},
\end{aligned} \tag{284}$$

and the various Φ terms are all functions of S_{ij}, A_{ij} and their derivatives (explicit forms given in Appendix A). Note here how they still account for the asymptotic freedom and linear confinement mentioned earlier. The scalar term goes to infinity as r goes to an infinite value, providing confinement, while the logarithm in the vector term becomes large at short distance, giving asymptotic freedom (this causes the vector term to behave more like $\tilde{\alpha}/r$). We now want to define a coordinate system such that the three coordinates r_{ij} become two relative coordinates that can be written in terms of the original r_{ij} distances between each quark pair. In direct analogy to what was done in the nonrelativistic case (Eq.(220), the coordinate system is defined as

$$\begin{aligned}
\boldsymbol{\rho} &= \mathbf{r}_2 - \mathbf{r}_3, \\
\boldsymbol{\lambda} &= \frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)\varepsilon_1}\mathbf{r}_2 + \frac{w\varepsilon_3}{(\varepsilon_2 + \varepsilon_3)\varepsilon_1}\mathbf{r}_3, \\
\mathbf{r}_1 - \mathbf{r}_2 &= -\frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\boldsymbol{\rho} - \boldsymbol{\lambda}, \\
\mathbf{r}_1 - \mathbf{r}_3 &= -\frac{\varepsilon_2}{\varepsilon_2 + \varepsilon_3}\boldsymbol{\rho} + \boldsymbol{\lambda}, \\
\mathbf{r}_2 - \mathbf{r}_3 &= \boldsymbol{\rho}, \\
\varepsilon_\rho &= \frac{\varepsilon_1(\varepsilon_2 + \varepsilon_3)}{w}, \quad \varepsilon_\lambda = \frac{\varepsilon_2\varepsilon_3}{\varepsilon_2 + \varepsilon_3},
\end{aligned} \tag{285}$$

63

where again, w is the total baryon energy eigenvalue and the epsilons are the individual c.m. energies of each quark, such that

$$w = \varepsilon_1 + \varepsilon_2 + \varepsilon, \tag{286}$$

and the ε_ρ, and ε_λ can be regarded as reduced energy terms (similar to reduced mass, but energy instead since we are writing most of our Hamiltonian in terms of energy). We choose this form (rather than a more standard one that Isgur uses) in order to maintain the proper form of our momentum terms from the two-body formalism. In this new system, the original Hamiltonian of Eq. (281) now becomes

$$
\begin{aligned}
H \quad = \quad & \frac{1}{F} \Big(\frac{p_\rho^2}{2\varepsilon_\rho(E, m_1, m_2, m_3)} + \frac{p_\lambda^2}{2\varepsilon_\lambda(E, m_1, m_2, m_3)} \\
& + \frac{\Phi_{12} + \Phi_{13}}{2\varepsilon_1(E, m_1, m_2, m_3)} + \frac{\Phi_{23} + \Phi_{12}}{2\varepsilon_2(E, m_1, m_2, m_3)} + \frac{\Phi_{31} + \Phi_{23}}{2\varepsilon_3(E, m_1, m_2, m_3)} \Big),
\end{aligned} \tag{287}
$$

so that the kinetic terms are now simple and easily worked out analytically. Note that this equation is very similar to the non-relativistic Hamiltonian Eq.(218) in the preceding section.

We use the variational principle

$$\langle \Psi_n | H | \Psi_m \rangle = \langle \Psi_n | E | \Psi_m \rangle, \tag{288}$$

to find the eigenvalues of our Hamiltonian. For greater accuracy, we expand our Hamiltonian basis into the matrix equation

$$\mathbb{H}\Psi = \lambda \mathbb{B}\Psi, \tag{289}$$

where $\mathbb{H}$ and $\mathbb{B}$ are both matrices given by

$$
\begin{aligned}
H_{nm} \quad &= \quad \langle \Psi_n | H | \Psi_m \rangle, \\
B_{nm} \quad &= \quad \langle \Psi_n | \Psi_m \rangle.
\end{aligned} \tag{290}
$$

We denote the wavefunction of total J,L and S, where total L is composed of the angular momenta associated with the ρ or λ coordinate and total S is composed of the individual spins of each of the three quarks, with S_1, S_2 and S_3 corresponding to the spins of quarks 1, 2 and 3, respectively, as

$$|JL(l_\rho l_\lambda)S(S_1 S_2 S_3) M n_\rho n_\lambda\rangle = |\Psi_n\rangle. \tag{291}$$

Therefore a matrix element of the Hamiltonian would be

$$\langle \Psi_n | H | \Psi_m \rangle = \langle JL(l_\rho l_\lambda)S(S_1 S_2 S_3)M n_\rho n_\lambda | H | JL'(l'_\rho l'_\lambda)S'(S'_1 S'_2 S'_3)M' n'_\rho n'_\lambda \rangle \tag{292}$$

One major advantage of this particular choice of coordinates is that the kinetic terms can be analytically evaluated. The matrix elements for the kinetic term

$$\frac{1}{F}\left(\frac{p_\rho^2}{2\varepsilon_\rho(E, m_1, m_2, m_3)} + \frac{p_\lambda^2}{2\varepsilon_\lambda(E, m_1, m_2, m_3)}\right) \tag{293}$$

are then

$$
\begin{aligned}
&\langle \Psi_n | T | \Psi_m \rangle \\
=\ & \Big\{ \frac{1}{F\varepsilon_\rho} \frac{\alpha_\rho^{(2l_\rho+3)}}{\Gamma[(2l_\rho+3)/2]} \sqrt{(n_\rho n'_\rho)^{(2l_\rho+3)/2}(n_\lambda n'_\lambda)^{(2l_\lambda+3)/2}} [l_\rho(l_\rho+1)\frac{\Gamma[(2l_\rho+1)/2]}{2([n_\rho + n'_\rho]\alpha_\rho^2/2)^{(2l_\rho+1)/2}} \\
& - (2l_\rho+3)n'_\rho\alpha_\rho^2 \frac{\Gamma[(2l_\rho+3)/2]}{2([n_\rho+n'_\rho]\alpha_\rho^2/2)^{(2l_\rho+3)/2}} + n'^2_\rho\alpha_\rho^4 \frac{\Gamma[(2l_\rho+5)/2]}{2([n_\rho+n'_\rho]\alpha_\rho^2/2)^{(2l_\rho+5)/2}}] \\
& + \frac{1}{F\varepsilon_\lambda} \frac{\alpha_\lambda^{(2l_\lambda+3)}}{\Gamma[(2l_\lambda+3)/2]} \sqrt{(n_\rho n'_\rho)^{(2l_\rho+3)/2}(n_\lambda n'_\lambda)^{(2l_\lambda+3)/2}} [l_\lambda(l_\lambda+1)\frac{\Gamma[(2l_\lambda+1)/2]}{2([n_\lambda + n'_\lambda]\alpha_\lambda^2/2)^{(2l_\lambda+1)/2}} \\
& - (2l_\lambda+3)n'_\lambda\alpha_\lambda^2 \frac{\Gamma[(2l_\lambda+3)/2]}{2([n_\lambda+n'_\lambda]\alpha_\lambda^2/2)^{(2l_\lambda+3)/2}} + n'^2_\lambda\alpha_\lambda^4 \frac{\Gamma[(2l_\lambda+5)/2]}{2([n_\lambda+n'_\lambda]\alpha_\lambda^2/2)^{(2l_\lambda+5)/2}}] \Big\} \\
& \times \sum_{m'_\rho m'_\lambda} \langle l'_\rho l'_\lambda m'_\rho m'_\lambda | LM \rangle \delta_{l'_\rho l'_\lambda} \delta_{m'_\rho m'_\lambda} \sum_{m_\rho m_\lambda} \langle l_\rho l_\lambda m_\rho m_\lambda | LM \rangle \delta_{l_\rho l_\lambda} \delta_{m_\rho m_\lambda}.
\end{aligned} \tag{294}
$$

The matrix elements for the potentials are not analytic though and must be evaluated numerically. However, with the current substitution, the r_{23} term is relatively simple since $r_{23} = \rho$. The matrix element for the spin-independent $\Phi_{23}(r_{23})$ potential then is

$$
\begin{aligned}
&\langle \Psi_n(\rho, \lambda) | \Phi_{23}(\rho) | \Psi_m(\rho, \lambda) \rangle \\
=\ & \int \Phi_{23}(\rho) \sqrt{\frac{(n_\rho\alpha_\rho^2)^{(2l_\rho+3)/2}(n_\lambda\alpha_\lambda^2)^{(2l_\lambda+3)/2}}{\Gamma[(2l_\rho+3)/2]\Gamma[(2l_\lambda+3)/2]}} \sqrt{\frac{(n'_\rho\alpha_\rho^2)^{(2l'_\rho+3)/2}(n'_\lambda\alpha_\lambda^2)^{(2l'_\lambda+3)/2}}{\Gamma[(2l'_\rho+3)/2]\Gamma[(2l'_\lambda+3)/2]}} \\
& \times \rho^{(l_\rho+l'_\rho+2)}\lambda^{(l_\lambda+l'_\lambda+2)}e^{-(n_\rho+n'_\rho)\alpha_\rho^2\rho^2/2 - (n_\lambda+n'_\lambda)\alpha_\lambda^2\lambda^2/2} \\
& \sum_{m_\rho m_\lambda} \langle l_\rho l_\lambda m_\rho m_\lambda | LM \rangle Y_{l_\rho}^{m_\rho} Y_{l_\lambda}^{m_\lambda} \sum_{m'_\rho m'_\lambda} \langle l'_\rho l'_\lambda m'_\rho m'_\lambda | L'M' \rangle Y_{l'_\rho}^{m'_\rho} Y_{l'_\lambda}^{m'_\lambda} d^3\rho d^3\lambda,
\end{aligned} \tag{295}
$$

which, due to there being no dependence on the λ variable in the potential $\Phi_{23}(\rho)$ and performing the angular

integrations, reduces down to

$$\langle \Psi_n(\rho,\lambda)|\Phi_{23}(\rho)|\Psi_m(\rho,\lambda)\rangle$$

$$= \int \Phi_{23}(\rho)\sqrt{\frac{(n_\rho\alpha_\rho^2)^{(2l_\rho+3)/2}(n_\lambda\alpha_\lambda^2)^{(2l_\lambda+3)/2}}{\Gamma[(2l_\rho+3)/2]\Gamma[(2l_\lambda+3)/2]}}\sqrt{\frac{(n'_\rho\alpha_\rho^2)^{(2l'_\rho+3)/2}(n'_\lambda\alpha_\lambda^2)^{(2l'_\lambda+3)/2}}{\Gamma[(2l'_\rho+3)/2]\Gamma[(2l'_\lambda+3)/2]}}$$

$$\frac{\Gamma[(l_\lambda+l'_\lambda+3)/2]}{2[(n_\lambda+n'_\lambda)\alpha_\lambda^2]^{(l_\lambda+l'_\lambda+3)/2}}\rho^{(l_\rho+l'_\rho+2)}e^{-(n_\rho+n'_\rho)\alpha_\rho^2\rho^2/2}$$

$$\times \sum_{m_\rho m_\lambda}\langle l_\rho l_\lambda m_\rho m_\lambda|LM\rangle\delta_{l_\rho l_\lambda}\delta_{m_\rho m_\lambda}\sum_{m'_\rho m'_\lambda}\langle l'_\rho l'_\lambda m'_\rho m'_\lambda|L'M'\rangle\delta_{l'_\rho l'_\lambda}\delta_{m'_\rho m'_\lambda}\,d\rho. \tag{296}$$

Thus we are left with a function of one variable that can easily be numerically integrated regardless of what $\Phi_{23}(\rho)$ happens to be. However, this is not the case for the other two terms. The matrix elements of the r_{12} and r_{13} interactions are, respectively

$$\langle \Psi_n(\rho,\lambda)|\Phi_{12}(\rho,\lambda)|\Psi_m(\rho,\lambda)\rangle$$

$$= \int \Phi_{12}(\rho,\lambda)\sqrt{\frac{(n_\rho\alpha_\rho^2)^{(2l_\rho+3)/2}(n_\lambda\alpha_\lambda^2)^{(2l_\lambda+3)/2}}{\Gamma[(2l_\rho+3)/2]\Gamma[(2l_\lambda+3)/2]}}\sqrt{\frac{(n'_\rho\alpha_\rho^2)^{(2l'_\rho+3)/2}(n'_\lambda\alpha_\lambda^2)^{(2l'_\lambda+3)/2}}{\Gamma[(2l'_\rho+3)/2]\Gamma[(2l'_\lambda+3)/2]}}$$

$$\times\rho^{(l_\rho+l'_\rho)}\lambda^{(l_\lambda+l'_\lambda)}e^{-(n_\rho+n'_\rho)\alpha_\rho^2\rho^2/2-(n_\lambda+n'_\lambda)\alpha_\lambda^2\lambda^2/2}$$

$$\sum_{m_\rho m_\lambda}\langle l_\rho l_\lambda m_\rho m_\lambda|LM\rangle Y_{l_\rho}^{m_\rho}Y_{l_\lambda}^{m_\lambda}\sum_{m'_\rho m'_\lambda}\langle l'_\rho l'_\lambda m'_\rho m'_\lambda|LM\rangle Y_{l'_\rho}^{m'_\rho}Y_{l'_\lambda}^{m'_\lambda}d^3\rho d^3\lambda, \tag{297}$$

$$\langle \Psi_n(\rho,\lambda)|\Phi_{13}(\rho,\lambda)|\Psi_m(\rho,\lambda)\rangle$$

$$= \int \Phi_{13}(\rho,\lambda)\sqrt{\frac{(n_\rho\alpha_\rho^2)^{(2l_\rho+3)/2}(n_\lambda\alpha_\lambda^2)^{(2l_\lambda+3)/2}}{\Gamma[(2l_\rho+3)/2]\Gamma[(2l_\lambda+3)/2]}}\sqrt{\frac{(n'_\rho\alpha_\rho^2)^{(2l'_\rho+3)/2}(n'_\lambda\alpha_\lambda^2)^{(2l'_\lambda+3)/2}}{\Gamma[(2l'_\rho+3)/2]\Gamma[(2l'_\lambda+3)/2]}}$$

$$\times\rho^{(l_\rho+l'_\rho)}\lambda^{(l_\lambda+l'_\lambda)}e^{-(n_\rho+n'_\rho)\alpha_\rho^2\rho^2/2-(n_\lambda+n'_\lambda)\alpha_\lambda^2\lambda^2/2}$$

$$\sum_{m_\rho m_\lambda}\langle l_\rho l_\lambda m_\rho m_\lambda|LM\rangle Y_{l_\rho}^{m_\rho}Y_{l_\lambda}^{m_\lambda}d^3\rho d^3\lambda\sum_{m'_\rho m'_\lambda}\langle l'_\rho l'_\lambda m'_\rho m'_\lambda|LM\rangle Y_{l'_\rho}^{m'_\rho}Y_{l'_\lambda}^{m'_\lambda}d^3\rho d^3\lambda, \tag{298}$$

and so as the potentials are now in terms of two variables, it is much more difficult and time-consuming to numerically evaluate this integral. We therefore wish to make another variable change in the r_{12} and r_{13} systems in order to rewrite them in terms of a single variable as well.

What now follows is a brief description of the variable change simplification; more explicit details can be found in Appendix D. The variable change used is based on properties of the spherical harmonics and how

they relate to spherical tensors. Since spherical harmonics are trigonometric functions, we can get

$$
\begin{aligned}
Y_1^0 &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta, \\
\rho\cos\theta &= z, \\
Y_1^0\rho &= \frac{1}{2}\sqrt{\frac{3}{\pi}}z, \\
(Y_1^0\rho)^*Y_1^0\rho &= \frac{1}{4}\frac{3}{\pi}z^2,
\end{aligned}
\tag{299}
$$

$$
\begin{aligned}
Y_1^{-1} &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta(\cos\phi - i\sin\phi), \\
x &= r\cos\theta\sin\phi, \\
Y_1^{-1}r &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}x - i\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\sin\phi\rho = \frac{1}{2}\sqrt{\frac{3}{2\pi}}(x - iy), \\
(Y_1^{-1}r)^*Y_1^{-1}r &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}(x + iy)\frac{1}{2}\sqrt{\frac{3}{2\pi}}(x - iy) = \frac{1}{4}\frac{3}{2\pi}(x^2 + y^2),
\end{aligned}
\tag{300}
$$

and similarly for the other spherical harmonics. Since we will be using no more than $l = 2$, we now specialize our discussion to $l = 1$ with a brief description of $l = 2$. Therefore, since part of our wavefunction is a spherical harmonic (which is a trigonometric function) and a coordinate, the wavefunction can be rewritten in spherical tensor form as

$$
\begin{aligned}
\Psi_n &= \frac{4}{3\sqrt{\pi}}n_\rho^{5/4}\alpha_\rho^{5/2}n_\lambda^{5/4}\alpha_\lambda^{5/2}\rho\lambda e^{-n_\rho\alpha_\rho^2\rho^2/2 - n_\lambda\alpha_\lambda^2\lambda^2/2}\sum_M\langle 11m_\rho m_\lambda|00\rangle Y_1^{m_\rho}Y_1^{m_\lambda} \\
&= \frac{4}{3\sqrt{\pi}}n_\rho^{5/4}\alpha_\rho^{5/2}n_\lambda^{5/4}\alpha_\lambda^{5/2}e^{-n_\rho\alpha_\rho^2\rho^2/2 - n_\lambda\alpha_\lambda^2\lambda^2/2}\sum_M\langle 11m_\rho m_\lambda|00\rangle\rho_{m_\rho}\lambda_{m_\lambda},
\end{aligned}
\tag{301}
$$

where

$$
\begin{aligned}
\rho_{m_\rho} &= \rho Y_1^{m_\rho}(\hat{\rho}) \\
\lambda_{m_\lambda} &= \lambda Y_1^{m_\lambda}(\hat{\lambda}).
\end{aligned}
\tag{302}
$$

Additional manipulations are still needed in order to work out the expectation values explicitly. For the r_{12}

integration, a new set of variables is defined as

$$\boldsymbol{\rho}' = \mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2, \tag{303}$$

$$\boldsymbol{\lambda}' = \frac{w\varepsilon_1}{\varepsilon_3(\varepsilon_1 + \varepsilon_2)}\mathbf{r}_1 + \frac{w\varepsilon_2}{\varepsilon_3(\varepsilon_1 + \varepsilon_2)}\mathbf{r}_2,$$

then are rewritten in terms of new primed variables and as tensors, using the same tensor substitution done above

$$\boldsymbol{\rho} = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\boldsymbol{\rho}' + \boldsymbol{\lambda}',$$

$$\rho_{m_\rho} = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\rho'_{m_\rho} + \lambda'_{m_\rho}, \tag{304}$$

$$\boldsymbol{\lambda} = \frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)}\boldsymbol{\rho}' - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\boldsymbol{\lambda}',$$

$$\lambda_{m_\lambda} = \frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)}\rho'_{m_\lambda} - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\lambda'_{m_\lambda}. \tag{305}$$

Note that this is not a new coordinate system but rather a change of integration variables. This means that, while we are currently working out the new integral for the r_{12} system, we can use a similar substitution for the r_{13} system and acquire a nearly identical equation with only a few constants changed (constants in terms of the integration variable, not overall constants for the full calculation). Before the new substitution of the primed coordinates, the expectation value of the potential $\Phi_{12}(\rho' = r_{12})$ is

$$\langle \Psi_n | \Phi_{12}(\mathbf{r}_{12}) | \Psi_m \rangle = \langle \Psi_n | \Phi_{12}(\boldsymbol{\rho}') | \Psi_m \rangle \tag{306}$$

$$= \frac{16}{9\pi}\alpha_\rho^5\alpha_\lambda^5(n_{\rho1}^{5/4}n_{\lambda1}^{5/4}n_{\rho2}^{5/4}n_{\lambda2}^{5/4})$$

$$\times \int \sum_{m_{\rho1}m_{\lambda1}} \langle 11m_{\rho1}m_{\lambda1}|00\rangle \rho^*_{m_{\rho1}}\lambda^*_{m_{\lambda1}} \sum_{m_{\rho2}m_{\lambda2}} \langle 11m_{\rho2}m_{\lambda2}|00\rangle \rho_{m_{\rho2}}\lambda_{m_{\lambda2}},$$

$$\times e^{-(n_{\rho1}+n_{\rho2})\alpha_\rho^2\rho^2 - (n_{\lambda1}+n_{\lambda2})\alpha_\lambda^2\lambda^2} d^3\rho d^3\lambda \Phi_{12}(\boldsymbol{\rho}'). \tag{307}$$

And so now, using the derived relationships between the primed and un-primed coordinates ρ and λ(Eq.302,303),

the expectation value becomes

$$\langle \Psi_n | \Phi_{12}(\boldsymbol{\rho}') | \Psi_m \rangle$$

$$= N \int e^{-c(\lambda' + \frac{b}{2c}\rho')^2 - (a - \frac{b^2}{4c})\rho'^2} d^3\rho' d^3\lambda' \Phi_{12}(\rho'),$$

$$\times \sum_M \langle 11 m_{\rho 1} m_{\lambda 1} | 00 \rangle \left(\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)} \rho'_{m_{\lambda 1}} - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} \lambda'_{m_{\lambda 1}} \right)^* \left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \rho'_{m_{\rho 1}} + \lambda'_{m_{\rho 1}} \right)^*,$$

$$\times \sum_M \langle 11 m_{\rho 2} m_{\lambda 2} | 00 \rangle \left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \rho'_{m_{\rho 2}} + \lambda'_{m_{\rho 2}} \right) \left(\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)} \rho'_{m_{\lambda 2}} - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} \lambda'_{m_{\lambda 2}} \right), \quad (308)$$

where for simplicity

$$N = \frac{16}{9\pi} \alpha_\rho^5 \alpha_\lambda^5 (n_{\rho 1}^{5/4} n_{\lambda 1}^{5/4} n_{\rho 2}^{5/4} n_{\lambda 2}^{5/4}),$$

$$a = \frac{(n_{\rho 1} + n_{\rho 2})\alpha_\rho^2}{2} \left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \right)^2 + \frac{(n_{\lambda 1} + n_{\lambda 2})\alpha_\lambda^2}{2} \left(\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)} \right)^2,$$

$$b = -\frac{(n_{\rho 1} + n_{\rho 2})\alpha_\rho^2}{2} \left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \right) + (n_{\lambda 1} + n_{\lambda 2})\alpha_\lambda^2 \frac{w\varepsilon_2 \varepsilon_3}{(\varepsilon_2 + \varepsilon_3)^2(\varepsilon_1 + \varepsilon_2)},$$

$$c = \frac{(n_{\rho 1} + n_{\rho 2})\alpha_\rho^2}{2} + \frac{(n_{\lambda 1} + n_{\lambda 2})\alpha_\lambda^2}{2} \left(\frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} \right)^2. \quad (309)$$

However, there is still a troublesome $\lambda' \cdot \rho'$ cross term in the Gaussian wavefunction itself. In order to eliminate this, one final coordinate change is used

$$\mathbf{x} = \boldsymbol{\lambda}' + \frac{b}{2c}\boldsymbol{\rho}',$$

$$\boldsymbol{\lambda}' = \mathbf{x} - \frac{b}{2c}\boldsymbol{\rho}',$$

$$\lambda'_m = x_m - \frac{b}{2c}\rho'_m. \quad (310)$$

The explicit details can be found in Appendix D; the end result is

$$\langle \Psi_n | \Phi_{12}(\boldsymbol{\rho}') | \Psi_m \rangle = \frac{1}{4} \sqrt{\pi} c^{-3/2} N \int e^{-(a - \frac{b^2}{4c})\rho'^2} \rho'^2 \phi_{12} d\rho'$$

$$\times (\rho'^2 \{ (\frac{\varepsilon_3^2(-2c\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3))^2}{4c^2(\varepsilon_2 + \varepsilon_3)^4} + \frac{(b(\varepsilon_1 + \varepsilon_2)\varepsilon_3 + 2c\varepsilon_2 w)^2}{4c^2(\varepsilon_1 + \varepsilon_2)^2(\varepsilon_2 + \varepsilon_3)^2})$$

$$+ \frac{\varepsilon_3(-2c\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3))(b(\varepsilon_1 + \varepsilon_2)\varepsilon_3 + 2c\varepsilon_2 w)}{4c^2(\varepsilon_1 + \varepsilon_2)(\varepsilon_2 + \varepsilon_3)^3} [2\delta_{m_{\lambda 1} m_{\rho 1}} + 2\delta_{m_{\lambda 2} m_{\rho 2}}] \})$$

$$+ N \int e^{-(a - \frac{b^2}{4c})\rho'^2} (\frac{1}{4} \sqrt{\pi} c^{-3/2} \rho'^4 + \frac{15}{16} \sqrt{\pi} c^{-7/2}) \rho'^2 \Phi_{12} d\rho'$$

$$\times \sum_{m_{\rho 1} m_{\lambda 1}} \langle 11 m_{\rho 1} m_{\lambda 1} | 00 \rangle [\sum_{m_{\rho 2} m_{\lambda 2}} (\langle 11 m_{\rho 2} m_{\lambda 2} | 00 \rangle$$

$$\times \sum_l \frac{4}{\pi} \langle 11 m_{\lambda 2} m_{\rho 2} | lm \rangle \langle 1100 | l0 \rangle \begin{pmatrix} 1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & l \\ m_{\rho 1} & m_{\lambda 1} & m \end{pmatrix})] \quad (311)$$

So now the expectation value is in terms of just one variable, which can be numerically evaluated easily. The r_{13} term simply requires changing the a, b, and c constants and the epsilons and the r_{23} term is trivial since the coordinate system is defined in terms of r_{23}. Again, these are listed in Appendix A. The potential is now in terms of just one variable, so almost regardless of what potential is used, the numerical calculations will be fairly straightforward. Thus, the coordinate system has been defined and transformed in such a way as to make a good deal of the problem analytic while keeping what is not analytic still easy to solve numerically.

We can follow a similar method for $l = 2$, by constructing new states as functions of the $l = 1$ states. The explicit forms of the spherical harmonics for $l = 0$ and $l = 1$ are

$$Y_0^0 = \frac{1}{2} \sqrt{\frac{1}{\pi}},$$

$$Y_1^{-1} = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{-i\phi},$$

$$Y_1^0 = \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta,$$

$$Y_1^1 = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{i\phi}, \quad (312)$$

which can be used to rewrite the $l = 2$ spherical harmonics as

$$
\begin{aligned}
Y_2^{-2} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{-2i\phi} = (Y_1^{-1})^2\sqrt{2}\sqrt{\frac{5\pi}{3}}, \\
Y_2^{-1} &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{-i\phi} = Y_1^0 Y_1^{-1}2\sqrt{\frac{5\pi}{3}}, \\
Y_2^0 &= \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1) = (Y_1^0)^2\frac{1}{\sqrt{3}}\sqrt{\frac{5\pi}{3}} - (Y_0^0)^2\sqrt{5\pi}, \\
Y_2^1 &= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{i\phi} = Y_1^0 Y_1^1 2\sqrt{\frac{5\pi}{3}}, \\
Y_2^{-2} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi} = (Y_1^1)^2\sqrt{2}\sqrt{\frac{5\pi}{3}}.
\end{aligned}
\tag{313}
$$

So then, since our wavefunction for $l_\rho = l_\lambda = 2$ is of the form

$$
\begin{aligned}
|\Psi_n(\boldsymbol{\rho},\boldsymbol{\lambda})\rangle &= \frac{8}{15\sqrt{\pi}}n_\rho^{7/4}\alpha_\rho^{7/2}n_\lambda^{7/4}\alpha_\lambda^{7/2}\rho^2\lambda^2 e^{-n_\rho\alpha_\rho^2\rho^2/2 - n_\lambda\alpha_\lambda^2\lambda^2/2} \\
&\quad \times \sum_{m_\rho m_\lambda}\langle 22m_\rho m_\lambda|LM\rangle Y_2^{m_\rho}Y_2^{m_\lambda},
\end{aligned}
\tag{314}
$$

we can use the original tensor substitution of Eq. (302) to define

$$
\begin{aligned}
\rho^2 Y_2^{-2} &= \rho^2(Y_1^{-1})^2\sqrt{2}\sqrt{\frac{5\pi}{3}} = \sqrt{2}\sqrt{\frac{5\pi}{3}}\rho_{-1}^2, \\
\rho^2 Y_2^{-1} &= \rho^2 Y_1^0 Y_1^{-1}2\sqrt{\frac{5\pi}{3}} = 2\sqrt{\frac{5\pi}{3}}\rho_{-1}\rho_0, \\
\rho^2 Y_2^0 &= \rho^2([Y_1^0]^2\frac{1}{\sqrt{3}}\sqrt{\frac{5\pi}{3}} - (Y_0^0)^2\sqrt{5\pi}] = \frac{1}{\sqrt{3}}\sqrt{\frac{5\pi}{3}}, \\
\rho^2 Y_2^1 &= \rho^2 Y_1^0 Y_1^1 2\sqrt{\frac{5\pi}{3}} = \rho_0\rho_1 2\sqrt{\frac{5\pi}{3}}, \\
\rho^2 Y_2^2 &= \rho^2(Y_1^1)^2\sqrt{2}\sqrt{\frac{5\pi}{3}} = \sqrt{2}\sqrt{\frac{5\pi}{3}}\rho_1^2.
\end{aligned}
\tag{315}
$$

Therefore, we can follow the same procedure outlined for $l = 1$, the only major difference being that the algebra is more complicated. However, since we're going to have to do at least one numerical integration anyway, we can simply use a program such as Mathematica (or Wolframalpha) to analytically integrate what is analytically integrable and then numerically integrate the rest without doing an absurd amount of algebra. This method should actually hold to allow for higher order l states to be done relatively simply, since essentially all that will become more complicated is the algebra arising from the substitutions (this may be part of future work on this subject).

With the matrix elements defined for a general potential and for analytic kinetic terms, we now need to explicitly define our potential model.

5 Three Body Potential

Conceptually speaking, the approach one would take to go from a two-body system with the formalism we have described to a three body one is straightforward. The problem is now treated as three two-body problems, so the overall form of the potentials remain mostly the same. However, the explicit calculations become considerably more complicated since there are three different sets of couplings to consider, three different sets of coordinates (which were reduced to two in the previous section), and three different operators for each potential term. This not only increases the magnitude of the calculations involved, it requires entirely new approaches as some methods that work very well for the two-body system can no longer be used. As everything is now in terms of the r_{23} variable, the other terms must be appropriately recoupled to properly use the operators. The two-body potential derived by Crater[4] can be written as

$$\Phi = 2m_w S + S^2 + 2\varepsilon_w A - A^2 + \Phi_D + \mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\Phi_{SO} + \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} \mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\Phi_{SOT}$$

$$+ \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \Phi_{SS} + (3\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\Phi_T + \mathbf{L} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\boldsymbol{\Phi}_{SOD} + i\mathbf{L} \cdot \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2 \Phi_{SOX}, \tag{316}$$

where the Φ terms are all functions of the vector potential, scalar potential, and quark masses (these are all defined explicitly in Appendix A). The three-body potential is of similar form and essentially just triples the number of terms, so instead of having 1 and 2 there are interactions for all three quarks. For the relatively simple vector (A), scalar (S) and Darwin (Φ_D) terms this is almost trivial, as there are no direct spin-dependent operators; however, the spin-spin, tensor, and spin-orbit terms require extensive reworking, which are described in the following sections.

5.1 State Couplings and Operator Methods

We will now describe how we set up our three-body states in such a way as to use the potential operators. In order to simplify our numerical calculations, it is helpful to note that the potential terms are products of a term involving the coupled angular momentum operators and a coordinate dependent terms that have trivial operator dependence . This allows us to use the operator angular momentum operators on a specified state and just get a number back that depends on the angular components of the state itself and not any radial components, so that the numerical integral itself does not involve any angular momentum operators. Thus, our potential terms separated into operator and non-operator pieces are given in table 2 and and the explicit forms of the Φ terms are given in Appendix A. For the purposes of working out the operators, it is sufficient to simply consider them to be radial functions that are unaffected by the operators.

Table 2: Potential terms, operators and non-operator components

Potential Term	Operator component	Non-operator component
Spin-Spin	$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$	$\Phi_{SS}(\mathbf{r}_{ij})$
Spin-Orbit	$\mathbf{L}_{ij} \cdot (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j)$	$\Phi_{SO}(\mathbf{r}_{ij})$
Spin-Orbit Difference	$\mathbf{L}_{ij} \cdot (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_j)$	$\Phi_{SOD}(\mathbf{r}_{ij})$
Tensor	$3\boldsymbol{\sigma}_i \cdot \hat{\mathbf{r}}_{ij}\boldsymbol{\sigma}_j \cdot \hat{\mathbf{r}}_{ij} - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$	$\Phi_T(\mathbf{r}_{ij})$
Spin-Orbit Cross	$i\mathbf{L}_{ij} \cdot \boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_j$	$\Phi_{SOX}(\mathbf{r}_{ij})$
Spin-Orbit Tensor	$\boldsymbol{\sigma}_i \cdot \hat{\mathbf{r}}_{ij}\boldsymbol{\sigma}_j \cdot \hat{\mathbf{r}}_{ij}\mathbf{L}_{ij} \cdot (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j)$	$\Phi_{SOT}(\mathbf{r}_{ij})$

Table 3: Spin-flavor wavefunctions for the baryon spectrum

N	J	L	S	SF	Ψ
8	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}(\phi'\chi' + \phi''\chi'')\psi_0$	Ψ_1
10	$\frac{3}{2}$	0	$\frac{3}{2}$	$\phi^s\chi^s\psi_0$	Ψ_2
8	$\frac{1}{2}, \frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}[(\phi'\chi'' + \phi''\chi')\psi' + (\phi'\chi' - \phi''\chi'')\psi'']$	$\Psi_3(J=\frac{1}{2}), \Psi_4(J=\frac{3}{2})$
8	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$	1	$\frac{3}{2}$	$\frac{1}{\sqrt{2}}[\phi'\chi^s\psi' + \phi''\chi^s\psi'']$	$\Psi_5(J=\frac{1}{2}), \Psi_6(J=\frac{3}{2}), \Psi_7(J=\frac{5}{2})$
10	$\frac{1}{2}, \frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}[\phi^s\chi'\psi' + \phi^s\chi''\psi'']$	$\Psi_8(J=\frac{1}{2}), \Psi_9(J=\frac{3}{2})$
1	$\frac{1}{2}, \frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}[\phi^a\chi''\psi' - \phi^a\chi'\psi'']$	$\Psi_{10}(J=\frac{1}{2}), \Psi_{11}(J=\frac{3}{2})$

For the baryons we have considered, there are a total of eleven different wavefunctions, which represent all possible spin-flavor couplings for the various particles. The form of these is given in table 3 where it has been split into three components: spin, flavor and space (represented by χ, ϕ, ψ, respectively), explicitly defined in Appendix B. As there are six interactions to consider and three couplings per interaction (we are using two-body operators, so there is a 1-2, 1-3, and 2-3 term for each operator), there are a total of 198 possible interactions to consider. Fortunately, many of these are similar or trivial and so the number that must actually be worked out explicitly drops considerably, but there still are a quite a large number that are non-trivial. The eleven wavefunctions are given by [47] where ψ_0 is a total $L = l_\rho = l_\lambda = 0$ wavefunction and ψ' and ψ'' are total $L = 1$ and $l_\rho = 1$ or $l_\lambda = 1$ wavefunctions, respectively ($L = 1$ state have parity of -1, so l_ρ and l_λ cannot both be 1). In addition, ϕ', ϕ'', ϕ^s and ϕ^a are all purely flavor wavefunctions and χ', χ'', χ^s are all purely spin wavefunctions, having total $S = 1/2, 1/2$ and $3/2$, respectively. These merely contain all possible combinations of flavor or spin so that the product of the two gives all possible spin-flavor couplings. All of these wavefunctions are orthogonal to the others in the set (that is, χ' is orthogonal to χ'' and χ^s etc.). Explicit forms of these terms are given in Appendix B. The quark flavor combination of the ϕ terms is different for each baryon, but since the operators we use do not affect the flavor, it does not matter what they are explicitly for the purposes of this calculation.

There are two methods we use to determine the affect of these operators. One is a simple ladder operator approach and the other involves use of the Wigner 9j recoupling coefficients. Both methods are always valid, but not necessarily always useful due to how the operator form affects each individual wavefunction for the ladder operators. It is worth noting that having two methods be viable also allows for a good check. The

ladder operator method works out simply for everything but the Spin-Orbit Cross term (Eq. (187)) due to how we can have a wavefunction independent of total M. Thus, we can set $M = J$ and force $M_s = S$. For the states $\Psi_1, \Psi_2, \Psi_4 \Psi_7, \Psi_9$ and Ψ_{11}, this means that any operator that changes total M_s will be orthogonal to the original wavefunction and thus we can eliminate any term that does change total M_s. As most readers are familiar with ladder operators, we relegate most of the details of this method to Appendix C. The other method (9j coupling) though is not as familiar and so we will describe it in detail. Due to having matrix elements for the two-body problem already defined by Crater [4] , the difficult part of this problem is recoupling the state into one which can use these matrix elements, since it is not readily clear how they operate on a three-body problem.

5.2 Spin-Spin

This section deals with how the spin-spin couplings are defined and worked out. As we will see in the spin-orbit term (and others as well), the use of 6j recoupling coefficients is necessary. The default state is a 2-3 coupling one due to how we have defined our coordinate system, so in order to work out the spin states for the other couplings (1-2 and 1-3) the state must be changed to reflect those couplings, since while the $\mathbf{S}_2 \cdot \mathbf{S}_3$ term simply yields a number, the $\mathbf{S}_1 \cdot \mathbf{S}_2$ and $\mathbf{S}_1 \cdot \mathbf{S}_3$ do not. The recoupling is done using 6j (Racah) coefficients. Here, we work out an example state (the proton) to show how this is done. Full calculations and results are in Appendix B.

Note on notation: the general quark spin state is

$$[s_1\{s_2 s_3\}^{s_{23}}]^S \tag{317}$$

where s_i represents each quark (1,2 or 3), S_{23} is the coupled spin of quark 2 and quark 3, and S is the total spin of the system. The curly brackets denote which quarks are sub-coupled (that is, which two are coupled first to a S_{ij} coupling before being coupled to total S). For example,

$$[s_1\{s_2 s_3\}^{s_{23}}]^S \tag{318}$$

would indicate that quarks 2 and 3 are coupled to S_{23} and quark 1 couples to that state for total S, while

$$[\{s_1 s_3\}^{s_{13}} s_2]^S \tag{319}$$

would indicate that quarks 1 and 3 are coupled to S_{13} and quark 2 couples to that state for total S.

The spin dependence of the wavefunction is contained entirely within the χ', χ'' and χ^S terms. These χ spin terms couple with a flavor term we call ϕ in a unique way for each set of baryons that share similar properties. Explicit forms of these are given in Appendix B. Therefore, all we are concerned with for the spin-spin operator are these χ terms. The various spin terms are written as

$$
\begin{aligned}
\chi'(m_S &= \frac{1}{2}) = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow), \\
\chi'(m_S &= -\frac{1}{2}) = \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow), \\
\chi''(m_S &= \frac{1}{2}) = \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow), \\
\chi''(m_S &= -\frac{1}{2}) = \frac{1}{\sqrt{6}}(-2\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow), \\
\chi^s(m_S &= 3/2) = \uparrow\uparrow\uparrow, \\
\chi^s(m_S &= \frac{1}{2}) = \frac{1}{\sqrt{3}}(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow), \\
\chi^s(m_S &= -\frac{1}{2}) = \frac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow).
\end{aligned}
\tag{320}
$$

These can be written in a Clebsch-Gordan sum form for the 1-2 coupling as

$$
\begin{aligned}
\chi'(m_S &= \frac{1}{2}) = \chi'(m_S = -\frac{1}{2}) = [\{s_1 s_2\}^0 s_3]^{1/2} \\
\chi''(m_S &= \frac{1}{2}) = \chi''(m_S = -\frac{1}{2}) = [\{s_1 s_2\}^1 s_3]^{1/2} \\
\chi^s(m_S &= \frac{1}{2}) = \chi^s(m_S = \frac{1}{2}) = \chi^s(m_S = -\frac{1}{2}) = [\{s_1 s_2\}^1 s_3]^{3/2}.
\end{aligned}
\tag{321}
$$

The easiest way now to convert these to 1-3 and 2-3 systems is to use a 6j recoupling coefficient, which is defined as

$$
|[\{ac\}^{J_{ac}} b]^J\rangle = \sum_{J_{ab}} (-1)^{j_b + j_c + J_{ab} + J_{ac}} \sqrt{(2J_{ab} + 1)(2J_{ac} + 1)} \begin{pmatrix} j_a & j_b & J_{ab} \\ J & j_c & J_{ac} \end{pmatrix} |[\{ab\}^{J_{ab}} c]^J\rangle.
\tag{322}
$$

where

$$
\begin{pmatrix} j_a & j_b & J_{ab} \\ J & j_c & J_{ac} \end{pmatrix}
\tag{323}
$$

is a 6j symbol, similar to 9j and 3j symbols. Like the 9j symbol, the 6j allows us to recouple a state into a desired form, which allows the operators to work simply. The difference between it and the 9j is simply that the 6j recouples three different angular momenta while the 9j recouples four (and in case the reader is curious, there are 12j symbols that change couplings among 5 angular momenta and 15j that change

couplings among 6, but those are not useful to us here). Thus, to go from 1-2 to 1-3, we have

$$
\begin{aligned}
\chi' &= [\{s_1s_2\}^0s_3]^{1/2} = \sum_{S_{13}}(-1)^{1+S_{13}+0}\sqrt{(2S_{13}+1)(1)}\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&S_{13}\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix}[\{s_1s_3\}^{S_{13}}s_2]^{1/2}\\
&= \frac{1}{2}[\{s_1s_3\}^0s_2]^{1/2} + \frac{\sqrt{3}}{2}[\{s_1s_3\}^1s_2]^{1/2}\\
\chi'' &= [\{s_1s_2\}^1s_3]^{1/2} = \sum_{S_{13}}(-1)^{1+S_{13}+1}\sqrt{(2S_{13}+1)(3)}\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&S_{13}\\\frac{1}{2}&\frac{1}{2}&1\end{pmatrix}[\{s_1s_3\}^{S_{13}}s_2]^{1/2}\\
&= \frac{\sqrt{3}}{2}[\{s_1s_3\}^0s_2]^{1/2} - \frac{1}{2}[\{s_1s_3\}^1s_2]^{1/2}\\
\chi^s &= [\{s_1s_2\}^1s_3]^{3/2} = \sum_{S_{13}}(-1)^{1+S_{13}+1}\sqrt{(2S_{13}+1)(3)}\begin{pmatrix}1/2&1/2&S_{13}\\3/2&1/2&1\end{pmatrix}[\{s_1s_3\}^{S_{13}}s_2]^{3/2}\\
&= [\{s_1s_3\}^1s_2]^{3/2}
\end{aligned}
\tag{324}
$$

and similarly for 1-2 to 2-3, we have

$$
\begin{aligned}
\chi' &= [\{s_1s_2\}^0s_3]^{1/2} = \sum_{S_{23}}(-1)^{1+S_{23}+0}\sqrt{(2S_{23}+1)(1)}\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&S_{23}\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix}[\{s_2s_3\}^{S_{23}}s_1]^{1/2}\\
&= \frac{1}{2}[\{s_2s_3\}^0s_1]^{1/2} + \frac{\sqrt{3}}{2}[\{s_2s_3\}^1s_1]^{1/2}\\
\chi'' &= [\{s_1s_2\}^1s_3]^{1/2} = \sum_{S_{23}}(-1)^{1+S_{23}+1}\sqrt{(2S_{23}+1)(3)}\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&S_{23}\\\frac{1}{2}&\frac{1}{2}&1\end{pmatrix}[\{s_2s_3\}^{S_{23}}s_1]^{1/2}\\
&= \frac{\sqrt{3}}{2}[\{s_2s_3\}^0s_1]^{1/2} - \frac{1}{2}[\{s_2s_3\}^1s_1]^{1/2}\\
\chi^s &= [\{s_1s_2\}^1s_3]^{3/2} = \sum_{S_{23}}(-1)^{1+S_{23}+1}\sqrt{(2S_{23}+1)(3)}\begin{pmatrix}1/2&1/2&S_{23}\\3/2&1/2&1\end{pmatrix}[\{s_2s_3\}^{S_{23}}s_1]^{3/2}\\
&= [\{s_2s_3\}^1s_1]^{3/2}
\end{aligned}
\tag{325}
$$

So now that spins i and j are coupled, the operator $\mathbf{S}_i\cdot\mathbf{S}_j$ can operate simply and just gives a number

$$
\begin{aligned}
\mathbf{S}_i\cdot\mathbf{S}_j[\{s_is_j\}^{S_{ij}=0}s_k]^{1/2} &= \frac{S_{ij}^2 - S_i^2 - S_j^2}{2} = \frac{0 - \frac{3}{4} - \frac{3}{4}}{2} = -\frac{3}{4}\\
\mathbf{S}_i\cdot\mathbf{S}_j[\{s_is_j\}^{S_{ij}=1}s_k]^{1/2} &= \frac{S_{ij}^2 - S_i^2 - S_j^2}{2} = \frac{2 - \frac{3}{4} - \frac{3}{4}}{2} = \frac{1}{4}.
\end{aligned}
\tag{326}
$$

Also note that in terms of ladder operators,

$$\mathbf{S}_i \cdot \mathbf{S}_j = \frac{1}{2}(S_{i+}S_{j-} + S_{i-}S_{j+}) + S_{iz}S_{jz}, \tag{327}$$

which works out simply for every Ψ_n listed above. The 6j recoupling is then not necessary for the spin-spin interaction, but it does become necessary in the others. Details and tables of all spin matrix elements are given in Appendix C. The spin-spin is, unfortunately, the simplest of our interactions. We have described two possible ways of obtaining the operator matrix elements here and now we continue on to the spin-orbit interaction.

5.3   9j Couplings

Here will we describe how the 9j recoupling coefficient is used for our three-body problem. We begin with a totally general state with four angular momenta of

$$|((j_1 j_4)j_7, (j_2 j_5)j_8)j_9 m_9\rangle,$$

where j_1 and j_4 couple to j_7, j_2 and j_5 couple to j_8 and finally j_7 and j_8 couple to a total $j_9 m_9$ state. If instead we wanted to couple the state to

$$|((j_1 j_2)j_3, (j_4 j_5)j_6)j_9 m_9\rangle,$$

then we can use a 9j recoupling coefficient. The definition of the transformation is (essentially a big Clebsch-Gordan coefficient)

$$|((j_1 j_4)j_7, (j_2 j_5)j_8)j_9 m_9\rangle$$
$$= \sum_{j_3}\sum_{j_6} |((j_1 j_2)j_3, (j_4 j_5)j_6)j_9 m_9\rangle\langle((j_1 j_2)j_3, (j_4 j_5)j_6)j_9 m_9|((j_1 j_4)j_7, (j_2 j_5)j_8)j_9 m_9\rangle, \tag{328}$$

where, like a CG coefficient

$$\langle((j_1 j_2)j_3, (j_4 j_5)j_6)j_9 m_9|((j_1 j_4)j_7, (j_2 j_5)j_8)j_9 m_9\rangle$$
$$= \sqrt{(2j_3+1)(2j_6+1)(2j_7+1)(2j_8+1)} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}. \tag{329}$$

Note that this 9j coefficient is the reason we have written our original state as $|((j_1j_4)j_7, (j_2j_5)j_8)j_9m_9\rangle$ instead of $|((j_1j_2)j_3, (j_4j_5)j_6)j_9m_9\rangle$, it allows greater clarity in the 9j symbol itself due to how the numbers read simply left to right (that is, $j_1, j_2, j_3, j_4...$).

So now, since our initial state is the total JLM state

$$|((S_iS_{jk})S, (l_il_j)L)JM\rangle,$$

and we want a state of total J coupled in such a way that the operators derived by Crater et al. [4] work simply

$$|((S_il_i)J_i, (S_{jk}l_j)J_{jk})JM\rangle,$$

we can use the 9j symbol to recouple the states as

$$
\begin{aligned}
&|((S_iS_{jk})S, (l_il_j)L)JM\rangle \\
&= \sum_{J_i}\sum_{J_{jk}} |((S_il_i)J_i, (S_{jk}l_j)J_{jk})JM\rangle\langle((S_il_k)J_i, (S_{jk}l_j)J_{jk})JM|((S_iS_{jk})S, (l_il_j)L)JM\rangle \\
&= \sum_{J_i}\sum_{J_{jk}} \sqrt{(2J_i+1)(2J_{jk}+1)(2S+1)(2L+1)} \\
&\quad \times \begin{Bmatrix} S_i & l_i & J_i \\ S_{jk} & l_j & J_{jk} \\ S & L & J \end{Bmatrix} |((S_il_i)J_i, (S_{jk}l_j)J_{jk})JM\rangle.
\end{aligned}
\tag{330}
$$

where $\begin{Bmatrix} S_i & l_i & J_i \\ S_{jk} & l_j & J_{jk} \\ S & L & J \end{Bmatrix}$ is a 9j symbol. For example, if our state was a 2-3 coupling, then the 9j recoupling would look like

$$
\begin{aligned}
|((S_1S_{23})S, (l_\lambda l_\rho)L)JM\rangle &= \sum_{J_1}\sum_{J_{23}} |((S_1l_\lambda)J_1, (S_{23}l_\rho)J_{23})JM\rangle\langle((S_1l_\lambda)J_1, (S_{23}l_\rho)J_{23})JM|((S_1S_{23})S, (l_\rho l_\lambda)L)JM\rangle \\
&= \sum_{J_1}\sum_{J_{23}} \sqrt{(2J_1+1)(2J_{23}+1)(2S+1)(2L+1)} \\
&\quad \times \begin{Bmatrix} S & l_\lambda & J_1 \\ S_{23} & l_\rho & J_{jk} \\ S & L & J \end{Bmatrix} |((S_1l_\lambda)J_1, (S_{23}l_\rho)J_{23})JM\rangle.
\end{aligned}
\tag{331}
$$

78

Every row and column of the 9j symbol itself must conform to the triangle condition as well (angular momentum Sudoku, basically), so while it appears to be a complicated and intimidating method at first glance, it quickly reduces down to something very manageable. It is important also to note that one of the advantages to using 9j symbols as opposed to combinations of 3j symbols is that the 9j is independent of any M values. Explicit calculations and tables of the 9j symbols are given in Appendix C.

5.4 Spin Orbit

This section defines and describes the spin orbit coupling for the three-body system. The focus is on how to handle the $L \cdot S$ coupling for the three-body system, as the other terms are simple variables. This is where the 9j symbols begin to come in very handy. In order to use the operator $L_{ij} \cdot S_{ij}$ in a simple way, we need the wavefunction to be in a total $J_{ij} = L_{ij} + S_{ij}$ state. Then, the spin-orbit operator working on a state $|l_{ij}S_{ij}J_{ij}\rangle$ gives

$$
\begin{aligned}
l_{ij} \cdot S_{ij}|l_{ij}S_{ij}J_{ij}\rangle &= [J_{ij}(J_{ij}+1) - l_{ij}(l_{ij}+1) - S_{ij}(S_{ij}+1)]|l_{ij}S_{ij}J_{ij}\rangle \\
l_{ij} \cdot S_{ij}|l_{ij}0J_{ij}\rangle &= 0 \\
l_{ij} \cdot S_{ij}|0S_{ij}J_{ij}\rangle &= 0
\end{aligned}
\tag{332}
$$

Since we have three couplings, we have three different l_{ij} terms. Due to how we are completely separating the operators from the spatial components of the wavefunction, these states can still work out relatively simply for the 1-3 and 1-2 couplings (recall that our "default" coupling is 2-3, defined in terms of the ρ variable). The momenta associated with these coordinate changes are, by definition, the momenta that are conjugate to each of them. So then, our three angular momentum terms would be (using the definitions given in Eq.303)

$$
\begin{aligned}
L_{12} &= r_{12} \times p_{12} = \rho' \times p_{\rho'} = l_{\rho'}, \\
L_{13} &= r_{13} \times p_{13} = \rho'' \times p_{\rho''} = l_{\rho''}, \\
L_{23} &= r_{23} \times p_{23} = \rho \times p_{\rho} = l_{\rho}.
\end{aligned}
\tag{333}
$$

Our substitutions for the 1-2 and 1-3 systems give a linear combination of wavefunctions-as the coordinate transform itself is a simple linear combination-of either $l = 1$ or $l = 0$, so the calculations for those states for the spin-orbit ends up being the same as that for the 2-3, just with a few more terms.

We can then apply the 9j symbols with the spin-orbit operator to obtain a value for the operator on a

79

given wavefunction. For a state of $l_{jk} = 0$ or $S_{jk} = 0$, the operator gives zero, which greatly simplifies calculations. Therefore, the spin-orbit operator $\mathbf{l}_{jk}\cdot\mathbf{S}_{jk}$ working on a state $|((S_iS_{jk})S,(l_il_j)L)JM\rangle$ gives

$$
\begin{aligned}
&\mathbf{l}_{jk}\cdot\mathbf{S}_{jk}|((S_iS_{jk})S,(l_il_j)L)JM\rangle \\
&= \sum_{J_i}\sum_{J_{jk}}\mathbf{l}_{jk}\cdot\mathbf{S}_{jk}\sqrt{(2J_i+1)(2J_{jk}+1)(2S+1)(2L+1)} \\
&\times \begin{Bmatrix} S_i & l_i & J_i \\ S_{jk} & l_j & J_{jk} \\ S & L & J \end{Bmatrix}|((S_il_i)J_i,(S_{jk}l_j)J_{jk})JM\rangle,
\end{aligned}
\tag{334}
$$

where we have used the 9j recoupling coefficients to change the state into one which the spin-orbit operator is merely a number. Details of these calculations are given in Appendix C.

As explained earlier, some of the wavefunctions allow the use of ladder operators on the base functions without any concern to couplings. Where this is doable it is considerably easier for the spin-orbit state since when the S_+ or S_- matrix elements are zero, the spin-orbit operator reduces to

$$
l_{jk} \cdot S_{jk} = l_{jk} \cdot (S_j + S_k) = l_{jkz}(S_{jz} + S_{kz})
\tag{335}
$$

or just the z components of the operators. Again, both methods are always viable, but the underlying structure of the wavefunction determines which is more useful for a given calculation.

The details of the ladder method are also given in appendix C. We have recoupled the state for a general wavefunction into one which the spin-orbit operator is simply a number by using the 9j symbols. Now that the two simplest terms are completed, we move on to the more complicated tensor interaction.

5.5 Tensor(Φ_T)

The so-called tensor term, $[3\boldsymbol{\sigma_1}\cdot\hat{\mathbf{r}}\boldsymbol{\sigma_2}\cdot\hat{\mathbf{r}} - \boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}]\Phi_T$ term involves couplings between different possible L states. Since our potentials are still two-body interactions, the effect of the $S_{Tij} = 3\boldsymbol{\sigma_i}\cdot\hat{\mathbf{r}}_{ij}\boldsymbol{\sigma_j}\cdot\hat{\mathbf{r}}_{ij}-\boldsymbol{\sigma_i}\cdot\boldsymbol{\sigma_j}$ operator remains the same for each interacting pair. Like the spin-orbit term, it involves total S and total L of the two-body system, so that in order to use the matrix elements from a two-body system derived by Crater et al. ([1, 2, 3]), we need to recouple it using the 9j symbols as well. From ([49]), we have the following definitions for the S_{Tij} operator working on an $|lsjm\rangle$ state, where l, s, j, and m are the two-body orbital angular momentum, spin angular momentum, total angular momentum and z component of the total angular momentum, respectively:

$$S_T|l0jm\rangle = 0, \tag{336}$$

$$S_T|j1jm\rangle = 2|jj1m\rangle, \tag{337}$$

$$
\begin{aligned}
S_T|j+11jm\rangle &= -\frac{2j+4}{2j+1}|j+11jm\rangle + 6\frac{\sqrt{j(j+1)}}{2j+1}|j-11jm\rangle, \\
S_T|j-11jm\rangle &= -\frac{2j-4}{2j+1}|j-11jm\rangle + |j+11jm\rangle 6\frac{\sqrt{j(j+1)}}{2j+1}.
\end{aligned}
\tag{338}
$$

Applying the tensor operator after using the 9j recoupling coefficients then gives

$$
[3\boldsymbol{\sigma}_j\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_k\cdot\hat{\mathbf{r}} - \boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k]|((S_iS_{jk})S,(l_il_j)L)JM\rangle
$$

$$
= \sum_{J_i}\sum_{J_{jk}}[3\boldsymbol{\sigma}_j\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_k\cdot\hat{\mathbf{r}} - \boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k]\sqrt{(2J_i+1)(2J_{jk}+1)(2S+1)(2L+1)}\begin{Bmatrix} S_i & l_i & J_i \\ S_{jk} & l_j & J_{jk} \\ S & L & J \end{Bmatrix}
$$

$$
\times|((S_il_i)J_i,(S_{jk}l_j)J_{jk})JM\rangle. \tag{339}
$$

Explicit forms of this equation change depending on which baryon state to which it is applied, so the details are again in Appendix C.

The ladder operator is also used for some states where it is useful. The full form of the operator is

$$
\begin{aligned}
3\mathbf{S}_i\cdot\mathbf{r}\mathbf{S}_j\cdot\mathbf{r} &= 3(S_{ix}r_x + S_{iy}r_y + S_{iz}r_z)(S_{jx}r_x + S_{jy}r_y + S_{jz}r_z) - \mathbf{S}_i\cdot\mathbf{S}_j \\
&= 3\Big(\frac{S_{i+}+S_{i-}}{2}\frac{S_{j+}+S_{j-}}{2}r_x^2 + \frac{S_{i+}+S_{i-}}{2}\frac{S_{j+}-S_{j-}}{2i}r_xr_y + \frac{S_{i+}+S_{i-}}{2}S_{jz}r_xr_z \\
&\quad + \frac{S_{i+}-S_{i-}}{2i}\frac{S_{j+}+S_{j-}}{2}r_yr_x + \frac{S_{i+}-S_{i-}}{2i}\frac{S_{j+}-S_{j-}}{2i}r_y^2 + \frac{S_{i+}-S_{i-}}{2i}S_{jz}r_yr_z \\
&\quad + \frac{S_{j+}+S_{j-}}{2}S_{iz}r_zr_x + S_{iz}\frac{S_{j+}-S_{j-}}{2i}r_zr_y + S_{iz}r_zS_{jz}r_z\Big) - \mathbf{S}_i\cdot\mathbf{S}_j,
\end{aligned}
\tag{340}
$$

which when we only consider terms that do not change total M_S becomes

$$
= 3\frac{S_{i+}S_{j-}+S_{i-}S_{j+}}{4}\sin^2\theta + 3S_{iz}S_{jz}\cos^2\theta - \mathbf{S}_i\cdot\mathbf{S}_j, \tag{341}
$$

that, as one may expect from, becomes simple in most cases where it can be applied because $\sin^2\theta+\cos^2\theta = 1$(and thus where this is useful the θ does not matter since it disappears from the equation). This dramatic simplification is only possible in the highest possible J states for each set of baryons, as when J is at a maximum value, we can set $M = J$ (since everything is independent of total M), which in turn forces total

M_S to be its highest possible value, meaning that any change to total M_S on the ket would cause it to be orthogonal to the bra and therefore those terms would not contribute.

Thus, the adaptation methods of the two-body tensor operator to a three-body system is now well-defined. We continue on to the fourth and fifth potential operators, the spin-orbit difference and the spin-orbit cross.

5.6 Spin-Orbit Difference and Spin-Orbit Cross (Φ_{SOD}, Φ_{SOX})

In a similar manner to the tensor term, here we describe how the spin-orbit difference and spin-orbit cross operators work on the total $\langle lsjm|$ state. From Eq. (186) we have that

$$
\begin{aligned}
\mathbf{L} \cdot (\boldsymbol{\sigma_i} - \boldsymbol{\sigma_j})\Phi_{SOD}|jl0n\rangle &= |jl1n\rangle[\{1 - (j-l)^2\}\{(j+l)(j+l+2)\}]\Phi_{SOD}, \\
\mathbf{L} \cdot (\boldsymbol{\sigma_i} - \boldsymbol{\sigma_j})\Phi_{SOD}|jl1n\rangle &= |jl0n\rangle[\{1 - (j-l)^2\}\{(j+l)(j+l+2)\}]\Phi_{SOD},
\end{aligned}
\tag{342}
$$

$$
\begin{aligned}
i\mathbf{L}\cdot\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_j\Phi_{SOX}|jl0n\rangle &= |jl1n\rangle[\{1 - (j-l)^2\}\{(j+l)(j+l+2)\}]\Phi_{SOX}, \\
i\mathbf{L}\cdot\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_j\Phi_{SOX}|jl1n\rangle &= -|jl0n\rangle[\{1 - (j-l)^2\}\{(j+l)(j+l+2)\}]\Phi_{SOX}.
\end{aligned}
\tag{343}
$$

As can be seen from the operator equations, these states will account for spin mixing effects, coupling spin zero states to spin one states. Recoupling using the 9j symbol gives

$$
\begin{aligned}
&\mathbf{L} \cdot (\boldsymbol{\sigma_j} - \boldsymbol{\sigma_k})|((S_iS_{jk})S, (l_il_j)L)JM\rangle \\
&= \sum_{J_i}\sum_{J_{jk}} \mathbf{L} \cdot (\boldsymbol{\sigma_j} - \boldsymbol{\sigma_k})\sqrt{(2J_i+1)(2J_{jk}+1)(2S+1)(2L+1)}
\begin{Bmatrix}
S_i & l_i & J_i \\
S_{jk} & l_j & J_{jk} \\
S & L & J
\end{Bmatrix} \\
&\times |((S_il_i)J_i, (S_{jk}l_j)J_{jk})JM\rangle.
\end{aligned}
\tag{344}
$$

These particular states also lend themselves very well to the ladder operator method, much like the spin-orbit state. For the spin orbit difference, the only non-zero operator (that doesn't change total M_S) is

$$
\mathbf{L} \cdot (\boldsymbol{\sigma_j} - \boldsymbol{\sigma_k}) = L_z(\sigma_{jz} - \sigma_{kz}),
\tag{345}
$$

and for the spin-orbit cross term we also get

$$
L_z(\frac{\sigma_{i-}\sigma_{j+} - \sigma_{i+}\sigma_{j-}}{4}).
\tag{346}
$$

82

These ladder operators are however only usable on about half of the possible states. Details of these and the other operators are given in Appendix C.

The spin-orbit cross and spin-orbit difference terms have been described from the purely two-body formalism to the three-body formalism, and so now we continue to the last term, the spin-orbit tensor term.

5.7 Spin-Orbit Tensor

The final piece of our potential is the spin-orbit tensor term,$\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} \mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$. Unlike the other terms, this one does not lend itself well to the ladder approach, so all of the states have been worked out using the 9j method. The operators on an $|lsjm\rangle$ state are ([49])

$$
\begin{aligned}
\boldsymbol{\sigma}_i \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_j \cdot \hat{\mathbf{r}} \mathbf{L} \cdot (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j)|j1jm\rangle &= [j(j+1) - l(l+1) - s(s+1)]|j1jm\rangle, \\
\boldsymbol{\sigma}_i \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_j \cdot \hat{\mathbf{r}} \mathbf{L} \cdot (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j)|j-11jm\rangle &= [j(j+1) - l(l+1) - s(s+1)] \\
&\quad \times (\frac{1}{2j+1}|j-11jm\rangle + 2\frac{\sqrt{j(j+1)}}{2j+1}|j+11jm\rangle), \\
\boldsymbol{\sigma}_i \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_j \cdot \hat{\mathbf{r}} \mathbf{L} \cdot (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j)|j+11jm\rangle &= [j(j+1) - l(l+1) - s(s+1)] \\
&\quad \times (-\frac{1}{2j+1}|j+11jm\rangle + 2\frac{\sqrt{j(j+1)}}{2j+1}|j-11jm\rangle). \quad (347)
\end{aligned}
$$

As with all of the potential terms, the individual values of these operators will vary, but we can write a completely general state as

$$
\begin{aligned}
&\boldsymbol{\sigma}_j \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_k \cdot \hat{\mathbf{r}} \mathbf{L} \cdot (\boldsymbol{\sigma}_j + \boldsymbol{\sigma}_k)|((S_i S_{jk})S, (l_i l_j)L)JM\rangle \\
&= \sum_{J_i} \sum_{J_{jk}} \boldsymbol{\sigma}_j \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_k \cdot \hat{\mathbf{r}} \mathbf{L} \cdot (\boldsymbol{\sigma}_j + \boldsymbol{\sigma}_k)\sqrt{(2J_i+1)(2J_{jk}+1)(2S+1)(2L+1)} \begin{Bmatrix} S_i & l_i & J_i \\ S_{jk} & l_j & J_{jk} \\ S & L & J \end{Bmatrix} \\
&\quad \times |((S_i l_i)J_i, (S_{jk} l_j)J_{jk})JM\rangle. \quad (348)
\end{aligned}
$$

Details of all calculations for these operators are in Appendix C.

This completes our derivation of the individual Hamiltonian matrix elements. We have written the three-body problem in terms of two relative coordinates and shown how they can be transformed for each interacting pair. This allows a description of the methods used to adapt the ([1, 2, 3]) two-body potential operators of Crater et al. derived for the meson spectrum to the three-body problem. We have used 9j symbols to recouple the wavefunctions into usable form and also have used ladder operators for some of the wavefunctions (the ones on which it works more simply than the 9j method). It now falls to numerical

83

calculations to obtain a fit for the spectrum using these equations.

6 Numerical Results and Comments

The expectation value of the Hamiltonian cannot be evaluated analytically, so it falls to numerical studies to acquire an explicit number. We use a Monte Carlo approach combined with a simple gradient method to obtain a best-fit chi-squared for the spectrum as a whole, as compared to current experimental data. The following sections describe the numerical methods used and give the results after using said methods.

6.1 Methods and Parameter Values

The numerical calculations were done using a Monte Carlo approach followed by a gradient method to obtain a least square fit for the spectrum as a whole. We originally attempted to use a more simplified gradient approach but it quickly became apparent that the functions are far too sensitive to changes and thus will get "stuck" in a local minimum much too easily without some other approach. So, we adopted a Monte Carlo routine that would trigger whenever the gradient approach found a new best fit in order to ensure we were reaching the best results for our theory. The integrations were done numerically using Gaussian Quadrature and the parameters α_ρ and α_λ were minimized by the Nelder-Mead simplex method, though it is worth noting that the α parameters do not generally vary much from the analytic result if one were to use a harmonic oscillator model and also that as the size of our matrix increases, the actual value of these parameters does not affect the fit as much, becoming irrelevant at an infinitely large matrix. Due to how we use the additional basis functions to essentially spread out the wavefunction, the results given in tables 4,5,6 and 7 are for $N = 3$, which corresponds to a 26×26 matrix, as beyond this we observed the overall energy and thus the fit would change very little, if at all.

Our model has a total of 8 parameters, given along with the values for Crater et al. where u,d,s,c and b are the masses of the up, down, strange, charm and bottom quarks, respectively, Λ, K and B are coupling constants in our model. It is worth noting that our model has significantly fewer parameters than most models, with only 8 total and 5 of those being universal to any model (the quark masses themselves). This is significant simply because of the way that a fitting routine will work; the more parameters one has, the easier it is to fit the data. Or as the mathematician John von Neumann put it: "with four parameters I can

Table 4: Parameter values

	This work		Reference 2
u	166.6 (MeV)	u	55.7 (MeV)
d	165.4 (MeV)	d	55.3 (MeV)
s	337.5 (MeV)	s	249.9 (MeV)
Λ	285.8 (MeV)	Λ	421.8 (MeV)
c	2050.3 (MeV)	c	1.476 (MeV)
b	5301.5 (MeV)	b	4.844(MeV)
K	18.1	K	4.198
B	100.6	B	0.05081

Table 5: Low lying baryon states

Baryon	J	Theoretical Mass (MeV)	Experimental Mass(MeV)	Exp-Theory(MeV)
p	1/2	961	938	-23
n	1/2	962	939	-22
Σ^+	1/2	1259	1189	-70
Σ^0	1/2	1261	1192	-69
Σ^-	1/2	1262	1197	-65
Ξ^0	1/2	1388	1314	-74
Ξ^-	1/2	1389	1321	-68
Λ^0	1/2	1093	1125	32
Δ^{++}	3/2	1266	1232	-34
Δ^+	3/2	1266	1232	-34
Δ^0	3/2	1267	1232	-35
Δ^-	3/2	1268	1232	-36
$\Sigma^+(1390)$	3/2	1393	1383	-10
$\Sigma^0(1390)$	3/2	1394	1384	-10
$\Sigma^-(1390)$	3/2	1395	1387	-8
$\Xi^0(1530)$	3/2	1505	1531	26
$\Xi^-(1530)$	3/2	1506	1535	29
Ω^-	3/2	1609	1672	63

fit an elephant, and with five I can make him wiggle his trunk."[10] The model of course, is still expected to be accurate regardless of the number of parameters, but it is worth noting in this work. In addition, there are only two parametric functions that define our potential model, so this further reduces the effect that a large number of parameters would have on our model.

6.2   Results and Comparison to Experiment

The complete results of our model are given in tables 5-7. As the purpose of this work is to test if the model used in the two-body case works well for the three-body, we are only using those baryons which have a three or four star rating by the Particle Data Group, meaning that they are fairly well-known.

The ground-state baryons are generally slightly high energy-wise for the first 8 and this is most likely to

---

[10]It should be mentioned though that due to the way our potentials are defined, simply increasing the number of parameters should not give a drastic increase in the quality of the fit
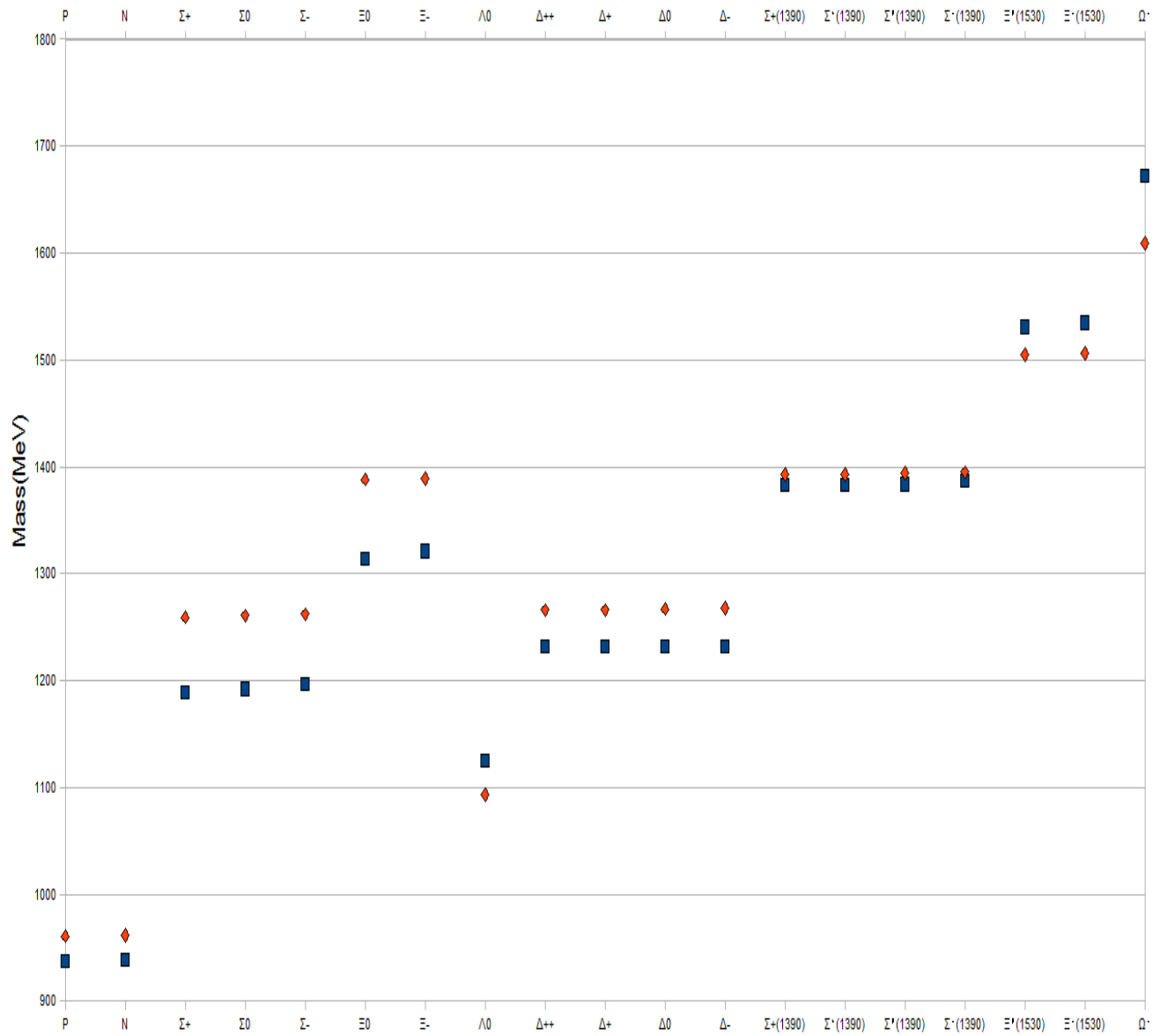
Figure 7: Low order baryons. Squares represent experimental data and diamonds are theoretical.

Table 6: Higher order baryons states

Baryon	J	Theoretical Mass (MeV)	Experimental Mass(MeV)	Exp-Theory
N(1440)	1/2	1508	1420-1470	-63
Λ(1600)	1/2	1684	1560-1700	-54
Σ(1660)	1/2	1660	1630-1690	0
Ξ(1690)	1/2	1787	1680-1700	-97
Δ(1600)	1/2	1535	1550-1700	90
Σ(1670)	3/2	1679	1665-1685	-4
N(1535)	1/2	1561	1525-1545	-26
Λ(1670)	1/2	1678	1660-1680	-8
Σ(1750)	1/2	1674	1730-1800	91
Σ(1775)	5/2	1673	1770-1780	102
N(1520)	3/2	1562	1515-1525	-42
Λ(1690)	3/2	1677	1685-1695	13
Ξ(1820)	3/2	1780	1818-1828	43
N(1650)	1/2	1549	1645-1670	108
Λ(1800)	1/2	1667	1720-1850	118
Σ(1880)	1/2	1679	1800-1960	201
N(1700)	3/2	1581	1650-1750	119
N(1675)	5/2	1561	1670-1680	114
Λ(1830)	5/2	1690	1810-1830	130
Ξ(1950)	5/2	1791	1935-1965	159
Δ(1620)	1/2	1554	1600-1660	76
Δ(1700)	3/2	1549	1670-1750	161
Λ(1405)	1/2	1410	1402-1410	-4
Λ(1520)	3/2	1680	1518-1521	-160

allow the following 10 to be fit relatively accurately. This is not a surprising result of our model due to the fact that since we are using no purely 3-body potentials, the only difference between these sets of baryons is the spin-spin interaction.

As for the higher order baryons, these are not as well-known and the experimental values are only given in a (sometimes wide) range, the difference between the experimental and theoretical values is given as the difference between the theoretical value and the average experimental value. This data is given in table 6.

The higher order baryons fall within an acceptable range on the whole, though there are a few outliers. Of important note is that our model does fit very well the often troublesome Λ(1405) particle. The other Λ particles however are, as before, missing some sort of interaction that will aid in differentiating among them.

In addition, we fit a the well-known charmed and bottom baryons, given in table 7. These agree relatively well with experimental data.

6.3   General Comments

The purpose of this work was to see if the two-body model used for the meson spectrum can be purely applied to a three-body problem while remaining in what is known as the "naive quark model," that is,
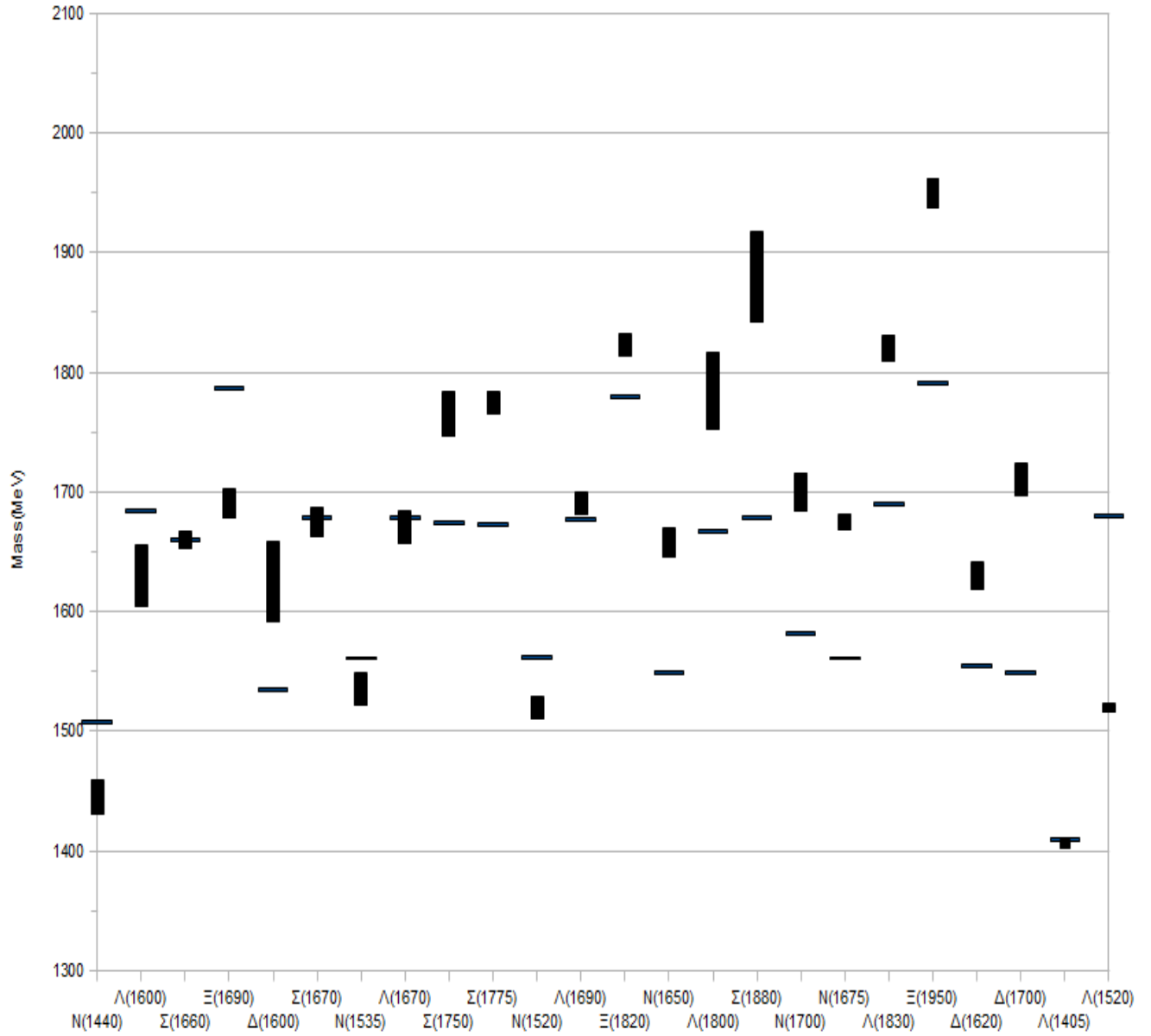
Figure 8: Higher order baryons. Theoretical values are given as horizontal lines and the range of experimental values for each baryon is shown as a vertical bar
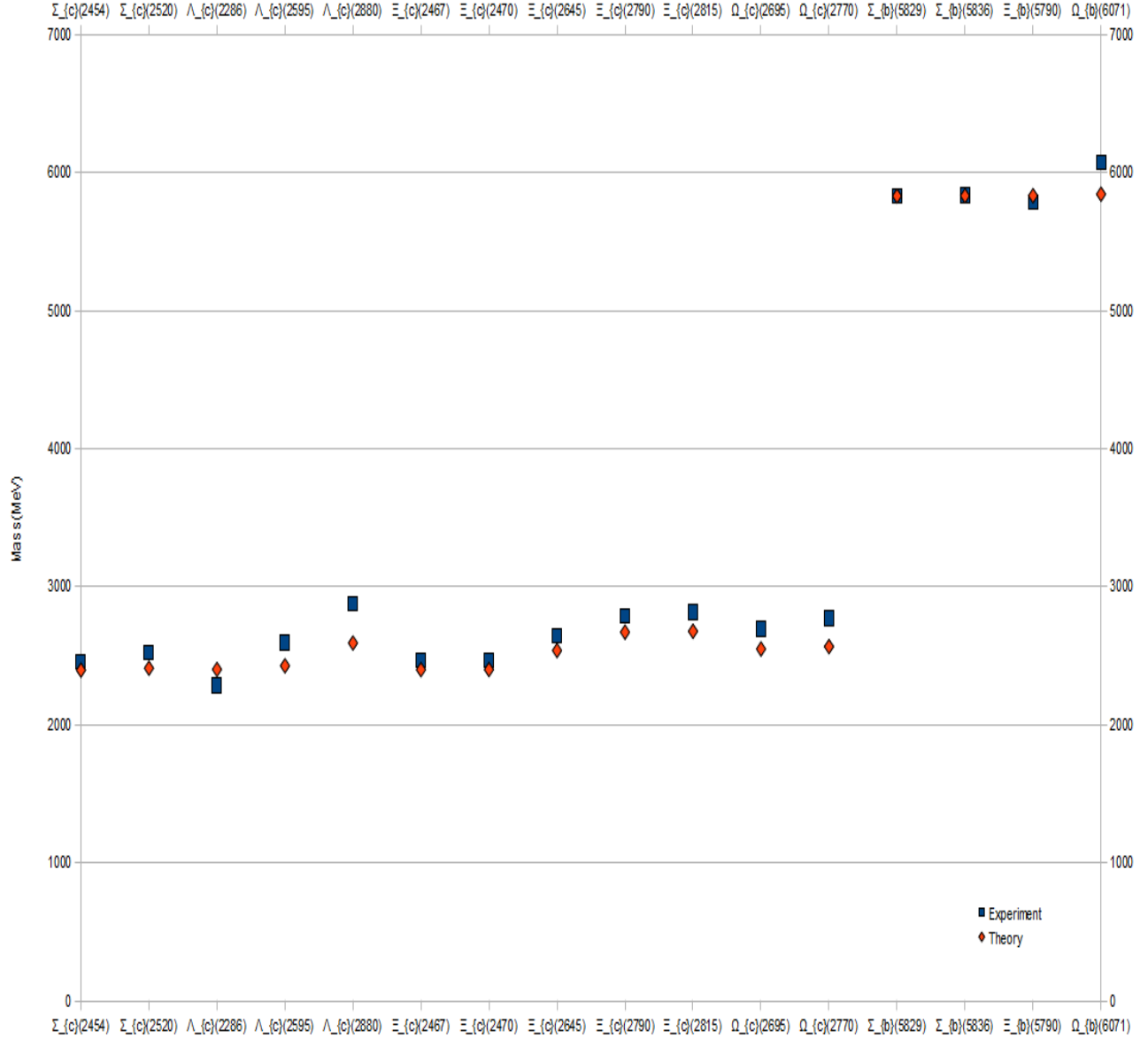
Figure 9: Charmed and bottom baryons, theoretical and experimental values. Squares represent experimental data and triangles are the results of our theoretical model.

Table 7: Charmed and bottom baryons

Baryon	J	Theoretical Mass (MeV)	Experimental Mass(MeV)	Exp-Theory(MeV)
$\Sigma_c(2454)$	1/2	2397	2454	57
$\Sigma_c(2520)$	1/2	2411	2520	109
$\Lambda_c(2286)$	1/2	2402	2286	-116
$\Lambda_c(2595)$	1/2	2428	2595	-167
$\Lambda_c(2880)$	1/2	2593	2880	-287
$\Xi_c(2467)$	1/2	2400	2467	67
$\Xi_c(2470)$	1/2	2401	2470	69
$\Xi_c(2645)$	1/2	2539	2645	106
$\Xi_c(2790)$	1/2	2672	2790	118
$\Xi_c(2815)$	1/2	2678	2815	137
$\Omega_c(2695)$	1/2	2550	2695	145
$\Omega_c(2770)$	1/2	2567	2770	203
$\Sigma_b(5829)$	1/2	5832	5832	0
$\Sigma_b(5836)$	1/2	5833	5836	3
$\Xi_b(5790)$	1/2	5834	5790	-44
$\Omega_b(6071)$	1/2	5845	6071	226

without considering fully three-body forces, ignoring any effects on the spectrum due to the instability of the particles and focusing only on interactions between one pair of quarks at a time. We have shown that this is indeed possible as our fit was relatively good compared to other approaches([9])([10])([52])([53]) and to the experimental data itself. Furthermore, the discrepancy between the theoretical and experimental results can be explained (qualitatively at least) as one overall missing piece that skews the fitting routine. We believe this to be the reason that the quark masses are higher than those done by Crater et al. [39] In general, though the initial octet is high, the splittings are correct among them. The masses of the up and down quark themselves do not match the work done by Crater, but there is a nearly negligible difference between them, which is an expected result and thus some theories assume this *ab initio*([9]). It is also interesting to note that the up and down quark masses are almost right in between the masses given by the two major sources of inspiration and material for this work, Crater and Isgur ([39, 6]).

On the whole though, the fit is nearly as accurate as others, most notably the work of Capstick and Isgur ([6]), which is generally regarded as one of the more valuable references for theoretical baryon spectroscopy. The only marked difference of the results of our model versus other models is that while our higher order baryons are generally fit fairly well, the lower ones are not fit as well, relatively speaking. However, as is discussed below, this may actually reinforce that the fundamental approach is sound.

6.4   Conclusion and Future Work

The model has shown that it is possible to use the purely two-body approach for a good fit of the baryon spectrum. However, some work remains to be done though to truly count this model as accurate for the purposes of making quark model predictions. This should not bee seen as a detriment to our model but rather a triumph of it as we are aware of interactions, such as how the interactions of two particles affects the third, that we did not consider and may be significant. The high masses of the octet can largely be attributed to our potential model not discriminating between such states as the N and the Δ with anything more than the spin-spin interactions, which may simply not be enough to account for approximately a 300 MeV difference between baryons. All of the other interactions require either a radial or an orbital excitation.

As for future work, it may be possible to introduce fully three-body forces in addition to the two-body ones and to use a fully three-body approach for a coordinate system and JLS couplings. Total JLS couplings for a three-body system are usually done in a mathematically rigorous fashion by coupling two particles together and then coupling their Clebsch-Gordan coupled two-body system to a third particle for a complete three-body system. A fully three-body approach to angular momentum couplings may at the very least yield a more elegant formalism and perhaps better overall results. A system derived purely for a three-body problem and including three-body JLS couplings may include additional interactions not seen in a two-body model. We believe this will solve the issue of the same family of particles (i.e. Σ, Λ, N) lacking in enough differentiation. Realistically speaking, it can be viewed as positive in this work that the lower order baryons are not fit as well as other models, but rather the model as a whole appears to be shifted. This lends credence to the theory as a whole being fundamentally sound, but merely incomplete. In other words, if the "missing piece" were added to the model in such a way as to affect all the baryons, it would be more likely to shift all of them in the correct direction, as opposed to if some of them were nearly exact in the current fit and thus an overall change/addition to the model would be likely to shift them away from values in good agreement with experiment.

In addition, it may be possible to extrapolate out to an infinitely large basis and thus get an exact value for the ground state, rather than simply truncating our basis size at an arbitrarily large value. This may yield a better overall fit, but it is worth noting that the truncation was done at the N=3 value because higher values did not appear to change substantially, so it may not give any more accurate results.

References

[1] Horace W. Crater and James Schiermeyer, Phys.Rev .D 82:094020, 2010

[2] Horace W. Crater, Jin-Hee Yoon and Cheuk-Yin Wong, Phys. Rev. D 79:034011,2009

[3] H. Crater and P. Van Alstine, Phys. Rev. D70:03402, 2004

[4] H. W. Crater and P. Van Alstine, Phys. Rev. D **30**, 2585 (1984).

[5] The following references detail the first applications of Dirac's constraint formalism to the relativistic two-body problem. M. Kalb and P. Van Alstine, Yale Reports, C00-3075-146 (1976),C00-3075-156 (1976); P. Van Alstine, Ph.D. Dissertation, Yale University, (1976); I. T. Todorov, Dubna Joint Institute for Nuclear Research No. E2-10175, 1976; Ann. Inst. H. Poincaré **A28**, 207 (1978); L.P. Horwitz and F. Rohrlich,   Phys. Rev. **D24**, 1528 (1981); F. Rohrlich, Phys. Rev. **D23** 1305, (1981); Nucl. Phys. **B161**, 469 (1979); A. Komar, Phys. Rev. **D18**, 1881, 1887 (1978); P. Droz-Vincent Rep. Math. Phys., **8** 79 (1975).

[6] S. Capstick and N Isgur, Phys. Rev. D, **34,** 2809 (1986)

[7] J.J.J. Kokkedee, *The Quark Model,* pp. 1-30, W. A. Benjamin (1969)

[8] Weinberg, Steven ; *The Quantum Theory of Fields - Foundations (vol. I)*, Cambridge University Press (1995)

[9] N Isgur and G. Karl, Phys. Rev. D, **20,** 1191 (1979)

[10] U. Löring, K. Kretzschmar, B.Ch. Metsch, and H.R. Petry, Eur. Phys. J. A 10, 309–346 (2001)

[11] H. Sazdjian, Annals of Physics 191,1(1989)

[12] H. W. Crater and P. Van Alstine, Phys. D70, 034026 (2004).

[13] C. W. Wong and C. Y. Wong Phys. Lett. **B301**,1 (1993)

[14] J. J. Sakurai, *Advanced Quantum Mechanics*, pp. 1-18, Addison Wesley; 1 edition (1967)

[15] P. A. Dirac, Proc R. Soc. London, Ser A **117**, 610 (1928).

[16] G. Breit, Phys. Rev. **34**, 553 (1929).

[17] G. Breit, Phys. Rev. **36**, 553 (1930).

[18] G. Breit, Phys. Rev. **39**, 553 (1932).

[19] H. A. Bethe and E.E. Salpeter, Phys. Rev. **82**, 309 (1951).

[20] N. Nakanishi, Phys. Rev. **113B**, 1182 (1965).

[21] I. T. Todorov, Phys. Rev. D **3**, 2351 (1971).

[22] V. A. Rizov and I. T Todorov, Fiz. Elem. Chastits At. Yadre **6**, 669 (1975) [translated in Sov. J. Part. Nucl. **6**, 269 (1975)].

[23] M. Kalb and P. Van Alstine, Yale Reports, C00-3075-146 (1976), C00-3075-156 (1976).

[24] H. W. Crater and P. Van Alstine, Phys. Lett. B **100**, 166 (1981).

[25] P. Van Alstine and H. W. Crater , J. Math. Phys. **23**, 1697 (1982).

[26] H. W. Crater and P. Van Alstine, Phys. Rev. D **36**, 3007 (1987).

[27] H. W. Crater and P. Van Alstine, Phys. Rev. D **37**, 1982 (1988).

[28] H. W. Crater and P. Van Alstine, J. Math. Phys. **31**, 1998 (1990).

[29] C. Wong and H. W. Crater, Phys. Rev. C **63**, 044907 (2001).

[30] H. W. Crater and D. Yang, J. Math. Phys. **32**, 2374 (1991)

[31] H. W. Crater and P. Van Alstine, Phys. Rev. **D46**, 476 (1992)

[32] H. Sazdjian, Phys. Lett. **156B**, 381 (1985)

[33] Bransden and Joachain,*Quantum Mechanics* (2nd Edition), Appendix 7, pp 640-641. Benjamin Cummings; 2 edition (2000)

[34] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).

[35] P. Long and H. W. Crater, J. Math. Phys. **39**, 124 (1998).

[36] N Isgur and G. Karl, Phys. Rev. D, **18,** 4187 (1986).

[37] K. Nakamura et al. (Particle Data Group), J. Phys. G 37, 075021 (2010) and 2011 partial update for the 2012 edition.

[38] H. W. Fricke and C.C. Noack, PRL 80, 3014 (1998)

[39] H. W. Crater and P. Van Alstine, Phys. **D70,** 034026, (2004).

[40] S. L. Adler and T. Piran, Phys. Lett., **117B**, 91 (1982) and references contained therein.

[41] E. Eichten and F. Feinberg, Phys. Rev. Lett. **D43** ,1205 (1979).

[42] T. Barnes and E.S. Swanson, Phys. Rev. D **46**, 131 (1992); T. Barnes, E.S. Swanson, and J. Weinstein, Phys. Rev. D **46**, 4868 (1992); T. Barnes, S. Capstick, M.D. Kovarik, and E.S. Swanson, Phys. Rev. C **48**, 539 (1993); T. Barnes, N. Black, D. J. Dean, and E. S. Swanson, Phys. Rev. C **60**, 045202 (1999).

[43] C. Y. Wong, E. S. Swanson, and T. Barnes, Phys. Rev. C **65**, 014903 (2002).

[44] J. Richardson, Phys. Lett. **82B,** 272 (1979).

[45] H. Crater and P. Van Alstine, Phys. Lett **100B**, 166 (1981).

[46] D.B. Lichtenberg, W. Namgung, J. G. Wills, and E. Predazzi, Z. Phys. **C19 ,**19 (1983)

[47] A.L. Yaouanc, L. Oliver, Olivier Peǹe, and J.C. Raynal, *Hadron Transitions in the Quark Model,* pp. 40-52. Routledge; (1987)

[48] H. W. Crater and P. Van Alstine, Phys. Rev. D46 766, (1992), H.W. Crater and D. Yang, J. Math. Phys. 32 2374, (1991)

[49] H. W. Crater, R. L. Becker, C. Y. Wong, and P. Van Alstine, Phys. Rev. D, Vol.46 ,5117 (1992)

[50] Private communication to Horace Crater from Boris Kupershmidt

[51] Matthew Duran, Master's Thesis, http://trace.tennessee.edu/utk_gradthes/354

[52] S. Capstick and W. Roberts, Quark Models of Baryon Masses and Decays, arXiv:nucl-th/0008028v1 (2001)

[53] J. Ferretti, A. Vassallo, and E. Santopinto, Relativistic quark-diquark model of baryons, Phys. Rev. C 83, 065204 (2011)

[54] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskii, *Quantum Theory of Angular Momentum,* World Scientific Pub Co Inc (1988)

7 Appendix

A Explicit forms of the Potentials

The full two-body potential is written as

$$\Phi_{ij} = 2m_{wij}S_{ij} + S_{ij}^2 + 2\varepsilon_{wij}A_{ij} - A_{ij}^2 + \Phi_{Dij} + \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\Phi_{S\,Si\,j} + \mathbf{L}\cdot(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)\Phi_{SOij} + (3\boldsymbol{\sigma}_i\cdot\hat{\mathbf{r}}_{ij}\boldsymbol{\sigma}_j\cdot\hat{\mathbf{r}}_{ij} - \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\Phi_{Tij}$$

$$+ \boldsymbol{\sigma}_i\cdot\hat{\mathbf{r}}_{ij}\boldsymbol{\sigma}_j\cdot\hat{\mathbf{r}}_{ij}\mathbf{L}_{ij}\cdot(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)\Phi_{SOTij} + \mathbf{L}_{ij}\cdot(\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_j)\boldsymbol{\Phi}_{SODij} + i\mathbf{L}_{ij}\cdot\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_j\Phi_{SOXij}, \tag{A1}$$

where the individual $i-j$ interacting potentials are given by (r being the distance between each pair in that particular $i-j$ combination)

$$\Phi_D = -\frac{2(\mathcal{F}' + 1/r)(\cosh 2\mathcal{K} - 1)}{r} + \mathcal{F}'^2 + \mathcal{K}'^2 + \frac{2\mathcal{K}'\sinh 2\mathcal{K}}{r} - \boldsymbol{\nabla}^2\mathcal{F} + m(r),$$

$$\Phi_{SO} = -\frac{\mathcal{F}'}{r} - \frac{(\mathcal{F}' + 1/r)(\cosh 2\mathcal{K} - 1)}{r} + \frac{\mathcal{K}'\sinh 2\mathcal{K}}{r},$$

$$\Phi_{SOD} = (l'\cosh 2\mathcal{K} - q'\sinh 2\mathcal{K}),$$

$$\Phi_{SOX} = (q'\cosh 2\mathcal{K} - l'\sinh 2\mathcal{K}),$$

$$\Phi_{SS} = k(r) + \frac{2\mathcal{K}'\sinh 2\mathcal{K}}{3r} - \frac{2(\mathcal{F}' + 1/r)(\cosh 2\mathcal{K} - 1)}{3r} + \frac{2\mathcal{F}'\mathcal{K}'}{3} - \frac{\boldsymbol{\nabla}^2\mathcal{K}}{3},$$

$$\Phi_T = \frac{1}{3}[n(r) + \frac{(3\mathcal{F}' - \mathcal{K}' + 3/r)\sinh 2\mathcal{K}}{r} + \frac{(\mathcal{F}' - 3\mathcal{K}' + 1/r)(\cosh 2\mathcal{K} - 1)}{r} + 2\mathcal{F}'\mathcal{K}' - \boldsymbol{\nabla}^2\mathcal{K}],$$

$$\Phi_{SOT} = -\mathcal{K}'\frac{\cosh 2\mathcal{K} - 1}{r} - \frac{\mathcal{K}'}{r} + \frac{(\mathcal{F}' + 1/r)\sinh 2\mathcal{K}}{r}. \tag{A2}$$

These in turn are made up of the functions[11]

$$k(r) = \frac{1}{3}\nabla^2(\mathcal{K} + \mathcal{G}) - \frac{2\mathcal{F}'(\mathcal{G}' + \mathcal{K}')}{3} - \frac{1}{2}\mathcal{G}'^2,$$

$$n(r) = \frac{1}{3}[\nabla^2\mathcal{K} - \frac{1}{2}\nabla^2\mathcal{G} + \frac{3(\mathcal{G}' - 2\mathcal{K}')}{2r} + \mathcal{F}'(\mathcal{G}' - 2\mathcal{K}')],$$

$$m(r) = -\frac{1}{2}\nabla^2\mathcal{G} + \frac{3}{4}\mathcal{G}'^2 + \mathcal{G}'\mathcal{F}' - \mathcal{K}'^2, \tag{A3}$$

---

[11] These are all two-body potentials acting in a three-body system, so the subscripts 1 or 2 refer to the $1st$ or $2nd$ particle in that pair, not the first or second particle in the overall system.

$$l'(r) = -\frac{1}{2r} \frac{E_2 M_2 - E_1 M_1}{E_2 M_1 + E_1 M_2} (\mathcal{L}' + \mathcal{J}'),$$

$$q'(r) = \frac{1}{2r} \frac{E_2 M_1 - E_1 M_2}{E_2 M_1 + E_1 M_2} (\mathcal{L}' + \mathcal{J}'). \tag{A4}$$

$$\mathcal{K} = \frac{(\mathcal{G} + \mathcal{L})}{2}$$

$$\mathcal{K}' = \frac{(\mathcal{G}' + \mathcal{L}')}{2}$$

$$\nabla^2 \mathcal{K} = \frac{(\nabla^2 \mathcal{G} + \nabla^2 \mathcal{L})}{2} \tag{A5}$$

$$\mathcal{F} = \frac{1}{2} \log \frac{\mathcal{D}}{\varepsilon_2 m_1 + \varepsilon_1 m_2} - \mathcal{G}$$

$$\mathcal{F}' = \frac{(L' - \mathcal{G}')(E_2 M_2 + E_1 M_1)}{2(E_2 M_1 + E_1 M_2)} - \mathcal{G}'$$

$$\nabla^2 \mathcal{F} = \frac{(\boldsymbol{\nabla}^2 L - \boldsymbol{\nabla}^2 \mathcal{G})(E_2 M_2 + E_1 M_1)}{2(E_2 M_1 + E_1 M_2)} - (\mathcal{L}' - \mathcal{G}')^2 \frac{(m_1^2 - m_2^2)^2}{2(E_2 M_1 + E_1 M_2)^2} - \boldsymbol{\nabla}^2 \mathcal{G}$$

$$\mathcal{D} = E_2 M_1 + E_1 M_2$$

$$\mathcal{D}' = E_2' M_1 + E_1' M_2 + E_2 M_1' + E_1 M_2' \tag{A6}$$

$$M_i = \sqrt{m_i^2 + \frac{2 m_w S + S^2}{(w - 2A)/w}}$$

$$M_i' = \frac{(m_w S' + S')}{\sqrt{m_i^2 + 2 m_w S + S^2}} = \frac{(m_w S' + S')}{M_i}$$

$$M_i'' = -\frac{(m_w S' + S')}{(m_i^2 + 2 m_w S + S^2)^{3/2}} = -\frac{M_i'}{M_i^2}$$

$$E_i = \frac{\varepsilon_i - A}{\sqrt{(w - 2A)/w}} \tag{A7}$$

$$E_i' = \frac{(-\varepsilon_w A' + A')}{\sqrt{\varepsilon_i^2 - 2\varepsilon_w A + A^2}} = \frac{(-\varepsilon_w A' + A')}{E_i}$$

$$E_i'' = \frac{(-\varepsilon_w A'' + A'')}{\sqrt{\varepsilon_i^2 - 2\varepsilon_w A + A^2}} - \frac{(-\varepsilon_w A' + A')^2}{(\varepsilon_i^2 - 2\varepsilon_w A + A^2)^{3/2}} = \frac{(-\varepsilon_w A'' + A'')}{E_i} - \frac{E_i'^2}{E_i}$$

$$\mathcal{G} = \frac{Log(\frac{1}{(1 - 2A/w)})}{2} \tag{A8}$$

$$\mathcal{G}' = \frac{A'}{w - 2A}$$

$$\nabla^2 \mathcal{G} = \frac{\boldsymbol{\nabla}^2 A}{w - 2A} + 2\mathcal{G}'^2$$

$$S' = \frac{4\pi\Lambda^2}{27}$$

$$A' = -\frac{e_1 e_2}{4\pi r^2} + \frac{8\pi}{27 r^2 log(e^2 K + \frac{B}{r^2 \Lambda^2})} - \frac{16 B\pi}{27 r^4 (e^2 K + \frac{B}{r^2 \Lambda^2})\Lambda^2 (log(e^2 K + \frac{B}{r^2 \Lambda^2}))^2}$$

$$\nabla^2 A = \frac{16 B\pi(-4B + (B + 3e^2 K r^2 \Lambda^2) log(e^2 K + \frac{B}{r^2 \Lambda^2}))}{27 r^3 (B + e^2 K r^2 \Lambda^2)^2 (log(e^2 K + \frac{B}{r^2 \Lambda^2}))^3} \tag{A9}$$

$$m_w = \frac{m_1 m_2}{w} \quad , \quad \varepsilon_w = \frac{(w^2 - m_1^2 - m_2^2)}{2w}$$

$$\mathcal{L} = log\left(\frac{\sqrt{m_1^2 + \exp(2\mathcal{G})(2m_w S + S^2)} + \sqrt{m_2^2 + \exp(2\mathcal{G})(2m_w S + S^2)}}{m_1 + m_2}\right)$$

$$\mathcal{L}' = \frac{w}{M_1 M_2}\left(\frac{S'(m_w + S)}{w - 2A} + \frac{(2m_w S + S^2)A'}{(w - 2A)^2}\right)$$

$$\nabla^2 \mathcal{L} = \frac{-\mathcal{L}'^2 (M_1^2 + M_2^2)}{M_1 M_2} + \frac{w}{M_1 M_2}$$
$$\times \left(\frac{\boldsymbol{\nabla}^2 S(m_w + S) + S'^2}{w - 2A} + \frac{4S'(m_w + S)A' + (2m_w S + S^2)\boldsymbol{\nabla}^2 A}{(w - 2A)^2} + \frac{4(2m_w S + S^2)A'^2}{(w - 2A)^3}\right) \tag{A10}$$

The scalar and vector invariants $A(r)$ and $S(r)$ form the basis for all of the above functions. They are given by

$$S = \frac{4\pi\Lambda^2 r}{27}$$

$$A = -\frac{8\pi}{27 r \log(Ke^2 + \frac{B}{(\Lambda r)^2})} + \frac{e_1 e_2}{4\pi r} \tag{A11}$$

B   Spin-Flavor-Space States

This appendix contains the derivations of the spin-flavor states for all the baryons in our fit. They are composed of products of spin wavefunctions, denoted χ, and flavor wavefunctions, denoted as ϕ [47] . The spin wavefunctions are, explicitly

$$\chi^S(S_z = \frac{3}{2}) = \uparrow\uparrow\uparrow,$$

$$\chi'(S_z = \frac{1}{2}) = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow),$$

$$\chi''(S_z = \frac{1}{2}) = \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow). \tag{B1}$$

Table 8: Baryon flavor wavefunctions

	ϕ^s	ϕ'	ϕ''
p		$\frac{1}{\sqrt{2}}(udu-duu)$	$\frac{1}{\sqrt{6}}(2uud-duu-udu)$
n		$\frac{1}{\sqrt{2}}(udd-dud)$	$\frac{1}{\sqrt{6}}(dud-udd-2ddu)$
Λ		$\frac{1}{2\sqrt{3}}(usd+sdu-sud-dsu-2dus+2uds)$	$\frac{1}{2}(sud+usd-sdu-dsu)$
Δ^{++}	uuu		
Δ^+	uud		
Δ^0	udd		
Δ^-	ddd		
Σ^+	uus	$\frac{1}{\sqrt{2}}(suu-usu)$	$\frac{1}{\sqrt{6}}(suu-usu-2uus)$
Σ^0	uds	$\frac{1}{2}(sud+sdu-usd-dsu)$	$\frac{1}{2\sqrt{3}}(usd+sdu+sud+dsu-2dus-2uds)$
Σ^-	dds	$\frac{1}{\sqrt{2}}(sdd-dsd)$	$\frac{1}{\sqrt{6}}(sdd-dsd-2dds)$
Ξ^0	uss	$\frac{1}{\sqrt{2}}(sus-uss)$	$\frac{1}{\sqrt{6}}(2ssu-sus-uss)$
Ξ^-	dss	$\frac{1}{\sqrt{2}}(sds-dss)$	$\frac{1}{\sqrt{6}}(2ssd-sds-dds)$
Ω^-	sss		
Σ_c	uuc	$\frac{1}{\sqrt{2}}(cuu-ucu)$	$\frac{1}{\sqrt{6}}(cuu-ucu-2uuc)$
Σ_b	uub	$\frac{1}{\sqrt{2}}(buu-ubu)$	$\frac{1}{\sqrt{6}}(buu-ubu-2uub)$
Λ_c		$\frac{1}{2\sqrt{3}}(ucd+cdu-cud-dcu-2duc+2udc)$	$\frac{1}{2}(cud+ucd-cdu-dcu)$
Λ_b		$\frac{1}{2\sqrt{3}}(ubd+bdu-bud-dbu-2dub+2udb)$	$\frac{1}{2}(bud+ubd-bdu-dbu)$
Ξ_c	usc	$\frac{1}{\sqrt{2}}(suc-usc)$	$\frac{1}{\sqrt{6}}(2scu-suc-usc)$
Ξ_b	usb	$\frac{1}{\sqrt{2}}(sub-usb)$	$\frac{1}{\sqrt{6}}(2sbu-sub-usb)$
Ω_b	ssb	$\frac{1}{\sqrt{2}}(ssc-scs)$	$\frac{1}{\sqrt{6}}(css-scs-2ssc)$

Table 9: Total spin-flavor-space wavefunctions

N	J	L	S	Total State	Ψ
8	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}(\phi'\chi'+\phi''\chi'')\psi_0$	Ψ_1
10	$\frac{3}{2}$	0	$\frac{3}{2}$	$\phi^s\chi^s\psi_0$	Ψ_2
8	$\frac{1}{2},\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}[(\phi'\chi''+\phi''\chi')\psi'+(\phi'\chi'-\phi''\chi'')\psi'']$	$\Psi_3(J=\frac{1}{2}),\Psi_4(J=\frac{3}{2})$
8	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	1	$\frac{3}{2}$	$\frac{1}{\sqrt{2}}[\phi'\chi^s\psi'+\phi''\chi^s\psi'']$	$\Psi_5(J=\frac{1}{2}),\Psi_6(J=\frac{3}{2}),\Psi_7(J=\frac{5}{2})$
10	$\frac{1}{2},\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}[\phi^s\chi'\psi'+\phi^s\chi''\psi'']$	$\Psi_8(J=\frac{1}{2}),\Psi_9(J=\frac{3}{2})$
1	$\frac{1}{2},\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}[\phi^a\chi''\psi'-\phi^a\chi'\psi'']$	$\Psi_{10}(J=\frac{1}{2}),\Psi_{11}(J=\frac{3}{2})$

There are four different flavor wavefunctions, denoted as $\phi',\phi'',\phi^s,\phi^a$,

and the singlet state

$$\phi^a = \frac{1}{\sqrt{6}}(uds+dsu+sud-dus-usd-sdu) \tag{B2}$$

There are eleven possible combinations of these spin and flavor states for (most of) the known baryons.

These wavefunctions then define a grouping of baryons and the individual baryon itself is defined by the flavor state from there, as given below

Table 10: Baryons and their corresponding spin-flavor wavefunctions

$\Psi_1 \rightarrow P, N, \Lambda, \Sigma^+, \Sigma^0, \Sigma^-, \Xi^0, \Xi^-, N(1440), \Lambda(1600), \Sigma(1660), \Xi(1690), \Sigma_c^+(2455), \Sigma_b^+, \Sigma_b^+, \Lambda_c^+, \Lambda_c^+(2595), \Lambda_b^0$

$\Psi_2 \rightarrow \Delta^{++}, \Delta^+, \Delta^0, \Delta^-, \Sigma^+(1385), \Sigma^0(1388), \Sigma^-(1390), \Xi^0(1530), \Xi^-(1535), \Omega^-, \Delta(1600), \Sigma(1690)$

$\Psi_3 \rightarrow N(1535), \Lambda(1670), \Sigma(1750), \Sigma(1880)$

$\Psi_4 \rightarrow N(1520), \Lambda(1690), \Sigma(1670), \Xi(1820)$

$\Psi_5 \rightarrow N(1650), \Lambda(1800), \Sigma(1750)$

$\Psi_6 \rightarrow N(1700), \Sigma(1940)$

$\Psi_7 \rightarrow N(1675), \Lambda(1830), \Sigma(1775), \Xi(1950)$

$\Psi_8 \rightarrow \Delta(1620)$

$\Psi_9 \rightarrow \Delta(1700)$

$\Psi_{10} \rightarrow \Lambda(1405)$

$\Psi_{11} \rightarrow \Lambda(1520)$

C   Operator and 9j Details

The eleven possible wavefunctions [47] are

N	J	L	S	Total State	Ψ
8	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}(\phi'\chi' + \phi''\chi'')\psi_0$	Ψ_1
10	$\frac{3}{2}$	0	$\frac{3}{2}$	$\phi^s\chi^s\psi_0$	Ψ_2
8	$\frac{1}{2}, \frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}[(\phi'\chi'' + \phi''\chi')\psi' + (\phi'\chi' - \phi''\chi'')\psi'']$	$\Psi_3(J = \frac{1}{2}), \Psi_4(J = \frac{3}{2})$
8	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$	1	$\frac{3}{2}$	$\frac{1}{\sqrt{2}}[\phi'\chi^s\psi' + \phi''\chi^s\psi'']$	$\Psi_5(J = \frac{1}{2}), \Psi_6(J = \frac{3}{2}), \Psi_7(J = \frac{5}{2})$
10	$\frac{1}{2}, \frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}[\phi^s\chi'\psi' + \phi^s\chi''\psi'']$	$\Psi_8(J = \frac{1}{2}), \Psi_9(J = \frac{3}{2})$
1	$\frac{1}{2}, \frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}[\phi^a\chi''\psi' - \phi^a\chi'\psi'']$	$\Psi_{10}(J = \frac{1}{2}), \Psi_{11}(J = \frac{3}{2})$

where ψ_0 is a ground state Gaussian and ψ' and ψ'' are orbitally excited Gaussians given by [47] and each have different coefficients for the 9j recoupling (radial excitations are accounted for in the explicit forms of the Gaussians themselves). The tables below give the 9j coefficients for every possible coupling that we use. They are constructed in a similar manner to a clebsch-gordan table, where we have([54])

	possible spin for $S_{jk} \rightarrow$	0	1
possible l for $l_j \downarrow$			
0		9j coefficient(J_i, J_{jk})	9j coefficient(J_i, J_{jk})
1		9j coefficient(J_i, J_{jk})	9j coefficient(J_i, J_{jk})

where J_i is the total J for the "spectator" particle (i.e. particle one in the 2-3 interaction) and J_{jk} is the total J for the interaction in question, which is what we are really concerned with for working out the state explicitly.

$$J = \frac{1}{2}, L = 0, S = \frac{1}{2}, l_i = l_j$$

	S_{jk}	0	1
l_j			
0		$1(\frac{1}{2},0)$	$1(\frac{1}{2},1)$
1		$-\sqrt{2}(\frac{1}{2},1) + \sqrt{\frac{2}{3}}(\frac{3}{2},1)$	$\frac{1}{3}(\frac{1}{2},0) + \frac{\sqrt{2}}{3}(\frac{1}{2},1) + \frac{1}{3}(\frac{3}{2},1) + \frac{\sqrt{5}}{3}(\frac{3}{2},2)$

$$J = \frac{1}{2}, L = 1, S = \frac{1}{2}, l_i = 1 - l_j$$

	S_{jk}	0	1
l_j			
0		$1(\frac{1}{2},0)$	$-\frac{1}{3}(\frac{1}{2},1) - \frac{2}{3}\sqrt{2}(\frac{3}{2},1)$
1		$1(\frac{1}{2},1)$	$-\frac{1}{\sqrt{3}}(\frac{1}{2},0) + \sqrt{\frac{2}{3}}(\frac{1}{2},1)$

$$J = \frac{1}{2}, L = 1, S = \frac{3}{2}, l_i = 1 - l_j$$

	S_{jk}	0	1
l_j			
0		0	$-\frac{2}{3}\sqrt{2}(\frac{1}{2},1) + \frac{1}{3}(\frac{3}{2},1)$
1		0	$\sqrt{\frac{2}{3}}(\frac{1}{2},0) + \frac{1}{\sqrt{3}}(\frac{1}{2},1)$

$$J = \frac{3}{2}, L = 0, S = \frac{3}{2}, l_i = l_j$$

	S_{jk}	0	1
l_j			
0		0	$1(\frac{1}{2},1)$
1		0	$\frac{1}{3}(\frac{3}{2},0) - \frac{1}{6}\sqrt{(2)}(\frac{1}{2},1) + \frac{\sqrt{5}}{3\sqrt{2}}(\frac{3}{2},1) - \frac{1}{6}\sqrt{10}(\frac{1}{2},2) + \frac{\sqrt{5}}{3\sqrt{2}}(\frac{3}{2},2)$

$$J = \frac{3}{2}, L = 1, S = \frac{1}{2}, l_i = 1 - l_j$$

	S_{jk}	0	1
l_j			
0		$1(\frac{3}{2},0)$	$\frac{2}{3}(\frac{1}{2},1)+\frac{\sqrt{5}}{3}(\frac{3}{2},1)$
1		$1(\frac{1}{2},1)$	$-\frac{1}{\sqrt{6}}(\frac{1}{2},1)+\frac{1}{3}\sqrt{\frac{15}{2}}(\frac{1}{2},2)$

$$J=\frac{3}{2}, L=1, S=\frac{3}{2}, l_i=1-l_j$$

	S_{jk}	0	1
l_j			
0		0	$\frac{\sqrt{5}}{3}(\frac{1}{2},1)-\frac{2}{3}(\frac{3}{2},1)$
1		0	$\sqrt{\frac{5}{6}}(\frac{1}{2},1)+\frac{1}{\sqrt{6}}(\frac{1}{2},2)$

$$J=\frac{5}{2}, L=1, S=\frac{3}{2}, l_i=1-l_j$$

	S_{jk}	0	1
l_j			
0		0	$1(\frac{3}{2},1)$
1		0	$1(\frac{1}{2},2)$

Since the states are all independent of total M, we can set $M = J$ for the higher J states in each set of wavefunctions. This allows the use of the much simpler ladder operators as opposed to the 9j method, which has the side benefit of being a check on both approaches (many were done both ways to ensure accuracy). The states on which this is possible are Ψ_1, Ψ_2, Ψ_3, Ψ_7, Ψ_9, and Ψ_{11}. The reason it is simple is because any state with $M = J = |L + S|$ will only have the maximum possible value for M_S and so any operator that changes total M_S will have a different set of quantum numbers from the state to which it is coupling and will thus give a matrix element of zero. The simplified versions of the ladder operators are (where we have used said property of M and M_S)

$$
\begin{aligned}
\mathbf{S}_i\cdot\mathbf{S}_j &= \frac{S_{i+}S_{j-}+S_{i-}S_{j+}}{2}+S_{iz}S_{jz}, \\
SO &= L\cdot S = L_z(S_{iz}+S_{jz}), \\
T &= 3\frac{S_{i+}S_{j-}+S_{i-}S_{j+}}{4}\sin^2\theta+3S_{iz}S_{jz}\cos^2\theta-S_i\cdot S_j, \\
SOD &= L_z(S_{iz}-S_{jz}), \\
SOX &= L_z(\frac{S_{i-}S_{j+}-S_{i+}S_{j-}}{2}).
\end{aligned}
\tag{C1}
$$

Note both that the spin-spin operator is simple regardless of the state and thus we can use it either way on any state and that the spin-orbit tensor operator does not appear. This is due to the fact that the spin-orbit tensor operator does not have a simple form, even with the simplification that any change in total M_S will be zero. Using both of these approaches, we can reduce all of our operators to simple numerical factors onto the radial wavefunction, which vary depending on the state. The tables are written in the format

total wavefunction	interaction
quark coupling	numerical coefficient

So that the table for Ψ_1 (for example) would read that all interactions but spin-spin are zero and all spin-spin interactions give back a coefficient of $-\frac{1}{4}$.

Ψ_1	SS	SO	T	SOT	SOD	SOX	Ψ_2	SS	SO	T	SOT	SOD	SOX
12	$-\frac{1}{4}$	0	0	0	0	0	12	$\frac{1}{4}$	0	0	0	0	0
13	$-\frac{1}{4}$	0	0	0	0	0	13	$\frac{1}{4}$	0	0	0	0	0
23	$-\frac{1}{4}$	0	0	0	0	0	23	$\frac{1}{4}$	0	0	0	0	0

Ψ_3	SS	SO	T	SOT	SOD	SOX	Ψ_4	SS	SO	T	SOT	SOD	SOX
12	$-\frac{1}{4}$	$-\frac{4}{3}$	0	0	0	0	12	$-\frac{1}{4}$	$\frac{2}{3}$	0	0	0	0
13	$-\frac{1}{4}$	$-\frac{4}{3}$	0	0	0	0	13	$-\frac{1}{4}$	$\frac{2}{3}$	0	0	0	0
23	$-\frac{1}{4}$	$-\frac{2}{3}$	0	0	0	0	23	$-\frac{1}{4}$	$\frac{2}{3}$	0	0	$-\frac{1}{2}$	0

Ψ_5	SS	SO	T	SOT	SOD	SOX	Ψ_6	SS	SO	T	SOT	SOD	SOX
12	$-\frac{1}{4}$	$-\frac{10}{3}$	-2	1	0	0	12	$-\frac{1}{4}$	$-\frac{4}{3}$	$\frac{8}{5}$	$-\frac{8}{5}$	0	0
13	$-\frac{1}{4}$	$-\frac{10}{3}$	-2	1	0	0	13	$-\frac{1}{4}$	$-\frac{4}{3}$	$\frac{8}{5}$	$-\frac{8}{5}$	0	0
23	$-\frac{1}{4}$	$-\frac{5}{3}$	-1	$\frac{1}{2}$	0	0	23	$-\frac{1}{4}$	$-\frac{2}{3}$	$\frac{4}{5}$	$-\frac{4}{5}$	0	0

Ψ_7	SS	SO	T	SOT	SOD	SOX	Ψ_8	SS	SO	T	SOT	SOD	SOX
12	$\frac{1}{4}$	2	$-\frac{2}{5}$	$\frac{2}{5}$	0	0	12	$-\frac{1}{4}$	$-\frac{4}{3}$	0	0	0	0
13	$\frac{1}{4}$	2	$-\frac{2}{5}$	$\frac{2}{5}$	0	0	13	$-\frac{1}{4}$	$-\frac{4}{3}$	0	0	0	0
23	$\frac{1}{4}$	1	$-\frac{1}{5}$	$\frac{1}{5}$	0	0	23	$-\frac{1}{4}$	-1	0	0	1	0

Ψ_9	SS	SO	T	SOT	SOD	SOX	Ψ_{10}	SS	SO	T	SOT	SOD	SOX
12	$-\frac{1}{4}$	$\frac{2}{3}$	0	0	0	0	12	$-\frac{1}{4}$	$-\frac{4}{3}$	0	0	0	0
13	$-\frac{1}{4}$	$\frac{2}{3}$	0	0	0	0	13	$-\frac{1}{4}$	$-\frac{4}{3}$	0	0	0	0
23	$-\frac{1}{4}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	23	$-\frac{1}{4}$	$-\frac{4}{3}$	0	0	-1	0

Ψ_{11}	SS	SO	T	SOT	SOD	SOX
12	$-\frac{1}{4}$	$\frac{2}{3}$	0	0	0	0
13	$-\frac{1}{4}$	$\frac{2}{3}$	0	0	0	0
23	$-\frac{1}{4}$	$\frac{1}{6}$	0	0	$-\frac{1}{2}$	0

D Coordinate System Detailed Derivations

This appendix contains explicit details for the coordinate transform derivation. First, we define our relative coordinates

$$
\begin{aligned}
\boldsymbol{\rho} &= \mathbf{r}_2 - \mathbf{r}_3 \\
\boldsymbol{\lambda} &= \frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)\varepsilon_1}\mathbf{r}_2 + \frac{w\varepsilon_3}{(\varepsilon_2 + \varepsilon_3)\varepsilon_1}\mathbf{r}_3 \\
\mathbf{r}_1 - \mathbf{r}_2 &= -\frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\boldsymbol{\rho} - \boldsymbol{\lambda} \\
\mathbf{r}_1 - \mathbf{r}_3 &= -\frac{\varepsilon_2}{\varepsilon_2 + \varepsilon_3}\boldsymbol{\rho} + \boldsymbol{\lambda} \\
\mathbf{r}_2 - \mathbf{r}_3 &= \boldsymbol{\rho}
\end{aligned}
\tag{D1}
$$

where the $\varepsilon's$ are the energies of each quark and

$$
w = \varepsilon_1 + \varepsilon_2 + \varepsilon_3,
\tag{D2}
$$

as usual. Our base wavefunction is

$$
\Psi_n = N\rho^{l_\rho}\lambda^{l_\lambda}e^{-n_\rho\alpha_\rho^2\rho^2/2 - n_\lambda\alpha_\lambda^2\lambda^2/2}Y_{l_\rho}^{m_\rho}Y_{l_\lambda}^{m_\lambda}
\tag{D3}
$$

which must be normalized. As usual, the spherical harmonics are already normalized

$$Y_l^m = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi} \tag{D4}$$

$$\int Y_l^m Y_{l'}^{m'} = \delta_{ll'}\delta_{mm'}$$

so we just have to normalize the radial part of the wavefunction, which gives

$$\langle\Psi_n||\Psi_n\rangle = 1 = \delta_{l_\rho l_\rho'}\delta_{m_\lambda m_\lambda'}N^2\int\rho^{2l+2}\lambda^{2l+2}e^{-n_\rho\alpha_\rho^2\rho^2-n_\lambda\alpha_\lambda^2\lambda^2}d\lambda d\rho$$

$$= N^2\frac{\Gamma[(2l_\rho+3)/2]}{(n_\rho\alpha_\rho^2)^{(2l_\rho+3)/2}}\frac{\Gamma[(2l_\lambda+3)/2]}{(n_\lambda\alpha_\lambda^2)^{(2l_\lambda+3)/2}}$$

$$N = \sqrt{\frac{(n_\rho\alpha_\rho^2)^{(2l_\rho+3)/2}(n_\lambda\alpha_\lambda^2)^{(2l_\lambda+3)/2}}{\Gamma[(2l_\rho+3)/2]\Gamma[(2l_\lambda+3)/2]}} \tag{D5}$$

$$\Psi_n = \sqrt{\frac{(n_\rho\alpha_\rho^2)^{(2l_\rho+3)/2}(n_\lambda\alpha_\lambda^2)^{(2l_\lambda+3)/2}}{\Gamma[(2l_\rho+3)/2]\Gamma[(2l_\lambda+3)/2]}}\rho^{l_\rho}\lambda^{l_\lambda}e^{-n_\rho\alpha_\rho^2\rho^2/2-n_\lambda\alpha_\lambda^2\lambda^2/2}Y_{l_\rho}^{m_\rho}Y_{l_\lambda}^{m_\lambda} \tag{D6}$$

or more completely with the appropriate Clebsch-Gordan coefficient

$$\Psi_n = \sqrt{\frac{(n_\rho\alpha_\rho^2)^{(2l_\rho+3)/2}(n_\lambda\alpha_\lambda^2)^{(2l_\lambda+3)/2}}{\Gamma[(2l_\rho+3)/2]\Gamma[(2l_\lambda+3)/2]}}\rho^{l_\rho}\lambda^{l_\lambda}e^{-n_\rho\alpha_\rho^2\rho^2/2-n_\lambda\alpha_\lambda^2\lambda^2/2}$$

$$\times\sum_{m_\rho m_\lambda}\langle l_\rho l_\lambda m_\rho m_\lambda|LM\rangle Y_{l_\rho}^{m_\rho}Y_{l_\lambda}^{m_\lambda} \tag{D7}$$

This wavefunction is now sufficient to perform the $\mathbf{r}_{23} = \boldsymbol{\rho}$ integration and the kinetic terms (which are analytic). The kinetic terms are given by

$$
\begin{aligned}
\nabla^2 \Psi_\rho &= \frac{1}{\rho^2}\frac{\partial}{\partial\rho}[l_\rho\rho^{l_\rho+1}e^{-n_\rho\alpha_\rho^2\rho^2/2} - n_\rho\alpha_\rho^2\rho^{l_\rho+3}e^{-n_\rho\alpha_\rho^2\rho^2/2}] \\
&= \frac{1}{\rho^2}[l_\rho(l_\rho+1)\rho^{l_\rho}e^{-n_\rho\alpha_\rho^2\rho^2/2} - n_\rho\alpha_\rho^2 l_\rho\rho^{l_\rho+2}e^{-n_\rho\alpha_\rho^2\rho^2/2} \\
&\quad -(l_\rho+3)n_\rho\alpha_\rho^2\rho^{l_\rho+2}e^{-n_\rho\alpha_\rho^2\rho^2/2} + n_\rho^2\alpha_\rho^4\rho^{l_\rho+4}e^{-n_\rho\alpha_\rho^2\rho^2/2}] \\
\nabla^2 \Psi_\rho &= N(l_\rho(l_\rho+1)\rho^{l_\rho-2}e^{-n_\rho\alpha_\rho^2\rho^2/2} - (2l_\rho+3)n_\rho\alpha_\rho^2\rho^{l_\rho}e^{-n_\rho\alpha_\rho^2\rho^2/2} + n_\rho^2\alpha_\rho^4\rho^{l_\rho+2}e^{-n_\rho\alpha_\rho^2\rho^2/2}) \\
\langle\Psi_\rho|T|\Psi_{\rho'}\rangle &= \int \rho^{l_\rho+2}N_1N_2[l_\rho(l_\rho+1)\rho^{l_\rho-2}e^{-(n_\rho+n_\rho')\alpha_\rho^2\rho^2/2} \\
&\quad -(2l_\rho+3)n_\rho\alpha_\rho^2\rho^{l_\rho}e^{-(n_\rho+n_\rho')\alpha_\rho^2\rho^2/2} + n_\rho^2\alpha_\rho^4\rho^{l_\rho+2}e^{-(n_\rho+n_\rho')\alpha_\rho^2\rho^2/2}] \\
\langle\Psi_\rho|T|\Psi_{\rho'}\rangle &= \sqrt{\frac{(n_\rho\alpha_\rho^2)^{(2l_\rho+3)/2}n_\lambda^{(2l_\lambda+3)/2}}{\Gamma[(2l_\rho+3)/2]}}\sqrt{\frac{(n_\rho'\alpha_\rho^2)^{(2l_\rho+3)/2}n_\lambda'^{(2l_\lambda+3)/2}}{\Gamma[(2l_\rho+3)/2]}}[l_\rho(l_\rho+1)\frac{\Gamma[(2l_\rho+1)/2]}{2([n_\rho+n_\rho']\alpha_\rho^2/2)^{(2l_\rho+1)/2}} \\
&\quad -(2l_\rho+3)n_\rho\alpha_\rho^2\frac{\Gamma[(2l_\rho+3)/2]}{2([n_\rho+n_\rho']\alpha_\rho^2/2)^{(2l_\rho+3)/2}} + n_\rho^2\alpha_\rho^4\frac{\Gamma[(2l_\rho+5)/2]}{2([n_\rho+n_\rho']\alpha_\rho^2/2)^{(2l_\rho+5)/2}}] \tag{D8} \\
&\quad \times \sum_{m_\rho}\langle l_\rho l_\lambda m_\rho m_\lambda|LM\rangle Y_{l_\rho}^{m_\rho} \times \sum_{m_\rho'}\langle l_\rho' l_\lambda' m_\rho' m_\lambda'|LM\rangle Y_{l_\rho'}^{m_\rho'}. \tag{D9}
\end{aligned}
$$

In order to perform the $\mathbf{r}_{12}$ and $\mathbf{r}_{13}$ integrations, we redefine the wavefunction in what we call spherical tensor form as

$$
\begin{aligned}
\Psi_n &= N\rho^{l_\rho}\lambda^{l_\lambda}e^{-n_\rho\alpha_\rho^2\rho^2/2-n_\lambda\alpha_\lambda^2\lambda^2/2} \\
&\quad \times \sum_{m_\rho m_\lambda}\langle l_\rho l_\lambda m_\rho m_\lambda|LM\rangle Y_{l_\rho}^{m_\rho}Y_{l_\lambda}^{m_\lambda} \\
&= Ne^{-n_\rho\alpha_\rho^2\rho^2/2-n_\lambda\alpha_\lambda^2\lambda^2/2}\sum_{m_\rho m_\lambda}\langle l_\rho l_\lambda m_\rho m_\lambda|LM\rangle\rho_{m_\rho}\lambda_{m_\lambda} \tag{D10}
\end{aligned}
$$

where

$$
\begin{aligned}
\rho_{m_\rho} &= \rho Y_{l_\rho}^{m_\rho}(\hat\rho) \tag{D11} \\
\lambda_{m_\lambda} &= \lambda Y_{l_\lambda}^{m_\lambda}(\hat\lambda).
\end{aligned}
$$

For $\mathbf{r}_{12}$ integration, we will define

$$
\begin{aligned}
\mathbf{r}_{12} &= \mathbf{r}_1 - \mathbf{r}_2 = \boldsymbol{\rho}' \tag{D12} \\
\boldsymbol{\lambda}' &= \frac{w\varepsilon_1}{\varepsilon_3(\varepsilon_1+\varepsilon_2)}\mathbf{r}_1 + \frac{w\varepsilon_2}{\varepsilon_3(\varepsilon_1+\varepsilon_2)}\mathbf{r}_2
\end{aligned}
$$

in order to create an integration variable purely in terms of $\mathbf{r}_{12} = \boldsymbol{\rho}'$. Then we rewrite our wavefunction in terms of new variables and as tensors

$$
\begin{aligned}
\boldsymbol{\rho} &= \mathbf{r}_2 - \mathbf{r}_3 = \mathbf{r}_2 - \frac{\varepsilon_1 \mathbf{r}_1 + \varepsilon_2 \mathbf{r}_2}{\varepsilon_3} = \frac{\varepsilon_2 + \varepsilon_3}{\varepsilon_3} \mathbf{r}_2 + \frac{\varepsilon_1}{\varepsilon_3} \mathbf{r}_1 \\
&= \left(\frac{(\varepsilon_2 + \varepsilon_3)\varepsilon_1}{\varepsilon_3(\varepsilon_1 + \varepsilon_2)} + \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_3(\varepsilon_1 + \varepsilon_2)} \right) \boldsymbol{\rho}' + \left(\frac{\varepsilon_2 + \varepsilon_3}{\varepsilon} + \frac{\varepsilon_1}{\varepsilon} \right) \boldsymbol{\lambda}' \\
\boldsymbol{\rho} &= \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \boldsymbol{\rho}' + \boldsymbol{\lambda}' \\
\rho_{m_\rho} &= \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \rho'_{m_\rho} + \lambda'_{m_\rho}
\end{aligned}
\tag{D13}
$$

$$
\begin{aligned}
\boldsymbol{\lambda} &= \frac{w\varepsilon_2}{\varepsilon_1(\varepsilon_2 + \varepsilon_3)} \mathbf{r}_2 + \frac{w\varepsilon_3}{\varepsilon_1(\varepsilon_2 + \varepsilon_3)} \mathbf{r}_3 = \frac{w\varepsilon_2}{\varepsilon_1(\varepsilon_2 + \varepsilon_3)} \mathbf{r}_2 + \frac{w\varepsilon_3}{\varepsilon_1(\varepsilon_2 + \varepsilon_3)} \left(\frac{-\varepsilon_1 \mathbf{r}_1 - \varepsilon_2 \mathbf{r}_2}{\varepsilon_3} \right) \\
&= \frac{w}{\varepsilon_2 + \varepsilon_3} \mathbf{r}_1 + \left(\frac{w\varepsilon_2}{\varepsilon_1(\varepsilon_2 + \varepsilon_3)} - \frac{w\varepsilon_2}{\varepsilon_1(\varepsilon_2 + \varepsilon_3)} \right) \mathbf{r}_2 \\
\boldsymbol{\lambda} &= \frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)} \boldsymbol{\rho}' - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} \boldsymbol{\lambda}' \\
\lambda_{m_\lambda} &= \frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)} \rho'_{m_\lambda} - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} \lambda'_{m_\lambda}
\end{aligned}
\tag{D14}
$$

A general expectation value of any potential Φ_{12} which depends only on the coordinate $\mathbf{r}_{12}$ is

$$
\langle \Psi_n | \Phi_{12}(\mathbf{r}_{12}) | \Psi_m \rangle = \langle \Psi_n | \Phi_{12}(\boldsymbol{\rho}') | \Psi_m \rangle
\tag{D15}
$$

$$
= N_1 N_2 \int \sum_{m_\rho m_\lambda} \langle l_\rho 1 m_\rho m_\lambda | LM \rangle \rho^*_{m_\rho} \lambda^*_{m_\lambda} \sum_{m'_\rho m'_\lambda} \langle l'_\rho 1 m'_\rho m'_\lambda | L'M' \rangle \rho_{m_\rho} \lambda_{m_\lambda} e^{-\frac{1}{2}(n_\rho + n'_\rho)\alpha_\rho^2 \rho^2 - \frac{1}{2}(n_\lambda + n'_\lambda)\alpha_\lambda^2 \lambda^2} \Phi_{12}(\boldsymbol{\rho}') d^3\rho d^3\lambda
\tag{D16}
$$

Using above identities for ρ and λ(D13,D14,D12) we arrive at

$$
\begin{aligned}
&\langle \Psi_n | \Phi_{12}(\boldsymbol{\rho}') | \Psi_m \rangle \\
&= N_1 N_2 \int e^{-c(\lambda' + \frac{b}{2c}\rho')^2 + (a - \frac{b^2}{4c})\rho'^2} d^3\rho' d^3\lambda' \\
&\quad \times \sum_{m_\rho m_\lambda} \langle l_\rho l_\lambda m_\rho m_\lambda | LM \rangle \left(\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)} \rho'_{m_{\lambda 1}} - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} \lambda'_{m_{\lambda 1}} \right)^* \left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \rho'_{m_{\rho 1}} + \lambda'_{m_{\rho 1}} \right)^* \\
&\quad \times \sum_{m'_\rho m'_\lambda} \langle l'_\rho l_\lambda m'_\rho m'_\lambda | L'M' \rangle \left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \rho'_{m_{\rho 2}} + \lambda'_{m_{\rho 2}} \right) \left(\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)} \rho'_{m_{\lambda 2}} - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} \lambda'_{m_{\lambda 2}} \right)
\end{aligned}
\tag{D17}
$$

where for simplicity

$$
\begin{aligned}
a &= \frac{(n_\rho + n'_\rho)\alpha_\rho^2}{2}\left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\right)^2 + \frac{(n_\lambda + n'_\lambda)\alpha_\lambda^2}{2}\left(\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)}\right)^2 \\
b &= -\frac{(n_\rho + n'_\rho)\alpha_\rho^2}{2}\left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\right) + (n_\lambda + n'_\lambda)\alpha_\lambda^2 \frac{w\varepsilon_2\varepsilon_3}{(\varepsilon_2 + \varepsilon_3)^2(\varepsilon_1 + \varepsilon_2)} \\
c &= \frac{(n_\rho + n'_\rho)\alpha_\rho^2}{2} + \frac{(n_\lambda + n'_\lambda)\alpha_\lambda^2}{2}\left(\frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\right)^2
\end{aligned}
\tag{D18}
$$

In order to eliminate the cross term in the Gaussian, we make one final change of variables and essentially complete the square, so a variable $\mathbf{x}$ is defined as

$$
\begin{aligned}
\mathbf{x} &= \boldsymbol{\lambda}' + \frac{b}{2c}\boldsymbol{\rho}' \\
\boldsymbol{\lambda}' &= \mathbf{x} - \frac{b}{2c}\boldsymbol{\rho}' \\
\lambda'_m &= x_m - \frac{b}{2c}\rho'_m
\end{aligned}
\tag{D19}
$$

From this point on the calculation will continue for $l = 1$ as it becomes extremely complicated for general l, but the formalism is the same. The expectation value now is

$$
\begin{aligned}
&\langle\Psi_n|\Phi_{12}(\boldsymbol{\rho}')|\Psi_m\rangle \\
=\; & N_1 N_2 \int e^{-c(\lambda' + \frac{b}{2c}\rho')^2 + (a - \frac{b^2}{4c})\rho'^2} d^3\rho' d^3\lambda' \\
& \times \sum_{m_\rho m_\lambda} \langle 1 l_\lambda m_\rho m_\lambda | LM\rangle \left(\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)}\rho'_{m_{\lambda 1}} - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\lambda'_{m_\lambda}\right)^* \left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\rho'_{m_\rho} + \lambda'_{m_\rho}\right)^* \\
& \times \sum_{m'_\rho m'_\lambda} \langle 1 l'_\lambda m'_\rho m'_\lambda | L'M'\rangle \left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\rho'_{m_\rho} + \lambda'_{m_\rho}\right)\left(\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)}\rho'_{m_\lambda} - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\lambda'_{m_\lambda}\right)
\end{aligned}
\tag{D20}
$$

$$
\begin{aligned}
\langle\Phi_{12}\rangle =\; & N_1 N_2 \int e^{-c(\lambda' + \frac{b}{2c}\rho')^2 + (a - \frac{b^2}{4c})\rho'^2} d^3\rho' d^3x' \sum_{m_\rho m_\lambda} \langle 11 m_\rho m_\lambda | LM\rangle \\
& \times\left(\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)}\rho'_{m_\lambda} - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\left(x_{m_\lambda} - \frac{b}{2c}\rho'_{m_\lambda}\right)\right)^*\left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\rho'_{m_\rho} + x_{m_\rho} - \frac{b}{2c}\rho'_{m_\rho}\right)^* \\
& \times\sum_{m'_\rho m'_\lambda} \langle 1 l'_\lambda m'_\rho m'_\lambda | L'M'\rangle\left(\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)}\rho'_{m'_\lambda} - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\left(x_{m'_\lambda} - \frac{b}{2c}\rho'_{m'_\lambda}\right)\right) \\
& \times\left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\rho'_{m'_\rho} + x_{m'_\rho} - \frac{b}{2c}\rho'_{m'_\rho}\right)
\end{aligned}
\tag{D21}
$$

Expanding this out, we obtain

$$
\begin{aligned}
\langle\Phi_{12}\rangle &= N_1 N_2 \int e^{-c(\lambda'+\frac{b}{2c}\rho')^2+(a-\frac{b^2}{4c})\rho'^2} d^3\rho' d^3x' \sum_{m_\rho m_\lambda} \langle 11 m_\rho m_\lambda | LM\rangle \\
&\times\Big[\frac{w\varepsilon_2}{(\varepsilon_2+\varepsilon_3)(\varepsilon_1+\varepsilon_2)}\rho'^*_{m_\lambda}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}\rho'^*_{m_\rho} + \frac{w\varepsilon_2}{(\varepsilon_2+\varepsilon_3)(\varepsilon_1+\varepsilon_2)}\rho'^*_{m_\lambda}x^*_{m_\rho} - \frac{w\varepsilon_2}{(\varepsilon_2+\varepsilon_3)(\varepsilon_1+\varepsilon_2)}\rho'^*_{m_\lambda}\frac{b}{2c}\rho'^*_{m_\rho} \\
&-\frac{\varepsilon_3}{\varepsilon_2+\varepsilon_3}x^*_{m_\lambda}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}\rho'^*_{m_\rho} - \frac{\varepsilon_3}{\varepsilon_2+\varepsilon_3}x^*_{m_\lambda}x^*_{m_\rho} + \frac{\varepsilon_3}{\varepsilon_2+\varepsilon_3}x^*_{m_\lambda}\frac{b}{2c}\rho'^*_{m_\rho} \\
&+\frac{b}{2c}\rho'^*_{m_\lambda}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}\rho'^*_{m_\rho} + \frac{b}{2c}\rho'^*_{m_\lambda}x^*_{m_\rho} - \frac{b^2}{4c^2}\rho'^*_{m_\lambda}\rho'^*_{m_\rho}\Big] \quad\quad (D22)
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{m'_\rho m'_\lambda} \langle 1 l'_\lambda m'_\rho m'_\lambda | L'M'\rangle \\
&\Big[\frac{w\varepsilon_2}{(\varepsilon_2+\varepsilon_3)(\varepsilon_1+\varepsilon_2)}\rho'_{m'_\lambda}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}\rho'_{m'_\rho} + \frac{w\varepsilon_2}{(\varepsilon_2+\varepsilon_3)(\varepsilon_1+\varepsilon_2)}\rho'_{m'_\lambda}x_{m'_\rho} - \frac{w\varepsilon_2}{(\varepsilon_2+\varepsilon_3)(\varepsilon_1+\varepsilon_2)}\rho'_{m'_\lambda}\frac{b}{2c}\rho'_{m'_\rho} \\
&-\frac{\varepsilon_3}{\varepsilon_2+\varepsilon_3}x_{m'_\lambda}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}\rho'_{m'_\rho} - \frac{\varepsilon_3}{\varepsilon_2+\varepsilon_3}x_{m'_\lambda}x_{m'_\rho} + \frac{\varepsilon_3}{\varepsilon_2+\varepsilon_3}x_{m'_\lambda}\frac{b}{2c}\rho'_{m'_\rho} \\
&+\frac{b}{2c}\rho'_{m'_\lambda}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}\rho'_{m'_\rho} + \frac{b}{2c}\rho'_{m'_\lambda}x_{m'_\rho} - \frac{b^2}{4c^2}\rho'_{m'_\lambda}\rho'_{m'_\rho}\Big] \quad\quad (D23)
\end{aligned}
$$

Now that we have expanded it we will go backwards and replace the spherical tensor form with the explicit forms of spherical harmonics from 302 and the coordinates

$$
\begin{aligned}
\langle\Phi_{12}\rangle &= N_1 N_2 \int e^{-c(\lambda'+\frac{b}{2c}\rho')^2+(a-\frac{b^2}{4c})\rho'^2} d^3\rho' d^3x' \sum_{m_\rho m_\lambda} \langle 11 m_\rho m_\lambda | LM\rangle \\
&\times\Big[\frac{w\varepsilon_2}{(\varepsilon_2+\varepsilon_3)(\varepsilon_1+\varepsilon_2)}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}Y_1^{m_\lambda *}(\hat\rho')Y_1^{m_\rho *}(\hat\rho')\rho'^2 + \frac{w\varepsilon_2}{(\varepsilon_2+\varepsilon_3)(\varepsilon_1+\varepsilon_2)}Y_1^{m_\lambda *}(\hat\rho')Y_1^{m_\rho *}(\hat x)\rho' x \\
&-\frac{w\varepsilon_2}{(\varepsilon_2+\varepsilon_3)(\varepsilon_1+\varepsilon_2)}\frac{b}{2c}Y_1^{m_\lambda *}(\hat\rho')Y_1^{m_\rho *}(\hat\rho')\rho'^2 - \frac{\varepsilon_3}{\varepsilon_2+\varepsilon_3}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}Y_1^{m_\lambda *}(\hat x)Y_1^{m_\rho *}(\hat\rho')\rho' x \\
&-\frac{\varepsilon_3}{\varepsilon_2+\varepsilon_3}Y_1^{m_\lambda *}(\hat x)Y_1^{m_\rho *}(\hat x)x^2 + \frac{\varepsilon_3}{\varepsilon_2+\varepsilon_3}\frac{b}{2c}Y_1^{m_\lambda *}(\hat x)Y_1^{m_\rho *}(\hat\rho')\rho' x \\
&+\frac{b}{2c}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}Y_1^{m_\lambda *}(\hat\rho')Y_1^{m_\rho *}(\hat\rho')\rho'^2 + \frac{b}{2c}Y_1^{m_\lambda *}(\hat\rho')Y_1^{m_\rho *}(\hat x)\rho' x - \frac{b^2}{4c^2}Y_1^{m_\lambda *}(\hat\rho')Y_1^{m_\rho *}(\hat\rho')\rho'^2\Big] (D24)
\end{aligned}
$$

$$\times \sum_{m'_\rho m'_\lambda} \langle 1l'_\lambda m'_\rho m'_\lambda | L'M' \rangle$$

$$[\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)} \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} Y_{l'_\lambda}^{m'_\lambda}(\hat{\rho}')Y_1^{m'_\rho}(\hat{\rho}')\rho'^2 + \frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)} Y_1^{m'_\lambda *}(\hat{\rho}')Y_1^{m'_\rho *}(\hat{x})\rho'x$$

$$-\frac{w\varepsilon_2}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2)}\frac{b}{2c} Y_1^{m'_\lambda *}(\hat{\rho}')Y_1^{m'_\rho *}(\hat{\rho}')\rho'^2 - \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} Y_1^{m'_\lambda *}(\hat{x})Y_1^{m'_\rho *}(\hat{\rho}')\rho'x$$

$$-\frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} Y_1^{m'_\lambda *}(\hat{x})Y_1^{m'_\rho *}(\hat{x})x^2 + \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3}\frac{b}{2c} Y_1^{m'_\lambda *}(\hat{x})Y_1^{m'_\rho *}(\hat{\rho}')\rho'x$$

$$+\frac{b}{2c}\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} Y_1^{m'_\lambda *}(\hat{\rho}')Y_1^{m'_\rho *}(\hat{\rho}')\rho'^2 + \frac{b}{2c} Y_1^{m'_\lambda *}(\hat{\rho}')Y_1^{m'_\rho *}(\hat{x})\rho'x - \frac{b^2}{4c^2} Y_1^{m'_\lambda *}(\hat{\rho}')Y_1^{m'_\rho *}(\hat{\rho}')\rho'^2] \quad \text{(D25)}$$

We use the following identities of spherical harmonics to perform the angular integrations and simplify

$$Y_{j_1}^{m_1}(\theta, \phi)Y_{j_2}^{m_2}(\theta, \phi) = \sum_{jm} \sqrt{\frac{2(j_1 + 1)(2j_2 + 1)}{4\pi(2j + 1)}} \langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle \langle j_1 j_2 00 | j0 \rangle Y_j^m(\theta, \phi) \quad \text{(D26)}$$

$$\int Y_{l_1}^{m_1}(\theta, \phi)Y_{l_2}^{m_2}(\theta, \phi)Y_{l3}^{m_3}(\theta, \phi)d\theta d\phi = \sqrt{\frac{2(l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$
$$\text{(D27)}$$

So now

$$\langle \Phi_{12} \rangle = N_1 N_2 \int e^{-c(x)^2 + (a - \frac{b^2}{4c})\rho'^2} d^3x d^3\rho' \sum_{m_\rho m_\lambda} \langle 11 m_\rho m_\lambda | LM \rangle \sum_{m'_\rho m'_\lambda} \langle 11 m'_\rho m'_\lambda | L'M' \rangle \quad \text{(D28)}$$

$$\times (Y_1^{m_\lambda *}(\hat{\rho}')Y_1^{m'_\lambda}(\hat{\rho}')Y_1^{m_\rho *}(\hat{\rho}')Y_1^{m'_\rho}(\hat{\rho}')\rho'^4 \frac{(-2c\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3))^2(b(\varepsilon_1 + \varepsilon_2)\varepsilon_3 + 2c\varepsilon_2 w)^2}{16c^4(\varepsilon_1 + \varepsilon_2)^2(\varepsilon_2 + \varepsilon_3)^4}$$

$$+\rho'^2 x^2 Y_1^{m_\lambda *}(\hat{x})Y_1^{m'_\lambda}(\hat{x})Y_1^{m_\rho *}(\hat{\rho}')Y_1^{m'_\rho}(\hat{\rho}')\frac{\varepsilon_3^2(-2c\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3))^2}{4c^2(\varepsilon_2 + \varepsilon_3)^4}$$

$$+x^2 \rho'^2 \frac{\varepsilon_3(-2c\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3))(b(\varepsilon_1 + \varepsilon_2)\varepsilon_3 + 2c\varepsilon_2 w)}{c^2(\varepsilon_1 + \varepsilon_2)(\varepsilon_2 + \varepsilon_3)^3}$$

$$\times [Y_1^{m_\lambda *}(\hat{x})Y_1^{m_\rho *}(\hat{x})Y_1^{m_\lambda}(\hat{\rho}')Y_1^{m'_\rho}(\hat{\rho}') + Y_1^{m_\lambda *}(\hat{\rho}')Y_1^{m'_\rho}(\hat{\rho}')Y_1^{m_\lambda}(\hat{x})Y_1^{m_\rho *}(\hat{x})$$

$$+Y_1^{m_\lambda *}(\hat{x})Y_1^{m'_\rho}(\hat{x})Y_1^{m'_\lambda}(\hat{\rho}')Y_1^{m_\rho *}(\hat{\rho}') + Y_1^{m_\lambda *}(\hat{\rho}')Y_1^{m_\rho *}(\hat{\rho}')Y_1^{m'_\lambda}(\hat{x})Y_1^{m'_\rho}(\hat{x})]$$

$$+x^2 \rho'^2 Y_1^{m_\lambda *}(\hat{\rho}')Y_1^{m'_\lambda}(\hat{\rho}')Y_1^{m_\rho *}(\hat{x})Y_1^{m'_\rho}(\hat{x})\frac{(b(\varepsilon_1 + \varepsilon_2)\varepsilon_3 + 2c\varepsilon_2 w)^2}{4c^2(\varepsilon_1 + \varepsilon_2)^2(\varepsilon_2 + \varepsilon_3)^2}$$

$$+x^4 Y_1^{m_\lambda *}(\hat{x})Y_1^{m'_\lambda}(\hat{x})Y_1^{m_\rho *}(\hat{x})Y_1^{m'_\rho}(\hat{x})\frac{\varepsilon_3^2}{(\varepsilon_2 + \varepsilon_3)^2})$$

This is a gigantic equation when everything is worked out, so we will treat it term by term. The first term

gives

$$\sum_{m_\rho m_\lambda} \langle 11m_\rho m_\lambda | LM \rangle \sum_{m'_\rho m'_\lambda} \langle 11m'_\rho m'_\lambda | L'M' \rangle Y_1^{m_\lambda *}(\hat{\rho}') Y_1^{m'_\lambda}(\hat{\rho}') Y_1^{m_\rho *}(\hat{\rho}') Y_1^{m'_\rho}(\hat{\rho}')$$

$$\times \rho'^4 \frac{(-2c\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3))^2 (b(\varepsilon_1 + \varepsilon_2)\varepsilon_3 + 2c\varepsilon_2 w)^2}{16c^4(\varepsilon_1 + \varepsilon_2)^2(\varepsilon_2 + \varepsilon_3)^4}$$

$$= \sum_{m_\rho m_\lambda} \langle 11m_\rho m_\lambda | LM \rangle \sum_{m'_\rho m'_\lambda} \langle 11m'_\rho m'_\lambda | L'M' \rangle Y_1^{m_\lambda *}(\hat{\rho}') Y_1^{m_\rho *}(\hat{\rho}')$$

$$\times \sum_{lm} \sqrt{\frac{3}{\pi(2l+1)}} \langle 11m'_\lambda m'_\rho | 11lm \rangle \langle 1100 | 11l0 \rangle Y_l^m(\hat{\rho}') \rho'^4$$

$$\times \frac{(-2c\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3))^2 (b(\varepsilon_1 + \varepsilon_2)\varepsilon_3 + 2c\varepsilon_2 w)^2}{16c^4(\varepsilon_1 + \varepsilon_2)^2(\varepsilon_2 + \varepsilon_3)^4} \tag{D29}$$

Again from Eq.D28, we have

$$\sum_{m_\rho m_\lambda} \langle 11m_\rho m_\lambda | LM \rangle \sum_{m'_\rho m'_\lambda} \langle 11m'_\rho m'_\lambda | L'M' \rangle x^4 Y_1^{m_\lambda *}(\hat{x}) Y_1^{m'_\lambda}(\hat{x}) Y_1^{m_\rho *}(\hat{x}) Y_1^{m'_\rho}(\hat{x}) \frac{\varepsilon_3^2}{(\varepsilon_2 + \varepsilon_3)^2}$$

$$= x^4 \frac{\varepsilon_3^2}{(\varepsilon_2 + \varepsilon_3)^2} \sum_{m_\rho m_\lambda} \langle 11m_\rho m_\lambda | LM \rangle \sum_{m'_\rho m'_\lambda} \langle 11m'_\rho m'_\lambda | L'M' \rangle Y_1^{m_\rho *}(\hat{x}) Y_1^{m_\lambda *}(\hat{x})$$

$$\times \sum_{lm} \sqrt{\frac{3}{\pi(2l+1)}} \langle 11m'_\lambda m'_\rho | 11lm \rangle \langle 1100 | 11l0 \rangle Y_l^m(\hat{\rho}') \tag{D30}$$

So the expectation value is

$$\langle \Phi_{12} \rangle = N_1 N_2 \int e^{-c(x)^2 + (a - \frac{b^2}{4c})\rho'^2} d^3x d^3\rho' \sum_{m_\rho m_\lambda} \langle 11m_\rho m_\lambda | LM \rangle \sum_{m'_\rho m'_\lambda} \langle 11m'_\rho m'_\lambda | L'M' \rangle$$

$$\times (Y_1^{m_\lambda *}(\hat{\rho}') Y_1^{m_\rho *}(\hat{\rho}') \sum_{lm} \sqrt{\frac{3}{\pi(2l+1)}} \langle 11m'_\lambda m'_\rho | 11lm \rangle \langle 1100 | 11l0 \rangle Y_l^m(\hat{\rho}')$$

$$\times \rho'^4 \frac{(-2c\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3))^2 (b(\varepsilon_1 + \varepsilon_2)\varepsilon_3 + 2c\varepsilon_2 w)^2}{16c^4(\varepsilon_1 + \varepsilon_2)^2(\varepsilon_2 + \varepsilon_3)^4}$$

$$+ \rho'^2 x^2 Y_1^{m_\lambda *}(\hat{x}) Y_1^{m'_\lambda}(\hat{x}) Y_1^{m_\rho *}(\hat{\rho}') Y_1^{m'_\rho}(\hat{\rho}') \frac{\varepsilon_3^2(-2c\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3))^2}{4c^2(\varepsilon_2 + \varepsilon_3)^4}$$

$$+ (x^2 \rho'^2 + x^2 \rho'^2 + x^2 \rho'^2 + x^2 \rho'^2) \frac{\varepsilon_3(-2c\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3))(b(\varepsilon_1 + \varepsilon_2)\varepsilon_3 + 2c\varepsilon_2 w)}{4c^2(\varepsilon_1 + \varepsilon_2)(\varepsilon_2 + \varepsilon_3)^3}$$

$$\times [Y_1^{m_\lambda *}(\hat{x})Y_1^{m_\rho *}(\hat{x})Y_1^{m_\lambda}(\hat{\rho}')Y_1^{m'_\rho}(\hat{\rho}') + Y_1^{m_\lambda *}(\hat{\rho}')Y_1^{m'_\rho}(\hat{\rho}')Y_1^{m'_\lambda}(\hat{x})Y_1^{m_\rho *}(\hat{x})$$

$$+Y_1^{m_\lambda *}(\hat{x})Y_1^{m'_\rho}(\hat{x})Y_1^{m'_\lambda}(\hat{\rho}')Y_1^{m_\rho *}(\hat{\rho}') + Y_1^{m_\lambda *}(\hat{\rho}')Y_1^{m_\rho *}(\hat{\rho}')Y_1^{m'_\lambda}(\hat{x})Y_1^{m'_\rho}(\hat{x})]$$

$$+x^2\rho'^2 Y_1^{m_\lambda *}(\hat{\rho}')Y_1^{m'_\lambda}(\hat{\rho}')Y_1^{m_\rho *}(\hat{x})Y_1^{m'_\rho}(\hat{x})\frac{(b(\varepsilon_1+\varepsilon_2)\varepsilon_3+2c\varepsilon_2 w)^2}{4c^2(\varepsilon_1+\varepsilon_2)^2(\varepsilon_2+\varepsilon_3)^2}$$

$$+x^4\frac{\varepsilon_3^2}{(\varepsilon_2+\varepsilon_3)^2}Y_1^{m_\rho *}(\hat{x})Y_1^{m_\lambda *}(\hat{x})\sum_{lm}\sqrt{\frac{3}{\pi(2l+1)}}\langle 11m'_\lambda m'_\rho|11lm\rangle\langle 1100|11l0\rangle Y_l^m(\hat{\rho}')) \qquad (D31)$$

We can now perform the angular integrations for the spherical harmonics, which are identical for both the ρ'^4 and x^4 term as there is no dependence on the angular pieces in the potentials

$$\int \sum_{m_\rho m_\lambda}\langle 1l_\lambda m_\rho m_\lambda|LM\rangle \sum_{m'_\rho m'_\lambda}\langle 11m'_\rho m'_\lambda|L'M'\rangle Y_1^{m_\lambda *}(\hat{\rho}')Y_1^{m_\rho *}(\hat{\rho}')\sum_{lm}\sqrt{\frac{3}{\pi(2l+1)}}$$

$$\times\langle 11m'_\lambda m'_\rho|11lm\rangle\langle 1100|11l0\rangle Y_l^m(\hat{\rho}'))$$

$$= \sum_{m_\rho m_\lambda}\langle 11m_\rho m_\lambda|LM\rangle \sum_{m'_\rho m'_\lambda}\langle 11m'_\rho m'_\lambda|L'M'\rangle \sum_{lm}\sqrt{\frac{3}{\pi(2l+1)}}\langle 11m'_\lambda m'_\rho|11lm\rangle\langle 1100|11l0\rangle$$

$$\times\sqrt{\frac{2(1+1)(21+1)(2l+1)}{4\pi}}\begin{pmatrix} 1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & l \\ m_\lambda & m_\rho & m \end{pmatrix} \qquad (D32)$$

All other terms either integrate to zero (due to an odd number of Y_{lm}'s) or to Kronecker deltas, so the rest of the integral is just a polynomial, given by

$$\langle \Phi_{12}\rangle = N_1 N_2 \int e^{-c(x)^2+(a-\frac{b^2}{4c})\rho'^2}x^2\rho'^2 dx d\rho' \sum_{m_\rho m_\lambda}\langle 11m_\rho m_\lambda|LM\rangle \sum_{m'_\rho m'_\lambda}\langle 11m'_\rho m'_\lambda|L'M'\rangle$$

$$\times\sum_{lm}\sqrt{\frac{2(1+1)(2l'_\lambda+1)}{4\pi(2l+1)}}\langle 11m'_\lambda m'_\rho|11m\rangle\langle 1100|11l0\rangle$$

$$\times\sqrt{\frac{2(1+1)(21+1)(2l+1)}{4\pi}}\begin{pmatrix} 1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & l \\ m_\lambda & m_\rho & m \end{pmatrix}$$

$$\times\rho'^4\frac{(-2c\varepsilon_1+b(\varepsilon_2+\varepsilon_3))^2(b(\varepsilon_1+\varepsilon_2)\varepsilon_3+2c\varepsilon_2 w)^2}{16c^4(\varepsilon_1+\varepsilon_2)^2(\varepsilon_2+\varepsilon_3)^4}$$

$$+\rho'^2 x^2\frac{\varepsilon_3^2(-2c\varepsilon_1+b(\varepsilon_2+\varepsilon_3))^2}{4c^2(\varepsilon_2+\varepsilon_3)^4}$$

$$+4x^2\rho'^2\frac{\varepsilon_3(-2c\varepsilon_1+b(\varepsilon_2+\varepsilon_3))(b(\varepsilon_1+\varepsilon_2)\varepsilon_3+2c\varepsilon_2 w)}{4c^2(\varepsilon_1+\varepsilon_2)(\varepsilon_2+\varepsilon_3)^3}$$

$$\times[\delta_{m_\lambda m'_\rho}\delta_{m_\lambda m_\rho} + 2\delta_{m_\lambda m'_\rho}\delta_{m'_\lambda m_\rho} + \delta_{m'_\lambda m'_\rho}\delta_{m_\lambda m_\rho}]$$

$$+x^2\rho'^2\delta_{m'_\lambda m'_\lambda}\delta_{m_\rho m'_\rho}\frac{(b(\varepsilon_1+\varepsilon_2)\varepsilon_3+2c\varepsilon_2 w)^2}{4c^2(\varepsilon_1+\varepsilon_2)^2(\varepsilon_2+\varepsilon_3)^2}$$

$$+x^4\frac{\varepsilon_3^2}{(\varepsilon_2+\varepsilon_3)^2}\sum_{lm}\sqrt{\frac{2(1+1)(2l'_\lambda+1)}{4\pi(2l+1)}}\langle 11m'_\lambda m'_\rho|11lm\rangle\langle 1100|11l0\rangle$$

$$\times\sqrt{\frac{2(1+1)(21+1)(2l+1)}{4\pi}}\begin{pmatrix}1&1&l\\0&0&0\end{pmatrix}\begin{pmatrix}1&1&l\\m_\lambda&m_\rho&m\end{pmatrix}) \tag{D33}$$

Performing the x integration yields

$$\langle\Phi_{12}\rangle = N_1 N_2\int\Phi_{12}e^{-c(x)^2+(a-\frac{b^2}{4c})\rho'^2}\rho'^2 d\rho'\sum_{m_\rho m_\lambda}\langle 11m_\rho m_\lambda|LM\rangle\sum_{m'_\rho m'_\lambda}\langle 11m'_\rho m'_\lambda|L'M'\rangle\{$$

$$\times\sum_{lm}\sqrt{\frac{3}{\pi(2l+1)}}\langle 11m'_\lambda m'_\rho|11lm\rangle\langle 1100|11l0\rangle$$

$$\sqrt{\frac{2(1+1)(21+1)(2l+1)}{4\pi}}\begin{pmatrix}1&1&l\\0&0&0\end{pmatrix}\begin{pmatrix}1&1&l\\m_\lambda&m_\rho&m\end{pmatrix}$$

$$\times\rho'^4\frac{(-2c\varepsilon_1+b(\varepsilon_2+\varepsilon_3))^2(b(\varepsilon_1+\varepsilon_2)\varepsilon_3+2c\varepsilon_2 w)^2}{16c^4(\varepsilon_1+\varepsilon_2)^2(\varepsilon_2+\varepsilon_3)^4}\frac{\sqrt{\pi}}{4c^{3/2}}$$

$$+\rho'^2 x^2\frac{\varepsilon_3^2(-2c\varepsilon_1+b(\varepsilon_2+\varepsilon_3))^2}{4c^2(\varepsilon_2+\varepsilon_3)^4}\frac{3}{4}\sqrt{\pi}\frac{1}{2c^{5/2}}$$

$$+3\rho'^2\sqrt{\pi}\frac{1}{2c^{5/2}}\frac{\varepsilon_3(-2c\varepsilon_1+b(\varepsilon_2+\varepsilon_3))(b(\varepsilon_1+\varepsilon_2)\varepsilon_3+2c\varepsilon_2 w)}{4c^2(\varepsilon_1+\varepsilon_2)(\varepsilon_2+\varepsilon_3)^3}$$

$$\times[\delta_{m_\lambda m'_\rho}\delta_{m_\lambda m_\rho} + 2\delta_{m_\lambda m'_\rho}\delta_{m'_\lambda m_\rho} + \delta_{m'_\lambda m'_\rho}\delta_{m_\lambda m_\rho}]$$

$$+\frac{3}{4}\sqrt{\pi}\frac{1}{2c^{5/2}}\rho'^2\delta_{m'_\lambda m'_\lambda}\delta_{m_\rho m'_\rho}\frac{(b(\varepsilon_1+\varepsilon_2)\varepsilon_3+2c\varepsilon_2 w)^2}{4c^2(\varepsilon_1+\varepsilon_2)^2(\varepsilon_2+\varepsilon_3)^2}$$

$$+x^{1+1+1+1}\frac{\varepsilon_3^2}{(\varepsilon_2+\varepsilon_3)^2}\sum_{lm}\sqrt{\frac{2(1+1)(2l'_\lambda+1)}{4\pi(2l+1)}}\langle 1l'_\lambda m'_\lambda m'_\rho|1l'_\lambda lm\rangle\langle 1l'_\lambda 00|11l0\rangle$$

$$\times\sqrt{\frac{2(1+1)(21+1)(2l+1)}{4\pi}}\begin{pmatrix}1&1&l\\0&0&0\end{pmatrix}\begin{pmatrix}1&1&l\\m_\lambda&m_\rho&m\end{pmatrix}\} \tag{D34}$$

So now the expectation value is in terms of just one variable, which can be numerically integrated easily.

Doing the 13 just involves changing the a,b,c constants and the epsilons and the 23 is trivial since it doesn't require any additional substitutions.

We can follow a similar method for $l=2$, by constructing new states as functions of the $l=1$ states.

The explicit forms of the spherical harmonics for $l = 0$ and $l = 1$ are

$$
\begin{aligned}
Y_0^0 &= \frac{1}{2}\sqrt{\frac{1}{\pi}}, \\
Y_1^{-1} &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi}, \\
Y_1^0 &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta, \\
Y_1^1 &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi},
\end{aligned}
\tag{D35}
$$

which can be used to rewrite the $l = 2$ spherical harmonics as

$$
\begin{aligned}
Y_2^{-2} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{-2i\phi} = (Y_1^{-1})^2\sqrt{2}\sqrt{\frac{5\pi}{3}}, \\
Y_2^{-1} &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{-i\phi} = Y_1^0 Y_1^{-1} 2\sqrt{\frac{5\pi}{3}}, \\
Y_2^0 &= \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1) = (Y_1^0)^2\frac{1}{\sqrt{3}}\sqrt{\frac{5\pi}{3}} - (Y_0^0)^2\sqrt{5\pi}, \\
Y_2^1 &= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{i\phi} = Y_1^0 Y_1^1 2\sqrt{\frac{5\pi}{3}}, \\
Y_2^{-2} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi} = (Y_1^1)^2\sqrt{2}\sqrt{\frac{5\pi}{3}}.
\end{aligned}
$$

So then, since our wavefunction for $l_\rho = l_\lambda = 2$ is of the form

$$
\begin{aligned}
|\Psi_n(\boldsymbol{\rho}, \boldsymbol{\lambda})\rangle &= \frac{8}{15\sqrt{\pi}}n_\rho^{7/4}\alpha_\rho^{7/2}n_\lambda^{7/4}\alpha_\lambda^{7/2}\rho^2\lambda^2 e^{-n_\rho\alpha_\rho^2\rho^2/2 - n_\lambda\alpha_\lambda^2\lambda^2/2} \\
&\quad \times \sum_{m_\rho m_\lambda}\langle 22m_\rho m_\lambda|LM\rangle Y_2^{m_\rho}Y_2^{m_\lambda},
\end{aligned}
\tag{D36}
$$

we can use the original tensor substitution of Eq. (D11) to define

$$
\begin{aligned}
\rho^2 Y_2^{-2} &= \rho^2 (Y_1^{-1})^2\sqrt{2}\sqrt{\frac{5\pi}{3}} = \sqrt{2}\sqrt{\frac{5\pi}{3}}\rho_{-1}^2, \\
\rho^2 Y_2^{-1} &= \rho^2 Y_1^0 Y_1^{-1} 2\sqrt{\frac{5\pi}{3}} = 2\sqrt{\frac{5\pi}{3}}\rho_{-1}\rho_0, \\
\rho^2 Y_2^0 &= \rho^2([Y_1^0]^2\frac{1}{\sqrt{3}}\sqrt{\frac{5\pi}{3}} - (Y_0^0)^2\sqrt{5\pi}] = \frac{1}{\sqrt{3}}\sqrt{\frac{5\pi}{3}}, \\
\rho^2 Y_2^1 &= \rho^2 Y_1^0 Y_1^1 2\sqrt{\frac{5\pi}{3}} = \rho_0\rho_1 2\sqrt{\frac{5\pi}{3}}, \\
\rho^2 Y_2^2 &= \rho^2(Y_1^1)^2\sqrt{2}\sqrt{\frac{5\pi}{3}} = \sqrt{2}\sqrt{\frac{5\pi}{3}}\rho_1^2.
\end{aligned}
\tag{D37}
$$

113

Therefore, we can follow the same procedure outlined for $l = 1$, the only major difference being that the algebra is more complicated. However, since we're going to have to do at least one numerical integration anyway, we can use a program such as Mathematica (or Wolframalpha) to analytically integrate what is analytically integrable and then numerically integrate the rest without doing an absurd amount of algebra (this is what was done in this work). This method should actually hold to allow for higher order l states to be done relatively simply, since essentially all that will become more complicated is the algebra arising from the substitutions (this may be part of future work on this subject).

Vita

Joshua F. Whitney was born in Massachusetts on April 15, 1982 and grew up in Maine. He received his Bachelor of Science degree from King College in 2004 with a double major in physics and computer science and his Ph. D. in physics from the University of Tennessee in 2011. He married Heather Whitney née Barker, in 2005. He currently teaches in the physics department at Wheaton College in Wheaton, Illinois alongside his wife and is exploring research opportunities at places such at Fermilab and Argonne National Laboratory.