12-2010

Government Intervention and Economic Growth

Maria Sarigiannidou
University of Tennessee - Knoxville, msarigia@utk.edu

To the Graduate Council:

I am submitting herewith a dissertation written by Maria Sarigiannidou entitled "Government Intervention and Economic Growth." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Economics.

Matthew N. Murray, Major Professor

We have read this dissertation and recommend its acceptance:

Robert A. Bohm, William S. Neilson, Phillip Daves

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

To the Graduate Council:

I am submitting herewith a dissertation written by Maria Sarigiannidou entitled "Government Intervention and Economic Growth". I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Economics.

Matthew N. Murray, Major Professor

We have read this dissertation
and recommend its acceptance:

Robert A. Bohm

William S. Neilson

Phillip Daves

Accepted for the Council:

Carolyn R. Hodges
Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

Government Intervention and Economic Growth

A Dissertation Presented for
The Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Maria Sarigiannidou
December 2010

Dedication

In the memory of my beloved grandmother, Christina
and of my father, Constantine.

To Panos
for his presence in my life… for everything…

To an academic world in which the ethical integrity of a student and, sense of self-respect will never be sought to undermined, or exchanged.

Acknowledgement

I wish to express my heartfelt thank you to Professor Matthew N. Murray, and Professor Theodore Palivos, for the invaluable contribution of each in the completion of my dissertation. Each, in a unique and distinguished way contributed invaluably in bringing my dissertation to a successful end.

Dr. Theodore Palivos has been my advisor professor in the writing of the first essay of the dissertation, titled *A Modern Theory of Kuznets' Hypothesis*. Currently a professor in Economic Science at the University of Macedonia, in Greece, he is officially a *courtesy* member of my Ph.D. committee. I wish to thank Professor Palivos for welcoming me when I sought his advice, and guidance on a research essay of mine. I thank him for being on my side and for assisting me in the completion of my doctoral dissertation until the end. I hold a sense of deep appreciation and respect for who he stands to be. His ethic of character and his academic accomplishments lend him an admirable integrity.

Professor Matthew N. Murray is the chair professor in my Ph.D. committee, and a mentor. In this role, he has shaped my years of academic path in a decisive way. He holds a personality that is charismatically integrated, with an intelligence extending on very different dimensions. I owe it to this that he was so able to discern the truth, and stand still in inherently difficult situations. In this way, he enabled me to walk my academic path in an uncompromising way. I will always thank him for this, and for his strength to bring an ideal in the pragmatic world.

I owe a special thank you to Professors William S. Neilson and Philip Daves for their eager acceptance to join my Ph.D. committee. With distinctive kindness and professionalism they made the end of this process profoundly simple, and satisfactory. I wish them to know that their questions, and comments on my dissertation were substantial, and I have looked upon them with thorough attention.

I wish to conclude with an expression of thankfulness to Professor Robert A. Bohm. He has contributed an important role as main professor in my doctoral committee. In his silent but very thoughtful way, he has always revealed a trustful support on my side. I know that if he could see a way out of the difficulties, he would have made them vanish. Thank you Professor Bohm for the stability you have inspired, and have always given to me.

As I submit my dissertation, I ought to place an acknowledgment of thankfulness to the institution I have belonged in all years passed by. My expression of thank you therefore to the Department of Economics, of the University of Tennessee at Knoxville, for the opportunity to a doctoral dissertation in Economic Science. I acknowledge thankfully that I have received financial assistance from my department throughout the

entire program of my study. My journey in this academic pursue would not be possible to realize, should there be no financial support.

I could not end this section without saying a word of thank you to Mrs. Donna Kemper and Mrs. Susan McGee, the administrative assistants in our department. They should know that they are highly respected for their silent strength, and for all they give to us.

Abstract

The first essay constitutes a theory which lends truth to the *Kuznets hypothesis*. The attention is centered on the role of financial markets in defining the process of knowledge accumulation, and ultimately the distribution of income earning capabilities in a population of *ex ante* heterogeneous individuals. The provision of credit is hindered by one-sided lack of commitment embedded in the area of educational investment. Adaptation in the legislative system to accommodate a punishment scheme conditional on default is the critical requirement for the economy to be carried on a dynamic growth path, albeit one of higher and worsening inequality. Owing to the accumulation of human capital and the associated externality on future generations' knowledge productivity, the economy ultimately makes its transition to a state of lower income differentials.

The second essay is an enquiry on the role of monetary policy in determining the growth dynamics of a small open economy. We postulate that the possibility of intermediated credit does not exist, the intention of the assumption being to uncover the role of inflation as tax on private spending. The analysis brings a valid argument of the *superneutrality of money*. Inflation when operating as consumption tax has no impact on the growth rate of output. This is established irrespective of the labor supply be held fixed, or incorporated as endogenous decision. When imitating the role of capital taxation, inflationary policy has a negative effect on capital accumulation in a framework of fixed labor supply. However, the validity of the superneutrality result is once again reestablished in an environment accommodating the endogeneity of labor supply.

The third essay is a theoretical investigation of the long-run effects of tax and expenditure policies in an open economy framework. The aim is to establish an analytic basis for the factual evidence associated with the non-monotonic response of the current account to fiscal shocks. To this endeavor we sought two sources of *time non-separability* in the preference structure, habit-forming consumption in consumer durable goods. Optimal private choices induce non-monotonic dynamics on consumption behavior that are exactly consistent with the evidence on the current account.

Table of Contents

General Introduction

This thesis is a collection of three essays on economic growth, and development. The theme of our inquiry is the role of government intervention in defining the pace, and character of economic growth in the distant future. The analysis in the entire volume is undertaken at purely theoretical level. We proceed with a general summary of the three essays, as well as a brief overview of the thesis' arrangement. The chapters themselves contain individual introductory sections. We do not attempt to repeat them here, but wish simply to provide a concise introduction in the subject of our research.

In the first essay we seek to construct a theory which in a novel way lends truth to the proposition formed by Kuznets (1955), with respect to the causal non-monotonic relationship between aggregate prosperity and inequality of income distribution. Our attention centers on the role of financial markets in defining the process of economic development, and ultimately the distribution of income earning capabilities in a population of *ex ante* heterogeneous individuals. If the roots of development lie in human capital accumulation, the possibility to fund educational choices through private credit organizations may prove critical. The provision of credit in this market is hindered by one-sided lack of commitment, and particular enforcement issues embedded in the area of educational investment. Contract enforcement hinging on the nature of consequences following an act of default, ultimately is a matter of the legislative system. In the tradition of Kehoe and Levine (1993) we assume that legislation accommodates the complete and permanent exclusion of defaulting borrowers from financial markets. The prospect of being prohibited from investment in tangible assets induces agents to choose commitment to prrior agreements. Contract arrangements thus become enforceable, leading credit institutions to eagerly engage in educational funding. This is the critical requirement for the economy to be carried on a dynamic growth path, escaping the ever-sustained trap in poverty state. We trace out paths of development so constructed as to give an explicit proof of the *trickle-down* theory of economic growth. Initially, an equilibrium is taken to exist in which a particular group of individuals, those with the highest investment return only choose to engage in education. Owing to the accumulation of human capital and the associated externality on future generations' knowledge productivity, the economy ultimately makes its transition to a state where the aggregate of all agents invest in individual improvement. As endogenous technological knowledge takes off, the externality effect arising from knowledge spillovers gives rise to inverted-*U* dynamics in the evolution of income distribution. A pattern of worsening inequality prevails in the early stages of growth. However, as dynamics bring the economy on a more evolved stage income differentials appear to shrink. Hence, income convergence is established to be the signal that an advanced level has been attained on the development path.

The second essay, titled "*Money's Role in Determining Long-Run Growth*", is an enquiry on the theme of monetary effectiveness in determining the growth dynamics. The analysis is carried with reference to an economy being open, yet a price taker in the international capital markets. Exploiting the developments in the theory of endogenous growth, perpetual unbounded growth is sustained upon the accumulation of a broad concept of capital, encompassing both physical and human notions. In a framework of endogenously determined growth it is possible to analyze the effects of economic policy on the *growth rate* of the aggregate real variables. At this level, the analysis departs from the traditional approach in the literature which had been to focus on the impact of monetary policy on the steady state *levels* of real economic aggregates. Theorists in the monetary literature concentrated early on, on how developments in the economy's financial system in essence define the role of money, and therefore, the character of influence of monetary policy on real aggregates. In an influential paper, Stockman (1981) proposed that when a credit market for consumption and capital goods is missing, distortionary monetary policy interacts with private capital decisions, causing investment, and real output to fall at a lower steady state level. Examining the theoretical validity of Stockman's argument in the context of equilibrium growth is the aim of the present research. We postulate that the possibility of intermediated credit does not exist, with the intention of the assumption being to uncover the role of inflation as tax on private spending. Initially, the postulate applies on purchases of consumption goods solely. In an alternative version of the model, investment on capital goods is also being subjected to the constraint that real cash holdings are the only means of conducting the purchase transaction. In this latter case, inflation bears an evident analogy to a capital tax. The theory been constructed thus gives us an insight into how inflation is been conceived to imitating fiscal tax instruments. To elucidate the consequences of endogenously determined labor, the theory is initially built on models that abstract from the decision to allocate time between leisure and other productive activities. In the latter part of this essay the analysis is extended to account for the endogeneity of the time-allocation decision.

The third, and last, part of this dissertation constitutes a theoretical essay on the long run effects of tax and expenditure policies. The analysis is carried with reference to an open economy, yet a price taker in the international markets. Our interest lies in exploring the transitional dynamics of the current account in response to permanent fiscal shocks. The empirical literature in the international macroeconomics has established that the current account evolves non-monotonically along its adjustment path to the long run equilibrium. It has been the aim of this study to show that this empirical phenomenon is proved within the theory, thus been validated on the ground of acceptance of a mathematical proposition. To this endeavor we sought two sources of *time non-separability* in the preference structure, habit forming consumption in consumer durable goods. When households choose to maintain their habitual standard of living and consumption exhibits a degree of durability, optimal private choices induce non-monotonic dynamics on consumption (saving) behavior that are exactly consistent with the factual evidence on the current account.

The dissertation is concluded with a general discussion upon the results of the research analysis. At the end of the volume there is a bibliography, and an appendix. The bibliography is comprehensive, covering all the reference sources been used in the development of the three essays.

ESSAY I

A Modern Theory of Kuznets' Hypothesis

I. *Introduction*

The Kuznets hypothesis The character of evolution of the distribution of income along an economy's development process has been a theme with a long history in economic enquiry. The literature starts with the classic contribution of Simon Kuznets (1955), who was the first to identify economic growth as a determinant cause of long term changes in the distribution of income. Establishing his proposition on data from the time of industrialization of currently advanced nations, Kuznets (1955) initiated the idea that the inequality characterizing income distribution exhibits a non-monotonic trend along the process of economic development: it appears to widen during a society's transition from a pre-industrial to an industrial system, it remains stable for a while and narrows as more mature stages of growth are reached.[2] This systematic evolution of income distribution along a country's development path became known as the *Kuznets Curve* –an inverted *U*-shape relationship between income per capita and personal income inequality.

In his article, Kuznets lays out a simple model that places weight on the process of industrialization in driving the observed trends in the distribution of income. All developing countries are characterized by the coexistence of a traditional agricultural, and an industrial sector. The former is distinguished by its lower per capita income, and possibly narrower, but never wider inequality of distribution. Economic development proceeds by the rapid growth of industry, and the accompanied resource flows from agriculture. In earlier stages of this process, pronounced urban income inequalities exacerbate the countrywide magnitude of income variation. However, the rise over time in the relative weight of the industrial sector leads eventually to a narrowing of the overall inequality of distribution. A variety of forces interact to bolster the economic position of poorer segments of the population. As economic development proceeds,

[2] Kuznets (1955) formulates his proposition using available data from the industrialization period for the United States, England and Germany.

continually more individuals move from rural to urban areas, thus taking advantage of the opportunities of the relatively rich industrial sector. Furthermore, many workers who started out at the bottom rungs of the industrial sector walk up economically and socially. At the same time, a smaller size of the labor force is connected to agriculture, and this causes the relative wage rate in the rural sector to increase. These along with other, political and social considerations suggest a rise in the relative shares of lower-income groups.

The inverted-U: Evidence The subsequent literature evolved mainly in the direction of examining the robustness of the Kuznets curve on an empirical ground. Ideally, the evolution of inequality along the course of development should be examined in the historical context of individual countries. However, reliable time series data are scarce for most countries as we go back into the past. Consequently, the route has been to draw on cross country experience. Evidence on variations in inequality of countries that are at different stages along the development process provides information for exactly what is lacked for a single country. A bulk of cross-sectional studies has provided justification of the inverted-*U* hypothesis, leading to its acceptance in the 1970s as a stylized fact. This literature is represented by Paukert (1973), Ahluwalia (1976*a*,*b*), Adelman and Morris (1973), Chenery *et al*. (1974), Bacha (1977, 1979), Ahluwalia, Carter and Chenery (1979), and Adelman and Robinson (1989).

However, the alleged status of the Kuznets hypothesis was called into question by an array of subsequent studies. Papanek and Kyn (1986) challenged its empirical validity in an analysis of cross-section and time series data for 83 countries. They found that the support for the Kuznets relation is not strong, and may be weakening over time. In addition they point that there is considerable variability in income distribution at all levels of income, which is failed to be explained by the Kuznets effect. In a similar vein, Bourguignon and Morrison (1990) find a weak link between per capita income and income distribution in a cross-section study of developing countries. Anand and Kanbur (1993) have also suggested that the relation had weakened over time. Li, Squire and Zou (1998) suggest that the Kuznets hypothesis is generally in accord with cross-sectional observations obtained at a point in time. However, they present evidence that counter the validity of an inverted-*U* pattern over the course of evolution of individual economies. Their position is that the inequality of income distribution has remained relatively stable in the second half of the 20th century in a sample of 49 developed and developing countries. In a more recent contribution to the literature, Barro (2000) has re-established the inverted-*U* as a central theory in linking inequality to economic growth. In a panel study of closely 100 countries and covering from 1960 to 1995 the Kuznets hypothesis is established as a strong empirical regularity.

The 'trickle-down' theory Several theorists have concentrated more recently on extending the theoretical basis behind the Kuznets hypothesis. The proposed theories relate each in a different way to the notion of *trickle-down.* The idea is that with enough growth and little intervention to correct income inequality, the fruits of economic development will eventually filter or *trickle-down* to the poor, as the demands for what the latter can offer are magnified (Debraj Ray, 1998).

Greenwood and Jovanovic (1990) formulate a theory in which economic growth is inextricably linked to the development of financial markets and institutions. In their model intermediation structure is costly to build; hence, the level of financial development depends on the stage of the growth cycle. At the same time, a well-advanced financial system spurs economic growth by mitigating the effects of information and transaction costs, thus contributing to an efficient allocation of investment funds. Intermediaries provide savers with a distribution of returns on their investments that both is preferred and has a higher mean. However, investment through financial markets is costly, and relatively poor agents may not afford to use the superior technology. The theoretical validity of the Kuznets curve is rooted in the advancement of an economy's financial system, and the extent to which its services are spread across population. In its earlier stages, the process of economic growth is accompanied by the progressive development of financial intermediation. Since relatively rich individuals may only be able to take advantage of the developing financial markets, the variation of income initially widens. Along the course of development, the sustained improvement in the economic position of progressively more and more individuals translates into a distribution of higher initial endowments of capital. The economy approaches a state where the entire population may claim a share in the higher income prospects of the investment technology provided by the financial sector. Income disparities ultimately fade away as the benefits of development permeate more widely.

A closely related argument was developed by Aghion and Bolton (1997) in a model of endogenous income distribution that also generates the dynamics of the Kuznets curve. Individuals face two investment opportunities: a backyard activity that yields a deterministic rate of return, and an entrepreneurial technology with superior, yet uncertain revenue. The latter requires a minimum amount of capital investment, which agents may borrow in the capital market if endowed with sufficient initial wealth. In this model, it is the middle class that borrows to finance costly investment, whereas the very poor and rich agents act as lenders through their investment in the safe asset. The key feature that drives the relation between growth and wealth inequality is the endogenous determination of the cost of borrowing. In the early phases of development aggregate wealth, hence the supply of credit provided by the rich class of lenders, is small implying a high cost of capital. As capital is further accumulated, the wealth of rich lenders grows relatively faster, leading to widening wealth inequalities. However, as economic growth progresses, more and more funds become available to finance a progressively smaller pool of borrowers. The equilibrium lending terms shift in favor of borrowers, thus equalizing the distribution of wealth.

Another strand of literature emphasizes the role of technological progress in governing the pattern of income inequality. Studies within this field represent Galor and Tsiddon (1997*a*), Aghion and Howitt (1997), and Helpman (1997). Two technologies coexist, an old and a more advanced one, and individuals choose where they seek to be employed. Intergenerational mobility is represented by the choice of a different employment sector than that of one's parents. The model predicts that following periods of major inventions – the factory system, electrical power, computers– economies undergo a phase of rapid economic growth associated with enhanced intergenerational mobility and increased inequality. This outcome depends on characteristics of the technologically advanced sector, such as paying a higher marginal return to ability while a lower reward to the less able. Along the course of development, complex technologies gain accessibility to a wider range of individuals. This process has the effect of diminishing intergenerational mobility, hence reducing the inequality of income distribution.

In another study, Galor and Tsiddon (1997*b*) present a theory of trickle-down based on human capital accumulation and the expansion of technological knowledge that stems from it. The forces that drive economic growth in this setting are the accumulation of human capital and the advancement of technology with the former taking the leading role. Technological knowledge augments endogenously, and is the by-product of individuals' investment in enhancing own education. The vehicle through which technological progress contributes to growth is the accumulation of knowledge. The latter acts to enhance the marginal return to individual investment in education, thus feeding back in the accumulation of human capital. A key feature of the analysis is the postulate that the individual learning aptitude is determined, in addition to own investment of resources in education, by parental human capital and society's aggregate knowledge. The model yields the dynamics of an inverted-*U* path. An initially poor economy composed of an uneducated population is characterized by a highly equitable distribution of income. To its largest extent the entire population earns a fairly low income stemming from minimum skill and productivity levels. Investment in human capital is initially undertaken by individuals of high educational background, since they are the only ones with high effectiveness in own investment in education. The economy as a whole registers growth, but the benefits of this growth are highly concentrated in a relatively small number of individuals. Technical progress is likely to have a more uneven character at low to intermediate levels of income. As technological knowledge takes off, ultimately the gains find their way to everybody. And along with the growing educational status of the labor force, the economy enters a cycle of steadily declining inequality.

In this paper we seek to offer an additional contribution in the research elaborating on the theoretical underpinnings of the Kuznets hypothesis. Close in spirit to Galor and Tsiddon (1997*b*), our theory builds on the trickle-down hypothesis in a model where growth is driven by accumulation of human capital and the expansion of knowledge in society. Our attention is concentrated on the role of financial markets in determining the potential for acquiring education, and therefore the distribution of income earning capabilities. We explore what fundamental forces lead to the non-existence of credit institutions in the market for funding education, and show that the emergence of the latter may play a critical role in spurring economic development. As endogenous technological knowledge takes off, the externality effect arising from knowledge spillovers gives rise to inverted-*U* dynamics in the evolution of income distribution.

Human capital: A missing market We construct an overlapping-generations model in which private incentives induce agents to invest in education, and where non-rival inventions are the by-product of the education process. Pursuing to address the issue of income distribution we develop a model with heterogeneous agents distinguished on the level of innate learning aptitude. Individuals who belong in the same generation, and thus face the same social capital, are characterized by different human capital levels due to the postulated heterogeneity in the effectiveness of their investment in education. An individual's level of human capital upon entering the workforce determines her productivity of labor, hence her income at that period of life. The character of income distribution, and its evolution across time, is therefore governed by the distribution of human capital in society.

This paper may be viewed as contribution to the literature on the role of financial intermediation in determining the pace, and character of economic growth.[3] In line with the traditional view, we establish on theoretical ground that the development of financial markets constitutes an inextricable part of the process of economic development. In a model where the roots of development lie in human capital accumulation, our aim is to examine how intermediated credit may spur or hinder individual investment in education, hence economic growth. We assume that formal education is costly, in the sense that it incurs a direct pecuniary cost.[4] Individuals may not fund educational choices out of

[3] The relationship between financial development and economic growth has long been examined in the macroeconomic literature. Early research on the topic is associated with the work of Goldsmith (1969), McKinnon (1973), and Shaw (1973). Important contributions more recently include Greenwood and Jovanovic (1990), Bencivenga and Smith (1991), King and Levine (1993), Saint-Paul (1992) and Levine (1992).

[4] We abstract from consumption or other form of expenditure incurred in the period the investment is made.

retained earnings, wealth or any form of inherited bequests.[5] Such investment must be financed from human capital loans through formal credit organizations.[6] In economies lacking such institutions, individuals are entirely barred from productive educational choices; a consequence of the failure of the credit market.

That credit markets for education loans may not function perfectly, or be entirely missing, is an argument with a long recognition in macroeconomics. Early on, Friedman (1962) attributed the source of the failure of this market to the intrinsic nature of human capital, in being embodied in those who possess it. It is thus impossible for the return of the investment to be passed on to lenders, or serve as collateral in the event of failure to repay. Moreover, it is particularly difficult to monitor the productive use of the loan, and the effort put up by the investor. The ability to make use of human capital may be unknown even to the borrower. Genuine bankruptcies and strategic default may well occur, with there being little that a lender can do to get his money back. These issues make the provision of credit in this market problematic.

Quite some research in the macroeconomic literature has adopted this idea; however most have formally modeled it on an *ad hoc* foundation, by imposing some form of exogenous borrowing constraints. Loury (1981) has examined the dynamics of income distribution in a stochastic model with an absent market for educational loans. Ljungqvist (1993) emphasizes the role of missing markets for human capital in explaining the persistence of underdevelopment in a world with free trade in consumption goods and physical capital. Buiter and Kletzer (1992) argue that the inability to borrow may reduce human capital accumulation in a model where individuals must self-finance own training

[5] The analysis in version VII looks at the case where individuals receive an endowment in the retirement age, independent of their prior income.

[6] Our emphasis is placed on the existence of credit support for tuition type expenses in education. The structure of the model implies that individuals do not demand credit to fund consumption, or other investment purposes. This channel of effect of intermediated credit on human capital accumulation has been investigated by De Gregorio and Kim (2000). In an economy with heterogeneous agents, financial institutions through the provision of *consumption* credit allow high-ability individuals to abstain from productive work in their youth, and devote the whole time endowment in education. On the other hand, agents with low efficiency in human capital investment may find it optimal to specialize in market activities, and use the financial markets to engage in intertemporal smoothing. By providing these opportunities for specialization, credit markets enhance the economy's average efficiency of education, and consequently growth and welfare for all current and future generations. Evidence in support of the presence of this effect has been presented, for the United States and other OECD countries by Behrman *et al*. (1989) and De Gregorio (1996).

costs. And Barro, Mankiw, and Sala-i-Martin (1995) re-examine the theory of convergence in income across countries in the context of a model in which financing for human capital investment is not available.

Endogenous debt constraints The focus in the present paper centers on the forces that lie behind the observed credit market imperfections in education funding, as well as the development of institutional infrastructure that may lead to overcome this market failure. Our contribution lies in integrating the theory of endogenous credit constraints into an analysis of the relationship between economic growth and the dynamic evolution of income distribution.

There are two alternative theoretical approaches within which debt constraints may emerge endogenously. The first builds on the premise that credit rationing arises as an optimal response of lending institutions to issues of asymmetric information. The core argument consists of the claim that moral hazard and adverse selection are interwoven in the lender-borrower relationship, and interfere with market behavior leading to a variety of failures in loan markets. There are several different microeconomic theories relating to private information problems that imply a form of credit rationing. These are mostly based upon the non-observability of labor input (moral hazard)[7], physical output[8], and individual ability (adverse selection)[9]. In the context of human capital analysis with external financing of education, Zeira (1991) shows that as a result of asymmetric information, credit may be endogenously rationed as a precautionary measure against the possibility of moral hazard. In the growth area, Tsiddon (1992) relies on moral hazard issues in the educational market to provide an explanation of the long run divergence of income levels across countries.

The second approach is based not on underlying information problems, but on the inability of creditors to enforce a loan contract. The central idea draws from the work of Schechtman and Escudero (1977), Eaton and Gersovitz (1981), and Manuelli (1986) in the international literature on sovereign debt. The framework was originally formalized in

[7] The reader may see Jaffee and Russell (1976), Stiglitz and Weiss (1981, 1987), Aghion and Bolton (1997) for more information.
[8] An interested reader may further look at Banerjee and Newman (1993), Galor and Zeira (1993).
[9] Jaffee and Stiglitz (1990) provide information on this subject.

the area of partial insurance against idiosyncratic risk by Kehoe and Levine (1993). Their study marks a new tradition in modeling endogenous borrowing limits, and is the path taken in this study as well.[10] In their formulation, the system of creditors' legal rights allows the punishment of a borrower committing default in the form of her exclusion from future participation in formal financial markets. Defaulters are denied access to new loans in the credit market, and intermediate institutions have the legal right to seize the tangible assets in the debtor's possession. This renders lending through capital markets following default an irrational act. In this setting, *participation constraints* ensure that in equilibrium agents entering into a contract would at no time be better off contemplating default.

Outline of the essay A word about the essay's arrangement. The precise structure of the model is set out in the following section, in accompaniment of an elaborate discussion on how financial intermediation is embedded into the analysis. Section III presents an exposition of circumstances that may lead to a state of poverty persist over time, and the possibility of transition on a path of equilibrium unbounded growth. The dynamic impact of the process of development on the economy-wide income distribution is discussed in Section IV. The final two sections augment the core analysis each in a different direction. Section V allows for the economy's interest rate to be endogenously determined. In Section VI the analysis is extended to accommodate a higher, more general degree of heterogeneity. The analysis in Section VII is the exposition of a case where the effective punishment scheme imposes less stringent consequences on a borrower committing default. Although difficult and often very complex, the exposition offers complementary insight, which stands in need for delving deeper into the theory. A brief discussion in the end takes the role of final conclusion.

[10] This enforcement mechanism has also been adopted by Azariadis and Lambertini (2003) in a pure exchange framework with overlapping generations. Andolfatto and Gervais (2006) has been the first study to embody a similar enforcement mechanism in a model with endogenous human capital formation, in which costly education is financed through private credit markets. De la Croix and Michel (2007) extend the latter analysis in a general equilibrium framework, allowing for the interest rate be endogenously determined.

II. *The model*

Demographic composition The model is a variant of the overlapping-generations model with production introduced by Diamond (1965) and Samuelson (1968). Time is measured as discrete intervals, beginning at time t=0. Individuals have finite life spans of three periods, with a new generation born in each period. We call an individual *young* in the first, *adult* in the second and *old* in the last period of life. An equal number of individuals enter and leave the economy in each period, implying a stationary population. Each new generation is composed of a continuum of individuals, with total measure normalized to unity, $i \in [0,1]$. Generations are named after their birth date.

Human capital accumulation and heterogeneity Human capital is defined to refer to the skills and knowledge level possessed by an individual. It is an intangible and inalienable factor that cannot be treated separately from those who create it or possess it. We assume that young agents are born with a minimum level of human capital $h_{\min} > 0$, which can be thought of as the ability to talk and coordinate with each other. While young, one may make an investment on individual improvement by devoting real resources to formal education. Education is costly in the sense that it incurs a direct pecuniary cost equal to q units of output per person.[11]

The individual's level of human capital upon entering the labor force depends on the effectiveness of her investment given that she engages in the education process. We postulate that the return on education is determined by the stock of knowledge in society while the investment is undertaken, and by the individual's innate ability.[12] Consistent with Tamura (1991) we postulate that the investment sector is characterized by an external spillover effect of human capital. The human capital of the average citizen contributes to enhance any individual's ability to acquire knowledge.[13] The assumption is

[11] We abstract from other aspects of costly education, such as the sacrifice of leisure and the disutility from effort.

[12] Education may be considered as a form of vocational process, in which q can include the cost of tuition, books, tools as well as a subsistence level of consumption. Students learn from every adult who is currently alive. Thus, the level of human capital acquired by an individual who invests in education depends upon the average stock of human capital among all adults in the society, (H).

[13] The meaning attached to the concept of societal knowledge is that of being embodied in the human capital possessed by the members of the population. We do not make a conceptual distinction between the stock of disembodied knowledge, in other words knowledge in books, and that of being embodied in

further adopted that the magnitude of the external effect is strictly increasing in own talent.

In the words of Loury (1981), "…the term *ability* refers to all factors outside of the individual's control which affect his productive capacity". Allowing individuals be distinguished on the level of their innate aptitude provides a source of heterogeneity at a skill level.[14] With the exception of innate ability, hence their effective learning parameter, individuals share access to a common non-linear investment technology. At the moment we assume that there are two types of agents in the economy, with *high* and *low* ability respectively. Since all individuals of a given type are identical, henceforth we characterize an agent by her type. The human capital level of an agent born in period t with ability A_j is represented by the following function[15]

$$h_{t+1}^{j} = \begin{cases} A_j^{\delta} \, H_t^{1-\delta} & \text{if invest} \\ h_{\min} & \text{o.w.} \end{cases}, \tag{1}$$

human inputs. In studies that do adopt this distinction (see Stokey, 1991, Laing *et al*., 2003) it is the potential of unbounded increases in the former that provides the basis for persistent growth. In our setting, endogenous never-ending growth is made feasible due to an intergenerational external spillover effect in the process of knowledge creation.

[14] The role of innate ability in shaping one's acquired human capital has been addressed in several studies in the literature. Levhari and Weiss (1974) use the term *uncertain inputs* to refer to innate talent as a determinant factor of earning capacity (at the completion of one's education). Individual ability is modeled through a random variable reflecting in part the unpredictable component of innate aptitude. The stochastic nature of the variable has the interpretation that, at the time when making a choice about the investment in her education, an individual has imperfect knowledge of exogenous characteristics such as her actual ability. A similar type of uncertainty has also been accommodated in other studies such as Eaton and Rosen (1980), Loury (1981), Snow and Warren (1990), and Benabou (1996, 2002). In our model we allow individuals to have perfect knowledge of their own aptitude; thus we abstract from the stochastic aspect of the latter variable.

[15] The literature builds on two alternative ways of production of intangible human capital. In the standard formulation proposed by Lucas (1988) human capital is produced within the household, and is determined solely by the time-investment in education. According to this approach, the price of new human capital is the implicit price evaluated by the household's utility. The other formulation has been employed by King, Plosser, and Rebelo (1988) and Rebelo (1991) and assumes that there is a market for new human capital. Human capital is produced in the education-service industry, and physical, in addition to human capital may serve as an input. The price of new human capital is the market price of education [Mino, 1996]. In the present context, our objective is to investigate the role of credit markets in the growth process of human capital, hence the latter formulation is more appropriate to adopt. However, since the relevance of the factor intensity condition is limited, physical capital is not incorporated as an input in the technology of the investment sector.

where j denotes an individual's type, $j \in \{L, H\}$, $\delta \in (0,1)$ and $H(t)$, $\forall t \geq 0$, represents the society's aggregate (and average) stock of human capital at date t.[16] Evidently, we assume that $A_L < A_H$, with $A_L > 1$ [17] We further postulate the following condition

Assumption 1

$$A_L^{\delta} H_t^{1-\delta} > h_{\min} .$$

We assume that in every period agents with high- and low level of ability constitute fractions $\lambda_H = \lambda$ and $\lambda_L = 1 - \lambda$ of the population, respectively. Then, in any given period t, the stock of the economy's aggregate (and average) level of human capital is

$$H_t = \lambda h_t^H + (1-\lambda) h_t^L \qquad \lambda \in [0,1], \tag{2}$$

where we assume that in period $t = 0$ there exists an initial old generation with $H_0 > 0$. Without loss of generality we assume that $H_0 = h_{\min}$.[18] We note that $\lambda h_t^H \equiv H_t^H$ represents the human capital level of the group of high-ability agents in period t (this constitutes the high-ability fraction of individuals of generation t-1). Similarly, $(1-\lambda) h_t^L \equiv H_t^L$.

Equations (1) and (2) reveal that the investment sector is characterized by an (intergenerational) external spillover effect. Private human capital investment causes growth in the average stock of human capital, which increases the effectiveness of investment in education by later cohorts. Since individuals are finitely lived, the external effect is the only source of steady-state growth.[19] Growth can be sustained by continuing accumulation of the input that generates the positive externality. Since no individual

---

[16] It would be reasonable to postulate that an individual's level of human capital is a function of parental educational background. The role of quality of the home environment in human capital formation has been investigated theoretically, and empirically, by several authors (see Coleman *et al.* 1966, Becker and Tomes 1986, Bénabou 1996, Galor and Tsiddon 1997*a*, *b*). We abstract from such intergenerational linkage in human capital levels, so as to focus on technological spillovers across individual investors.

[17] That the low-level of ability must exceed unity is logically derived in footnote 18.

[18] The assumptions $A_L H_t > h_{\min}$ and $H_0 = h_{\min}$ combine to imply that $A_L > 1$.

[19] This externality may be distinguished from the conventional transmission of human capital within households in the literature (e.g. Becker and Tomes, 1986). In an overlapping generations model it has been often interpreted as intergenerational externality (e.g. Stokey, 1991; Bovenberg and van Ewijk, 1997; Hendricks, 1999). We do not assume externality in output production (e.g. Lucas, 1990), human capital production (e.g. Azariadis and Drazen, 1990) or training time among individuals within a generation (e.g. Chamley 1993; Benhabib and Perli 1994) in order to exclude the possible indeterminacy of equilibrium paths. Tamura (1990) suggests a similar spillover effect in the technology of the educational sector.

decisions affect in an appreciable way the average skill level, no one takes this effect into account when deciding whether to invest in education.

Structure of individual's life We lay out the decision-making process of a representative member of generation *t*. As mentioned, in the first period of life a choice must be made whether to enter the educational sector, and acquire human capital in excess of $h_{\min}$. If the individual decides to invest in education she must incur the cost *q*. In the absence of any initial wealth or labor income, the cost of education must be financed by borrowing in the credit market.

Denote first period's saving (borrowing) by $s_{1t} > 0 \ \ (< 0)$. Assuming that this period's consumption is not valued, the budget constraint is expressed by

$$s_{1t}^j = \begin{cases} -q & if\ invest \\ 0 & o.w. \end{cases} \qquad \forall j \in \{L, H\}, \tag{3}$$

in other words, the young borrows.

In the second period of life adults enter the labor market, supplying one unit of time inelastically.[20] We normalize units so that output produced is equal to the human capital employed. That is, the labor income of an individual with human capital h_{t+1}^j is given by

$$y_{t+1}^j = h_{t+1}^j, \tag{4}$$

where y_{t+1}^j stands for the individual wage income earned in period *t*+1. The hypothesis that income earning ability depends upon innate aptitude is consistent with evidence from the empirical literature. Griliches and Mason (1972) provide some direct evidence about the positive role of ability. The vast literature about returns to human capital supplies some indirect evidence, provided that education is positively correlated with ability [Galor & Tsiddon, 1997*a* p.365].

Let c_{2t+1}^j and s_{2t+1}^j denote respectively the second-period consumption and saving of a member of generation *t* with ability A_j, $j = L, H$. Earned income net of debt repayment is allocated between consumption and savings[21]

[20] We assume that workers do not acquire human capital through on-the-job training.

$$c^j_{2t+1} + s^j_{2t+1} = y^j_{t+1} + R\, s^j_{1t}. \tag{5}$$

Using (4) the budget constraint in the second period of life is expressed as

$$c^j_{2t+1} = h^j_{t+1} + R\, s^j_{1t} - s^j_{2t+1} \quad . \tag{5'}$$

And using (1) and (3), (5') is written as

$$c^j_{2t+1} = \begin{cases} A_j^{\delta}\, H_t^{1-\delta} - R\,q - s^j_{2t+1} & \text{if } q > 0 \\ h_{\min} - s^j_{2t+1} & \text{if } q = 0 \end{cases}. \tag{5''}$$

In the third period of life agents retire, using the entire return of savings for consumption

$$c^j_{3t+2} = R\, s^j_{2t+1}, \tag{6}$$

where c^j_{3t+2} denotes consumption in old age of a member of generation t who is of type j, $j \in \{L, H\}$.[22]

Individual's optimization problem All young agents share identical preferences, defined over consumption in the second and third period of their lives. The preferences of an individual of type j born at time t are represented by the intertemporally additive utility function

$$U^j_t = \beta \ln\left(c^j_{2t+1}\right) + (1-\beta)\ln\left(c^j_{3t+2}\right), \tag{7}$$

where U^j_t stands for the lifetime utility of a member of generation t, who is of type j. In the first period of life an agent decides whether to acquire education, a decision determining her gross lifetime income. At the same time, the individual shall decide whether to default on her debt or remain committed to her obligation. The joint decision determines the agent's *net* lifetime income, which she allocates between second- and

[21] We abstract from other forms of transferring consumption from one period to another, such as fiat money or types of storage technology. Income may be transferred from the second to third period only by lending through the financial system.

[22] Alternatively, one may assume that individuals receive in old age an endowment, ω_3, irrespective of their educational status. This may be thought of as a type of retirement income. The analysis of this case is demonstrated in Section VII.

third-period consumption so as to maximize her lifetime utility (equation 7) subject to the two budget constraints (equations 5'' and 6). There exists a unique and interior solution to the optimization problem that is expressed by

$$s_{2t+1}^{j*} = \begin{cases} (1-\beta)\left(A_j^{\delta} H_t^{1-\delta} - Rq\right) & \text{if } q > 0 \\ (1-\beta)h_{\min} & \text{if } q = 0 \end{cases} \quad \forall j = L, H \,. \tag{8}$$

Setting equation (8) into (5'') we obtain the optimal level of second-period consumption

$$c_{2t+1}^{j*} = \begin{cases} \beta\left(A_j^{\delta} H_t^{1-\delta} - Rq\right) & \text{if } q > 0 \\ \beta h_{\min} & \text{if } q = 0 \end{cases} \quad \forall j \in \{L, H\}. \tag{9}$$

Similarly, substitution of (8) into (6) yields the optimal level of third-period consumption, expressed by

$$c_{3t+2}^{j*} = \begin{cases} (1-\beta)R\left(A_j^{\delta} H_t^{1-\delta} - Rq\right) & \text{if } q > 0 \\ (1-\beta)R h_{\min} & \text{if } q = 0 \end{cases} \quad \forall j \in \{L, H\}. \tag{10}$$

Financial intermediation and contract enforcement We assume that there exists a market of financial institutions that allow individuals to trade in financial markets, as well as to obtain credit for human capital investment. Financial institutions intermediate economic activity between (adult) individuals who save to enhance third period consumption, and those who borrow. We postulate the existence of $\kappa = 1,...,K$ members within the financial system. As referred by Ljungqvist and Sargent (2004), and first espoused by Green (1987), we suppose that financial institutions have access to a capital market "outside" the economy, where they can borrow or lend at a riskless real interest rate. Individual households do not have access to this outside market, and they are prohibited from borrowing or lending with each other (Ljungqvist and Sargent, 2004). Should they engage in intertemporal trade, they must do so solely through the intermediary system. At the moment we assume that private contracts may be signed at a fixed (gross) real interest rate, R, charged on both deposits and borrowing.

We assume that young individuals, once obtaining credit they always invest in human capital. By assumption, an educational loan is not to be used in alternative ways. When adult, educated individuals have the option of going bankrupt, thus evading existing debt payments. A creditor cannot ensure that the borrower will meet her obligations.[23] In the terminology of Jaffee and Russell (1976), individuals in our model are *potentially dishonest* in the sense that when there are incentives to default they always choose to do so.[24] We shall consider that this lack of commitment in honoring a private contract is one-sided: deposit institutions by supposition always honor their promises to future payment streams. In an environment with this asymmetry in credible commitment, contracts must be *self-enforcing* in the sense that households are induced by their own self-interest to repay their creditors (Ljungqvist and Sargent, 2004).[25]

Started by the international literature on sovereign debt, invoking a punishment scheme on defaulting borrowers has seemed to be the only way out of the particular difficulties engendered by one- or two-sided lack of commitment on one's promises. In line with Kehoe and Levine (1993), the prospect of complete and permanent exclusion from the financial market is the credible threat that provides in our setting the motive for individual commitment to contract obligations. The punishment strategy following an act of default is twofold: on one hand, individuals have no access on financial assets as medium of saving. Creditors' legal rights allow them to seize the entire future savings in the debtor's possession implying that it is in the strongest interest of the latter to carry no savings in the formal financial sector. On the second hand, defaulters are denied access to new loans in the credit market. Given the structure of our model, enforcement may not be supported by long-term cutoff from further credit. It is never optimal for financial institutions to provide credit to adult individuals, since no debt is honored in the last

[23] The lender could ensure repayment of debt if workers were, for example, required to disclose information to become employed and reveal information while employed (*see* Fender and Wang 2003).

[24] Jaffee and Russell (1976) distinguish between two types of individuals, the honest, who are *pathologically* honest since they refuse to default even when there is an incentive to do so, and the dishonest that are *potentially* dishonest, since there are cases where they reveal only honest behavior [p.652].

[25] Recall that limited contractual enforcement in our setting is due to the inalienability of human capital, which cannot be seized and transferred to a creditor in the event of default. Lack of enforcement is strengthened by the assumption that existent individual endowments in old age consist of no collateral goods.

period of life (Kehoe and Levine, 1993). So, were they to default no payment could be imposed on them. This reads into the constraint

$$s_{2t+1}^{j*} \geq 0 \;\; \forall j = L, H \,, \tag{11}$$

which constitutes an *individual rationality constraint* for members of the banking system. Hence, the only individuals who may be able to borrow are the young who choose to enter the educational sector. Consequently, in our context the Kehoe and Levine enforcement mechanism reduces into the prohibition of placing savings within the formal financial system. The only cost of default is the loss of the ability to smooth consumption along the course of one's life.[26]

The full-exclusion scheme of Kehoe and Levine (1993) is structured on the ground of two critical assumptions that concern the legal rights of financial institutions, and the interrelationships among them. The ability to prohibit a borrower who repudiates on her debt from actively participating in the formal financial market rests on the implicit existence of a legal entity (legislation and judicial system), capable to observe any such trade actions and confiscate the relevant net payments. In the Kehoe and Levine model the economy's legislative system grants creditors the power to appropriate assets in the debtors' possession, and preclude their access to future contingent claims markets.[27] Extensive legal power of lenders is a necessary prerequisite to support a decentralized allocation in a structure of partial commitment. We should remark that the existence of a competitive equilibrium is established in an environment that abstracts from issues of

[26] The fact that in our environment the full-exclusion scheme is reducible to a simpler form is not material for the mechanism's effectiveness in punishing a defaulting borrower. The vital component of the Kehoe and Levine scheme is that agents are deprived of the opportunity to invest their savings. It has been shown by Bulow and Rogoff (1989), and later been confirmed by more contemporary studies (see Bond and Krishnamurthy, 2004) that the sole elimination to a borrower on default status of his right to further credit, has no impact on his incentive to remain loyal at first place. If individuals have access to a savings market irrespective of their contract commitment, the optimal loan size for banks is zero. We ought to make the remark that this proposition is guaranteed only insofar specific conditions are assumed.

[27] From the standpoint of the authors, individuals may not be barred from trade in spot markets, nor can their endowments be taxed due to unfulfilled obligations on private contracts. The view is justified on the basis that trades in spot markets are anonymous, and that there may not be a physical separation of private endowments from individual owners. At the same time, agents must identify themselves to make contracts and to collect on them in contingent claims markets. Therefore, creditors may seize the assets of a debtor who defaults on her debt, and may keep track of any future attempts of hers to enter contingent claims markets. As a consequence, they can exclude the borrower from engaging in intertemporal transactions, while tax her individual assets [Kehoe and Levine, 1993 p.869].

competition among financial institutions. Kehoe and Levine presuppose that members of the banking system form a stringent coalition, thus fully coordinating their decisions. [28] In like manner, we restrict our attention to the case of no competition within the financial market. For the sake of simplicity, we assume without loss that the borrower deals with a single bank, interpreted to represent the entity of financial sector.[29]

While we recognize the widely accepted status of the Kehoe and Levine arrangement in studies with contract design problems, we must make a remark on the criticism carried by Bond and Krishnamurthy (2004). The authors dealt thoroughly with the minimal practical value of the scheme, reasoning that the full-exclusion enforcement rule is hard to identify in observed institutions, or resemble laws governing borrowers' defaulting on debt and declaring bankruptcy [pp.691-2]. We argue that only an aspect of this criticism cherishes validity in our context, and that the Kehoe and Levine scheme is a good representation of the actual legal doctrine regulating educational loan markets. In connection with any type of non-educational credit, it is indeed the case that the *permanent* character of prohibitions imposed on delinquent borrowers is too stringent an assumption in keeping with a plausible theory. It is common to most if not all legislative systems that an individual is entitled to declaring bankruptcy on her debt, thus claiming an opportunity to a "fresh start". However, the ability to have one's debt waived does not extend in the area of educational loans. An individual debtor continues to be liable for all obligations on student loans even after her claim to bankruptcy is successfully pursued.[30] These debts are legally discharged only in the event of full repayment. The creditors' legal right to confiscate the private assets of the debtor being bound by obligation establishes the empirical validity of the full-exclusion scheme in the area of educational investment.

[28] The authors say nothing about how this coordination actually occurs. Formal consideration of the competition within the financial sector has been carried by Bond and Krishnamurthy (2004), in an analysis which in the main contributes a critic on the Kehoe and Levine absolute-exclusion scheme.

[29] The conclusions of the model are the same whether one assumes that the borrower deals with a representative bank, or whether the entire banking sector behaves as a coalition (monopolist). Bond and Krishnamurthy (2004) present an elaborate study on salient features of the Kehoe and Levine scheme.

[30] The US Bankruptcy Code does not exempt individuals from their obligation on educational debt. The legal doctrine was formalized in the United States in 1978. Subsequent amendments of legislation (the last passed on in 2005) extended the array of educational loans covering all credit aiming to higher education, funded by governmental or private source. Similarly, in the United Kingdom a debtor remains bound by her obligations on government educational loans after the claim to bankruptcy.

The second aspect of criticism concerns the Arrow-Debreu trading arrangement of complete markets. In the Kehoe and Levine structure competitive markets meet at date 0 to trade claims to consumption at all times $t > 0$, that are contingent on all possible realizations of events up to t. In this respect, Bond and Krishnamurthy (2004) posit that the implementation of debt constraints is in fact complex state- and date-contingent specifications of payments [p.691]. The crux of the criticism is that the computation of this system of payments being highly information intensive amounts to an insuperable task.[31] In the words of Ljungqvist and Sargent (2004) "…we are assigning a very demanding task to the *invisible hand* who must not only look for market-clearing prices but also check participation constraints for all agents and all states of the world" [p.740]. The acceptance of this criticism clearly does not lend power to the practical value of our theoretical construct.

III. *Equilibrium*

Poverty trap We have claimed that conditional on default, an individual may be barred forever from asset trading in financial markets. As the preceding analysis brings out, the character of such prohibition is defined by the law surrounding the creditors' right to loan repayment. When legislation supports absolute and permanent consequences upon default, as in the Kehoe and Levine scheme, we say that lenders come out with a strong legal position (or else *strong legal rights*). The legislative system entitles them to full loan repayment in every circumstance; a situation we stated describes educational credit in well-advanced financial systems. In ever weaker positions, full loan repayment may not be enforced; a situation we here term *weak legal rights*. Practically this is true when debt for higher education may be discharged after declaring bankruptcy, and when unsecured educational loans gain low priority in the event of property liquidation. As is generally supposed, the power to which creditors are potentially entitled has a critical effect on their readiness to finance investment (La Porta, *et al*, 1998). We postulate that when the legal system offers little or no protection to lenders, failing on loan repayment

[31] Reciting Bond and Krishnamurthy (2004) "… the computation procedure requires the central (judicial) authority to possess knowledge on agents' production and consumption possibilities; knowledge that is unknown how it could be obtained in a decentralized competitive environment." [p.691].

is in essence accompanied by no penalty. As a consequence, strategic default may well be expected, making credit institutions least eager to engage in loan financing. The optimal response is to deny the provision of any loan, resulting in an extreme form of credit rationing; an actual feature of educational credit markets in most economies of the world.

Weak legal protection on the account of creditors impedes the development of a private educational loan market leading to a competitive equilibrium characterized by no investment in higher education. In a setting where growth hinges on the accumulation of human capital economic development must then come to a halt. This is self-evident in a system where education may not be otherwise socially provided, an assumption that has been postulated at the outset of our analysis.

Given that a member of generation t receives education her intertemporal consumption if she chooses to default is described by the following equations

$$\left(c_{2t+1}^{j*}\right)^{WR,D} = \beta\, A_j^{\delta}\, H_t^{1-\delta} \qquad q>0,\ \forall\, j\in\{L,H\},\ \forall\, t\geq 0\,, \tag{12}$$

and

$$\left(c_{3t+2}^{j*}\right)^{WR,D} = (1-\beta)R\ A_j^{\delta}\, H_t^{1-\delta} \qquad q>0,\ \forall\, j\in\{L,H\},\ \forall\, t\geq 0\,, \tag{13}$$

where WR (SR) refers to a system of weak (strong) legal rights on creditors' account, and D (ND) stands for default (no-default) respectively. When remaining loyal to contract commitment optimal adult- and old-age consumption is given by, respectively

$$\left(c_{2t+1}^{j*}\right)^{ND} = \beta\left(A_j\, H_t - R\,q\right) \qquad q>0, \forall j\in\{L,H\}, \forall\, t\geq 0\,, \tag{14}$$

and

$$\left(c_{3t+2}^{j*}\right)^{ND} = (1-\beta)R\left(A_j\, H_t - R\,q\right) \qquad q>0, \forall\, j\in\{L,H\}, \forall t\geq 0\,. \tag{15}$$

where $\left(c_{\nu\, t+1}^{j*}\right)^{WR,ND} = \left(c_{\nu\, t+1}^{j*}\right)^{SR,ND} \equiv \left(c_{\nu\, t+1}^{j*}\right)^{ND}$ for $\nu = 2, 3$.

Evidently, utility is higher when evading debt obligations due to higher second-period consumption, and because agents can still engage in intertemporal smoothing through saving. Hence,

$$V_t^{WR,D}(j) > V_t^{WR,ND}(j) \qquad q > 0, \forall j \in \{L, H\}, \forall t \geq 0\,, \tag{16}$$

where

$$V_t^{WR,D}(j) = \ln\left\{\gamma A_j^{\delta} H_t^{1-\delta}\right\}, \quad q > 0, \forall j \in \{L, H\}, \forall t \geq 0\,, \tag{17}$$

and

$$V_t^{E,ND}(j) = \ln\left\{\gamma A_j^{\delta} H_t^{1-\delta} - Rq\right\} \quad q > 0, \forall j \in \{L, H\}, \forall t \geq 0\,, \tag{18}$$

with $V_t(\cdot)$ standing for the indirect utility function of a member of cohort *t*, while *E* denoting that the individual has received education when young. It obviously applies that $V_t^{WR,ND}(j) = V_t^{SR,ND}(j) \equiv V_t^{E,ND}(j) \quad \forall j = L, H$, while $\gamma \equiv \beta^{\beta}(1-\beta)^{1-\beta} R^{1-\beta}$. We come to the conclusion that an individual who acquires education shall always commit default on her debt.

Were one to receive no education she would earn the unskilled income $h_{\min}$, hence the lifetime utility

$$V_t^{NE}(j) \equiv V_t^{NE} = \ln\{\gamma h_{\min}\} \quad \forall j \in \{L, H\}. \tag{19}$$

Drawing on *Assumption* 1 we infer that remaining unskilled is never the preferred choice. It is evident that

$$V_t^{WR,D}(j) > V_t^{NE} \qquad \forall j \in \{L, H\}, \forall t \geq 0\,. \tag{20}$$

Obtaining credit, although desirable, in no circumstance is feasible. Due to the certainty of default behavior it is individually rational for members of the financial system to deny the provision of any loan.

Owing to the rationing of all credit, the total measure of population remains uneducated earning the minimum income of unskilled labor. On the assumption that a system of weak legal rights prevails in each period of the time interval $t \in [0, r-1]$ $r > 0$, the equilibrium path has the characteristics

$$h_{t+1}^{j} = h_{\min} \quad \forall j \in \{L, H\}, \forall t \in [0, r-1], \tag{21}$$

and

$$y_{t+1}^{j} = h_{\min} \qquad \forall j \in \{L, H\}, \forall t \in [0, r-1]. \tag{22}$$

The competitive outcome along this equilibrium path prescribes that the economy produces the time-invariant quantity

$$Y_{t+1}^{NE} = H_{t+1}^{NE} = h_{\min}, \tag{23}$$

where at the outset we postulated $y_0^j \equiv y_0 = h_{\min}, \forall j = L, H$, thus $Y_0 = h_{\min}$. The economy here is void of any growth, with production merely contributing to sustain the starting level of effective labor, and output.

Potential for growth: A fraction of population invests We proceed with defining equilibrium paths along which the potential for ever sustained growth is realized within the structure of this model. We consider two such equilibria, so constructed as to give an explicit proof of the *trickle-down* benefits of economic growth. As our basis, an equilibrium is taken to exist in which a particular group of individuals, those with the highest investment return, can only choose to engage in education. Owing to the accumulation of human capital and the associated externality on future generations' productivity, the economy reaches the state where the aggregate of all agents invest in individual improvement. The financial sector, eager to support educational decisions of all and any prospects, carries the economy on a new dynamic path on its way to development. With reference to existence issues, we shall stress that the critical requirement for perpetual growth is the provision of strong legal protection for creditors, should contract obligations be violated. Even though growth is an endogenous outcome in our model, its manifestation ultimately hinges on the ability of lenders to enforce loan repayment by imposing financial consequences on those who default. We take as our basis that effective on period $t = r$, $r > 1$, legislation entitles creditors to claim full loan repayment from delinquent borrowers (case of *strong legal rights*). Lenders may seize the entire assets of a debtor in default, effectively prohibiting the latter from any act of saving as time unfolds.

We introduce a formal definition of the postulated heterogeneity at an individual income level. We recall that individuals are distinguished on the level of their innate aptitude, and consequently their rate of return from education. Our supposition is that there exist two types of agents in the economy, those with *high* and *low ability* respectively. This is chosen to be our benchmark case, which we subsequently augment to allow for a (countable) infinite number of types. We employ the definition:

Definition 1 An agent born in time period t is said to be of *high type* if her (gross) return from education can support her honoring of debt obligations. Accordingly, *low type* agents are discerned by an earned income as low as not in the least covering loan repayment. This definition reads into the postulates

Assumption 2

$$A_H^{\delta}\, H_t^{1-\delta} > R\,q \qquad \forall t \geq 0 . \qquad (\alpha)$$

$$A_L^{\delta}\, H_t^{1-\delta} < R\,q \qquad \forall t \geq 0 . \qquad (\beta)$$

The meaning ascribed to the employed distinction thus has to do with the feasibility in carrying out one's contract commitments. Assumption (2β) is read to mean that investment in human capital does not pay off if one is to remain committed to her debt liability. Were a low-type agent to obtain credit, she would always default on her obligations, however honest in intention.

We remark here that while an individual knows her type when deciding whether to invest in education, innate ability has nevertheless an unobservable quality. The private information of one's own type is not to be publicly revealed, or otherwise obtained by a credit institution. The fact that individuals cannot be identified on the basis of their expected rate of return on education brings about the constraint that borrowers choose to conform to contract arrangements in equilibrium.[32] Such incentive is assuredly instilled by the stringent nature of consequences of our punishment scheme. With innate ability being non identifiable, the only possibility to deviate from an equilibrium outcome with rationing to everyone who applies for credit is with *self-selection* of individuals to different choices. Carrying the analysis in formalized language we introduce the following definition:

[32] It is to be noted that the value of innate ability for each type, hence educational productivities are known to moneylenders. The latter have knowledge of the feasibility constraints, as expressed by *Assumption* 2α, β. However, potential borrowers may not be discerned on the basis of their individual type. Innate aptitude is non-identifiable, thus rendering the explicit rationing of low-type agents impossible.

Definition 2 A contract is said to be *self-enforcing* if it is individually optimal for borrowers to conform to prior arrangements at every date and contingency.[33]

It occurs that within our context, the contract design with the aforementioned absolute consequences following default elicits only promise-keeping behavior. The present value of (indirect) utility associated with the consumption stream after repudiating on one's debt is given by

$$V_t^{SR,D}(j) = \beta \ln\left\{A_j^{\delta} H_t^{1-\delta}\right\} + (1-\beta)\ln\{0\} \to -\infty \qquad \forall j = L, H . \tag{24}$$

There results, consequently, that

$$V_t^{E,ND}(j) > V_t^{SR,D}(j) \qquad for\ j = H , \tag{25}$$

where the lifetime (indirect) utility when adhering to the contract is given by (18).

The high ability agents choose to obtain education and repay their debt for the reason that default is too costly. Condition (25) cannot possibly hold for low-type agents since the logarithmic function $V^{E,ND}(j)$ is non-definable on a negative argument (*low* income realization does not enable debt repayment). Insofar as the only possibility is to renege on the agreement, the household attains the lowest utility level associated with no consumption smoothing. $V^{E,ND}(L)$ in effect degenerates to the lowest value of individual welfare $V^{SR,D}(j) \to -\infty,\ \forall j = L, H$. Consequently, we quote the proposition:

Proposition 1 The contract arrangement (q, R) offered in a system where borrowers have no access to savings opportunities conditional on default is a self-enforcing contract. Given the feasibility of loan repayment, it is never optimal to renege on agreed obligations.

Further, we may prove without difficulty

Proposition 2 A private credit market for human capital investment is *sustainable* if and only if

[33] The definition is borrowed from Ljungqvist and Sargent (2004, p.640).

- Given the feasibility of loan repayment agents seeking credit are offered a self-enforcing contract.
- Individuals for whom debt repayment is non-feasible prefer to receive no education.

Proof

We have established that high-ability agents, once obtaining credit always adhere to the contract agreement. The second necessary condition requires that the low ability agents choose (optimally) to remain unskilled. For individuals of this type, investment in human capital comes at the cost of sacrificing the opportunity to smooth consumption. Therefore, it holds true that they prefer earning the low income of unskilled labor, while maintaining their ability to ensure their old age consumption via saving in tangible assets. Upon invoking that $V^{E,ND}(L) \rightarrow V^{SR,D}(L) \rightarrow -\infty$, and recalling expression (19) $V_t^{NE} = \ln\{\gamma h_{\min}\} > 0$, it is trivially proved that low-talented individuals indeed prefer to seek no education.[34] Within the present context on the basis of the postulate that last-period consumption consists exclusively of previous savings, the postulated definitions of agent types (*Assumption* 2), and with the use of the logarithmic utility function (7) we have formally arrived at the sentence that the credit educational market is privately sustainable.‖

We conclude with the following proposition

Proposition 3 A competitive equilibrium with a subset of the population acquiring privately financed education exists on the condition of occurrence of the following requirements

- A positive measure of individuals choose optimally to obtain education.
- The credit market is sustainable.
- Savings be non-negative for both H and L types.

Proof

- The first condition constitutes the *participation constraint* for the borrower side. The high-ability agent can always guarantee herself the present value of utility V^{NE} by

[34] The proof is obviously weakened as soon as we remove the assumption of last-period consumption being exhaustively determined of prior savings. This case is presented in section VII.

supplying unskilled labor. The contract must offer her at least this utility level. Therefore, it must be satisfied that

$$V_t^{E,ND}(H) > V_t^{NE}, \tag{26}$$

which, drawing upon equations (18) and (19), applies if and only if

$$A_H^{\delta} H_t^{1-\delta} > R\,q + h_{\min}. \tag{26'}$$

Expression (26') is also referred as the *individual rationality constraint* for the high-type agent (IR_H). To ensure the validity of *Assumption* 2*α* and of condition (26') it suffices to impose their intersection, which is expressed by the latter relationship. We are certain of the truth of (26') in all periods $t \geq r$ given that

$$H_r = h_{\min}, \tag{27}$$

and the constraint applying solely in period $t = r$

$$A_H^{\delta} h_{\min}^{1-\delta} > R\,q + h_{\min}. \tag{26''}$$

- Drawing on *Proposition* 2 we transfer the conclusion that the postulated definitions of agents types (*Assumption* 2) suffice as proof of the sentence that the educational credit market is privately sustainable.
- The last statement of *Proposition* 3 imposes the individual rationality constraints for the banking system, as expressed by relationships (11). Invoking equation (8) we obtain

$$s_{2t+1}^{H*} = (1-\beta)\left(A_H^{\delta} H_t^{1-\delta} - R\,q\right) > 0 \qquad \forall t \geq 0, \tag{28}$$

which is strictly positive on the basis of the definition of the high-type agent (*Assumption* 2*α*). With respect to the low-talented individuals saving is represented by

$$s_{2t+1}^{L*} = (1-\beta)\,h_{\min} > 0 \qquad \forall t \geq 0. \tag{29}$$

The proof of *Proposition* 3 consists basically of imposing *Assumption* 2*β* along with constraint (26''). The validation of the remaining relationships is inferred by means of logical reasoning.║

The dynamic evolution of the society's stock of human capital along this equilibrium path is governed by the first order non-linear difference equation

$$H_{t+1} = (1-\lambda)h_{\min} + \lambda A_H^{\delta} H_t^{1-\delta} \qquad \forall t > r\,. \tag{30}$$

The solution to the linear difference equation $(i.e.\ \delta = 1)$ characterizes the current stock of knowledge as a function of society's historically given H_0, the level of human capital of unskilled labor, and the ability as well as the measure of those who invest. Having presupposed $H_0 = h_{\min}$, the solution is described by

$$H_t = \begin{cases} h_{\min}\left[\dfrac{(1-\lambda)+\lambda(\lambda A_H)^t(1-A_H)}{1-\lambda A_H}\right] & if\ \lambda A_H \neq 1 \\ [1+(1-\lambda)t]\, h_{\min}, & if\ \lambda A_H = 1 \end{cases} \qquad \forall\, t \geq 1\,. \tag{31}$$

It is easily observed that in the case of $\lambda A_H \neq 1$ the numerator of the term in brackets may most likely be negative. This calls to impose the relation $1-\lambda A_H < 0$, or equivalently $A_H > 1/\lambda$. Evidently, this is stricter compared to the initial assumption $A_H > 1$.

Sustained growth carried by the entire population The equilibrium we described involves a constant fraction of each generation (the measure of high-type agents) acquiring education, and earning income $A_H^{\delta} H_t^{1-\delta}$. The remaining population chooses to remain unskilled and earn the minimum income level $h_{\min}$. As long as the validity of *Assumption* 2β and constraint (26'') is ensured, the composition of the labor force between educated and uneducated individuals is analogous to the fraction of each generation being genetically of high aptitude. The latter is established *a priori* to be a stationary variable across all time periods.

It has been an initial motivation to prove the existence of possibility that the educational status is affected dynamically as the economy evolves along the path of perpetual unbounded growth. Within the structure of this model, sustained growth carries the potential that the measure of population who find it optimal to invest in education changes endogenously. We prove that due to perpetual growth the stock of aggregate knowledge reaches a threshold level above which individuals of the low type as well choose optimally to invest in the acquisition of human capital. We establish specific generality of this result by running through the case of linear human capital technology, $\left(i.e.\ \delta = 1\right)$. We do not take the foregoing proof beyond the linear case due to the particular complexity in solving non-linear difference equations in abstract form.

The theorem states that

Proposition 4 There exists a time period $\tau > 0$, where $\tau \in \left[r+1, \infty\right)$, in which the income realization of educated low-type agents exceeds the threshold level that defines education the optimal choice. In other words,

$$A_L^{\delta} H_t^{1-\delta} \begin{cases} \geq R\,q + h_{\min} & \forall\, t \in \left[\tau, \infty\right) \\ < R\,q + h_{\min} & \forall\, t \in \left[0, \tau - 1\right] \end{cases}. \tag{32}$$

Considering the linear human capital technology, τ is defined as

$$\tau = \begin{cases} \dfrac{\ln\left[\Theta\right]}{\ln\left[\lambda A_H\right]}, & if\ \lambda A_H > 1 \\ \dfrac{Rq + h_{\min}\left(1 - A_L\right)}{A_L h_{\min}\left(1 - \lambda\right)}, & if\ \lambda A_H = 1 \end{cases}, \tag{33}$$

with $\Theta \equiv \dfrac{\left(Rq + h_{\min}\right)\left(1 - \lambda A_H\right) - A_L h_{\min}\left(1 - \lambda\right)}{A_L h_{\min}\, \lambda \left(1 - A_H\right)}$.

Proof

Relationship (32) written in linear form yields

$$A_L H_t \geq Rq + h_{\min}\,. \tag{34}$$

We use the solution of the aggregate human capital stock as given by (31), to substitute for H_t. Equation (33) is then derived in a straightforward manner.

We can prove the existence of time period τ only on the condition that the latter is greater to unity. More precisely, the theorem is true if and only if

- *Case* $\lambda A_H > 1$

$$\Theta > \lambda A_H , \tag{35}$$

which leads to a standard second-order polynomial

$$\left(\lambda^2 A_L h_{\min}\right)A_H^2 - \lambda\left[Rq + h_{\min}\left(1+\lambda A_L\right)\right]A_H + Rq + h_{\min}\left[1 - A_L\left(1-\lambda\right)\right] > 0 . \tag{35'}$$

The expression is positive insofar as, either

$$A_H < \frac{Rq + h_{\min}\left(1+\lambda A_L\right) - \sqrt{\Omega}}{2\lambda A_L h_{\min}} \equiv A_H^1 , \tag{36α}$$

or,

$$A_H > \frac{Rq + h_{\min}\left(1+\lambda A_L\right) + \sqrt{\Omega}}{2\lambda A_L h_{\min}} \equiv A_H^2 , \tag{36β}$$

where we note that $\Omega \equiv \left[Rq + h_{\min}\left(1+\lambda A_L\right)\right]^2 - 4A_L h_{\min}\left[Rq + h_{\min}\left(1 - A_L\left(1-\lambda\right)\right)\right] \geq 0$, and $A_H^1 > 0$. Being more intuitive plausible, we choose to employ condition (38*β*).

- *Case* $\lambda A_H = 1$

The condition $\tau > 1$ implies that

$$A_L < \frac{Rq + h_{\min}}{\left(2-\lambda\right)h_{\min}} . \tag{37}$$

The right-hand side exceeds unity, as is required, given the imposition of

$$Rq > \left(1-\lambda\right)h_{\min} . \tag{38}$$ ‖

We proceed to construct the mathematical conditions framing the equilibrium path along the interval $t \in \left[\tau, \infty\right)$. Upon noting that condition (26) is now met for both types of agents

$$V_t^{E,ND}\left(j\right) > V_t^{SR,D}\left(j\right) \qquad \textit{for } j = L, H ,\ \forall t \in \left[\tau, \infty\right). \tag{39}$$

Proposition 2 is rephrased to read as follows

Proposition 5 A private credit market for human capital investment is *sustainable* in the time interval $t \in [\tau, \infty)$ if and only if agents seeking credit are always offered a self-enforcing contract.

Proof

We have proved in the previous section that in this context with last-period consumption being exclusively determined by one's savings, the postulated feasibility of carrying out loan repayment, the use of a logarithmic utility function, and creditors backed by a system of strong legal rights, the contract arrangement (q, R) is self-enforcing (see *Proposition* 1). It follows that it takes only to impose the feasibility conditions for the two types to ensure that the educational credit market is privately sustainable. But feasibility is in fact established by *Proposition* 4 from period τ onward for both types. It follows that the proof of *Proposition* 5 entails the validity of[35]

$$A_L^{\delta} H_t^{1-\delta} > R\,q + h_{\min} \qquad \forall\, t \in [\tau, \infty), \tag{32'}$$

which is established as

$$A_L H_t \geq Rq + h_{\min}\,. \qquad \forall\, t \in [\tau, \infty). \tag{34'}$$ ‖

Corollary 1 The private credit market for educational investment is *sustainable* in each and all time periods of the interval $t \in [\tau, \infty)$ given the validity of condition (34').

The proof of existence of the equilibrium in which the entire population invests in education is enclosed in the following proposition

Proposition 6 A competitive equilibrium where the entire population acquires privately financed education exists on the condition of occurrence of the following requirements

- All types choose optimally to invest in individual improvement.
- The credit market is privately sustainable.
- Individual saving be non-negative for both H and L types.

[35] It is only apparent that imposing the feasibility constraint for the low-type is sufficient to establish the analogous constraint for the high-ability agent.

Proof

- We recall that the first condition constitutes the *participation constraint* for the borrower side. It is optimal to obtain education if and only if

$$V_t^{E,ND}(j) > V_t^{NE} \qquad \forall\, j = L, H \,, \tag{40}$$

which equivalently states

$$A_j^{\delta} H_t^{1-\delta} > R\,q + h_{\min} \qquad \forall\, j = L, H \,. \tag{40'}$$

It is simply evident that imposing the individual rationality constraint for the low-type is sufficient to establish the analogous constraint for the high-ability agent. *Proposition* 4 establishes the validity of optimality condition (34') in the time interval $t \in [\tau, \infty)$, for the case of linear human capital technology.

- The second condition of the theorem involves the requirement that the credit market be sustainable. In light of *Corollary* 1 we recall that sustainability is established upon the validity of the optimality condition (34').
- In connection with the last condition, we impose the individual rationality constraints for the banking system, entailing that individual saving be positive for all types of agents. Invoking equation (8) we have

$$s_{2t+1}^{j*} = (1-\beta)\left(A_j^{\delta} H_t^{1-\delta} - R\,q\right) > 0 \qquad \forall\, j \in \{L, H\}. \tag{41}$$

In analogy with our previous reasoning, we need only establish that saving be positive for the low-type agent. Evidently, this is already met in linear form under constraint (34').

It follows that the sole thing we must postulate to establish *Proposition* 6 for the case of the linear human capital technology is the optimality condition (34'). The truth of the remaining relationships is then logically inferred.║

The dynamic evolution of the economy's aggregate stock of knowledge is governed by the first order non-linear difference equation:

$$H_{t+1} = \overline{A}\, H_t^{1-\delta} \qquad \forall \in [\tau, \infty), \tag{42}$$

where $\overline{A} \equiv (1-\lambda) A_L^{\delta} + \lambda A_H^{\delta}$. In the linear case $(i.e.\ \delta = 1)$ the solution to the difference equation is expressed by

$$H_t = \begin{cases} B^t H_\tau, & if\ B \neq 1 \\ H_\tau, & if\ B = 1 \end{cases} \qquad \forall t \geq \tau + 1 \tag{43}$$

where $B \equiv \lambda\, A_H + (1-\lambda)\, A_L$. The aggregate human capital H_τ is governed by equations (33).

IV. *Income distribution*

It has been the aim of this study to establish an analytic basis for the factual evidence lending truth to the Kuznets hypothesis. We show in the present section that this empirical phenomenon is proved within the theory, and is thus validated on the ground of acceptance of a mathematical proposition. Our proof procedure rests on the concept of *Lorenz ordering*, a notion which has been formally introduced by Chatterjee and Ravikumar (1999).

Following Kuznets (1955) we define the *income share* of a particular segment of society as the ratio of real per capita income within the specific group over the corresponding average for the entire population. In our context, individuals fall into two groups, with all agents within a class earning identical income. Therefore, it becomes relevant to specify the respective shares of the two distinct income groups (equivalently of the representative low- and high-type agents). The income share of group *j* at time period *t* is defined as $sh_t^j \equiv y_t^j / Y_t$, $\forall j \in \{L, H\}$. The economy-wide distribution is represented by the set of individual shares of all income classes. We define $\mathbf{sh}_{t+1}^{\zeta} = \{sh_{t+1}^{L\,\zeta}, sh_{t+1}^{H\,\zeta}\}$ for all $t \geq 0$, where ζ denotes the equilibrium type, $\zeta \in \{I, II, III\}$. Excerpted by Chatterjee and Ravikumar (1999) the definition of *Lorenz superiority* reads as follows

Definition 3 The low- and high-income groups (accordingly the entire measure of individuals) are arranged in a form of increasing order. Let there be two different economy-wide distributions of income shares, represented by $\mathbf{sh}$ and $\mathbf{sh}'$ respectively. It is said that **sh** is Lorenz superior to, or Lorenz dominates distribution $\mathbf{sh}'$ if $\sum_{j=L}^{\mu} \lambda_j sh_t'^{j} \le \sum_{j=L}^{\mu} \lambda_j sh_t^{j}$ for all $\mu \in \{L, H\}$ and $t \in [0, \infty)$, with the inequality holding strictly for at least one μ.

The underlying logic of the notion of Lorenz superiority is that the distribution possessing this property exhibits lower inequality compared to any other income distribution. As is further brought out by Chatterjee and Ravikumar (1999), a Lorenz superior distribution is consistently ascribed a higher degree of equality by each and every conventionally used measure of inequality, such as the Gini coefficient, the coefficient of variation and the standard deviation of the logarithms.

We proceed to establish that the economy-wide income distribution in the state of poverty (equilibrium of type-*I*) Lorenz dominates the income distribution along the equilibrium path on which the economy develops due a segment of population engaging in human capital accumulation (type-*II* equilibrium). Subsequently, we demonstrate that the income distribution along the last stage of development, where the entire population participates in human capital accumulation (denoted equilibrium type-*III*), is Lorenz superior to the corresponding distribution of the precedent phase (equilibrium type-*II*).

Invoking the proposed definition by Kuznets (1955), and equations describing individual and average income, we obtain the following expression for the share of each income class in the poverty equilibrium

$$sh_{t+1}^{j\,I} = 1 \quad \forall j \in \{L, H\}, \forall t \in [0, r-1]. \tag{44}$$

In the phase of underdevelopment individuals of all types earn per capita income equal to the economy-wide average, $h_{\min}$. Genetic differences in learning aptitude *vanish* in the sense that they are not reflected in the income earning ability of agents. Absent heterogeneity in educational status, innate differences do not matter. The potential for differing earning productivities remains unrealized, with all workers being *trapped* in the choice of a single occupation, and therefore identical earnings. We call attention that perfect equality is an *endogenous* outcome in this context, caused by a *deficiency* in the economy's legislative system, namely the provision of insufficient legal power to financial institutions when faced with the possibility of default. Extreme credit rationing to educational investment, hence a missing credit market is an optimal response to a lack of commitment problem.

We have presupposed that adaptations in the legislative and judicial systems to accommodate *strong* legal protection of creditors are effective on period $t = r$, a date taken to be given exogenously.[36] The building of such *infrastructure* suffices to carry the economy out of its low income trap.[37] Along the growth path following such development, the different classes of agents earn respectively the income shares

$$sh_{t+1}^{L\,II} = h_{\min} \,/\, \left\{(1-\lambda)h_{\min} + \lambda\, A_H^{\delta}\, H_t^{II\,1-\delta}\right\} \qquad \forall t \in [r, \tau - 1] \;. \tag{45α}$$

$$sh_{t+1}^{H\,II} = A_H^{\delta}\, H_t^{II\,1-\delta} \,/\, \left\{(1-\lambda)h_{\min} + \lambda\, A_H^{\delta}\, H_t^{II\,1-\delta}\right\} \qquad \forall t \in [r, \tau - 1]. \tag{45β}$$

Logically, the high-type agents represent the rich class earning the higher income share in the labor force. Applying the criterion of Lorenz superiority, we obtain that the income distribution $\mathbf{sh}_{t+1}^{I}$ Lorenz dominates the distribution $\mathbf{sh}_{t+1}^{II}$ under the condition that the following requirements be met

- $(1-\lambda)sh_{t+1}^{L\,II} \le (1-\lambda)sh_{t+1}^{L\,I}$, (46)

implying

$$h_{\min} \le A_H^{\delta}\, H_t^{II\,1-\delta} \qquad \forall t \ge r\,, \tag{47}$$

and,

- $(1-\lambda)sh_{t+1}^{L\,II} + \lambda\, sh_{t+1}^{H\,II} \le (1-\lambda)sh_{t+1}^{L\,I} + \lambda\, sh_{t+1}^{H\,I}$, (48)

whose elementary validation can be easily proved (each side equals unity). We are certain of the truth of relationship (47) given our assumption that the income of educated individuals may not to be exceeded by (or be equal to) the earnings of unskilled labor

[36] We recall that the *switch* takes place in time period $t = r$, with this being the first period in which high-ability agents may invest in education. Since the return on education is realized on the subsequent date, $t = r + 1$ is the first relevant period for computing the income shares of the development stage *II*.

[37] Absent a system of public education, this is also the sole way to guide the economy out of stagnation.

(*Assumption* 1).[38] We conclude that distribution $\mathbf{sh}_{t+1}^{I}$ is Lorenz superior to the income distribution $\mathbf{sh}_{t+1}^{II}$ on the condition of *Assumption* 1.

The logic underlying the conditions of Lorenz dominance relates to the growth pattern of the income shares of economic classes. We have established that the share of the group of low-type agents is smaller in each period of the interval $t \in [r, \tau - 1]$ compared to the group's share in dates of the stagnant equilibrium (relationship 46). Conversely, the income share of the high-type class is greater when the latter are able to invest in education compared to when they are constrained not to.[39] It immediately follows that condition (48) may be true only if the growth rate of the share of low-type class *upon transition* be larger (in absolute magnitude) to the respective rate of the share of talented ones. It is simple enough to show that this is in fact the case upon *Assumption* 1 being true. We have

$$g_T^{II}\left(sh_{t+1}^{L}\right) = \frac{\lambda\left(h_{\min} - A_H^{\delta}\, H_t^{II\,1-\delta}\right)}{(1-\lambda)h_{\min} + \lambda\, A_H^{\delta}\, H_t^{II\,1-\delta}} < 0\,, \tag{49}$$

and

$$g_T^{II}\left(sh_{t+1}^{H}\right) = \frac{(1-\lambda)\left(A_H^{\delta}\, H_t^{II\,1-\delta} - h_{\min}\right)}{(1-\lambda)h_{\min} + \lambda\, A_H^{\delta}\, H_t^{II\,1-\delta}} > 0\,, \tag{50}$$

established to have a negative, and positive sign respectively on the basis of *Assumption* 1. In our chosen notation, $g_T^{\varsigma}\left(sh_{t+1}^{j}\right)$ denotes the rate of change of the income share of type-*j* agents as the economy attains equilibrium path $\varsigma \in \{II, III\}$ (*T* stands for transition). We write $g_T^{II}\left(sh_{t+1}^{j}\right) \equiv \left(sh_{t+1}^{j\,II} - sh_t^{j\,I}\right) / sh_t^{j\,I}$, where by definition $t \equiv r$. Low-ability individuals become *relatively* poorer in the transition to a higher stage of development, experiencing their income share to shrink. In addition, as the potential of high-aptitude agents has the opportunity to materialize, this class becomes richer in the economy's escape of poverty. We may easily prove that

[38] We note that under *Assumption* 1 relationship (46) holds as strict inequality only. Hence, all conditions are met for ascribing $\mathbf{sh}_{t+1}^{I}$ the property of Lorenz dominance on distribution $\mathbf{sh}_{t+1}^{II}$.

[39] This may be shown to be true upon inequality (47), hence on the already imposed *Assumption* 1.

$$\left(1-\lambda\right)g_{T}^{II}\left(sh_{t+1}^{L}\right) > \lambda\left| g_{T}^{II}\left(sh_{t+1}^{H}\right)\right| \quad , \tag{51}$$

which holds upon *Assumption* 1 being true. The poor become poorer at a faster rate than the income of the rich is amplified. It is logically deduced that

$$\frac{y_{t+1}^{H\,II}}{y_{t+1}^{L\,II}} > \frac{y_{t}^{H\,I}}{y_{t}^{L\,I}} = 1\,, \tag{52}$$

where $y_{t+1}^{j\,II}$, $j \in \{L,H\}$, is evaluated on the interval of date $t=r$, while $y_{t+1}^{j\,I}$ evaluated on $t \in [0, r-1]$ has a constant value. It is only evident that $sh_{t+1}^{H\,\zeta} / sh_{t+1}^{L\,\zeta} \equiv y_{t+1}^{H\,\zeta} / y_{t+1}^{L\,\zeta}$, $\forall \zeta \in \{I, II, III\}$, and $t \geq 0$. Clearly, it applies $y_{t+1}^{H\,II} / y_{t+1}^{L\,II} = A_{H}^{\delta}\, H_{t}^{II\,1-\delta} / h_{\min}$.

The pattern of worsening inequality prevails so long as the economy evolves along this *intermediate* stage of development. The poor become always poorer relatively to the society's average income, while the income of the rich is continuously amplified. It is plain that this is reflected in the direction of change of the respective income shares, as expressed by

$$g\left(sh_{t+1}^{L\,II}\right) = \frac{\lambda\, A_{H}^{\delta}\left(H_{t}^{II\,1-\delta} - H_{t+1}^{II\,1-\delta}\right)}{\left(1-\lambda\right)h_{\min} + \lambda\, A_{H}^{\delta}\, H_{t+1}^{II\,1-\delta}} < 0 \qquad t \in [r+1, \tau-1]. \tag{53}$$

$$g\left(sh_{t+1}^{H\,II}\right) = \frac{\left(1-\lambda\right)h_{\min}\left(H_{t+1}^{II\,1-\delta} - H_{t}^{II\,1-\delta}\right)}{\left(1-\lambda\right)h_{\min} + \lambda\, A_{H}^{\delta}\, H_{t+1}^{II\,1-\delta}} > 0 \qquad t \in [r+1, \tau-1]. \tag{54}$$

In fact, the same conclusion could be drawn from observing that the differential of per capita earnings widens in proportion to the rate of human capital accumulation realized in the time passed by. We point that

$$g\left(\frac{y_{t+1}^{H\,II}}{y_{t+1}^{L\,II}}\right) = \frac{H_{t}^{II\,1-\delta} - H_{t-1}^{II\,1-\delta}}{H_{t-1}^{II\,1-\delta}} > 0\,, \qquad t \in [r+1, \tau-1], \tag{55}$$

where we define $g\left(\frac{y_{t+1}^{H\,\zeta}}{y_{t+1}^{L\,II}}\right) \equiv \frac{\left(y_{t+1}^{H\,\zeta} / y_{t+1}^{L\,\zeta}\right) - \left(y_{t}^{H\,\zeta} / y_{t}^{L\,\zeta}\right)}{y_{t}^{H\,\zeta} / y_{t}^{L\,\zeta}}$, along any equilibrium path ζ, $\zeta \in \{I, II, III\}$.

We have claimed in the previous section that due to perpetual growth educational investment may be consistent with optimal incentives eventually for the array of all types of agents. Income convergence is a characteristic that signals the economy has made its transition to this *advanced* stage in the growth path. Along this phase of development various economic classes earn the respective shares of aggregate output

$$sh_{t+1}^{j\,III} = A_j^{\delta} \big/ \left(1-\lambda\right) A_L^{\delta} + \lambda\, A_H^{\delta} \quad \forall j \in \{L, H\}, \forall t \in [\tau, \infty). \tag{56}$$

We establish that distribution $\mathbf{sh}_{t+1}^{II}$ is characterized by greater income inequality, in terms of Lorenz superiority, compared to the income distribution of the subsequent equilibrium phase, $\mathbf{sh}_{t+1}^{III}$. As it is known, the proof entails the requirement

- $(1-\lambda) sh_{t+1}^{L\,II} \leq (1-\lambda) sh_{t+1}^{L\,III}$. (57)

Upon *Assumption* 1 relation (57) is satisfied as strict inequality.

$$h_{\min} < A_L^{\delta} H_t^{II\,1-\delta}. \tag{58}$$

It must further be met that

- $(1-\lambda) sh_{t+1}^{L\,II} + \lambda\, sh_{t+1}^{H\,II} \leq (1-\lambda) sh_{t+1}^{L\,III} + \lambda\, sh_{t+1}^{H\,III}$, (59)

be always valid. It is intuitively clear that relation (59) is established upon the prerequisite that the decrease in the share of the rich class on the impact of transition be exceeded by the positive growth in the share of the poor. It is straightforward to show that

$$g_T^{III}\left(sh_{t+1}^{L}\right) = \frac{\lambda\, A_H^{\delta} \left(A_L^{\delta} H_t^{II\,1-\delta} - h_{\min}\right)}{h_{\min} \left[(1-\lambda) A_L^{\delta} + \lambda\, A_H^{\delta}\right]} > 0, \tag{60}$$

while

$$g_T^{III}\left(sh_{t+1}^{H}\right) = \frac{(1-\lambda)\left(h_{\min} - A_L^{\delta} H_t^{II\,1-\delta}\right)}{H_t^{II\,1-\delta} \left[(1-\lambda) A_L^{\delta} + \lambda\, A_H^{\delta}\right]} < 0, \tag{61}$$

are established to have a positive, and negative sign respectively on the basis of *Assumption* 1. We write $g_T^{III}\left(sh_{t+1}^{j}\right) \equiv \left(sh_{t+1}^{j\,III} - sh_t^{j\,II}\right) / sh_t^{j\,II}$, where by definition $t+1 \equiv \tau$. It is simple to show that

$$\left(1-\lambda\right) g_T^{III}\left(sh_{t+1}^{L}\right) > \lambda \left| g_T^{III}\left(sh_{t+1}^{H}\right) \right|, \tag{62}$$

with its validity resting upon *Assumption* 1.

It becomes evident from relations (60) and (61) that the income differential of the two classes shrinks as dynamics bring the economy on the more evolved stage in date $t = \tau + 1$. In consequence, we have

$$\frac{y_{t+1}^{H\,II}}{y_{t+1}^{L\,II}} > \frac{y_{t+1}^{H\,III}}{y_{t+1}^{L\,III}}, \tag{63}$$

with $y_{t+1}^{j\,II}$, $j \in \{L, H\}$, being evaluated on the interval $t \in [r, \tau - 1]$, while $y_{t+1}^{j\,III}$ on $t \in [\tau, \infty)$. We remark that the narrowing in earnings divergence consists of a discrete discontinuous jump occurring in consequence of the transition. The latter, we recall, is effected in the length of a sole time period, on date $\tau + 1$. Continuous, persistent fall in the inequality of income distribution does not follow ever sustained growth along this *final* stage of development. Economic classes claim each a constant share of the economy's output. Yet the rich remain always richer on this path, an event intrinsically plausible. It is only apparent that $y_{t+1}^{H\,III} / y_{t+1}^{L\,III} = A_H^{\delta} / A_L^{\delta}$, subject to no endogenous impact.

The evolution of inequality in aggregate distribution clearly exhibits a non-monotonic trend along the economy's path to development. The following theorem acknowledges the proposition formed by Kuznets (1955) as conclusively established.

Proposition 7 Should *Assumption* 1 be imposed, the relationship between inequality in the economy-wide income distribution and aggregate prosperity resembles an *inverse-U* shaped curve. Starting from perfect equality in a state of stagnation, income inequality exhibits a smooth upward trend as growth progressively takes off. Along this process, a critical threshold of development level is reached causing a qualitative change in dynamics. A sudden discontinuous fall in earnings' inequality is accompanied by a constant wage differential as unbounded growth is sustained perpetually.

We return our attention to the proposition of Chatterjee and Ravikumar (1999), that a Lorenz superior distribution is consistently ascribed a higher degree of equality by each and every conventionally used measure of inequality. The use of a simple example validates this argument, and lends confirmation to our conclusion that the dynamic pattern of the inequality of income distribution resembles an inverse-*U* curve. We choose to apply the widely used measure of *Gini* coefficient. The formal definition of the index reads as follows

$$G = \frac{1}{2N^2\overline{Y}} \sum_{j=1}^{J}\sum_{k=1}^{J} n_j n_k \left| y_j - y_k \right| . \tag{64}$$

where we recall that N denotes the measure of aggregate population, $\overline{Y}$ represents the economy's average income, and J is the number of distinct incomes. Finally, subscripts j and k each represent an economic class, with $j,k \in \{1,2,\ldots,J\}$. The population measure of each income group is hereby denoted by n_j .[40] In a recent study, Palivos and Yip (2007) have proved that for the simple case of $J = 2$, the aforementioned definition is written in the following simplified form

$$G = \frac{n_1}{N}\left(1 - \frac{y_1}{\overline{Y}}\right), \tag{65}$$

where evidently $n_1 + n_2 = N$, and $y_1 = \min\{y_j\}_{j=1}^{J}$. In the context of our analysis, the *Gini* measure is clearly defined as

$$G_{t+1} = (1-\lambda)\left(1 - \frac{h_{t+1}^{L}}{H_{t+1}}\right) \quad . \tag{65'}$$

It can easily be proved that the value of the index in each equilibrium state is given by

$$G_{t+1}^{I} = 0 \qquad \forall t \in [0, r-1], \tag{66}$$

[40] The definition is excerpted from Debraj Ray (1998).

$$G_{t+1}^{II} = \frac{\lambda(1-\lambda)\left(A_H^\delta H_t^{II\,1-\delta} - h_{\min}\right)}{(1-\lambda)h_{\min} + \lambda A_H^\delta H_t^{II\,1-\delta}} \qquad \forall t \in [r, \tau-1], \tag{67}$$

with the latter being strictly positive on the account of *Assumption* 1. Further,

$$G_{t+1}^{III} = \frac{\lambda(1-\lambda)\left(A_H^\delta - A_L^\delta\right)}{(1-\lambda)A_L^\delta + \lambda A_H^\delta} \qquad \forall t \in [\tau, \infty], \tag{68}$$

clearly, being a strictly positive constant. Theoretically, we confirm a positive growth measure on the account of *Assumption* 1, and positive growth for aggregate knowledge; conditions that do apply for both cases of linear and non-linear human capital technology. It has proved difficult within this framework to obtain a definable prediction of the rate of change of the growth measure of the *Gini* coefficient in the course of the *intermediate* stage of development. We have

$$g\left(G_{t+1}^{II}\right) = \frac{A_H^\delta h_{\min}\left(H_t^{1-\delta} - H_{t-1}^{1-\delta}\right)}{\left(A_H^\delta H_{t-1}^{1-\delta} - h_{\min}\right)H_t} \qquad \forall t \in [r+1, \tau-1]. \tag{69}$$

where we have defined $g\left(G_{t+1}^{II}\right) \equiv \left(G_{t+1}^{II} - G_t^{II}\right)/G_t^{II}$, and $\delta \in (0,1)$. In the linear case, the result is expressed as following

$$g\left(G_{t+1}^{II}\right) = \frac{A_H h_{\min} H_{t-1}^{II}}{\left(A_H H_{t-1} - h_{\min}\right)H_t^{II}}\, g\left(H_t^{II}\right) \qquad \forall t \in [r+1, \tau-1]. \tag{70}$$

where in analogous manner we have defined $g\left(H_t^{II}\right) \equiv \left(H_t^{II} - H_{t-1}^{II}\right)/H_{t-1}^{II}$. Using the solution for H_t as given by equation (31), the growth result is written

$$g\left(G_{t+1}^{II}\right) = \frac{(A_H - 1)(\lambda A_H - 1)^2 (\lambda A_H)^t}{\Omega_1 + \Omega_2 (\lambda A_H)^t + \Omega_3 (\lambda A_H)^{2t}} \qquad \forall t \in [r+1, \tau-1], \tag{70'}$$

where $\Omega_1 \equiv A_H(1-\lambda)^2 + \lambda(1-A_H)^2 A_H^{-1} - (1-\lambda A_H)(1-\lambda)$,
$\Omega_2 \equiv A_H \lambda(1-\lambda)(1-A_H) + \lambda(1-\lambda A_H)(A_H - 1)$, and $\Omega_3 \equiv \lambda^2(1-A_H)^2$. Performing a simulation analysis would enable us to sign the expression, and allow us to conclude about the form of upward trend of the inequality of income distribution.

It is evident that the following holds true

$$G_{t+1}^{I} < G_{t+1}^{II}, \tag{71}$$

and

$$G_{t+1}^{III} < G_{t+1}^{II}, \tag{72}$$

with the latter being valid on the basis of *Assumption* 1. The results lead us to infer once again an inverted-*U* pattern of evolution in personal income inequality. Due to strong legal rights be established for credit institutions, growth prospects start to materialize carrying the economy to a path of higher and worsening inequality. Society escapes a poverty loophole, albeit a state of perfect equality. It can be logically guaranteed that inequality will take eventually a downward jump to a fixed computable level at a high measure of probability (see equation 68). We have remarked upon the existence of the critical time period in which this occurs in *Proposition* 4.

The *Kuznets Curve* that our theory implies is presented in the following diagram:

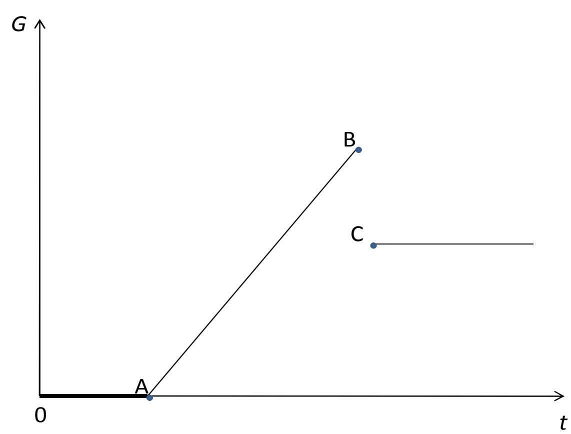

Figure I.1: *Kuznets Curve (benchmark version of the model)*

It is evident to the reader that points *A* and *B* correspond to the critical time periods *r* and τ, respectively, at which the transition takes place to a new equilibrium path. Point *C* marks the start of the *advanced* phase of development at period $\tau + 1$.

V. *Extension to general equilibrium*

In our previous environment financial intermediaries had access to a hypothesized credit market *outside* of the economy. The sole participants in this market were financial institutions, able to borrow and invest any amount at an exogenously fixed interest rate. It is within our scope to extend the preceding analysis in a way that allows the interest rate being endogenously determined. Equilibrating forces in the credit market require that loan demand be equated to credit supply. Making use of Walras's law, we choose to employ the economy's equilibrium resource constraint, that aggregate investment be equal to domestic saving.

We proceed with a concise presentation of our theory, as is reformulated to account for the endogeneity of the interest rate, R. The model's previous analytical structure is followed in near precision, while the linguistic accompaniment of mathematics is justly omitted.

The budget constraint in the working period of life is now given by the expression

$$c_{2t+1}^{j} = \begin{cases} A_j^{\delta} H_t^{1-\delta} - R_{t+1} q - s_{2t+1}^{j} & \text{if } q > 0 \\ h_{\min} - s_{2t+1}^{j} & \text{if } q = 0 \end{cases}. \tag{73}$$

Accordingly, consumption in the retirement age is given by

$$c_{3t+2}^{j} = R_{t+2} s_{2t+1}^{j}. \tag{74}$$

The optimization problem yields that individual optimal saving is

$$s_{2t+1}^{j*} = \begin{cases} (1-\beta)\left(A_j^{\delta} H_t^{1-\delta} - R_{t+1} q\right) & \text{if } q > 0 \\ (1-\beta) h_{\min} & \text{if } q = 0 \end{cases} \quad \forall j = L, H. \tag{75}$$

Substituting for optimal saving in equations (73) and (74) we obtain the respective expressions for optimal second- and third period consumption

$$c_{2t+1}^{j*} = \begin{cases} \beta\left(A_j^{\delta} H_t^{1-\delta} - R_{t+1} q\right) & \text{if } q > 0 \\ \beta h_{\min} & \text{if } q = 0 \end{cases} \quad \forall j \in \{L, H\}. \tag{73'}$$

$$c_{3t+2}^{j*} = \begin{cases} (1-\beta)R_{t+2}\left(A_j^{\delta}\, H_t^{1-\delta} - R_{t+1}\, q\right) & \text{if } q > 0 \\ (1-\beta)R_{t+2}\, h_{\min} & \text{if } q = 0 \end{cases} \quad \forall\, j \in \{L, H\}. \tag{74'}$$

State of poverty The preceding analysis established that a state of underdevelopment, accompanied by perfect equality in low income earnings, is a prospect actualized indefinitely when credit institutions are entitled to limited rights in claiming debt repayment. Financial entities lack the incentive to provide credit for human capital investment, with the consequence of agents' preclusion from the opportunity to privately financed education. In that environment, the possibility of saving for retirement age is served through the private financial system, construed to be composed of a form of deposit institutions. Saving yields a positive rate of return, modeled as an exogenous fixed variable, which may not be otherwise endogenously determined.[41]

A segment of the society invests We proceed to generalize within the context of general equilibrium the path along which potential ever sustained growth is realized. Owing to appropriate transformations in the economy's legislative system, a prior missing market for human capital investment may function on time period r, where $r \geq 2$.

The postulated heterogeneity on innate aptitude level becomes critical for individual decision making, and consequently for the economy's growth pattern and inequality of income distribution. We redefine formally individual heterogeneity in income earning ability, recognizing that the interest rate is now endogenously determined in each period. *Definition* 1 is mathematically expressed as follows:

Assumption 3 An agent born at time period *t* is said to be of *high type* if and only if

$$A_H^{\delta}\, H_t^{1-\delta} > R_{t+1}^{*II}\, q \qquad \forall t \geq 0. \tag{α}$$

On the other hand, it is not feasible for *low type* agents to carry out their contract commitment, however honest in intention. Mathematically, this reads into

[41] An alternative way to retain the possibility of saving for retirement age would be to postulate the existence of a storage technology, taken to provide a positive return on investment. The conception of such technology ought to be considered an arbitrarily chosen device, with the meaning to supplement the role of absent saving institutions.

$$A_L^{\delta}\, H_t^{1-\delta} < R_{t+1}^{*II}\, q \qquad \forall t \geq 0\,. \tag{β}$$

The variable R_{t+1}^{*II} expresses the endogenous value of the interest rate as determined by the economy's equilibrium resource constraint. The latter reads that aggregate saving be equal to the private demand for investment, which solely consists of the credit financing human capital accumulation. Formally expressed, we have

$$S_{2\,t+1}^{II} = \lambda\, q \qquad \forall t \geq r\,, \tag{76}$$

where S_{2t+1}^{II} denotes domestic private saving accumulated in the working period of life from members of generation *t*. Evidently, $\lambda\, q$ represents the private demand for educational investment. On the other side, aggregate saving is to be defined as

$$S_{2\,t+1}^{II} = (1-\lambda)(1-\beta)h_{\min} + \lambda\,(1-\beta)\left(A_H^{\delta}\, H_t^{II\,1-\delta} - R_{t+1}^{*II}\, q\right) \qquad \forall t \geq r\,. \tag{77}$$

We obtain that the equilibrium expression of the interest rate price is given by

$$R_{t+1}^{*\,II} = \frac{(1-\lambda)h_{\min}}{\lambda\, q} + \frac{A_H^{\delta}\, H_t^{II\,1-\delta}}{q} - \frac{1}{1-\beta} \qquad \forall t \geq r\,. \tag{78}$$

The function conveys that the rate of return on financial assets is derived by the technology of human capital accumulation.

We confine our attention to establishing the proof of *Proposition* 3, which forms the essence of this analysis. Similarly worded, the statement of the theorem is the following

Proposition 8 A competitive equilibrium with a subset of population acquiring privately financed education exists on the condition of occurrence of the following requirements

- A positive measure of individuals optimally choose to obtain education
- The credit market is sustainable
- Savings be non-negative for both *H* and *L* types.

Proof

- The measure of high-type agents must be induced to *participate* in the contract arrangement. This translates to mean that the contract must offer a high-ability individual at least the utility level she would obtain if she chose to remain unskilled. In other words,

$$V_t^{E,ND}(H) > V_t^{NE,II} \qquad \forall t \geq r\,, \tag{79}$$

where it holds that

$$V_t^{E,ND}(j) = \ln\left\{\beta^{\beta}(1-\beta)^{1-\beta} R_{t+2}^{*\,1-\beta}\left(A_j^{\delta} H_t^{1-\delta} - R_{t+1}^{*}\, q\right)\right\} \quad \forall j \in \{L, H\}. \tag{80}$$

$$V_t^{NE}(j) \equiv V_t^{NE} = \ln\left\{\beta^{\beta}(1-\beta)^{1-\beta} R_{t+2}^{*\,1-\beta}\, h_{\min}\right\} \quad \forall j \in \{L, H\}. \tag{81}$$

Relation (79) is expressed as

$$A_H^{\delta} H_t^{II\,1-\delta} > h_{\min} + q\, R_{t+1}^{*} \qquad \forall t \geq r\,, \tag{79'}$$

which, evaluated in period $t = r$ yields

$$A_H^{\delta} h_{\min}^{1-\delta} > h_{\min} + q\, R_{t+1}^{*\,II} \qquad \forall t \geq r\,. \tag{79''}$$

Evidently, it suffices to impose relationship (79'') to ensure the validity of (79') in all periods $t > r$.

- Drawing upon *Proposition* 2 we transfer the conclusion that the postulated definitions of agent types (*Assumption* 3) count as proof of the sentence that the educational credit market is privately sustainable.
- The individual rationality constraint for the members of the financial system amounts to imposing that individual saving is positive for both types. Invoking the optimal saving function (equation 8), we have that

$$s_{2t+1}^{H\,*} = (1-\beta)\left(A_H^{\delta} H_t^{1-\delta} - R_{t+1}^{*\,II}\, q\right) > 0\,, \tag{82}$$

which is strictly positive on the basis of the definition of the high-type agent (*Assumption* 3α). With respect to the low-type individuals saving is represented by

$$s_{2t+1}^{L\,*} = (1-\beta)\, h_{\min} > 0 \qquad \forall t \geq 0\,, \tag{83}$$

which is strictly positive for $\beta \in (0,1)$, and $h_{\min} > 0$.

The proof of *Proposition* 8 clearly consists of imposing the definitional condition *Assumption* 3*β*, as well as the constraint (79''), along with the equilibrium expression of the interest rate (equation 77). The truth of the remaining relations is inferred by simple reasoning.‖

All types acquire education In analogy with the analysis in the core version the following theorem proves the transition of the economy to a higher stage of development, where education is accommodated for all types of agents. The following is a restatement of *Proposition* 4, carried in the general equilibrium context

Proposition 9 There exists a time period $\tilde{\tau} > 0$, where $\tilde{\tau} \in [r+1, \infty)$, in which the income realization of educated low-type agents exceeds the threshold level that defines education the optimal choice. This reads as follows

$$A_L^{\delta} H_t^{1-\delta} \begin{cases} \geq R_{t+1}^{*} q + h_{\min} & \forall t \in [\tilde{\tau}, \infty) \\ < R_{t+1}^{*} q + h_{\min} & \forall t \in [0, \tilde{\tau} - 1] \end{cases}. \tag{84}$$

If we limit ourselves to the linear case $(\delta = 1)$ we can prove that $\tilde{\tau}$ is defined as

$$\tilde{\tau} = \begin{cases} \dfrac{\ln[\tilde{\Theta}]}{\ln[\lambda A_H]}, & \text{if } \lambda A_H > 1 \\ & \\ \dfrac{R_{t+1}^{*III} q + h_{\min}(1 - A_L)}{A_L h_{\min}(1-\lambda)}, & \text{if } \lambda A_H = 1 \end{cases}, \tag{85}$$

with $\tilde{\Theta} \equiv \dfrac{\left(R_{t+1}^{*III} q + h_{\min}\right)(1 - \lambda A_H) - A_L h_{\min}(1-\lambda)}{A_L h_{\min} \lambda (1 - A_H)}$. The expression is identical to the definition of τ in the baseline version with the obvious exception that R is now endogenously determined.

Proof

The reasoning of the proof is completely analogous to that of *Proposition* 4. Relationship (75) yields in linear form

$$A_L H_t \geq R^{*III}_{t+1} q + h_{\min} \qquad \forall t \in [\tilde{\tau}, \infty). \tag{86}$$

We use the solution of aggregate human capital stock (equations 31) to substitute for H_t. Deriving expression (85) is then a straightforward task.

We can prove the existence of time period $\tilde{\tau}$ only on the condition that the latter is greater to unity. More precisely, the theorem is true if

- *Case* $\lambda A_H > 1$

$$\tilde{\Theta} > \lambda A_H, \tag{87}$$

which leads to a standard second-order polynomial

$$\left(\lambda^2 A_L h_{\min}\right) A_H^2 - \lambda\left[R^{*III}_{t+1} q + h_{\min}\left(1 + \lambda A_L\right)\right] A_H + R^{*III}_{t+1} q + h_{\min}\left[1 - A_L(1-\lambda)\right] > 0. \tag{87'}$$

The expression is positive insofar as, either

$$A_H < \frac{R^{*III}_{t+1} q + h_{\min}\left(1 + \lambda A_L\right) - \sqrt{\Omega}}{2\lambda A_L h_{\min}} \equiv A_H^1, \tag{88α}$$

or, alternatively

$$A_H > \frac{R^{*III}_{t+1} q + h_{\min}\left(1 + \lambda A_L\right) + \sqrt{\Omega}}{2\lambda A_L h_{\min}} \equiv A_H^2, \tag{88β}$$

where we note that $\Omega \equiv \left[R^{*III}_{t+1} q + h_{\min}\left(1 + \lambda A_L\right)\right]^2 - 4 A_L h_{\min}\left[R^{*III}_{t+1} q + h_{\min}\left(1 - A_L(1-\lambda)\right)\right] \geq 0$, and $A_H^1 > 0$.

Being more intuitive plausible, we choose to employ condition (88*β*).

- *Case* $\lambda A_H = 1$

The condition $\tilde{\tau} > 1$ implies that

$$A_L < \frac{R^{*III}_{t+1} q + h_{\min}}{(2-\lambda) h_{\min}}. \tag{89}$$

The right-hand side exceeds unity, as is required, given the imposition of

$$R^{*III}_{t+1}\, q > (1-\lambda)h_{\min}\,. \qquad (90)\ \|$$

We recall, R^{*}_{t+1} expresses the endogenous value of the interest rate as being determined by the economy's equilibrium resource constraint. Along the time interval $t \in [\widetilde{\tau}, \infty)$, the latter is defined to read as follows

$$S^{III}_{2\,t+1} = q \qquad \forall t \in [\widetilde{\tau}, \infty). \qquad (91)$$

where we recall that S^{III}_{2t+1} denotes the domestic private saving accumulated in the working period of life from members of generation *t*. The latter is defined as

$$S^{III}_{2\,t+1} = (1-\beta)\{(1-\lambda)A_L^{\delta} + \lambda\, A_H^{\delta}\}H_t^{1-\delta} - (1-\beta)q\, R^{*}_{t+1} \qquad \forall t \in [\widetilde{\tau}, \infty). \qquad (92)$$

The private demand for educational investment is expressed by variable q. It is straightforward to obtain the equilibrium expression for the interest rate, being defined as

$$R^{*III}_{t+1} = \frac{\{(1-\lambda)A_L^{\delta} + \lambda\, A_H^{\delta}\}H_t^{1-\delta}}{q} - \frac{1}{1-\beta} \qquad \forall t \in [\widetilde{\tau}, \infty). \qquad (93)$$

The proof of existence of the equilibrium state along which growth is supported by human capital investment of all types is enclosed in a restatement of *Proposition* 6

Proposition 10 A competitive equilibrium where the entire population acquires privately financed education exists on the condition of occurrence of the following requirements

- All types optimally choose to invest in individual improvement
- The credit market is privately sustainable
- Individual saving be non-negative for both *H* and *L* types

Proof

- Individuals of either type must be induced to engage in human capital investment. This is guaranteed only insofar as

$$V_t^{E,ND}(j) > V_t^{NE} \qquad \forall\, j = L, H\,, \forall t \in [\widetilde{\tau}, \infty), \qquad (94)$$

which is equivalent to

$$A_j^{\delta}\, H_t^{1-\delta} > R_{t+1}^{*III}\, q + h_{\min} \qquad \forall\, j = L, H\,, \forall t \in \left[\widetilde{\tau}, \infty\right), \tag{94'}$$

given that $V_t^{E,ND}(j)$, and V_t^{NE} are defined by equations (80) and (81) respectively. Evidently, it suffices to impose the sole relation

$$A_L^{\delta}\, H_t^{1-\delta} > R_{t+1}^{*III}\, q + h_{\min} \qquad\qquad t = \widetilde{\tau}\ . \tag{94''}$$

- In light of *Corollary* 1, the sustainability of credit market is established upon the validity of condition (94").
- In connection with the last condition, we impose once more the individual rationality constraints for the banking system (equations 11). Invoking equation (8), we obtain

$$s_{2t+1}^{j*} = (1-\beta)\left(A_j^{\delta}\, H_t^{1-\delta} - R_{t+1}^{*III}\, q\right) > \ 0 \qquad\qquad \forall\, j \in \{L, H\}, \forall t \in \left[\widetilde{\tau}, \infty\right) \tag{95}$$

We need only establish that saving be positive for the low-type agent, which evidently is met under condition (94").

It follows that the sole thing we must postulate to establish *Proposition* 10 is inequality (94"). The truth of the remaining relations is logically inferred.‖

VI. *Higher degree of heterogeneity*

An appropriate extension of the basic construct of the model would be to augment the set of values in the domain of innate ability, A_j. Such a generalization is, in and of itself, a noteworthy task in that we lay down the theory in higher mathematical abstraction. Yet its practical value is that it provides us with a way of obtaining a *smooth* inverted-*U* curve. Taking heterogeneity to the highest degree of generality, we postulate that the domain of variable A_j forms a district measure of (countable) infinite types. In specific, we assert that the measure of heterogeneous types is defined on the bounded interval

$\forall j \in [1,\ldots,J]$, where $J \geq 2$.[42] Each class of type-*j* individuals constitutes a fraction λ_j of the population measure, $N = 1$. It is evident that $\sum_{j=1}^{J} \lambda_j = 1$.

It may easily be seen that the structure of the model set out in section *II*, as well as the analysis on the stationary equilibrium path, becomes no different when the general case $J > 2$ is applied.[43] Carrying not an unfruitful repetition, we proceed with the analysis on equilibrium growth.

We recall that our definition of *individual type* is intrinsically connected to an agent's innate aptitude towards knowledge acquisition, a factor being genetically, or otherwise exogenously determined. Nevertheless, human capital productivity does not critically determine the ability to carry out one's contract commitment.[44] Being of a certain type is accompanied by no idiosyncratic feature determining the feasibility of loan repayment. The definition of the latter concept formally reads as follows[45]

Definition 4 Loan repayment is *feasible* for an agent born in time *t* if the following holds to be true

$$A_j^{\delta}\, H_t^{1-\delta} > R\, q\,. \tag{96}$$

The following assumption is adopted

Assumption 4 Individuals of type $j \in [1,\ldots,k]$, where $k \in [1,\ldots,J-1]$, are discerned by an income earning ability as low as not in the least covering debt repayment. On the other hand, the earned income of an agent of type $j \in [k+1,\ldots,J]$ supports her honoring of debt liability. Hence, we explicitly postulate

$$A_j^{\delta}\, H_t^{1-\delta} < R\, q \qquad j \in [1,\ldots,k]. \tag{α}$$

[42] We postulate that for any two cardinal numbers l and m of the set $j \in [1,2,\ldots,J]$, with $1 \leq l < m \leq J$, the corresponding members in the set of heterogeneous abilities $A_j \in [A_1, A_2, \ldots, A_m]$ are of the same cardinal ordering, i.e. $A_l < A_m$.

[43] The present analysis refers to the case of exogenously determined interest rate. Hence, it is relevant to observe equations (1) to (23).

[44] This was a natural feature of the simple two-type case. In that setting, differentiation in income earning ability was inescapably synonymous to critical differences in the feasibility of carrying out contract obligations.

[45] The definition of the concept has already been introduced in *Assumption* 2.

$$A_j^{\delta} H_t^{1-\delta} > R\,q \qquad j \in [k+1,\ldots,J]. \tag{β}$$

Assumption 4(α) says that investment in human capital does not pay off for types $j \in [1,\ldots,k]$ if one is to remain committed to contract liability. Default on debt is an inescapable consequence, irrespective of an inherent honest intention. The assumption is reducible to the following expression

Assumption 4'

$$A_k^{\delta} H_t^{1-\delta} < R\,q \tag{α}$$

$$A_{k+1}^{\delta} H_t^{1-\delta} > R\,q \tag{β}$$

We are now able to prove, by means of what has already been said, that an equilibrium state may exist with the economy being placed on a path of ever-sustained growth. As has been stated previously, this potential is realized as a result of the private financing for education being made feasible, if only initially for the segment of society with relatively high investment return (namely, the types $j \in [k+1,\ldots,J]$). Similarly worded, the foregoing theorem forms an exact generalization of *Proposition* 3 to the multiple *J*-type case.

Proposition 11 A competitive equilibrium with a subset of the population acquiring privately financed education exists on the condition of occurrence of the following requirements

- A positive measure of individuals optimally choose to obtain education,
- The credit market is sustainable,
- Savings be non-negative for all agent types.

Proof

- The *participation constraint* for the borrower side consists of the following condition

$$V_t^{E,ND}(j) > V_t^{NE} \qquad \forall t \geq r\,,\, j \in [k+1,\ldots,J], \tag{97}$$

where we recall, $V^{E,ND}(j)$, and V^{NE} are given by equations (18) and (19) respectively. Relation (88) is satisfied so long as

$$A_j^{\delta}\, H_t^{II\,1-\delta} > h_{\min} + Rq \qquad \forall t \geq r\,,\, j \in [k+1,\ldots,J], \tag{97'}$$

which, evaluated in period $t = r$, for type $j = k+1$ yields

$$A_{k+1}^{\delta}\, h_{\min}^{1-\delta} > h_{\min} + Rq\,. \tag{97''}$$

Evidently, it suffices to impose relationship (97'') to ensure the validity of (97') for all types $j \in [k+1,\ldots,J]$, in each and all time periods $t \geq r$.

- Drawing on *Proposition* 2, it is purely logically proved that the financial market for human capital investment is privately sustainable. The requirement of proof is fully satisfied upon the postulated definitions of *Assumption* 4'.
- In order to prove that it is individually rational for credit entities to engage in loan provision we need to impose that saving be positive for all types of agents. Invoking the optimal saving function (equation 8) we obtain

$$s_{2t+1}^{j\,*} = (1-\beta)\left(A_j^{\delta}\, H_t^{II\,1-\delta} - R\,q\right) > 0 \qquad \forall t \geq r\,,\, j \in \{k+1,\ldots,J\}, \tag{98}$$

which is strictly positive on the basis of *Assumption* 4'(β). With respect to the individuals of relatively low productivity saving is given by

$$s_{2t+1}^{j\,*} = (1-\beta)\, h_{\min} > 0 \qquad \forall t \geq r\,,\, j \in \{1,\ldots,k\}, \tag{99}$$

obviously being strictly positive for $\beta \in (0,1)$, and $h_{\min} > 0$. In summary, the proof of *Proposition* 11 requires the validity of *Assumption* 4'(α), and of constraint (97''). The truth of the remaining relations is then logically inferred.‖

The dynamic evolution of the society's stock of human capital along this equilibrium path is governed by the first order non-linear difference equation

$$H_{t+1} = h_{\min} \sum_{j=1}^{k} \lambda_j + H_t^{II\,1-\delta} \sum_{j=k+1}^{J} \lambda_j\, A_j^{\delta} \qquad \forall t \geq r\,. \tag{100}$$

The solution to the linear form $(i.e.\ \delta = 1)$ is described as

$$H_t = \begin{cases} h_{\min}\left[(1-\varphi)\theta^t + \varphi\right] & \text{if } \theta \equiv \sum_{j=k+1}^{J} \lambda_j A_j \neq 1 \\ h_{\min}\left(1 + t\sum_{j=1}^{k}\lambda_j\right) & \text{if } \theta \equiv \sum_{j=k+1}^{J} \lambda_j A_j = 1 \end{cases} \quad \forall t \geq r+1. \tag{101}$$

where we define $\varphi \equiv \dfrac{\sum_{j=1}^{k}\lambda_j}{1-\theta}$.

We prove the existence of a critical level of development which, once reached, type k has an optimal incentive to invest in education.

Proposition 12 There exists a time period $\tau_1 > 0$, where $\tau_1 \in [r+1, \infty)$, in which the income realization of educated type-k agents exceeds the threshold level that defines education the optimal choice. In other words,

$$A_k^{\delta} H_t^{1-\delta} \begin{cases} \geq Rq + h_{\min} & \forall t \in [\tau_1, \infty) \\ < Rq + h_{\min} & \forall t \in [0, \tau_1 - 1] \end{cases}. \tag{102}$$

Considering the linear human capital technology, τ_1 is defined as

$$\tau_1 = \begin{cases} \dfrac{\ln[\Phi]}{\ln[\theta]}, & \text{if } \theta > 1 \\ \dfrac{Rq + h_{\min}(1 - A_k)}{A_k h_{\min}\sum_{j=1}^{k}\lambda_j}, & \text{if } \theta = 1 \end{cases}, \tag{103}$$

with $\Phi \equiv \dfrac{Rq + h_{\min}(1 - \varphi A_k)}{A_k h_{\min}(1-\varphi)}$.

Proof

Relationship (93) is written in linear form

$$A_k H_t \geq Rq + h_{\min}. \tag{104}$$

We use the solution of aggregate human capital stock as given by (101), to substitute for H_t. Equations (103) are then derived in a straightforward manner.

The requirement must be imposed that τ_1 be greater than one. More precisely, it must hold true that

- $$\frac{Rq + h_{\min}(1-\varphi A_k)}{A_k\, h_{\min}(1-\varphi)} > \sum_{j=k+1}^{J} \lambda_j\, A_j \qquad \text{if } \theta > 1\,. \tag{105}$$

- $$A_k < \frac{Rq + h_{\min}}{\left(1+\sum_{j=1}^{k} \lambda_j\right) h_{\min}}\,. \tag{106}$$ ‖

We prove the following theorem

Proposition 13 A competitive equilibrium exists in each and all time periods $t \in [\tau_1, \infty)$ having the following characteristics: the measure of population with learning abilities ranging in the interval $A_j \in [A_k, A_J]$ acquire privately financed education, whereas the remaining subset with abilities $A_j \in [A_1, A_{k-1}]$ choose to remain unskilled. The existence of the equilibrium is established on the condition of occurrence of the following requirements

- The measure of the population with ability types $j \in [k, J]$ optimally choose to obtain education.
- The credit market is sustainable.
- Savings be non-negative for all agent types.

Proof

- As has already been mentioned, the *participation constraint* for the borrower side consists of the condition

$$V_t^{E,ND}(j) > V_t^{NE} \qquad \forall t \geq \tau_1,\, j \in [k,\ldots,J], \tag{107}$$

Drawing upon equations (18) and (19), we obtain that relation (107) is satisfied so long as

$$A_j^{\delta}\, H_t^{1-\delta} > h_{\min} + Rq \qquad \forall t \geq \tau_1,\, j \in [k,\ldots,J], \tag{107'}$$

Evidently, it suffices to impose relationship (107') for type $j = k$ to ensure its validity for all remaining types $j = k+1,\ldots,J$. Evaluated in the linear case, expression (107') yields

$$A_k > \frac{h_{\min} + Rq}{H_t} \qquad \forall t \geq \tau_1 . \tag{107''}$$

- Drawing on *Proposition* 2, it is logically proved that the financial market for human capital investment is privately sustainable. The requirement of proof is fully satisfied upon the feasibility conditions

$$A_{k-1}^{\delta} H_t^{1-\delta} < R\,q \qquad \forall t \geq \tau_1 , \tag{108}$$

$$A_k^{\delta} H_t^{1-\delta} > R\,q \qquad \forall t \geq \tau_1 . \tag{109}$$

- Finally, we need to impose that saving is positive for all types of agents. Invoking the optimal saving function (equation 8) we have

$$s_{2t+1}^{j\,*} = (1-\beta)\left(A_j^{\delta} H_t^{II\,1-\delta} - R\,q\right) > 0 \qquad \forall t \geq r \, , \, j \in \{k\ldots,J\}, \tag{110}$$

which is strictly positive on the basis of condition (107''). With respect to the individuals of relatively low productivity, saving is given by

$$s_{2t+1}^{j\,*} = (1-\beta)\, h_{\min} > 0 \qquad \forall t \geq r \, , \, j \in \{1,\ldots,k-1\}, \tag{111}$$

obviously being strictly positive for $\beta \in (0,1)$, and $h_{\min} > 0$. In summary, the proof of *Proposition* 13 requires the validity of conditions (107''), and (108). The truth of the remaining relations is then logically inferred.$\|$

The dynamic evolution of the society's stock of human capital along the time path $t \in\geq \tau_1$ is governed by

$$H_{t+1} = h_{\min} \sum_{j=1}^{k-1} \lambda_j + H_t^{1-\delta} \sum_{j=k}^{J} \lambda_j\, A_j^{\delta} \qquad t \in\geq \tau_1 . \tag{112}$$

The solution to the linear form of difference equation (112) $(i.e.\ \delta = 1)$ is described by

$$H_t = \begin{cases} h_{\min}\left[(1-\widetilde{\varphi})\widetilde{\theta}^{\,t} + \widetilde{\varphi}\right] & if\ \widetilde{\theta} \equiv \sum_{j=k}^{J} \lambda_j A_j > 1 \\ h_{\min}\left(1 + t\sum_{j=1}^{k-1}\lambda_j\right) & if\ \widetilde{\theta} \equiv \sum_{j=k}^{J} \lambda_j A_j = 1 \end{cases} \quad \forall t \geq \tau_1 + 1 . \tag{113}$$

where we define $\widetilde{\varphi} \equiv \dfrac{\sum_{j=1}^{k-1}\lambda_j}{1-\widetilde{\theta}}$.

The following theorem proves that per capita income reaches a critical threshold which, once surpassed, agents of type *k*-1 optimally choose to invest in education.

Proposition 14 Let there exist a time period $\tau_2 > 0$, with $\tau_2 \in [\tau_1 + 1, \infty)$. The income realization of educated individuals of type *k*-1 exceeds the threshold level that defines education the optimal choice. In other words,

$$A_{k-1}^{\delta} H_t^{1-\delta} \begin{cases} \geq Rq + h_{\min} & \forall t \in [\tau_2, \infty) \\ < Rq + h_{\min} & \forall t \in [0, \tau_2 - 1] \end{cases} . \tag{114}$$

Considering the linear human capital technology, τ_2 is defined as

$$\tau_2 = \begin{cases} \dfrac{\ln[\widetilde{\Phi}]}{\ln[\widetilde{\theta}]}, & if\ \widetilde{\theta} > 1 \\ \dfrac{Rq + h_{\min}(1 - A_{k-1})}{A_{k-1} h_{\min} \sum_{j=1}^{k-1}\lambda_j}, & if\ \widetilde{\theta} = 1 \end{cases} , \tag{115}$$

with $\widetilde{\Phi} \equiv \dfrac{Rq + h_{\min}(1 - \widetilde{\varphi} A_{k-1})}{A_{k-1} h_{\min}(1-\widetilde{\varphi})}$.

Proof

Equation (115) is derived by following the same procedure as in *Proposition* 12. We write relationship (114) in linear form

$$A_k H_t \geq Rq + h_{\min} . \tag{116}$$

Using the solution of aggregate human capital stock as given by (110) to substitute for H_t, we obtain the expression for τ_2 (115). Once again, the requirement must be imposed that τ_2 be greater to unity. Precisely, it must be

- $$\frac{Rq + h_{\min}\left(1-\widetilde{\varphi}\,A_{k-1}\right)}{A_{k-1}\,h_{\min}\left(1-\widetilde{\varphi}\right)} > \sum_{j=k}^{J}\lambda_j\,A_j \qquad if\ \widetilde{\theta} > 1, \tag{117}$$

- $$A_{k-1} < \frac{Rq + h_{\min}}{\left(1+\sum_{j=1}^{k-1}\lambda_j\right)h_{\min}}. \tag{118}$$ ‖

It is straightforward to prove

Proposition 15 A decentralized equilibrium exists in time periods $t \geq \tau_2$ having the following characteristics: the subset of the population with learning abilities ranging in the interval $A_j \in \left[A_{k-1}, A_J\right]$ acquire privately financed education, whereas the remaining set of individuals with abilities $A_j \in \left[A_1, A_{k-2}\right]$ choose to remain unskilled. The existence of the equilibrium is established upon the condition of occurrence of the following requirements

- The measure of population with ability types $j \in \left[k-1, J\right]$optimally choose to obtain education.
- The credit market is sustainable.
- Savings be non-negative for all agent types.

Proof

The proof procedure bears an evident analogy to that of *Proposition* 13. We forgo an extensive step-by-step analysis, and plainly assert our argument. *Proposition* 15 is established upon the condition that education be the optimal choice for type *k*-1 agents, as well as the assumption that loan repayment be non-feasible for agent type *k*-2. Hence,

$$A_{k-1}^{\delta}\,H_t^{1-\delta} > h_{\min} + Rq \qquad \forall t \geq \tau_2\,, \tag{119}$$

being reducible in the linear case to

$$A_{k-1} > \frac{h_{\min} + Rq}{H_t} \qquad \forall t \geq \tau_2 . \tag{119'}$$

and

$$A_{k-2}^{\delta} H_t^{1-\delta} < Rq \qquad \forall t \geq \tau_2 \tag{120}$$ ‖

The dynamic evolution of the society's stock of human capital along the time path $t \in \geq \tau_2$ is governed by the first-order difference equation

$$H_{t+1} = h_{\min} \sum_{j=1}^{k-2} \lambda_j + H_t^{1-\delta} \sum_{j=k-1}^{J} \lambda_j A_j^{\delta} \qquad t \in \geq \tau_2 . \tag{121}$$

The solution to the linear form of equation (121) is given by

$$H_t = \begin{cases} h_{\min} \left[(1-\hat{\varphi})\hat{\theta}^t + \hat{\varphi} \right] & \text{if } \hat{\theta} \equiv \sum_{j=k-1}^{J} \lambda_j A_j > 1 \\ & \\ h_{\min} \left(1 + t \sum_{j=1}^{k-2} \lambda_j \right) & \text{if } \hat{\theta} \equiv \sum_{j=k-1}^{J} \lambda_j A_j = 1 \end{cases} \qquad \forall t \geq \tau_2 + 1 . \tag{122}$$

where we define $\hat{\varphi} \equiv \dfrac{\sum_{j=1}^{k-2} \lambda_j}{1-\hat{\theta}}$.

The economy finally attains a level of per-capita income that defines education the optimal choice for each and every agent type.

Proposition 16 There exists a time period $\tau_n > 0$, where $\tau_n \in [\tau_1, \infty)$, in which the income realization of educated individuals of the lowest ability exceeds the threshold level that defines education the optimal choice. Namely,

$$A_1^{\delta} H_t^{1-\delta} \begin{cases} \geq Rq + h_{\min} & \forall t \in [\tau_n, \infty) \\ < Rq + h_{\min} & \forall t \in [0, \tau_n - 1] \end{cases} . \tag{123}$$

where $A_1 \equiv \min \{A_j\}_{j=1,\dots,J}$. Taking the case of linear technology, τ_n is defined by the following expression

$$\tau_n = \begin{cases} \dfrac{\ln\left[\overline{\Phi}\right]}{\ln\left[\overline{\theta}\right]}, & \text{if } \overline{\theta} \equiv \sum_{j=2}^{J} \lambda_j A_j > 1 \\ \dfrac{Rq + h_{\min}\left(1 - A_1\right)}{\lambda_1 A_1 h_{\min}}, & \text{if } \overline{\theta} \equiv \sum_{j=2}^{J} \lambda_j A_j = 1 \end{cases}, \tag{124}$$

where $\overline{\Phi} \equiv \dfrac{Rq + h_{\min}\left(1 - \overline{\varphi} A_1\right)}{A_1 h_{\min}\left(1 - \overline{\varphi}\right)}$, and $\overline{\varphi} \equiv \dfrac{\lambda_1}{1 - \overline{\theta}}$.

Proof

The proof of existence of time period τ_n is exactly analogous to the proof procedure of *Propositions* 12, and 13. Once again, the requirement must be imposed that τ_n be greater to unity. In specific, it must hold true that

- $$\frac{Rq + h_{\min}\left(1 - \overline{\varphi} A_1\right)}{A_1 h_{\min}\left(1 - \overline{\varphi}\right)} > \sum_{j=2}^{J} \lambda_j A_j \qquad \text{if } \overline{\theta} > 1. \tag{125}$$

- $$A_1 < \frac{Rq + h_{\min}}{\left(1 + \lambda_1\right) h_{\min}} \qquad \text{if } \overline{\theta} = 1. \tag{126}$$ ∥

The proof of existence of the equilibrium path on which growth is supported by human capital investment of each and all types is enclosed in the following theorem

Proposition 17 A competitive equilibrium where the entire population acquires privately financed education exists in each and all time periods of the interval $t \in \left[\tau_n, \infty\right)$ on the condition of occurrence of the following requirements

- All types optimally choose to invest in individual improvement.
- The credit market is privately sustainable.
- Individual saving be non-negative for each and all individual types.

Proof

- Each and every individual type must be induced to engage in human capital investment. Once again, this is guaranteed only insofar as

$$A_j^{\delta} H_t^{1-\delta} > h_{\min} + Rq \qquad \forall j \in \{1, \ldots, J\}, \forall t \in \left[\tau_n, \infty\right). \tag{127}$$

Evidently, it suffices to impose relationship (127) for type $j=1$ to ensure its validity for all remaining types $j=2,\ldots,J$. Hence,

$$A_1^{\delta}\,H_t^{1-\delta} > h_{\min} + R\,q \qquad \forall\, t\in\left[\tau_n,\infty\right) \tag{127'}$$

Evaluated in the linear case, expression (127') yields

$$A_1 > \frac{h_{\min}+Rq}{H_t} \qquad \forall\, t\in\left[\tau_n,\infty\right). \tag{127''}$$

- In light of Corollary 1, the human capital credit market is sustainable upon the condition that feasibility is established for the lowest-ability agents. The condition translates into

$$A_1^{\delta}\,H_t^{1-\delta} > R\,q \qquad \forall\, t\in\left[\tau_n,\infty\right). \tag{128}$$

- Saving must be positive for all types of agents. Invoking the optimal saving function (equation 8) this means

$$s_{2t+1}^{j\,*} = \left(1-\beta\right)\left(A_j^{\delta}\,H_t^{1-\delta} - R\,q\right) \qquad \forall j\in\left[1,\ldots,J\right], \forall\, t\in\left[\tau_n,\infty\right), \tag{129}$$

which is strictly positive on the basis of optimality conditions (127). Once again, we need only establish that saving be positive for the lowest-type agents, which evidently is met under the condition (127'). It is trivial to show that *Proposition* 17 is established for the linear case upon the imposition of condition (127''). The validity of the remaining relations is logically implied.‖

Our argument on the non-monotonic dynamics of the economy-wide distribution ought not to be qualitatively sensitive to an analysis of higher degree of heterogeneity. Following the same proof procedure, the essence of *Proposition* 7 is here established in the form of a more general argument.

We recall that the income share of the class of type-*j* individuals, at time period *t*, is defined as $sh_t^j \equiv y_t^j / Y_t$, $\forall j\in\{1,\ldots,J\}$. The economy-wide income distribution is represented by the set of shares of all income classes $\mathbf{sh}_{t+1}^{\ell} = \left\{sh_{t+1}^{1\,\ell},\ldots,sh_{t+1}^{J\,\ell}\right\}$, for $t\geq 0$, with ℓ denoting the stage of equilibrium growth, $\ell\in\{I,II,\ldots,J-k+1\}$. We now proceed to establish that the economy-wide income distribution in the state of poverty Lorenz dominates the distribution of the equilibrium where the subset of the population with abilities ranging in the interval $j \in [k+1, J]$ invests in human capital accumulation.

Invoking the aforementioned definition of income share, as well as the equations on individual and average income (22) and (23) respectively, we obtain the following expression for the share of income classes in the poverty equilibrium:

$$sh_{t+1}^{j\,I} = 1 \qquad j \in [1, J], \forall t \in [0, r-1]. \tag{130}$$

As has been previously stated, in the phase of underdevelopment genetic differences in learning aptitude *vanish* in the sense that they are not reflected in the income earning ability of agents. This is a state of perfect income equality, with all agents earning the minimum average income, $h_{\min}$.

We recall that adaptations in the legislative system to accommodate *strong* legal protection of creditors are effective on period $t = r$. Along the growth path following such development the various types of agents earn the respective income shares

$$sh_{t+1}^{j\,II} = h_{\min} \Big/ \left\{ h_{\min} \sum_{j=1}^{k} \lambda_j + H_t^{II\,1-\delta} \sum_{j=k+1}^{J} \lambda_j A_j^{\delta} \right\} \; j \in [1, k], \forall t \in [r, \tau_1 - 1]. \tag{131}$$

$$sh_{t+1}^{j\,II} = A_j^{\delta} H_t^{II\,1-\delta} \Big/ \left\{ h_{\min} \sum_{j=1}^{k} \lambda_j + H_t^{II\,1-\delta} \sum_{j=k+1}^{J} \lambda_j A_j^{\delta} \right\}$$

$$j \in [k+1, J], \forall t \in [r, \tau_1 - 1]. \tag{132}$$

Applying the criterion of Lorenz superiority, we obtain that the income distribution $\mathbf{sh}_{t+1}^{I}$ Lorenz dominates distribution $\mathbf{sh}_{t+1}^{II}$ under the condition that the following requirements be met

- $\lambda_1 sh_{t+1}^{1\,II} \leq \lambda_1 sh_{t+1}^{1\,I} \qquad \forall t \in [r, \tau - 1],$ (133)

implying

$$h_{\min} \sum_{j=k+1}^{J} \lambda_j \leq H_t^{II\,1-\delta} \sum_{j=k+1}^{J} \lambda_j A_j^{\delta} \qquad \forall t \in [r, \tau_1 - 1]. \tag{133'}$$

- $\lambda_1 sh_{t+1}^{1\,II} + \lambda_2 sh_{t+1}^{2\,II} \leq \lambda_1 sh_{t+1}^{1\,I} + \lambda_2 sh_{t+1}^{2\,I} \qquad \forall t \in [r, \tau_1 - 1],$ (134)

which also reduces to relation (133').

- $\lambda_1 sh_{t+1}^{1\,II} + \lambda_2 sh_{t+1}^{2\,II} + \ldots + \lambda_k sh_{t+1}^{k\,II} \le \lambda_1 sh_{t+1}^{1\,I} + \lambda_2 sh_{t+1}^{2\,I} + \ldots + \lambda_k sh_{t+1}^{k\,I}$

$$\forall t \in [r, \tau_1 - 1], \qquad (135)$$

similarly being true on the basis of relation (133').

- $\lambda_1 sh_{t+1}^{1\,II} + \ldots + \lambda_k sh_{t+1}^{k\,II} + \lambda_{k+1} sh_{t+1}^{k+1\,II} \le \lambda_1 sh_{t+1}^{1\,I} + \ldots + \lambda_k sh_{t+1}^{k\,I} + \lambda_{k+1} sh_{t+1}^{k+1\,I}$

$$\forall t \in [r, \tau_1 - 1], \qquad (136)$$

which reads into

$$h_{\min} \sum_{j=1}^{k} \lambda_j + \lambda_{k+1} A_{k+1}^{\delta} H_t^{II\,1-\delta} \le \sum_{j=1}^{k+1} \lambda_j \left\{ h_{\min} \sum_{j=1}^{k} \lambda_j + H_t^{II\,1-\delta} \sum_{j=k+1}^{J} \lambda_j A_j^{\delta} \right\}$$

$$\forall t \in [r, \tau_1 - 1]. \qquad (136')$$

- $\lambda_1 sh_{t+1}^{1\,II} + \ldots + \lambda_k sh_{t+1}^{k\,II} + \ldots + \lambda_J sh_{t+1}^{J\,II} \le \lambda_1 sh_{t+1}^{1\,I} + \ldots + \lambda_k sh_{t+1}^{k\,I} + \ldots + \lambda_J sh_{t+1}^{J\,I}$

$$\forall t \in [r, \tau_1 - 1]. \qquad (137)$$

whose validation can be much too easily proved (each side equals unity).

In the last stage of development, $\bar{\ell} \equiv J - k + 1$, educational investment is consistent with optimal incentives for the array of all types of individuals. Along this equilibrium state, the various classes of agents earn the following share of aggregate output

$$sh_{t+1}^{j\,\bar{\ell}} = A_j^{\delta} \Big/ \sum_{j=1}^{J} \lambda_j A_j^{\delta} \qquad j \in [1, J], \forall t \in [\tau_n, \infty). \qquad (138)$$

We establish that distribution $\mathbf{sh}_{t+1}^{II}$ is characterized by greater income inequality in terms of Lorenz superiority compared to the income distribution of the last phase of development, $\mathbf{sh}_{t+1}^{\bar{\ell}}$. As it is known, the proof entails the following requirements

- $\lambda_1 sh_{t+1}^{1\,II} \le \lambda_1 sh_{t+1}^{1\,\bar{\ell}} \qquad \forall t \in [\tau_n, \infty), \qquad (139)$

which is satisfied upon the truth of the following relation

$$h_{\min} \sum_{j=1}^{J} \lambda_j A_j^{\delta} \le A_1^{\delta} \left\{ h_{\min} \sum_{j=1}^{k} \lambda_j + H_t^{II\,1-\delta} \sum_{j=k+1}^{J} \lambda_j A_j^{\delta} \right\} \qquad \forall t \in [\tau_n, \infty). \qquad (139')$$

It must further be met

- $$\lambda_1 sh_{t+1}^{1\,II} + \lambda_2 sh_{t+1}^{2\,II} \le \lambda_1 sh_{t+1}^{1\,\bar{\ell}} + \lambda_2 sh_{t+1}^{2\,\bar{\ell}} \qquad \forall t \in [\tau_n, \infty), \tag{140}$$

being validated on the basis of the condition

$$(\lambda_1 + \lambda_2) h_{\min} \sum_{j=1}^{J} \lambda_j A_j^{\delta} \le (\lambda_1 A_1^{\delta} + \lambda_2 A_2^{\delta}) \left\{ h_{\min} \sum_{j=1}^{k} \lambda_j + H_t^{II\,1-\delta} \sum_{j=k+1}^{J} \lambda_j A_j^{\delta} \right\} \qquad \forall t \in [\tau_n, \infty). \tag{140'}$$

- $$\lambda_1 sh_{t+1}^{1\,II} + \lambda_2 sh_{t+1}^{2\,II} + \ldots + \lambda_k sh_{t+1}^{k\,II} \le \lambda_1 sh_{t+1}^{1\,\bar{\ell}} + \lambda_2 sh_{t+1}^{2\,\bar{\ell}} + \ldots + \lambda_k sh_{t+1}^{k\,\bar{\ell}} \qquad \forall t \in [\tau_n, \infty), \tag{141}$$

being equivalent to

$$h_{\min} \sum_{j=1}^{k} \lambda_j \sum_{j=1}^{J} \lambda_j A_j^{\delta} \le \sum_{j=1}^{k} \lambda_j A_j^{\delta} \left\{ h_{\min} \sum_{j=1}^{k} \lambda_j + H_t^{II\,1-\delta} \sum_{j=k+1}^{J} \lambda_j A_j^{\delta} \right\} \qquad \forall t \in [\tau_n, \infty). \tag{141'}$$

- $$\lambda_1 sh_{t+1}^{1\,II} + \ldots + \lambda_k sh_{t+1}^{k\,II} + \lambda_{k+1} sh_{t+1}^{k+1\,II} \le \lambda_1 sh_{t+1}^{1\,\bar{\ell}} + \ldots + \lambda_k sh_{t+1}^{k\,\bar{\ell}} + \lambda_{k+1} sh_{t+1}^{k+1\,\bar{\ell}} \qquad \forall t \in [\tau_n, \infty), \tag{142}$$

implying

$$\left\{ h_{\min} \sum_{j=1}^{k} \lambda_j + \lambda_{k+1} A_{k+1}^{\delta} H_t^{II\,1-\delta} \right\} \sum_{j=1}^{J} \lambda_j A_j^{\delta} \le \sum_{j=1}^{k+1} \lambda_j A_j^{\delta} \left\{ h_{\min} \sum_{j=1}^{k} \lambda_j + H_t^{II\,1-\delta} \sum_{j=k+1}^{J} \lambda_j A_j^{\delta} \right\}, \qquad \forall t \in [\tau_n, \infty). \tag{142'}$$

Lastly,

- $$\lambda_1 sh_{t+1}^{1\,II} + \ldots + \lambda_k sh_{t+1}^{k\,II} + \ldots + \lambda_{J-1} sh_{t+1}^{J-1\,II} \le \lambda_1 sh_{t+1}^{1\,\bar{\ell}} + \ldots + \lambda_k sh_{t+1}^{k\,\bar{\ell}} + \ldots + \lambda_{J-1} sh_{t+1}^{J\,\bar{\ell}} \qquad \forall t \in [\tau_n, \infty), \tag{143}$$

which yields

$$\left\{ h_{\min} \sum_{j=1}^{k} \lambda_j + H_t^{II\,1-\delta} \sum_{j=k+1}^{J-1} \lambda_j A_j^{\delta} \right\} \sum_{j=1}^{J} \lambda_j A_j^{\delta} \le \sum_{j=1}^{J-1} \lambda_j A_j^{\delta} \left\{ h_{\min} \sum_{j=1}^{k} \lambda_j + H_t^{II\,1-\delta} \sum_{j=k+1}^{J} \lambda_j A_j^{\delta} \right\}$$

$$\forall t \in [\tau_n, \infty). \qquad (143')$$

The reader may find the *Kuznets curve* of this version of the model in the Appendix.

VII. *Less stringent punishment scheme*

In this section we pursue an extension of the model in which individual consumption in old age is expanded to include a fixed retirement income. By assumption, all economic agents receive the same real endowment, irrespective of one's educational status. The objective underlying this approach is to determine the *ability* of the economy to sustain the existence of a private credit market in the education area when the punishment scheme is effectively weakened. This is examined by actually rendering default less costly (whereas previously $V^{E,D}(j) \to -\infty$, in the present case $V^{E,D}(j) > 0$). We show that a market for education loans may be privately sustained in this context, albeit at the cost of stricter conditions.

The budget constraint in the last period of life is modified to be

$$c^j_{3t+2} = \omega_3 + R\, s^j_{2t+1}, \qquad (144)$$

where ω_3 denotes the endowment in real units. Adopting the utility function of logarithmic form (7), optimal saving is now expressed as

$$s^{j*}_{2t+1} = \begin{cases} \dfrac{(1-\beta)R\left(A_j^{\delta}\, H_t^{1-\delta} - Rq\right) - \beta\omega_3}{R} & \text{if } q > 0 \\[2ex] \dfrac{(1-\beta)R\, h_{\min} - \beta\omega_3}{R} & \text{if } q = 0 \end{cases} \qquad \forall j = L, H. \qquad (145)$$

Upon substitution of the saving function (145) into the budget constraints (5") and (144), we obtain the optimal second- and third-period consumption, respectively given by

$$c^{j*}_{2t+2} = \begin{cases} \dfrac{\beta\left\{R\left(A_j^{\delta}\, H_t^{1-\delta} - R\,q\right) + \omega_3\right\}}{R} & \text{if } q > 0 \\[2ex] \dfrac{\beta\left\{R\, h_{\min} + \omega_3\right\}}{R} & \text{if } q = 0 \end{cases} \qquad \forall j \in \{L, H\}. \qquad (146)$$

$$c_{3t+2}^{j*} = \begin{cases} (1-\beta)\left\{R\left(A_j^{\delta}\, H_t^{1-\delta} - R\,q\right) + \omega_3\right\} & \text{if } q > 0 \\ (1-\beta)\left(R\,h_{\min} + \omega_3\right) & \text{if } q = 0 \end{cases} \quad \forall j \in \{L, H\}. \tag{147}$$

State of underdevelopment Given that a member of generation t receives education her intertemporal consumption if she chooses to default is described by equations

$$\left(c_{2t+1}^{j*}\right)^{WR,D} = \frac{\beta\left(R\,A_j^{\delta}\, H_t^{1-\delta} + \omega_3\right)}{R} \qquad \forall j \in \{L, H\}, \tag{148}$$

and

$$\left(c_{3t+2}^{j*}\right)^{WR,D} = (1-\beta)\left(R\;A_j^{\delta}\; H_t^{1-\delta} + \omega_3\right) \qquad \forall j \in \{L, H\}, \tag{149}$$

When remaining loyal to contract commitment optimal adult- and old-age consumption is given by, respectively

$$\left(c_{2t+1}^{j*}\right)^{ND} = \frac{\beta\left\{R\left(A_j^{\delta}\, H_t^{1-\delta} - R\,q\right) + \omega_3\right\}}{R} \qquad q > 0\,,\ \forall j \in \{L, H\}, \tag{150}$$

and

$$\left(c_{3t+2}^{j*}\right)^{ND} = (1-\beta)\left\{R\left(A_j^{\delta}\; H_t^{1-\delta} - Rq\right) + \omega_3\right\} \qquad q > 0\,, \forall j \in \{L, H\}, \tag{151}$$

where $\left(c_{\nu\, t+1}^{j\,*}\right)^{WR,ND} = \left(c_{\nu\, t+1}^{j\,*}\right)^{SR,ND} \equiv \left(c_{\nu\, t+1}^{j\,*}\right)^{ND}$ for ν = 2, 3. Evidently, utility is higher when evading debt obligations due to higher second-period consumption, and because agents can still engage in intertemporal smoothing through saving. Hence,

$$V_t^{WR,D}(j) > V_t^{ND}(j) \qquad q > 0\,, \forall j \in \{L, H\}, \tag{152}$$

where

$$V_t^{WR,D}(j) = \ln\left\{\vartheta\left(R\,A_j^{\delta}\; H_t^{1-\delta} + \omega_3\right)\right\} \qquad \forall j \in \{L, H\}, \tag{153}$$

and

$$V_t^{E,ND}(j) = \ln\left\{\vartheta\left(R\left(A_j^{\delta}\; H_t^{1-\delta} - Rq\right) + \omega_3\right)\right\} \qquad q > 0\,, \forall j \in \{L, H\}, \tag{154}$$

with $\vartheta \equiv \beta^{\beta}(1-\beta)^{1-\beta}/R^{\beta}$. Obviously, it applies $V_t^{WR,ND}(j)=V_t^{SR,ND}(j)\equiv V_t^{E,ND}(j)$ $\forall j=L,H$. The conclusion is reached that an individual who acquires education shall always commit default on her debt. Were one to receive no education she would earn the unskilled income $h_{\min}$, and hence lifetime utility

$$V_t^{NE}(j)\equiv V_t^{NE}=\ln\left[\vartheta\left(R\,h_{\min}+\omega_3\right)\right]\quad \forall j\in\{L,H\}. \tag{155}$$

Drawing upon *Assumption* 1, we infer that remaining unskilled is never the preferred choice. It is evident that

$$V_t^{WR,D}(j)>V_t^{NE}\qquad \forall j\in\{L,H\} \tag{156}$$

Similarly to the core version (where $\omega_3=0$) the model predicts that financial institutions engage in no educational funding as a result of the expected lack of commitment on part of borrowers. Once again, owing to the rationing of all credit, the entire population remains uneducated earning the minimum income of unskilled labor. On the assumption that a system of weak legal rights prevails in each period of the time interval $t\in[0,r-1]$, $r>0$, the equilibrium path has the characteristics of a poverty trap

$$y_{t+1}^{j}=h_{t+1}^{j}=h_{\min}\quad \forall j\in\{L,H\},\forall t\in[0,r-1]. \tag{157}$$

The competitive outcome along this equilibrium path prescribes that the economy produces the time-invariant quantity

$$Y_{t+1}^{NE}=H_{t+1}^{NE}=h_{\min}, \tag{158}$$

where we recall $y_0^j\equiv y_0=h_{\min},\forall j$, thus $Y_0=h_{\min}$.

A segment of the society invests Once again, we take as our basis that effective on period $t=r$, $r>1$, legislation entitles creditors to seize the entire assets of a debtor in default, effectively prohibiting the latter from any act of saving as time unfolds.

The contract design elicits promise-keeping behavior for the high-type agent only insofar as

$$V_t^{E,ND}(j)>V_t^{SR,D}(j)\qquad for\ j=H\ ,\forall t\geq r+1. \tag{159}$$

The present utility value associated with repudiating on one's debt is given by

$$V_t^{SR,D}(j) = \ln\left\{\left(A_j^{\delta} H_t^{1-\delta}\right)^{\beta} \omega_3^{1-\beta}\right\} \qquad \forall j = L, H, \forall t \geq r+1. \tag{160}$$

Drawing upon equation (154), there results that the following condition must be imposed

$$\vartheta\left\{R\left(A_H^{\delta} H_t^{1-\delta} - Rq\right) + \omega_3\right\} > \left(A_H^{\delta} H_t^{1-\delta}\right)^{\beta} \omega_3^{1-\beta} \qquad \forall t \geq r+1. \tag{159'}$$

Taking as our basis *Assumption* 2, we note again that relation (159) cannot possibly hold for the low-type agents, since the logarithmic function $V^{E,ND}(j)$ is non-definable on a negative argument. Insofar as the only possibility is to renege on the agreement, the household attains the utility level associated with no consumption smoothing. $V^{E,ND}(L)$ in effect degenerates to individual welfare $V^{SR,D}(L)$. We quote the proposition

Proposition 18 The contract arrangement (q, R) offered in a system where borrowers have no access to savings opportunities conditional on default may be supported as a self-enforcing contract. Granting the feasibility of loan repayment, if consumption in old age falls not below the fixed threshold ω_3, constraint (159') is required in order for borrowers to renege not on agreed obligations.

The following theorem is proved without difficulty

Proposition 19 A private credit market for human capital investment is *sustainable* if and only if

- Given the feasibility of loan repayment agents seeking credit are offered a self-enforcing contract.
- Individuals for whom debt repayment is non-feasible prefer to receive no education.

Proof

- We have established that upon the validity of relationship (159'), high-ability agents optimally choose to adhere to the contract agreement.
- The second necessary condition requires that low ability agents optimally choose to remain unskilled. On the basis of a plausible condition, it holds that low-type agents indeed prefer to earn the low income of unskilled labor, while maintaining their ability to

enhance consumption in the retirement age beyond the fixed endowment ω_3. The optimality condition states:

$$V_t^{NE} > V_t^{E,D}(L) \qquad \forall t \geq r+1. \tag{161}$$

Upon invoking equations (155) and (160), we obtain that the following hypothesis must be imposed:

$$\vartheta(Rh_{\min} + \omega_3) > \left(A_L^{\delta} H_t^{1-\delta}\right)^{\beta} \omega_3^{1-\beta} \qquad \forall t \geq r+1. \tag{161'} \quad \|$$

We conclude this section with the following proposition:

Proposition 20 A competitive equilibrium with a subset of population acquiring privately financed education exists on the condition of occurrence of the following requirements

- The measure of high-type agents optimally chooses to obtain education.
- The credit market is privately sustainable.
- Individual savings is non-negative for both *H* and *L* types.

Proof

- The measure of high-type agents is induced to participate in the contract arrangement if the following condition is met

$$V_t^{E,ND}(H) > V_t^{NE} \qquad \forall t \geq r. \tag{162}$$

which, upon invoking equations (154) and (155), translates into

$$A_H^{\delta} H_t^{1-\delta} > R\,q + h_{\min} \qquad \forall t \geq r. \tag{162'}$$

Evaluated in period $t = r$, we have

$$A_H^{\delta} h_{\min}^{1-\delta} > R\,q + h_{\min}. \tag{162''}$$

Evidently, it suffices to impose relationship (162'') to ensure the validity of (162') in all forthcoming periods, $t > r$.

- In light of *Proposition* 19, the human capital credit market is privately sustainable upon the validity of *Assumption* 2, as well as conditions (159') and (161').

- The individual rationality constraint for financial entities entails that saving be positive for both types of individuals, which reads into the conditions

$$A_H^{\delta}\, H_t^{1-\delta} \geq Rq + \frac{\beta\omega_3}{(1-\beta)R} \qquad \forall t \geq r\ . \tag{163}$$

and

$$h_{\min} \geq \frac{\beta\omega_3}{(1-\beta)R} \qquad \forall t \geq r\ . \tag{164}$$

The proof of *Proposition* 20 consists of imposing *Assumption* 2(β), and relations (159'), (161'), (162'') and (164). The truth of the remaining conditions is then logically inferred.‖

The dynamic evolution of the society's stock of human capital along the aforementioned equilibrium path is governed by the first-order non-linear difference equation (30). Similarly to the baseline model, the solution to the linear version is given by the expressions (31).

All types invest Due to perpetual growth the stock of aggregate knowledge reaches a critical threshold, above which individuals of both types have an optimal incentive to invest. Similarly to the core version of the model, we establish this proposition for the case of the linear human capital technology. A restatement of *Proposition* 4 reads as follows

Proposition 21 There exists a time period $\hat{\tau} > 0$, where $\hat{\tau} \in [r+1, \infty)$, in which the income realization of educated low-type agents exceeds the threshold level that defines education the optimal choice. Mathematically, this reads into

$$A_L^{\delta}\, H_t^{1-\delta} \begin{cases} \geq R\,q + h_{\min} & \forall\, t \in [\hat{\tau}, \infty) \\ < R\,q + h_{\min} & \forall\, t \in [0, \hat{\tau}-1] \end{cases}. \tag{165}$$

Considering the linear human capital technology, $\hat{\tau}$ is defined as

$$\hat{\tau} = \begin{cases} \dfrac{\ln[\Theta]}{\ln[\lambda A_H]}, & if\ \lambda A_H > 1 \\ \dfrac{Rq + h_{\min}(1 - A_L)}{A_L h_{\min}(1-\lambda)}, & if\ \lambda A_H = 1 \end{cases}, \tag{166}$$

with $\Theta \equiv \dfrac{(Rq + h_{\min})(1 - \lambda A_H) - A_L h_{\min}(1-\lambda)}{A_L h_{\min} \lambda (1 - A_H)}$.

The condition that defines education the optimal choice (relation 162), is written as

$$A_L^{\delta} H_t^{1-\delta} \geq R q + h_{\min}. \tag{165'}$$

This is identical to the corresponding condition in the baseline version of the model $(\omega_3 = 0)$. It is only evident that the two versions imply identical solutions for the critical time period τ, hence $\tau = \hat{\tau}$. The proof of the theorem being identical to that of *Proposition* 4 is here omitted.

Proposition 1 is rephrased in our context to read as follows

Proposition 22 The contract arrangement (q, R) offered in a system where borrowers have no access to savings opportunities conditional on default, may be supported as a self-enforcing contract. Granting the feasibility of loan repayment for all types of agents, conditions (167') ought to be imposed in order for borrowers to renege not on agreed obligations.

Proof

The feasibility of loan repayment is ensured for both types of agents under the fulfillment of relation (165'). It goes without saying, then, that $V^{E,ND}(j)$ is a definable function, and positive for both H and L types (see equation 154). The contract arrangement (q, R) is self-enforcing for agents of both ability levels on condition that relation (159) applies for $\forall j \in \{L, H\}$. Evidently, this reads into

$$\vartheta\left\{R\left(A_L^{\delta} H_t^{1-\delta} - Rq\right) + \omega_3\right\} > \left(A_L^{\delta} H_t^{1-\delta}\right)^{\beta} \omega_3^{1-\beta} \qquad \forall t \geq r+1, \tag{167α}$$

$$\vartheta\left\{R\left(A_H^{\delta} H_t^{1-\delta} - Rq\right) + \omega_3\right\} > \left(A_H^{\delta} H_t^{1-\delta}\right)^{\beta} \omega_3^{1-\beta} \qquad \forall t \geq r+1, \tag{167β}$$

The theorem logically follows:

Proposition 23 The credit market for educational investment is privately *sustainable* in each and all time periods $t \in [\hat{\tau}, \infty)$ given the validity of condition (165'), and optimality constraints (167).

The proof of existence of the equilibrium in which the entire population invests is enclosed in the following proposition

Proposition 24 A competitive equilibrium where the entire population acquires privately financed education exists on the condition of occurrence of the following requirements

- Both types make the optimal decision to invest in individual improvement.
- The credit market is privately sustainable.
- Individual saving is non-negative for both *H* and *L* types.

Proof

- The *participation constraint* for the borrower side entails that condition (162) applies for both types of agents. This translates into

$$A_j^{\delta} H_t^{1-\delta} > R\,q + h_{\min} \qquad \forall j = L, H\,, \tag{168}$$

which is reducible to the optimality condition (165').

- Drawing on *Proposition* 24, we assert that the financial market for human capital investment is privately sustainable upon the validity of optimality conditions (165'), and (167*α*, *β*).

- Loan provision ought to be individually rational for credit institutions, which implies the conditions

$$s_{2t+1}^{j*} = (1-\beta)\left(A_j^{\delta} H_t^{1-\delta} - R\,q\right) \qquad \forall j \in \{L, H\}, \forall t \in [\hat{\tau}, \infty). \tag{169}$$

It is evident that individual saving is strictly positive for both types on the basis of optimality condition (165'). In summary, the proof of this proposition requires the validity of optimality conditions (165'), and (167*α*, *β*).‖

As in the core version of the model, the dynamic evolution of the economy's aggregate stock of knowledge is governed by the non-linear difference equation (42). The solution to the linear case is given by equations (43). An analysis on income distribution is omitted here due to being identical to that of the benchmark version (see Section IV).

VIII. *Concluding remarks*

Let us cast a glance backward on the course of this essay. We sought to construct a theory which in a novel way lends truth to the proposition formed by Kuznets (1955), with respect to the non-monotonic relationship between prosperity and inequality of income distribution. Our attention centered on the role of financial markets in defining the process of economic development, and ultimately the distribution of income earning capabilities in a population of *ex ante* heterogeneous individuals. If the roots of development lie in human capital accumulation, the possibility to fund educational choices through private credit organizations is critical in its own right. The theory abstracts from the possibility of education be publicly provided, and of alternative means of financing human capital investment, through wealth possessions or forms of inherited bequests. Owing to this confinement it is a consequence of the failure of the credit market that individuals may be entirely barred from productive educational choices. In this circumstance, the potential for differing earning productivities remains unrealized, with all workers being *trapped* in the choice of a single low-income occupation, and therefore identical earnings. The provision of credit in this market is hindered by one-sided lack of commitment, and particular enforcement issues embedded in the area of educational investment. Contract enforcement hinging on the nature of consequences following an act of default ultimately is a matter of the legislative system. In the tradition of Kehoe and Levine (1993) we assume that legislation accommodates the complete and permanent exclusion of defaulting borrowers from financial markets. The prospect of being prohibited to invest in tangible assets induces agents to choose commitment to previous agreements. Contract arrangements thus become enforceable, leading credit institutions to eagerly engage in educational funding. This is the critical requirement for the economy to be carried on a dynamic path of ever sustained growth, escaping a poverty loophole. We trace out paths of development so constructed as to give an explicit proof of the *trickle-down* theory of economic growth. Initially, an equilibrium is taken to exist in which a particular group of individuals, those with the highest investment return, only choose to engage in education. Owing to the accumulation of human capital and the associated externality on future generations' knowledge productivity, the economy ultimately makes its transition to a state where the aggregate of all agents invest in individual improvement. As endogenous technological knowledge takes off, the externality effect arising from knowledge spillovers gives rise to inverted-*U* dynamics in the evolution of income distribution. A pattern of worsening inequality prevails in early stages of growth. However, as dynamics bring the economy on a more evolved stage, income differentials appear to shrink. Income convergence is established to be the signal that an advanced level of development has been attained.

Following the mainstream tradition, our enquiry on the dynamic pattern of wage income distribution was accommodated in the context of a deterministic theory. Implicit in our model is the assumption that no unpredictable change ever occurs, or at least economic agents believe that it doesn't. Consequently, the study remains silent about the impact of uncertainty on the actions of individual agents, and thus on the dynamics of the system. A model denying that economic variables may be inherently unpredictable leaves valid questions quite unanswered, thus weakening the position of proven theorems. An endeavor sought in future work is to extend the existing analysis in a way that uncertainty is embedded into investment decisions in human capital.

ESSAY II

*Money's Role in Determining Long-Run Growth**

**The present essay was written under the supervision of Dr. Mohammed Mohsin, during the academic semester of spring 2004, at the University of Tennessee, Knoxville. The research analysis is the product of the author's collaborated work with Dr. Mohsin. The literature review in this essay (introductory section pp.81-103) is the sole work of the author of the doctoral dissertation, written and edited at a later time.*

Introduction

Theme and outline of the essay This essay constitutes a contribution in the theoretical research elaborating on the importance of monetary policy in determining an economy's long run growth prospects. Exploiting the developments in the theory of endogenous growth, perpetual unbounded growth is sustained upon the accumulation of a broad concept of capital, encompassing both physical and human notions. In a framework of endogenously determined growth it is possible to analyze the effects of economic policy on the *growth rate* of aggregate real variables. At this level, the analysis departs from the traditional approach in the literature, which had been to focus on the impact of monetary policy on the steady state *levels* of real economic aggregates. Theorists in the monetary literature concentrated early on, on how developments in financial intermediation by imposing a role for money in the payment system, in essence define the way monetary policy exerts its influence on the real economy. In an influential paper, Stockman (1981) proposed that when a credit market for consumption and capital goods is missing, distortionary monetary policy interacts with private capital decisions, causing investment, and real output to *fall* at a lower steady state level. The crucial feature of the model that generates this result is that money is required to purchase capital in addition to consumption goods. In this case investment is taxed twice; once through the effect of inflation on current capital purchases, and secondly, through the impact of future inflation on future consumption spending. However, if money is not required in transactions involving capital, then there is a gain from additional investment in the form of lower current money holdings that offsets the lower utility yield from investment due to higher future inflation. In this case, the *superneutrality* proposition reclaims validity on theoretical ground. A number of authors dealt thoroughly with the applicability of Stockman's (1981) argument in the context of endogenous growth, and this is the area this study belongs as well. Relying on different mechanisms -Jones and Manuelli (1993), Marquis and Reffett (1995) and De Gregorio (1993) specify different applications of the *investment channel* within the context of constant-returns-to scale equilibrium growth, whereas Marquis and Reffett (1991*a*) employ a human capital channel- they all get across the same idea: distortionary monetary policy affects the prospects of growth negatively. Thus, the cited studies all bring a valid argument in favor of the *nonneutrality of money*; the latter yet, is established in the context of closed-economy settings. Herein lays the contribution of the following research essay. The analysis is carried with reference to an economy being open, yet a price taker in the international capital markets. The aim is once again to examine the theoretical validity of Stockman's (1981) argument in the context of equilibrium growth remains.

It goes without saying that when credit markets do not function perfectly, or are entirely missing, money is assigned the role of the primary, or sole, medium of exchange. We postulate that the possibility of intermediated credit does not exist, the intention of the assumption being to uncover the role of inflation as tax on private spending. Initially, the postulate applies on purchases of consumption goods only. In an alternative version of the model (titled *Model II*) the investment on capital goods is also being subjected to the constraint that cash holdings are the only means of conducting the transaction. In this latter case, inflation bears an evident analogy to a capital tax. The theory been constructed thus gives us an insight into how inflation is been conceived to imitating fiscal tax instruments.

To elucidate the consequences of endogenously determined labor, the theory is initially built on models that abstract from the decision to allocate time between leisure and other productive activities. Using a model of inelastic labor to analyze the consequences of policy changes may prove limiting, since a tax on consumption and wage income operate as non-distortionary taxes in such an environment. Further, there is merit in the task itself of exploring how the decision to allocate time between various activities interacts with the intertemporal allocation of consumption to determine the dynamic and long-term growth behavior of an economy. The analysis is been extended to account for the endogeneity of the time-allocation decision in the latter part of this essay (titled *Models III* and *IV*).[46]

A word about the essay's arrangement. The following section presents an elaborate review of the literature on monetary growth theory. The subsequent section proceeds with the exposition of the research analysis, comprising of the aforementioned (four) analytical models. A brief discussion in the end takes the role of final conclusion.

Theory of money and growth The effects of monetary growth on the real side of the economy have been the subject of research of an enormous amount of the theoretical literature in macroeconomics. This literature starts with the classic contributions of Tobin (1965), and Sidrauski (1967). Using the framework of the conventional exogenous growth model, Tobin emphasized the portfolio substitution effect, according to which agents, as a result of higher inflation, reallocate their savings in favor of capital and away from nominal assets. Thus, he argued that monetary growth, and therefore inflation, is positively related to the economy's long run capital stock. Sidrauski (1967) took up the same question by developing a model where savings and money demand functions are derived from the optimizing behavior of agents, rather than being postulated and held fixed as in Tobin's framework. The major path through which money affects the workings of the economy in the Tobin model is through its effect on the real disposable income, which in turn determines the consumption (or equivalently savings) behavior of the individuals [Levhari and Patinkin, 1968, p.714]. In Sidrauski's model, it is assumed that expectations are adaptive; in other words, they are induced from the past history of changes of the relevant variables, and when they do not materialize individuals partially revise them. The model concludes that the long-run stock of capital depends only on the latter's depreciation rate, the population's growth rate, and the representative agent's subjective discount rate. Thus, money is found to be superneutral in the sense that the steady state capital stock is independent of changes in the rate of money growth and inflation. Fischer (1979) complemented the work of Sidrauski by examining whether the

[46] The last two models differ with respect to the exchange function of money, in the same vein as *Models I* and *II* respectively.

superneutrality result obtains also on the transition path towards the long run equilibrium. He found that even in the original Sidrauski model, the path the economy takes to the steady state is not invariant to the rate of monetary growth. The Tobin effect may prevail at every point along the transition path but the steady state.

The subsequent literature evolved in several directions with the main aim to analyze the robustness of the superneutrality result in different frameworks. Brock (1974) extended Sidrauski's work by developing an intertemporal optimizing growth model where expectations are endogenously determined so that perfect foresight obtains. In addition, labor supply is no longer assumed to be perfectly inelastic, as is the case in Sidrauski's model. Brock showed that the neutrality result is challenged as long as the marginal utility of consumption and leisure are not independent of money, which, he argues is a plausible assumption.

The superneutrality result is contingent on the assumption of Ricardian debt neutrality, which in Sidrauski's model is ensured by assuming an infinite planning horizon of the economic unit, in other words the birth and death rates are zero. In order to depart from debt neutrality one has to assume that there is entry of new generations in the model and that there is no operational bequest motive, so that the burden of government debt can be passed on to future generations. A number of studies have analyzed the effects of macroeconomic policies in a framework where debt neutrality does not apply. Weil (1986) obtains non-neutrality of money in a model of population growth. Marini and Van der Ploeg (1988) developed a model of finite lifetimes and no intergenerational bequest motive. They assume that the birth and death rates are the same in order to abstract from population growth. Considering the effects of monetary policy under a tax-finance regime, they show that an increase in monetary growth leads in the long run to an increase in capital, output and consumption of physical goods. This effect is very similar to the Tobin effect, yet it is derived from a general equilibrium model with micro foundations. All of the above papers discuss the effects of monetary growth on capital accumulation in closed economy settings. Van der Ploeg (1991) takes the same issue in the context of a two-country optimizing model with uncertain lifetimes, population growth and no intergenerational bequest motive. His findings support the link between

Ricardian debt neutrality and Sidrauski superneutrality. More specifically, as long as the birth rate is positive, both joint and unilateral increases in tax-financed monetary growth lead to global increases in capital accumulation and output. The main result of this paper is therefore to provide a micro foundation for the Tobin effect.

Several earlier theorists concentrated on the question of how the way money is introduced into a model affects the predicted relationship between money growth and capital. Generally speaking there are several alternatives to introduce money into an optimizing model: the money-in-the-utility function, the money-in-the-production function approach, the transactions-costs and cash-in-advance approaches. Examples of the first approach are Sidrauski (1967), Brock (1974), and Fischer (1979) among others.

The sensitivity of the superneutrality result has also been examined within the money in the production-function framework. One of the pioneering studies in this context is Levhari and Patinkin (1968). Using the conventional neoclassical growth model where money provides productive services, they show that superneutrality does not prevail in the level sense. The money-in-the-production-function approach, as is also emphasized by Fischer (1974), essentially recognizes the role of money as a medium of exchange: real money balances provide 'shopping services' in the sense that they enable the economic unit in question to acquire a quantity of commodities. In this way, at an economy-wide level, real balances free resources –labor and capital- for the production of commodities that would otherwise be devoted to sustaining the exchange system in an economy without money. This is the exact meaning lying behind the introduction of money in the production function. Real balances are not described as a factor because they directly increase physical production but rather because they free resources that would otherwise be tied up in transactions. The real effects of alternative rates of inflation have also been analyzed in a similar framework by Dornbusch and Frenkel (1973). Although the interpretation of the role of money in their model is the same, real money balances do not enter directly the production function of physical output, but that of 'delivered' consumption [Dornbusch and Frenkel, 1973 p.152]. The fraction of output that reflects the real costs of 'delivering' output to consumers is assumed to be a decreasing function of real balances, since by definition, the latter are a substitute for the real resources needed to sustain transactions.

Incorporating money in the utility or the production function have been two widely used approaches in the literature of monetary models. One of their main advantages is the generality they bring in producing a demand for money. The use of money in these models is essentially postulated, or better imposed, sometimes even arbitrarily. In this spirit, Clower (1967) has criticized the money-in-the-utility approach that it does not yield a theory where money plays a special role in transactions. In the same vein, Kareken and Wallace (1980) have opposed at the implicit theorizing that takes place in this approach and have argued that underlying consistency cannot be checked. The lack of microfoundations in the use of money was a source of discomfort for many macroeconomists. In response, a number of studies pursued models in which the reasons for the use of money are explicitly described, so that the demand for money emerges from within the model.

Early on, in a seminal study, Saving (1971) objecting to the use of money as an argument in the utility function attempted to 'remedy' this issue by developing a model of transactions costs. The key feature of the transactions technology is that a scarce resource, the agent's time, is used up in transacting. This implies that any change in the economic environment that alters the time spent in transacting directly alters the resources available for work and leisure. What motivates the agent to hold money in this framework is that the transactions time required for each unit of consumption depends negatively on the ratio of real money holdings to his nominal consumption expenditure. It is evident that on this foundation the demand for money is derived from within the model, through the optimizing decisions of the economic agent. Using a similar model, where money balances reduce the costs of transactions, Kimbrough (1986) examined the real effects of an inflationary policy. His model predicts an inverse relationship between inflation and both output and employment, in the level sense. However, due to its analytical difficulty, his model focuses on the analysis of employment and abstracts from capital accumulation. Wang and Yip (1991) fill this gap by developing a tractable shopping-time model of money with capital and endogenous labor-leisure choice which enables them to examine the effects of money growth on both employment and capital accumulation. It is shown that higher rates of money growth have a negative effect on the economy's long run capital stock, employment level, consumption and welfare.

One popular alternative to the transactions cost approach that is often used to motivate a transactions-based demand for money is the cash-in-advance constraint. This is an extreme case of the transactions cost technology described above (the time spent transacting is a decreasing function of the ratio of real balances to consumption expenditure) where the transactions costs are infinite when the ratio of real balances to consumption is smaller than one, and zero otherwise. The cash-in-advance type constraint most commonly adopted in monetary models, and as has been advocated by Clower (1967), states that nominal consumption in the current period cannot exceed nominal money balances carried over from the previous period. An economy with this feature in the money-and-growth literature has been studied by Stockman (1981). In his model, he incorporates into the conventional exogenous growth model the constraint that the individual must be able to finance his purchases of consumption and gross investment out of his current-period money balances. The model reaches the different and surprising –for the time- result that a permanent increase in the rate of monetary growth leads to a decrease in the steady state capital stock. The crucial feature of the model that generates this result is that money is required to purchase capital in addition to consumption goods. In this case investment is taxed twice; once through the effect of inflation on current capital purchases, and second, through the impact of future inflation on future consumption spending. However, if money is not required in transactions involving capital, then there is a gain from additional investment in the form of lower current money holdings that offsets the lower utility yield from investment due to higher future inflation. In this case, it is shown that the steady state capital stock is neutral with respect to higher inflation.[47] Abel (1985) examined the dynamic behavior of the economy along the linearized transition path in Stockman's model, in the same way Fischer (1979) analyzed the transition path in the Sidrauski (1967) model. In particular, he focused on

[47] This is not identical with the result obtained in Sidrauski's (1967) model because in the later the steady state real money balances fall as a result of higher monetary growth, while in the present model they remain unchanged.

the effect of a permanent (unanticipated) increase in monetary growth on the speed of adjustment of the economy towards the steady state. In the case where cash is required in advance for consumption but not for investment purchases, it is shown that money is superneutral along the transition path as well as in the long run. If the cash-in-advance constraint applies to both consumption and investment, Abel finds that the dynamic behavior of the economy is not independent of the rate of monetary growth. The effect on the speed of adjustment, however, can differ dramatically depending on a certain simple function of parameters of preferences and technology. An extension to Abel's work (1985) has been provided by Carmichael (1989). The latter's contribution lies in the fact that he further examines the effects of perfectly *anticipated* monetary policy, whereas Abel is limited to the effects of unexpected changes in the rate of monetary growth. In addition, since his primary interest is to characterize co-movements between output, interest rates, and stock-market prices he abstracts completely from capital accumulation, which is the main consideration in Abel's model. Using a model with endogenous labor supply and money introduced via a cash-in-advance constraint, Carmichael shows that an unanticipated increase in the growth rate of money supply induces agents to substitute leisure for consumption, leading to a negative effect on output. An anticipated increase in the money growth rate has similar real effects by influencing, in this case, inflationary expectations alone.

A different response to the dissatisfaction at the arbitrary use of money in macro models came by Feenstra (1986). In his seminal study, Feenstra (1986) showed that there exists an exact equivalence between a general class of models with liquidity costs appearing in the budget constraint and the money-in-the-utility framework. [48] The former class of models captures many of the conventional models of money demand as special cases, such as the Baumol (1952) and Tobin (1956) transactions models, generalized transactions and precautionary models[49], the cash-in-advance and money-in-the-

[48] Dornbusch and Frenkel (1973) were the first to develop a comparison between the approaches where the demand for real money balances is assumed, pioneered by Tobin (1965), and the money-in-the-utility framework initiated by Sidrauski (1967). The equivalence between these two approaches was first indicated by a simple example in Brock (1974).

[49] In this class of models money is held to finance consumption and there are penalty costs associated with a cash shortfall.

production-function framework. The last is treated as a case of negative liquidity costs. Using the studies of Calvo (1979) and Obstfeld (1984) which investigate multiple stable equilibria in a monetary growth model, Feenstra demonstrates the equivalence between the approaches of entering money in the production and utility function. Proceeding with the case of cash-in-advance constraints, he shows that models that adopt the Clower (1967) constraint that consumption purchases should be financed by money holdings carried over from the previous period, have similar qualitative properties to models where real balances enter as an argument in the utility function and the cross derivative of the latter between goods and money is positive. This result is significant since it is generally known that the sign of this derivative affects the properties of monetary equilibria. The main insight drawn from Feenstra's study is that the superneutrality result obtained by Sidrauski (1967), and others, in optimizing models gains additional validity.

Theory of endogenous growth The studies that we discussed so far all build on the neoclassical theory of economic growth as was developed by Robert Solow (1956). A volume of literature evolved in the 1960s as a response to Solow's (1956) seminal paper aiming to explore variations of the latter. These models are consistent with the premise that the forces that drive economic growth are technology and population growth, with the former taking the leading role. Technological change provides the incentive for continued capital accumulation, and together, capital accumulation and technological change account for much of the increase in output [Romer, 1990 p.72]. The vehicle through which technological progress contributes to growth is its accumulative effect on the economy's stock of knowledge. Knowledge, through its nonrival[50] character, possesses the important feature that it can be accumulated without bound on a per capita basis, thus making possible the occurrence of sustainable growth.

Formally, the standard approach is to incorporate into the production process a separate argument that represents the stock of technological knowledge or more generally, the level of nonrival inputs. The issue that arises in this case, where the nonrival input has a productive value, is that the production technology cannot be a

[50] A purely nonrival good has the property that its use by one firm or person in no way precludes or limits its use by another.

constant-returns-to-scale function of all its inputs taken together. Because of the properties of homogeneous functions it follows that a firm with these kinds of production possibilities could not survive as a price taker, since if all inputs –including technology- were paid their value marginal product, the firm would suffer losses [Romer, 1990 p.76]. The neoclassical growth literature dealt with this issue by treating technology as a public input that is exogenous to the economy (Solow 1956) or is provided by the government (Shell 1966, 1967). Technology is viewed in these models as a purely nonrival and nonexcludable input, whose stock is free to be exploited at zero cost by every individual and firm in the economy.

Clearly, treating the technological factor as public good has the virtue of reconciling the non-convexity in the production possibilities with price-taking behavior. On the other hand, the exogenous specification of technological progress is a technically useful device that offers coherence to a theory of growth that nonetheless does not attempt to analyze the source of technical change.[51] The latter neglect constitutes a major shortcoming of the neoclassical theory of growth. As Lucas (2002) emphasizes

> "Treating exogenous technical change as an engine of growth… is a partial equilibrium argument that simply evades the question of the source of technical change".

He then adds

> "In growth theory, exogenous technological change is just a euphemism for unanalyzed production externalities" [Lucas, 2002 pp.6-7].

A second weakness of the neoclassical growth theory, which evidently follows from its aim not to explain the accumulation of knowledge, is that there is no place in it for individual purposeful behavior in the growth process [Romer, 1990 p.76]. It is undoubtedly true that the creation of knowledge in society is the product of intentional investment of resources to inventive activities on the part of profit-maximizing firms and entrepreneurs (Romer 1990, Grossman and Helpman 1994). An early reaction to this

[51] The exogenous specification of the accumulation of knowledge eliminates the need to incorporate external effects into growth theory. Therefore it provides a framework where of optimal and competitive equilibrium paths are equivalent (Lucas, 2002).

unsatisfactory situation came from Kenneth Arrow in 1962, with his innovative work on learning-by-doing. In his work, he attempts to provide a theory of knowledge creation, and to incorporate the latter into a growth model such that steady technological change emerges endogenously from the dynamics of the model. Arrow based his argument on two established premises: The first is that learning is the product of experience. Knowledge is acquired through the attempt to 'solve a problem', and therefore takes place only during activity (Arrow 1962, p.155). The second premise involves the existence of diminishing returns during a subject's attempt to solve the same problem repeatedly. For any given stimulus, learning decreases with repetition sharply until it reaches a state of 'equilibrium'. Therefore, the assertion is that steady increases in learning, and as a consequence performance, require continuing development of new stimulus situations. In Arrow's model, the variable that represents experience is cumulative gross investment. New capital goods change continuously the environment where production takes place, providing the stimuli for new knowledge, and steady growth in productivity, to emerge. In accordance to the models proposed by Solow and Shell, Arrow maintains the assumption that knowledge is a public good, therefore not compensated by the market.

The learning-by-doing model has been a prominent attempt to make the evolution of technological change endogenous, and responsive to market incentives. However, in certain respects the formulation of the model is inadequate. First, the assumption of fixed proportionality between new physical capital and new knowledge is restrictive. Second, the model does not allow for intentional private investment in research and development. Individual optimizing behavior has rather a more indirect role in generating new technical knowledge, as the latter is merely a side effect of the production of capital goods. An attempt to fill the second gap came from Romer (1986). He proposed an equilibrium model of endogenous technological change which builds on Arrow's learning-by-doing formulation, but it departs from it in that the accumulation of knowledge is driven by firms' intentional optimizing behavior. The technology for a firm is a function of the firm-specific stock of knowledge, the economy-wide aggregate knowledge, and the level of all other inputs. New knowledge can be created by investing resources in research. It is assumed that newly produced private knowledge although it cannot be patented it can be partially kept secret. It is this partial excludability of the benefits of research and development that ensures the intentional private investment of resources in R&D. The concept of knowledge used in this model is that of disembodied knowledge, e.g. knowledge in books (Romer, 1986). However, the formulation of the model can be compatible with a concept of knowledge as being embodied in some form of tangible capital, such as physical or human capital. In this case, knowledge and capital are assumed to be used in fixed proportions in production, and the variable in question is reinterpreted to represent a composite good that is made up of both capital and intangible knowledge. As a result, the dynamics of the model are similar to those of Arrow's learning-by-doing model, and the mathematical equations can be interpreted in terms of learning-by-doing that is incidental to capital production [Romer, 1990 p.77].

Romer's (1986) paper has been one of the main contributions that challenged the prominent role of the neoclassical growth theory, and marked the advent of the so-called *new theories of growth*. The new theories of endogenous growth represent a class of models sharing the distinctive characteristic that the engine of growth comes from the model itself; no exogenous technological progress or population growth is required. Even more, the models in this body of literature possess the feature that the source for sustaining growth lies in some form of increasing returns to scale or the externality effect arising from knowledge spillovers. There are several alternative devices considered in the literature through which endogenous growth is generated. The first basic approach is represented by Romer's (1986) model and, as it was previously mentioned, it attributes the leading role in the growth process to a natural externality created from investing in new knowledge. In this economy the rate of growth of per capita output, and consumption, is monotonically increasing over time approaching an upper bound asymptotically. The key feature of the model that generates this result is the presence of increasing returns to scale in the production of output, and more importantly the presence of an increasing marginal product of knowledge.[52] It is this latter assumption that ensures

[52] The assumption of an increasing marginal productivity of knowledge from a social point of view is what distinguishes the production function adopted in the paper of Romer (1986) from the one used in Arrow (1962) [Romer, 1986, pp.1015-6].

the unbounded growth of knowledge on all possible efficient and competitive equilibrium paths. The marginal product of knowledge never reaches a level that is low enough such that it is optimal on the part of firms not to undertake further research, and therefore stop at a steady state where knowledge is constant.

An alternative approach to endogenous growth theory argues that the prime engine of economic growth is the accumulation of human capital. The latter is defined to refer to the skills and knowledge level possessed by the labor force. The theory of human capital was introduced in the literature with the influential paper of Lucas (1988). [53] His theory builds on the premise that knowledge cannot be treated separately from the human inputs that create it or possess it [Lucas, 1099 p.15]. The dominant hypothesis is that the accumulation of human capital results in the production of technological knowledge, which in turn is the source of increases in macroeconomic productivity and increasing returns in technology. The knowledge externality takes the form of the positive effect of the economy's average level of human capital on the productivity of all factors of production. Since no individual decisions affect in an appreciable way the average skill level, although all benefit from it, no one takes the latter into account when deciding how to allocate his time. In such a model, growth can be sustained by continuing accumulation of the input that generates the positive externality (Grossman and Helpman, 1994). As in the learning-by-doing formulation, the production of nonrival knowledge is an unintentional side effect of the production of a conventional good, which is human capital [Romer, 1990 p.77]. However, there is a role, even though indirect, for private decision in this process. This stems from the fact that individual agents can make intentional

[53] Perhaps the earliest study that proposes a model of endogenous human capital accumulation is that of Uzawa (1965). In his model, all changes in technological knowledge are embodied in labor, and are reflected in the efficiency of the labor force. Improvements in the latter occur as a result of various activities in the 'educational sector', which represent a larger share of resources employed in education, health, construction and maintenance of public goods, etc. on the part of some public authority. These activities have a uniform impact over the whole economy. The important feature of his solution is that growth is sustained without the need of an external engine of growth. Instead, the growth rate of the economy is endogenously determined from the rate of labor that is allocated at the educational sector – labor is the only factor employed in this sector.

investment of resources, a share of their working time and physical capital, into the process of formal education in order to enhance their future levels of human capital.[54]

The theory of human capital was developed in an attempt to assign an important role as a source of growth to factors other than technology. Although the theory has succeeded in this regard, it is unsatisfactory in one respect. There is a logical difficulty that stems from the prediction of this theory that the growth rate of the economy is equal to the rate of the accumulation of human capital, or a linear function of it. This implies that never-ending growth requires never-ending increases in human capital. However, for such a variable, never-ending growth is implausible because human capital skills are possessed by individual human beings and so are not automatically passed on to workers in succeeding generations [McCallum 1996]. McCallum extends this argument by stating that the real force behind sustaining growth is the accumulation of some form of knowledge, not human capital. The former is possessed by society in general, and can be passed on from generation to generation; therefore, it can be accumulated without limit, providing the basis for never-ending growth.

In response to the various shortcomings of the learning-by-doing and human capital formulations, Romer (1990) developed a formal model that fulfils two objectives. First, in accordance with the neoclassical economists and contrary to the human capital approach, he assigns the role of the primary engine of growth to technological change. Second, he goes a step forward into filling the theoretical gap in the literature by making improvements in technology the explicit product of the intentional investment in research on the part of profit-maximizing firms. The contribution of this paper compared to Romer (1986) lies mainly in the refinement of the concept of knowledge. The latter is redefined in a way that supports its properties of being a nonrival, yet excludable input. New knowledge is the product of research and is defined to be embodied, or codified, in the designs of new products. Designs, or 'blueprints' are assumed to be protected by patents in their use in the production of new goods, therefore generating monopoly profits to the private firms who undertake the research and development activities that render their

[54] The growth rate of consumption and per capita capital is linearly determined by the rate of growth of human capital. It is through the latter variable that the parameter indicating the effectiveness of investment in human capital, and the rate of time preference affect the growth rate along the balanced path of both the socially optimal and competitive equilibria.

creation. It is this effective excludability of the benefits of new research that provides the incentive to private firms to invest in the development of new knowledge. Apart from their productive role in the production of output, designs contribute through an externality effect in the process of research itself. Designs of new goods are not protected by patents over their use in research. Therefore, any inventor has free access to the entire stock of new research. This implies that as the total stock of designs, and new knowledge grows larger, the productivity of human capital in the research sector increases as well. It is this non-excludable part of the benefits of research that creates the spillover effects in the process of knowledge creation, and provides the mechanism for endogenous never-ending growth.

The models that were described so far emphasize increasing returns to scale as the source of endogenous growth. In a classical paper, Rebelo (1991) showed that increasing returns to scale and externalities are not necessary to generate endogenous growth. The latter can be compatible with production technologies that exhibit constant returns to scale as long as there are constant returns to the factors that can be accumulated. This implies that labor and non-reproducible factors are not used in production. If the latter are essential to production then sustained growth is made feasible only by assuming that the technology displays increasing returns to scale. In addition, Rebelo shows that the special case where the production function is linear in a measure of capital broadly defined to encompass both physical and human inputs, and everything is reproducible, captures all the main qualitative features of the class of more complicated endogenous growth models with convex technologies. This framework, which is the so called '*AK*' model, has been commonly adopted in the literature of monetary growth due to its main advantage of preserving a theory's analytical simplicity.

Money and endogenous growth The developments in the theory of endogenous growth have been exploited to analyze the long run interactions between monetary policy and the real sector of an economy. The endogenous generation of growth in these models, without relying on the occurrence of technical progress or population changes, provides the appropriate framework to analyze the effects of economic policies on the *growth* rate of the aggregate real variables. This has made it possible to depart from the traditional approach in the literature which had been to focus on the impact of monetary policy on the steady state *levels* of real economic aggregates. The literature addressing the long run effects of inflation on growth does so using various mechanisms of endogenous growth, and various channels of linkage between monetary policy and long term economic performance. One of the main transmission mechanisms in the core volume of literature is the so-called 'investment channel', by which it is meant that monetary expansion sets in motion a chain of economic events that ultimately affect private investment decisions, hence real economic performance. There are two avenues through which this mechanism operates: one is by altering the effective relative price of capital, and the other through affecting the real net return on investment. Returning to the insight of the early papers of Stockman (1981) and Abel (1985), the former was the first to identify the role of imperfect credit markets for investment on the way monetary policy exerts its influence on real economy. His argument defines an example of the investment channel, as it works through an effect on the effective relative price of capital. As has already been mentioned in the previous section, the central idea of these studies is that the existence of cash-in-advance constraints on purchases of physical capital goods translates under an inflationary policy into a higher effective relative price of the latter, henceforth into lower investment and real output. If only consumption purchases are subject to cash-in-advance constraint there is no channel through which higher inflation can affect investment or other private decisions, hence the real sector of an economy. The theoretical validity of Stockman's argument was examined in the context of equilibrium growth by Jones and Manuelli (1993). Adopting the simplest technology that embodies endogenous growth, and allowing for a cash-in-advance constraint on purchases of consumption goods only, the model yields the conclusion that monetary policy continues to be impotent in having an impact on either the level or the growth rate of real output. The latter, as is standard in the endogenous growth literature, is solely determined by parameters of taste and technology. The paper proceeds with providing further insight into how the investment channel applies to this endogenous-growth framework. If the model is extended such that in addition to consumption spending, purchases of investment goods are also subject to a cash-in-advance constraint, then the resulted prediction accords with Stockman's proposition that inflationary policy raises the effective relative price of capital, hence acting like a tax on investment. In contrast to the neoclassical setting, however, the effects of the decline in the rate of accumulation of capital extend beyond the level of output to its growth rate, thus establishing the claim that money matters for growth.[55] The latter argument can also be the outcome of employing different routes of the investment channel. Potential ways through which this could be achieved involve the introduction in the model of nominally denominated rigidities in the tax code. Specific examples would be the nominally denominated tax credits, imperfectly indexed tax bracketing and nominally denominated depreciation allowances [Jones and Manuelli, 1993]. Exploring this last case is the focus of attention of the second part of the paper of Jones and Manuelli (1993). The prediction of the model is once again that a higher rate of growth of the money supply has a negative effect on the economy's growth rate. The argument that supports this conclusion is that higher inflation, through the following increase in the nominal interest rate, causes a reduction in the present value of tax credits that correspond to the depreciation allowance. On this view, the future capital stock becomes more expensive to acquire, thus leading to a reduction in current investment spending, and consequently real output in both the level and growth sense. A natural extension of the existing analysis would be to consider two forms of capital, specifically to incorporate human in addition to physical capital into the model. This would allow the operation of an additional result that moderates the overall negative effect of inflation on growth. As in the previous setting, the increase in the inflation rate results in an increase in the cost of acquiring physical capital by decreasing the value of the depreciation allowance in real terms. Considering that depreciation allowances for human capital are not generally predicted by most tax codes, there is a force that promotes investment in human compared to physical capital. This effect generates a positive impact on growth allowing for the final -negative- effect of inflation to be moderated.

Marquis and Reffett (1995) have also taken this line and have investigated, on theoretical ground, the applicability of Stockman's argument in the context of endogenous growth. The production technology in their model is constructed to

[55] This is a stated argument. The derivations of this particular extension of the basic model are not included in the paper.

accommodate the neoclassical technology adopted in Stockman (1981) and Abel (1985), as well as to exhibit the potential for asymptotic equilibrium growth.[56] Maintaining compatibility with the former papers, the same trading environment is adopted. Thus, economic agents are required to hold cash in advance of their purchases of consumption, as well as investment goods in order to finance those expenditures. The conclusions drawn from the analysis regarding the long run effects of monetary policy are consistent with the argument of Jones and Manuelli (1993), and the predicted implication of the aforementioned investment channel at work. In specific, expansionary monetary policy interacts with private capital decisions, causing investment and real output to settle at a lower steady state level. The model suggests that as a result of the inflation tax, asymptotic endogenously determined growth decreases as well. The size of the decrease depends crucially upon the level of the monetary distortion, as it is measured by the magnitude of the inflation tax, or alternatively, the change in the nominal interest rate.[57] The core argument of the paper consists of the claim that as the monetary distortion grows larger the growth effect it is associated with is amplified. Moreover, there exists a threshold level of the nominal interest rate beyond which long run growth is eliminated altogether. At this level, the model connects with the papers of Stockman (1981) and Abel (1985) from the perspective that they yield the same proposition: A higher rate of monetary growth produces 'level' but not growth effects.[58] It is evident that within the course of this analysis, which is broad enough to encompass both the neoclassical growth setup and the potential for asymptotic equilibrium growth, the predictions of Stockman and Abel are contained as special cases.

[56] This is a one-sector version of the technology developed in Jones and Manuelli (1990). As in the latter, the production function consists of a linear growth part, and a concave, constant-returns-to-scale technology in both capital and labor. Imposing a certain condition on preferences and technology (Condition G, Jones and Manuelli, 1990 p.1014) ensures that the linear growth term is sufficiently large such that, the existence of a competitive equilibrium balanced growth path is guaranteed, along which endogenously determined growth is displayed (Marquis and Reffett, 1995 p.111).
[57] The ability of monetary policy to affect the economy's growth rate (the standard result in endogenous growth theory applies, that consumption, investment and output all grow at the same rate) depends, in addition to the monetary distortion, on preferences (as described by the intertemporal elasticity of substitution, and the discount factor) and technology (the linear growth parameter, and the depreciation rate of capital) [Marquis and Reffett, 1995 p.116]. For example, a higher intertemporal elasticity of substitution implies a lower rate of discount for households, and more pronounced growth effects of monetary policy.
[58] These consist of a reduction in the rate of investment, and consequently, of a lower steady state level of both capital and real output.

Additional insight into how the investment mechanism operates in the context of endogenous growth has also been provided by De Gregorio (1993).[59] In the same line as Jones and Manuelli (1993) and Marquis and Reffett (1995), this paper seeks to examine the ability of monetary policy to generate growth effects in the framework of an *AK* technology. However, in terms of modeling the role of money, the author pursues a different direction. The modeling strategy he adopts is built on the transactions costs approach developed by William Baumol (1952), James Tobin (1956) and later Robert Barro (1976). In specific, holdings of real money balances are assumed to facilitate transactions by reducing the cost of contacting them.[60] Both the purchases of consumption and investment goods, undertaken on the part of households and firms respectively, are subject to this type of financing constraint.[61] The analysis establishes the standard sequence of results of the investment mechanism as it operates within the context of *AK* technology interacting with imperfect credit markets for investment. In support of the argument expressed in Marquis and Reffett (1995), De Gregorio (1993) emphasizes the negative role of monetary policy in affecting the level and growth rate of real output. The process in the interim is similar to that taking place in the case where investment spending is subject to a cash-in-advance constraint. Firms' reaction to higher inflation is to economize on their holdings of real money balances, thus incurring an increase in their transactions costs. This in turn causes a reduction in the private return on investment by raising the latter's effective relative price, thereby creating a strong disincentive to invest. As is standard in the *AK* framework, the lower rate of capital

[59] This study presents two distinct models. The discussion here refers to the first model. The second model will be discussed further below in this section.

[60] The cash-in-advance constraint can be seen as a special case of the transactions costs approach, in which case liquidity costs equal infinity when real money balances are lower than the consumption or investment expenditure they are intended to finance, and zero otherwise. These conditions result in the optimal rule that real money holdings exactly match the amount of consumption, or investment expenditure [De Gregorio 1993 p.276]. In contrast, the assumption in the general transactions costs formulation is that the cost of implementing transactions is positive and finite irrespective of the value of the real money balances to expenditure ratio. Although this liquidity cost can be reduced by increasing one's real money holdings, for no amount of the latter can it be eliminated to zero [Feenstra, 1986 p.278].

[61] As in Barro (1976) liquidity costs are defined to be a non-linear function of the ratio of real money balances to consumption. In specific, the liquidity cost function for both consumption and investment purchases is decreasing and convex in the ratio of real money holdings to consumption, and investment, expenditure respectively [De Gregorio, 1993 pp.274-5].

accumulation is followed by an adverse effect on growth, in addition to the level effects of lower steady state capital and real output.[62]

Building on a different aspect of the endogenous growth literature, the human capital accumulation mechanism, Marquis and Reffett (1991*a*) contribute an additional study in the literature that investigates the role of money in determining the growth rate of economic aggregates. The central feature of their model is the absence of credit markets for investment in human capital. Given that the accumulation of human capital is the engine of growth, this hypothesis provides an alternative channel through which distortionary monetary policy generates growth effects. The objective of the authors is to construct a model with more general characteristics than the others in the literature, which allows for the possibility the implications of inflation taxes to be simultaneously processed through a 'human capital channel', in addition to the standard investment mechanism. This more general setting serves to establish a richer set of theoretical predictions, in which the superneutrality result and the Stockman and Abel propositions range as potential outcomes.[63] The analysis contains different cases of credit-constrained markets, and suggests the following answers with respect to the determinants of long run growth: When no cash-in-advance constraint applies on purchases of either physical or human capital, whereas a cash-in-advance constraint applies on consumption goods, monetary policy is found to have no effect on output. This outcome, in favor of the conventional superneutrality result, provides a generalization to an endogenous growth framework of Stockman's (1981) and Abel's (1985) proposition that, when the inflation tax acts on consumption decisions only, steady state capital stock is unaltered.[64] The same theoretical consistency is obtained when Marquis and Reffett (1991*a*) examine the case of cash-in-advance constraints applying on purchases of physical but not of human

---

[62] It is important to emphasize that this result stems from the assumption that purchases of capital goods can only be financed through money; thereby making firms subject to the inflation tax. If only consumers faced transactions costs, the only channel through which monetary policy can affect growth, namely the private return of capital, is constant and invariant to the rate of inflation. In this case, consumer behavior with respect to inflation has no effects on growth [De Gregorio, 1993 p.278].

[63] This paper provides an extension of a previous work of the same authors (Marquis and Reffett, 1991*b*). In the latter working paper they show that when cash-in-advance constraints apply on investment in human capital, monetary policy distorts private decisions with respect to investment in education, and leads to a reduction in growth rate. The present article introduces a more general setting, with the possibility of cash-in-advance constraints to apply on investment in both physical and human capital.

[64] This is emphasized by the authors in Marquis and Reffett, (1991*a*) p.108.

capital: The monetary distortion produces adverse level effects on steady state capital and output. This is the exact claim of Stockman's (1981) second proposition, only that now it is obtained in the context of endogenous growth. As is standard in human capital methodology, in both cases the economy's growth rate is determined by the preference parameter, the term measuring the quality of education, and the rates of depreciation of the two types of capital. Within this framework, the only way for macroeconomic policy to cause an effect on growth is if it distorts decisions that interact with the accumulation of human capital. Imposing cash-in-advance constraints on purchases of human capital provides a means to achieve this. This is the subsequent focus of the paper, and results in the prediction that an increase in the inflation rate generates a negative effect on growth. The mechanism at work here is that higher inflation increases the cost of acquiring human capital, thereby leading individuals to reallocate their time in favor of productive activities. This comes at the cost of devoting less time- and other resources, in education resulting in a lower steady state level of human capital and rate of growth. The present analysis shares, therefore, the same conclusion with the equilibrium growth studies we previously mentioned. Relying on different mechanisms –Jones and Manuelli (1993), Marquis and Reffett (1995) and De Gregorio (1993) specify different applications of the investment channel within the context of constant-returns-to scale equilibrium growth, whereas Marquis and Reffett (1991*a*) employ a human capital channel- they all get across the same idea: distortionary monetary policy affects the prospects of growth negatively. The paper ends with providing some insight on the case where cash-in-advance constraints apply on purchases of both physical and human capital. Since the latter constraint alone is responsible for generating negative growth effects, it works to intensify the human capital effects on growth.[65]

The effectiveness of the above model to connect the growth process with public policy lies not simply on the fact that accumulating human capital is a matter of individual choice; for this is the underlying feature of all human capital models. It lies in

[65] In addition, the Stockman level effect on output is obtained due to the cash-in-advance constraint on physical capital. However, in contrast to the previous cases, the overall impact of inflation on the real interest rate becomes ambiguous. This is due to the opposite effects on the latter displayed as a result of the decrease in steady state (physical) capital and growth rate. The former leads to higher marginal productivity of capital, and hence real interest rate, while the lower growth rate implies the exact opposite effect.

that it contains a theory linking policy with the individual decisions to obtain new skills. It is evident then that the role of money into a model is not complementary to the economy's real side, but plays an equally critical role in providing answers to policy issues. For instance, in a similar framework where money has the role of providing productive services, the growth neutrality results cannot be nullified.[66]

Labor-leisure choice The research in the monetary-growth literature in its greatest extent treats labor as being inelastic. Thereby it builds on models that abstract from the decision to allocate time between leisure and other productive activities. This approach, although common, is not adequate on the ground of several perspectives. First, the endogeneity of labor supply introduces an important aspect of realism into the model. This applies with equal significance to optimal growth models, as well as to models of business cycle theory.[67] Second, leisure is relevant in the theory of taxation. It is generally true that a tax on labor affects the time allocated to productive occupations only if there is the possibility of substitution towards untaxed leisure activities. Hence, using a model of inelastic labor to analyze the consequences of policy changes is limiting, since a tax on consumption and wage income operate as non-distortionary taxes in such an environment. Lastly, there is merit in the task itself of exploring how the decision to allocate time between various activities interacts with the intertemporal allocation of consumption to determine the dynamic and long-term growth behavior of an economy.[68]

One of the earlier studies on optimal growth theory with endogenous labor supply is that of Brock (1974). His work offers insight on the conditions under which the result of money neutrality applies within the standard neoclassical growth framework.[69] But classic growth theory does not really help in understanding the connection between

---

[66] This case is examined in Wang and Yip (1992). When money is incorporated in the production function via Hicks neutral technology, monetary policy is non-neutral only in the level sense. Using a modified setup, Pecorino (1995) restores the connection of money and growth rate. The key assumption and novel feature of this model is that physical capital is an input in the process of human capital accumulation. Monetary policy affects growth through altering implicit taxation on inputs in the physical capital/output sector.

[67] Leisure is a key variable in modern business cycle theory, since around two-thirds of the output variation over the business cycle can be accounted for by fluctuations in hours worked (see Ladron-De-Guevara *et al.*, 1999).

[68] See Ladron-De-Guevara *et al.* (1999).

[69] Elsewhere in this essay we refer in specific on the predictions of Brock's study.

employment and sustained growth in endogenous terms. In view of this limitation, a few authors worked out models of endogenous growth that allow for variability in employment levels to emerge from within the model's dynamics. One such effort was taken by De Gregorio (1993). In his model, the marginal productivity of capital is specified to be a function of the economy's employment ratio.[70] As long as higher inflation leads agents to economize on real balances by substituting toward leisure activities, this framework is successful in offering an alternative channel through which monetary policy can affect labor variability, hence growth.[71]

An alternative mechanism that establishes on the distortionary impact of inflation on labor supply is provided in models of endogenous human capital accumulation. In the particular setting where money has a positive value in reducing the time costs in transacting, monetary policy has the ability to cause a negative effect on income growth. This possibility is examined in Wang and Yip (1993), where the time-effort spent in transactions directly affects the resources devoted in accumulation of new knowledge.[72] Love and Wen (1999) further extend this approach to allow for leisure to be endogenous, in addition to the time invested in the market and education sector.[73] The purpose of this

[70] The model assumes full employment of the number of workers; the latter however are assumed to be employed a variable number of hours. Thus the employment ratio represents labor employed -measured in hours worked- per unit of the economy's labor endowment. The latter is more closely related to economically active population than to the labor force (in the second case the ratio would be referred to as the employment rate). Under the proper normalization of labor endowment to unity, the labor input denotes both the overall level of employment, as well as the employment ratio, while per capita quantities are defined as 'per units of labor endowment'. It should be further noted that this is the only modeling strategy that allows the marginal product of capital to depend on employment, while at the same time we avoid the scale effects that arise from defining per capita variables in terms of quantities per unit of employed labor (see De Gregorio, 1993 pp.280-1).

[71] Using Romer's (1986) specification of endogenous growth and emphasizing the transactions motive for holding money, the model implies that inflation reduces the overall level of employment, and hence the marginal productivity of capital. The former is both a result of a fall in labor demand due to increased labor costs, and a reduction in labor supply due to an increase in the time-cost of transactions, and a subsequent substitution towards leisure. As a result of lower employment, inflation in this economy has a negative impact on investment, and rate of income growth.

[72] In this framework, the level of real money balances relative to consumption expenditure has an inverse impact on transactions costs. Individuals respond to higher inflation by economizing on money holdings. This has the effect of raising the time cost of transactions, causing them to cut back on their time invested in education.

[73] In their model, time is allocated among productive services in the market sector, investment in acquisition of human capital, and leisure. This is different from the remaining literature, which places emphasis on the decision between the former two activities, and thus assumes that non-leisure time is inelastic.

is to explore how the intertemporal substitution between consumption and leisure interacts with the growth mechanism through labor supply. The implication is that the endogeneity of leisure intensifies the negative effect of inflation on the time supplied in both productive and knowledge-accumulating activities.[74] The return to both physical and human capital in turn declines, with a consequent fall in the growth rate.[75]

Money and endogenous growth in an open economy In this section we carry the theoretical question of the long run effects of monetary policy within the international macroeconomic framework. All the previous studies that we have considered explore different aspects of this issue for closed economies. Little emphasis has been placed in the literature on the extension of monetary growth theory in the open economy setting. One such effort has been made by Palokangas (1997), which is to our knowledge the first attempt at such an endeavor. The distinctive characteristic of his model is that inflationary policy is adopted as an optimal response by public authorities in order to ameliorate the distortionary impact of 'ordinary' taxation.[76] By providing an alternative means of raising revenue, seigniorage allows the possibility to reduce taxation, hence raise the real rate of return to capital. In the present model, in which the particular structure of endogenous growth is that of Rebelo's (1991), the engine of growth is provided by the accumulation of a broad concept of capital, defined to encompass both physical and human capital. A

---

[74] Economic agents respond to the increase in the effective price of consumption due to higher inflation by substituting leisure for consumption. Labor supply is thus reduced (indirect effect) in addition to the decrease due to less available time caused by the increase in transaction costs (direct effect). The size of the former effect depends positively on the elasticity of labor supply, and the elasticity of transaction costs with respect to consumption.

[75] The possibility for monetary policy to impact growth through level employment effects is also examined in Jones and Manuelli (1993). In his model with human capital utilization inflationary monetary policy has growth effects if and only if it affects asymptotically the level of labor supply. Although Jones and Manuelli employ Lucas's (1988) technology, their model does not distinguish between productive and education activities. Investment in knowledge accumulation is not based therefore on the allocation of time between the two activities, but is modeled directly in terms of consumption good equivalents. This distinction separates the model's structure and monetary policy mechanism from the models of Wang and Yip (1993), and Love and Wen (1999).

[76] In the open economy setting the feature that renders taxation potentially distortionary is the ability of agents to transfer resources abroad. On the same principle, closed economy models would require the incorporation of a non-taxable sector, which would enable agents to substitute away form tax-burdened activities. Palokangas (1997) uses the former strategy to put forth a model with an elastic tax base with respect to the various tax rates. It is this latter property that creates the space for monetary policy –in particular the creation of money through seigniorage- to have a potential positive effect on aggregate welfare by relieving the economy of part of the deadweight burden caused by 'ordinary' taxation.

potential route thus exists for public policy to enhance growth, through altering the mix of revenue-raising policies away from distortionary taxation. However, depending on the role of money that is emphasized in the model, this effect can be reversed. The approach of this paper is to use the transactions-cost theory of holding money, which predicts a negative relationship between the growth rate of money and rate of investment in human capital.[77]

Model I. *Inflation imitating a consumption tax. The case of fixed labor supply*[78]

The economy consists of a constant number (N) of identical individuals, each of whom has an infinite planning horizon and possesses perfect foresight. Population remains fixed over time. We shall denote individual quantities by lowercase letters, so that $X = Nx$. We assume that the economy produces a single traded commodity, the foreign price of which is given in the world market. In the absence of any impediments to trade, purchasing power parity (*PPP*) is assumed to hold. Expressed in percentage terms, the latter is described by the following expression

$$\pi = \pi^* + \varepsilon , \tag{I.1}$$

where π denotes the inflation rate of the good in terms of the domestic currency, π^* is the inflation rate of the traded commodity in terms of the foreign currency assumed to be exogenously given to the small open economy. Finally, ε denotes the rate of exchange depreciation of the domestic currency.

The model we shall examine is that of a small open economy that operates in a world of perfect capital markets. This implies that the real rate of interest earned on foreign bond holdings is constant, and exogenously given for the small economy irrespective of its international transactions. Furthermore, we assume for simplicity that there is no

[77] Following Kimbrough (1986) and De Gregorio (1993) this modeling approach suggests that transactions costs depend negatively on the money-to-expenditure ratio. In the present framework the accumulation of human capital can only be financed by individual savings; it is this assumption that allows inflation to have an adverse impact on the growth process.

[78] The following remark is ought to be made. The analytical framework of the model has a similar character with the theory presented in Turnovsky (1996). Turnovsky (1996) investigates the effects of tax and expenditure policies on a small open economy in a model with linear production technology that exhibits ongoing, endogenously determined growth. Similarly to the present framework, the model abstracts from effects on the employment side of the economy by assuming that labor is totally fixed or, alternatively, that it grows at some exogenously determined rate.

foreign inflation. Normalizing the foreign price level to a constant[79], its rate of increase becomes zero and equation (I.1) is written as

$$\pi = \varepsilon \,. \tag{I.2}$$

The model's production side is built on Rebelo's (1991) approach of modeling endogenous growth. The economy has one sector of production, the output of which can be used both as a consumption and capital good. The production process involves one factor of production, which represents a composite of various types of physical and human capital. Labor and non-reproducible factors (e.g. land) do not play a role in production. Constant-returns-to-scale imply that the production function takes the simple linear form of an *AK* technology[80]

$$y = A_0 k \,. \tag{I.3}$$

where y and k denote the individual firm's output and capital stock respectively. Combining (I.3) with $Y = Ny$, aggregate output in the economy is given by

$$Y = A_0 K \qquad A_0 > 0 \,. \tag{I.3´}$$

Thus aggregate output is proportional to the aggregate capital stock, thereby leading to an equilibrium having ongoing, endogenously determined, growth.

The individual firm accumulates physical capital, which is assumed to be infinitely durable. The expenditure on a given increase in the capital stock is represented by $i \equiv \frac{I}{N}$. An important feature of the model is the assumption that the investment process involves costs of adjustment, or installation,[81] which are represented by the convex component of the following cost function

[79] For reasons of tractability the foreign price level is assumed to equal one.
[80] Rebelo (1991) shows that this linear model in which only reproducible factors are incorporated into the production technology captures all the essential features of the class of endogenous growth models that exhibit increasing returns to scale, or embody some form of knowledge externality. Models in which 'everything is capital', in the sense that all factors of production can be accumulated over time, have also been studied early on in the economic literature (Knight, 1935, 1944, and Hagen, 1942) [Rebelo, 1991 p.507].
[81] The model of a small open economy that faces perfect world capital markets, a constant rate of time preference and investment is tradable, is characterized by degenerate dynamics. In particular, in steady-state equilibrium, the condition of optimal intertemporal allocation of consumption requires that the rate of time preference be equal to the given (real) world interest rate. This implies the existence of a steady state

$$\Phi\left(i,k\right)=i+H(i,k)=i+\frac{h}{2}\frac{i^2}{k}=i\left(1+\frac{h}{2}\frac{i}{k}\right). \tag{I.4}$$

where the addition of i units of capital requires the use of $H(i, k)$ units of output. The function $H(i, k)$ is assumed to be (*a*) nonnegative; (*b*) linearly homogeneous, and (*c*) convex in investment; i.e. $H' \geq 0$, and $H'' > 0$. The assumption of non-negativity implies that disinvestment at the rate $I < 0$ involves positive dismantling costs, also represented by $H(\cdot)$. The homogeneity assumption is made largely for convenience; in addition it ensures that the market value of the capital stock is invariant with respect to changes in the scale of the economy. We also specify that the total cost of zero investment is zero and the marginal cost of the initial installation is unity; thus, it is assumed that $H\left(0,k\right)=0$, and $H_i(0,k)=0$.[82]

Aggregating over the N individual firms, leads to

$$\Phi\left(I,K\right)=I+\frac{h}{2}\frac{I^2}{K}=I\left(1+\frac{h}{2}\frac{I}{K}\right). \tag{I.4$'$}$$

The representative agent's welfare is given by the intertemporal isoelastic utility function

$$U=\int_0^{\infty}\frac{1}{\gamma}\left(\frac{C}{N}\right)^{\gamma}e^{-\rho t}dt \qquad -\infty<\gamma\leq 1, \tag{I.5}$$

at this limiting case only, and the absence of dynamics that restore the equilibrium condition once there is divergence from it. Several approaches have been used in the literature to 'remedy' this situation. One, proposed by Uzawa (1981) has been to endogenize the rate of time preference by specifying the latter as a function of the level of utility. Another way to circumvent this problem is to assume imperfect substitutability between domestic and foreign bonds by imposing quadratic costs on holdings of foreign bonds (see Turnovsky, 1985). An alternative approach, subject to less criticism than the previous two is to introduce the uncertain lifetime assumption of Blanchard (1985), or assume a growing population of overlapping infinitely lived households as in Weild (1989). Other possible ways are to introduce some form of nominal price or wage rigidity into the model (see e.g. van der Klundert and van der Ploeg, 1988), or assume the small economy faces an upward sloping schedule for debt (Bhandari, Haque and Turnovsky, 1990). The most commonly used approach to solve the degeneracy of dynamics, and important source of sluggishness into the model is the assumption that the accumulation of physical capital is subject to adjustment, or installation, costs. In the absence of such costs, and under the assumption of a perfect world capital market the small economy can import an unconstrained quantity of capital from abroad. In response to a change in the market value of capital, the stock of physical capital can thus adjust instantaneously to the new steady-state level with no new investment taking place (see Turnovsky, 2000).

[82] For a more detailed discussion on the properties of the adjustment cost function see Turnovsky (2000).

where C denotes aggregate private consumption. The parameter γ is related to the intertemporal elasticity of substitution, s, by $s = 1/(1-\gamma)$.

We assume that the individual agent holds two assets, domestic money, which is not held by foreigners and net foreign bonds. The latter pays the exogenously given world interest rate, r.[83], The individual's total assets, a, are therefore defined as follows

$$a = b + m_i, \tag{I.6}$$

where b denotes the individual's real stock of foreign bonds, and m_i is the individual i's money holdings. Aggregating over the N individuals, we obtain an expression for the aggregate stock of assets, A,

$$A = B + m, \tag{I.6′}$$

where B stands for the aggregate stock of net foreign bonds, and m for the aggregate real money holdings. Differentiating this equation yields

$$\dot{A} = \dot{B} + \dot{m}. \tag{I. 6″}$$

The accumulation of assets by the aggregate economy is described by the following equation

$$\dot{A} = Y + rB - C - I\left(1 + \frac{h}{2}\frac{I}{K}\right) + \tau - \varepsilon m, \tag{I.7}$$

where τ represents real transfers received from the government. It is assumed that this is identical among all agents, received independently of economic behavior.

Money is incorporated into the model by means of the Clower (1967) constraint, interpreted to mean that 'only money buys goods'. We impose the assumption that this rule applies on consumption, but not on investment expenditure. Therefore, at any time period an agent can acquire goods only to the value of his current money stock (m_i). The latter is equal to money carried over from the previous period, plus any current transfer

[83] Since we abstract from the possibility of foreign inflation, the world interest rate represents both the nominal and real rate of return on foreign bonds.

receipts from the government. The cash-in-advance constraint for individual agent i is given by

$$m_i = c\,. \tag{I.8}$$

Aggregating over the N individuals leads to

$$m = C\,. \tag{I.8$'$}$$

Substituting equations (I.3$'$), (I.6$'$) and (I.8$'$) into (I.7) yields the following asset accumulation equation for the aggregate economy

$$\dot{A} = A_0 K + rA - (1 + r + \varepsilon)C - I\left(1 + \frac{h}{2}\frac{I}{K}\right) + \tau\,. \tag{I.9}$$

For simplicity we assume that capital does not depreciate. Therefore, the economy faces the physical capital accumulation constraint

$$\dot{K} = I\,. \tag{I.10}$$

Using the fact that in the absence of distortions the competitive equilibrium is a Pareto optimum, we solve for the competitive equilibrium for this economy by computing the solution to a central planner's problem. Taking this approach, we consider a central planning authority which chooses the values of aggregate capital (K), consumption (C) and investment (I) that maximize the utility of the representative agent, subject to the aggregate resource constraint of the economy (I.9), and the capital accumulation equation (I.10). The problem is stated as follows

$$\max_{K,C,I} \qquad U = \int_0^\infty \frac{1}{\gamma}\left(\frac{C}{N}\right)^\gamma e^{-\rho t} dt\,,$$

subject to $\dot{A} = A_0 K + rA - (1 + r + \varepsilon)C - I\left(1 + \frac{h}{2}\frac{I}{K}\right) + \tau$,

and $\dot{K} = I$.

The Hamiltonian for this optimization problem is given by

$$H = \frac{1}{\gamma}\left(\frac{C}{N}\right)^{\gamma} + \lambda\left[A_0 K + rA - (1+r+\varepsilon)C - I\left(1+\frac{h}{2}\frac{I}{K}\right) + \tau\right] + q'I ,$$

where λ and q' are the current-value Lagrange multipliers. The optimality conditions are given by the following expressions

- $H_C = 0 \Rightarrow N^{-\gamma}C^{\gamma-1} = \lambda(1+r+\varepsilon)$, (I.11)
- $H_I = 0 \Rightarrow \dfrac{I}{K} = \dfrac{q-1}{h} \equiv \phi$, (I.12)
- $\dot{\lambda} = -\dfrac{\partial H}{\partial A} + \lambda\rho \Rightarrow \dfrac{\dot{\lambda}}{\lambda} = \rho - r$, (I.13)
- $\dot{q}' = -\dfrac{\partial H}{\partial K} + q'\rho \Rightarrow \dot{q}' = -\lambda\left[A_0 + \dfrac{h}{2}\left(\dfrac{I}{K}\right)^2\right] + q'\rho$. (I.14)

Taking the time derivative of (I.11) we obtain an expression for the growth rate of aggregate consumption

$$\frac{\dot{C}}{C} = \frac{r-\rho}{1-\gamma} \equiv \psi , \tag{I.15}$$

so that starting from an initial level C_0, aggregate consumption at time *t* is

$$C_t = C_0 e^{\psi t} . \tag{I.15′}$$

Equation (I.12) is an expression for the growth rate of aggregate capital $\dfrac{\dot{K}}{K} = \varphi$, which can be solved to yield

$$K_t = K_0 \exp\{\phi t\} = K_0 \exp\left\{\frac{q-1}{h}t\right\}. \tag{I.16}$$

Combining equations (I.12), (I.14) and $q = q'/\lambda$ leads to

$$\frac{A_0}{q} + \frac{\dot{q}}{q} + \frac{(q-1)^2}{2hq} = r . \tag{I.17}$$

Equation (I.17) equates the net rate of return on domestic capital to the rate of return on the traded bond. The former consists of three components. The first is the output per unit of installed capital (valued at the price q), while the second is the rate of capital gain. The third element reflects the fact that an additional source of benefits of higher capital stock is to reduce the installation costs associated with new investment.

Finally, the following transversality conditions must be imposed:

$$\lim_{t \to \infty} \lambda A_t e^{-\rho t} = 0, \tag{I.18a}$$

$$\lim_{t\to\infty} q' K e^{-\rho t} = 0 \ . \tag{I.18b}$$

The critical determinant of the growth rate of capital is the market price of installed capital, q, the path of which is determined by the arbitrage condition (I.17). In order for the capital stock to follow a path of steady growth (or decline), the stationary solution to this equation, attained when $\dot{q} = 0$, must have at least one real solution.

Setting $\dot{q} = 0$ in (I.17) implies that the steady state value of q ($\tilde{q}$) must be a solution to the quadratic equation

$$A_0 + \frac{(\tilde{q}-1)^2}{2h} = r\tilde{q}\,. \tag{I.19}$$

The necessary and sufficient condition for the capital stock to converge to a steady growth path is that this equation has real roots. This will be the case if and only if

$$r\left(1+\frac{hr}{2}\right) \geq A_0\,. \tag{I.20}$$

Equation (I.20) implies that the smaller the adjustment costs (h) are, the smaller the marginal physical product of capital (A_0) must be, in order for a balanced growth path for capital to exist. The reason this holds is that there is a tradeoff between the first and third components of the rates of return to capital given by the left-hand side of (I.17). The smaller the adjustment cost (h) the greater the return to capital due to valuation differences between installed capital and the embodied resources, and the greater the incentive to transform new output to capital. If for a given h, A_0 is sufficiently large to reverse (I.20), the returns to capital dominate the returns to bonds, irrespective of the price of capital, so that no long-run balanced equilibrium can exist where the returns on the two assets are bought into equality.

The formal solutions for the two real roots are

$$\tilde{q}_1, \tilde{q}_2 = (1 + rh) \pm h\sqrt{r\left(1 + \frac{hr}{2}\right) - A_0} \ . \tag{I.21}$$

With equation (I.19) having two real roots, the potential arises for two steady equilibrium growth rates for capital to exist. Two cases can be identified:

Case I: $\tilde{q}_2 > 1 > \tilde{q}_1 > 0$,

Case II: $\tilde{q}_2 > \tilde{q}_1 > 1$.

In order to identify the dynamics of $\dot{q}$, we use (I.17) to obtain

$$\frac{\partial \dot{q}}{\partial q} = r - \frac{q-1}{h} . \tag{I.22}$$

Therefore, we have the following cases:

- $$\frac{\partial \dot{q}}{\partial q} \geq 0 \Rightarrow q \leq rh + 1 , \tag{I.23a}$$

- $$\frac{\partial \dot{q}}{\partial q} < 0 \Rightarrow q > rh + 1 . \tag{I.23b}$$

Figure 1 below illustrates the phase diagram for (I.17) in the case (I.20) holds, so that a steady state growth path for capital exists. It is seen from the diagram that the equilibrium point *A*, which corresponds to the smaller equilibrium value $\tilde{q}_1$ (negative root) is an unstable equilibrium, while point *B* corresponding to the larger value $\tilde{q}_2$ (positive root) is locally stable. However, it can be shown that any time path for q converging to *B* violates the transversality condition (I.18*b*), which is required to be met.

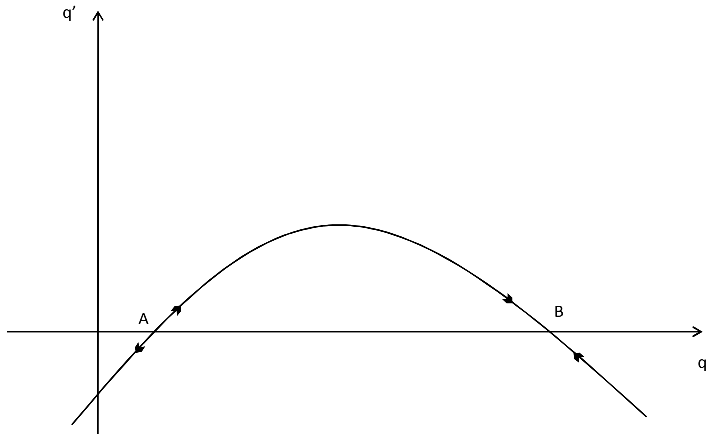

Figure 1. *Phase diagram.*

Substituting equation (I.16) into (I.18*b*) one obtains

$$\lim_{t\to\infty} \lambda_0 q_t K_0 \exp\left\{\left(\frac{\tilde{q}-1}{h}-r\right)t\right\} = 0\,. \tag{I.24}$$

It is clearly seen that when $\dfrac{\tilde{q}-1}{h}-r>0 \Rightarrow \tilde{q}>rh+1$ this limit diverges. Thereby, the larger root $\tilde{q}_2$ violates the transversality condition on the capital stock. Similarly, the smaller root $\tilde{q}_1$ ensures that the required transversality condition holds. The behavior of q can thus be summarized by the following proposition:

Proposition 1 The only solution for q which is consistent with the transversality condition is that q always be at the unstable steady-state solution $\tilde{q}_1$, given by the negative root to (I.21). Consequently there are no transitional dynamics in the market price of capital q. In response to any shock, q immediately jumps to its new equilibrium value.

Note that this result is identical with that derived in Turnovsky (1996). Therefore, the predictions of the model regarding the behavior of the market price of capital, q, and the capital stock are not altered with the inclusion of money in the form of a cash-in-advance constraint.

The description of the economy is completed with the introduction of the government sector. We shall assume for simplicity that the only expenditure the government engages into is the distribution of lump-sum transfers to the private sector. The sole source of revenue for the government is by printing new money.[84] We abstract from ordinary taxation and alternative modes of financing the government deficit, such as the issuing of domestic government bonds. The budget constraint of the domestic government, assumed to be maintained continuously balanced, is expressed by

$$\dot{m} + \varepsilon m = \tau . \tag{I.25}$$

Substituting equations $(\text{I.3}')$ and $(\text{I.6}'')$ in (I.7) yields

$$\dot{m} + \dot{B} = A_0 K + rB + \tau - C - I\left(1 + \frac{h}{2}\frac{I}{K}\right) - \varepsilon m . \tag{I.26}$$

Combining (I.25) and (I.26) we obtain an expression for the net rate of accumulation of traded bonds by the private sector

$$\dot{B} = A_0 K - C - I\left(1 + \frac{h}{2}\frac{I}{K}\right) + rB . \tag{I.27}$$

The expression states that the rate of accumulation of traded bonds equals the balance of payments on current account, which in turn equals the balance of trade plus the net interest earned on the traded bonds. Substituting the expressions for $C(t)$ from $(\text{I.15}')$, for I from (I.12) and for $K(t)$ from (I.16), the accumulation equation (I.27) can be written in the form

$$\dot{B} = rB + \theta K_0 \exp\{\phi t\} - C_0 \exp\{\psi t\}, \tag{I.28}$$

[84] Seignorage represents the real revenue a government acquires by using newly issued money to buy goods and nonmoney assets. Seignorage revenue consists of the change in the economy's real money holdings ($\dot{m}$), plus the proceeds of the inflation tax. The latter represents the *devaluation* of the previous period's stock of real money balances due to the higher level of prices. Mathematically, it is equal to the inflation rate over the previous period's stock of real money balances. In continuous-time framework it is expressed as εm.

where φ and ψ are defined in (I.12) and (I.15), respectively, and

$$\theta \equiv A_0 - \frac{q^2 - 1}{2h}. \tag{I.29}$$

Solving (I.28) we obtain the solution for the net stock of traded bonds

$$B_t = \left(B_0 + \frac{\theta K_0}{r - \phi} - \frac{C_0}{r - \psi} \right) e^{rt} - \frac{\theta K_0}{r - \phi} e^{\phi t} + \frac{C_0}{r - \psi} e^{\psi t}. \tag{I.30}$$

The variable q appearing in (I.29) is the negative root q_1 given by (I.21), though for notational convenience the subscript 1 will henceforth be omitted.

In order to ensure national intertemporal solvency, the transversality condition given by equation (I.18a), $\lim_{t \to \infty} \lambda A_t e^{-\rho t} = 0$ must be satisfied. Solving the differential equation (I.13) yields

$$\lambda_t = \lambda_0 e^{(\rho - r)t}. \tag{I.31}$$

Using (I.31) equation (I.18a) is written as

$$\lim_{t \to \infty} \lambda_0 e^{(\rho - r)t} A_t e^{-\rho t} = 0. \tag{I.32}$$

Substituting for A_t from (I.6′) and using the cash-in-advance constraint given by (I.8′) we obtain

$$\lim_{t \to \infty} \lambda_0 e^{(\rho - r)t} \left(B_t + C_t \right) e^{-\rho t} = 0. \tag{I.32′}$$

Using the solutions for B_t and C_t given by (I.30) and (I.15′) respectively yields

$$\lim_{t \to \infty} \lambda_0 \left\{ \left(B_0 + \frac{\vartheta K_0}{r - \phi} - \frac{C_0}{r - \psi} \right) - \frac{\vartheta K_0}{r - \phi} e^{(\phi - r)t} + \frac{C_0}{r - \psi} e^{(\psi - r)t} + C_0 e^{(\psi - r)t} \right\} = 0. \tag{I.33}$$

In order for (I.33) to be satisfied the following conditions must hold

$$C_0 = \left(r - \psi \right) \left(B_0 + \frac{\vartheta K_0}{r - \phi} \right), \tag{I.34a}$$

$$r - \varphi > 0\,, \tag{I.34b}$$

$$r - \psi > 0\,. \tag{I.34c}$$

Condition (I.34*a*) determines the feasible initial level of consumption and ensures the convergence of the first term. Condition (I.34*b*) ensures the convergence of the second term, and the last condition is necessary for the third and fourth terms to converge. Substituting (I.34*a*) into (I.30) gives us the equilibrium stock of traded bonds

$$B_t = \left(B_0 + \frac{\vartheta K_0}{r - \phi} \right) e^{\psi t} - \frac{\vartheta K_0}{r - \phi} e^{\phi t}\,. \tag{I.35}$$

Equations (I.12), (I.15), (I.15$'$), and (I.35), together with the solution for q and the initial condition (I.34*a*) comprise a closed form solution describing the evolution of the small open economy starting from given initial stocks of traded bonds (B_0), and the capital stock K_0.

There are several properties of the competitive equilibrium of this economy that are worth noting: First, an important feature of the simple linear technology that we adopted is that domestic output, and capital grow at the same long-run rate $g_Y = g_K = \dfrac{\tilde{q} - 1}{h} \equiv \tilde{\varphi}$, where $\tilde{q}$ stands for the negative root given by (I.21). Substituting from (I.21) we obtain:

$$\tilde{\varphi} = r - \sqrt{r\left(1 + \frac{hr}{2}\right) - A_0}\,. \tag{I.36}$$

The equilibrium growth rate of capital is thus determined by the technological conditions in the domestic economy, represented by the marginal physical product of capital (A_0) and the adjustment costs (h), as well as of the nominal return on foreign bonds. The influence of technology on the rate of growth is rather intuitive. The latter is higher the larger the marginal product of capital, and the lower the installation costs in investment.

A second implication of this model is that it can sustain differential growth rates of consumption and output. The former is driven by the difference between the rate of return on traded bonds and the domestic rate of time preference, as is described by equation

(I.15). The intuition behind the effects of these variables is straightforward: the higher the rate of interest on foreign bonds, or alternatively the lower the rate of time preference (in other words, the more patient the domestic country) the greater is the fraction of their wealth that domestic agents invest in foreign assets. The income generated from the increased accumulation of foreign bonds provides the source that sustains an increase in the future rate of growth of domestic consumption.[85]

The ability of the economy to sustain differential growth rates of consumption and output is a consequence of it being open. If the economy is closed, consumption has to grow at the rate of growth of domestic output; thus being restricted by the economy's own productive capabilities [Turnovsky, 1996 p.51]. It should be further noted that the model's predictions regarding the equilibrium behavior of growth rates of all real variables are fully consistent with the corresponding predictions in the endogenous growth literature (see for example Rebelo, 1991, and Turnovsky, 1996).

An additional property of the equilibrium in this economy is that it has no transitional dynamics; consumption and output expand *always* at their steady state growth rates. The former (given by ψ) is determined by the constant preference parameters, and the given interest rate on foreign bonds. The latter is driven by the market price of capital ($\tilde{q}$), which adjusts instantaneously in response to any disturbance to ensure that capital and output always lie on an equilibrium growth path. This characteristic of the equilibrium path is a straightforward implication of employing Rebelo's (1991) approach of endogenous growth.

[85] In a version of the model with a richer tax structure, the growth rate of consumption is a function of the after-tax interest rate on foreign bonds. This implies that a policy that reduces the tax rate on income from foreign assets has parallel growth consequences for domestic aggregate consumption, as does an increase in the given world interest rate, and growing patience of domestic consumers. This indicates that the principal determinants of the consumption growth rate include government policy variables, in addition to economic fundamentals (parameters of tastes). While this version of the model has been examined byTurnovsky (1996), the theme is part of a larger literature that aims to identify the extent to which cross-country differences in per-capita growth rates is attributed to corresponding variations in tax policy regimes. In this endeavor Jones and Manuelli (1990) use a convex technology that allows for sustained growth to examine the impact of taxation on capital income on the long-run behavior of the economy. They prove that proportional taxes on capital can potentially move the economy from the region of sustained equilibrium growth to one in which there is no growth in the long-run along the equilibrium path (p.1023). In general, it is shown that positive tax rates result in a decrease in the asymptotic growth rate relative to the no-tax situation. In a similar study, Rebelo (1988) analyzes the role of government policies in determining the growth properties of a convex model of endogenous growth.

Another important characteristic of this equilibrium relates to the effect of the inflation rate on economic performance. The use of the cash-in-advance approach enables us to analyze the effects of inflation on the real decisions of firms and households, through the 'taxing' effect on money holdings. In particular, the assumption that money is required to finance purchases of consumption goods has the implication that inflation acts as a distortionary tax on consumption spending. The question of the impact of various personal and corporate taxes on the real decisions of firms and households has been examined by Turnovsky (1996) in a similar endogenous growth framework. One of the conclusions to emerge from the author's analysis is that the equilibrium growth rates of consumption and capital (output) are completely neutral with respect to the consumption tax, given that the proceeds of the latter are rebated back in a lump-sum fashion.[86] In this case, the tax on consumption does not affect the marginal rate of substitution between consumption at different dates, thus acting as a lump-sum tax. Hence, it is neutral with respect to growth consequences. Our analysis supports this prediction, for we observe that neither the growth rate of consumption (ψ) nor that of capital ($\tilde{\varphi}$) are affected by the inflation rate (ε).[87]

The fact that inflation does not have a role in the determination of the economy's growth rate of output, is hardly surprising in an economy where inflation imitates the role of a distortionary tax on consumption expenditure. This result bears analogy to the conclusion arisen in Barro (1990)'s theory of a closed economy. Owing to being closed, the economy may never sustain differential growth rates on domestic production, and consumption. The latter both grow at a common rate defined by the difference between the marginal productivity of capital and the rate of time preference, multiplied by the intertemporal elasticity of substitution. In an environment that lacks an endogenous labor – leisure choice, a consumption tax (similarly, a flat – rate income tax) proves to be

[86] Three taxes are considered in his analysis; namely a consumption tax, a tax on income from physical capital and from holdings of foreign bonds. The equilibrium growth rate of consumption is unaffected by any tax rate, while the growth rate of capital (and output) is neutral with respect to the consumption tax only. In particular, the growth rate of capital is negatively impacted by the tax rate on assets of physical capital.

[87] In a setting where the supply of labor is elastic, the consumption tax affects the trade-off between consumption and leisure, and hence the consumption tax is distortionary. Depending on preferences, however, the intertemporal decision may not change, resulting in no effect on growth rates (see Jones and Manuelli, 1990 p.1034).

equivalent to lump – sum taxation. Therefore, it exerts no effect on either private decisions, or the path of steady – state growth. It has to be said that the validity of this result rests heavily on the labor supply being held fixed. Proceeding in the way of endogenizing the labor – leisure choice invalidates the above *neutrality* proposition. A higher tax on consumption, or wage income, leads to a fall in the equilibrium consumption – to – output ratio, the employment level, and subsequently, the rate of output growth.

We are led to the insight that if inflation does have a determining effect on the economy's growth rate it must be through channels that we have not explored yet. A natural extension of the present model is to incorporate such a mechanism, by assuming that the acquisition of capital goods is constrained by the absence of credit markets for their financing. In consequence, firms are subject to a cash-in-advance constraint on their investment expenditure on physical capital. In such a context, inflation acts as *tax* on investment, thus imitating the role of ordinary taxation on assets of physical capital. This task is pursued in the model of following section.

Model II. *Inflation acting as a capital tax in a model of fixed labor supply*[88]

The purpose of this section is to examine the effects of the inflation tax on the investment path of the economy. To accomplish this, the exchange role of money is expanded along the direction of allowing investment to be a "cash" good. Specifically, credit markets for investment are assumed to be imperfect, or completely absent. Either of these assumptions implies that in each period investment purchases are constrained by available cash balances. The structure of the economy parallels that developed in the previous section. Therefore, we consider a small open economy populated by a representative agent, who consumes and produces a single traded commodity. We assume that the foreign price of the good is given in the world market. In the absence then of any impediments to trade, the purchasing power parity condition implies

[88] The following note is remarked. The analytical framework of the model has a similar character with the theory presented in Turnovsky (1996). Turnovsky (1996) investigates the effects of tax and expenditure policies on a small open economy in a model with linear production technology that exhibits ongoing, endogenously determined growth. Similarly to the present framework, the model abstracts from effects on the employment side of the economy by assuming that labor is totally fixed or, alternatively, that it grows at some exogenously determined rate.

$$\pi = \pi^* + \varepsilon \,, \tag{II.1}$$

where the notation is similar to that of *Model I*.

Imposing the assumption that the foreign price level is constant and normalized to one, $\pi^* = 0$ and equation (II.1) yields

$$\pi = \varepsilon \,. \tag{II.2}$$

It is convenient in the present framework to distinguish between the consumption and production activities of the representative agent. We will therefore treat the optimization problems of the typical household and firm as separate.[89] We assume that the domestic resident holds three assets. The first is domestic money, which is not held by foreigners. The other two assets are domestic and foreign bonds, which earn the same real interest rate, *r*. The individual's total assets, A^h, are therefore defined as follows

$$A^h = B^h + m^h \,. \tag{II.3}$$

The representative consumer is assumed to choose her level of consumption, *C*, and total assets, A^h, by solving the following intertemporal optimization problem

$$\text{Maximize } U = \int_{t=0}^{\infty} \frac{1}{\gamma} C^{\gamma} e^{-\rho t} dt \qquad -\infty < \gamma \leq 1 \,, \tag{II.4}$$

subject to the budget constraint, which expressed in real terms is given by

$$C + \dot{B}^h + \dot{m}^h + \varepsilon m^h = w + D + rB^h + \tau \,, \tag{II.5}$$

where B^h represents the real stock of domestic and foreign bonds held by the representative agent, m^h denotes the real money holdings of the individual consumer, *w* is the real wage rate, *D* depicts the real profit paid out to the individual consumer, and τ denotes the government transfer in real terms.

We assume that labor is fixed, or is supplied inelastically, and therefore we normalize it to unity. The agent's labor income is thus given by *w*. We also impose the additional

[89] The corresponding variables will be distinguished by superscripts *h* and *f* respectively.

condition that the individual holds money in order to finance her consumption expenditure. This implies the cash-in-advance constraint

$$m^h = C . \tag{II.6}$$

Combining equations (II.3), (II.5) and (II.6), the individual's budget constraint is written as follows

$$\dot{A}^h = w + D + rA^h + \tau - \left(1 + r + \varepsilon\right) C . \tag{II.7}$$

Therefore, the representative consumer's optimization problem is to choose *C*, and *A*, to maximize (II.4) subject to (II.7). The Hamiltonian for this problem is

$$H = \frac{1}{\gamma} C^{\gamma} + \lambda \left[w + D + rA + \tau - (1 + r + \varepsilon) C \right]. \tag{II.8}$$

The optimality conditions for this problem are the following

- $H_C = 0 \Rightarrow C^{\gamma - 1} = \lambda (1 + r + \varepsilon)$, (II.9)
- $\dot{\lambda} = -\dfrac{\partial H}{\partial A^h} + \lambda \rho \Rightarrow \dfrac{\dot{\lambda}}{\lambda} = \rho - r$, (II.10)
- The Transversality Condition: $\lim_{t \to \infty} \lambda A_t^h \exp\left\{-\rho t\right\} = 0$. (II.11)

Taking the time derivative of (II.9) we obtain an expression for the growth rate of individual (as well as aggregate) consumption

$$\frac{\dot{C}}{C} = \frac{r - \rho}{1 - \gamma} \equiv \varphi . \tag{II.12}$$

This is a first-order differential equation, which can be solved to yield

$$C_t = C_0 e^{\phi t} . \tag{II.13}$$

The representative firm's formal optimization problem is to choose the level of investment, *I*, and capital, *K*, to maximize real profit

$$\Pi = \int_{t=0}^{\infty} De^{-rt} dt \,, \tag{II.14}$$

subject to the accumulation equation

$$\dot{K} = I \,. \tag{II.15}$$

The firm's profit, *D,* is defined as follows

$$D = A_0 K - w - \Phi(I, K) - rB^f + \dot{B}^f - \frac{\dot{M}^f}{P} \,. \tag{II.16}$$

As in the previous model, the production side is built on Rebelo's (1991) approach of modeling endogenous growth. The economy has one sector of production, the output of which can be used both as a consumption and capital good. The production process involves one production factor, which represents a composite of various types of physical and human capital. Labor and non-reproducible factors (e.g. land) do not play a role in production. Constant-returns-to-scale implies that the production function of domestic output Y takes the simple linear form of an AK technology

$$Y = A_0 K \qquad A_0 > 0 \,. \tag{II.17}$$

It is also assumed that the expenditure on a given increase in the capital stock, I, involves adjustment (installation) costs, which are represented by the following quadratic function

$$\Phi(I, K) = I + h\frac{I^2}{2K} = I\left(1 + \frac{h}{2}\frac{I}{K}\right). \tag{II.18}$$

Furthermore, we assume that firms can finance their investment plans either by issuing bonds, or by using retained earnings. Thus, at any period t the firm issues private securities at the rate $\dot{B}^f$, and pays interest rate r on the stock of previously issued bonds, B^f.

We also impose the restriction that investment expenditure can only be financed by cash balances carried over from previous periods. We thus have the following cash-in-advance constraint

$$m^f = \Phi(I,K) = I\left(1+\frac{h}{2}\frac{I}{K}\right), \tag{II.19}$$

where m^f is the typical firm's real money holdings. In other words,

$$m^f = \frac{M^f}{P}. \tag{II.20}$$

Differentiating (II.19) we obtain an expression for the rate of change of the firm's money holdings in real terms

$$\frac{\dot{M}^f}{P} = \dot{m}^f + \varepsilon m^f, \tag{II.21}$$

where εm^f is the cost of inflation tax.

The firm's total assets are defined as

$$A^f = m^f - B^f, \tag{II.22}$$

which implies

$$\dot{A}^f = \dot{m}^f - \dot{B}^f. \tag{II.23}$$

Combining equations (II.16), (II.19), (II.21), (II.22) and (II.23) we obtain[90]

$$D = A_0 K - w - (1+r+\varepsilon)I\left(1+\frac{h}{2}\frac{I}{K}\right) - \left(\dot{A}^f - rA^f\right). \tag{II.24}$$

Therefore, the firm's optimization problem becomes

$$\text{Maximize} \int_{t=0}^{\infty}\left[A_0 K - w - (1+r+\varepsilon)I\left(1+\frac{h}{2}\frac{I}{K}\right) - \left(\dot{A}^f - rA^f\right)\right]e^{-rt}dt,$$

subject to the capital accumulation equation $\dot{K} = I$.

It can be shown that this is equivalent to solving[91]

[90] The detailed derivation may be found in *Appendix* II.
[91] The reader may find the derivation in *Appendix* II.

$$\max_{I,K}\int_0^\infty\left[A_0K-w-(1+r+\varepsilon)I\left(1+\frac{h}{2}\frac{I}{K}\right)\right]e^{-rt}dt\,,$$

subject to $\dot{K}=I$.

The Hamiltonian for this problem is

$$H=A_0K-w-(1+r+\varepsilon)I\left(1+\frac{h}{2}\frac{I}{K}\right)+qI\,, \tag{II.25}$$

and the optimality conditions are given by

- $$H_I=0\Rightarrow\frac{I}{K}=\frac{q-(1+r+\varepsilon)}{h(1+r+\varepsilon)}=\psi\,, \tag{II.26}$$

- $$\dot{q}=-\frac{\partial H}{\partial K}+qr\Rightarrow\frac{\dot{q}}{q}+\frac{A_0}{q}+\frac{(q-(1+r+\varepsilon))^2}{2hq(1+r+\varepsilon)}=r\,, \tag{II.27}$$

- The Transversality Condition: $\lim\limits_{t\to\infty}q_tK_te^{-rt}=0$ (II.28)

Equation (II.21) is an expression for the growth rate of aggregate capital: $\frac{\dot{K}}{K}=\psi$. Solving this differential equation yields

$$K_t=K_0\exp\{\psi t\}=K_0\exp\left\{\frac{q-(1+r+\varepsilon)}{h(1+r+\varepsilon)}t\right\}. \tag{II.29}$$

The critical determinant of the growth rate of capital is the market price of installed capital, q, the path of which is determined by the arbitrage condition (II.22). In order for the capital stock to follow a path of steady growth (or decline), the stationary solution to this equation, attained when $\dot{q}=0$, must have at least one real solution.

Setting $\dot{q}=0$ in (II.22) implies that the steady state value of q ($\tilde{q}$) must be a solution to the quadratic equation

$$A_0+\frac{(\tilde{q}-(1+r+\varepsilon))^2}{2h(1+r+\varepsilon)}=r\tilde{q}\,. \tag{II.30}$$

The necessary and sufficient condition for the capital stock to converge to a steady growth path is that this equation has real roots. This will be the case if and only if

$$r\left(1+\frac{hr}{2}\right) \geq \frac{A_0}{1+r+\varepsilon}. \tag{II.31}$$

The formal solutions for the two real roots are

$$\tilde{q}_1, \tilde{q}_2 = (1+rh)(1+r+\varepsilon) \pm h(1+r+\varepsilon)\sqrt{r\left(1+\frac{hr}{2}\right)-\frac{A_0}{1+r+\varepsilon}}. \tag{II.32}$$

With equation (II.25) having two real roots, the potential arises for two steady equilibrium growth rates for capital to exist. Two cases can be identified:

Case I: $\tilde{q}_2 > 1 > \tilde{q}_1 > 0$

Case II: $\tilde{q}_2 > \tilde{q}_1 > 1$

In order to identify the dynamics of $\dot{q}$, we use (II.22) to obtain

$$\frac{\partial \dot{q}}{\partial q} = r - \frac{q}{h(1+r+\varepsilon)} + \frac{1}{h}. \tag{II.33}$$

We have the following cases

- $\dfrac{\partial \dot{q}}{\partial q} \geq 0 \Rightarrow q \leq (1+r+\varepsilon)(rh+1)$,
- $\dfrac{\partial \dot{q}}{\partial q} < 0 \Rightarrow q > (1+r+\varepsilon)(rh+1)$.

Figure 2 illustrates the phase diagram for (II.22) in the case (II.26) holds, so that a steady state growth path for capital exists. It is seen from the diagram that the equilibrium point *C*, which corresponds to the smaller equilibrium value $\tilde{q}_1$ (negative root), is an unstable equilibrium, while point *D*, which corresponds to the larger value, $\tilde{q}_2$ (positive root), is locally stable. However, it can be shown that any time path for q which converges to equilibrium point *D* violates the transversality condition (II.23), which is required to be met.

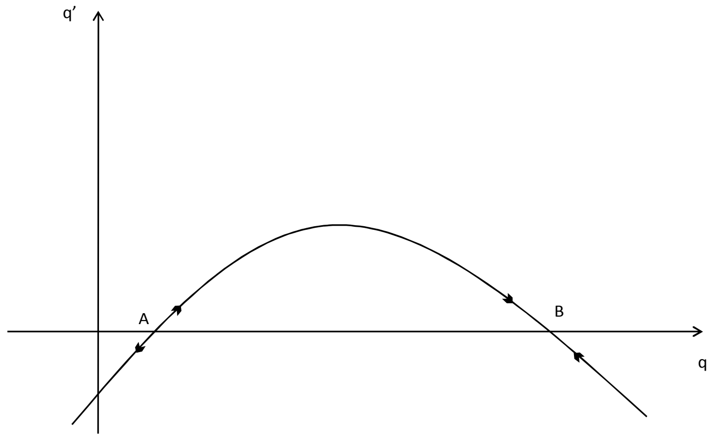

Figure 2. Phase diagram.

Substituting (II.24) into (II.23) we get

$$\lim_{t\to\infty} q_t K_0 \exp\left\{\left(\frac{q-(1+r+\varepsilon)}{h(1+r+\varepsilon)}-r\right)t\right\}=0\,. \tag{II.34}$$

It is clearly seen that when $\dfrac{q-(1+r+\varepsilon)}{h(1+r+\varepsilon)}-r>0 \Rightarrow q>\left(1+r+\varepsilon\right)\left(1+rh\right)$ this limit diverges. Thereby, the larger root $\tilde{q}_2$ violates the transversality condition on the capital stock. Similarly, the smaller root $\tilde{q}_1$ ensures that the required transversality condition holds. The behavior of the q can thus be summarized by

Proposition 1 The only solution for q which is consistent with the transversality condition is that q always be at the unstable steady-state solution $\tilde{q}_1$, given by the negative root to (II.27). Consequently there are no transitional dynamics in the market price of capital q. In response to any shock, q immediately jumps to its new equilibrium value.

The domestic government is assumed to maintain a continuously balanced budget, given by

$$\dot{m}^h + \dot{m}^f + \varepsilon\left(m^h + m^f\right) = \tau . \tag{II.35}$$

Defining $m \equiv m^h + m^f$ (II.30) is written as

$$\dot{m} + \varepsilon m = \tau . \tag{II.36}$$

Combining equations (II.3), (II.6), (II.7), (II.16) and (II.21) with the government's budget constraint (II.31) implies that the rate of accumulation of net foreign bonds by the private sector, the current account balance, is described by[92]

$$\dot{B} = rB + A_0 K - C - I\left(1 + \frac{h}{2}\frac{I}{K}\right), \tag{II.37}$$

where B is the stock of net foreign bonds defined as $B \equiv B^h - B^f$.

Substituting the expressions for $C(t)$ from (II.13), for I from (II.21) and $K(t)$ from (II.24), the accumulation equation (II.32) can be written in the form

$$\dot{B} = rB + \theta K_0 e^{-\psi t} - C_0 e^{-\phi t}, \tag{II.38}$$

where φ and ψ are defined in (II.12) and (II.21), respectively, and

$$\theta \equiv A_0 - \frac{q^2 - (1 + r + \varepsilon)^2}{2h(1 + r + \varepsilon)^2}. \tag{II.39}$$

The q appearing in (II.34) is the negative root q_1 given by (II.27), though for notational convenience the subscript 1 will henceforth be omitted.

The final step is to solve (II.33). Starting from a given initial stock B_0, the stock if traded bonds at time t is given by

$$B_t = \left(B_0 + \frac{\theta K_0}{r - \psi} - \frac{C_0}{r - \phi}\right)e^{-rt} - \frac{\theta K_0}{r - \psi}e^{\psi t} + \frac{C_0}{r - \phi}e^{\phi t}. \tag{II.40}$$

Equations (II.13), (II.24) and (II.35) together with the solution for q describe the evolution of the small open economy starting from given initial stocks of traded bonds B_0 and capital stock K_0. An important characteristic of this equilibrium is that

[92] The reader may find the detailed derivation in *Appendix* II.

consumption and physical capital are always on their steady state growth paths, growing at the rates φ and ψ respectively. The former is driven by the difference between the rate of return on foreign (and domestic) bonds and the domestic rate of time preference, as is described by equation (II.12). The growth rate of output (and capital) is driven by q, which is determined by the technological conditions in the domestic economy, as represented by the marginal physical product of capital, A_0, and adjustment costs h, as well as the return on foreign (and domestic bonds). Substituting the negative root from equation (II.27) into (II.21) we obtain the following expression for the equilibrium growth rate of capital

$$\tilde{\psi} = r - \sqrt{r\left(1 + \frac{rh}{2}\right) - \frac{A_0}{1 + r + \varepsilon}}\ . \qquad \text{(II.41)}$$

For the simple linear production function the rate of growth of capital also determines the equilibrium growth rate of domestic output. Therefore, an important feature of this equilibrium is that it can sustain differential growth rates of consumption and domestic output.

Another important characteristic of this equilibrium is that the growth rate of consumption, φ, is completely neutral with respect to the inflation (or equivalently depreciation) rate, ε. Given that money holdings are used to finance investment expenditure, inflation can be interpreted as an investment tax. The neutrality result found above is consistent with Turnovsky (1996), where the rate of growth of consumption is immune to changes in the capital income tax. On the contrary, equation (II.36) implies that the inflation rate affects negatively the equilibrium growth rate of capital (and output). This result is qualitatively no different than that obtained in Turnovsky (1996), where a higher tax on capital reduces the growth rate of capital.

Model III. *Inflation imitating a consumption tax in a model of endogenous labor*[93]

The objective of the present section is to extend the benchmark framework (*Model* I) in terms of allowing labor to be endogenously determined. Apart from enriching the model's realistic plausibility,[94] this extension offers us insight on the validity of the superneutrality result in an environment where employment channels are at work. The model's previous analytical structure is followed in near precision, while it reformulated to account for the endogeneity of the time-allocation decision. The linguistic explanation of mathematics is justly omitted, in order to avoid the unfruitful repetition of details.

The economy is populated by *N* identical individuals, each of who has an infinite planning horizon and possesses perfect foresight. Population remains stationary over time. Once again, we assume that the economy produces a single traded commodity, the foreign price of which is given in the world market. In the absence of any impediments to trade, the purchasing power parity condition is expressed in percentage terms as[95]

$$\pi = \pi^{*} + \varepsilon \, . \tag{III.1}$$

Assuming that the foreign price level is constant and equal to one, π^{*} is zero; hence equation (III.1) implies

$$\pi = \varepsilon \, . \tag{III.2}$$

Output of the individual firm, *y*, is determined using an *AK* technology

$$y = A_0 \left(1-l\right)^{\beta} k \qquad A_0 > 0 \, , \;\; 0 < \beta < 1 \, , \tag{III.3}$$

We assume that the representative agent is endowed with one unit of time that can be allocated either to leisure, *l*, or to work, 1-*l*, $\left[0 < l < 1\right]$. The individual firm faces diminishing returns to scale in labor, and constant returns to scale in the factors that can be accumulated (capital).

Combining (III.3) with $Y = Ny$, aggregate output in the economy is given by

[93] The following remark is ought to be made. The analytical framework of the model bears an evident similarity with the theory presented in Turnovsky (1999). Turnovsky (1999) investigates the effects of tax and expenditure policies on a small open economy in a model with linear production technology that exhibits endogenously determined growth.
[94] The terms '*model*', or '*benchmark model*' refer to *Model* I.
[95] Unless otherwise specified, the notation of variables is identical to that of the previous models.

$$Y = A_0 \left(1-l\right)^{\beta} K \qquad A_O > 0, \ 0 < \beta < 1. \tag{III.3′}$$

Thus aggregate output is proportional to the aggregate capital stock, thereby leading to an equilibrium having ongoing, endogenously determined, growth. The aggregate output is an *AK* technology, in which the productivity of the aggregate capital stock depends positively upon the fraction of time devoted to work.

The individual firm also accumulates physical capital, with expenditure on a given increase in the capital stock, $i \equiv \frac{I}{N}$, involving adjustment (installation) costs which we incorporate in the quadratic function

$$\Phi\left(i,k\right) = i + \frac{h}{2}\frac{i^2}{k} = i\left(1 + \frac{h}{2}\frac{i}{k}\right). \tag{III.4}$$

Aggregating over the *N* individual firms, leads to

$$\Phi\left(I,K\right) = I + \frac{h}{2}\frac{I^2}{K} = I\left(1 + \frac{h}{2}\frac{I}{K}\right). \tag{III.4′}$$

Preferences are modeled in the conventional time-separable way using an intertemporal isoelastic utility function. In addition to the optimal consumption level, the individual chooses at each period the optimal allocation of time between leisure and work.[96]

$$U = \int_0^{\infty} \frac{1}{\gamma}\left(\left(\frac{C}{N}\right) l^{\theta}\right)^{\gamma} e^{-\rho t} dt, \tag{III.5}$$

$$\theta > 0, \ -\infty < \gamma \le 1, \quad 1 > \gamma(1+\theta), \quad 1 > \gamma\theta,$$

where the parameter θ measures the impact of leisure on the welfare of the private agent. The remaining constraints on the coefficients are required to ensure that the utility function is concave in the *C* and *l*.

[96] There are two classes of time-separable preferences for which the endogenous treatment of leisure is consistent with steady-state growth. In the first class, the utility function takes the form $U(C, L)$, where U is concave, twice differentiable and homogeneous of degree k. The second class of utility functions has been proposed by King, Plosser and Rebelo (1988). This takes the form $U(C, L) = \log(C) + v(L)$, in the case of unit elasticity of intertemporal substitution. [Rebelo, 1991 pp.513-4].

We assume that the individual holds two assets. The first is domestic money, which is not held by foreigners. The other is net foreign bonds that pay an exogenously given world interest rate, *r*. The individual's total assets, *a*, are therefore defined as follows

$$a = b + m_i \,, \tag{III.6}$$

Aggregating over the *N* individuals, we obtain an expression for the aggregate stock of assets

$$A = B + m \,, \tag{III.6´}$$

Differentiating this equation yields

$$\dot{A} = \dot{B} + \dot{m} \,. \tag{III.6′}$$

The accumulation of assets by the aggregate economy is described by the following equation

$$\dot{A} = Y + rB - C - I\left(1 + \frac{h}{2}\frac{I}{K}\right) + \tau - \varepsilon m \,, \tag{III.7}$$

Finally, we also impose the cash-in-advance constraint

$$m_i = c \,, \tag{III.8}$$

which, aggregating over the *N* individuals leads to

$$m = C \,. \tag{III.8´}$$

Substituting equations $(\text{III.3}')$, $(\text{III.6}')$, and $(\text{III.8}')$ into (III.7) yields the following asset accumulation equation for the aggregate economy

$$\dot{A} = A_0 \left(1 - l\right)^{\beta} K + rA - \left(1 + r + \varepsilon\right)C - I\left(1 + \frac{h}{2}\frac{I}{K}\right) + \tau \,. \tag{III.9}$$

For simplicity we assume that capital does not depreciate, so that the economy faces the physical capital accumulation constraint

$$\dot{K} = I \,. \tag{III.10}$$

We consider the equilibrium generated in a centrally planned economy in which the planner chooses K, C, I and l to maximize the utility of the representative agent, subject to the aggregate resource constraint of the economy (III.9), and the capital accumulation equation (III.10)

$$\max_{K,C,I,l} \qquad U=\int_0^\infty \frac{1}{\gamma}\left(\left(\frac{C}{N}\right)l^{\theta}\right)^{\gamma} e^{-\rho t}dt\,,$$

subject to: $\dot{A}=A_0\left(1-l\right)^{\beta}K+rA-\left(1+r+\varepsilon\right)C-I\left(1+\frac{h}{2}\frac{I}{K}\right)+\tau\,,$

and $\qquad \dot{K}=I\,.$

The Hamiltonian for this optimization problem is given by

$$H=\frac{1}{\gamma}\left(\frac{C}{N}\right)^{\gamma}l^{\theta\gamma}+\lambda\left[A_0\left(1-l\right)^{\beta}K+rA-\left(1+r+\varepsilon\right)C-I\left(1+\frac{h}{2}\frac{I}{K}\right)+\tau\right]+q'I\,,$$

where λ and q' denote the current value Lagrange multipliers. The optimality conditions are the following

- $H_C=0\Rightarrow N^{-\gamma}C^{(\gamma-1)}l^{\vartheta\gamma}=\lambda\left(1+r+\varepsilon\right),$ (III.11)

- $H_l=0\Rightarrow N^{-\gamma}C^{\gamma}\theta l^{(\theta\gamma-1)}=\lambda A_0K\beta\left(1-l\right)^{(\beta-1)},$ (III.12)

- $H_I=0\Rightarrow \frac{I}{K}=\frac{q-1}{h}\equiv\psi\,,$ (III.13)

- $\dot{\lambda}=-\frac{\partial H}{\partial A}+\lambda\rho\Rightarrow\frac{\dot{\lambda}}{\lambda}=\rho-r\,,$ (III.14)

- $\dot{q}'=-\frac{\partial H}{\partial K}+q'\rho\Rightarrow\dot{q}'=-\lambda\left[A_0\left(1-l\right)^{\beta}+\frac{h}{2}\left(\frac{I}{K}\right)^2\right]+q'\rho\,.$ (III.15)

Combining (III.15), with (III.13), (III.14) and $q=\frac{q'}{\lambda}$ leads to

$$\frac{A_0\left(1-l\right)^{\beta}}{q}+\frac{\dot{q}}{q}+\frac{(q-1)^2}{2hq}=r\,. \qquad \text{(III.16)}$$

Equation (III.16) equates the net rate of return on domestic capital to the rate of return on the traded bond. The former consists of three components. The first is the output per unit of installed capital (valued at the price q), while the second is the rate of capital gain. The third element reflects the fact that an additional source of benefits of higher capital stock is to reduce the installation costs associated with new investment.

Finally, the following transversality conditions must be imposed

$$\lim_{t\to\infty} \lambda A_t e^{-\rho t} = 0, \tag{III.17a}$$

$$\lim_{t\to\infty} q' K e^{-\rho t} = 0 \ . \tag{III.17b}$$

Taking the time derivative of (III.11) we obtain the following expression

$$(\gamma - 1)\frac{\dot{C}}{C} + \theta\gamma\frac{\dot{l}}{l} = \frac{\dot{\lambda}}{\lambda} = \rho - r\,. \tag{III.18}$$

Equation (III.13) is a differential equation for the growth rate of capital which can be solved to yield

$$K_t = K_0 \exp\{\psi t\} = K_0 \exp\left\{\frac{q-1}{h}t\right\}. \tag{III.19}$$

Taking the time derivative of equation (III.12) and using equations $(\mathrm{III.3}')$, (III.11) and (III.14) we get the following equation

$$(1+r+\varepsilon)\,\theta\left[\gamma\frac{\dot{C}}{C} + (\theta\gamma - 1)\frac{\dot{l}}{l}\right]\frac{C}{l} = \frac{\beta}{1-l}Y\left[\rho - r + \frac{\dot{K}}{K} - (\beta - 1)\left(\frac{l}{1-l}\right)\left(\frac{\dot{l}}{l}\right)\right]. \tag{III.20}$$

Solving equation (III.12) and using equation (III.11) we obtain the equilibrium consumption-leisure ratio

$$\frac{C}{l} = Y\frac{\beta}{1-l}\cdot\frac{1}{(1+r+\varepsilon)\theta}\,. \tag{III.21}$$

Substituting equation (III.21) into equation (III.20) we obtain

$$\gamma\frac{\dot{C}}{C}+(\theta\gamma-1)\frac{\dot{l}}{l}=\rho-r+\frac{\dot{K}}{K}-(\beta-1)\left(\frac{l}{1-l}\right)\left(\frac{\dot{l}}{l}\right). \tag{III.22}$$

Substituting for $\frac{\dot{C}}{C}$ from equation (III.18), and for $\frac{\dot{K}}{K}$ from equation (III.13) yields a differential equation for leisure

$$\dot{l}=\frac{1}{F(l)}\left[r-\rho-\frac{(1-\gamma)(q-1)}{h}\right], \tag{III.23}$$

where $F(l)=\left[1-\gamma(1+\theta)\left(\frac{1}{l}\right)+(1-\gamma)(1-\beta)\left(\frac{1}{1-l}\right)\right]>0$.

The macroeconomic equilibrium can be expressed by the pair of differential equations in q and l, as given by equations (III.16) and (III.23)

$$\begin{cases}\dot{q}=rq-\dfrac{(q-1)^2}{2h}-A_0(1-l)^\beta \\ \dot{l}=\dfrac{1}{F(l)}\left[r-\rho-\dfrac{(1-\gamma)(q-1)}{h}\right]\end{cases}. \tag{III.24}$$

The steady state is described by setting $\dot{q}=\dot{l}=0$ and is therefore characterized by the relative price of capital, q, and the fraction of time devoted to leisure, l, both being constant. Linearizing system (III.24) around the steady state, we can show that the two eigenvalues to the linearized approximation are both real and positive[97]. Hence, we conclude that the only bounded equilibrium is one in which both q and l adjust instantaneously to ensure that the economy is always on its balanced growth path given by

$$\frac{A_0(1-\tilde{l})^\beta}{\tilde{q}}+\frac{(\tilde{q}-1)^2}{2h\tilde{q}}=r, \tag{III.25a}$$

$$\tilde{\psi}=\frac{\tilde{q}-1}{h}=\frac{r-\rho}{1-\gamma}, \tag{III.25b}$$

[97] The reader may find the derivation in Appendix II.

where $\tilde{\psi}$ is the equilibrium growth rate of capital.

Taking the time derivative of the aggregate production function (III.3′) and setting $\dot{l} = 0$, it is shown that the rate of growth of capital also determines the equilibrium growth rate of domestic output

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{r-\rho}{1-\gamma} \equiv \tilde{\psi} . \tag{III.26}$$

Setting $\dot{l} = 0$ in equation (III.18) we obtain an expression for the equilibrium growth rate of consumption

$$\frac{\dot{C}}{C} = \frac{r-\rho}{1-\gamma} \equiv \tilde{\psi} \tag{III.27}$$

One may observe that the equilibrium of this model is one in which domestic output, capital and consumption all grow at a common rate. The latter is determined by the difference between the world rate of interest and the domestic rate of time preference, multiplied by the intertemporal elasticity of substitution. From equation (III.25*b*) we may obtain the equilibrium price of capital, $\tilde{q}$; this is the capital value that once attained, it is ensured that aggregate capital grows at the equilibrium rate, $\tilde{\psi}$. Having obtained $\tilde{q}$, equation (III.25*a*) then determines the fraction of time devoted to leisure (employment). We conclude that in this small open economy with elastic labor supply the growth rate of output, and capital is independent of production characteristics, such as the productivity parameter, A_0, and the marginal adjustment cost, *h*. Changes in these parameters are only reflected in the individual's labor-leisure choice.

The domestic government is assumed to maintain a continuously balanced budget, which is expressed by the following equation

$$\dot{m} + \varepsilon m = \tau . \tag{III.28}$$

Substituting equation (III.6″) into (III.7), and using the government's budget constraint, we obtain the current account balance equation

$$\dot{B} = Y + rB - C - I\left(1 + \frac{h}{2}\frac{I}{K}\right), \tag{III.29}$$

or $$\dot{B} = rB + \left[\left(1 - \frac{C}{Y}\right)\frac{Y}{K} - \frac{I}{K}\left(1 + \frac{h}{2}\frac{I}{K}\right)\right]K. \tag{III.29′}$$

We substitute $\frac{C}{Y}$, $\frac{I}{K}$, and $K(t)$ in (III.29′) for the equivalent expressions from equations (III.21), (III.13), and (III.19) respectively. We obtain

$$\dot{B} = rB + \left[\left(1 - \frac{\beta}{\theta(1+r+\varepsilon)} \cdot \frac{\tilde{l}}{1-\tilde{l}}\right)\frac{\tilde{Y}}{\tilde{K}} - \frac{\tilde{q}^2 - 1}{2h}\right]K_0 e^{\tilde{\psi} t}. \tag{III.30}$$

Solving this equation we get an expression for the nation's intertemporal resource constraint

$$B_0 + \frac{K_0}{r - \tilde{\psi}}\left[\left(1 - \frac{\beta}{\theta(1+r+\varepsilon)} \cdot \frac{\tilde{l}}{1-\tilde{l}}\right)\frac{\tilde{Y}}{\tilde{K}} - \frac{\tilde{q}^2 - 1}{2h}\right] = 0. \tag{III.31}$$

The initial value of the nation's foreign bonds plus the capitalized value of the current account surplus along the balanced growth path must sum to zero. Having determined the equilibrium values of $\tilde{l}$, $\tilde{q}$ and $\tilde{Y}/\tilde{K}$, the intertemporal constraint (III.31) then determines the combination of the initial capital stock, K_0, and the initial stock of foreign bonds, B_0, necessary for the equilibrium to be intertemporally viable. If the inherited stocks of these assets violate (III.31) it is assumed that the central planner can engage in an initial trade, described by $dB_0 + \tilde{q}dK_0 = 0$ to bring about the correct ratio.

The above model with a simple cash-in-advance constraint imposed on individual consumption purchases behaves very similarly with the model developed in Turnovsky (1999). Domestic consumption, capital, and output all grow at a common rate determined by taste parameters, together with the rate of return on foreign bonds. The long-run growth rate was found to be independent of the inflation rate. This result is consistent with the findings of Turnovsky (1999), according which the growth rate is completely neutral with respect to any fiscal instruments, including the tax rate on consumption.

Model IV. *Inflation imitating capital taxation in a model of endogenous labor*[98]

We consider a small open economy populated by a representative agent, who consumes and produces a single traded commodity, the foreign price of which is given in the world market. In the absence of any impediments to trade, purchasing power parity is assumed to hold. Expressed in percentage terms, it is described by

$$\pi = \pi^* + \varepsilon , \tag{IV.1}$$

Assuming that the foreign price level is constant and equal to one, π^* is equal to zero, and equation (IV.1) is written

$$\pi = \varepsilon . \tag{IV.2}$$

We assume that the domestic resident holds three assets, domestic money, which is not held by foreigners, and domestic and foreign bonds. The latter two earn the same real interest rate, *r*. The individual's total assets, A^h are therefore defined as

$$A^h = B^h + m^h , \tag{IV.3}$$

where B^h denotes the real stock of domestic and foreign bonds, and m^h the real money holdings of the individual consumer. It follows that

$$\dot{A}^h = \dot{B}^h + \dot{m}^h . \tag{IV.3′}$$

Once again, we assume that the representative agent is endowed with a unit of time that can be allocated either to leisure, *l*, or to work, 1-*l*, $[0 < l < 1]$. The representative consumer chooses her level of consumption, *C*, the fraction of time allocated to leisure (work) *l,* and total assets, A^h, by solving the following intertemporal optimization problem

$$\text{Max} \quad U = \int_0^\infty \frac{1}{\gamma}\left(Cl^\theta\right)^\gamma e^{-\rho t} dt , \tag{IV.4}$$

[98] The following note is ought to be remarked. The analytical framework of the model bears an evident similarity with the theory presented in Turnovsky (1999). Turnovsky (1999) investigates the effects of tax and expenditure policies on a small open economy in a model with linear production technology that exhibits endogenously determined growth.

where $\theta > 0$, $-\infty < \gamma \le 1$, $1 > \gamma(1+\theta)$, $1 > \gamma\theta$,

subject to the budget constraint, expressed in real terms as

$$C + \dot{B}^h + \dot{m}^h + \varepsilon m^h = w(1-l) + D + rB^h + \tau, \tag{IV.5}$$

The constraints on coefficients are required to ensure that the utility function is concave in the C and l. We recall that w is the real wage rate, D is the real profit paid out to the individual consumer, and τ denotes real government transfers.

We impose the additional condition that the individual holds money in order to finance her consumption expenditure. This implies the cash-in-advance constraint

$$m^h = C. \tag{IV.6}$$

Combing equations (IV.3′), (IV.5) and (IV.6), the individual's budget constraint is written as follows

$$\dot{A}^h = w(1-l) + D + rA^h + \tau - (1+r+\varepsilon)C. \tag{IV.7}$$

The representative consumer's optimization problem is to choose C, l and A^h, to maximize (IV.4) subject to the budget constraint (IV.7). The Hamiltonian for this problem is

$$H = \frac{1}{\gamma}\left(Cl^\theta\right)^\gamma + \lambda\left[w(1-l) + D + rA^h + \tau - (1+r+\varepsilon)C\right]. \tag{IV.8}$$

The optimality conditions for this problem are the following

- $H_C = 0 \Rightarrow C^{(\gamma-1)} l^{\theta\gamma} = \lambda(1+r+\varepsilon),$ (IV.9)

- $H_l = 0 \Rightarrow \theta C^\gamma l^{\theta\gamma-1} = \lambda w,$ (IV.10)

- $\dot{\lambda} = -\dfrac{\partial H}{\partial A^h} + \lambda\rho \Rightarrow \dfrac{\dot{\lambda}}{\lambda} = \rho - r,$ (IV.11)

- The Transversality Condition: $\lim\limits_{t\to\infty} \lambda A_t^h e^{-\rho t} = 0.$ (IV.12)

Taking the time derivative of equation (IV.9) we obtain an expression for the growth rate of individual (as well as aggregate) consumption

$$(\gamma-1)\frac{\dot{C}}{C}+\theta\gamma\frac{\dot{l}}{l}=\frac{\dot{\lambda}}{\lambda}=\rho-r\,. \tag{IV.13}$$

The representative firm's formal optimization problem is to choose the level of investment, I, and capital, K, to maximize real profit

$$\Pi=\int_{t=0}^{\infty} D e^{-rt}dt\,, \tag{IV.14}$$

subject to the accumulation equation

$$\dot{K}=I\,. \tag{IV.15}$$

The firm's profit, D, is defined as follows

$$D=A_0(1-l)^{\beta}K-w(1-l)-\Phi(I,K)-rB^f+\dot{B}^f-\frac{\dot{M}^f}{P}\,. \tag{IV.16}$$

Domestic output of the commodity, Y, is determined by the domestic capital stock, K, and employment level, $(1-l)$, using the technology

$$Y=A_0(1-l)^{\beta}K \qquad A_0>0\,,\ \ 0<\beta<1\,. \tag{IV.17}$$

Evidently, we have assumed that the representative firm faces diminishing returns to scale in labor, and constant returns to scale in capital. Thus aggregate output is proportional to the aggregate capital stock, leading to an equilibrium having ongoing, endogenously determined, growth. The aggregate output is an AK technology, in which the productivity of the aggregate capital stock depends positively upon the fraction of time devoted to work.

We also assume that the expenditure on a given increase in the capital stock, I, involves adjustment costs (installation costs) which we incorporate in the quadratic function

$$\Phi(I,K) = I + h\frac{I^2}{2K} = I\left(1+\frac{h}{2}\frac{I}{K}\right). \tag{IV.18}$$

In addition, at any period *t,* the firm issues corporate bonds at the rate $\dot{B}^f$, and it pays interest on the existing stock of previously issued bonds, B^f.

We impose the cash-in-advance constraint that the individual firm holds money in order to finance its investment expenditure. This implies the additional constraint

$$m^f = \Phi(I,K) = I\left(1+\frac{h}{2}\frac{I}{K}\right), \tag{IV.19}$$

where, we recall m^f is the firm's real money holdings. In other words,

$$m^f = \frac{M^f}{P}. \tag{IV.20}$$

Differentiating equation (IV.19) we obtain an expression for the rate of change of the firm's money holdings in real terms

$$\frac{\dot{M}^f}{P} = \dot{m}^f + \varepsilon m^f, \tag{IV.21}$$

where εm^f is the cost of inflation tax.

The firm's total assets are defined as

$$A^f = m^f - B^f, \tag{IV.22}$$

implying that

$$\dot{A}^f = \dot{m}^f - \dot{B}^f. \tag{IV.23}$$

Combining equations (IV.16), (IV.19), (IV.21), (IV.22) and (IV.23) we obtain

$$D = A_0(1-l)^\beta K - w(1-l) - (1+r+\varepsilon)I\left(1+\frac{h}{2}\frac{I}{K}\right) - \left(\dot{A}^f - rA^f\right). \tag{IV.24}$$

The firm's optimization problem becomes

$$\max \int_{t=0}^{\infty} \left[A_0 (1-l)^{\beta} K - w(1-l) - (1+r+\varepsilon) I \left(1 + \frac{h}{2} \frac{I}{K} \right) - \left(\dot{A}^f - rA^f \right) \right] e^{-rt} dt .$$

subject to the capital accumulation equation $\dot{K} = I$.

It can be shown that this is equivalent to solving

$$\max_{I,K} \int_{0}^{\infty} \left[A_0 (1-l)^{\beta} K - w(1-l) - (1+r+\varepsilon) I \left(1 + \frac{h}{2} \frac{I}{K} \right) \right] e^{-rt} dt ,$$

subject to the capital accumulation equation $\dot{K} = I$.

The Hamiltonian for this problem is

$$H = A_0 (1-l)^{\beta} K - w(1-l) - (1+r+\varepsilon) I \left(1 + \frac{h}{2} \frac{I}{K} \right) + qI , \tag{IV.25}$$

and the optimality conditions are given by

- $$H_I = 0 \Rightarrow \frac{I}{K} = \frac{q-(1+r+\varepsilon)}{h(1+r+\varepsilon)} \equiv \psi , \tag{IV.26}$$
- $$H_{(1-l)} = 0 \Rightarrow A_0 \beta K (1-l)^{\beta-1} = w , \tag{IV.27}$$
- $$\dot{q} = -\frac{\partial H}{\partial K} + qr \Rightarrow \frac{\dot{q}}{q} + \frac{A_0 (1-l)^{\beta}}{q} + \frac{\left(q-(1+r+\varepsilon)\right)^2}{2hq(1+r+\varepsilon)} = r , \tag{IV.28}$$
- The Transversality Condition: $\lim_{t \to \infty} q K_t e^{-rt} = 0$. (IV.29)

Equation (IV.26) is an expression for the growth rate of aggregate capital, which can be solved to yield

$$K_t = K_0 \exp\{\psi t\} = K_0 \exp\left\{ \frac{q-(1+r+\varepsilon)}{h(1+r+\varepsilon)} t \right\}. \tag{IV.30}$$

Combining equations (IV.10) and (IV.27) we have

$$\theta C^{\gamma} l^{\theta\gamma-1} = \lambda A_0 \beta K (1-l)^{\beta-1} . \tag{IV.31}$$

Taking the time derivative of equation (IV.31) and combining with equations (IV.9), (IV.11) and (IV.26) we obtain

$$(1+r+\varepsilon)\theta\left[\gamma\frac{\dot{C}}{C}+(\theta\gamma-1)\frac{\dot{l}}{l}\right]\frac{C}{l}=\frac{\beta}{1-l}Y\left[\rho-r+\frac{q-(1+r+\varepsilon)}{h(1+r+\varepsilon)}-(\beta-1)\left(\frac{\dot{l}}{1-l}\right)\right]. \quad \text{(IV.32)}$$

Combining equation (IV.31) with equations (IV.9) and (IV.17) we obtain the equilibrium consumption-leisure ratio

$$\frac{C}{l}=\frac{\beta}{\theta(1+r+\varepsilon)}\cdot\frac{Y}{1-l}. \quad \text{(IV.33)}$$

Setting equation (IV.33) into (IV.32) yields

$$\gamma\frac{\dot{C}}{C}+(\theta\gamma-1)\frac{\dot{l}}{l}=\rho-r+\frac{q-(1+r+\varepsilon)}{h(1+r+\varepsilon)}-(\beta-1)\left(\frac{\dot{l}}{1-l}\right). \quad \text{(IV.34)}$$

Substituting for $\dot{C}/C$ from equation (IV.13) yields the differential equation for leisure

$$\dot{l}=\frac{1}{F(l)}\left[r-\rho-\frac{(1-\gamma)\left[q-(1+r+\varepsilon)\right]}{h(1+r+\varepsilon)}\right], \quad \text{(IV.35)}$$

where $F(l)=\left[1-\gamma(1+\theta)\left(\frac{1}{l}\right)+(1-\gamma)(1-\beta)\left(\frac{1}{1-l}\right)\right]>0$.

The macroeconomic equilibrium can be expressed by the pair of differential equations in q and l, as given by equations (IV.28) and (IV.35) respectively

$$\begin{cases}\dot{q}=rq-\dfrac{\left[q-(1+r+\varepsilon)\right]^2}{2h(1+r+\varepsilon)}-A_0(1-l)^{\beta}\\[2ex]\dot{l}=\dfrac{1}{F(l)}\left[r-\rho-\dfrac{(1-\gamma)\left[q-(1+r+\varepsilon)\right]}{h(1+r+\varepsilon)}\right]\end{cases}. \quad \text{(IV.36)}$$

The steady state is described by setting $\dot{q}=\dot{l}=0$, hence characterized by the relative price of capital, q, and the fraction of time devoted to leisure, l, both being constant. Linearizing system (IV.36) around the steady state, we can show that the two eigenvalues to the linearized approximation are both real and positive[99]. Therefore, the conclusion is reached that the only bounded equilibrium is one in which both q and l adjust instantaneously to ensure that the economy is always on its balanced growth path given by

$$\frac{A_0\left(1-\tilde{l}\right)^{\beta}}{\tilde{q}}+\frac{\left(\tilde{q}-\left(1+r+\varepsilon\right)\right)^2}{2h\left(1+r+\varepsilon\right)\tilde{q}}=r\,, \tag{IV.37a}$$

$$\tilde{\psi}=\frac{\tilde{q}-\left(1+r+\varepsilon\right)}{h\left(1+r+\varepsilon\right)}=\frac{r-\rho}{1-\gamma}, \tag{IV.37b}$$

where $\tilde{\psi}$ is the equilibrium growth rate of capital.

Taking the time derivative of the aggregate production function (IV.17), and setting $\dot{l}=0$, we can obtain the equilibrium growth rate of capital (and output)

$$\frac{\dot{Y}}{Y}=\frac{\dot{K}}{K}=\frac{r-\rho}{1-\gamma}\equiv\tilde{\psi}\,. \tag{IV.38}$$

Setting $\dot{l}=0$ in equation (IV.13) we obtain an expression for the equilibrium growth rate of consumption

$$\frac{\dot{C}}{C}=\frac{r-\rho}{1-\gamma}\equiv\tilde{\psi}\,. \tag{IV.39}$$

The conclusion is reached that in equilibrium domestic output, capital and consumption all grow at a common rate determined by the world interest rate, the domestic rate of time preference, and the intertemporal elasticity of substitution. From equation (IV.37*b*) we may obtain the equilibrium price of capital, $\tilde{q}$; this is the value of capital that once attained, it is ensured that domestic capital grows at the equilibrium rate, $\tilde{\psi}$. Having obtained $\tilde{q}$, equation (IV.37*a*) then determines the equilibrium employment time. Finally, it is observed that in this small open economy with elastic labor supply, the growth rate of output (and capital) is independent of production characteristics, such as

[99] The reader may find the detailed derivation in Appendix II.

the productivity parameter, A_0, and the marginal adjustment cost, *h*. Changes in these parameters are reflected only in the equilibrium labor-leisure choice.

The domestic government is assumed to maintain a continuously balanced budget, given by

$$\dot{m}^h + \dot{m}^f + \varepsilon\left(m^h + m^f\right) = \tau \,. \tag{IV.40}$$

Defining $m \equiv m^h + m^f$ equation (IV.40) is written as

$$\dot{m} + \varepsilon m = \tau \,. \tag{IV.41}$$

Combining equations (IV.3), (IV.7), (IV.16), (IV.17), (IV.23) and the government's budget constraint (IV.41) we obtain the rate of accumulation of net foreign bonds by the private sector (the current account balance)

$$\dot{B} = Y + rB - C - I\left(1 + \frac{h}{2}\frac{I}{K}\right), \tag{IV.42}$$

or $$\dot{B} = rB + \left[\left(1 - \frac{C}{Y}\right)\frac{Y}{K} - \frac{I}{K}\left(1 + \frac{h}{2}\frac{I}{K}\right)\right]K\,, \tag{IV.42$'$}$$

where we note that the stock of net foreign bonds is defined as $B \equiv B^h - B^f$.

We substitute for C/Y, I/K and $K(t)$, in equation (IV.42′), with the equivalent expressions from equations (IV.33), (IV.26) and (IV.30) respectively. We obtain

$$\dot{B} = rB + \left[\left(1 - \frac{\beta}{\theta(1+r+\varepsilon)}\frac{\tilde{l}}{1-\tilde{l}}\right)\frac{\tilde{Y}}{\tilde{K}} - \frac{\tilde{q}^2 - (1+r+\varepsilon)^2}{2h(1+r+\varepsilon)^2}\right]K_0 e^{\tilde{\psi} t}\,. \tag{IV.43}$$

Solving this equation we get an expression for the nation's intertemporal resource constraint

$$B_0 + \frac{K_0}{r-\tilde{\psi}}\left[\left(1 - \frac{\beta}{\theta(1+r+\varepsilon)}\cdot\frac{\tilde{l}}{1-\tilde{l}}\right)\frac{\tilde{Y}}{\tilde{K}} - \frac{\tilde{q}^2 - (1+r+\varepsilon)^2}{2h(1+r+\varepsilon)^2}\right] = 0\,. \tag{IV.44}$$

The initial value of the nation's foreign bonds plus the capitalized value of the current account surplus along the balanced growth path must sum to zero. Having determined the equilibrium values of $\tilde{l}$, $\tilde{q}$ and $\tilde{Y}/\tilde{K}$, the intertemporal resource constraint (IV.44) determines the combination of the initial capital stock, K_0, and the initial stock of foreign bonds, B_0, necessary for the equilibrium to be intertemporally viable.

The above cash-in-advance model, and the model developed in the previous section (*Model* III) behave very similarly with the model developed in Turnovsky (1999). In all three models the equilibrium is such that domestic consumption, capital, and output all grow at a common rate, determined by taste parameters, together with the return on foreign bonds (the after-tax rate of return on foreign bonds in Turnovsky, 1999). We found the long-run growth rate to be independent of the inflation rate. This result is consistent with the findings of Turnovsky (1999), according which the growth rate is completely neutral with respect to fiscal instruments, such as the tax rate on consumption, and capital income.

Concluding remarks

The present study has been an enquiry on an old theme in the theory of macroeconomics, namely, how far may be carried the confidence in monetary policy's importance. The question is reopened, would economic dynamics possibly take monetary effectiveness too far afield, to determine the pace, and character of the process of economic growth? The analysis is carried with reference to an economy being open, yet a price taker in the international capital markets. That financial intermediation constitutes an inextricable part of the process of economic growth is an idea with a long recognition in macroeconomic literature. The theory we sketched brings a valid argument of this view, with the role of monetary policy be interwoven into the process. It goes without saying that when credit markets do not function perfectly, or are entirely missing, money is assigned the role of being the primary, or sole, medium of exchange. We postulate that the possibility of intermediated credit does not exist, with the intention of the assumption being to uncover the role of inflation as tax on private spending. Initially, the postulate applies on purchases of consumption goods only. In an alternative version of the model, the investment on capital goods is being subjected to the constraint that cash balances carried from the previous period are the only means of conducting the transaction. In this latter case, inflation bears an evident analogy to a capital tax. The theory been constructed thus gives us an insight into how inflation is been conceived to imitating fiscal tax instruments. To elucidate the consequences of endogenously determined labor, the theory is initially built on models that abstract from the decision to allocate time between leisure and other productive activities. The analysis has been extended to account for the endogeneity of the time-allocation decision in the second part of this essay.

Owing to access in world capital markets, the economy may sustain differential growth rates in consumption expenditure, and domestic output production (capital). In all outlined models, the former is determined by the difference between the rate of return on traded bonds and domestic rate of time preference, multiplied by the intertemporal elasticity of substitution. It needs to be said that this definition is logically derived from a set of properties, and basic assumptions of the theory. A form of specific generality has been established here, in the sense that the expression of consumption's growth rate is shown to be common to all four individual cases. As a consequence, it is independent of monetary aggregates irrespective of the role of inflation as consumption, or capital income tax.

With reference to the issue of *money superneutrality* the analysis has brought up the following propositions: (*i*) Inflation in the role of a consumption tax has no impact on the growth rate of an open economy's output (capital). This is true in both cases of fixed, and elastic labor supply. (*ii*) When inflation performs the role of a capital tax the superneutrality result breaks down in a framework of fixed labor supply. Monetary expansion acts to raise the effective relative price of capital, thus having a negative distortionary effect on the growth rate of output. (*iii*) In an environment that accommodates an endogenous labor – leisure choice, the validity of the superneutrality result is reestablished. This is in contrast to a closed economy, where the adverse effect is also prevalent in the case of elastic labor supply.

ESSAY III

*Habit Formation in Durable Consumption and the Current Account: The Dynamic Effects of Fiscal Policies**

**The present essay was written under the supervision of Dr. Mohammed Mohsin, during the time period October 2004 to February 2005, at the University of Tennessee, Knoxville. The research analysis is the product of the author's collaborated work with Dr. Mohsin.*

Introduction

Theme This essay constitutes a theory on the long run effects of fiscal tax and expenditure policies. The analysis is carried with reference to an open economy, yet a price taker in the international markets. Our interest lies in exploring the transitional dynamics of the current account in response to permanent fiscal shocks. The empirical literature in the international macroeconomics has established that the current account evolves non-monotonically along its adjustment path to the long run equilibrium. It has been the aim of this study to show that this empirical phenomenon may be proved within the theory, thus be validated on the ground of acceptance of a mathematical proposition. To this endeavor we sought two sources of *time non-separability* in the preference structure, habit forming consumption in consumer durable goods. When households choose to maintain their habitual standard of living and consumption exhibits a degree of durability, optimal private choices induce the non-monotonic dynamics on consumption, hence saving, behavior that are exactly consistent with the factual evidence on the current account. It ought to be said that adopting the aforementioned source of time non-separability is a critical task in its own right, irrespective of the role in inducing the aimed dynamics. Empirical studies in macroeconomics have continuously argued in favor of habit-forming patterns in consumption behavior, as well as of significant private expenditure in goods with durable character. Clearly, accommodating these aspects of individual behavior into the model vitally enhances the realistic plausibility of the theory, thus establishing its practical value. Omitting them from a theory constructed to explain a phenomenon pertaining to saving behavior may justly become a point of valid criticism.

Few studies have attempted to examine the implications of government, or monetary policies in an open economy framework using a time dependent, yet an exogenous preference structure. Examples include Obstfeld (1992), Mansoorian (1993, 1996, and 1998), and Ikeda and Gombi (1998), with the last being the sole study in the area of fiscal macroeconomics. The theory outlined in this essay constitutes an extension of the research in Ikeda and Gombi (1998) on two dimensions. In contrast to the latter study, consumption possesses a character of durability in our analysis. Secondly, we consider a broader array of fiscal tax instruments. Whereas Ikeda and Gombi (1998) examine the implications of capital taxation and government spending, the analysis herein accommodates the impact of taxation on consumption expenditure, income from holdings of foreign bonds, and finally lump-sum taxation.

Habit formation In a seminal paper, Ryder and Heal (1973) addressed the issue of complementarity between consumption at successive moments, and proposed a new, at the time, more realistic formulation of the utility function. The essential feature of their approach is that a new variable is introduced into the utility function, interpreted as the customary level of consumption. Instantaneous satisfaction depends both on instantaneous consumption and on the customary consumption level, implying a form of the utility function $U\big(c(\cdot)\big) \equiv \int_0^\infty e^{-\delta t} u\big[c(t), z(t)\big] dt$. The variable $z(t)$ represents the habitual standard of living, and is defined as the weighted average of past consumption levels, with the weights declining exponentially into the past. The justification for including such a variable is obvious: the amount of satisfaction that a person derives from consuming a given bundle of goods depends not only on that bundle, but also on her past consumption and on her general social environment. This approach has considerable intuitive plausibility: For example, it is not uncommon for sociologists concerned with political changes during economic development to remark that a period of historically high consumption levels followed by a drop in consumption is more likely to cause social discontent than is a period of uniformly low consumption levels: in the former case, the period of high consumption builds up high customary or expected consumption levels, and the decline, though it may be to levels that are historically high, produces a sharp fall in satisfaction [Ryder and Heal, 1973 pp.1-2].

Models in which habits develop over the flow of services provided by consumption have been used by a number of authors to explain several macroeconomic and financial regularities and puzzles. Constantinides (1990) uses the habit-persistence model of Ryder and Heal (1973) to solve the Mehra and Prescott (1985) equity premium puzzle. He is able to solve the puzzle because the habit persistence model can smooth consumption over and above the smoothing implied by the usual time separable preference structure. Backus, Gregory, and Telmer (1993) show that habit persistence helps to account for the high variation in the expected returns on the forward relative to spot markets for currencies. Moreover, Mansoorian (1993) uses the habit-persistence model to reexamine the Harberger-Laursen-Metzler effect. More recently, Mansoorian (1996) examines the policy implications of the habit-persistence model in a small open economy framework. Heaton (1993), Ferson and Constantinides (1991), and Fuhrer (2000) among others provide empirical evidence in favor of habit persistence.

Durability in consumption expenditure Dunn and Singleton (1986), Eichenbaum, Hansen and Singleton (1988) and Eichenbaum and Hansen (1990) have documented their results as evidence of significant consumption expenditures in durable goods. In addition, Ferson and Constantinides (1991), Heaton (1993, 1995) among others have clearly confirmed that the introduction of durable goods helped improve the empirical performance of asset pricing models. It is important to note that durables and semi-durables make up about 20 percent of total consumption expenditures in industrial countries. Moreover durable goods are known to be a big part of business cycles. Yet, almost all theoretical models with a single-good, intertemporal optimizing framework, have paid scant attention to these facts and incorporated only non-durable goods. In a single-good model, a possible approximation of the reality would be to inject a certain degree of durability.

Outline of the essay The precise structure of the model is set out in the following section, in accompaniment of an elaborate analysis on the equilibrium dynamics of the model. A note in the end takes the role of final conclusion.

The model

Structure of the model The model is one of a small open economy populated by infinitely lived identical agents. There is a single traded good, which can be used for consumption and investment. Given the market wage, w_t, households supply one unit of labor inelastically in each point in time *t*. They hold non-human wealth in the form of foreign bonds b_t. Bonds can be purchased in the international market at a constant interest rate r.

The household's consumption behavior is habit-forming. As in Ryder and Heal (1973) we assume that the habitual standard of living is a weighted average of past consumption of the services of consumer durables $\left(c_j + s_j, j < t\right)$, with exponentially declining weights given to more distant values of $c_j + s_j$. We have

$$z_t = \rho \int_{-\infty}^{t} (c_\tau + s_\tau) \exp(-\rho(t-\tau)) d\tau \, , \tag{1}$$

where z_t represents the habitual standard, and ρ (> 0) is a parameter determining the relative weights of consumption at different times. Equation (1) may be re-written as follows

$$\dot{z}_t = \rho(c_t + s_t - z_t) \, . \tag{1$'$}$$

It is noted that c_t denotes the consumption rate (the amount of consumer durables purchased at time *t*), and s_t is the stock of durable goods, assumed to have been inherited from the past. It is further assumed that durable goods depreciate at the rate δ. Therefore, we write

$$s_t = \int_{-\infty}^{t} exp(\delta(\tau - t)) c_\tau d\tau \, . \tag{2}$$

It follows that the evolution of s_t is given by

$$\dot{s}_t = c_t - \delta s_t \, . \tag{2$'$}$$

The consumer's lifetime utility function is specified as follows

$$U_0 = \int_0^{\infty} U(c_t + s_t, z_t)\exp(-\theta t)dt . \tag{3}$$

where θ is the rate of time preference. In accordance to Ryder and Heal (1973) the utility function $U(\cdot)$ satisfies the following regularity conditions:

(*i*) $U_1 > 0$;

(*ii*) $U_2 \leq 0$;

(*iii*) $U_1(c+s, c+s) + U_2(c+s, c+s) > 0$;

(*iv*) U is concave in $(c+s, z)$; and

(*v*) $\lim_{c+s \to 0}[U_1(c+s, c+s) + U_2(c+s, c+s)] = \infty$.

Intertemporal complementarities in consumption are defined as $U_{12}(c,c) + (\rho/(\theta + 2\rho))U_{22}(c,c) < (>) 0$, where preferences are said to display distant (adjacent) complementarity respectively. The meaning is that present consumption is complementary to consumption in the distant (adjacent) future respectively (see Ryder and Heal, 1973) [Ikeda and Gombi, 1998 p.366].

The representative household's optimization problem is choosing the set of variables $C_0 = \{c_t, z_t, s_t, b_t\}_{t=0}^{\infty}$ so as to maximize equation (3), subject to the equations of motion for z_t, and s_t, as given by equations (1) and (2) respectively, the flow budget constraint expressed by

$$\dot{b}_t = r(1 - \tau_b)b_t + w_t + \pi_t - (1 + \tau_c)c_t - T_t, \tag{4}$$

the non-Ponzi game condition $\lim_{t \to \infty} b_t e^{-rt} > 0$, the path of $\{T_t\}$ taken as given, and the initial conditions (b_0, z_0, s_0). It is noted that π_t is the profit that the representative household receives as the owner of the firm.

The shadow prices of saving, habit formation, and durable goods are represented respectively by the variables $\lambda_t(\geq 0)$, $\xi_t(\leq 0)$ and $\mu_t(\geq 0)$. The Hamiltonian for the representative agent's problem is expressed as

$$H \equiv U(c_t + s_t, z_t) + \lambda_t \left[r(1-\tau_b) b_t + w_t + \pi_t - (1+\tau_c) c_t - T_t \right] + \xi_t \left[\rho (c_t + s_t - z_t) \right] + \mu_t (c_t - \delta s_t) .$$

The optimality conditions for this problem are given by the following expressions

$$U_1(c_t + s_t, z_t) - \lambda_t (1+\tau_c) + \xi_t \rho + \mu_t \equiv 0 , \tag{5}$$

$$\dot{\lambda}_t = \lambda_t (\theta - r(1-\tau_b)) , \tag{6}$$

$$\dot{\xi}_t = -U_2(c_t + s_t, z_t) + \xi_t (\rho + \theta) , \tag{7}$$

$$\dot{\mu}_t = -U_1(c_t + s_t, z_t) + \mu_t (\delta + \theta) - \rho \xi_t . \tag{8}$$

The transversality conditions are given by

$$\lim_{t \to \infty} e^{-\theta t} \lambda_t b_t = 0 , \tag{9}$$

$$\lim_{t \to \infty} e^{-\theta t} \xi_t z_t = 0 , \tag{10}$$

$$\lim_{t \to \infty} e^{-\theta t} \mu_t s_t = 0 . \tag{11}$$

Equation (6) implies that the only way for λ_t to be at steady state is for

$$\theta = r(1-\tau_b) . \tag{12}$$

It is taken as given in the foregoing analysis that equation (12) holds true, implying that λ_t is always at its steady state value.

The representative firm chooses the time profiles of labor demand and the rate of net investment $\{l_t, I_t\}_{t=0}^{\infty}$ so as to maximize the present discounted value of its future net cash flows. The output of the individual firm, y_t, is determined using the following technology

$$y_t = AF(k_t, l_t) = Ak^a l^{1-a} \qquad a < 1 , \tag{13}$$

where A is the productivity parameter, and k_t denotes the firm's capital stock, assumed to be infinitely durable. Further, the production function $AF(k_t, l_t)$ is linearly homogeneous in k_t and l_t.

It is a major assumption of the model that the individual firm's expenditure on a given increase in the capital stock, I_t, involves adjustment (or installation) costs. The latter are incorporated in the non-negative quadratic function $G(I)$, that satisfies the properties $G(0) = 0$ and $G'(0) = 0$. Convexity implies $G' \geq 0$ and $G'' > 0$.

The representative firm's optimization problem is therefore described by

$$V_0 = \max \int_0^\infty \pi_t \exp(-(r - \tau_b))t \,, \tag{14}$$

$$\text{subject to } \dot{k}_t = I_t \,, \tag{15}$$

where

$$\pi_t = AF(k_t, l_t) - w_t l_t - \tau_k k_t - I_t - G(I_t) \,. \tag{16}$$

Considering capital taxation, τ_k is specified as the tax levied on each unit of k_t. Letting $q_t (> 0)$ represent the shadow price of investment, in other words, the marginal *q*, the Hamiltonian for the firm's problem is

$$H \equiv AF(k_t, l_t) - w_t l_t - \tau_k k_t - I_t - G(I_t) + q_t I_t \,. \tag{17}$$

The equilibrium behavior of the firm satisfies the following optimality conditions

$$AF_l(k_t, 1) = w_t \,, \tag{18}$$

$$q_t = 1 + G'(I_t)$$

$$\Rightarrow I_t = G'^{-1}(q_t - 1) = I(q_t) \,, \tag{19}$$

$$\dot{q}_t = -AF_k(k_t, 1) + \tau_k + (r - \tau_b) q_t \,, \tag{20}$$

where, it is noted that labor supply has been normalized to unity. Finally, the transversality condition is given by

$$\lim_{t\to\infty} q_t k_t e^{-(r-\tau_b)t} = 0\,. \tag{21}$$

Throughout the analysis it is assumed that domestic government maintains continuously a balanced budget given by

$$g_t = T_t + \tau_b r b_t + \tau_c c_t + \tau_k k_t\,. \tag{22}$$

Thus, government expenditure g_t, is equal to the tax revenues from lump-sum taxation, and the taxes on holdings of the foreign bond, physical capital, and consumption. For exogenously given levels of g_t, and the tax rates τ_b, τ_c and τ_k, the lump-sum tax T_t is residually determined by equation (22).

Equilibrium dynamics and the steady state

The dynamic system for $(z_t, \xi_t, \mu_t, k_t, q_t)$ is derived from equations (1), (5), (7)-(12), and (19)-(20). To this goal, equation (5) is linearized around the steady state to obtain

$$Q_t - \overline{Q} = -\frac{U_{12}^*}{U_{11}^*}(z_t - \overline{z}) - \frac{\rho}{U_{11}^*}(\xi_t - \overline{\xi}) - \frac{1}{U_{11}^*}(\mu_t - \overline{\mu})\,, \tag{23}$$

where $Q_t = c_t + s_t$, and stars denote steady state values. Linearizing equation (1) around the steady state and using equation (23) one obtains

$$\dot{z}_t = -\rho\left(\frac{U_{12}^*}{U_{11}^*} + 1\right)(z_t - \overline{z}) - \frac{\rho^2}{U_{11}^*}(\xi_t - \overline{\xi}) - \frac{\rho}{U_{11}^*}(\mu_t - \overline{\mu})\,. \tag{24}$$

Similarly, the linearization of equation (7) yields

$$\dot{\xi}_t = \frac{U_{12}^{*2} - U_{11}^* U_{22}^*}{U_{11}^*}(z_t - \overline{z}) + \left[\rho\left(\frac{U_{21}^*}{U_{11}^*} + 1\right) + \theta\right](\xi_t - \overline{\xi}) + \frac{U_{21}^*}{U_{11}^*}(\mu_t - \overline{\mu})\,, \tag{25}$$

where equation (23) has been used to substitute for $(Q_t - \overline{Q})$. Combining equations (8) and (5) one gets

$$\dot{\mu}_t = -\lambda_t(1 + \tau_c) + (1 + \delta + \theta)\mu_t\,, \tag{26}$$

which, after linearization becomes

$$\dot{\mu}_t = (1+\delta+\theta)(\mu_t - \bar{\mu}). \tag{26'}$$

The dynamic system is completed with equations (27) and (28), which are obtained after linearizing equations (19) and (20), respectively.

$$\dot{k}_t = \left[G''^{-1}(\bar{q}-1) \right](q_t - \bar{q}), \tag{27}$$

$$\dot{q}_t = -AF_{kk}(\bar{k},1)(k_t - \bar{k}) + (r-\tau_b)(q_t - \bar{q}). \tag{28}$$

The system describing the dynamics of the economy consists of equations (24), (25), (26′), (27) and (28). Therefore it is given by

$$\begin{pmatrix} \dot{z}_t \\ \dot{\xi}_t \\ \dot{\mu}_t \\ \dot{k}_t \\ \dot{q}_t \end{pmatrix} = \begin{pmatrix} -\rho\left(\dfrac{U_{12}^*}{U_{11}^*}+1\right) & -\dfrac{\rho^2}{U_{11}^*} & -\dfrac{\rho}{U_{11}^*} & 0 & 0 \\ \dfrac{U_{12}^{*2} - U_{11}^* U_{22}^*}{U_{11}^*} & \rho\left(\dfrac{U_{12}^*}{U_{11}^*}+1\right)+\theta & \dfrac{U_{12}^*}{U_{11}^*} & 0 & 0 \\ 0 & 0 & 1+\delta+\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & G''^{-1}(\bar{q}-1) \\ 0 & 0 & 0 & -AF_{kk}(\bar{k},1) & r-\tau_b \end{pmatrix} \cdot \begin{pmatrix} z_t - \bar{z} \\ \xi_t - \bar{\xi} \\ \mu_t - \bar{\mu} \\ k_t - \bar{k} \\ q_t - \bar{q} \end{pmatrix},$$

where the coefficient matrix is evaluated at the steady state point. The linear dynamic system has a block structure, which implies that the characteristic roots may be obtained by solving separately the two sub-systems of dimensions (3×3) and (2×2) respectively.

The smaller root for the first sub-matrice is given by the expression

$$\omega = \frac{\theta - \sqrt{(\theta+2\rho)^2 + \dfrac{4\rho(\theta+2\rho)}{U_{11}^*}\left(U_{12}^* + \dfrac{\rho}{\theta+2\rho}U_{22}^*\right)}}{2}, \tag{29}$$

which is strictly negative by assumption.[100] This system has two positive and one negative eigenvalues, therefore it exhibits saddle point stability. The saddle-path is given by the following equations

$$z_t - \bar{z} = (z_0 - \bar{z})e^{\omega t}, \tag{30}$$

[100] The reader may see Obstfeld (1992) and Mansoorian (1993, 1996).

$$\xi_t - \overline{\xi} = (z_0 - \overline{z})\left(\frac{\omega - a_{11}}{a_{12}}\right)e^{\omega t}, \tag{31}$$

$$\mu_t - \overline{\mu} = 0, \tag{32}$$

where $a_{11} = -\rho\left(\frac{U_{12}^*}{U_{11}^*} + 1\right)$ and $a_{12} = -\frac{\rho^2}{U_{11}^*}$. The stable root ω specifies a saddle trajectory for the optimal consumption dynamics. Differentiating equation (29) one may derive

$$\dot{z}_t = \omega(z_t - \overline{z}). \tag{33}$$

Substituting equation (33) into equation (1) a saddle trajectory is obtained for $(Q_t - \overline{Q})$ and $(z_t - \overline{z})$

$$(Q_t - \overline{Q}) = \left(\frac{\omega + \rho}{\rho}\right)(z_t - \overline{z}). \tag{34}$$

From equation (29) it is implied that $\omega + \rho$ is negative in the case of distant complementarity, and thus trajectory (34) is negatively sloping, whereas under adjacent complementarity $\omega + \rho$ is positive implying a trajectory with a positive slope.

In order to derive the solution for s_t equation (2) is linearized around the steady state. Using the equations (23) and (30)–(32) it is straightforward to obtain

$$s_t - \overline{s} = -\frac{\Gamma}{1+\delta+\omega}(z_0 - \overline{z})e^{\omega t} + \left[(s_0 - \overline{s}) + \frac{\Gamma}{1+\delta+\omega}(z_0 - \overline{z})\right]e^{-(1+\delta)t} \tag{35}$$

The adjustment of c_t along the optimal path derived in a similar way from equations (23), (30)-(32), and (35), is expressed by

$$c_t - \overline{c} = -\frac{\Gamma(\delta+\omega)}{1+\delta+\omega}(z_0 - \overline{z})e^{\omega t} - \left[(s_0 - \overline{s}) + \frac{\Gamma}{1+\delta+\omega}(z_0 - \overline{z})\right]e^{-(1+\delta)t}, \tag{36}$$

where $\Gamma = -(\omega + \rho)/\rho$. For the (k, q) sub-matrice, the stable root is given by

$$\chi = \frac{(r-\tau_b) - \sqrt{(r-\tau_b)^2 - 4AF_{kk}^* / G''}}{2} \ (<0)\,, \tag{37}$$

where $G'' = G''(\overline{q}-1)$. The saddle path is given by the following equations

$$k_t - \overline{k} = (k_0 - \overline{k})e^{\chi t}\,, \tag{38}$$

$$q_t - \overline{q} = (k_0 - \overline{k})\chi G'' e^{\chi t}\,. \tag{39}$$

Combining equations (38) and (39) one obtains the following saddle trajectory

$$k_t - \overline{k} = \frac{1}{\chi G''}(q_t - \overline{q})\,. \tag{40}$$

In order to derive the current account identity of the economy the household's flow budget constraint, as given by equation (4), is combined with the definition of the profit function, as expressed by equation (16). It is noted that the latter is evaluated at the equilibrium level of labor. The following equation is obtained

$$\dot{b}_t = r(1-\tau_b)b_t + AF(k_t,1) - \tau_k k_t - I_t - G(I_t) - (1+\tau_c)c_t - T_t\,. \tag{41}$$

Substituting equation (19) into equation (41), yields

$$\dot{b}_t = r(1-\tau_b)b_t + AF(k_t,1) - \tau_k k_t - I(q_t) - G(I_t) - (1+\tau_c)c_t - T_t\,. \tag{42}$$

By linearizing equation (42) around the steady state, and subsequently using equations (36), (38) and (39) one obtains the following differential equation

$$\begin{aligned} \dot{b}_t = {} & r(1-\tau_b)(b_t - \overline{b}) + (AF_k^* - \tau_k - \chi)(k_0 - \overline{k})e^{\chi t} + \frac{(1+\tau_c)\Gamma(\delta+\omega)}{1+\delta+\omega}(z_0 - \overline{z})e^{\omega t} + \\ & + (1+\tau_c)\left[(s_0 - \overline{s}) + \frac{\Gamma}{1+\delta+\omega}(z_0 - \overline{z})\right]e^{-(1+\delta)t} \end{aligned}\,. \tag{43}$$

The solution of the current account is

$$b_t - \overline{b} = \frac{AF_k^* - \tau_k - \chi}{\chi - r(1-\tau_b)}(k_0 - \overline{k})e^{\chi t} + \frac{(1+\tau_c)\Gamma(\delta+\omega)}{(1+\delta+\omega)(\omega - r(1-\tau_b))}(z_0 - \overline{z})e^{\omega t} -$$
$$- \frac{(1+\tau_c)}{(1+\delta+r(1-\tau_b))}\left[(s_0 - \overline{s}) + \frac{\Gamma}{1+\delta+\omega}(z_0 - \overline{z})\right]e^{-(1+\delta)t} + \qquad , \tag{44}$$
$$+\left\{(b_0 - \overline{b}) - \Psi_1(k_0 - \overline{k}) - \Psi_2(z_0 - \overline{z}) - \Psi_3(s_0 - \overline{s})\right\}e^{r(1-\tau_b)t}$$

where

$$\Psi_1 = \frac{AF_k^* - \tau_k - \chi}{\chi - r(1-\tau_b)},\ \Psi_2 = \frac{\Gamma(1+\tau_c)(\delta + r(1-\tau_b))}{(\omega - r(1-\tau_b))(1+\delta+r(1-\tau_b))} \text{ and } \Psi_3 = -\frac{1+\tau_c}{1+\delta+r(1-\tau_b)}.$$

For equation (42) to converge, the coefficient of $e^{r(1-\tau_b)t}$ has to be zero. This implies the following condition

$$b_0 - \overline{b} = \Psi_1(k_0 - \overline{k}) + \Psi_2(z_0 - \overline{z}) + \Psi_3(s_0 - \overline{s}). \tag{45}$$

The steady-state equilibrium $(\overline{b}, \overline{c}, \overline{z}, \overline{s}, \overline{\lambda}, \overline{\xi}, \overline{\mu}, \overline{k}, \overline{q})$ is determined by the following system

$$\overline{c} + \overline{s} = \overline{z}, \tag{46}$$

$$\overline{c} = \delta\overline{s}, \tag{47}$$

$$U_1(\overline{c} + \overline{s}, \overline{z}) = \overline{\lambda}(1+\tau_c) - \rho\overline{\xi} - \overline{\mu}, \tag{48}$$

$$(\rho + \theta)\overline{\xi} = U_2(\overline{c} + \overline{s}, \overline{z}), \tag{49}$$

$$(1+\delta+\theta)\overline{\mu} = \overline{\lambda}(1+\tau_c), \tag{50}$$

$$\overline{q} = 1, \tag{51}$$

$$AF_k(\overline{k}, 1) - \tau_k = r - \tau_b, \tag{52}$$

$$r(1-\tau_b)\overline{b} + AF(\overline{k}, 1) - \tau_k\overline{k} = (1+\tau_c)\overline{c} + T, \tag{53}$$

$$b_0 - \overline{b} = \Psi_1(k_0 - \overline{k}) + \Psi_2(z_0 - \overline{z}) + \Psi_3(s_0 - \overline{s}). \tag{54}$$

It has to be noted that in deriving the above system no assumption was made with respect to the government maintaining a balanced budget. The presumption is that the government runs a deficit (surplus) This system is used to examine the steady state effects of changes in the consumption tax τ_c, the capital tax τ_k, the tax on foreign bonds τ_b, the lump-sum tax T and productivity parameter A. Unanticipated and permanent changes are considered, thought to take place at an arbitrarily chosen point in time.

Examining the effect of changes in government expenditure requires the use of a differentiated steady state system, which incorporates the assumption of a continuously balanced government budget. In this system, the current account identity is derived by integrating the household's flow budget constraint as given by equation (4) with the profit function, as given by equation (16) evaluated at the equilibrium level of labor, and the government's budget constraint given by equation (22). The current account balance in this case is given by the equation

$$\dot{b}_t = rb_t + AF(k_t,1) - c_t - g_t - I_t - G(I_t). \tag{55}$$

Using the firm's optimality condition (19) to substitute I_t for $I(q_t)$ and linearizing equation (55) around the steady state the current account identity is written as

$$\dot{b}_t = r(b_t - \bar{b}) + AF_k^*(k_t - \bar{k}) - (c_t - \bar{c}) - I'(\bar{q})(q_t - \bar{q}), \tag{56}$$

where $F_k^* = F_k(\bar{k},1)$. Using equations (36), (38) and (39), equation (56) is written

$$\dot{b}_t - rb_t = \Lambda_0 + \Lambda_1 e^{\chi t} + \Lambda_2 e^{\omega t} + \Lambda_3 e^{-(1+\delta)t}, \tag{57}$$

where $\Lambda_0 = -r\bar{b}$, $\Lambda_1 = [AF_k^* - \chi](k_0 - \bar{k})$, $\Lambda_2 = \dfrac{\Gamma(\delta+\omega)}{1+\delta+\omega}(z_0 - \bar{z})$, and

$$\Lambda_3 = (s_0 - \bar{s}) + \frac{\Gamma}{1+\delta+\omega}(z_0 - \bar{z}).$$

Solving the differential equation (57) the adjustment of the current account along the optimal path is derived. This is expressed by the following equation

$$b_t - \overline{b} = \frac{AF_k^* - \chi}{\chi - r}(k_0 - \overline{k})e^{\chi t} + \frac{\Gamma}{1+\delta+\omega}\cdot\frac{\delta+\omega}{\omega - r}(z_0 - \overline{z})e^{\omega t}$$
$$-\frac{1}{1+\delta+r}\cdot\left[(s_0 - \overline{s}) + \frac{\Gamma}{1+\delta+\omega}(z_0 - \overline{z})\right]e^{-(1+\delta)t} + .$$
$$\left\{(b_0 - \overline{b}) - \frac{AF_k^* - \chi}{\chi - r}(k_0 - \overline{k}) - \frac{\Gamma(\delta+\omega)}{(1+\delta+\omega)(\omega - r)}(z_0 - \overline{z}) + \frac{1}{1+\delta+r}\cdot\left[(s_0 - \overline{s}) + \frac{\Gamma}{1+\delta+\omega}(z_0 - \overline{z})\right]\right\}e^{rt} \tag{58}$$

In order for equation (56) to converge, the coefficient of e^{rt} has to be zero. This implies that the following condition must be met

$$b_0 - \overline{b} = \Omega_1(k_0 - \overline{k}) + \Omega_2(z_0 - \overline{z}) + \Omega_3(s_0 - \overline{s}), \tag{59}$$

where

$$\Omega_1 = \frac{AF_k^* - \chi}{\chi - r}(<0),\ \Omega_2 = \frac{\Gamma(\delta + r)}{(\omega - r)(1+\delta+r)},\ \text{and}\ \Omega_3 = -\frac{1}{1+\delta+r}(<0).$$

We note that $\Omega_2 = \dfrac{-\dfrac{\omega+\rho}{\rho}\cdot(\delta+r)}{(\omega - r)(1+\delta+r)}$. Since $sign(\omega - r) < 0$, then

$$sign\{\Omega_2\} = sign\{\omega + \rho\} \Rightarrow \begin{cases} >0 & \text{if adjacent complementarity} \\ <0 & \text{if distant complementarity} \end{cases}.$$

The steady-state equilibrium $(\overline{b}, \overline{c}, \overline{z}, \overline{s}, \overline{\lambda}, \overline{\xi}, \overline{\mu}, \overline{k}, \overline{q})$ is determined by the following system:

$$\overline{c} + \overline{s} = \overline{z}, \tag{60}$$

$$\overline{c} = \delta\overline{s}, \tag{61}$$

$$U_1(\overline{c} + \overline{s}, \overline{z}) = \overline{\lambda}(1+\tau_c) - \rho\overline{\xi} - \overline{\mu}, \tag{62}$$

$$(\rho + \theta)\overline{\xi} = U_2(\overline{c} + \overline{s}, \overline{z}), \tag{63}$$

$$(1+\delta+\theta)\overline{\mu} = \overline{\lambda}(1+\tau_c), \tag{64}$$

$$\overline{q} = 1, \tag{65}$$

$$AF_k(\bar{k},1) - \tau_k = r - \tau_b, \tag{66}$$

$$r\bar{b} + AF(\bar{k},1) = \bar{c} + g, \tag{67}$$

$$b_0 - \bar{b} = \Omega_1(k_0 - \bar{k}) + \Omega_2(z_0 - \bar{z}) + \Omega_3(s_0 - \bar{s}), \tag{68}$$

where equation (60) represents $\dot{z} = 0$ (see equation 1), equation (61) represents $\dot{s} = 0$ (see equation 2), equation (62) is the optimality condition for c evaluated at the steady state (see equation 5), equation (63) represents $\dot{\xi} = 0$ (see equation 7), equation (64) represents $\dot{\mu} = 0$ (see equation 26), equation (65) represents $\dot{k} = 0$ (see equation 19), equation (66) represents $\dot{q} = 0$ (see equation 20), and equation (67) represents $\dot{b} = 0$ (equation 55). Equation (68), which is a restatement of equation (59), characterizes the steady-state equilibrium since initial values b_0, k_0, z_0 and s_0 are exogenously given.

Concluding remarks

The purpose of this essay has been to investigate on theoretical grounds the long run effect of fiscal policies in an open economy framework, in which private consumption exhibits a degree of durability, and households optimally choose to maintain their habitual standard of living. Particular attention has been aimed at the dynamics of the current account, empirically observed to exhibit a non-monotonic trend. Due to the model's mathematical complexity, the solutions to the steady state system are analytically intractable. One would like to obtain a quantitative measure of comparative static results, such as the direction and the rate of response of economic aggregates to changes in policy variables. This information is not possible to obtain in the context of graphical and analytical methods. Numerical integration would enable to approximate the solutions, and visualize the dynamics of the intractable steady state system. Due to particular difficulties bound up with the development of numerical integration analysis, the latter is left to remain an open subject for future research.

The first essay constitutes a theory which in a novel way lends truth to the proposition formed by Kuznets (1955), with respect to the non-monotonic relationship between prosperity and inequality of income distribution. The attention is centered on the role of financial markets in defining the process of economic development, and ultimately the distribution of income earning capabilities in a population of *ex ante* heterogeneous individuals. In a model in which the roots of development lie in human capital accumulation, the possibility to fund educational choices through private credit organizations may prove critical. The provision of credit in this market is hindered by one-sided lack of commitment, and particular enforcement issues embedded in the area of educational investment. A consequence of the failure of the credit market is that individuals are initially barred from productive educational choices. In the tradition of Kehoe and Levine (1993) we assume that legislation accommodates the complete and permanent exclusion of defaulting borrowers from financial markets. The prospect of being prohibited to invest in tangible assets induces agents to choose commitment to previous agreements. Contract arrangements thus become enforceable, leading credit institutions to eagerly engage in educational funding. This is the critical requirement for the economy to be carried on a dynamic growth path, escaping the trap in an ever-sustained poverty state. We trace out paths of development so constructed as to give an explicit proof of the *trickle-down* theory of economic growth. Initially, an equilibrium is taken to exist in which a particular group of individuals, those with the highest investment return, only choose to engage in education. Owing to the accumulation of human capital and the associated externality on future generations' knowledge productivity, the economy ultimately makes its transition to a state where the aggregate of all agents invest in individual improvement. As endogenous technological knowledge takes off, the externality effect arising from knowledge spillovers gives rise to inverted-*U* dynamics in the evolution of income distribution.

The second essay is an enquiry on an old theme in the theory of macroeconomics, namely the role of monetary policy in determining an economy's long run growth dynamics. The analysis is carried with reference to an economy being open, yet a price taker in the international capital markets. We postulate that the possibility of intermediated credit does not exist, the intention of the assumption being to uncover the role of inflation as tax on private spending. Initially, the postulate applies on purchases of consumption goods only. In an alternative version of the model investment on capital goods is also being subjected to the constraint that cash holdings are the only means of conducting the transaction. In this latter case, inflation bears an evident analogy to a capital tax. To elucidate the consequences of endogenously determined labor, the theory is initially built on models that abstract from the decision to allocate time between leisure and other productive activities. The analysis is extended to account for the endogeneity of the time-allocation decision in the latter part of the essay. With reference to the issue of *money superneutrality* the analysis has brought up the following propositions: Inflation when operating as consumption tax has no impact on the growth rate of output. This is established irrespective of the labor supply be held fixed, or incorporated as endogenous decision. When imitating the role of capital taxation, inflationary policy has a negative effect on capital accumulation in the model of fixed labor supply. However, in an environment that accommodates the endogeneity of labor–leisure choice, the validity of the superneutrality result is once again reestablished.

The third, and last, part of this dissertation constitutes a theoretical essay on the long run effects of tax and expenditure policies. The analysis is carried with reference to an open economy, yet a price taker in the international markets. Our interest leans heavily in the transitional dynamics of the current account in response to permanent fiscal shocks. The empirical literature in the international macroeconomics has established that the current account evolves non-monotonically along its adjustment path to the long run equilibrium. It has been the aim of this study to show that this empirical phenomenon may be proved within the theory, thus be validated on the ground of acceptance of a mathematical proposition. To this endeavor we sought two sources of *time non-separability* in the preference structure, habit forming consumption in consumer durable goods. When households choose to maintain their habitual standard of living and consumption exhibits a degree of durability, optimal private choices induce non-monotonic dynamics on consumption, hence saving, behavior that are exactly consistent with the factual evidence on the current account.

References

- Abel, A. (1985) “Dynamic Behavior of Capital Accumulation in a Cash-In-Advance Model” *Journal of Monetary Economics* 16, pp.55-71.
- Adelman, I. and C. T. Morris (1973) *Economic Growth and Social Equity in Developing Countries*, Stanford University Press, Stanford, CA.
- Adelman, I. and Robinson, S. (1989) *Income Distribution and Development* in Handbook of Development Economics, vol. II (*ed*. H. Chenery and T. N. Srinivasan), pp.949-1003. Amsterdam: North-Holland.
- Aghion, P. and P. Bolton (1997) “A Theory of Trickle-Down Growth and Development” *Review of Economic Studies* 64 pp.151-172.
- Aghion, P. and P. Howitt (1997) *Endogenous Economic Growth* Cambridge MA: MIT Press.
- Ahluwalia, M. S. (1976*a*) “Income Distribution and Development: Some Stylized Facts” *American Economic Review* 66 pp.128-135.
- Ahluwalia, M. S. (1976*b*) “Inequality, Poverty and Development” *Journal of Development Economics* 3 pp.307-342.
- Ahluwalia, M. S., N. G. Carter and H. B. Chenery (1979) “Growth and Poverty in Developing Countries” *Journal of Development Economics* 6 pp.299-341.
- Alvarez, F. and U. J. Jermann (2000) “Efficiency, Equilibrium, and Asset Pricing with Risk of Default” *Econometrica* 68 pp.775-797.
- Anand, S. and S. M. R. Kanbur (1993) “The Kuznets Process and the Inequality-Development Relationship” *Journal of Development Economics* 40 pp.25-52.
- Andolfatto, D. and M. Gervais (2006) “Human Capital Investment and Debt Constraints” *Review of Economic Dynamics* 9 pp.52-67.
- Arrow, K. J. (1962) “The Economic Implications of Learning by Doing” *Review of Economic Studies* 29, pp.155-73.

- Azariadis, C. and A. Drazen (1990) “Threshold Externalities in Economic Development” *Quarterly Journal of Economics* 105, pp.501-525.
- Azariadis, C. and L. Lambertini (2003) “Endogenous Debt Constraints in Lifecycle Economies” *Review of Economic Studies* 70 pp.461-487.
- Bacha, E. L. (1977) *The Kuznets Curve and Beyond: Growth and Changes in inequalities* Development Discussion Papers no.29, Harvard Institute for International Development, Harvard University, Cambridge, MA.
- Bacha, E. L. (1979) “Notes on the Brazilian Experience with Mini-devaluations, 1968-1976” *Journal of Development Economics* 6 pp.463-481.
- Banerjee, A. and A. Newman (1993) “Occupational Choice and the Process of Development” *Journal of Political Economy* 101, pp.274-299.
- Barro, R. J. (1990) “Government Spending in a Simple Model of Endogenous Growth” *Journal of Political Economy* vol.98, pp.S103-25.
- Barro, R. J. (2000) “Inequality and Growth in a Panel of Countries” *Journal of Economic Growth* 5, pp.5-32.
- Barro, R. J., N. G. Mankiw and X. Sala-i-Martin (1995) “Capital Mobility in Neoclassical Models of Growth” *American Economic Review* vol.85 pp.103-115.
- Baumol, W. J. (1952) “The Transactions Demand for Cash: An Inventory Theoretic Approach” *Quarterly Journal of Economics* 66, pp.545-556.
- Becker, G. (1964) *Human Capital*, New York: NBER and Columbia University Press.
- Becker, G. S. and N. Tomes (1986) “Human Capital and the Rise and Fall of Families” *Journal of Labor Economics* 4 pp.S1-S39.
- Benabou, R. (1996) “Heterogeneity, Stratification, and Growth: Macroeconomic Implications of Community Structure and School Finance” *American Economic Review* 86 pp. 584-609.

- Benabou, R. (2002) “Tax and Education Policy in a Heterogeneous-Agent Economy: What Levels of Redistribution Maximize Growth and Efficiency” *Econometrica* 70 pp. 481-517.
- Bencivenga, V. R., and B. D. Smith (1991) “Financial Intermediation and Endogenous Growth” *Review of Economic Studies* 58 pp.195-209.
- Behrman, J. R., R. A. Pollak, and P. Taubman (1989) “Family Resources, Family Size, and Access to Financing for College Education” *Journal of Political Economy* 97 pp.398-419.
- Benhabib, J. and R. Perli (1994) “Uniqueness and Indeterminacy: On the Dynamics of Endogenous Growth” *Journal of Economic Theory* 63 pp.113-142.
- Blackburn, K. and V. Hung (1991) “Equilibrium Growth in a Monetary Economy”, Discussion Paper No. 9114, University of Southampton.
- Bond, P. and A. Krishnamurthy (2004) “Regulating Exclusion from Financial Markets” *Review of Economic Studies* 71 pp.681-707.
- Bourguignon, F. and C. Morrison (1990) “Income Distribution, Development and Foreign Trade: A Cross Sectional Analysis” *European Economic Review* 34 pp.1113-1132.
- Bovenberg, A. L., van Ewijk, C. (1997) “Progressive Taxes, Equity and Human Capital Accumulation in an Endogenous Growth Model with Overlapping Generations” *Journal of Public Economics* 64 pp.153-179.
- Brock, W. A. (1974) “Money and Growth: The Case of Long Run Perfect Foresight” *International Economic Review* 15, pp.750-777.
- Buiter, W. and K. Kletzer (1992) “Permanent International Productivity Growth Differentials in an Integrated Global Economy”, NBER Working Paper no.4220.
- Calvo, G. A. (1979) “On Models of Money and Perfect Foresight” *International Economic Review* 20, pp.83-103.

- Carmichael, B. (1989) “Anticipated Monetary Policy in a Cash-In-Advance Economy” *The Canadian Journal of Economics* 22(1), pp.93-108.
- Chamley, C. (1993) “Externalities and Dynamics in Models of Learning or Doing” *International Economic Review* 34 pp.583-609.
- Chang and Lai (2000) “Anticipated Inflation in a Monetary Economy with Endogenous Growth” *Economica* 67, pp.399-417.
- Chatterjee, S. and B. Ravikumar (1999) “Minimum Consumption Requirements: Theoretical and Quantitative Implications for Growth and Distribution” *Macroeconomic Dynamics* 3 pp.482-505.
- Chenery, H. B., M. S. Ahluwalia, C. L. G. Bell, J. H. Duloy and R. Jolly (1974) “Redistribution with Growth: An Approach to Policy” Oxford University Press, Oxford.
- Clower, R.W. (1967) “A Reconsideration of the Microfoundations of Monetary Theory” *Western Economic Journal* 6, pp.1-9.
- Coleman, J. S. *et al*. (1966) *Equality of Educational Opportunity* Washington, D.C. US. GPO.
- Coleman, W. J. II (1990) “Money and Capital in a Cash-in-Advance Economy” Unpublished paper, Federal Research Board of Governors.
- Cooley, T.F. and G.D. Hansen (1989) “The Inflation Tax in a Real Business Cycle Model” *American Economic Review* 79, pp.733-748.
- Cooley, T.F. and G.D. Hansen (1991) “The Welfare Costs of Moderate Inflations” *Journal of Money, Credit, and Banking* 23, pp.483-503.
- Cooley, T.F. and G.D. Hansen (1997) “Unanticipated Money Growth and the Business Cycle Reconsidered” *Journal of Money, Credit and Banking* 29(4), pp.624-648.

- Danthine, J.P., J.B. Donaldson, and L. Smith (1987) "On the Superneutrality of Money in a Stochastic Dynamic Macroeconomic Model" *Journal of Monetary Economics* 20(3), pp.475-99.
- De Gregorio J. (1991) "Inflation, Taxation, and Long Run Growth" Mimeo, IMF. *Also found as* J. De Gregorio (1993).
- De Gregorio, J. (1992) "The Effects of Inflation on Economic Growth: Lessons from Latin America" *European Economic Review* 36, pp.417-25.
- De Gregorio, J. (1993) "Inflation, Taxation, and Long-Run Growth" *Journal of Monetary Economics* 31, pp.271-298.
- De Gregorio, J. (1996) "Borrowing Constraints, Human Capital Accumulation, and Growth" *Journal of Monetary Economics* 37 pp.49-71.
- De Gregorio, J. and S. Kim (2000) "Credit Markets with Differences in Abilities: Education, Distribution, and Growth" *International Economic Review* 41 pp.579-607.
- De la Croix, D. and P. Michael (2007) "Education and Growth with Endogenous Debt Constraints" *Economic Theory* 33 pp.509-530.
- Denison, E. (1961) "The Sources of Economic Growth in the United States (Committee for Economic Development, New York).
- Diamond, P. A. (1965) "National Debt in a Neoclassical Growth Model" *American Economic Review* 55 pp.1126-1150.
- Djankov, S., C. McLiesh and A. Shleifer (2007) "Private Credit in 129 Countries" *Journal of Financial Economics* 84 pp.299-329.
- Dornbusch, R. and J. Frenkel (1973) "Inflation and Growth: Alternative Approaches" *Journal of Money, Credit and Banking* 5(1), pp.141-156.
- Eaton, J. and M. Gersovitz (1981) "Debt with Potential Repudiation: Theoretical and Empirical Analysis" *Review of Economic Studies* 48 pp.289-309.

- Eaton, J. and H.S. Rosen (1980) "Taxation, Human Capital, and Uncertainty" *The American Economic Review* 70(4) pp.705-715.
- Feenstra, R.C. (1986) "Functional Equivalence between Liquidity Costs and the Utility of Money" *Journal of Monetary Economics* 17, pp.271-91.
- Fender, J. and P. Wang (2003) "Educational Policy in a Credit Constrained Economy with Skill Heterogeneity" *International Economic Review* 44 pp.939-964.
- Fischer, S. (1974) "Money and the Production Function" *Economic Inquiry* 12, pp.517-533.
- Fischer, S. (1979) "Capital Accumulation on the Transition Path in a Monetary Optimizing Model" *Econometrica* 47(6), pp.1433-1440.
- Friedman, M. (1962) *Capitalism and Freedom* University of Chicago Press, Chicago, IL.
- Galor, O. and D. Tsiddon (1997*a*) "Technological Progress, Mobility, and Economic Growth" *American Economic Review* 87 pp.363-382.
- Galor, O. and D. Tsiddon (1997*b*) "The Distribution of Human Capital and Economic Growth" *Journal of Economic Growth* 2 pp.93-124.
- Galor, O. and J. Zeira (1993) "Income Distribution and Macroeconomics" *Review of Economic Studies* 60 pp.35-52.
- Goldsmith, R.W. (1969) *Financial Structure and Development*. New Haven, Conn.: Yale Univ. Press.
- Gomme, P. (1993) "Money and growth revisited: measuring the costs of inflation in an endogenous growth model" *Journal of Monetary Economics* 32, pp.51-78.
- Green, E.J. (1987) "Lending and the Smoothing of Uninsurable Income." In Edward C. Prescott and Neil Wallace (*eds*.), *Contractual Arrangements for Intertemporal Trade*, *Minnesota Studies in Macroeconomics Series*, *Vol. 1*. Minneapolis: University of Minnesota Press, pp.3-25.

- Greenwood, J. and B. Jovanovic (1990) "Financial Development, Growth, and the Distribution of Income" *Journal of Political Economy* 98 pp.1076-1107.
- Griliches, Z. and W.M. Mason (1972) "Education, Income and Ability" *Journal of Political Economy* 80(3) pp.S74-S103.
- Grossman, G.M. and E. Helpman (1994) "Endogenous Innovation in the Theory of Growth" *Journal of Economic Perspectives* 8(1), pp.23-44.
- Helpman, E. (1997) *General Purpose Technologies and Economic Growth* Cambridge, MA: MIT Press.
- Hendricks, L. (1999) "Taxation and Long-run Growth" *Journal of Monetary Economics* 43 pp.411-434.
- Jaffee, D. and T. Russell (1976) "Imperfect Information, Uncertainty, and Credit Rationing" *Quarterly Journal of Economics* 90 pp.651-666.
- Jaffee, D. and J. Stiglitz (1990) "Credit Rationing" in B. Friedman and F. Hahn, *Handbook of Monetary Economics* 2 (Amsterdam: North-Holland).
- Jappelli, T. and M. Pagano (2002) "Information Sharing, Lending, and Defaults: Cross-country Evidence" *Journal of Banking and Finance* 26 pp.2017-2045.
- Jones, L.E. and R.E. Manuelli (1990) "A Convex Model of Equilibrium Growth: Theory and Policy Implications" *Journal of Political Economy* 98, pp.1008-1038.
- Jones, L.E. and R.E. Manuelli (1993) "Growth and Effects of Inflation" NBER Working Paper #4523 (November). *Also found as* L.E. Jones, and R.E. Manuelli (1993).
- Jones, L.E. and R.E. Manuelli (1993) "Growth and the effects of inflation". Mimeo, Northwestern and Stanford Universities. *Also found as* L.E. Jones, and R.E. Manuelli (1995).
- Jones, L.E. and R.E. Manuelli (1995) "Growth Effects of Inflation" *Journal of Economic Dynamics and Control* 19, pp.1405-1428.

- Ireland, P. (1994) "Money and Growth: An Alternative Approach". *American Economic Review* 84, pp.47-65.
- Kareken, J.H. and N. Wallace, (1980) "Models of Monetary Economies" Federal Reserve Bank, Minneapolis, MN.
- Kehoe, T. J. and D.K. Levine (1993) "Debt-Constrained Asset Markets" *Review of Economic Studies* 60 pp.865-888.
- Kimbrough, Kent P. (1986) "Inflation, Employment, and Welfare in the Presence of Transactions Costs" *Journal of Money, Credit and Banking* 18(2), pp.127-140.
- King, R.G. and R. Levine (1993) "Finance Entrepreneurship and Growth" *Journal of Monetary Economics* 32 pp.513-542.
- Kuznets, S. (1955) "Growth and Income Inequality" *American Economic Review* 45 pp.1-28.
- La Porta, R., F. Lopez-de-Silanes, A. Shleifer and R.W. Vishny (1998) "Law and Finance" *Journal of Political Economy* 106 pp.1113-1155.
- Ladron-De-Guevara A., S. Ortigueira and M. S. Santos (1999) "A Two-Sector Model of Endogenous Growth with Leisure" *Review of Economic Studies* 66, pp.609-631.
- Laing, D., T. Palivos and P. Wang (2003) "The Economics of 'New Blood" *Journal of Economic Theory* 112 pp.106-156.
- Levhari, D. and D. Patinkin (1968) "The Role of Money in a Simple Growth Model" *American Economic Review* 58, pp.713-753.
- Levhari, D. and Y. Weiss (1974) "The Effect of Risk on the Investment in Human Capital" *The American Economic Review* 64(6) pp. 950-963.
- Levine, R. (1992) "Financial Intermediary Services and Growth" *Journal of Japanese and International Economics* 6 pp.383-405.

- Li, H., L. Squire, and H. Zou (1998) “Explaining International and Intertemporal Variations in Income Inequality” *Economic Journal* 108 pp.26-43.
- Loury, G.C. (1981) “Intergenerational Transfers and the Distribution of Earnings” *Econometrica* 49 pp.843-867.
- Ljungqvist, L. (1993) “Economic Underdevelopment: The Case of a Missing Market for Human Capital” *Journal of Development Economics* 40 pp.219-239.
- Ljungqvist, L. and T J. Sargent *Recursive Macroeconomic Theory* edited by M.I.T. Press, Cambridge MA (2004)
- Love, D. and Jean-Francois Wen (1999) “Inflation, Welfare, and the Time-Costs of Transacting” *The Canadian Journal of Economics* 32(1), pp.171-94.
- Lucas, R.E. (1988) "On the Mechanics of Economic Development" *Journal of Monetary Economics* 22 pp.3-42.
- Lucas, R.E. (1990) “Supply-side Economics: An Analytical Review” *Oxford Economic Papers* 42 pp.293-316.
- Lucas, R.E. (2002)_“Lectures on Economic Growth” Harvard: Harvard University Press.
- Marini G. and F. van der Ploeg (1988) “Monetary and Fiscal Policy in an Optimizing Model with Capital Accumulation and Finite Lives” *The Economic Journal* 98(392), pp. 772-786.
- Marquis, M. and K. Reffett (1991*a*) “Real Interest Rates and Endogenous Growth in a Monetary Economy” *Economics Letters* 37, pp.105-109.
- Marquis, M. and K. Reffett (1991*b*) “Endogenous Growth with Liquidity-Constrained Human Capital Acquisition” Florida State University Working Paper no. 91-04-7. *Also found as*: M. Marquis, and K. Reffett (1992*b*).
- Marquis, M. and K. Reffett (1992*a*). “Anticipated Inflation and Capital in a Multisector Monetary Economy” *International Review of Economics and Finance*

- Marquis, M. and K Reffett (1992*b*) “Endogenous Growth with Liquidity-Constrained Human Capital Acquisition” Unpublished Manuscript, Florida State University, Tallahassee, FL.
- Marquis M. and K. Reffett (1994) “New Technology Spillovers into the Payment System” *Economic Journal* 104 (September), pp.1123-1138.
- Marquis M. and K. Reffett (1995) “The Inflation Tax in a Convex Model of Equilibrium Growth” *Economica* 62, pp.109-121.
- Manuelli, R. (1986) “Topics in Intertemporal Economics”, University of Minnesota Ph.D. Dissertation.
- McCallum, B.T. (1996) “Neoclassical versus Endogenous Growth Analysis: An Overview” Federal Reserve Bank of Richmond *Economic Quarterly* Volume 82/4.
- McKinnon, R.I. (1973) *Money and Capital in Economic Development*. Washington: Brookings Institute.
- Mino, K. (1996) “Analysis of a Two-Sector Model of Endogenous Growth with Capital Income Taxation” *International Economic Review* 37 pp.227-251.
- Mino, K. and A. Shibata (1995) “Monetary Policy, Overlapping Generations, and Patterns of Growth” *Economica* 62(246) (May), pp.179-194.
- Obstfeld, M. (1984) “Multiple Stable Equilibria in an Optimizing Perfect-Foresight Models” *Econometrica* 52, pp.223-228.
- Pagano, M. and T. Jappelli (1993) “Information Sharing in Credit Markets” *Journal of Finance* 43 pp.1693-1718.
- Palivos, T. and C.K. Yip (2007) “Illegal Immigration in a Heterogeneous Society” Discussion Paper Series WP 2007-02, Department of Economics, University of Macedonia Greece.
- Palokangas, T. (1997) “Inflation and Growth in an Open Economy” *Economica* 64(255), pp.509-518.

- Papanek, G.F. and O. Kyn (1986) “The Effect on Income Distribution of Development, the Growth Rate and Economic Strategy” *Journal of Development Economics* 23 pp.55-65.
- Paukert, F. (1973) “Income Distribution at different levels of development: A survey of evidence” *International Labor Review* 108.
- Pecorino, P. (1995) “Inflation, Human Capital Accumulation and Long-Run Growth”, *Journal of Macroeconomics* 17(3), pp.533-542.
- Ploeg, F. van der. (1991) “Money and Capital in Interdependent Economies with Overlapping Generations” *Economica* 58, pp.233-256.
- Prescott, E. (1987) “A Multiple Means-of-Payment Model”, in William A. Barnett and Kenneth J. Singleton, eds., *New Approaches to Monetary Economics*. Cambridge: Cambridge University Press, pp.42-51.
- Ray, D. (1998) *Development Economics*. Princeton University Press.
- Rebelo (1991) “Long-Run Policy Analysis and Long-Run Growth” *Journal of Political Economy* 99 pp.500-521.
- Romer, P.M. (1986) “Increasing Returns and Long-Run Growth” *Journal of Political Economy* 94, pp.1002-1037.
- Romer, P.M. (1990) “Endogenous Technological Change” *Journal of Political Economy* 98(5), pp.S71-S102.
- Saint-Paul, G. (1992) “Technological Choice, Financial Markets and Economic Development” *European Economic Review* 36 pp.763-781.
- Samuelson, P.A. (1968) “The Two-Part Golden Rule Deduced as the Asymptotic Turnpike of Catenary Motions” *Western Economic Journal* 6, pp.85-89.
- Sapienza, P. (2002) “The Effects of Banking Mergers on Loan Contracts” *Journal of Finance* 57 pp.329-368.

- Saving, T. R. (1971) "Transactions Costs and the Demand for Money" *American Economic Review* 61(3) p.407-420.
- Schechtman, J., and V. Escudero (1977) "Some Results on an 'Income Fluctuation Problem' " *Journal of Economic Theory* 16 pp.151-166.
- Shaw, E.S. (1973) *Financial Deepening in Economic Development*. New York: Oxford University Press.
- Shell, K. (1966) "Toward A Theory of Inventive Activity and Capital Accumulation" *A.E.R. Papers and Proceedings* 56, pp.62-68.
- Shell, K. (1967) "A Model of Inventive Activity and Capital Accumulation" in *Essays on the Theory of Optimal Economic Growth*, edited by Karl Shell. Cambridge, Mass.: MIT Press.
- Sidrauski, M. (1967) "Rational Choice and Patterns of Growth in a Monetary Economy" *American Economic Review* 57, pp.534-544.
- Snow, A. and R.S. Warren, Jr. (1990) "Human Capital Investment and Labor Supply under Uncertainty" *International Economic Review* 31 pp.195-206.
- Solow, R.M. (1956) "A Contribution to the Theory of Economic Growth" *The Quarterly Journal of Economics* 70(1), pp.65-94.
- Stiglitz, J.E. and A. Weiss (1981) "Credit Rationing in Markets with Imperfect Information" *American Economic Review* 71 pp.393-410.
- Stiglitz, J.E. and A. Weiss (1987) "Macroeconomic Equilibrium and Credit Rationing" Working Paper No.2164, National Bureau of Economic Research.
- Stockman, A.C. (1981) "Anticipated Inflation and the Capital Stock in a Cash-In-Advance Economy" *Journal of Monetary Economics* 8, pp.387-393.
- Stokey, N.L. (1991) "Human Capital, Product Quality, and Growth" *The Quarterly Journal of Economics* 106 pp. 587-616.

- Tamura, R. (1991) "Income Convergence in an Endogenous Growth Model" *Journal of Political Economy* 99 pp.522-540.
- Tobin, J. (1956) "The Interest Elasticity of Transactions Demand for Cash" *Review of Economics and Statistics* 38, pp.241-247.
- Tobin, J. (1965) "Money and Economic Growth". *Econometrica* 33, pp.671-684.
- Tsiddon, D. (1992) "A Moral Hazard Trap to Growth" *International Economic Review* 33 pp.299-321.
- Turnovsky, S. (1996) "Fiscal policy, Growth, and Macroeconomic Performance in a Small Open Economy" *Journal of International Economics* 40, pp.41-66.
- Turnovsky, S. (1999) "Fiscal Policy and Growth in a Small Open Economy with Elastic Labor Supply" *Canadian Journal of Economics* 32(5), pp.1191-1214.
- Turnovsky, S. (2000) *Methods of Macroeconomic Dynamics*. Cambridge, MA: MIT Press (*Second edition*.)
- Van Der Ploeg, F. and G. Alogoskoufis (1994) "Money and Endogenous Growth" *Journal of Money, Credit, and Banking* 26, pp.771-791.
- Wang, P. and C.K. Yip (1991*a*) "Transactions Cost, Endogenous Labor, and the Superneutrality of Money" *Journal of Macroeconomics* 13, pp. 183-191.
- Wang, P. and C. K. Yip (1991*b*) "Real effects of money and welfare costs of inflation in an endogenously growing economy with transaction costs" Unpublished manuscript, Pennsylvania State University. *Also found as* P. Wang, and C.K. Yip (1993).
- Wang, P. and C.K. Yip (1992) "Examining the Long Run Effect of Money on Economic Growth" *Journal of Macroeconomics* 14, pp.359-369.
- Wang, P. and C.K. Yip (1993) "Real Effects of Money and Welfare Costs of Inflation in an Endogenously Growing Economy with Transaction Costs" Federal Reserve Bank of Dallas Research Paper No.9311 (March). *Also found as* Ping Wang, and Chong K. Yip (1994).

- Wang, Ping, and Chong K. Yip (1994) "Real effects of money and welfare costs of inflation in an endogenously growing economy with transaction costs" Manuscript, Georgia State University.
- Weil, P. (1986) "Monetary Policy and Budget Deficits with Overlapping Families of Infinitely-Lived Agents" Mimeo Harvard University.
- Weil, P. (1987) "Permanent deficits and inflation" *Journal of Monetary Economics* 20, pp.393-410.
- Weil, P. (1991) "Is money net wealth?" *International Economic Review* 32, pp.37-53.
- Zeira, J. (1991) "Credit Rationing in an Open Economy" *International Economic Review* 32 pp.959-972.

The extended model of a countable infinite set of types, $j \in [1, J]$, allows the *Kuznets Curve* to obtain a *smooth*-like form (see *Figure* I.2). Evidently, points *A* and *B* correspond to critical time periods r and τ respectively, at which the transition takes place to a new equilibrium path.

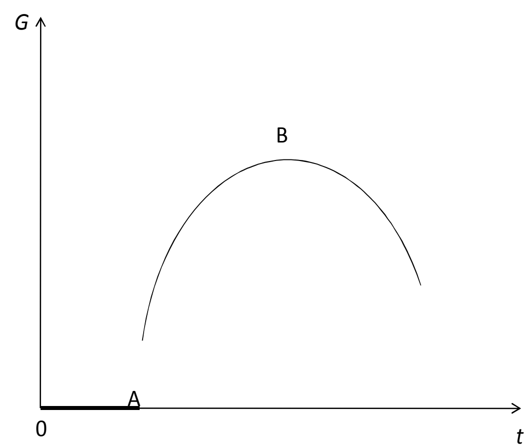

Figure 2: *Kuznets Curve for the multiple-type case*

Model II

- Recall from the text that the firm's optimization problem is to choose I, and K so as to maximize

$$\Pi = \int_{t=0}^{\infty} De^{-rt}dt \,, \tag{A1}$$

subject to the accumulation equation

$$\dot{K} = I \,, \tag{A2}$$

where

$$D = A_0 K - w - \Phi(I,K) - rB^f + \dot{B}^f - \frac{\dot{M}^f}{P}, \tag{A3}$$

and

$$\frac{\dot{M}^f}{P} = \dot{m}^f + \varepsilon m^f \,. \tag{A4}$$

We recall the equations

$$A^f = m^f - B^f \,, \tag{A5}$$

$$\dot{A}^f = \dot{m}^f - \dot{B}^f \,, \tag{A6}$$

$$m^f = \Phi(I,K) = I\left(1 + \frac{h}{2}\frac{I}{K}\right). \tag{A7}$$

Substituting equations (A4) and (A5) into (A3), we obtain

$$\begin{aligned} D &= A_0 K - w - \Phi(I,K) - r\left(m^f - A^f\right) + \dot{B}^f - \left(\dot{m}^f + \varepsilon m^f\right) \\ &= A_0 K - w - \Phi(I,K) - rm^f + rA^f - \left(\dot{m}^f - \dot{B}^f\right) - \varepsilon m^f \,. \end{aligned}$$

Using (A6) we get

$$D = A_0 K - w - \Phi(I, K) - (r+\varepsilon) m^f + r A^f - \dot{A}^f ,$$

and combining with (A7) we obtain

$$D = A_0 K - w - (1 + r + \varepsilon) I \left(1 + \frac{h}{2} \frac{I}{K}\right) - \left(\dot{A}^f - r A^f\right). \qquad \text{(A3′)}$$

- The firm's objective function is

$$\max \int_{t=0}^{\infty} \left[A_0 K - w - (1 + r + \varepsilon) I \left(1 + \frac{h}{2} \frac{I}{K}\right) - \left(\dot{A}^f - r A^f\right) \right] e^{-rt} dt$$

$$= \int_{t=0}^{\infty} \left[A_0 K - w - (1 + r + \varepsilon) I \left(1 + \frac{h}{2} \frac{I}{K}\right) \right] e^{-rt} dt - \int_{t=0}^{\infty} \left[\dot{A}^f - r A^f \right] e^{-rt} dt \qquad \text{(A8)}$$

Solving the second term yields

$$\begin{aligned} &\int_0^{\infty} \frac{\partial\left(A^f(t) e^{-rt}\right)}{\partial t} \\ &= A^f(t) e^{-rt} \Big|_0^{\infty} \\ &= A^f(\infty) \cdot 0 - A^f(0) \cdot 1 \\ &= -A^f(0) = -A_0^f . \end{aligned}$$

where we used the assumption that the $\lim_{t \to \infty} A^f(t)$ is a finite number.

The firm's optimization problem therefore becomes

$$\text{Max} \qquad \int_{t=0}^{\infty} \left[A_0 K - w - (1 + r + \varepsilon) I \left(1 + \frac{h}{2} \frac{I}{K}\right) \right] e^{-rt} dt + \left(A_0^f\right),$$

subject to $\dot{K} = I$,

which is equivalent to maximizing

$$\int_{t=0}^{\infty}\left[A_0 K - w - (1+r+\varepsilon) I\left(1+\frac{h}{2}\frac{I}{K}\right)\right]e^{-rt}dt\,,$$

subject to $\dot{K} = I$.

- Recall the text equations (II.3), (II.6) and (II.7) re-written respectively as

$$A^h = B^h + m^h\,, \tag{A9}$$

$$m^h = C\,, \tag{A10}$$

$$\dot{A}^h = w + D + rA^h + \tau - (1+r+\varepsilon)C\,. \tag{A11}$$

Substituting (A9) and (A10) into (A11) we have

$$\dot{A}^h = w + D + r\left(B^h + m^h\right) + \tau - C - (r+\varepsilon)m^h\,,$$

$$\Rightarrow \dot{A}^h = w + D + rB^h + \tau - C - \varepsilon m^h\,. \tag{A12}$$

Recall equations (II.16) and (II.21) respectively

$$D = A_0 K - w - \Phi(I,K) - rB^f + \dot{B}^f - \frac{\dot{M}^f}{P}\,, \tag{A13}$$

$$\frac{\dot{M}^f}{P} = \dot{m}^f + \varepsilon m^f\,. \tag{A14}$$

Substituting (A14) into (A13) we have

$$D = A_0 K - w - \Phi(I,K) - rB^f + \dot{B}^f - \dot{m}^f - \varepsilon m^f\,. \tag{A15}$$

Substituting (A15) into (A12) yields

$$\dot{A}^h = A_0 K - \Phi(I,K) + r\left(B^h - B^f\right) + \dot{B}^f - \dot{m}^f - \varepsilon\left(m^h + m^f\right) + \tau - C\,. \tag{A16}$$

Using the definitions for the stock of net foreign bonds (B) and total money balances (m)

$$B \equiv B^h - B^f ,$$

$$m \equiv m^h + m^f ,$$

(A16) is written

$$\dot{A}^h = A_0 K - \Phi(I, K) + rB + \dot{B}^f - \dot{m}^f - \varepsilon m + \tau - C . \tag{A17}$$

(A9) implies

$$\dot{A}^h = \dot{B}^h + \dot{m}^h . \tag{A18}$$

Setting (A18) into (A19) yields

$$\left(\dot{B}^h - \dot{B}^f\right) + \left(\dot{m}^h + \dot{m}^f\right) = A_0 K - \Phi(I, K) + rB - \varepsilon m + \tau - C ,$$

$$\Rightarrow \dot{B} + \dot{m} = A_0 K - \Phi(I, K) - C + rB + \tau - \varepsilon m . \tag{A19}$$

Using the government's budget constraint

$$\dot{m} + \varepsilon m = \tau , \tag{A20}$$

we obtain the current account balance

$$\dot{B} = A_0 K - \Phi(I, K) - C + rB . \tag{A21}$$

Model III

- The linearized approximation of system (III.24) is given by

$$\begin{pmatrix} \dot{q} \\ \dot{l} \end{pmatrix} = \underbrace{\begin{pmatrix} r - \dfrac{\tilde{q} - 1}{h} & A_0 \beta \left(1 - \tilde{l}\right)^{\beta - 1} \\ -\dfrac{1 - \gamma}{hF(\tilde{l})} & 0 \end{pmatrix}}_{\Omega} \cdot \begin{pmatrix} q - \tilde{q} \\ l - \tilde{l} \end{pmatrix} .$$

The determinant and the trace of Ω are both real and positive, and are given by

$$\Delta = \left(\frac{1-\gamma}{hF(\tilde{l})}\right)\left(\frac{\tilde{Y}}{\tilde{K}}\right)\left(\frac{\beta}{1-\tilde{l}}\right) > 0,$$

$$Tr(\Omega) = r - \frac{\tilde{q}-1}{h} > 0.$$

Therefore, the system is locally unstable and the only bounded equilibrium is one in which both q and l adjust instantaneously to ensure that the economy is on its balanced growth path.

Model IV

- The linearized approximation of system (IV.36) is given by

$$\begin{pmatrix}\dot{q}\\ \dot{l}\end{pmatrix} = \underbrace{\begin{pmatrix} r - \dfrac{\tilde{q}-(1+r+\varepsilon)}{h(1+r+\varepsilon)} & A_0\beta\left(1-\tilde{l}\right)^{\beta-1} \\ -\dfrac{1-\gamma}{h(1+r+\varepsilon)F(\tilde{l})} & 0 \end{pmatrix}}_{\Omega} \cdot \begin{pmatrix} q-\tilde{q} \\ l-\tilde{l}\end{pmatrix}.$$

The determinant and the trace of Ω are both real and positive, and are given by

$$\Delta = \left(\frac{1-\gamma}{h(1+r+\varepsilon)F(\tilde{l})}\right)\left(\frac{\tilde{Y}}{\tilde{K}}\right)\left(\frac{\beta}{1-\tilde{l}}\right) > 0,$$

$$Tr(\Omega) = r - \frac{\tilde{q}-(1+r+\varepsilon)}{h(1+r+\varepsilon)} > 0.$$

Therefore, the system is locally unstable and the only bounded equilibrium is one in which both q and l adjust instantaneously to ensure that the economy is on its balanced growth path.

Vitae

Maria Sarigiannidou was born in Thessaloniki, Greece on April 25, 1975. She received a Bachelor degree in economic science from the University of Macedonia at Thessaloniki, Greece, in 1999. She obtained a Master of Science degree in economics in 2001 from the same university. The same year she entered the graduate program in economics at the University of Tennessee, Knoxville and in 2003 she received a Master of Arts degree in economic science. She specialized in macroeconomic theory, and specifically in monetary economics. In 2007 she accepted a position as visiting professor with the department of economics at the University of Arkansas, Fayetteville. At year 2008 she engaged in research collaboration with Dr. Theodore Palivos, Professor at the department of economics at the University of Macedonia, Greece. She was awarded the Doctor of Philosophy degree in economic science in December, 2010.