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Schubert Numbers

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To the Graduate Council:

I am submitting herewith a dissertation written by Masato Kobayashi entitled "Schubert Numbers." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Shasikant B. Mulay, Major Professor

We have read this dissertation and recommend its acceptance:

David Anderson, Pavlos Tzermias, Michael Langston

Accepted for the Council:

Carolyn R. Hodges

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Schubert Numbers

A Dissertation Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Masato Kobayashi
May 2010

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Dedication

Dedicated to my father

Acknowledgments

My thanks must go to my mother Kikue, a new couple Naoya & Kazue, a new great-grandmother Saku, a new mother Chiharu, my dear Miho, Kawamura-san, Seki-san, a new couple John & Emily, Tomoya, Professor Iitaka, Professor Sakai and Professor Kachi. There is no doubt that I cannot finish the thesis without them. Last, but not least, I would like to express my sincere gratitude to my advisor, Professor Mulay for cordial encouragement and valuable help.

Abstract

This thesis discusses intersections of the Schubert varieties in the flag variety associated to a vector space of dimension n. The Schubert number is the number of irreducible components of an intersection of Schubert varieties. Our main result gives the lower bound on the maximum of Schubert numbers. This lower bound varies quadratically with n. The known lower bound varied only linearly with n. We also establish a few technical results of independent interest in the combinatorics of strong Bruhat orders.

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Introduction

This thesis articulates some recent progress made by its author in the area of algebraic combinatorics closely related to the algebraic geometry of Schubert subvarieties of the flag manifold. This introduction attempts to explain some historical lineage of the subject and thereby allow the reader a quick glimpse into the motivations behind our study.

Algebraic geometry is the study of solutions of polynomial equations; in other words, a natural extension of the ancient quest commonly known as the Theory of Equations. Roots of polynomials of a single variable have the geometric nature of points. Solutions of $f(x, y) = 0$, where $f(x, y)$ is a polynomial, in independent variables x, y, can be visualized as a curve in the xy-plane. Simultaneous solutions of $f(x, y) = 0$ and $g(x, y) = 0$, where $g(x, y)$ is also a polynomial, can be viewed as the points of intersection between the two corresponding plane curves. An eighteenth century theorem called Bézout's Theorem, named in honor of its discoverer Bézout, tells us that when $f(x, y)$ and $g(x, y)$ do not share a nonconstant common factor the corresponding curves intersect in as many points as the product of the degrees of f and g. It is not elementary to prove this simple statement due to the fact that for its claim to hold one has to count the intersection-points "properly". Of course it is even better to actually locate the points of intersection than to know only their number. Bézout did precisely that by determining the equation of intersection from the given polynomials $f(x, y)$ and $g(x, y)$ (called the "resultant of f, g."). In his work began the intersection theory of algebraic varieties which is still being developed.

In about the middle of the nineteenth century, Grassmann and (his pupil) Schubert formulated an elaborate theory to undertake the study of intersections of the simplest class of algebraic varieties, namely the linear varieties. To capture points at infinity as well as the imaginary points, it is advantageous to work in projective space over an algebraically closed field. Henceforth, our reference to "space" tacitly assumes it to be a projective. The first encounter with nontrivial intersection problems involving lines takes place in three dimensions. Schubert posed the following problem: (in 3-space) what is the number of lines that meet each of the given 4 lines in general position? This is a stunning question since its simplicity hides the difficulties involved in coming up with an educated guess about the possible answer, let alone justifying that answer rigorously. The Grassmann - Schubert theory provides a convenient set-up for solving such problems. The set of $d-1$ dimensional linear subvarieties of a $n-1$ dimensional space can be naturally identified with the set of all d-dimensional subspaces of an n-dimensional vector space. This set, usually denoted by $\mathrm{Gr}(d,\ n)$, was shown to be a projective algebraic variety by Grassmann. So, in his honor, it is called a “Grassmannian”. For example, the set of lines in 3-space is the Grassmannian $\mathrm{Gr}(2,\ 4)$. Given a sequence (called a full flag) $\mathcal{F} : V_1 \subset V_2 \subset \cdots$ of n subspaces of our n-dimensional space, where V_i has dimension i, and an increasing sequence $s : s_1 < s_2 < \cdots < s_d$ of positive integers $\leq n$, let $X(\mathcal{F},\ s)$ denote the set of all d-dimensional subspaces W such that the dimension of each $W \cap V_j$ is at least $\#\{i \mid s_i \leq j\}$. It is a fact that $X(\mathcal{F},\ s)$ is an algebraic subvariety of $\mathrm{Gr}(d,\ n)$. To honor Schubert, $X(\mathcal{F}, s)$ is called a Schubert variety. Coming back to our four lines, note that they correspond to four distinct 2-dimensional subspaces W_1, W_2, W_3, W_4 of a (fixed) 4-dimensional vector space. Including each one of them in some full flag, we get four flags $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$. Now the lines (in our 3-space) that meet the line corresponding W_i constitute the Schubert variety $X(\mathcal{F}_i,\ 2 < 4)$. So the Schubert-question about lines amounts to understanding the intersection

$$X(\mathcal{F}_1,\ 2 < 4) \cap X(\mathcal{F}_2,\ 2 < 4) \cap X(\mathcal{F}_3,\ 2 < 4) \cap X(\mathcal{F}_1,\ 2 < 4).$$

Somewhat surprisingly, this intersection almost always consists of just 2 members. In other words, there are exactly 2 lines that meet the given 4 lines in general position. Moreover, the defining equations, and hence the actual lines, of this intersection can also be determined.

There are other problems of a somewhat different type. Consider a fixed triple (x, L, P), where P is a plane in the 3-space, L a line contained in P and x is a point of L. Suppose we are interested in the family of triples (y, Λ, π), where π is a plane, Λ is a line contained in π, y is a point on Λ such that π passes through x and Λ meets L. To parametrize such families algebraically, or realize them as algebraic varieties, we have to introduce Schubert subvarieties of the manifold of full flags. Let $\mathrm{FL}(n)$ denote the set of full flags in an n-dimensional vector space. Fix a full flag $\mathcal{F}$ as above and fix a permutation σ of the numbers $1, 2, \cdots, n$. The Schubert subvariety of $\mathrm{FL}(n)$ associated to σ is the set $X(\mathcal{F},\ \sigma)$ consisting of all the full flags $W_1 \subset \cdots \subset W_n$ such that for each pair (i, j) the space $W_i \cap V_j$ has dimension at least $\#(\{1, \cdots, j\} \cap \{\sigma(1), \cdots, \sigma(i)\})$. In this set-up, the family mentioned above is realized as the subvariety $X(\mathcal{F},\ \alpha) \cap X(\mathcal{F}\ \beta)$ of $\mathrm{FL}(4)$, where $\mathcal{F}$ corresponds to the fixed triple (x, L, P), α is the cycle (1423) and β is the transposition (14). Observe that in the Schubert-problem for 4 lines there were 4 different flags involved and only one defining sequence $2 < 4$, whereas here we have the same flag but different defining permutations α, β. Thus, intersections of Schubert varieties, in the Grassmannian or in the manifold of full flags, either with respect to varying flags or with respect to a fixed flag, play a natural role in intersection theory. Since the family of hypersurfaces of a fixed degree also has the structure of a projective space, intersection problems involving their special subfamilies get translated as linear intersection problems of the above type. Hopefully our discussion so far has hinted at the significance of studying intersections of Schubert varieties.

Starting from its ancestral home in algebraic geometry, the theory of Schubert varieties has, by now, established deep and wide connections with combinatorics, representation theory of algebraic groups, systems of differential equations, *etc.* For example, the study of singularities of Schubert varieties and computation of the associated Kazdan - Lusztig polynomials is of high importance in Representation theory. In the study of Fuchsian systems of (nonlinear) differential equations Schubert varieties enter in a natural way. Ramification properties of algebraic functions are naturally related to Schubert varieties. These diverse disciplines pose various different type of problems and shed new light on this hundred and fifty year old subject. Hilbert astutely realized the importance of the Schubert theory and incorporated its study as the fifteenth problem in his famous list of problems disclosed at the very outset of the twentieth century.

In the present thesis, we deal with the following specific question: what is the maximal number of irreducible components of an intersection of (a set of) Schubert varieties in $\mathrm{FL}(n)$? In particular, what is the maximal number of irreducible components of the intersection of *two* Schubert subvarieties of $\mathrm{FL}(n)$? This maximum is called the *Schubert number*. Although at present we have no answer to the question, our main theorem here provides a lower bound for the Schubert number. We conjecture this lower bound to be the actual maximum attained even in the case of an intersection of two Schubert varieties. A noteworthy feature of our lower bound is that it is quadratic in n whereas the earlier known bound is linear in n. En route, we establish a few new technical results which are of independent interest in the combinatorics of the so called "strong Bruhat order". The organization of the thesis is as follows: the first chapter deals with various combinatorial concepts, definitions, properties, and known results that will be tacitly used throughout the rest of the thesis. The second chapter deals with the algebraic geometry part; namely, definitions of the Grassmann, flag and Schubert varieties, their basic properties and some (known) theorems which constitute the foundations of our exploration. At the end of the second chapter, we articulate the main results in a technically precise form. The last two chapters, *i.e.*, chapters 3 and 4, provide the proofs of our main results.

Chapter 1

Symmetric groups, Poset, Bruhat order, Monotone triangles

This chapter is devoted to various preliminaries about symmetric groups, posets, Bruhat order and monotone triangles. The concepts and the objects introduced in this chapter prepare the groundwork for later chapters.

1.1 Symmetric Groups

Symmetric groups are ubiquitous in mathematics. Since they play an important role in many branches of mathematics, it is essential to understand their rich structure. This section aims to provide a quick introduction to the study of symmetric groups with focus on those properties which will be needed in the later chapters. The notation introduced here, some of which is less commonly encountered, will be tacitly employed hereafter.

1.1.1 Definition and Notation

Definition 1.1. For a positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. A *permutation of degree* n is a bijection of $[n]$ onto itself. The set of all permutations of degree n is denoted by S_n.

Example 1.2. The mapping $w : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$ is a permutation on [3]. One natural way to denote w is to use the matrix notation: $w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. We prefer the more streamlined notation: $w := 231$.

In general, this can be defined as follows.

Notation. By $w := i_1 i_2 \cdots i_n$, where $i_k \in [n]$ for all $k \in [n]$, we mean the permutation $w \in S_n$ given by $w(k) := i_k$.

The set S_n forms a group under composition (of permutations). The identity element of the group S_n is the identity permutation $e : 12 \cdots n$. The inverse of an

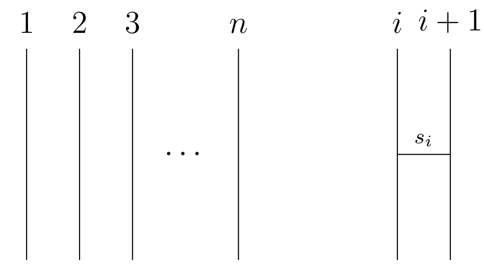


Figure 1.1: Numbered strings and connectors

element $w := i_1 i_2 \cdots i_n$ of S_n is the permutation $w^{-1} := j_1 j_2 \cdots j_n$ where $j_r := k$ provided $i_k = r$. The well known fact that $\#S_n = n!$ can be easily established by induction on n. It is customary to call the binary operation of a group as a 'product'.

1.1.2 String Expressions and Coxeter relations

The structure of the symmetric groups has been studied for more than a hundred years. From the group-theoretic perspective, there is a way of express permutations that is better suited for studying algebraic properties. This is the so called *string expression* of a permutation. We use n vertical strings and a sufficiently large number of horizontal *connectors* (Figure 1.1). The vertical strings are labeled by the numbers 1 to n and the horizontal connectors are labeled by the labels $s_1, s_2, \ldots, s_{n-1}$. For each label s_i an infinite supply of horizontal connectors labeled s_i is assumed to be available. We define a *diagram* to be a configuration of the vertical strings together with finitely many connectors such that the vertical strings appear in their labeling order starting from 1 on the left and ending with n on the right, a connector labeled s_i is used exclusively to connect the i- and the $i+1$-th vertical strings, and no two connectors appear along the same horizontal line (or level). Think of s_i as denoting the permutation that interchanges i and $i+1$ while fixing everything else. Such an s_i is called a *simple reflection.* Let $S := \{s_1, s_2, \ldots, s_{n-1}\}$ be the set of simple reflections of S_n.

Each $w \in S_n$ can be expressed by a diagram which is called a *string expression of* w, and each diagram is a string expression of some permutation in S_n. First, we describe an inductive procedure to construct a string expression of a $w \in S_n$. If $n = 1$, then the lone string 1 constitutes the string expression of w. Assume $n \geq 2$ and $w(k) = n$. If $k = n$, then regarding w as an element of $S_{(n-1)}$ we construct its string expression and simply append the string n to it. If $k \neq n$, connect each pair of adjacent strings between the string k and the string n using an appropriately labeled connector in such a way that the levels of the connectors descend successively downwards going from k towards n. Call this diagram D. Now $w_1 := ws_k \cdots s_{(n-1)}$ clearly fixes n and thus can be thought of as in $S_{(n-1)}$. Say a diagram D_1 is obtained by appending the string n to a string expression of w_1. Then, a string expression of w is obtained by joining the bottom-end-points of the strings in D to the top-end-points of the corresponding strings in D_1. It is easy to show that the diagram D is

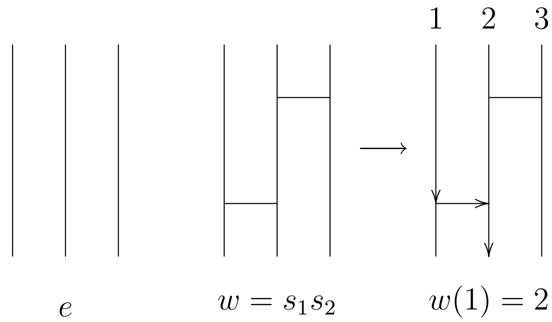


Figure 1.2: string expression

indeed a string expression of the product $s_{(n-1)} \cdots s_k$. Moreover, w is seen to be the composition of the simple reflections, *i.e.*, S generates S_n.

Conversely, given such a diagram we associate a unique permutation w to it as follows: to determine $w(i)$, start going down from the top of a vertical string labeled i, at the first instance of an encounter with a horizontal connector, move to the adjacent string. On any vertical string only the downward travel is permitted. Following these rules continue to travel until say, the j-th (possibly $j = i$) vertical string so that no further (allowed) horizontal migration is permissible. Then $w(i) = j$. Applying this procedure to the diagram D of the above paragraph, we recover $s_{(n-1)} \dots s_k$. Thus, by induction, this procedure is verified to be inverse to the construction described in the above paragraph. In particular, no horizontal connectors appear in a string expression if and only if it is a string expression of the identity permutation. Below Figure 1.1.2 is an illustration of a string expression of the identity permutation and a string expression of the permutation $w = s_1 s_2 = 231$.

It is straightforward to verify that, from the given string expressions of permutations x, y a string expression of the composition xy can be obtained by connecting the top-end-points of the vertical strings in the string expression of x to the bottom-end-points of the corresponding vertical strings in a string expression of y. In particular, to realize a permutation w as a composition of the simple reflections, we chop a string expression of w into several pieces using such horizontal cuts that each of the resulting pieces has at most one horizontal connector and each connector from the original expression appears in some piece. Then, $w = s_{i_1} \dots s_{i_r}$ where the simple reflections appearing in the product, from left to right, are the connectors from the bottom piece to the top piece respectively.

Likewise, a string expression of the inverse of a permutation w is obtained by turning a given string expression of w upside down. The Figure 1.1.2 below provides an illustration.

The set S of simple reflections is not a free set of generators of S_n. Obviously, $s_i^2 = e$ for each i. Distinct simple reflections s_i and s_j commute if $|j - i| > 1$. Observe that $s_i s_{i+1} \neq s_{i+1} s_i$ but $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. This last equality is called a *braid relation*. String expressions provide the following visual illustration.

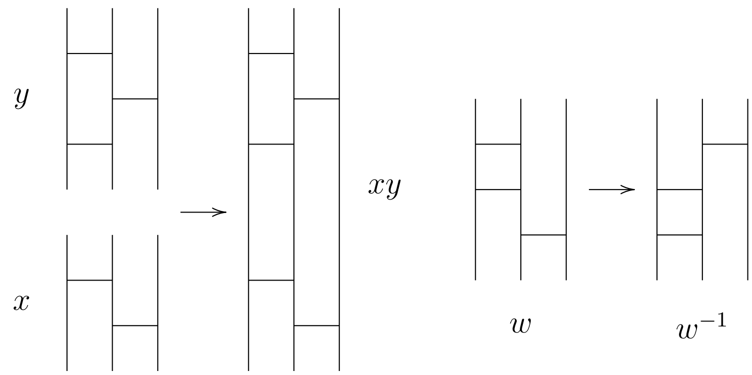


Figure 1.3: multiplication and inverse

Summarizing, simple reflections satisfy the following for all i and j such that $|j-i|>1$:

$$\begin{aligned} s_i^2 &= e && \text{(involution)} \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{(braid relation)} \\ s_i s_j &= s_j s_i && \text{(commutation)} \end{aligned}$$

These are the so called *Coxeter relations* (Figure 1.4). It is known that the generating set S together with the above Coxeter relations furnish an abstract presentation of the group S_n (see [6, Chapter 1] for a proof).

1.1.3 Reduced words and Length of a permutation

As seen in the previous section, for each $w \in S_n$ there exist a nonnegative integer d and a sequence $i_1, i_2, \ldots, i_d$ of integers in $[n-1]$ such that $w = s_{i_1} s_{i_2} \ldots s_{i_d}$. If $d = 0$, then the empty product $s_{i_1} s_{i_2} \ldots s_{i_d}$ is (by convention) the identity permutation. Such a d-tuple $(i_1, i_2, \ldots, i_d)$ is called a *word of* w having d simple reflections. Of course, any $w \in S_n$ has infinitely many words. Of special importance are the words that have the least number of simple reflections, *e.g.*, the empty word is clearly the shortest word of the identity permutation e.

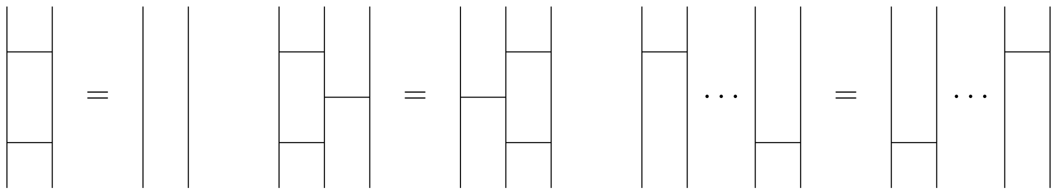
Definition 1.3.

(1) The *length* of w, denoted by $\ell(w)$, is defined to be the least nonnegative integer d such that there is a word of w having d simple reflections. In particular, $\ell(e) = 0$.

(2) If $(i_1, i_2, \ldots, i_d)$ is a word of w and $d = \ell(w)$, then we say that $(i_1, i_2, \ldots, i_d)$ is a *reduced word (or expression)* of w.

The following properties are well known and easy to prove (see [6, Sections 1.4-5]).

(1) $\ell(s_i) = 1$ for all i.



$s_i^2 = e$ $\qquad$ $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ $\qquad$ $s_i s_j = s_j s_i \quad |j-i| > 1$

Figure 1.4: Coxeter relations

(2) Let $t_{ij} \in S_n$ denotes the transposition which interchanges i and j while keeping the remaining members of $[n]$ fixed. If $i < j$, then it is straightforward to see that $t_{ij} = s_i s_{i+1} \ldots s_{j-2} s_{j-1} s_{j-2} \ldots s_{i+1} s_i$ (*e.g.* $t_{25} = s_2 s_3 s_4 s_3 s_2$). Later we prove $\ell(t_{ij}) = 2(j-i) - 1$ and hence the above word is indeed a reduced word of t_{ij}.

(3) For all $w \in S_n$ and $s_i \in S$, we have $\ell(ws_i) = \ell(w) \pm 1$.

1.1.4 Descents

Definition 1.4. Define the *left descent* and the *right descent* of w to be the sets $D_L(w)$ and $D_R(w)$ (respectively) given by

$$D_L(w) = \{1 \leq i \leq n-1 \mid w^{-1}(i) > w^{-1}(i+1)\},$$
$$D_R(w) = \{1 \leq i \leq n-1 \mid w(i) > w(i+1)\}.$$

Example 1.5.

$$D_L(634971285) = \{2, 5, 8\},$$
$$D_R(634971285) = \{1, 4, 5, 8\}.$$

Proposition 1.6. *We have $D_L(w) = D_R(w^{-1})$. Moreover, $D_R(w) = \varnothing \iff w = e$. Consequently, if $\ell(w) > 0$, then $w(i) > w(i+1)$ for some i.*

Proof. The first equality follows from the definition of the descents. Now $D_R(w) = \varnothing$ if and only if $w(1) < w(2) < \cdots < w(n)$ if and only if $w(i) = i$ for all $i \in [n]$ if and only if $w = e$. $\square$

1.1.5 Inversions

Let w be a permutation of $[n]$.

Definition 1.7. An ordered pair of integers (i, j) is said to be an *inversion of w* if $1 \leq i < j \leq n$ and $w(i) > w(j)$. The *set of inversions of w* is defined to be the set

$I(w) = \{(i,j) \mid 1 \le i < j \le n, w(i) > w(j)\}$. Let $\mathrm{inv}(w) = \#I(w)$ (the cardinality of $I(w)$).

The following basic properties can be easily established.

(1) $\mathrm{inv}(w) \le n(n-1)/2$.

(2) $\mathrm{inv}(w) = 0$ if and only if $I(w) = \varnothing$ if and only if $w = e$.

(3) $(i,k) \in I(w) \iff (w(k),\ w(i)) \in I(w^{-1})$.

(4) We have

$$\mathrm{inv}(ws_i) = \begin{cases} \mathrm{inv}(w) + 1 & \text{if } w(i) < w(i+1), \\ \mathrm{inv}(w) - 1 & \text{if } w(i) > w(i+1). \end{cases}$$

Proposition 1.8. $\ell(w) = \mathrm{inv}(w)$.

Proof. From the last basic property listed above, it follows that if there is a word of w having d simple reflections, then $\mathrm{inv}(w) \le d$. Hence $\mathrm{inv}(w) \le \ell(w)$. We establish the opposite inequality by induction on $\mathrm{inv}(w)$. If $\mathrm{inv}(w) = 0$, then $w = e$ and hence $\ell(w) = 0$. Suppose $\mathrm{inv}(w) > 0$. Then, $w \ne e$. Choose $i \in [n-1]$ such that $w(i) > w(i+1)$ (existence of such an i is ensured by the previous proposition). Note that $\mathrm{inv}(ws_i) = \mathrm{inv}(w) - 1$. By the induction hypothesis $\ell(ws_i) \le \mathrm{inv}(ws_i)$. Thus $\ell(w) \le \ell(ws_i) + 1 \le (\mathrm{inv}(w) - 1) + 1 = \mathrm{inv}(w)$. □

1.1.6 Reduction

Let T denote the subset of S_n consisting of the transpositions (also called *reflections*). Note that T has cardinality $n(n-1)/2$.

Definition 1.9. Let $w \in S_n$. By a *reduction of* w we mean a member x of the set $\{wt \mid t \in T\} \cup \{tw \mid t \in T\}$ such that $\ell(w) < \ell(x)$. We write $w \xrightarrow[t]{t'} x$ if $t'w = wt = x$ and $\ell(w) < \ell(x)$. Either of the labels t' or t may be omitted for the sake of notational simplicity.

Remarks 1.10.

(1) Note that $x = t'w$ for some $t' \in T$ if and only if $x = wt$ for some $t \in T$ since $t = w^{-1}t'w$ ($t' = wtw^{-1}$) is also a transposition.

(2) In general, from $w \longrightarrow x$ it does not follow that $\ell(x) - \ell(w) = 1$.

It is useful to look at some concrete examples of compositions of a permutation w and a transposition.

Example 1.11. Let $w = 34251$ and $t = t_{35}$. Since $w^{-1}(3) = 1, w^{-1}(5) = 4$, we have

$$\begin{aligned}
t_{35}(34251)(1) &= t_{35}(3) = 5,\\
t_{35}(34251)(4) &= t_{35}(5) = 3,\\
(34251)t_{35}(3) &= 34251(5) = 1,\\
(34251)t_{35}(5) &= 34251(3) = 2.
\end{aligned}$$

Therefore, $tw = 54231$ and $wt = 34152$. Hence

$$\begin{array}{ll}
t_{35}(34251) = \underline{5}42\underline{3}1 & \text{switch values 3 and 5}\\
(34251)t_{35} = 34\underline{1}5\underline{2} & \text{switch positions 3 and 5}
\end{array}$$

Proposition 1.12. *Let $x \in S_n$ and $t_{ik} \in T$. Suppose $(i, k) \in I(x)$. Let*

$$r = \#\{j \mid i < j < k \text{ and } x(i) > x(j) > x(k)\}.$$

Then $\ell(x) - \ell(xt_{ik}) = 2r + 1$. In particular, $\ell(t_{ik}) = 2|i - k| + 1$.

Proof. Set $w = xt_{ik}$. Recall that $\mathrm{inv}(w) = \ell(w)$. We investigate the correspondence of inversion pairs occurring in $I(w)$ and $I(x)$ to count the numbers $\mathrm{inv}(w)$ and $\mathrm{inv}(x)$. First note that $(i, k) \in I(x) \setminus I(w)$. Consider the list of subsets of $[n]^2$ defined below.

$$\begin{array}{llll}
I_0 & = \{(i,k)\}, & I_1 & = \{(\alpha,i) \mid \alpha < i\},\\
I_2 & = \{(\alpha,k) \mid \alpha < i\}, & I_3 & = \{(\alpha,k) \mid i < \alpha < k\},\\
I_4 & = \{(i,\beta) \mid i < \beta < k\}, & I_5 & = \{(i,\beta) \mid k < \beta\},\\
I_6 & = \{(k,\beta) \mid k < \beta\}, & I_7 & = \{(\alpha,\beta) \mid \alpha,\beta \notin \{i,k\}\}.
\end{array}$$

Define $I_m(w) = I_m \cap I(w)$ and $I_m(x) = I_m \cap I(x)$ for $0 \le m \le 7$. Then $I(w), I(x)$ are partitioned by the sets $I_m(w)$ and the sets $I_m(x)$ respectively. It is straightforward to verify that

$$\begin{aligned}
(\alpha, i) \in I_1(x) &\iff (\alpha, k) \in I_2(w),\\
(\alpha, k) \in I_2(x) &\iff (\alpha, i) \in I_1(w),\\
(i, \beta) \in I_5(x) &\iff (k, \beta) \in I_6(w),\\
(k, \beta) \in I_6(x) &\iff (i, \beta) \in I_6(w),\\
(\alpha, \beta) \in I_7(x) &\iff (\alpha, \beta) \in I_7(w),\\
I_3(w) &\subseteq I_3(x),\\
I_4(w) &\subseteq I_4(x).
\end{aligned}$$

To compute $\ell(x) - \ell(w)$, we need to know the cardinalities of sets $I_3(x) \setminus I_3(w)$ and $I_4(x) \setminus I_4(w)$. An easy verification shows that $\#(I_3(x) \setminus I_3(w)) = \#(I_4(x) \setminus I_4(w)) = 2r$. Since $(i, k) \in I(x) \setminus I(w)$, we conclude that $\ell(x) - \ell(w) = 2r + 1$. By taking $x = t_{ik}$ we at once get $\ell(t_{ik}) = 2|i - k| + 1$. $\square$

Example 1.13. Here are some exemplary applications of the above Proposition.

$$\begin{aligned}
&\ell(6253\overset{56}{41}) - \ell((625341)t_{56}) = 1,\\
&\ell(\mathbf{6}2\underline{5}\mathbf{3}41) - \ell(t_{36}(625341)) = 3,\\
&\ell(62\mathbf{5}\underline{34}\mathbf{1}) - \ell(t_{15}(625341)) = \ell(62\overset{3}{5}\underline{34}\overset{6}{1}) - \ell((625341)t_{36}) = 5.
\end{aligned}$$

1.1.7 Permutation-matrices

It is well known that S_n can be thought of as a subgroup of $\mathrm{GL}_n(C)$ for any commutative ring C with $1 \neq 0$. For later use we wish to fix such a representation of S_n and define the 'rank matrix' of a permutation.

Definition 1.14. Let $M_n(C)$ be the set of $n \times n$ matrices over C. Given $w \in S_n$ the *permutation-matrix associated to* w is the matrix $A := (a_{ij}) \in M_n(C)$ defined by $a_{ij} = \begin{cases} 0 & \text{if } j \neq w(i), \\ 1 & \text{if } j = w(i). \end{cases}$ for all $1 \leq i, j \leq n$.

By abuse of notation, we identify the above defined permutation-matrix A with the permutation w. The set of all such permutation-matrices is a subgroup of $\mathrm{GL}_n(C)$ which is naturally isomorphic to S_n by above defined association.

Definition 1.15. Let $w \in S_n$ and $1 \leq p, q \leq n$.

(1) By w_{pq} denote the submatrix of w consisting of its first p rows and first q columns.

(2) The *rank function of* w *at* (p, q) is defined by $r_w(p, q) = \#\{i \mid i \leq p, w(i) \leq q\}$ (this is the number of nonzero entries in w_{pq}).

(3) The *rank matrix of* w is the $n \times n$ matrix $R(w)$ whose (p, q)-th entry is $r_w(p, q)$.

Example 1.16. $312 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $R(312) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

1.2 Posets

This section introduces basic terminology in the study of posets and the important concept of the MacNeille Completion of a poset. These will play a key role in our investigation.

1.2.1 Basics

By a *poset* we mean a nonempty set P endowed with a partial-order $\leq$. It is denoted by the ordered pair $(P, \leq)$. If the underlying set P is finite, then $(P, \leq)$ is said to be a finite poset. When dealing with a fixed partial-order $\leq$ on a set P, we often use P to denote the poset $(P, \leq)$.

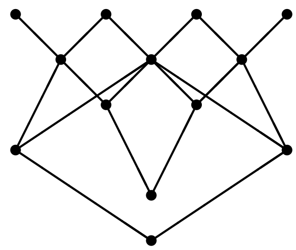


Figure 1.5: A Hasse diagram

Definition 1.17. Let $(P, \le)$ be a finite poset, $x, y \in P$ and $Q \subseteq P$. If $x \le y$ and $x \ne y$, then we write $x < y$.

(1) y is said to *cover* x if $x < y$ and there is no $z \in P$ such that $x < z < y$. We use the notation $x \lhd y$ to indicate that y covers x.

(2) The *Hasse diagram* of P is a directed graph whose vertices are the elements of P and whose edges are the ordered pairs (x, y) with $x \lhd y$. By convention, in any planar Hasse diagram, a vertex y is placed at a level above x whenever $x \lhd y$. If there is no edge-path between x and y, then x is incomparable to y with respect to $\le$ (Figure 1.5).

(3) The set Q is called a *chain* if any two members of Q are comparable with respect to $\le$, *i.e.*, given $x, y \in Q$, either $x \le y$ or $y \le x$. In such case, $\#Q - 1$ is called its *length*.

(4) P is said to be a *graded poset of rank* r if (i) every maximal chain has the same length r and (ii) there exists a map $\rho : P \to \{0, 1, 2, \dots, r\}$ (called a *rank function*) such that $\rho(y) - \rho(x) = 1$ for all $x, y \in P$ with $x \lhd y$. Such a poset is usually denoted by the ordered-triple $(P, \le, \rho)$.

(5) Define the *up-set* and the *down-set* of x to be $U[x] = \{y \in P \mid y \ge x\}$ and $D[x] = \{y \in P \mid y \le x\}$ respectively.

(6) Let $U[Q]$ (resp. $D[Q]$) denote the intersection of the sets $U[x]$ (resp. $D[x]$) as x ranges over Q.

(7) Let $Q \subseteq P$. If $U[Q]$ (resp. $D[Q]$) has a unique minimal (resp. maximal) element, it is denoted by $\vee Q$ (resp. $\wedge Q$). By $x \vee y$ (resp. $x \wedge y$) we mean $\vee\{x, y\}$ (resp. $\wedge\{x, y\}$).

(8) If $\vee Q$ and $\wedge Q$ exist for every subset $Q \subseteq P$, then P is called a *lattice*.

(9) P is said to be a *distributive lattice* if P is a lattice such that

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in P$.

(10) Let $(P', \leq)$ be a poset. A map $f : P \to P'$ is said to be an *order-preserving (resp. order-reversing) morphism* if for all $x, y \in P$ we have $x \leq y \iff f(x) \leq f(y)$ (resp. $f(x) \geq f(y)$). Additionally, if f is bijective, then it is called an *isomorphism*. An order-preserving (resp. order-reversing) *automorphism* of P is an order-preserving (resp. order-reversing) isomorphism $f : P \to P$.

(11) By the *lower neighborhood of* x we mean the set $C(x) := \{p \in P \mid p \lhd x\}$.

(12) The *down degree of* x is $d_-(x) := \#C(x)$.

Remark 1.18. If f is an order-preserving automorphism of a graded poset $(P, \leq, \rho)$, then $\rho(f(x)) = \rho(x)$ for all $x \in P$ since f preserves the covering relations.

1.2.2 MacNeille Completions

Let $(P, \leq)$ be a poset. Consider the set $\Lambda(P)$ consisting of all ordered couples $(\mathcal{L}, \alpha)$ such that $\mathcal{L}$ is a lattice, $\alpha : P \to \mathcal{L}$ is an injective, order-preserving morphism of posets and $\mathcal{L}$ is the only sub-lattice of $\mathcal{L}$ containing the image of α.

Definition 1.19. A pair (L, ι) in $\Lambda(P)$ is said to be a *MacNeille Completion of P* if given any pair (M, β) in $\Lambda(P)$ there exists an injective, order-preserving morphism $\phi : L \to M$ of posets with $\phi\iota = \beta$.

Proposition 1.20. *Every poset has a MacNeille Completion which is unique up to an order-preserving isomorphism of posets.*

Proof. The asserted unique-ness up to isomorphism follows readily from the definition. For the existence of the MacNeille Completion of a poset, see [48, Section 6]. □

Notation. the MacNeille Completion of a poset P is denoted by $L(P)$ and P is thought of as a sub-poset of $L(P)$.

1.3 Bruhat Order

We proceed to define an important partial order on the group S_n.

Definition 1.21. (*Bruhat Order*) Let $w, y \in S_n$. Define $w \leq y$ if there exists a sequence $x_0, x_1, \ldots, x_m$ in S_n such that $w = x_0$, $x_m = y$ and $x_i \longrightarrow x_{i+1}$ for all $0 \leq i \leq m - 1$.

In the literature, this is often called the *strong Bruhat Order* since there is a corresponding notion of the *weak Bruhat order* [6, Section 3.1]. Since we have no occasion to deal with the weak Bruhat order, we restrict our considerations to the strong Bruhat order and for simplicity call it the Bruhat order.

Proposition 1.22.

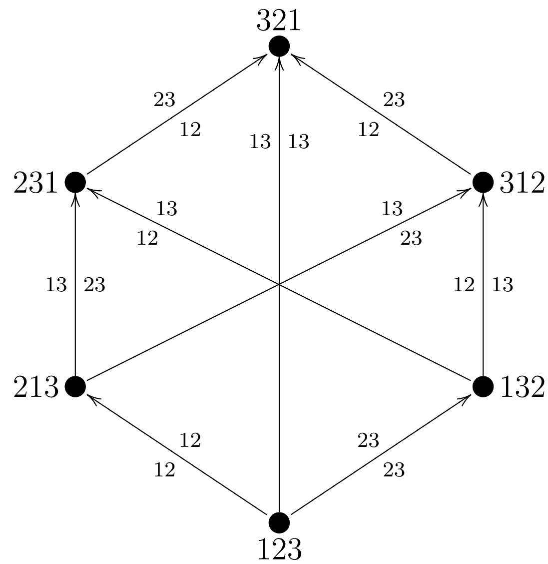


Figure 1.6: Bruhat order structure on S_3

(1) *$(S_n, \le, \ell)$ is a graded poset of rank $n(n-1)/2$.*

(2) *The identity permutation e is the unique minimal element of $(S_n, \le)$.*

Proof. Readily follows from the definitions. □

Figure 1.6 is the Hasse diagram (Bruhat graph) of $(S_3, \le)$. Labels on the edges indicate left and right reflections at a reduction with ij standing for the transposition t_{ij}.

1.3.1 Automorphisms of $(S_n, \le, \ell)$

Definition 1.23. Let $w_0 \in S_n$ denote the permutation $i \mapsto n - i + 1$.

Proposition 1.24. *$\ell(w_0) = n(n-1)/2$ and $w_0^{-1} = w_0$.*

The proof is clear.

Proposition 1.25. *Let $x, y \in S_n$ and $t, t' \in T$.*

(1) $\ell(x) = \ell(x^{-1}) = \ell(w_0 x w_0)$.

(2) $\ell(x w_0) = \ell(w_0 x) = \ell(w_0) - \ell(x)$.

(3) *The following are equivalent:*

(a) $x \xrightarrow[t']{t} y$,

(b) $x^{-1} \xrightarrow[t]{t'} y^{-1}$,

(c) $w_0xw_0 \xrightarrow[w_0t'w_0]{w_0tw_0} w_0yw_0$,

(d) $w_0y \xrightarrow[t']{w_0tw_0} w_0x$,

(e) $yw_0 \xrightarrow[w_0t'w_0]{t} xw_0$.

Proof. Left to the reader. $\square$

Proposition 1.26. *The maps $x \mapsto x^{-1}$, $x \mapsto w_0xw_0$ and $x \mapsto w_0x^{-1}w_0$ are order-preserving automorphisms of $(S_n, \leq, \ell)$. The maps $x \mapsto w_0x$, $x \mapsto xw_0$ are order-reversing automorphisms of $(S_n, \leq, \ell)$. In particular, w_0 is the unique maximal element of the poset $(S_n, \leq)$.*

Proof. The proof is a straightforward [6, Section 2.3]. $\square$

1.3.2 Subword property

The following property is well-known. The reader is referred to [6, Theorem 2.2.2] for a proof.

Proposition 1.27. *Let x, w be a permutations of $[n]$. Let $(j_1, \ldots, j_d)$ be a reduced word of x. Then, $w \leq x$ in the Bruhat order of S_n if and only if there is a reduced word $(i_1, \ldots, i_c)$ of w which is a subsequence of $(j_1, \ldots, j_d)$.*

Example 1.28. In S_n with $n \geq 7$, the words $w := s_2s_1s_4s_6$ and $x := s_2s_3s_2s_5s_1s_4s_6s_5$ are indeed reduced words. Since 2146 appears as a proper subsequence of 23251465, from the subword property it follows that $w < x$.

1.3.3 Rank-matrix property

Here, we think of S_n as the set of $n \times n$ permutation matrices as identified previously. We need the notion of the rank matrix and the associated rank function.

Proposition 1.29. *For $w, x \in S_n$ we have $w \leq x$ in the Bruhat order of S_n if and only if $r_w(p, q) \geq r_x(p, q)$ for all $1 \leq p, q \leq n$.*

Proof. First suppose $w = xt_{ab}$ where $1 \leq a < b \leq n$ and $w(a) = x(b) < x(a) = w(b)$. We prove that $r_w(p, q) \geq r_x(p, q)$ for all $p, q \in [n]$. Our proof is divided in several cases.

(1) $p < a$. Then, for all $i \leq p, w(i) = x(i)$. Thus $r_w(p, q) = \#\{i \mid i \leq p, w(i) \leq q\} = \#\{i \mid i \leq p, x(i) \leq q\} = r_x(p, q)$.

(2) $a \leq p < b$. Then, $w(i) = x(i)$ for all $i \leq p$ except $w(a) < x(a)$.

   (a) If $q < w(a)$, then $q < w(a) = x(b) < x(a)$. Thus both of $w(a), x(a) \nleq q$ Thus $r_w(p, q) = r_x(p, q)$.

   (b) If $w(a) \leq q < w(b)$, then $q < w(b) = x(a) \nleq q$. Thus $r_w(p, q) = r_x(p, q)+1$.

(c) If $w(b) \leq q$, then $w(a) < w(b) = x(a) \leq q$. Thus $r_w(p,q) = r_x(p,q)$.

(3) $b \leq p$. Then $w(i) = x(i)$ for all $i \leq p$ except $x(b) = w(a) < w(b) = x(a)$.

(a) If $q < w(a)$, then $x(b) = w(a) \not\leq q$ and $w(b) = x(a) \not\leq q$. Thus $r_w(p,q) = r_x(p,q)$.

(b) If $w(a) \leq q < w(b)$, then $x(a) = w(b) \not\leq q$ but $x(b) = w(a) \leq q$. This means that these equalities contribute exactly 1 to both of $r_w(p,q)$ and $r_x(p,q)$. Thus $r_w(p,q) = r_x(p,q)$.

(c) If $w(b) \leq q$, then $x(b) = w(a) < x(a) = w(b) \leq q$. This means that these equalities contribute exactly 2 to both of $r_w(p,q)$ and $r_x(p,q)$. Thus $r_w(p,q) = r_x(p,q)$.

For the converse the reader is referred to [9, p.174, Lemma 11]. □

Example 1.30. Consider $w := 213$ and $x := 312$ in S_3. Then, it is clear that $R(213) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} > \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = R(312)$. Hence $213 < 312$.

1.4 Monotone Triangles

1.4.1 Ferrers diagrams and Young tableaux

Definition 1.31.

(1) A *Ferrers diagram* is a weakly increasing finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_i \leq \lambda_j$ for $1 \leq i < j \leq k$. It is convenient to regard λ as a planar array of cells in which the i-th row consists of λ_i cells (Figure 1.7).

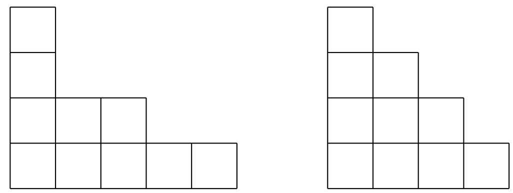
(2) Define $|\lambda| := \lambda_1 + \cdots + \lambda_k$. Then, the Ferrers diagram λ is a *partition* of the integer $|\lambda|$.

(3) For each positive integer n, the Ferrers diagram $(1, 2, \ldots, n)$ is denoted by δ_n.

(4) Let λ be a Ferrers diagram. A *Young tableau* of *shape* λ is an assignment of a positive integers to each cell of λ such that in each row, the assigned integers form a strictly increasing sequence (from left to right). The assigned numbers are said to be the *entries* of this Young tableau. A formal expression of a Young tableau of shape λ is an array

$$x \;:=\; (x_{ij} \mid 1 \leq j \leq \lambda_i \;\; 1 \leq i \leq k)$$

such that (i) each x_{ij} is a positive integer and (ii) $x_{ij} < x_{im}$ for all (i,j,m) with $1 \leq j < m \leq \lambda_i$, $1 \leq i \leq k$. The i-th row of a Young tableau x is denoted by $x[i]$ and by $|x[i]|$ we denote the set of entries in $x[i]$. Usually, a Young tableau is written by omitting the cell-boundaries of its underlying shape.



$\lambda = (1, 1, 3, 5)$ $\qquad$ $\delta_4 = (1, 2, 3, 4)$

Figure 1.7: Ferrers diagrams

(5) A *standard Young tableau* is a Young tableau each of whose column-entries form a nonincreasing sequence (from top to bottom), *i.e.*, $x_{ij} \geq x_{nj}$ for all (i, j, n) with $1 \leq i < n \leq k$ and $1 \leq j \leq \lambda_i$.

(6) By $\mathrm{YT}(\lambda, n)$ we denote the set of all Young tableaux of shape λ and with entries in the set $[n]$. By $\mathrm{SYT}(\lambda, n)$ we mean the subset of $\mathrm{YT}(\lambda, n)$ consisting (exclusively) of the standard Young tableaux.

(7) Define a partial order on $\mathrm{YT}(\lambda, n)$ as follows: for $x, y \in \mathrm{YT}(\lambda, n)$, define $x \leq y$ if $x_{ij} \leq y_{ij}$ for each (i, j).

(8) To a permutation $x \in S_n$ we associate a Young tableau (understood to be empty if $n = 1$), which (by abuse of notation) is also denoted by x, as follows: x is defined to be the (unique) Young tableau of shape δ_{n-1} such that

$$|x[i]| := \{x(1), \ldots, x(i)\} \qquad \text{for } 1 \leq i \leq n - 1.$$

Example 1.32. Below are examples of Young tableaux. Note that the middle tableau is a standard Young tableau (in fact, it is associated to the permutation $x := 42513$).

$$\begin{array}{cccc} 210 & & & \\ 802 & & & \\ 2000 & 2005 & & \\ 34 & 1027 & 1230 & 2009, \end{array} \qquad \begin{array}{cccc} 4 & & & \\ 2 & 4 & & \\ 2 & 4 & 5 & \\ 1 & 2 & 4 & 5 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} 5 & 7 & \\ 2 & 4 & \\ 2 & 7 & 9. \end{array}$$

Proposition 1.33. *Let x, y be Young tableaux of the same shape $\lambda := (\lambda_1, \ldots, \lambda_k)$.*

(1) *The arrays $x \vee y$ and $x \wedge y$ defined by the assignments $(x \vee y)_{ij} := \max(x_{ij}, y_{ij})$ and $(x \wedge y)_{ij} := \min(x_{ij}, y_{ij})$ respectively, are Young tableaux of shape λ.*

(2) *If x and y both are standard Young Tableaux, then $x \vee y$ and $x \wedge y$ are also standard Young tableaux.*

(3) *If x is (associated to) a permutation of $[n]$, then x is a standard Young tableau. Furthermore, $x_{ij} \leq x_{(i+1)(j+1)}$ for pairs (i, j) satisfying $1 \leq j \leq i \leq n - 2$.*

(4) *The poset* $(\mathrm{YT}(\lambda, n),\ \leq)$ *is a lattice with join operation* $\vee$ *and meet operation* $\wedge$. *Moreover, the poset* $(\mathrm{SYT}(\lambda, n),\ \leq)$ *is a sub-lattice of* $(\mathrm{YT}(\lambda, n),\ \leq)$.

Proof. Consider an integer triple (i, j, m) with $1 \leq j < m \leq \lambda_i$, $1 \leq i \leq k$. Then, clearly $\max(x_{ij}, y_{ij}) < \max(x_{im}, y_{im})$ and $\min(x_{ij}, y_{ij}) < \min(x_{im}, y_{im})$. Hence $x \vee y$ and $x \wedge y$ are Young tableaux of shape λ. Suppose x and y both are standard. Let (i, j, r) be an integer triple such that $1 \leq i < r \leq k$ and $1 \leq j \leq \lambda_i$. Since by assumption $x_{ij} \geq x_{rj}$ and $y_{ij} \geq y_{rj}$, it follows that $\max(x_{ij}, y_{ij}) \geq \max(x_{rj}, y_{rj})$ and $\min(x_{ij}, y_{ij}) \geq \min(x_{rj}, y_{rj})$. Hence $x \vee y$ as well as $x \wedge y$ is a standard Young Tableau of shape λ.

Next, suppose x is in S_n. If $n \leq 2$, there is nothing to prove. Henceforth assume $n \geq 3$. Then, $\lambda = \delta_{n-1}$ and $|x[i]| \subset |x[i+1]|$ for $1 \leq i \leq n-2$. Fix a pair (i, j) with $1 \leq i \leq n-2$ and $1 \leq j \leq i$. It suffices to show that $x_{ij} \geq x_{(i+1)j}$. Let $\{\theta\} := |x[i+1]| \setminus |x[i]|$ and let m be the greatest integer in $[i]$ such that $x_{im} < \theta$. Then, $x_{(i+1)j} = x_{ij}$ for $1 \leq j \leq m$, $x_{(i+1)(m+1)} = \theta < x_{i(m+1)}$ and $x_{(i+1)j} = x_{i(j-1)} < x_{ij}$ for $m+2 \leq j \leq i$. Thus x is a standard Young tableau. Also, observe that since $x_{(i+1)(j+1)} = x_{i(j+1)} > x_{ij}$ for $1 \leq j \leq m-1$, $x_{(i+1)(m+1)} = \theta > x_{im}$ and $x_{(i+1)(j+1)} = x_{ij}$ for $m+1 \leq j \leq i$, we have $x_{ij} \leq x_{(i+1)(j+1)}$ for pairs (i, j) satisfying $1 \leq j \leq i \leq n-2$.

The last assertion is an obvious consequence of the first two assertions. □

1.4.2 Monotone Triangles

Definition 1.34.

(1) By a *monotone triangle* of order n we mean an element x of $\mathrm{SYT}(\delta_{n-1}, n)$ such that

$$x_{ab} \leq x_{(a+1)(b+1)} \quad \text{for } 1 \leq b \leq a \leq n-2\ .$$

(2) Let $(\mathrm{MT}(n),\ \leq)$ denote the sub-poset of $(\mathrm{SYT}(\delta_{n-1},\ n),\ \leq)$ consisting of all monotone triangles of order n.

Proposition 1.35. *Given* $x, y \in \mathrm{MT}(n)$, *the Young tableaux* $x \vee y$ *and* $x \wedge y$ *are monotone triangles of order* n. *In particular,* $\mathrm{MT}(n)$ *is a sub-lattice of* $\mathrm{SYT}(\delta_{n-1}, n)$.

Proof. We have already seen that $x \vee y$ and $x \wedge y$ are standard Young tableaux of shape δ_{n-1}. Clearly the entries of $x \vee y$ and $x \wedge y$ belong to $[n]$. Consider an integer pair (a, b) with $1 \leq b \leq a \leq n-2$. Since $x_{ab} \leq x_{(a+1)(b+1)}$ as well as $y_{ab} \leq y_{(a+1)(b+1)}$, we have

$$\begin{aligned} &\max(x_{ab}, y_{ab}) \leq \max(x_{(a+1)(b+1)}, y_{(a+1)(b+1)}) \quad \text{as well as} \\ &\min(x_{ab}, y_{ab}) \leq \min(x_{(a+1)(b+1)}, y_{(a+1)(b+1)}). \end{aligned}$$

Thus $x \vee y$, $x \wedge y$ are monotone triangles of order n. □

Remark 1.36. As established in the previous subsection, the Young tableau associated to a permutation of the set $[n]$ is a monotone triangle of order n. This provides a natural embedding of S_n in $\mathrm{MT}(n)$. We tacitly regard S_n as a subset of $\mathrm{MT}(n)$ via this embedding. It is clear that the identity permutation $e \in S_n$ is the unique minimal member of the lattice $(\mathrm{MT}(n),\ \leq)$ and the permutation $w_0 \in S_n$, defined by

$w_0(i) := n - i + 1$ for $i \in [n]$, is the unique maximal element of $(\mathrm{MT}(n), \preceq)$. Since the monotone triangles e, w_0 are given by $e_{ab} = b$ and $(w_0)_{ab} = n - a + b$ respectively, we have $b \le x_{ab} \le n - a + b$ for all $x \in \mathrm{MT}(n)$.

Proposition 1.37. *The above described set-theoretic injection of S_n into* $\mathrm{MT}(n)$ *is an order-preserving morphism from the poset $(S_n, \le)$ (where $\le$ stands for the Bruhat order) into* $(\mathrm{MT}(n), \preceq)$.

Proof. It is enough to show that if $w \le x$ in the Bruhat order $\le$, then $w \preceq x$ where $\preceq$ is the order on monotone triangles. In view of the definition of the Bruhat order, we may assume (without loss) that $w \underset{t_{ij}}{\longrightarrow} x$ for some $1 \le i < j \le n$. Then $x = wt_{ij}$ and $x(j) = w(i) < w(j) = x(i)$. Note that if either $1 \le a \le i-1$ or $j \le a \le n-1$, then $w[a] = x[a]$. Henceforth suppose $i \le a \le j-1$. Then $\{w(i)\} = |w[a]| \setminus |x[a]|$ and $\{x(i)\} = |x[a]| \setminus |w[a]|$. If $a = 1$, then $w[a] = w(i)$ and $x[a] = x(i)$. Since $w(i) < x(i)$, we get the desired in this case. Assume $a > 1$ and arrange the $a - 1$ members of $|x[a]| \cap |w[a]|$ in increasing order as $m_1 < \cdots < m_{a-1}$. If either $m_{a-1} < w(i)$ or $x(i) < m_1$, then we also have $m_{a-1} < x(i)$ or $w(i) < m_1$ (respectively) and hence either $m_1 < \cdots < m_{a-1} < w(i) = w[a] \preceq x[a] = m_1 < \cdots < m_{a-1} < x(i)$ or $w(i) < m_1 < \cdots < m_{a-1} = w[a] \preceq x[a] = x(i) < m_1 < \cdots < m_{a-1}$. Otherwise, let k be the least positive integer such that $w(i) < m_k$ and let r be the greatest positive integer such that $m_r < x(i)$. Clearly $r \ge k-1$, $w_{ak} = w(i)$ and $x_{a(r+1)} = x(i)$. Now $w_{ap} = m_p = x_{ap}$ for $1 \le p \le k-1$, $w_{ap} = m_{p-1} = x_{ap}$ for $r+2 \le p \le a$ and $w_{ap} = m_{p-1} < m_p = x_{ap}$ for $k+1 \le p \le r$. Lastly, $w_{ak} = w(i) < \min m_k, x(i) = x_{ak}$ and $w_{a(r+1)} = \max w(i), m_r < x(i) = x_{a(r+1)}$. Thus $w[a] \prec x[a]$. $\square$

Example 1.38. Let $w = 34251, x = 54231$. Then $w \underset{t_{14}}{\longrightarrow} x$. It is easy to see that

$$w = \begin{array}{cccc} 3 & & & \\ 3 & 4 & & \\ 2 & 3 & 4 & \\ 2 & 3 & 4 & 5 \end{array} \quad \prec \quad \begin{array}{cccc} 5 & & & \\ 4 & 5 & & \\ 2 & 4 & 5 & \\ 2 & 3 & 4 & 5 \end{array} \quad = x$$

as monotone triangles.

Proposition 1.39.

(1) $(\mathrm{MT}(n), \preceq)$ *is a finite distributive lattice with the join operation $\vee$ and the meet operation $\wedge$.*

(2) $(\mathrm{MT}(n), \preceq)$ *is a MacNeille completion of $(S_n, \le)$.*

(3) *For all $n \ge 3$,*

$$\#\mathrm{MT}(n) = \frac{1!4!7!10!\ldots(3n-2)!}{n!(n+1)!(n+2)!\ldots(2n-1)!}.$$

Proof. The first assertion is a a straightforward consequence of the definitions. The proofs of the next two assertions are substantilly more difficult. For their proofs the reader is referred to [38] and [32, 45] respectively. □

Remark 1.40. The values of $\#\mathrm{MT}(n)$ for $1 \le n \le 8$ are $1, 2, 7, 42, 429, 7436, 218348$, 10850216, respectively. Evidently, they increase very rapidly. It would be nice to have an elementary inductive proof for counting the monotone triangles. Currently, the only available proof is technically involved [32] as referred above.

Notation. Since $\mathrm{MT}(n)$ is the MacNeille completion of $(S_n, \le)$, we can also denote it by $L(S_n)$.

Example 1.41.

(1) S_n is not closed under $\vee, \wedge$. Let $x = 42531, y = 51324$. Then

$$x \vee y = \begin{array}{llll} 4 & & & \\ 2 & 4 & & \\ 2 & 4 & 5 & \\ 2 & 3 & 4 & 5 \end{array} \vee \begin{array}{llll} 5 & & & \\ 1 & 5 & & \\ 1 & 3 & 5 & \\ 1 & 2 & 3 & 5 \end{array} = \begin{array}{llll} 5 & & & \\ 2 & 5 & & \\ 2 & 4 & 5 & \\ 2 & 3 & 4 & 5 \end{array} = 52431 \in S_5,$$

$$x \wedge y = \begin{array}{llll} 4 & & & \\ 2 & 4 & & \\ 2 & 4 & 5 & \\ 2 & 3 & 4 & 5 \end{array} \wedge \begin{array}{llll} 5 & & & \\ 1 & 5 & & \\ 1 & 3 & 5 & \\ 1 & 2 & 3 & 5 \end{array} = \begin{array}{llll} 4 & & & \\ 1 & 4 & & \\ 1 & 3 & 5 & \\ 1 & 2 & 3 & 5 \end{array} \notin S_5.$$

(2) $L(S_3)(= \mathrm{MT}(3))$ is obtained from S_3 by adding the monotone triangle $\begin{array}{ll} 2 & \\ 1 & 3 \end{array}$ which equals $231 \wedge 312$ as well as $213 \vee 132$.

Figures 1.4.2, 1.4.2 in pages 21, 22 show the Hasse diagram of $L(S_4)$ and four equivalent characterizations of Bruhat order on S_3.

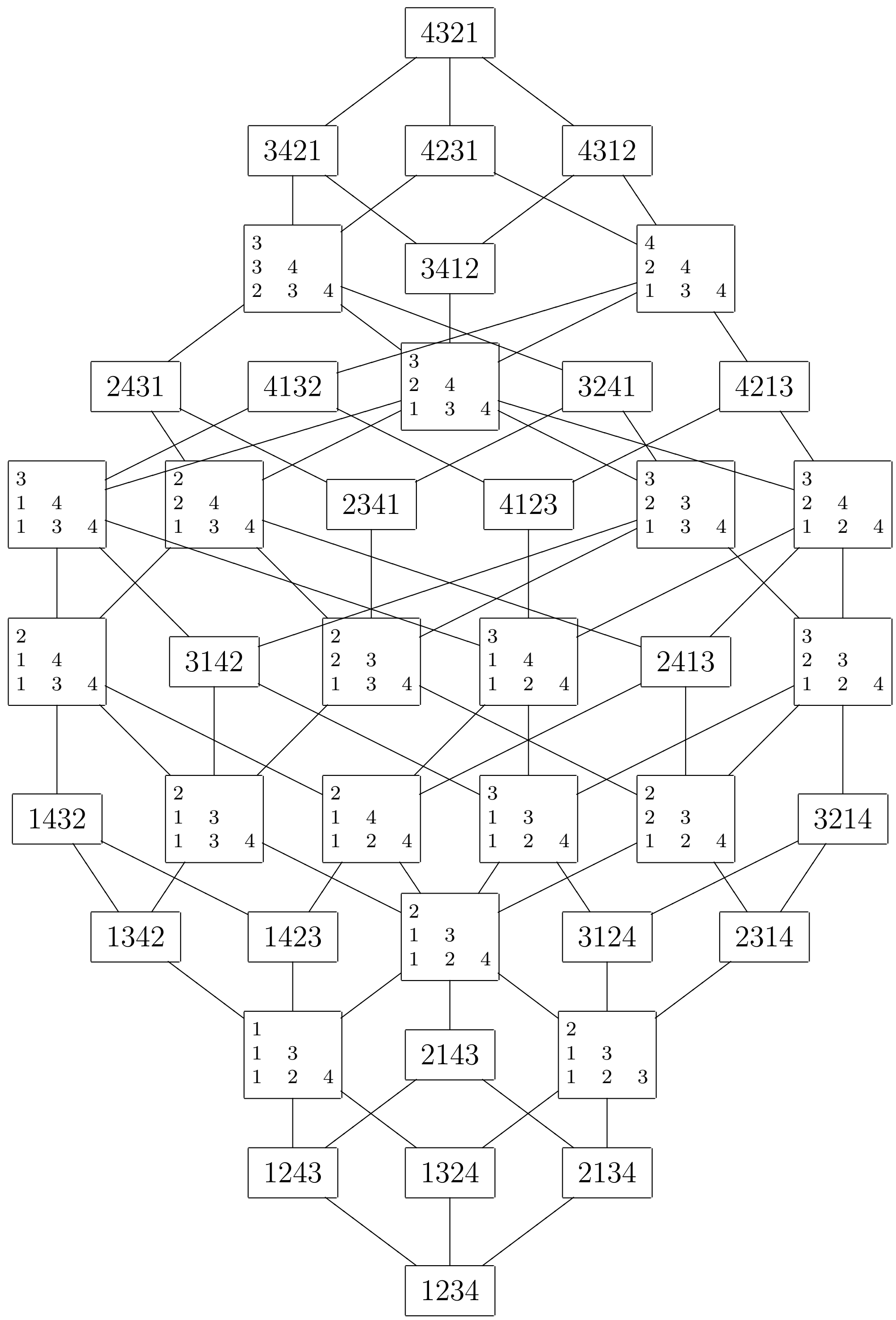


Figure 1.8: MT(4) (*i.e.*, $L(S_4)$)

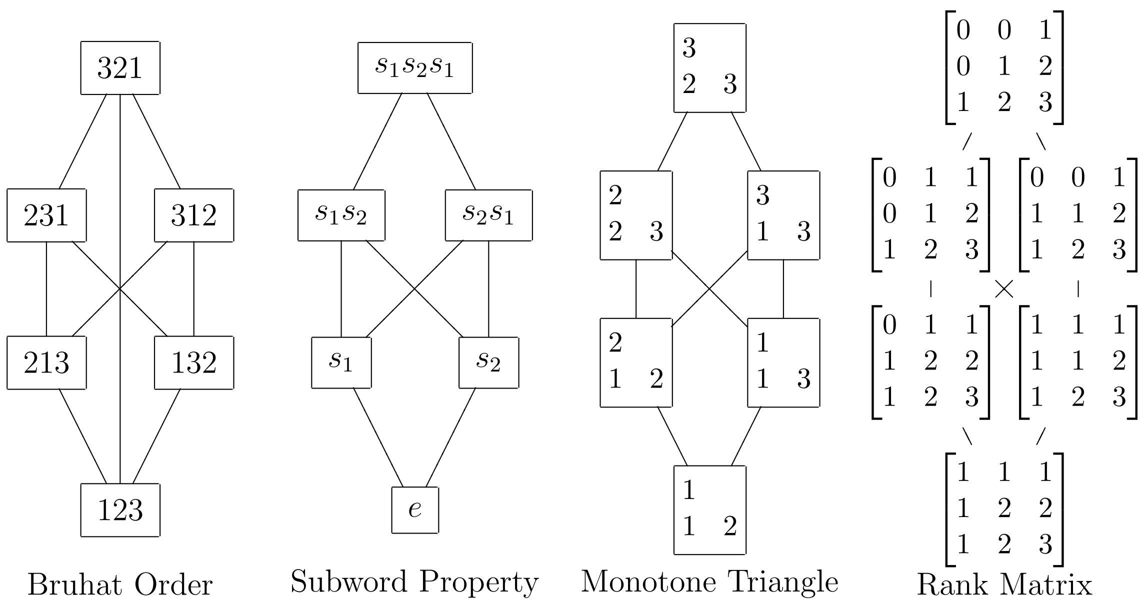


Figure 1.9: Four equivalent orders on S_3

Chapter 2

Flag varieties and their Schubert subvarieties

2.1 Grassmannians and Flag Varieties

Let n be a positive integer. Henceforth we tacitly assume that $n \geq 2$. Let V be an n-dimensional vector space over a field k.

Definition 2.1.

(1) Given a strictly increasing sequence $d : d_1 < d_2 < \cdots < d_r$ of integers with $d_i \in [n]$ for $1 \leq i \leq r$, by a *d-flag* in V we mean a sequence

$$V_1 \subset \cdots \subset V_r$$

of k-subspaces of V such that $\dim V_i = d_i$ for $1 \leq i \leq r$. The set of all d-flags in V is denoted by $\mathrm{FL}(d, V)$ (*a space of partial flags*).

(2) If $r = 1$, *i.e.*, when the sequence d consists of a single integer (also denoted by d), the corresponding set $\mathrm{FL}(d, V)$ is denoted by $\mathrm{Gr}(d, V)$. It is called the *Grassmannian* of d-dimensional subspaces of V.

(3) If $r = n$, *i.e.*, when the d is the sequence $1 < 2 < \cdots < n$, the corresponding flag is called a *full flag* in V. Thus, a full flag in V is a sequence

$$V_1 \subset \cdots \subset V_n = V$$

of k-subspaces of V with $\dim V_i = i$ for $1 \leq i \leq n$. The space of full flags in V is denoted by $\mathrm{FL}(V)$.

For positive integers r, s let $\mathbb{M}(r, s, k)$ be the vector space (over k) of all $r \times s$ matrices with entries in k. Let $\mathbb{M}(r, k) := \mathbb{M}(r, r, k)$ and as usual let $\mathrm{GL}(r, k)$ be the group of units of the k-algebra $\mathbb{M}(r, k)$, *i.e.*, the multiplicative group of $r \times r$ invertible matrices with entries in k. Let $\mathcal{R}(n, d, k) \subset \mathbb{M}(n, d, k)$ be the subset of all matrices of rank d. If $A \in \mathcal{R}(n, d, k)$, then the column-space of A is a d-dimensional k-subspace of the vector space k^n. Conversely, any d-dimensional k-subspace of k^n is in fact the column-space of some $A \in \mathcal{R}(n, d, k)$. Moreover, given $B \in \mathcal{R}(n, d, k)$, the column-space of B equals the column-space of A if and only if $A = Bg$ for some $g \in \mathrm{GL}(d, k)$. Thus there is a bijective correspondence between the Grassmannian $\mathrm{Gr}(d, k^n)$ and the set of orbits of $\mathcal{R}(n, d, k)$ under the right-multiplication action of the group $\mathrm{GL}(d, k)$. Let $\mathrm{Gr}(n, d, k)$ denote this orbit-set. Via a fixed k-linear isomorphism $V \equiv k^n$ the Grassmannian $\mathrm{Gr}(d, V)$ can be identified with $\mathrm{Gr}(d, k^n)$ and hence also with the set $\mathrm{Gr}(n, d, k)$.

Let $S_d[n]$ denote the set of all d-element subsets of $[n]$. Fix a labeling of $S_d[n]$ by the integers $1, 2, \dots, \binom{n}{d}$. Say $S := \{s_i \mid 1 \le i \le \binom{n}{d}\}$ where each s_i is regarded as a strictly increasing sequence $j_1 < \dots < j_d$ of integers in $[n]$. Given $A \in \mathbb{M}(n, d, k)$ and some $s \in S_d[n]$, where $s : j_1 < \dots < j_d$, let $q(s, A)$ denote the $d \times d$ minor of A determined by the rows $j_1 < \dots < j_d$. Let $q_i(A) := q(s_i, A)$ and let $q(A) := (q_1(A), q_2(A), \dots)$ denote the resulting $\binom{n}{d}$-tuple of elements of k. Note that, $q(A)$ is a nonzero vector if and only if A has rank d. Results from elementary Linear Algebra suffice to verify that for $A, B \in \mathcal{R}(n, d, k)$ we have $q(A) = c\, q(B)$ with $0 \ne c \in k$ if and only if $A = Bg$ for some $g \in \mathrm{GL}(d, k)$. Consequently, we get a well-defined (set-theoretic) injection from $\mathrm{Gr}(n, d, k)$ to the $\binom{n}{d} - 1$ dimensional projective space over k which maps the orbit of an $A \in \mathcal{R}(n, d, k)$ to the point having homogeneous coordinates $q(A)$. In this manner, to a k-linear isomorphism $\phi : V \equiv k^n$ corresponds a map

$$\phi_{(d)} :\ \mathrm{Gr}(d, V) \to \mathbb{P}_k^{\binom{n}{d}-1}.$$

More generally, if d denotes the sequence $d_1 < d_2 < \dots < d_r$, then by regarding $\mathrm{FL}(d, V)$ as a subset of the product $\mathrm{Gr}(d_1, V) \times \dots \times \mathrm{Gr}(d_r, V)$, we let $\phi_{(d)}$ denote the restriction of $\phi_{(d_1)} \times \dots \times \phi_{(d_r)}$ to $\mathrm{FL}(d, V)$. This is known as the 'Plücker embedding' of $\mathrm{FL}(d, V)$. The proof of the following well known proposition shows that the image of $\phi_{(d)}$ is a Zariski-closed subset.

Proposition 2.2. *Given an integer-sequence $d : d_1 < d_2 < \dots < d_r$ with $d_i \in [n]$ for $1 \le i \le r$, the set $\mathrm{FL}(d, V)$ is a smooth, projective variety over k. In particular, $\mathrm{Gr}(d, V)$ is a smooth projective variety of dimension $d(n-d)$ and $\mathrm{FL}(V)$ is a smooth projective variety of dimension $n(n-1)/2$.*

Proof. See [9, Chapters 9,10]. □

Remark 2.3. Let $\mathrm{GL}(V)$ denote the group of k-linear automorphisms of the vector space V. The set $\mathrm{Gr}(d, V)$ is clearly an invariant set under the induced action of $\mathrm{GL}(V)$ on the set of subsets of V. Likewise, under the natural extension of this action to $\mathrm{Gr}(d_1, V) \times \dots \times \mathrm{Gr}(d_r, V)$ the set $\mathrm{FL}(d, V)$ is invariant. The resulting action of $\mathrm{GL}(V)$ on $\mathrm{FL}(d, V)$ can be easily seen to be transitive. Furthermore, $\mathrm{GL}(V)$ acts as a group of automorphisms of the variety $\mathrm{FL}(d, V)$ thereby making $\mathrm{FL}(d, V)$ a *homogeneous space.* The smoothness asserted in the above proposition follows from the basic fact that a homogeneous space is necessarily smooth.

2.2 Cell-decompositions of $\mathbf{Gr}(d,V)$ and $\mathbf{FL}(V)$

We continue to use the above notation. From now on we deal only with the Grassmannians and the variety of full flags in V. Let $G := \mathrm{GL}(n,k)$ and let B (resp. B^t) denote the subgroup of G consisting of the upper-triangular (resp. lower-triangular) matrices. Note that each of B, B^t acts on $\mathcal{R}(n,d,k)$ by left multiplication and this yields a well-defined induced action on $\mathrm{Gr}(n,d,k)$.

Definition 2.4. Let $s : i_1 < \cdots < i_d$ be a sequence of integers in $[n]$ and let $M \in \mathcal{R}(n,d,k)$.

(1) Define $A_s := [a_{ij}] \in \mathcal{R}(n,d,k)$ by setting

$$a_{ij} \;=\; \begin{cases} 1 & \text{if } i = i_j, \\ 0 & \text{if } i \neq i_j \end{cases}$$

for all $(i,j) \in [n] \times [d]$.

(2) $M := [m_{ij}]$ is said to be in *s-reduced form* provided

$$m_{ij} \;=\; \begin{cases} 1 & \text{if } i = i_j, \\ 0 & \text{if } i = i_p \text{ for } p < j \text{ and} \\ 0 & \text{if } i > i_j \end{cases}$$

for all $(i,j) \in [n] \times [d]$.

(3) $M := [m_{ij}]$ is said to be in *anti-s-reduced form* provided

$$m_{ij} \;=\; \begin{cases} 1 & \text{if } i = i_j, \\ 0 & \text{if } i = i_p \text{ for } p < j \text{ and} \\ 0 & \text{if } i < i_j \end{cases}$$

for all $(i,j) \in [n] \times [d]$.

(4) M is said to be an s-reduced form (resp. anti-s-reduced form) of $N \in \mathcal{R}(n,d,k)$ if M is in s-reduced form (resp. in anti-s-reduced form) and there exists a matrix $g \in \mathrm{GL}(d,k)$ such that $M = Ng$.

(5)
$$\Omega(n,d,s) \;:=\; \{bA_sg \mid b \in B, \quad g \in \mathrm{GL}(d,k)\}.$$

(6)
$$\widehat{\Omega}(n,d,s) \;:=\; \{LA_sg \mid L \in B^t, \quad g \in \mathrm{GL}(d,k)\}.$$

Proposition 2.5. *The following holds.*

(1) *If $M \in \mathcal{R}(n,d,k)$ is in s-reduced (resp. anti-s-reduced) form for an $s \in S_d[n]$, then $M = CA_s$ for some $C \in B$ (resp. $C \in B^t$).*

(2) *Each member of* $\mathcal{R}(n,d,k)$ *has an s-reduced form for some* $s \in S_d[n]$. *Also, each member of* $\mathcal{R}(n,d,k)$ *has an anti-s-reduced form for some* $s \in S_d[n]$.

(3) *If* $s \in S_d[n]$, $M \in \mathcal{R}(n,d,k)$ *and* $g \in \mathrm{GL}(d,k)$ *are such that* M, Mg *both are in s-reduced form (resp. in anti-s-reduced form), then* g *is the identity matrix.*

(4) $\mathrm{Gr}(n,d,k)$ *is partitioned by the sets* $\Omega(n,d,s)$ *as* s *ranges over* $S_d[n]$. *Thus*

$$\mathrm{Gr}(n,d,k) \;=\; \bigsqcup_{s \in S_d[n]} \Omega(n,d,s).$$

(5) $\mathrm{Gr}(n,d,k)$ *is partitioned by the sets* $\widehat{\Omega}(n,d,s)$ *as* s *ranges over* $S_d[n]$. *Thus*

$$\mathrm{Gr}(n,d,k) \;=\; \bigsqcup_{s \in S_d[n]} \widehat{\Omega}(n,d,s).$$

Proof. Straightforward. □ □

Choose an ordered k-basis $E := \{e_1, \cdots, e_n\}$ of V. Regard k^n as the set of n-rowed columns with entries in k. Identify each vector $v \in V$ with its coordinate-column with respect to the ordered basis E. This yields a corresponding bijection

$$\phi(E) : \; \mathrm{Gr}(d,V) \to \mathrm{Gr}(n,d,k).$$

Now consider a full flag $\mathcal{F}$ in V. Say

$$\mathcal{F} \;:\; V_1 \subset V_2 \subset \cdots \subset V_n = V.$$

Let $\mathbb{E}(\mathcal{F})$ denote the set of all ordered k-bases $\{e_1, \cdots, e_n\}$ of V such that $\{e_1, \cdots, e_i\}$ is a k-basis of V_i for $1 \le i \le n$. Also, let $\widehat{\mathbb{E}}(\mathcal{F})$ denote the set of all ordered k-bases $\{e_1, \cdots, e_n\}$ of V such that the set $\{e_n, \cdots, e_1\}$ (obtained by reversing the order) belongs to $\mathbb{E}(\mathcal{F})$. Fix a member $E \in \mathbb{E}(\mathcal{F})$ (resp. $E \in \widehat{\mathbb{E}}(\mathcal{F})$). For any $F \in \mathbb{E}(\mathcal{F})$ (resp. any $F \in \widehat{\mathbb{E}}(\mathcal{F})$) letting $C \in B$ (resp. $C \in B^t$) be the transition matrix between E and F we have

$$\phi(F)(W) \;=\; C\psi(E)W \qquad \text{for all } W \in \mathrm{Gr}(d,V).$$

Hence the inverse image of $\Omega(n,d,s)$ (resp. $\widehat{\Omega}(n,d,s)$) under $\phi(E)$ depends only on the chosen flag $\mathcal{F}$. Let $\Omega(\mathcal{F},s)$ (resp. $\widehat{\Omega}(\mathcal{F},s)$) denote this inverse image. Then, via $\phi(E)$, we obtain the decomposition

$$\mathrm{Gr}(d,V) \;=\; \bigsqcup_{s \,\in\, S_d[n]} \Omega(\mathcal{F},s)$$

which is referred to as the *canonical cell-decomposition of* $\mathrm{Gr}(d,V)$ with respect to

$\mathcal{F}$. Also, we have the decomposition

$$\mathrm{Gr}(d, V) \;=\; \bigsqcup_{s \,\in\, S_d[n]} \widehat{\Omega}(\mathcal{F}, s)$$

which is referred to as the *anti-canonical cell-decomposition of* $\mathrm{Gr}(d, V)$ with respect to $\mathcal{F}$. Each $\Omega(\mathcal{F},\ s)$ (resp. $\widehat{\Omega}(\mathcal{F}, s)$) is called a *canonical cell* (resp. an *anti-canonical cell*) of $\mathrm{Gr}(d, V)$. The term "cell" is justified (or motivated) by the following : if $s : i_1 < \cdots < i_d$ and

$$l(s) \;:=\; (i_1 + \cdots + i_d) - \frac{d(d+1)}{2},$$

then $\Omega(n, d, s)$ is easily identified (set-theoretically) with the set $k^{l(s)}$. Observe that $l(s)$ attains its maximum $d(n-d)$ when $s = \nu : n-d+1 < \cdots < n$. For this reason $\Omega(\mathcal{F}, \nu)$ is called the *canonical big cell* (with respect to $\mathcal{F}$). Likewise, letting

$$\lambda(s) \;:=\; (n - i_1) + \cdots + (n - i_d) - \frac{d(d-1)}{2},$$

$\widehat{\Omega}(n, d, s)$ is identified (set-theoretically) with the set $k^{\lambda(s)}$ and $\widehat{\Omega}(\mathcal{F}, \iota)$, where $\iota : 1 < \cdots < d$, is called the *anti-canonical big cell* (with respect to $\mathcal{F}$). The following proposition follows readily from the preceding discussion.

Proposition 2.6. *A big cell (either canonical or anti-canonical) is Zariski-dense in* $\mathrm{Gr}(d, V)$.

Next, we describe cell decompositions of $\mathrm{FL}(V)$. Let G/B denote the set of all left-cosets of B in G. Choose an ordered k-basis $E := \{e_1, \cdots, e_n\}$ of V. As before, regard k^n as the set of n-rowed columns with entries in k and identify each vector $v \in V$ with its coordinate-column with respect to the ordered basis E. This way V is identified with k^n and $\mathrm{GL}(V)$ is identified with G. Given a full flag

$$\mathcal{G} \;:\; W_1 \subset W_2 \subset \cdots \subset W_n = V$$

in V, let $\mathrm{mat}(\mathcal{G})$ be the set of matrices $M \in G$ with the property that the first i columns of M form a k-basis of W_i for $1 \le i \le n$. It is easy to verify that matrices M_1, M_2 are in $\mathrm{mat}(\mathcal{G})$ if and only if $M_1 = M_2 b$ for some $b \in B$. In other words, $\mathrm{mat}(\mathcal{G})$ is a left-coset of B in G. Conversely, given any such left-coset gB, for each $i \in [n]$ letting W_i be the k-subspace of V spanned by the first i columns of g, we obtain a full flag $\mathcal{G}$ in V such that $\mathrm{mat}(\mathcal{G}) = gB$. Thus, the choice of E leads to a bijection

$$\psi(E) :\ \mathrm{FL}(V) \to\ G/B.$$

Consider a full flag $\mathcal{F}$ in V. Fix a member $E \in \mathbb{E}(\mathcal{F})$. For any $F \in \mathbb{E}(\mathcal{F})$, letting $b \in B$ to be the transition matrix between E and F we have

$$\psi(F)(\mathcal{G}) \;=\; b\psi(E)\mathcal{G} \quad \text{for all } \mathcal{G} \in \mathrm{FL}(V).$$

Clearly, the converse also holds. Hence the inverse image of the double-coset BgB

under $\psi(E)$ depends only on the chosen flag $\mathcal{F}$. It is well known (and easy to verify) that the double-coset decomposition of G with respect to the subgroup-pair $(B,\ B)$ is given by the double-cosets $B\theta B$ as θ ranges over S_n (this is a so called *Bruhat decomposition* of G). In particular, we have

$$G/B \;=\; \bigsqcup_{\theta\,\in\, S_n} B\theta B$$

and correspondingly (via $\psi(E)$) the decomposition

$$\mathrm{FL}(V) \;=\; \bigsqcup_{\theta\,\in\, S_n} W(\mathcal{F},\ \theta).$$

We call this the *canonical cell-decomposition of* $\mathrm{FL}(V)$ with respect to $\mathcal{F}$ (where $W(\mathcal{F},\ \theta)$ is called a *canonical cell*). As in the case of a Grassmannian, the term "cell" is justified (or motivated) by the fact that set-theoretically

$$W(\mathcal{F},\ \theta) \;\simeq\; k^{\ell(\theta)}.$$

Since $\ell(\theta)$ attains its maximum $n(n-1)/2$ when θ is the maximum of the poset $(S_n,\ \leq)$, the corresponding canonical cell is called the *canonical big cell* (with respect to $\mathcal{F}$). At the other extreme, the identity permutation is the only permutation of length 0. The corresponding canonical cell being 0-dimensional, it is referred to as the *distinguished point* (with respect to $\mathcal{F}$).

For $E,\ F \in \hat{\mathbb{E}}(\mathcal{F})$, letting $\lambda \in B^t$ be the transition matrix between E and F we have

$$\psi(F)(\mathcal{G}) \;=\; \lambda\psi(E)\mathcal{G} \qquad \text{for all } \mathcal{G} \in \mathrm{FL}(V).$$

The double-coset decomposition of G with respect to the subgroup-pair $(B^t,\ B)$ is given by the double-cosets $B^t\theta B$ as θ ranges over S_n (a Bruhat decomposition of G). Thus, in a manner similar to the one above, we obtain the decomposition

$$\mathrm{FL}(V) \;=\; \bigsqcup_{\theta\,\in\, S_n} \widehat{W}(\mathcal{F},\ \theta).$$

This is referred to as the *anti-canonical cell-decomposition of* $\mathrm{FL}(V)$ with respect to $\mathcal{F}$ (where $\widehat{W}(\mathcal{F},\ \theta)$ is an *anti-canonical cell*). Here too we have the set-theoretic identification

$$\widehat{W}(\mathcal{F},\ \theta) \simeq\; k^{\frac{n(n-1)}{2}-\ell(\theta)}.$$

The two cell-decompositions discussed above are said to be each others *Opposite.* The anti-canonical cell corresponding to the identity permutation is called the *Opposite big cell.* Similar to the case of the Grassmannian, we have the following.

Proposition 2.7. *A big cell (either canonical or anti-canonical) is Zariski-dense in* $\mathrm{FL}(V)$.

Remark 2.8. For $c \in k$ let $\delta_n(c)$ denote the $n \times n$ matrix resulting from multiplying the first row of the identity matrix by c. For $g \in G$ let $\det g$ denote the determinant of g and let $\eta(g) := g\delta_n(1/\det g)$. Thus we have a set-theoretic surjection $\eta : G \to \mathrm{SL}(n,k)$ whose restriction to $\mathrm{SL}(n,k)$ is the identity mapping of $\mathrm{SL}(n,k)$. Moreover, if $\mathrm{SL}(n,k)/B$ denotes the set of left cosets of $B \cap \mathrm{SL}(n,k)$ in $\mathrm{SL}(n,k)$, then η induces a bijection from G/B to $\mathrm{SL}(n,k)/B$. Consequently the above explained Bruhat decompositions of G/B induce similar Bruhat decompositions of $\mathrm{SL}(n,k)$ by double cosets corresponding to $\eta(\theta)$ as θ ranges over S_n. It should be noted that for $n \geq 2$ the image of S_n under η is *not* a subgroup of $\mathrm{SL}(n,k)$.

2.3 Schubert Varieties

There is an important family of subvarieties of $\mathrm{FL}(d,V)$ called *Schubert varieties*. In this section we state their definition as well as their basic properties restricting our exposition to $\mathrm{Gr}(d,V)$ and $\mathrm{FL}(V)$.

Observe that the set $S_d[n]$, defined in the last section, is the set $\mathrm{SYT}((d),n)$ defined in 1.4.1. Also, we have the partial order $\leq$ defined on (see 1.4.2) $\mathrm{SYT}((d),n)$. Thus $(S_d[n], \leq)$ is a poset.

Definition 2.9. As before, fix a full flag

$$\mathcal{F} \ : \ V_1 \subset V_2 \subset \cdots \subset V_n = V.$$

(1) For $\sigma \in S_d[n]$, where $\sigma : i_1 < \cdots < i_d$, and $p \in [n]$ define

$$r_\sigma(p) \ = \ \#\{m \ \mid \ m \in [d], \ \text{ and } \ p \geq i_m\}.$$

Define $X(\mathcal{F},\ \sigma)$ to be the subset of $\mathrm{Gr}(d,V)$ consisting of subspaces W such that $\dim(W \cap V_p) \geq r_\sigma(p)$ for $p \in [n]$.

(2) For $\theta \in S_n$ and $(p,q) \in [n] \times [n]$, let

$$r_\theta(p,q) = \#\{i \ \mid \ i \in [p] \ \text{ and } \ \theta(i) \leq q\}.$$

Define $X(\mathcal{F},\ \theta)$ to be the set of all full flags $\mathcal{G} : W_1 < \cdots < W_n$ in V such that $\dim(W_p \cap V_q) \geq r_\theta(p,q)$ for all $(p,q) \in [n] \times [n]$.

Proposition 2.10. *With the notation as in the above definition, the following holds.*

(1) *For each $\sigma \in S_d[n]$ the above defined $X(\mathcal{F},\ \sigma)$ is an $l(\sigma)$-dimensional irreducible subvariety of $\mathrm{Gr}(d,V)$. Furthermore, we have the cell-decomposition*

$$X(\mathcal{F},\ \sigma) \ = \ \bigsqcup_{s \,\leq\, \sigma} \Omega(\mathcal{F},s).$$

(2) *$X(\mathcal{F},\ \sigma)$ is the Zariski-closure of $\Omega(\mathcal{F},\sigma)$ in $\mathrm{Gr}(d,V)$.*

(3) *For each* $\theta \in S_n$ *the above defined* $X(\mathcal{F},\ \theta)$ *is an* $\ell(\theta)$*-dimensional irreducible subvariety of* $\mathrm{FL}(V)$*. Furthermore, we have the cell-decomposition*

$$X(\mathcal{F},\ \theta)\ =\ \bigsqcup_{\alpha \le \theta} W(\mathcal{F},\ \alpha).$$

(4) $X(\mathcal{F},\ \theta)$ *is the Zariski-closure of* $W(\mathcal{F},\ \theta)$ *in* $\mathrm{FL}(V)$.

Proof. See [9, Chapter 9]. □

Proposition 2.11. *For* $x, y \in S_n$*, the following are equivalent.*

(1) $r_x(p,q) \ge r_y(p,q)$ *for all* $(p,q) \in [n] \times [n]$.

(2) $x \le y$ *(where* $\le$ *is the strong Bruhat order).*

(3) $X(\mathcal{F},\ x) \subseteq X(\mathcal{F},\ y)$.

Proof. From the definition of Schubert varieties (or the cell-decomposition of a Schubert variety displayed in the third assertion of the above proposition) and the fact that the partial order on the set of $n \times n$ rank-matrices $[r_x(i,j)]$ is equivalent to the strong Bruhat order in the S_n, the desired equivalence follows □

Remark 2.12. Similar equivalence holds in the case of Schubert subvarieties of the Grassmannian.

Definition 2.13. Let $x \in L(S_n)$. The down-set of x in the poset $(L(S_n),\ \le)$ is denoted by $D[x]$. Recall that for any subset Q of the lattice $L(S_n)$ there is a well defined element $\wedge Q$ of $L(S_n)$.

(1) The set $\Gamma(x) := \mathrm{Max}\,(D[x] \cap S_n)$ is called the *lower permutation neighborhood* of x.

(2) For a subset $Q \subseteq S_n$, the *Schubert number* of Q is $\mathrm{sc}(Q) := \#\Gamma(\wedge Q)$. If $Q = \{v, w\}$, then we simply write $\mathrm{sc}(v,w)$ in place of $\mathrm{sc}(Q)$.

Remark 2.14. It is easy to see that $\mathrm{sc}(Q) = 1$ if and only if $\wedge Q \in S_n$.

Proposition 2.15. *Let* $Q \subseteq S_n$*. Then,* $\mathrm{sc}(Q)$ *equals the number of irreducible components of* $\bigcap_{q \in Q} X_q$.

Proof. Follows from the previous proposition. □

2.4 Ladder-determinantal Schubert varieties

In this section we focus solely on the Schubert subvarieties of $\mathrm{FL}(V)$. Via the transitive action of $\mathrm{GL}(V)$ on $\mathrm{FL}(V)$, the Schubert subvarieties $X(\mathcal{F},\ \theta)$ and $X(\mathcal{G},\ \theta)$ are seen to be isomorphic for any two full flags $\mathcal{F}$ and $\mathcal{G}$ in $\mathrm{FL}(V)$. Hence, in studying the individual Schubert varieties we may a fix a flag at the very outset. After fixing a full

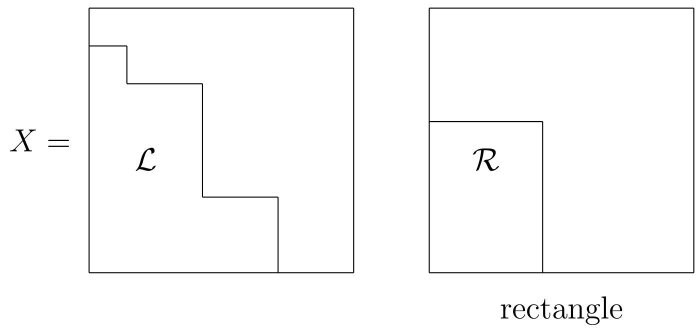


Figure 2.1: Ladder

flag $\mathcal{F}$ in V we choose an ordered basis $E \in \mathbb{E}(\mathcal{F})$ and thereby identify V with k^n. To simplify the notation these choices are henceforth kept in the background. So, denote $\mathrm{FL}(V)$ by $\mathrm{FL}(n)$, $W(\mathcal{F},\ \theta)$ by W_θ, $\widehat{W}(\mathcal{F},\ \theta)$ by $\widehat{W}_\theta$ and $X(\mathcal{F},\ \theta)$ by X_θ. Note that $\widehat{W}_{\mathrm{id}}$ (where id stands for the identity permutation) is identified as the set of $n \times n$ lower-triangular matrices $[a_{ij}]$ ($a_{ij} \in k$) with $a_{ii} = 1$ for all $i \in [n]$. Thus, as a variety, $\widehat{W}_{\mathrm{id}}$ is can be regarded as the $n(n-1)/2$-dimensional affine space over k.

Definition 2.16. Assume $n \geq 2$ and let $\widehat{W} := \widehat{W}_{\mathrm{id}}$.

(1) For $\theta \in S_n$, let $V(\theta) := \widehat{W} \cap X_\theta$.

(2) Let $Z = [Z_{ij}]$ be an $n \times n$ matrix whose entries are indeterminates over k. By $k[Z]$ we denote the polynomial ring in the n^2 indeterminates Z_{ij} over k.

(3) A subset (or subregion) $\mathcal{L}$ of Z is called a *ladder* provided there is a sequence $\{j_r\}_{r\in[n]}$ of nonnegative integers with $0 = j_1 \leq \cdots \leq j_n \leq n-1$ such that $X_{rj} \in \mathcal{L}$ for all $(r, j) \in [n] \times [j_r]$ (Figure 2.1). If there are positive integers p, q such that $j_i = 0$ for $1 \leq i \leq n-p$ and $j_i = q$ for $n-p < i \leq n$, then $\mathcal{L}$ is called a *rectangular ladder.* Further, a ladder is said to be (n, p)*-admissible* provided $j_r < r + p - 1$ for all $r \in [n]$.

(4) If a ladder $\mathcal{L}$ is properly contained in a ladder $\mathcal{L}^*$, then we write $\mathcal{L} < \mathcal{L}^*$.

(5) For a positive integer $p \leq n-1$ and a ladder $\mathcal{L}$ contained in Z, let $I_p(\mathcal{L})$ denote the ideal of $k[Z]$ generated by all $p \times p$ minors of Z contained in $\mathcal{L}$. By $V(p, \mathcal{L})$ we mean the affine variety defined by the ideal $I_p(\mathcal{L})$.

(6) Define L to be the affine linear variety defined by the prime ideal

$$P \ := \ \{Z_{ij} - \delta_{ij} \mid 1 \leq i \leq j \leq n\}k[Z]$$

where δ_{ij} is 1 or 0 according to whether $i = j$ or $i \neq j$ respectively.

Remarks 2.17.

(1) When the set $\{j_1, \cdots, j_n\}$ (as in the definition of a ladder) has $t+1$ (distinct) members, the corresponding ladder is said to be a t-step ladder.

(2) A ladder $\mathcal{L}$ in Z is (n,p)-admissible if and only if $L \cap V(p,\mathcal{L})$ is a nonempty set.

The assertions listed in the following proposition are due to Mulay, for their proofs the reader is referred to [42]. In connection with the last assertion of the proposition, see [43].

Proposition 2.18. *Identify $\widehat{W}$ with L. Then the following holds.*

(1) *For each (n,p)-admissible ladder $\mathcal{L}$ in Z there exists $\tau \in S_n$ such that*

$$V(\tau) = L \cap V(p,\mathcal{L}).$$

(2) *Corresponding to each $\tau \in S_n$, there is a sequence of positive integers $p_1 < p_2 < \cdots < p_m$ of length $m \leq n-1$ and a sequence of ladders $\mathcal{L}_1 < \cdots < \mathcal{L}_m$ in Z such that $\mathcal{L}_i$ is (n,p_i)-admissible for $1 \leq i \leq m$ and*

$$V(\tau) \ = L \cap V(p_1,\mathcal{L}_1) \cap \cdots \cap V(p_m,\mathcal{L}_m).$$

Moreover, the radical ideal of $k[Z]$ defining $V(\tau)$ is the ideal

$$P + I_{p_1}(\mathcal{L}_1) + \cdots + I_{p_m}(\mathcal{L}_m).$$

(3) *For $\tau \in S_n$, the variety $V(\tau)$ is 1-codimensional in $\hat{W}$ (and hence also* 1-*codimensional in* FL(n)*) if and only if $V(\tau) = V(p,\mathcal{L})$ where $\mathcal{L}$ is an (n,p)-admissible $p \times p$ rectangular ladder (i.e., a square ladder of size p). There are exactly $n-1$ Schubert divisors (i.e., Schubert subvarieties of codimension 1) in* FL(n).

(4) *For each $\tau \in S_n$ we have $V(\tau) = \cap V(\theta)$ where the intersection ranges over permutations θ covering τ (in the poset $(S_n, \leq)$).*

(5) *Call a Schubert variety $V(\tau)$* determinantal-type *if $V(\tau) = L \cap V(p,\mathcal{R})$ for some (n,p)-admissible rectangular ladder $\mathcal{R}$ in Z. Then, there are exactly $\binom{n+1}{3}$ determinantal-type Schubert subvarieties of* FL(n) *and each of these has codimension $\leq n^2/4$.*

(6) *Each Schubert subvariety $V(\tau)$ is an intersection of determinantal-type Schubert varieties.*

(7) *Let*

$$B_n \ = \ \{(s,t,p) \in \mathbf{N}^3 \ | \ 1 \leq p \leq s,t \leq n \text{ and } s+t-p+1 \leq n\}.$$

Let $\mathcal{R}$ be an $s \times t$ rectangular ladder in Z. Then, $\mathcal{R}$ is (n,p)-admissible if and *only if* $(s,t,p) \in B_n$. *For* $(s,t,p) \in B_n$, *define*

$$\pi(s|t,p) = \begin{cases} n-i+1 & \text{if } 1 \le i \le p-1 \\ n-s+p-i & \text{if } p \le i \le t \\ n+t-p+2-i & \text{if } t+1 \le i \le s+t-p+1 \\ n-i+1 & \text{if } s+t-p+2 \le i \le n. \end{cases}$$

Then $\pi(s|t,p) \in S_n$ *and* $V(\pi(s|t,p)) \;=\; L \cap V(p,\mathcal{R})$.

(8) *Let* (s,t,p) *and* (s',t',p') *be members of* B_n *such that* $t \le t'$. *Set* $\alpha := \pi(s|t,p)$ *and* $\beta := \pi(s'|t',p')$. *Then,* X_α, X_β *are determinantal-type Schubert varieties and* $X_\alpha \cap X_\beta$ *has* $\max\{1, \nu+1\}$ *irreducible components where* ν *denotes the minimum of the following nine integers:* $t'-t$, $s'-s$, $p'-p$, $t'-p'+1$, $t-p+1$, $s'-p'+1$, $s-p+1$, $(s'+t'-p')-(s+t-p)$, $s+t-p-p'+2$.

2.5 Open problems: a terse overview

The Grassmannian and the variety of full flags play a major role in the intersection-theory of algebraic varieties. The study of Schubert varieties was initially motivated by enumerative algebraic geometry (see [9]). By now it is linked, in a substantial way, to the theory of characteristic classes, the study of determinantal loci, representation theory of linear algebraic groups, algebraic combinatorics *etc*. Thus, quite naturally, there are many aspects of the study of Schubert varieties that remain open for further exploration. These form a source of unresolved problems which are investigated by researchers at present. In this section we comment on a few of these questions. For broader perspective, the reader is referred to [1, 5, 9, 15, 28, 34, 35, 36, 37, 42, 46, 49, 50]. Also, there is a nice survey in [41, Introduction].

- Let $\alpha, \beta \in S_n$ (resp. $\alpha, \beta \in S_d[n]$) and let $\mathcal{F}, \mathcal{G} \in \mathrm{FL}(V)$. Understanding the precise nature of the variety $X(\mathcal{F},\alpha) \cap X(\mathcal{G},\beta)$ for general full flags $\mathcal{F}, \mathcal{G}$ is still a major problem (first posed by Chevalley). Even the restricted problem of characterizing pairs (α,β) such that $X(\mathcal{F},\alpha) \cap X(\mathcal{G},\beta) \neq \emptyset$ for all pairs $(\mathcal{F},\mathcal{G})$, is yet to be fully settled. There are various recently established algorithms, mostly in the case of the Grassmannian, that have shed some light on this issue. This has led to a rich combinatorial theory involving "games", "puzzles" *etc*; we refer the reader to [16, 17, 18, 46, 50].

- The problem mentioned in the above item should be viewed as the *dynamic* intersection problem. In contrast to this there is a *static* intersection problem where we fix a full flag $\mathcal{F}$ and investigate $X_\alpha \cap X_\beta$ or equivalently $V(\alpha) \cap V(\beta)$. To begin with we would like to count the number of irreducible components of $V(\alpha) \cap V(\beta)$. In particular, determine the maximum possible number of such irreducible components. Obviously we may generalize to $\cap_{\tau \in Q} X_\tau$ where Q is an arbitrary finite family. From the properties exposed in the previous section it is clear that this is a purely combinatorial problem in the poset $(S_n, \leq)$ (or $(S_d[n], \leq)$).

- In the previous section it was seen that the affine varieties $V(\tau)$ where $\tau \in S_n$, have determinantal nature, namely

$$V(\tau) \ = L \cap V(p_1, \mathcal{L}_1) \cap \cdots \cap V(p_m, \mathcal{L}_m).$$

  In [42], there is an algorithm which determines the data on the left side starting from the given permutation τ. Conversely, given a sequence

$$(p_1, \mathcal{L}_1), \cdots\cdots, (p_m, \mathcal{L}_m)$$

  where $\mathcal{L}_i$ is an (n, p_i)-admissible ladder in Z, $p_1 < \cdots < p_m$ are positive integers and $\mathcal{L}_1 < \cdots < \mathcal{L}_m$, there is the natural problem of determining (algorithmically) the family $Q \subset S_n$ such that

$$L \cap V(p_1, \mathcal{L}_1) \cap \cdots \cap V(p_m, \mathcal{L}_m) \ = \ \bigcap_{\tau \in Q} V(\tau).$$

  In [43] just the first step in this direction is demonstrated. Even the more restricted question as to when the family Q consists of just one member (*i.e.*, the intersection is irreducible) remains open in general.

- The study of singularities of the Schubert subvarieties of $\mathrm{FL}(n)$ have implications in geometry as well as in representation theory. The singular locus of X_τ is itself a union of a certain subfamily (determined completely by τ) of Schubert varieties. Since each Schubert variety passes through the distinguished point, it suffices to study the singular loci of the affine varieties $V(\tau)$. Although a permutation-description of the irreducible components of the singular locus of X_τ has been recently obtained in various forms by [4, 10, 33], a ladder-determinantal description is not yet known in full generality. Likewise, the question of equidimensionality (or non-equidimensionality) of the singular locus, is also not fully settled. In [44] there are partial results in this direction.

- Detailed knowledge of the singularities of Schubert varieties requires good understanding of the tangent cone to $V(\tau)$ at the distinguished point. This includes a description of the ideal defining the tangent cone and a computation of (or a useful formula for) the Hilbert function of $V(\tau)$ at the distinguished point. In the existing literature there are some interesting formulas for the Hilbert functions of the ladder-determinantal varieties; we refer the reader to [1, 2, 12, 13, 29, 30, 31]. Computations of the Kazhdan-Lusztig polynomials at singular points of X_τ and interpretation of their coefficients is also an important issue (see [5, 7]). There are related combinatorial problems concerning Schubert polynomials and their coefficients (see [8, 39, 40]).

2.6 Main Results

In this section we state the main results which our study has uncovered and on which rests the novelty of the present thesis. Their proofs occupy the following chapters. We also mention our conjecture which serves as a near-term goal of our ongoing study. It is necessary to begin with some definitions.

Definition 2.19.

(1) Let $(P, \leq)$ be a poset. An element $x \in P$ is said to be *join-irreducible* if for each subset $Q \subset P$ with $x = \vee Q$ we have $x \in Q$. Let $\mathrm{Jirr}(P)$ denote the set of join-irreducible members of P.

(2) For $x \in L(S_n)$ (as seen in the last chapter, $L(S_n) = \mathrm{MT}(n)$), let $j(x)$ denote the number of join-irreducible permutations $\tau \in S_n$ such that $\tau \leq x$, *i.e.*, $j(x)$ denotes the cardinality of $D[x] \cap \mathrm{Jirr}(S_n)$.

Theorem (See Theorem 3.13)**.** *Let w_0 denote the maximum of the poset $(S_n, \leq)$. For $\tau \in S_n$ we have $\tau \in \mathrm{Jirr}(S_n)$ if and only if the Schubert variety X_α, where $\alpha := w_0\tau$, is of determinantal-type.*

Theorem (See Theorems 4.4, 4.8)**.** *Let $x \in S_n$.*

(1) $j(x) \;=\; j(x^{-1})$.

(2) $j(x) \;=\; \sum_{1 \leq b \leq a \leq n-1} (x_{ab} - b)$.

(3)

$$j(x) \;=\; \sum_{a=1}^{n-1} (x(a) - a)(n - a) \;=\; \frac{1}{2} \sum_{a=1}^{n} (x(a) - a)^2.$$

(4) *If $I(x)$ is the set of inversions of x, then, we have*

$$j(x) = \sum_{(i,\, k) \in I(x)} (x(i) - x(k)).$$

Remark 2.20. The need to investigate properties of join-irreducible permutations can be explained thus: the above theorem in conjuction with the known fact that $\lfloor n^2/4 \rfloor$ is precisely the maximum of $\{\ell(v) \mid v \in \mathrm{Jirr}(S_n)\}$ helps us establish the following theorem.

Theorem (See Theorem 4.13)**.** *We have*

$$\max\{\mathrm{sc}(Q) \mid Q \subseteq S_n\} \;\geq\; \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Conjecture (See Conjecture 4.14):

$$\max\{\mathrm{sc}(v,w) \mid \{v,w\} \subseteq S_n\} \;=\; \max\{\mathrm{sc}(Q) \mid Q \subseteq S_n\} \;=\; \left\lfloor \frac{n^2}{4} \right\rfloor .$$

Remark 2.21. The last assertion of Proposition 2.18 furnishes a count of the number of irreducible components of an intersection of two determinantal-type Schubert varieties. The number ν defined there, can be easily estimated. Observe that $s \geq \nu+p-1$, $t \geq \nu + p - 1$ and hence $2\nu + p - 2 \leq s + t - p$. Also,

$$(s' + t' - p') \;\geq\; \nu + (s + t - p) \;\geq\; 3\nu + p - 2$$

and since (s', t', p') is in B_n, we have $s' + t' - p' \leq n - 1$. It follows that the number of irreducible components is at most $1 + \lfloor n/3 \rfloor$. If n is an integer ≥ 3, then letting $s = t = \lfloor n/3 \rfloor$, $p = 1$, $s' = t' = 2\lfloor n/3 \rfloor$, and $p' = 1 + \lfloor n/3 \rfloor$ we have exactly $1 + \lfloor n/3 \rfloor$ irreducible components. Thus the Schubert number for the determinantal-type pairs in S_n is seen to be $1 + \lfloor n/3 \rfloor$. Clearly this number does not exceed $\lfloor n^2/4 \rfloor$.

Chapter 3

Join-irreducible permutations

In this chapter we provide various characterizations of join-irreducible permutations [38, 48]. Join-irreducible elements of a poset are key to understanding the order-structure of the poset.

3.1 Definitions

Definition 3.1. Let P be a poset and let x be an element of P.

(1) x is said to be *join-irreducible in* P if whenever $x = \vee Q$ for a subset $Q \subseteq P$, we have $x \in Q$.

(2) Let $\mathrm{Jirr}(P)$ denote the set of all join-irredducible members of P.

(3) Define $D_j[x] := D[x] \cap \mathrm{Jirr}(P)$.

Remark 3.2. [48, Section 2]. If P has a unique minimal element $\widehat{0}$, then by convention $\widehat{0} = \vee \varnothing$ and $\widehat{0}$ is *not* a join-irreducible element of P. On the other hand, if P has more than one minimal element, then each of these is join-irreducible in P.

Figure 3.1 illustrates examples. The x's appearing in the leftmost and the rightmost diagrams are not join-irreducible whereas the x's appearing in the middle are join-irreducible.

Proposition 3.3. *Recall that $L(Q)$ denotes the MacNeille completion of the poset Q.*

(1) *A finite distributive lattice L is a graded poset whose rank equals $\#\mathrm{Jirr}(L)$ and whose canonical rank function j is given by $j(x) := \#D_j[x]$. In other words, $x \lhd y \iff D_j[y] = D_j[x] \cup \{z\}$ for some $z \in D_j[y] \setminus D_j[x]$. In particular, $j(\max L) = \#\mathrm{Jirr}(L)$.*

(2) *We have $\mathrm{Jirr}(P) = \mathrm{Jirr}(L(P))$ and $L(\mathrm{Jirr}(P)) = L(P)$.*

Proof. For proof of the first assertion see [3, Proposition 2.10]. For the second assertion, see [48, Section 6]. □

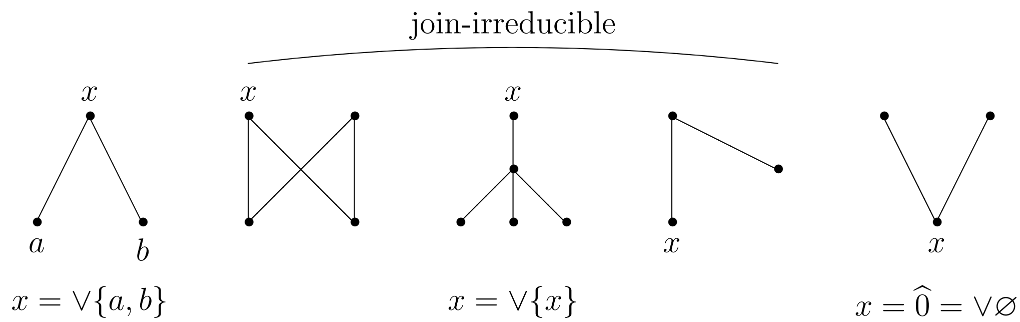


Figure 3.1: join-irreduciblity

Following [48, Section 8], we introduce the permutations J_{abc}.

Definition 3.4.

(1) Let $\mathcal{A}_n = \{(a,b,c) \in \mathbf{N}^3 \mid 1 \leq b \leq a \leq n-1 \text{ and } b+1 \leq c \leq n-a+b\}$.

(2) For $(a,b,c) \in \mathcal{A}_n$, define $J_{abc} \in S_n$ by

$$J_{abc}(i) = \begin{cases} i & \text{if } 1 \leq i \leq b-1, \\ i+c-b & \text{if } b \leq i \leq a, \\ i-a+b-1 & \text{if } a+1 \leq i \leq a-b+c, \\ i & \text{if } a-b+c+1 \leq i \leq n. \end{cases}$$

Table 3.1 shows $\mathrm{Jirr}(S_5)$.

Proposition 3.5. *Recall that w_0 is the maximal element of $(S_n, \leq)$.*

(1) $(J_{abc})^{-1} = J_{rst}$ *where* $(r,s,t) := (c-1,\ b,\ a+1)$.

(2) $w_0\, J_{abc}\, w_0 = J_{rst}$ *where* $(r,s,t) := (n-a,\ n-a+b-c+1,\ n+2-c)$.

(3) $\ell(J_{abc}) = (a-b+1)(c-b)$.

(4) *The correspondence* $(a,b,c) \longleftrightarrow J_{abc}$ *yields a bijection from* $\mathcal{A}_n$ *onto* $\mathrm{Jirr}(S_n)$.

(5) $\#\mathrm{Jirr}(S_n) = (n-1)n(n+1)/6$.

Proof. (1), (2) are straightforward to verify. To prove (3), note that $(i,j) \in I(J_{abc})$ if and only if $b \leq i \leq a$ and $a+1 \leq j \leq a-b+c$. Hence $\ell(J_{abc}) = \#I(J_{abc}) = (a-b+1)(c-b)$. For the proof of the remaining two assertions the reader is referred to [48, Section 8]. □

J_{abc}	one-line	canonical expression	ℓ	j
J_{112}	21345	s_1	1	1
J_{223}	13245	s_2	1	1
J_{334}	12435	s_3	1	1
J_{445}	12354	s_4	1	1
J_{113}	31245	s_2s_1	2	3
J_{212}	23145	s_1s_2	2	3
J_{224}	14235	s_3s_2	2	3
J_{323}	13425	s_2s_3	2	3
J_{335}	12534	s_4s_3	2	3
J_{434}	12453	s_3s_4	2	3
J_{114}	41235	$s_3s_2s_1$	3	6
J_{312}	23415	$s_1s_2s_3$	3	6
J_{225}	15234	$s_4s_3s_2$	3	6
J_{423}	13452	$s_2s_3s_4$	3	6
J_{213}	34125	$s_2s_1s_3s_2$	4	8
J_{324}	14523	$s_3s_2s_4s_3$	4	8
J_{115}	51234	$s_4s_3s_2s_1$	4	10
J_{412}	23451	$s_1s_2s_3s_4$	4	10
J_{214}	45123	$s_3s_2s_1s_4s_3s_2$	6	15
J_{313}	34512	$s_2s_1s_3s_2s_4s_3$	6	15

Table 3.1: $\mathrm{Jirr}(S_5)$

3.2 Extremal monotone triangles

Join-irreducible permutations can be nicely characterized in the poset of monotone triangles.

Lemma 3.6. *For $(a, b, c) \in \mathcal{A}_n$, we have $J_{abc} = \min\{y \in \mathrm{MT}(n) \mid y_{ab} \geq c\}$.*

Remark 3.7. Before proceeding to the formal proof, it is useful to see an example. Let $n = 5$. According to the lemma, J_{323} is the least among monotone triangle whose $(3, 2)$ entry is at least 3. Of course the $(3, 2)$ entry of (J_{323}) is indeed 3. Now we try to recover the monotone triangle J_{323} using only this property. First, put 3 at the $(3, 2)$ slot in δ_4.

$$\begin{array}{cccc}
__ & & & \\
__ & __ & & \\
__ & \boxed{3} & __ & \\
__ & __ & __ & __
\end{array}$$

Next, we put in other numbers in such a way that the third row stays as small as possible while keeping in view the requirements of constructing a monotone triangle.

So the $(3,1)$ entry should be $\min\{1,2\} = 1$, and the $(3,3)$ entry should be $\min\{4,5\} = 4$.

$$
\begin{array}{llll}
- & & & \\
- & - & & \\
1 & \boxed{3} & 4 & \\
- & - & - & -
\end{array}
$$

While constructing the second row, we should neglect the maximal entry of the third row (*i.e.*, 4). Thus the second row should be $1 < 3$.

$$
\begin{array}{llll}
- & & & \\
1 & 3 & & \\
1 & \boxed{3} & 4 & \\
- & - & - & -
\end{array}
$$

In contrast, in the construction of the fourth row we should use the least possible among the available numbers (*i.e.*, 2) to get

$$
\begin{array}{llll}
- & & & \\
1 & 3 & & \\
1 & \boxed{3} & 4 & \\
1 & 2 & 3 & 4
\end{array}
$$

Likewise, 1 should appear in the first row. Thus we end up with

$$
J_{323} = \begin{array}{llll}
1 & & & \\
1 & 3 & & \\
1 & \boxed{3} & 4 & \\
1 & 2 & 3 & 4
\end{array}
$$

as our monotone triangle.

Proof. Let $[m,\ n]$ be the set of positive integers $m \le i \le n$. As before, $[n]$ stands for $[1,\ n]$. Let $x = J_{abc}$. Regarding x as a monotone triangle, let x_{ij} denote the (i,j) entry of x. Note that

$$x(1) < x(2) < \cdots < x(b) = c < x(b+1) < \cdots < x(a).$$

Hence $x_{ab} = c$. Let $y \in \mathrm{MT}(n)$ be such that $y_{ab} \ge c$. We proceed to show that $x \le y$, *i.e.*, $x_{\alpha\beta} \le y_{\alpha\beta}$ for all (α,β). Suppose $y_{\alpha\beta} < x_{\alpha\beta}$ for some (α,β). Fix such a pair (α,β). Now we consider various cases.

(1) Suppose $1 \le \alpha \le b-1$. Then, $|x[\alpha]| = \{x(1) < x(2) < \cdots < x(\alpha)\} = [\alpha]$ and hence $x_{\alpha\beta} = \beta$. But then $\beta \le y_{\alpha\beta} < x_{\alpha\beta} = \beta$, a clear contradiction.

(2) Suppose $b \leq \alpha \leq a$. Then,

$$|x[\alpha]| \;=\; \{x(1) < x(2) < \cdots < x(\alpha)\} = [b-1] \cup [c,\; c+\alpha-b]$$

and hence

$$x_{\alpha\beta} \;=\; \begin{cases} \beta & \text{if } 1 \leq \beta \leq b-1, \\ c+\beta-b & \text{if } b \leq \beta \leq \alpha. \end{cases}$$

(a) If $1 \leq \beta \leq b-1$, then $\beta \leq y_{\alpha\beta} < x_{\alpha\beta} = \beta$, a contradiction.

(b) Since y is a monotone triangle, we have $y_{ab} \leq y_{\alpha b}$ and $y_{\alpha b} + \beta - b \leq y_{\alpha\beta}$. Thus, if $b \leq \beta \leq \alpha$, then

$$x_{\alpha\beta} = c + \beta - b \leq y_{ab} + \beta - b \leq y_{\alpha b} + \beta - b \leq y_{\alpha\beta} < x_{\alpha\beta},$$

a contradiction.

(3) Suppose $a+1 \leq \alpha \leq a-b+c$. Then $|x[\alpha]| = [\alpha - a + b - 1] \cup [c,\; a-b+c]$. Hence

$$x_{\alpha\beta} \;=\; \begin{cases} \beta & \text{if } 1 \leq \beta \leq \alpha-a+b-1, \\ c+\beta-(\alpha-a+b) & \text{if } \alpha-a+b \leq \beta \leq \alpha. \end{cases}$$

(a) If $1 \leq \beta \leq \alpha-a+b-1$, then we have $\beta \leq y_{\alpha\beta} < x_{\alpha\beta} = \beta$, a contradiction.

(b) If $\alpha - a + b \leq \beta \leq \alpha$, then

$$x_{\alpha\beta} \;=\; x_{\alpha,\alpha-a+b} + \beta - (\alpha-a+b) \;=\; c + \beta - (\alpha - a + b).$$

Consequently, we have

$$\begin{aligned} c + \beta - (\alpha-a+b) = x_{\alpha\beta} &> y_{\alpha\beta} \\ &\geq y_{\alpha,\alpha-a+b} + \beta - (\alpha-a+b) \\ &= c + \beta - (\alpha-a+b), \end{aligned}$$

a contradiction. Note that we must have $y_{\alpha,\alpha-a+b} \geq c$ otherwise, $c \leq y_{ab} \leq y_{\alpha,\alpha-a+b} < c$, an impossibility.

(4) Suppose $a-b+c+1 \leq \alpha \leq n$. Then $|x[\alpha]| = [\alpha]$ and hence $x_{\alpha t} = t$ for all $1 \leq t \leq \alpha$. We have $\beta \leq y_{\alpha\beta} < x_{\alpha\beta} = \beta$, a contradiction.

□

Corollary 3.8.

(1) *J_{abc} is join-irreducible.*

(2) $J_{abc} \in D_j[x] \iff x_{ab} \geq c$.

(3) *For all c, d in $[b+1,\; n-a+b]$, we have*

$$J_{abc} \;<\; J_{abd} \iff c \;<\; d.$$

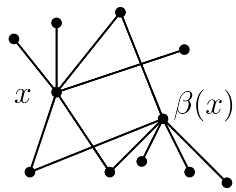


Figure 3.2: dissector pair

Proof.

(1) Let $J_{abc} = \vee Q$, say $Q = \{y_1, \ldots, y_m\} \neq \varnothing$. Then $c = (J_{abc})_{ab} = (\vee Q)_{ab} = \min\{(y_1)_{ab}, \ldots, (y_m)_{ab}\}$. Thus $(y_k)_{ab} = c$ for some k. The above lemma implies $J_{abc} = y_k \in Q$.

(2) Both statements are equivalent to $J_{abc} \leq x$.

(3) If $J_{abc} < J_{abd}$, then $c \leq (J_{abd})_{ab} = d.$. But if $c = d$, then $J_{abc} = J_{abd}$; a contradiction. Thus $c < d$. Conversely, if $c < d$, then $c < d = (J_{abd})_{ab}$ and hence $J_{abc} < J_{abd}$.

□

3.3 Dissector pairs

In the present section, we characterize $\mathrm{Jirr}(S_n)$ in yet another way. For this, we need to introduce new terminology which rarely appears in the literature (for example, [48]). Results of this section will be later used to establish a lower bound for the Schubert number.

Definition 3.9. Let x, y be elements of a poset P.

(1) An ordered pair (x, y) is called a *dissector pair* of P if $P = U[x] \cup D[y]$ with $U[x] \cap D[y] = \varnothing$ (Figure 3.2).

(2) An element x is called an *upper dissector* if there exists some $\beta(x) \in P$ such that $(x, \beta(x))$ is a dissector pair.

(3) Similarly y is called a *lower dissector* if there exists some $\alpha(y) \in P$ such that $(\alpha(y), y)$ is a dissector pair.

(4) We denote the set of all upper, lower dissectors of P by $\mathrm{Udis}(P)$, $\mathrm{Ldis}(P)$ (may be empty).

Proposition 3.10. $\mathrm{Udis}(P) \subseteq \mathrm{Jirr}(P)$.

Proof. If $\mathrm{Udis}(P) = \varnothing$, there is nothing to prove. Let $x \in \mathrm{Udis}(P)$. We have $P = U[x] \cup D[\beta(x)]$ with $U[x] \cap D[\beta(x)] = \varnothing$. Now suppose x is not join-irreducible. There exists $Q \subseteq P$ such that $x = \vee Q$, and $x \notin Q$. For all $y \in Q$, we have $y \leq x$ by the definition of join. But the equality is now impossible, so $y < x$ for all $y \in Q$. Thus $y \in D[\beta(x)]$ for all $y \in Q$ since $y \notin U[x]$. By the definition of join, x is the minimum of elements z such that $y \in D[z]$ for all $y \in Q$, we must have $x \leq \beta(x)$, *i.e.*, $x \in U[x] \cap D[\beta(x)]$, a contradiction. □

Lemma 3.11. $\mathrm{Udis}(S_n) = \mathrm{Jirr}(S_n)$.

Proof. See [38]. □

It follows that a join-irreducible permutation J_{abc} is also an upper dissector. It is natural to ask : what is $\beta(J_{abc})$?

Definition 3.12. For $(a, b, c) \in \mathcal{A}_n$, define

$$M_{abc}(i) = \begin{cases} n - i + 1 & \text{if } 1 \leq i \leq a - b, \\ c + a - b - i & \text{if } a - b + 1 \leq i \leq a, \\ n + b + 1 - i & \text{if } a + 1 \leq i \leq n + b - c + 1, \\ n - i + 1 & \text{if } n + b - c + 2 \leq i \leq n. \end{cases}$$

Theorem 3.13. *Let w_0 denote the maximum of the poset $(S_n, \leq)$. For $\tau \in S_n$ we have $\tau \in \mathrm{Jirr}(S_n)$ if and only if the Schubert variety X_α, where $\alpha := w_0\tau$, is of determinantal-type.*

Proof is an easy consequence of the following lemma.

Lemma 3.14. *For all $(a, b, c) \in \mathcal{A}_n$ and $(s, t, q) \in B_n$, we have*

(1) $w_0 J_{abc} = M_{a,a-b+1,n+2-c}$.

(2) $M_{abc} = \pi(n - c + 1 | a, a - b + 1)$.

(3) $\pi(s|t, q) = M_{t,t-q+1,n-s+1}$.

Proof. Left to the reader. □

Proposition 3.15.

(1) $M_{abc} = \max\{y \in \mathrm{MT}(n) \mid y_{ab} \leq c - 1\}$.

(2) *M_{abc} is meet-irreducible.*

(3) *For c, d in $[b + 1,\ n - a + b]$, we have $M_{abc} < M_{abd} \iff c < d$.*

Proof. Entirely similar to the proof of Lemma 3.6 and 3.8. □

Proposition 3.16. *For each $(a, b, c) \in \mathcal{A}_n$, the pair (J_{abc}, M_{abc}) is a dissector pair of the poset S_n and also of the poset $L(S_n)$.*

Proof. Let $x \in S_n$ and $(a, b, c) \in \mathcal{A}_n$. Then, either $x_{ab} \geq c$ or $x_{ab} \leq c-1$. Equivalently, $x \in U[J_{abc}] \cup D[M_{abc}]$ and the union is disjoint. Hence (J_{abc}, M_{abc}) is a dissector pair of S_n. Exactly the same holds if we replace S_n by $L(S_n)$. □

Remarks 3.17.

(1) Join-irreducilbe permutations play an important role in the strong Bruhat order on Coxeter groups other than S_n [11].

(2) There are other characterizations of join-irreducible permutations such as : bi-grassmannians, rectangular canonical expressions, reduced words without braid relations *etc* [48].

(3) It is well-known that for each non-minimum element in a distributive lattice, there exists a unique irredundant join-decomposition. In [21] the author establishes an algorithm to construct such a decomposition in the case of $L(S_n)$. This decomposition can be used to extend the poset-automorphisms of S_n to those of $L(S_n)$. For example, we can define x^{-1} for all $x \in L(S_n)$ in such a way that Theorem 4.4(1) holds for all $x \in L(S_n)$ (see [25]). Such automorphisms help us discover several combinatorial, algebraic and enumerative properties of the poset $\mathrm{Jirr}(S_n)$ (for example, $\mathrm{Jirr}(S_n)$ is a graded poset). Details can be found in [25].

(4) Using join-irreducible elements it is possible to construct an interval in the poset S_n which gives the maximum number of atoms [23].

(5) There is also a notion of a join-"prime" element. Call x a *join-prime* if whenever $x \leq \vee Q$ for some subset of the poset (under consideration), then there exists $y \in Q$ such that $x \leq y$. A join-prime element is necessarily join-irreducible. The converse also holds if the poset is a distributive lattice [3, p.42].

Chapter 4

Schubert Numbers

As mentioned in Proposition 1.39, $\mathrm{MT}(n) \cong L(S_n)$ is a finite distributive lattice and hence possesses a canonical rank function. In section 4.1, we establish some properties of this rank function. In sections 4.2 and 4.3, we develop formulas to compute the canonical rank function in the case of S_n. Finally, in section 4.4 we prove the theorem mentioned at the end of the second chapter, which provides a lower bound for the maximum of Schubert numbers.

4.1 A rank function for $L(S_n)$

Definition 4.1. For $x \in L(S_n)$, define

$$\Sigma(x) = \sum_{a=1}^{n-1} \sum_{b=1}^{a} x_{ab},$$
$$j(x) = \#D_j[x].$$

Proposition 4.2. *Let* id *denote the identity permutation and w_0 stand for the maximum of $(S_n, \leq)$.*

(1) *We have $j(\mathrm{id}) = 0$ and $j(w_0) = (n-1)n(n+1)/6$.*

(2) *If a, b in $[n-1]$ are such that $b \leq a$, then there exists a chain of join-irreducible elements:*

$$J_{a,b,b+1} < J_{a,b,b+2} < \cdots < J_{a,b,n-a+b}.$$

Consequently, for $x \in L(S_n)$,

$$\underbrace{J_{a,b,b+1}, J_{a,b,b+2}, \ldots, J_{abx_{ab}}}_{x_{ab}-b} \in D_j[x].$$

and $\#\{c \mid J_{abc} \in D_j[x]\} = x_{ab} - b$.

(3) *For $x \in L(S_n)$, we have $j(x) = \Sigma(x) - \Sigma(\mathrm{id})$.*

Proof.

(1) This follows from the fact that $\mathrm{id} < J_{abc} < w_0$ for all $J_{abc} \in \mathrm{Jirr}(S_n)$ (Proposition 1.22, 26 and Remark 3.5(5)).

(2) Follows from Corollary 3.8(3).

(3) Recall that $\mathrm{id}_{ab} = b$ for all a, b (Remark 1.36). Thus

$$\begin{aligned} j(x) &= \#D_j[x] \\ &= \sum_{a=1}^{n-1} \sum_{b=1}^{a} \#\{c \mid J_{abc} \in D_j[x]\} \\ &= \sum_{a=1}^{n-1} \sum_{b=1}^{a} (x_{ab} - b) \\ &= \sum_{a=1}^{n-1} \sum_{b=1}^{a} x_{ab} - \sum_{a=1}^{n-1} \sum_{b=1}^{a} \mathrm{id}_{ab} \;=\; \Sigma(x) - \Sigma(\mathrm{id}). \end{aligned}$$

$\square$

Example 4.3. Let $x = \begin{pmatrix} 3 & & & \\ 2 & 4 & & \\ 2 & 4 & 5 & \\ 1 & 2 & 4 & 5 \end{pmatrix}$. Then

$$\begin{aligned} j(x) &= \Sigma \begin{pmatrix} 3 & & & \\ 2 & 4 & & \\ 2 & 4 & 5 & \\ 1 & 2 & 4 & 5 \end{pmatrix} - \Sigma \begin{pmatrix} 1 & & & \\ 1 & 2 & & \\ 1 & 2 & 3 & \\ 1 & 2 & 3 & 4 \end{pmatrix} \\ &= \Sigma \begin{pmatrix} 2 & & & \\ 1 & 2 & & \\ 1 & 2 & 2 & \\ 0 & 0 & 1 & 1 \end{pmatrix} = 12. \end{aligned}$$

4.2 Rank Function for Permutations

In the last section, we saw a formula for $j(x)$ in the case of a monotone triangle x. In particular, when x is a permutation, this formula becomes somewhat simpler on account of the additional property that $|x[a]| \subset |x[a+1]|$ for all $a \in [n-1]$ (Definition 1.31(8)).

Theorem 4.4. *Let $x \in S_n$. Then the following holds.*

(1) $j(x) = j(x^{-1})$.

(2)

$$\Sigma(x) \;=\; \sum_{a=1}^{n-1} x(a)(n-a).$$

(3)

$$j(x) \;=\; \sum_{a=1}^{n-1}(x(a)-a)(n-a) \;=\; \frac{1}{2}\sum_{a=1}^{n}(x(a)-a)^2.$$

Proof.

(1) Using Proposition 1.26 and Corollary 3.8(2) we see that

$$J_{abc} \in D_j[x] \iff J_{abc} \leq x \iff {J_{abc}}^{-1} \leq x^{-1} \iff {J_{abc}}^{-1} \in D_j[x^{-1}].$$

(2) Since $|x[a]| \subset |x[a+1]|$ for $1 \leq a \leq n-1$, $x(a)$ appears $(n-a)$ times in the monotone triangle x.

(3) The first equality is obtained by observing that

$$\begin{aligned}\sum_{a=1}^{n-1}(x(a)-a)(n-a) &= \sum_{a=1}^{n-1} x(a)(n-a) - \sum_{a=1}^{n-1} a(n-a)\\ &= \;\Sigma(x) - \Sigma(\mathrm{id}) \;=\; j(x).\end{aligned}$$

To establish the second equality first note that,

$$\begin{aligned}\sum_{a=1}^{n-1}(x(a)-a)(n-a) &= \;\sum_{a=1}^{n}(x(a)-a)(n-a)\\ &= \;n\sum_{a=1}^{n} x(a) - \sum_{a=1}^{n} ax(a) - n\sum_{a=1}^{n} a + \sum_{a=1}^{n} a^2\\ &= \;\sum_{a=1}^{n} a^2 - \sum_{a=1}^{n} ax(a),\end{aligned}$$

and secondly note that

$$\begin{aligned}\frac{1}{2}\sum_{a=1}^{n}(x(a)-a)^2 &= \;\frac{1}{2}\sum_{a=1}^{n}(x(a)^2 - 2ax(a) + a^2)\\ &= \;\sum_{a=1}^{n} a^2 - \sum_{k=1}^{n} ax(a).\end{aligned}$$

Since $\sum_{a=1}^{n} x(a)^2 = \sum_{a=1}^{n} a^2$, our assertion is proved.

□

Example 4.5. Let $x = 42513$. Then

$$j(x) = (x(1)-1)4 + (x(2)-2)3 + (x(3)-3)2 + (x(4)-4)1 \; = \; 13,$$
$$j(x) = \frac{1}{2}[(x(1)-1)^2 + (x(2)-2)^2 + (x(3)-3)^2 + (x(4)-4)^2 + (x(5)-5)^2] \; = \; 13.$$

Next lemma explores the change in $j(x)$ when we pass to a reduction of x in S_n.

Lemma 4.6. *Let $x \in S_n$, t_{ik} a transposition with $i < k$ and $w = xt_{ik}$. Then, we have*

$$j(x) \; = \; j(w) + (k-i)(x(i) - x(k)).$$

Proof. Note that $w(m) = x(m)$ for all $m \neq i, k$. Also, $w(i) = x(k)$ and $w(k) = x(i)$. Applying the formula proved in Theorem 4.4(3) to w and x we get

$$\begin{aligned}
j(x) - j(w) = \; & \sum_{a=1}^{n-1}(x(a)-a)(n-a) - \sum_{a=1}^{n-1}(w(a)-a)(n-a) \\
= \; & \sum_{a=1}^{n}(x(a)-a)(n-a) - \sum_{a=1}^{n}(w(a)-a)(n-a) \\
= \; & \sum_{a=1}^{n}(x(a)-w(a))(n-a) \\
= \; & (x(i)-w(i))(n-i) + (x(k)-w(k))(n-k) \\
= \; & (x(i)-x(k))(n-i) + (x(k)-x(i))(n-k) \\
= \; & (k-i)(x(i)-x(k)).
\end{aligned}$$

□

Corollary 4.7. *In particular, if $k = i+1$, i.e., $t_{ik} = s_i$, then*

$$j(x) \; = \; j(xs_i) + x(i) - x(i+1).$$

4.3 Rank and inversions

In this section, we establish the exact relation between $j(x)$ and the inversions of our permutation x.

Theorem 4.8. *For $x \in S_n,$, we have*

$$j(x) \; = \sum_{(i,k)\,\in\, I(x)} (x(i) - x(k))$$

where $I(x)$ denotes the set of inversions of x.

Proof. Our proof is by induction on $\ell(x)$. If $\ell(x) = 0$, then $x = \mathrm{id}$ and hence $j(x) = 0$ (Proposition 4.2(1)) as well as $I(x) = \emptyset$. Henceforth assume $\ell(x) > 0$. Then, there exists an index a such that $(a, a+1) \in I(x)$ (Proposition 1.6). So $x(a+1) < x(a)$. Setting $w := xs_a$, we have $\ell(w) = \ell(x) - 1$ (Proposition 1.12) and $(a, a+1) \notin I(w)$ since

$$w(a) = (xs_a)(a) = x(a+1) < x(a) = (xs_a)(a+1) = w(a+1).$$

For $y \in S_n$, define

$$\begin{aligned}
I_1(y) &:= \{(i, a) \in I(y) \mid 1 \le i \le a-1\},\\
I_2(y) &:= \{(i, a+1) \in I(y) \mid 1 \le i \le a-1\},\\
I_3(y) &:= \{(a, k) \in I(y) \mid a+2 \le k \le n\},\\
I_4(y) &:= \{(a+1, k) \in I(y) \mid a+2 \le k \le n\},\\
I_5(y) &:= \{(i, k) \in I(y) \mid i, k \notin \{a, a+1\}\}.
\end{aligned}$$

Clearly

$$\begin{aligned}
I(w) &= \bigcup_{p=1}^{5} I_p(w) \quad \text{and}\\
I(x) &= \bigcup_{p=1}^{5} I_p(x) \cup \{(a, a+1)\}.
\end{aligned}$$

where the unions are disjoint. Observe that for all $i \in [a-1]$, we have

$$(i, a) \in I_1(w) \iff w(a) < w(i) \iff x(a+1) < x(i) \iff (i, a+1) \in I_2(x).$$

Therefore

$$\sum_{(i,\,a)\,\in\, I_1(w)} (w(i) - w(a)) \;=\; \sum_{(i,\,a+1)\,\in\, I_2(x)} (x(i) - x(a+1)).$$

Similarly,

$$\begin{aligned}
(i, a+1) \in I_2(w) &\iff (i, a) \in I_1(x),\\
(a, k) \in I_3(w) &\iff (a+1, k) \in I_4(x),\\
(a+1, k) \in I_4(w) &\iff (a, k) \in I_3(x),\\
(i, k) \in I_5(w) &\iff (i, k) \in I_5(x).
\end{aligned}$$

Since $\ell(w) = \ell(x) - 1$,, the induction hypothesis implies

$$j(w) \;=\; \sum_{(i,\,k) \in I(w)} (w(i) - w(k)).$$

Now thanks to Corollary 4.7, we conclude that

$$\begin{aligned} j(x) &= j(w) + (x(a) - x(a+1)) \\ &= \sum_{(i,k)\,\in\, I(w)} (w(i) - w(k)) + (x(a) - x(a+1)) \\ &= \sum_{p=1}^{5} \sum_{(i,k)\,\in\, I_p(x)} (x(i) - x(k)) + (x(a) - x(a+1)) \\ &= \sum_{(i,k)\,\in\, I(x)} (x(i) - x(k)). \end{aligned}$$

□

Corollary 4.9.

(1) *For $x \in S_n$, we have*

$$j(x) = \sum_{(i,k)\,\in\, I(x)} (k - i).$$

(2) $j(J_{abc}) \;=\; (a - b + 1)(c - b)(a - 2b + c + 1)/2.$

Proof. (1) We have

$$\begin{aligned} j(x) &= j(x^{-1}) && \text{(Theorem 4.4(1))} \\ &= \sum_{(i',\,k') \in I(x^{-1})} \{x^{-1}(i') - x^{-1}(k')\} && \text{(Apply the above theorem to } x^{-1}) \\ &= \sum_{(x(k),\, x(i))\,\in\, I(x^{-1})} (k - i) && (\text{let } i = x^{-1}(k'), k = x^{-1}(i')) \\ &= \sum_{(i,k)\,\in\, I(x)} (k - i). && \text{(Definition 1.7(3))} \end{aligned}$$

(2) Observe that if $(i, k) \in I(J_{abc})$, then $b \le i \le a$ and $a+1 \le k \le a-b+c$. Then,

$$\begin{aligned} j(J_{abc}) &= \sum_{k=a+1}^{a-b+c} \sum_{i=b}^{a} [J_{abc}(i) - J_{abc}(k)] \\ &= \sum_{k=a+1}^{a-b+c} \sum_{i=b}^{a} [(i + c - b) - (k - a + b - 1)] \\ &= \frac{1}{2}(a - b + 1)(c - b)(a - 2b + c + 1). \end{aligned}$$

□

Example 4.10.

(1) Let $x = 42513$. Then $I(x) = \{(1,2),(1,4),(1,5),(2,4),(3,4),(3,5)\}$. We have

$$\begin{aligned} j(x) &= \sum_{(i,k)\,\in\, I(x)} (x(i) - x(k)) \\ &= x(1) - x(2) + x(1) - x(4) + x(1) - x(5) + x(2) - x(4) + x(3) - x(4) \\ &\quad + x(3) - x(5) \\ &= 2 + 3 + 1 + 1 + 4 + 2 \\ &= 13. \end{aligned}$$

(2) $j(J_{313}) \;=\; (3-1+1)(3-1)(3-2+3+1)/2 \;=\; 15.$

4.4 A lower bound for $\max\{\mathrm{sc}(Q) \mid Q \subseteq S_n\}$

In this section, we establish a lower bound for the the maximum of the Schubert numbers of subsets of S_n. At the end we state a related conjecture. We begin with a lemma. Recall that $\lfloor n^2/4 \rfloor$ is the maximum of $\{\ell(v) \mid v \in \mathrm{Jirr}(S_n)\}$.

Lemma 4.11. *Let $x \in \mathrm{Jirr}(S_n)$. Then*

(1) $d_-(x) \;=\; \ell(x) \;=\; \mathrm{sc}(x, \beta(x))$.

(2) *Let $k = \lfloor n/2 \rfloor$. Then $d_-(J_{k,1,n-k+1}) \;=\; \lfloor n^2/4 \rfloor$.*

Proof.

(1) Note that

$$d_-(x) \;=\; \#\{t \in T \mid xt \lhd x\} \;\le\; \#\{t \in T \mid xt < x\} \;=\; \ell(x).$$

Let $t := t_{ik}$ where $i < k$ and suppose $xt_{ik} < x$. Since x is join-irreducible, there does not exist j such that $x(i) > x(j) > x(k)$. Thanks to Proposition 1.12, we have $xt \lhd x$ (in S_n). Hence $d_-(x) = \ell(x)$. Let $C(x) := \{w \in S_n \mid w \lhd x\}$. Since $(x, \beta(x))$ is a dissector pair of $S_n = U[x] \cup D[\beta(x)]$ and $x \parallel \beta(x)$, given $w \in C(x)$ we have $w \le \beta(x)$. Since

$$\Gamma(x \wedge \beta(x)) \;=\; \mathrm{Max}(D[x \wedge \beta(x)] \cap S_n) \;=\; C(x),$$

we infer that $\mathrm{sc}(x, \beta(x)) = \#\Gamma(x \wedge \beta(x)) = \#C(x) = \ell(x)$. This establishes our first assertion (Figure 4.1).

(2) Using Proposition 3.5(3) we get

$$d_-(J_{k,1,n-k+1}) \;=\; \ell(J_{k,1,n-k+1}) \;=\; (n-k)k \;=\; \left\lfloor n^2/4 \right\rfloor.$$

$\square$

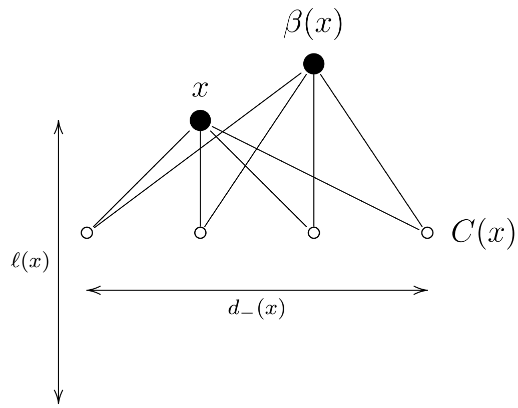


Figure 4.1: Schubert numbers and dissector pairs

Example 4.12. If $n = 4$, then $J_{213} = 3412$, $\beta(J_{213}) = 4231$, and $C(3412) = \{1432, 2413, 3142, 3214\}$. Hence in FL(4), the intersection

$$X_{3412} \cap X_{4231} = X_{1432} \cup X_{2413} \cup X_{3142} \cup X_{3214}$$

has 4 irreducible components with $\ell(3412) = d_-(3412) = 4$, the maximum of the Schubert numbers for S_4.
If $n = 5$, then $J_{313} = 34512, \beta(J_{313}) = 54231$. We have

$$X_{34512} \cap X_{54231} \;=\; X_{14532} \cup X_{24513} \cup X_{31542} \cup X_{32514} \cup X_{34152} \cup X_{34215}$$

with $\ell(34512) = d_-(34512) = 6$.

Theorem 4.13. *We have*

$$\max\{\mathrm{sc}(Q) \mid Q \subseteq S_n\} \;\geq\; \left\lfloor \frac{n^2}{4} \right\rfloor .$$

Proof: It suffices to exhibit a pair of permutations in S_n for which the corresponding Schubert number is $\lfloor n^2/4 \rfloor$. In view of the above lemma, we do have

$$\mathrm{sc}(J_{k,1,n-k+1},\ \beta(J_{k,1,n-k+1})) \;=\; \ell(J_{k,1,n-k+1}) \;=\; \left\lfloor n^2/4 \right\rfloor . \square$$

Conjecture 4.14. (1) $\max\{\mathrm{sc}(x, y) \mid x, y \in S_n\} = \lfloor n^2/4 \rfloor$.

(2) *Moreover, we have* $\mathrm{sc}(x,\ y) = \lfloor n^2/4 \rfloor$ *if and only if*

$$(x, y) \in \{(J_{k,1,n-k+1},\ \beta(J_{k,1,n-k+1})),\ ({J_{k,1,n-k+1}}^{-1},\ \beta({J_{k,1,n-k+1}}^{-1}))\}.$$

Summary and Future Directions

(1) We hope to gain more understanding of the poset structure of monotone triangles with the objective of converting our conjecture about the Schubert numbers into a theorem.

(2) It is possible to extend some of our results to other Coxeter groups. In particular for type B Coxeter groups, the join-irreducible elements can also be expressed in terms of signed monotone triangles [47, Section 4.9].

(3) There is a bijective correspondence between monotone triangles and alternating sign matrices [45]. What corresponds to join-irreducible permutations and the function j in the language of alternating sign matrices? This question seems to be interesting and approachable.

(4) Here is another characterization of join-irreducilbe permutations:

$$x \in \mathrm{Jirr}(S_n) \iff x \text{ has a "rectangular" canonical word.}$$

There is a bijective order-preserving correspondence between determinantal-type Schubert varieties and join-irreducible permutations [24].

(5) Most of the determinantal-type Schubert varieties are singular. It remains to discover the exact relationship between these singularities and their associated Kazhdan-Lusztig polynomials [14].

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Vita

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