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Small and Large Scale Limits of Multifractal Stochastic Processes with Applications

Jennifer Laurie Sinclair
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Jennifer Laurie Sinclair entitled "Small and Large Scale Limits of Multifractal Stochastic Processes with Applications." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Jan Rosinski, Major Professor

We have read this dissertation and recommend its acceptance:

William Wade, Suzanne Lenhart, Diego del-Castillo-Negrete

Accepted for the Council:

Carolyn R. Hodges

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Small and large scale limits of multifractal stochastic processes with applications

A Dissertation
Presented for the
Doctor of Philosophy
Degree
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Jennifer Laurie Sinclair
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Abstract

Various classes of multifractal processes, that is processes that display different properties at different scales, are studied. Most of the processes examined in this work exhibit stable trends at small scales and Gaussian trends at large scales, although the opposite can also occur. Many natural phenomena exhibit a fractal structure depending on some scaling factor, such as space or time. Thus, these types of processes have many useful modeling applications, including Biology and Economics. First, generalized tempered stable processes are defined and studied, following the original work on tempered stable processes by Jan Rosinski [16]. Generalized tempered stable processes encompass the modern variations on tempered stable distributions that have been introduced in the field, including "Modified tempered stable distributions [10]," "Layered stable distributions [8]," and "Lamperti stable processes [2]." This work shows generalized tempered stable processes exhibit multifractal properties at different scales in the space of cadlag functions equipped with the Skorokhod topology and investigates other properties, such as series representations and absolute continuity. Next, processes driven by generalized tempered stable processes involving a certain Volterra kernel are defined and short and long term behavior is established, following the work of Houdré and Kawai [7]. Finally, inspired by the work of Pipiras and Taqqu [13], the multifractal behavior of more general infinitely divisible processes is established, based on the Lévy-Itô representation of infinitely divisible processes. Numerous examples are given throughout the entire text to exemplify the strong presence of processes of this type in current literature.

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Chapter 1

Introduction

This work is in the field of probability and involves modeling multifractal structures that appear to have a normal distribution over large scales and a highly variable distribution over small scales. Many natural phenomena exhibit a fractal structure that varies depending on some scaling factor, such as space or time. For example, the edges of a cloud seem smooth from a distance, but can be jagged if inspected closely. A wave hitting the sand may appear to have a smooth structure if viewed from the air, but displays sharp, irregular patterns if viewed from the beach. Internet traffic can display completely different patterns over short periods of time than over long periods of time. A stock price can have dramatic changes by day, but will tend to have a normal structure over years. Fractals are also helpful in modeling flight turbulence, which exhibits sharp movements over small scales, and smooth movements over large scales. Multifractal Lévy processes can be used to model some phenomena of this type.

1.1 Background Information

"Stochastic processes are mathematical models of time evolution of random phenomena [19]." Lévy processes are an important subclass of stochastic processes. Roughly speaking, they are stochastic processes that have independent and stationary increments. Basic examples of Lévy processes are Brownian motions and Poisson processes. Assume the existence of a probability space $(\Omega, \mathcal{F}, P)$ on which all of the stochastic processes and random variables throughout are defined. A process, $\{X_t : t \geq 0\}$, has independent increments if for any $n \geq 1$ and any sequence of times, $0 \leq t_0 < t_1 < t_2 < \ldots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent. And, a process has stationary increments if the distribution of $X_{s+t} - X_s$ does not depend on s. Below is a precise definition of a Lévy process.

Definition 1.1.1. [19] A stochastic process, $\{X_t : t \geq 0\}$, on $\mathbb{R}^d$ is called a Lévy process if the following conditions hold:

1. The process has independent increments
2. The process has stationary increments
3. $X_0 = 0$ a.s.
4. The process is stochastically continuous, that is for every $t \geq 0$ and $\epsilon > 0$,

$$\lim_{s \to t} P[|X_s - X_t| > \epsilon] = 0$$

5. There is an $\Omega_0 \in \mathcal{F}$ with $P[\Omega_0] = 1$ such that for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

A Lévy process, $\{X_t : t \geq 0\}$, has the basic property that for every t, the distribution of X_t is infinitely divisible.

Definition 1.1.2. [19] A probability measure μ on $\mathbb{R}^d$ is infinitely divisible if for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^n$.

In other words, a probability measure is infinitely divisible if it is possible to take an n-th root of its characteristic function. Gaussian, Poisson, and stable distributions are examples of infinitely divisible distributions, but uniform and binomial distributions are not infinitely divisible.

There is an important known relationship between infinitely divisible distributions and Lévy processes in law ("Lévy process in law" means that condition (5) is omitted). As was mentioned before, if $\{X_t : t \geq 0\}$ is a Lévy process in law, then for any $t \geq 0$, the distribution of X_t is infinitely divisible. But a converse statement is also true. That is, if μ is an infinitely divisible distribution, then there exists a Lévy process in law such that the distribution of X_1 equals μ (see Theorem 7.10 in [19]).

Another fundamental property of Lévy processes is that fact that, because of their independent and stationary increments, they are characterized by their distributions at time 1. That is if $\{X_t : t \geq 0\}$ and $\{X_t' : t \geq 0\}$ are Lévy processes in law such that the distribution of X_1 equals the distribution of X_1', then $\{X_t : t \geq 0\}$ and $\{X_t' : t \geq 0\}$ are identical in law (meaning that their finite dimensional distributions are the same). This result is also found in Theorem 7.10 of [19].

The Lévy-Khintchine formula gives a representation of the characteristic functions of all infinitely divisible distributions. It states that if $\mathcal{L}(X)$ is infinitely divisible, then its characteristic function is given by

$$E\exp\{i\langle u, X\rangle\}$$
$$= \exp\left\{-\frac{1}{2}\langle u, Au\rangle + i\langle \gamma, u\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_{\|x\|\le 1}(x)\right)\nu(dx)\right\}$$

where $u, \gamma \in \mathbb{R}^d$, A is a symmetric nonnegative-definite d x d matrix, and ν is a measure satisfying

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}^d}(\|x\|^2 \wedge 1)\nu(dx) < \infty$$

where throughout this paper, define $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

The parameter γ is called the drift, A is called the Gaussian covariance matrix, and ν is called the Lévy measure. The generating triplet, (A, ν, γ) uniquely characterizes the distribution. Moreover, note that if $\{X_t : t \ge 0\}$ is a Lévy process and $\mathcal{L}(X_1)$ has generating triplet (A, ν, γ), then for any t, $\mathcal{L}(X_t)$ has generating triplet $(tA, t\nu, t\gamma)$. This follows from the fact that Lévy processes are characterized by their distributions at time 1 (See Theorem 8.1 and Corollary 8.3 of [19]). If $\gamma = 0$, there is no "drift part." If $A = \mathbf{0}$, "there is no Gaussian part." If $\nu = 0$, there is no "jump component."

This paper is primarily concerned with Lévy process with no Gaussian part, that is, the processes that are characterized only by the parameters γ and ν. The parameter ν can be most helpful in many applications varying from moment properties to convergence criteria. For example,

$$\int_{\|x\|>1}\|x\|\nu(dx) < \infty$$

if and only if X_t has finite first moment for any t and

$$\int_{\|x\|>1}\|x\|^2\nu(dx) < \infty$$

if and only if X_t has finite second moment for any t (see Theorem 25.3 of [19]).

A Lévy process is a specific example of an infinitely divisible random measure, which will be defined at this time. Let $\mathcal{S}$ be a set and $\mathcal{S}_0$ be a σ-ring of subsets of $\mathcal{S}$.

Definition 1.1.3. [17] A stochastic process $M = \{M(A)\}_{A \in \mathcal{S}_0}$ is called an infinitely divisible random measure (IDRM) if

1. $M(\emptyset) = 0$ a.s.
2. For every $\{A_i\} \in \mathcal{S}_0$, $\{M(A_i)\}$ forms a sequence of independent random variables and if $\bigcup_{i=1}^{n} A_i \in \mathcal{S}_0$, then

$$M(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} M(A_i) \quad \text{a.s.}$$

3. For every $A \in \mathcal{S}_0$, $M(A)$ has in infinitely divisible distribution.

In particular, if the infinitely divisible distribution is a Poisson distribution, then M is a Poisson random measure. And, if the infinitely divisible distribution is a Gaussian distribution, then M is a Gaussian random measure. Also, a Lévy process can be viewed as an IDRM where the indexing sets have the form $A = (0, t]$ so that $X_t = M((0, t])$. See Rosinski's text [17] for a more through commentary on IDRM's.

Definition 1.1.4. A process $\{X_t : t \in \mathbb{R}\}$ is an infinitely divisible stochastic process if for any sequence of times, $t_1, \ldots, t_k \in \mathbb{R}$,

$$(X_{t_1}, \ldots, X_{t_k})$$

has an infinitely divisible distribution.

The following is a powerful theorem that involves representing infinitely divisible processes as stochastic integrals of deterministic functions with respect to Poisson random measures.

Theorem 1.1.5 (Generalized Lévy-Itô representation [17])**.** *Let $\{X_t : t \in \mathbb{R}\}$ be a stochastically continuous infinitely divisible process with no Gaussian or drift component. Let N be a Poisson random measure on a measurable space $(S, \mathcal{S})$ with intensity η. Then,*

$$X_t^{'} = \int_S f_t(s) \left(N(ds) - \frac{\eta(ds)}{1 \vee \|f_t(s)\|} \right)$$

is a version of $\{X_t : t \in \mathbb{R}\}$ where $\{f_t(s)\}$ are measurable deterministic functions in $\mathbb{R}^d$.

Corollary 1.1.6 (Lévy-Itô representation [17]). *Let $\{X_t : t \in \mathbb{R}\}$ be a Lévy process with no Gaussian or drift component. Let N be a Poisson random measure on a measurable space $(S, \mathcal{S})$ with intensity η. Then,*

$$S = \mathbb{R} \times \mathbb{R}^d, \quad \eta(du, dw) = du\, \nu(dw),$$

and

$$f_t(u, w) = 1_{[0,t)}(u)w$$

so

$$X_t' = \int_{\mathbb{R}\times\mathbb{R}^d} 1_{[0,t)}(u)w \left(N(ds) - \frac{du\,\nu(dw)}{1 \vee \|w\| 1_{[0,t)}(u)} \right)$$

is a version of $\{X_t : t \in \mathbb{R}\}$ where ν is the Lévy measure of X_1.

Stochastic integrals of this type will be investigated later in this paper. Now, we turn our focus to notions of convergence of stochastic processes. This turns out to much more simple for Lévy processes because of their independent and stationary increments. The following theorem relates convergence of Lévy processes in the space of cadlag functions with convergence of the marginals at time 1.

Theorem 1.1.7 (Skorohod (Theorem 15.17 of [9])). *Let $X, X^1, X^2, \ldots$ be Lévy processes in $\mathbb{R}^d$ with $X_1^n \xrightarrow{d} X_1$. Then, there exist some processes Y^n such that $X^n = Y^n$ in distribution and $sup_{s \leq t}|Y_s^n - X_s^n| \xrightarrow{P} 0$ for all $t \geq 0$.*

This means that in the special case of Lévy processes, in order to check the convergence in the space of cadlag (right continuous with left limits existing) functions equipped with the Skorokhod topology, one only needs to verify the convergence of the marginals at time 1. Note that convergence in this space makes sense for Lévy process since one can take a version with cadlag paths by condition (5) of the definition of a Lévy process.

However, when the processes in question are not Lévy process, more work is needed to investigate convergence. Suppose X and X^n are random processes. Denote convergence of the finite-dimensional distributions as "$\xrightarrow{fdd}$." Thus, $X^n \xrightarrow{fdd} X$ means that

$$(X_{t_1}^n, \ldots, X_{t_k}^n) \xrightarrow{d} (X_{t_1}, \ldots, X_{t_k}) \qquad t_1, \ldots, t_k \in \mathbb{R}$$

where "$\xrightarrow{d}$" denotes convergence in distribution.

Now, for the process to convergence in the space of continuous functions, it first must have continuous paths. Next, a criteria is presented that can be checked to determine if a process has continuous paths.

Theorem 1.1.8 (Kolmogorov [9]). *Let X be a process on $\mathbb{R}^d$ with values in a complete metric space (S, ρ) and assume for some $a, b > 0$ that*

$$E\{\rho(X_s, X_t)\}^a \leq C|s-t|^{d+b} \qquad s, t \in \mathbb{R}^d$$

for some finite constant C. Then, X has a continuous version, and for any $c \in (0, b/a)$ the latter is a.s. locally Hölder continuous with exponent c.

Now, let $C[0,1]$ be the space of continuous functions on $[0,1]$ with the metric given by

$$\rho(x, y) := \sup_{0 \leq t \leq 1} |x(t) - y(t)|.$$

To have weak convergence in this space, it is not enough to have convergence of the finite dimensional distributions (see Example 2.5 of Billingsley [1] for a counterexample). But, as is mentioned in Chapter 2 of Billingsley [1], "it does in the presence of relative compactness." And, tightness implies relative compactness by Prohorov's Theorem. So, in summary, to show weak convergence in the space of continuous functions, one needs to prove the convergence of the finite dimensional distributions and the tightness of the sequence. Here is a useful tool to verify if a sequence is tight.

Theorem 1.1.9 ([9]). *Let $X^1, X^2, X^3, \ldots$ be continuous processes on $\mathbb{R}^d$ with values in a separable, complete metric space (S, ρ). Assume that $\{X_0^n\}$ is tight in S and that for some constants $a, b > 0$*

$$E\{\rho(X_s^n, X_t^n)\}^a \leq C|s-t|^{d+b} \qquad s, t \in \mathbb{R}^d,\ n \in \mathbb{N}$$

uniformly in n. Then, $\{X^n\}$ is tight in $C(\mathbb{R}^d, S)$, and for every $c \in (0, b/a)$ the limiting processes are a.s. locally Hölder continuous with exponent c.

Obtaining this upper bound is often more simple than proving tightness directly, which makes this theorem quite useful in practice.

Series representations are another important tool used to investigate properties of stochastic processes, as they lead to simulations of the processes. The idea is to take well known random variables, such as gamma or uniform, that are easy to simulate. Then, take certain combinations of them to approximate the new process in question. Rosinski's "Series representations of Lévy processes from the perspective of point processes [15]" provides a useful and highly recognized general theory on series representations of Lévy processes. The main theorem of the paper is described here.

Let $\{V_i\}$ be a sequence of random elements in a measurable space S with a common distribution. Let $\{\Gamma_i\}$ be a sequence of partial sums of exponential (1) random variables. Assume $\{V_i\}$ and $\{\Gamma_i\}$ are independent and let $H : (0, \infty) \times S \to \mathbb{R}^d$ be a measurable function that is nonincreasing in r. Also, let

$$A(s) := \int_0^s E(H(r, v))1_{|H(r,v)| \le 1} dr.$$

Theorem 1.1.10 ([15]). $\sum_{i=1}^{\infty} H(\Gamma_i, V_i)$ *converges a.s. if and only if*

(*i*)

$$\int_0^{\infty} E\left(|H(r, V)|^2 \wedge 1\right) dr < \infty$$

(*ii*)

$$a := \lim_{s \to \infty} A(s) \quad \textit{exists.}$$

Moreover, $\sum_{i=1}^{\infty} H(\Gamma_i, V_i)$ *has an infinitely divisible distribution with generating triplet* $(0, \nu, a)$ *where*

$$\nu(B) = \int_0^{\infty} P(H(r, V) \in B \setminus \{0\}) dr.$$

This theorem proves the convergence of the sequence, but you must first know the proper H function to use to get the desired infinitely divisible random variable with Lévy measure ν. Methods to generate H are also described in [15]. Examples include LePage's method and the rejection method. A series representation may still be obtained even if condition (*ii*) fails. In this case, one simply needs to introduce the correct centers to ensure proper convergence. After introducing the centers, if necessary, the results extend easily to Lévy processes on $[0, T]$ (see Theorem 5.1 of [15]).

This collection of background information relies heavily on known results and is not intended to be exhaustive. The reader is encouraged to refer to Sato's text [19] for extra information on Lévy processes and infinitely divisible distributions and to refer to Billingsley's text [1] for insight into convergence results for probability measures. Also, Kallenberg [9] has written an excellent reference on topics in modern probability theory.

1.2 Motivation

Multifractal processes have been given much attention recently. Tempered stable processes (TS processes) are a subclass of multifractal Lévy processes which have been heavily referenced in Mathematics, Biology, and Economics. The processes are obtained by tilting the

Lévy measure associated with the process. The idea of tilting the Lévy measure corresponds to the statistical notion of tilting a density. This is described in Rosinski's paper on tempered stable processes [16] and is summarized in the next few paragraphs.

Recall that if X is an infinitely divisible random variable with no Gaussian or drift term, then its characteristic function is given by the Lévy-Khintchine formula:

$$Ee^{i\langle u,X\rangle} = \exp\left\{\int_{\mathbb{R}^d}\left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle 1_{\|x\|\leq 1}(x)\right)\nu(dx)\right\}$$

where ν is a Lévy measure satisfying:

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}^d}(\|x\|^2 \wedge 1)\nu(dx) < \infty.$$

Suppose X is infinitely divisible on $\mathbb{R}$ and has a density, f. Also, suppose that X is nice enough such that it can be expressed with the Laplace transform:

$$L(\lambda) := Ee^{-\lambda x} = \int_0^\infty e^{-\lambda x} f(x)dx = \exp\left\{\int_{\mathbb{R}^d}\left(e^{-\lambda x} - 1\right)\nu(dx)\right\}.$$

Now, consider a tilted density given by:

$$f_\theta(x) := \frac{1}{L(\theta)}e^{-\theta x}f(x) \quad \text{for} \quad \theta > 0.$$

Then, its Laplace transform is given by:

$$\begin{aligned}
L_\theta(\lambda) = &\int_0^\infty e^{-\lambda x} f_\theta(x)dx = \frac{1}{L(\theta)}\int_0^\infty e^{-\lambda x}e^{-\theta x}f(x)dx \\
=&\frac{1}{L(\theta)}\int_0^\infty e^{-(\lambda+\theta)x}f(x)dx = \frac{1}{L(\theta)}L_(\lambda+\theta) \\
=&\exp\left\{-\int_{\mathbb{R}^d}\left(e^{-\theta x} - 1\right)\nu(dx)\right\}\exp\left\{\int_{\mathbb{R}^d}\left(e^{-(\lambda+\theta)x} - 1\right)\nu(dx)\right\} \\
=&\exp\left\{-\int_{\mathbb{R}^d}\left(e^{-\theta x} - 1\right)\nu(dx) + \int_{\mathbb{R}^d}\left(e^{-(\lambda+\theta)x} - 1\right)\nu(dx)\right\} \\
=&\exp\left\{\int_{\mathbb{R}^d}\left(e^{-\lambda x} - 1\right)e^{-\theta x}\nu(dx)\right\}.
\end{aligned}$$

So, it can be observed that tilting a density is analogous to tilting the Lévy measure in the infinitely divisible case.

This idea of tilting the Lévy measure can be extended to Lévy processes, which have independent and stationary increments and infinitely divisible laws. For example, in stochastic finance, the CGMY [3] model is obtained by tilting the Lévy measure associated with stable distributions (which exhibits infinite activity, but does not account for small jumps) and has been used to model assets that exhibit both small and large jumps. The CGMY model is the Lévy process, $\{X_t^{CGMY}\}$ with Lévy measure given by

$$\nu(dx) = k(x)dx$$

where

$$k(x) = \begin{cases} Ce^{-G|x|}|x|^{-1-Y} & \text{if} \quad x < 0 \\ \\ Ce^{-M|x|}|x|^{-1-Y} & \text{if} \quad x > 0 \end{cases}$$

The parameter $C > 0$ measures the overall activity and can provide control over kurtosis. $G \geq 0$ and $M \geq 0$ control the rate of the exponential decay of the left and right tails, respectively (skewness). And, Y, which in order for $\{X_t^{CGMY}\}$ to remain a Lévy process, must satisfy $Y < 2$, is used to describe the fine structure of the process where activity refers to the number of jumps in any given time interval. $Y < 0$ implies finite activity, $0 < Y < 1$ implies infinite activity, but finite variance, and $1 < Y < 2$ implies infinite activity and infinite variance. And remember, this is simply one example of an application of TS processes.

As was mentioned earlier, there are many variations of TS processes, which has motivated this work to provide a generalization of TS processes. And, following the paths of others, the next logical step is to study processes driven by this generalized TS processes. However, this family of TS like processes are not the only processes that exhibit short and long term behavior. There is a more general class of infinitely divisible processes that exemplify multifractal short and long term behavior. That is, they tend to an infinite variance limit as the scaling factor, β, goes to 0, and they tend to a Gaussian limit as β tends to ∞, which will be examined in this work.

1.3 Overview

Chapter 2 is devoted to multidimensional GTS Lévy processes. This involves uniting all of the variations of TS processes, thus creating a generalized tempered stable (GTS) process and proving it displays multifractal properties at different scales in the space of cadlag functions equipped with the Skorokhod topology. Conditions under which the process is absolutely continuous with respect to the underlying stable process are of great interest, as are the series representations that lead to simulations of the process.

As an analog of fractional Brownian motion, Houdré and Kawai have introduced "Fractional tempered stable motion" [7] in their 2006 paper by taking the integral of a Volterra kernel, $K_{H,\alpha}(t, s)$, with respect to a TS process:

$$X_t := \int_0^t K_{H,\alpha}(t, s) dX_s^{TS}, \quad t \geq 0.$$

They showed that in short time, it is close to fractional stable Levy motion, but in the long term is similar to fractional Brownian motion. In chapter 3, the short and long term behavior of the processes driven by a GTS process are studied, which characterizes the class of processes driven by the various TS modifications, that would previously have to be done on a case by case basis.

Chapter 4 involves the generalization of a previous work by Pipiras and Taqqu [13] in 2006. In this paper, the authors examine processes with finite first moment given by the integral representations:

$$X_t := \int_S f_t(s) N(ds), \quad t \in \mathbb{R},$$

where N is a compensated Poisson random measure on a measurable space S. They describe general conditions for the normalized and time-scaled process to converge to a limit. In chapter 4 we remove the moment condition imposed on the original process and study the short and long term behavior. Without the moment condition, every infinitely divisible process without Gaussian component can be represented as a stochastic integral of this type. Moreover, the results are enlarged to the multidimensional case. Thus, our work expands the scope of applications of this limit theory to include all classes of multidimensional infinitely divisible processes without Gaussian part.

Chapter 2

Generalized tempered stable processes

Tempered stable processes (TS processes) were originally used as models in physics [12] and and mathematical finance [3]. TS processes were studied rigorously by Koponen [11] and Rosinski [16]. Recently, many new variations on TS distributions have been introduced in the field, including "Modified tempered stable distributions [10]," "Layered stable distributions [8]," and "Lamperti stable processes [2]." Each variation displays a different method of tilting the stable distribution, but all exemplify multifractal short and long term behavior. That is, they tend to an infinite variance limit as the scaling factor, β, goes to 0, and they tend to a Gaussian limit as $\beta \to \infty$. All of the modifications enjoy computable characteristics and moment generating functions. This chapter involves uniting all of these variations, thus creating a generalized tempered stable (GTS) process and proving it displays multifractal properties at different scales in the space of cadlag functions equipped with the Skorokhod topology. Conditions under which the process is absolutely continuous with respect to the underlying stable process are given, as are the series representations that lead to simulations of the process.

2.1 Preliminaries

Definition 2.1.1. A Lévy process is called a generalized tempered stable process if its Lévy measure at time 1 is given in polar coordinates as

$$\nu^{GTS}(B) = \int_{S^{d-1}} \int_0^\infty 1_B(r\xi)\, q(r,\xi)\, r^{-\alpha-1}\, dr\, \sigma(d\xi) \tag{2.1}$$

where B is a Borel set in $\mathbb{R}_0^d$, $\alpha \in (0,2)$, σ is a finite measure on S^{d-1}, and q is a measurable function from $(0,\infty) \times S^{d-1}$ to $(0,\infty)$ such that:

$$\lim_{r\to 0} q(r,\cdot) = c_1(\cdot) \qquad \text{in} \quad L^1(S^{d-1},\sigma).$$

Note: We write ν_σ^α to denote an α-stable Lévy measure with spectral measure σ. Let $\{X_t^\alpha\}_{t\geq 0} \sim S_\alpha(\sigma,\eta)$ denote a d-dimensional stable process with Lévy measure ν_σ^α at time 1 and drift η. And, let $\{X_t^{GTS}\}_{t\geq 0} \sim GTS_\alpha(\sigma,q,\eta)$ denote a d-dimensional generalized tempered stable process with Lévy measure ν^{GTS} at time 1 and drift η. Recall that "$\xrightarrow{d}$" denotes convergence in the space of cadlag functions equipped with the Skorokhod topology. Also, let

$$\sigma_1(d\xi) := c_1(\xi)\sigma(d\xi).$$

It is well known that the characteristic function of X_1^α is given by

$$E\exp\{i\langle u, X_1^\alpha\rangle\} = \begin{cases} \exp\left\{\int_{\mathbb{R}_0^d}\left(e^{i\langle u,x\rangle}-1\right)\nu_\sigma^\alpha(dx)\right\} & if \quad \alpha\in(0,1) \\ \exp\left\{\int_{\mathbb{R}_0^d}\left(e^{i\langle u,x\rangle}-1-i\langle u,x\rangle 1_{\|x\|\leq 1}(x)\right)\nu_\sigma^1(dx)\right\} & if \quad \alpha=1 \\ \exp\left\{\int_{\mathbb{R}_0^d}\left(e^{i\langle u,x\rangle}-1-i\langle u,x\rangle\right)\nu_\sigma^\alpha(dx)\right\} & if \quad \alpha\in(1,2). \end{cases}$$

The characteristic function of X_1^{GTS} is given by

$$E\exp\{i\langle u, X_1^{GTS}\rangle\} = \exp\left\{\int_{\mathbb{R}_0^d}\left(e^{i\langle u,x\rangle}-1-i\langle u,x\rangle 1_{\|x\|\leq 1}(x)\right)\nu^{GTS}(dx)\right\}.$$

2.2 Examples

Example 2.2.1. (*Tempered Stable Processes*) A Lévy process is called a Tempered stable process if its Lévy measure at time 1 is given in polar coordinates as equation 2.1 where B is a Borel set in $\mathbb{R}_0^d$, $\alpha\in(0,2)$, σ is a finite measure on S^{d-1}, and q is a measurable function from $(0,\infty)\times S^{d-1}$ to $(0,\infty)$ such that $q(\cdot,\xi)$ is completely monotone and $q(\infty,\xi)=0$ for each $\xi\in S^{d-1}$. It is called proper if, in addition, $q(0+,\xi)=1$ for each $\xi\in S^{d-1}$.

Complete monotonicity means that $(-1)^n\frac{d^n}{dr^n}q(r,\xi) > 0$ for all $r>0$, $\xi\in S^{d-1}$, and $n=0,1,2,\ldots$. In particular, $q(\cdot,\xi)$ is strictly decreasing and convex. So, for example, $q(r,\xi)=e^{-r}$ is the q-function of a Tempered stable process.

Example 2.2.2. (*Modified Tempered Stable (MTS) Distribution*) An infinitely divisible distribution is called a Modified tempered stable distribution if its Lévy measure is given in polar coordinates as equation 2.1 where B is a Borel set in $\mathbb{R}_0^d$, $\alpha \in (0,2)$, σ is a finite measure on S^{d-1}, and q is a measurable function from $(0,\infty) \times S^{d-1}$ to $(0,\infty)$ such that

$$q(r,\xi) = \begin{cases} 2^{\frac{1-\alpha}{2}} \left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1} (\lambda_+ r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ r) & \text{if} \quad \xi = 1 \\ 2^{\frac{1-\alpha}{2}} \left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1} (\lambda_- r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- r) & \text{if} \quad \xi = -1 \end{cases}$$

where $\lambda_+, \lambda_- > 0$, $\alpha < 2$, and K_p is the modified Bessel function of the second kind with an integral representation given by

$$K_p(x) = \frac{1}{2}\left(\frac{x}{2}\right)^p \int_0^\infty e^{-t-\frac{x^2}{4t}} t^{-p-1} dt.$$

Example 2.2.3. (*Lamperti Stable Processes*) A Lévy process is called a Lamperti stable process if its Lévy measure at time 1 is given in polar coordinates as equation 2.1 where B is a Borel set in $\mathbb{R}_0^d$, $\alpha \in (0,2)$, σ is a finite measure on S^{d-1}, and q is a measurable function from $(0,\infty) \times S^{d-1}$ to $(0,\infty)$ such that

$$q(r,\xi) = \frac{e^{rf(\xi)}}{(e^r - 1)^{\alpha+1}} r^{\alpha+1}$$

where $f : S^{d-1} \to \mathbb{R}$ is such that $\sup_{\xi \in S^{d-1}} f(\xi) < \alpha + 1$.

2.3 Long and short term behavior

In this section, the short and long term behavior of a GTS is studied. In small scales, the process exhibits stable trends, while it exhibits Gaussian trends in large scales.

Theorem 2.3.1. *Consider a generalized tempered stable process with no drift term, $X_t^{GTS} \sim GTS_\alpha(\sigma, q, 0)$. And, let*

$$n_\alpha := \begin{cases} -\int_{\{\|x\| \le 1\}} x\,\nu^{GTS}(dx) & if \quad \alpha \in (0,1) \\ 0 & if \quad \alpha = 1 \\ \int_{\{\|x\| > 1\}} x\,\nu^{GTS}(dx) & if \quad \alpha \in (1,2) \end{cases}$$

Then,

Case(i): (Small Scales) *as* $h \to 0$,

$$h^{-\frac{1}{\alpha}}\{X_{ht}^{GTS} - ht\, n_\alpha\}_{t\geq 0} \xrightarrow{d} \{X_t^\alpha\}_{t\geq 0}$$

where $\{X_t^\alpha\}_{t\geq 0} \sim S_\alpha(\sigma_1, 0)$ *is a stable process of index* α.

Case(ii): (Large Scales) *assuming additionally that*

$$\int_{\{\|x\|>1\}} \|x\|^2\, \nu^{GTS}(dx) < \infty,$$

it follows that as $h \to \infty$,

$$h^{-\frac{1}{2}}\{X_{ht}^{GTS} - htn_\alpha\} \xrightarrow{d} \{B_t\}_{t\geq 0}$$

where $\{B_t\}_{t\geq 0}$ *is a centered Brownian motion with covariance matrix*

$$\int_{\mathbb{R}_0^d} xx'\nu^{GTS}(dx).$$

Remark 2.3.2. The assumptions on the behavior of the q function near zero are inspired by the behavior of the q function associated with layered stable processes. The assumptions made in this paper appear weaker, however, there is a slight error in the assumptions made in Theorem 3.1 [8] and a counterexample to is described after a brief review of layered stable processes.

Now, A Lévy process is called a layered stable process if its Lévy measure at time 1 is given by

$$\nu^{LAY}(B) = \int_{S^{d-1}} \int_0^\infty 1_B(r\xi) q(r,\xi) r^{-\alpha-1} dr \sigma(d\xi)$$

where B is a Borel set in $\mathbb{R}_0^d$ and σ is a finite positive measure on S^{d-1} and q is a locally integrable function from $(0,\infty) \times S^{d-1}$ to $(0,\infty)$ such that:

$$q(r,\xi) \sim c_1(\xi) \qquad as\ r \to 0$$

and

$$r^{\beta-\alpha}q(r,\xi) \sim c_2(\xi) \qquad as\ r \to \infty$$

for σ-almost every $\xi \in S^{d-1}$ where c_1 and c_2 are positive integrable with respect to σ functions on S^{d-1} and $\alpha \in (0,2)$ and $\beta \in (0,\infty)$. And, where $f(x) \sim g(x)$ as $x_0 \to x_0$ means that $f(x)/g(x) \to 1$ as $x_0 \to x_0 \in [-\infty,\infty]$.

Now, for σ-almost every $\xi \in S^{d-1}$, it is given that

$$q(r,\xi) \sim c_1(\xi) \qquad as\ r \to 0$$

which is equivalent to

$$\lim_{r\to 0} \frac{q(r,\xi)}{c_1(\xi)} = 1.$$

And, this is equivalent to

$$\lim_{r\to 0} q(r,\xi) = c_1(\xi)$$

for σ-almost every $\xi \in S^{d-1}$.

So, for the counterexample, take $d = 2$ so $S^1 = \{\xi = e^{i\theta} : 0 \leq \theta < 2\pi\}$ is a representation of all the points on the unit circle. And, for any set $B \in S^1$ of the form: $\{\xi = e^{i\theta} : 0 \leq a \leq \theta \leq b < 2\pi\}$, let $\sigma(B) = b - a$. Now, let

$$q(r,\xi) := \begin{cases} \frac{1}{r^2} & \text{if} \quad \theta \in (0,r] \\ \\ 1 & \text{else} \end{cases}$$

So,

$$\lim_{r\to 0} q(r,\xi) = c_1(\xi)$$

for σ-almost every $\xi \in S^{d-1}$ where $c_1(\xi) := 1$ and q is locally integrable. But, in case (a) in the proof of the small scales result, it is necessary to have:

$$\int_{\mathbb{R}_0^d} g(x)\, h(T_{h^{-\frac{1}{\alpha}}}\nu^{GTS})(dx) \to \int_{\mathbb{R}_0^d} g(x)\, \nu_{\sigma_1}^{\alpha}(dx) \qquad \text{as} \quad h \to 0.$$

But, taking $g(x) := 1_{[\epsilon,\infty)}(x)$, one can see that

$$\lim_{h\to 0}\int_{\mathbb{R}_0^d} g(x)\, h(T_{h^{-\frac{1}{\alpha}}}\nu^{GTS})(dx) = \lim_{h\to 0}\; h\int_{S^{d-1}}\int_{\epsilon}^{\infty} \; q(h^{\frac{1}{\alpha}}r,\xi)\,(h^{\frac{1}{\alpha}}r)^{-1-\alpha}\, d(h^{\frac{1}{\alpha}}r)\,\sigma(d\xi)$$

$$= \lim_{h\to 0}\int_{S^{d-1}}\int_{\epsilon}^{\infty} \; q(h^{\frac{1}{\alpha}}r,\xi)\, r^{-1-\alpha}\, dr\,\sigma(d\xi)$$

$$= \lim_{h\to 0}\int_{0}^{2\pi}\int_{\epsilon}^{\infty} \frac{1}{(h^{\frac{1}{\alpha}}r)^2} 1_{(0,h^{\frac{1}{\alpha}}r]}(\theta) r^{-\alpha-1}\, dr\, d\theta$$

which by simple calculation equals:

$$\lim_{h\to 0}\; \frac{1}{h^{\frac{1}{\alpha}}}\int_{\epsilon}^{\infty} r^{-2-\alpha} dr = \infty.$$

Thus, an application of Lebesgue Dominated Convergence Theorem in the proof of layered stable is not valid. This is precisely why convergence in L^1 for small scales is assumed for the q function associated with generalized tempered stable processes. Now, for the proof of Theorem 1.

2.4 Proof of the main theorem

For convenience, define a transformation of a measure ρ by $(T_r\rho)(B) = \rho(r^{-1}B)$ for any $r > 0$ and each Borel set B.

Case(i): Small Scales ($h \to 0$).

By a theorem of Skorokhod (see Theorem 15.17 of Kallenberg) we only need to check the weak convergence of the marginals at time 1. Now, $h^{-\frac{1}{\alpha}}(X_h^{GTS} - h\, n_\alpha)$ is an infinitely divisible random variable with cumulant function given by:

$$\int_{\mathbb{R}_0^d}\left(e^{i\left\langle u, h^{-\frac{1}{\alpha}}x\right\rangle} - 1 - i\left\langle u, h^{-\frac{1}{\alpha}}x\right\rangle 1_{\|x\|\le 1}(x)\right) h\,\nu^{GTS}(dx) - i\left\langle u, h^{1-\frac{1}{\alpha}}\, n_\alpha\right\rangle$$

$$= \int_{\mathbb{R}_0^d}\left(e^{i\left\langle u, (h^{-\frac{1}{\alpha}}x)\right\rangle} - 1 - i\left\langle u, (h^{-\frac{1}{\alpha}}x)\right\rangle 1_{\|h^{-\frac{1}{\alpha}}x\|\le 1}(h^{-\frac{1}{\alpha}}x)\right) h\,\nu^{GTS}(dx) - i\left\langle u, h^{1-\frac{1}{\alpha}}\, n_\alpha\right\rangle$$

$$= \int_{\mathbb{R}_0^d}\left(e^{i\langle u,x\rangle} - 1 - i\,\langle u,x\rangle\, 1_{\|x\|\le 1}(x)\right) h(T_{h^{-\frac{1}{\alpha}}}\nu^{GTS})(dx) - i\left\langle u, h^{1-\frac{1}{\alpha}}\, n_\alpha\right\rangle.$$

Thus, $h^{-\frac{1}{\alpha}}(X_h^{GTS} - h\,n_\alpha)$ is an infinitely divisible random variable with Levy measure $h(T_{h^{-\frac{1}{\alpha}}}\nu^{GTS})$, drift $h^{1-\frac{1}{\alpha}}\,n_\alpha$, and no Gaussian part. Now, Theorem 15.14 of Kallenberg states that we only need to check that for some $k \in (0,1)$, each of the following holds:

$$\text{a)}\quad h(T_{h^{-\frac{1}{\alpha}}}\nu^{GTS}) \text{ converges vaguely towards } \nu^{\alpha}_{\sigma_1} \text{ on } \bar{\mathrm{R}}^d_0 \quad \text{as } h \to 0$$

$$\text{b)}\quad a_h^k := \int_{\|x\|\le k} xx'\,h(T_{h^{-\frac{1}{\alpha}}}\nu^{GTS})(dx) \to \int_{\|x\|\le k} xx'\,\nu^{\alpha}_{\sigma_1}(dx) \quad \text{as } h \to 0$$

$$\text{c)}\quad b_h^k := -\int_{k\le\|x\|\le 1} x\,h(T_{h^{-\frac{1}{\alpha}}}\nu^{GTS})(dx) \to -\int_{k\le\|x\|\le 1} x\,\nu^{\alpha}_{\sigma_1}(dx) \quad \text{as } h \to 0.$$

For case (a), to show the vague convergence of measures, it is necessary to show

$$\int_{\mathbb{R}^d_0} g(x)\,h(T_{h^{-\frac{1}{\alpha}}}\nu^{GTS})(dx) \to \int_{\mathbb{R}^d_0} g(x)\,\nu^{\alpha}_{\sigma_1}(dx) \qquad \text{as} \quad h \to 0$$

where $g : \mathbb{R}^d_0 \to \mathbb{R}$ is any bounded continuous function vanishing in a neighborhood of the origin with $\lim_{n\to\infty} g(x_n)$ existing for every $\{x_n\}$ in $\mathbb{R}^d_0$ such that $\|x_n\| \to \infty$. Since g is bounded and vanishes on a neighborhood near the origin, there exists a C and an $\epsilon > 0$ such that $|g(x)| \le C < \infty$ for all x and $g(x) = 0$ on $\{x \in \mathbb{R}^d_0 : \|x\| \le \epsilon\}$. Now,

$$\int_{\mathbb{R}^d_0} g(x)\,h(T_{h^{-\frac{1}{\alpha}}}\nu^{GTS})(dx) = h\int_{S^{d-1}}\int_{\epsilon}^{\infty} g(r\xi)\ q(h^{\frac{1}{\alpha}}r,\xi)\,(h^{\frac{1}{\alpha}}r)^{-1-\alpha}\,d(h^{\frac{1}{\alpha}}r)\,\sigma(d\xi)$$

$$= \int_{S^{d-1}}\int_{\epsilon}^{\infty} g(r\xi)\ q(h^{\frac{1}{\alpha}}r,\xi)\,r^{-1-\alpha}\,dr\,\sigma(d\xi),$$

which by a change of variable equals

$$\int_{S^{d-1}}\int_{h^{\frac{1}{\alpha}}\epsilon}^{\infty} g(h^{-\frac{1}{\alpha}}r\xi)\ q(r,\xi)\,(h^{-\frac{1}{\alpha}}r)^{-1-\alpha}\,d(h^{-\frac{1}{\alpha}}r)\,\sigma(d\xi)$$

$$= h\int_{S^{d-1}}\int_{h^{\frac{1}{\alpha}}\epsilon}^{\infty} g(h^{-\frac{1}{\alpha}}r\xi)\ q(r,\xi)\,r^{-1-\alpha}\,dr\,\sigma(d\xi)$$

$$= h\int_{S^{d-1}}\int_{h^{\frac{1}{\alpha}}\epsilon}^{\epsilon} g(h^{-\frac{1}{\alpha}}r\xi)\ q(r,\xi)\,r^{-1-\alpha}\,dr\,\sigma(d\xi)$$

$$+\, h\int_{S^{d-1}}\int_{\epsilon}^{\infty} g(h^{-\frac{1}{\alpha}}r\xi)\ q(r,\xi)\,r^{-1-\alpha}\,dr\,\sigma(d\xi)$$

And, we will consider each integral separately. Now, notice, the second integral is bounded independently of h since,

$$\int_{S^{d-1}} \int_{\epsilon}^{\infty} |g(h^{-\frac{1}{\alpha}} r\xi)| \; q(r,\xi)\, r^{-1-\alpha}\, dr\, \sigma(d\xi)$$

$$\leq \int_{S^{d-1}} \int_{\epsilon}^{\infty} C \; q(r,\xi)\, r^{-1-\alpha}\, dr\, \sigma(d\xi)$$

$$= C \int_{\|x\|>\epsilon} \nu^{GTS}(dx)$$

which is finite since ν^{GTS} is a Lévy measure. Thus, the second integral,

$$h \int_{S^{d-1}} \int_{\epsilon}^{\infty} g(h^{-\frac{1}{\alpha}} r\xi) \; q(r,\xi)\, r^{-1-\alpha}\, dr\, \sigma(d\xi) \to 0 \qquad \text{as} \quad h \to 0.$$

So, the second integral vanishes with the passage to the limit. Now, consider the first integral,

$$h \int_{S^{d-1}} \int_{h^{\frac{1}{\alpha}}\epsilon}^{\epsilon} g(h^{-\frac{1}{\alpha}} r\xi) \; q(r,\xi)\, r^{-1-\alpha}\, dr\, \sigma(d\xi)$$

which, by change of variable, equals

$$h \int_{S^{d-1}} \int_{\epsilon}^{h^{-\frac{1}{\alpha}}\epsilon} g(r\xi) \; q(h^{\frac{1}{\alpha}} r,\xi)\, (h^{\frac{1}{\alpha}} r)^{-1-\alpha}\, d(h^{\frac{1}{\alpha}} r)\, \sigma(d\xi)$$

and simplifies to

$$\int_{S^{d-1}} \int_{\epsilon}^{\infty} g(r\xi) \; q(h^{\frac{1}{\alpha}} r,\xi)\, 1_{\{r \leq h^{-\frac{1}{\alpha}}\epsilon\}}(r) \; r^{-1-\alpha}\, dr\, \sigma(d\xi)$$

which, by Fubini, is equal to

$$\int_{\epsilon}^{\infty} \left\{ \int_{S^{d-1}} g(r\xi) \; q(h^{\frac{1}{\alpha}} r,\xi)\, 1_{\{r \leq h^{-\frac{1}{\alpha}}\epsilon\}}(r)\, \sigma(d\xi) \right\} \; r^{-1-\alpha}\, dr$$

$$:= \int_{\epsilon}^{\infty} Q_h(r) \; r^{-1-\alpha}\, dr.$$

Now, the goal is to use the Lebesgue Dominated Convergence Theorem to show the above integral converges to

$$\int_{\epsilon}^{\infty} \int_{S^{d-1}} g(r\xi)\, c_1(\xi) \; \sigma(d\xi) \; r^{-1-\alpha}\, dr$$

$$= \int_{S^{d-1}} \int_{\epsilon}^{\infty} g(r\xi)\ r^{-1-\alpha}\, dr\sigma_1(d\xi)$$

$$= \int_{\mathbb{R}_0^d} g(x) \nu_{\sigma_1}^{\alpha}(dx).$$

So, it is necessary to show the show that $Q_h(r)$ is bounded by an integrable function and converges pointwise to

$$\int_{S^{d-1}} g(r\xi)\, c_1(\xi)\ \sigma(d\xi).$$

So, first, it is necessary to show there exists a function $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$|Q_h(r)| \leq f(r) \qquad \text{for all } h > 0 \text{ and } r > 0$$

and

$$\int_{\epsilon}^{\infty} |f(r)|\ r^{-1-\alpha}\, dr < \infty.$$

To see this, notice that since

$$\int_{S^{d-1}} q(r,\xi)\, \sigma(d\xi) \to \int_{S^{d-1}} c_1(\xi)\, \sigma(d\xi) \qquad \text{as} \quad r \to 0,$$

there exists an r_0 such that:

$$\sup_{r \leq r_0} \int_{S^{d-1}} q(r,\xi)\, \sigma(d\xi) = C_1 < \infty.$$

Now, we may assume $r_0 \leq \epsilon$, so $r \leq \epsilon h^{-\frac{1}{\alpha}}$ implies that $h^{\frac{1}{\alpha}} r \leq \epsilon \leq r_0$. Hence,

$$\int_{S^{d-1}} q(r,\xi)\, 1_{\{r \leq h^{-\frac{1}{\alpha}}\epsilon\}}(r)\, \sigma(d\xi) = C_1 < \infty$$

and this implies

$$\left| \int_{S^{d-1}} g(r\xi)\, q(r,\xi)\, 1_{\{r \leq h^{-\frac{1}{\alpha}}\epsilon\}}(r)\ \sigma(d\xi) \right| \leq \int_{S^{d-1}} |g(r\xi)|\, q(r,\xi)\, 1_{\{r \leq h^{-\frac{1}{\alpha}}\epsilon\}}(r)\ \sigma(d\xi)$$

$$|g(r\xi)| \int_{S^{d-1}} q(r,\xi)\, 1_{\{r \leq h^{-\frac{1}{\alpha}}\epsilon\}}(r)\ \sigma(d\xi) \leq CC_1.$$

So, using $f(r) := CC_1$, it follows that:

$$\left| \int_{\epsilon}^{\infty} \left\{ \int_{S^{d-1}} g(r\xi)\ q(h^{\frac{1}{\alpha}} r, \xi)\, 1_{\{r \leq h^{-\frac{1}{\alpha}}\epsilon\}}(r)\, \sigma(d\xi) \right\}\ r^{-1-\alpha}\, dr \right|$$

$$\leq \int_{\epsilon}^{\infty} |f(r)| \, r^{-1-\alpha} \, dr = CC_1 \int_{\epsilon}^{\infty} \, r^{-1-\alpha} \, dr < \infty.$$

Thus, it is clear that Q is dominated by an integrable function. And, now it is necessary to show that for each $\xi \in S^{d-1}$,

$$Q_h(r) \to \int_{S^{d-1}} g(r\xi) \, c_1(\xi) \; \sigma(d\xi) \qquad \text{as} \quad h \to 0 \quad \text{for all } r > 0.$$

Well, $\lim_{r \to 0} q(r, \cdot) = c_1(\cdot)$ in $L^1(S^{d-1}, \sigma)$, implies that given any $\delta > 0$, we can choose r_0 such that:

$$\sup_{r \leq r_0} \int_{S^{d-1}} |q(r, \xi) - c_1(\xi)| \, \sigma(d\xi) < \delta.$$

And, for $h \in (0, 1)$, we have $h^{\frac{1}{\alpha}} r \leq h^{\frac{1}{\alpha}} r_0 \leq r_0$. So,

$$\int_{S^{d-1}} |q(h^{\frac{1}{\alpha}} r, \xi) - c_1(\xi)| \, 1_{\{r \leq h^{-\frac{1}{\alpha}} \epsilon\}}(r) \, \sigma(d\xi) < \delta.$$

Thus,

$$\int_{S^{d-1}} |g(r\xi) q(h^{\frac{1}{\alpha}} r, \xi) - g(r\xi) c_1(\xi)| \; 1_{\{r \leq h^{-\frac{1}{\alpha}} \epsilon\}}(r) \, \sigma(d\xi)$$

$$\leq C \int_{S^{d-1}} \; |q(h^{\frac{1}{\alpha}} r, \xi) - c_1(\xi)| \; 1_{\{r \leq h^{-\frac{1}{\alpha}} \epsilon\}}(r) \, \sigma(d\xi) \leq \frac{\delta}{C}$$

and since δ can be made arbitrarily small, it follows that $\lim_{h \to 0} g(r, \cdot) \; q(h^{\frac{1}{\alpha}} r, \cdot) = g(r, \cdot) \; c_1(\cdot)$ in $L^1(S^{d-1}, \sigma)$. So,

$$\int_{S^{d-1}} g(r\xi) \; q(h^{\frac{1}{\alpha}} r, \xi) \, \sigma(d\xi) \to \int_{S^{d-1}} g(r\xi) \; c_1(\xi) \, \sigma(d\xi) \qquad \text{as} \quad h \to 0$$

and, so,

$$Q_h(r) \to \int_{S^{d-1}} g(r\xi) \; c_1(\xi) \, \sigma(d\xi) = \int_{\mathbb{R}_0^d} g(x) \, \nu_{\sigma_1}^{\alpha}(dx) \qquad \text{as } h \to 0.$$

For case (b), notice

$$
\begin{aligned}
a_h^k &= \int_{\|x\|\leq k} xx'\, h(T_{h^{-\frac{1}{\alpha}}}\nu^{GTS})(dx) \\
&= h \int_{S^{d-1}} \int_0^k r^2\, \xi\xi'\; q(h^{\frac{1}{\alpha}}r, \xi)\, (h^{\frac{1}{\alpha}}r)^{-1-\alpha}\, d(h^{\frac{1}{\alpha}}r)\, \sigma(d\xi) \\
&= \int_{S^{d-1}} \int_0^k r^2\, \xi\xi'\; q(h^{\frac{1}{\alpha}}r, \xi)\, r^{-1-\alpha}\, dr\, \sigma(d\xi) \\
&= \int_0^\infty \left\{ \int_{S^{d-1}} \xi\xi'\; q(h^{\frac{1}{\alpha}}r, \xi)\, 1_{\{r\leq k\}}(r)\, \sigma(d\xi) \right\} r^{1-\alpha}\, dr \\
&:= \int_0^\infty Q_h(r)\, r^{1-\alpha}\, dr
\end{aligned}
$$

And, with intent to use the Lebesgue Dominated Convergence Theorem again, it must be shown that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$
|Q_h(r)| \leq f(r) \qquad \text{for all } h > 0 \text{ and } r > 0
$$

where

$$
\int_0^\infty |f(r)|\; r^{1-\alpha}\, dr < \infty.
$$

And,

$$
Q_h(r) \to \int_{S^{d-1}} \xi\xi'\, 1_{\{r\leq k\}}(r)\, c_1(\xi)\; \sigma(d\xi) \qquad \text{as} \quad h \to 0 \quad \text{for all } r > 0.
$$

Well, to see that $Q_h(r)$ is bounded by an integrable function, notice

$$
\begin{aligned}
|Q_h(r)| &= \left| \int_{S^{d-1}} \xi\xi'\; q(h^{\frac{1}{\alpha}}r, \xi)\, 1_{\{r\leq k\}}(r)\, \sigma(d\xi) \right| \\
&\leq \int_{S^{d-1}} \|\xi\xi'\|\; q(h^{\frac{1}{\alpha}}r, \xi)\, 1_{\{r\leq k\}}(r)\, \sigma(d\xi) \\
&= \int_{S^{d-1}} q(h^{\frac{1}{\alpha}}r, \xi)\, 1_{\{r\leq k\}}(r)\, \sigma(d\xi).
\end{aligned}
$$

And, as in case (a), there exists an r_0 such that:

$$
\sup_{r\leq r_0} \int_{S^{d-1}} q(r, \xi)\, \sigma(d\xi) = C_1 < \infty.
$$

Now, $r \le k$, implies $h^{\frac{1}{\alpha}} r \le h^{\frac{1}{\alpha}} k$, so, choose h small enough so that $h^{\frac{1}{\alpha}} k \le r_0$. Then, we have

$$|Q_h(r)| \le \int_{S^{d-1}} q(h^{\frac{1}{\alpha}} r, \xi)\, 1_{\{r \le k\}}(r)\, \sigma(d\xi)$$
$$\le C_1 1_{\{r \le k\}}(r) := f(r)$$

and

$$\int_0^\infty f(r)\, r^{1-\alpha}\, dr = \int_0^\infty C_1 1_{\{r \le k\}}(r)\, r^{1-\alpha}\, dr$$
$$= C_1 \int_0^k r^{1-\alpha}\, dr = \frac{k^{2-\alpha}}{2-\alpha} < \infty$$

since $\alpha \in (0, 2)$.

And, to see the pointwise convergence of $Q_h(r)$, recall $\lim_{r \to 0} q(r, \cdot) = c_1(\cdot)$ in $L^1(S^{d-1}, \sigma)$, which implies that given any $\delta > 0$, we can choose r_0 such that:

$$sup_{r \le r_0} \int_{S^{d-1}} |q(r, \xi) - c_1(\xi)|\, \sigma(d\xi) < \delta.$$

Now, choosing h small enough such that $h^{\frac{1}{\alpha}} k \le r_0$ implies

$$\int_{S^{d-1}} |q(h^{\frac{1}{\alpha}} r, \xi) - c_1(\xi)|\, 1_{\{r \le k\}}(r)\, \sigma(d\xi) < \delta.$$

So,

$$\int_{S^{d-1}} |\xi\xi' q(h^{\frac{1}{\alpha}} r, \xi) - \xi\xi' c_1(\xi)|\; 1_{\{r \le k\}}(r)\, \sigma(d\xi)$$
$$\le \int_{S^{d-1}} \|\xi\xi'\|\; |q(h^{\frac{1}{\alpha}} r, \xi) - c_1(\xi)|\; 1_{\{r \le k\}}(r)\, \sigma(d\xi)$$
$$= \int_{S^{d-1}} |q(h^{\frac{1}{\alpha}} r, \xi) - c_1(\xi)|\; 1_{\{r \le k\}}(r)\, \sigma(d\xi) < \delta$$

and since δ can be made arbitrarily small, it follows that $\lim_{h \to 0} f(\cdot)\; q(h^{\frac{1}{\alpha}} r, \cdot) = f(\cdot)\; c_1(\cdot)$ in $L^1(S^{d-1}, \sigma)$ where $f(\xi) = \xi\xi'$. So,

$$\int_{S^{d-1}} \xi\xi'\; q(h^{\frac{1}{\alpha}} r, \xi)\, \sigma(d\xi) \to \int_{S^{d-1}} \xi\xi'\; c_1(\xi)\, \sigma(d\xi) \qquad \text{as} \quad h \to 0$$

and so,

$$Q_h(r) \to \int_{S^{d-1}} \xi\xi'\; c_1(\xi)\, \sigma(d\xi) \qquad \text{as } h \to 0.$$

For case (c), ($\alpha = 1$ case only) we have:

$$\begin{aligned}
b_h^k =& \int_{k \le \|x\| \le 1} x\, h(T_{h^{-\frac{1}{\alpha}}} \nu^{GTS})(dx) \\
=& h \int_{S^{d-1}} \int_k^1 r\, \xi\ q(h^{\frac{1}{\alpha}} r, \xi)\, (h^{\frac{1}{\alpha}} r)^{-1-\alpha}\, d(h^{\frac{1}{\alpha}} r)\, \sigma(d\xi) \\
=& \int_{S^{d-1}} \int_k^1 r\, \xi\ q(h^{\frac{1}{\alpha}} r, \xi)\, r^{-1-\alpha}\, dr\, \sigma(d\xi) \\
=& \int_0^\infty \left\{ \int_{S^{d-1}} \xi\ q(h^{\frac{1}{\alpha}} r, \xi)\, 1_{\{k \le r \le 1\}}(r)\, \sigma(d\xi) \right\} r^{-\alpha}\, dr \\
:=& \int_0^\infty Q_h(r)\, r^{-\alpha}\, dr
\end{aligned}$$

And, again to apply the Lebesgue Dominated Convergence Theorem, it must be shown that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$|Q_h(r)| \le f(r) \qquad \text{for all } h > 0 \text{ and } r > 0$$

where

$$\int_0^\infty |f(r)|\, r^{1-\alpha}\, dr < \infty.$$

And,

$$Q_h(r) \to \int_{S^{d-1}} \xi \xi'\, 1_{\{r \le k\}}(r)\, c_1(\xi)\ \sigma(d\xi) \qquad \text{as} \quad h \to 0 \quad \text{for all } r > 0.$$

Now, to see the boundedness by an integrable function, notice

$$\begin{aligned}
|Q_h(r)| =& \left| \int_{S^{d-1}} \xi\ q(h^{\frac{1}{\alpha}} r, \xi)\, 1_{\{k \le r \le 1\}}(r)\, \sigma(d\xi) \right| \\
\le& \int_{S^{d-1}} \|\xi\|\ q(h^{\frac{1}{\alpha}} r, \xi)\, 1_{\{k \le r \le 1\}}(r)\, \sigma(d\xi) \\
=& \int_{S^{d-1}} q(h^{\frac{1}{\alpha}} r, \xi)\, 1_{\{k \le r \le 1\}}(r)\, \sigma(d\xi).
\end{aligned}$$

And, as in case (a), there exists an r_0 such that:

$$\sup_{r \le r_0} \int_{S^{d-1}} q(r, \xi)\, \sigma(d\xi) = C_1 < \infty.$$

Now, $r \leq 1$, implies $h^{\frac{1}{\alpha}} r \leq h^{\frac{1}{\alpha}}$, so for h small enough so that $h^{\frac{1}{\alpha}} \leq r_0$, we have

$$|Q_h(r)| \leq \int_{S^{d-1}} q(h^{\frac{1}{\alpha}} r, \xi)\, 1_{\{k \leq r \leq 1\}}(r)\, \sigma(d\xi)$$

$$\leq C_1 1_{\{k \leq r \leq 1\}}(r) := f(r)$$

and

$$\int_0^\infty f(r)\, r^{-\alpha}\, dr = \int_0^\infty C_1 1_{\{k \leq r \leq 1\}}(r)\, r^{-\alpha}\, dr$$

$$= C_1 \int_k^1 r^{-\alpha}\, dr = C_1 \int_k^1 r^{-1}\, dr$$

$$= C_1(-\ln\, k) < \infty.$$

And for the pointwise convergence recall $\lim_{r \to 0} q(r, \cdot) = c_1(\cdot)$ in $L^1(S^{d-1}, \sigma)$, which implies that given any $\delta > 0$, we can choose r_0 such that:

$$\sup_{r \leq r_0} \int_{S^{d-1}} |q(r, \xi) - c_1(\xi)|\, \sigma(d\xi) < \delta.$$

which, if we choose h small enough such that $h^{\frac{1}{\alpha}} \leq r_0$, implies

$$\int_{S^{d-1}} |q(h^{\frac{1}{\alpha}} r, \xi) - c_1(\xi)|\, 1_{\{k \leq r \leq 1\}}(r)\, \sigma(d\xi) < \delta.$$

So,

$$\int_{S^{d-1}} |\xi q(h^{\frac{1}{\alpha}} r, \xi) - \xi c_1(\xi)|\ 1_{\{k \leq r \leq 1\}}(r)\, \sigma(d\xi)$$

$$\leq \int_{S^{d-1}} \|\xi\|\ |q(h^{\frac{1}{\alpha}} r, \xi) - c_1(\xi)|\ 1_{\{k \leq r \leq 1\}}(r)\, \sigma(d\xi)$$

$$= \int_{S^{d-1}} |q(h^{\frac{1}{\alpha}} r, \xi) - c_1(\xi)|\ 1_{\{k \leq r \leq 1\}}(r)\, \sigma(d\xi) < \delta$$

and since δ can be made arbitrarily small, it follows that $\lim_{h \to 0} f(\cdot)\ q(h^{\frac{1}{\alpha}} r, \cdot) = f(\cdot)\ c_1(\cdot)$ in $L^1(S^{d-1}, \sigma)$ where $f(\xi) = \xi$. So,

$$\int_{S^{d-1}} \xi\ q(h^{\frac{1}{\alpha}} r, \xi)\, \sigma(d\xi) \to \int_{S^{d-1}} \xi\ c_1(\xi)\, \sigma(d\xi) \qquad \text{as} \quad h \to 0$$

and, so,

$$Q_h(r) \to \int_{S^{d-1}} \xi \; c_1(\xi)\, \sigma(d\xi) \qquad \text{as } h \to 0$$

as desired and the small scales case is, thus, proven.

Case (ii): Large Scales ($h \to \infty$).

Similarly to the proof of the small scales case we conclude, $h^{-\frac{1}{2}}(X_h^{GTS} - h\, n_\alpha)$ is an infinitely divisible random variable with L evy measure $h(T_{h^{-\frac{1}{2}}} \nu^{GTS})$, drift $h^{1-\frac{1}{2}}\, n_\alpha$, and no Gaussian part. So, again by Theorem 15.14 of Kallenberg, it is only necessary to check the following convergences:

a) $h(T_{h^{-\frac{1}{2}}} \nu^{GTS})$ converges vaguely towards 0 on $\bar{\mathrm{R}}_0^d$ as $h \to \infty$

b) $a_h^k := \int_{\|x\| \le k} xx'\, h(T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx) \to \int_{\mathbb{R}_0^d} xx'\, \nu^{GTS}(dx)$ as $h \to \infty$ for some $k \in (0,1)$

c) $b_h^k := -\int_{k \le \|x\| \le 1} x\, h(T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx) \to 0$ as $h \to \infty$ for some $k \in (0,1)$

where

$$h(T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx) = h\;\, q(h^{\frac{1}{2}} r, \xi)\, (h^{\frac{1}{2}} r)^{-1-\alpha}\, d(h^{\frac{1}{2}} r)\, \sigma(d\xi).$$

For case (a), let $g : \mathbb{R}_0^d \to \mathbb{R}$ be a bounded continuous function vanishing in a neighborhood of the origin. So there exists a C and an $\epsilon > 0$ such that $|g| \le C < \infty$ and $g(x) = 0$ on $\{x \in \mathbb{R}_0^d : \|x\| \le \epsilon\}$. Also, suppose that $\lim_{n\to\infty} g(x_n)$ exists for every $\{x_n\}$ in $\mathbb{R}_0^d$ such that $\|x_n\| \to \infty$.

Now,

$$\left| \int_{\mathbb{R}_0^d} g(x)\, h(T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx) \right| \le \int_{\|x\| > \epsilon} |g(x)|\; h(T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx)$$

$$\le C \int_{\|x\| > \epsilon} h\; \frac{\|h^{\frac{1}{2}} x\|^2}{\|h^{\frac{1}{2}} x\|^2}\; (T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx)$$

which equals

$$C \int_{\|x\| > \epsilon} \frac{1}{\|x\|^2} \|h^{\frac{1}{2}} x\|^2\; (T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx)$$

$$\leq C \int_{\|x\|>\epsilon} \frac{1}{\epsilon^2}\|h^{\frac{1}{2}}x\|^2 \ (T_{h^{-\frac{1}{2}}}\nu^{GTS})(dx),$$

which by change of variable equals

$$\frac{C}{\epsilon^2}\int_{\|x\|>h^{\frac{1}{2}}\epsilon} \|x\|^2 \ \nu^{GTS}(dx)$$

$$= \frac{C}{\epsilon^2}\int_{\mathbb{R}_0^d} 1_{\{\|x\|>h^{\frac{1}{2}}\epsilon\}}(x)\|x\|^2 \ \nu^{GTS}(dx)$$

which goes to zero as $h \to \infty$ by the Lebesgue Dominated Convergence Theorem since

$$\int_{\mathbb{R}_0^d} \|x\|^2 \ \nu^{GTS}(dx) < \infty$$

and $1_{\{\|x\|>h^{\frac{1}{2}}\epsilon\}}(x) \to 0$ as $h \to \infty$ for each x.

For case (b), the convergence of the Gaussian part, notice

$$a_h^k = \int_{\|x\|\leq k} xx' \, h(T_{h^{-\frac{1}{2}}}\nu^{GTS})(dx)$$

$$= \int_{\|x\|\leq k} (h^{\frac{1}{2}}x)(h^{\frac{1}{2}}x)' \ (T_{h^{-\frac{1}{2}}}\nu^{GTS})(dx)$$

which by change of variables equals

$$\int_{\|x\|\leq h^{\frac{1}{2}}k} xx' \ \nu^{GTS}(dx)$$

$$= \int_{\mathbb{R}_0^d} xx' \ 1_{\{\|x\|\leq h^{\frac{1}{2}}k\}}(x) \ \nu^{GTS}(dx)$$

$$\to \int_{\mathbb{R}_0^d} xx' \ \nu^{GTS}(dx)$$

as $h \to \infty$ by the Lebesgue Dominated Convergence Theorem since

$$\int_{\mathbb{R}_0^d} \|x\|^2 \ \nu^{GTS}(dx) < \infty$$

and $1_{\{\|x\|\leq h^{\frac{1}{2}}k\}}(x) \to 1$ as $h \to \infty$ for each x.

And, lastly for part (c), the convergence of the drift term ($\alpha = 1$ case only), we have

$$
\begin{aligned}
|b_h^k| &= \left| - \int_{k<\|x\|\le 1} x\, h(T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx) \right| \\
&\le \int_{k<\|x\|\le 1} h\|x\|\, (T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx) \\
&= \int_{k<\|x\|\le 1} h^{\frac{1}{2}}\, \frac{\|h^{\frac{1}{2}} x\|}{\|h^{\frac{1}{2}} x\|} \|h^{\frac{1}{2}} x\|\ (T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx)
\end{aligned}
$$

which by change of variables equals

$$
\int_{k<\|x\|\le 1} \frac{1}{\|x\|} \|h^{\frac{1}{2}} x\|^2\ (T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx)
$$

$$
\begin{aligned}
&\le \frac{1}{k} \int_{k<\|x\|\le 1} \|h^{\frac{1}{2}} x\|^2\ (T_{h^{-\frac{1}{2}}} \nu^{GTS})(dx) \\
&= \frac{1}{k} \int_{h^{\frac{1}{2}} k<\|x\|\le h^{\frac{1}{2}}} \|x\|^2\ \nu^{GTS}(dx) \\
&\le \frac{1}{k} \int_{\mathbb{R}_0^d} 1_{\{\|x\|>h^{\frac{1}{2}} k\}}(x) \|x\|^2\ \nu^{GTS}(dx)
\end{aligned}
$$

which goes to zero as $h \to \infty$ by the Lebesgue Dominated Convergence Theorem since

$$
\int_{\mathbb{R}_0^d} \|x\|^2\ \nu^{GTS}(dx) < \infty
$$

and $1_{\{\|x\|>h^{\frac{1}{2}} k\}}(x) \to 0$ as $h \to \infty$ for each x.

2.5 Absolute continuity with respect to a stable process

In this section, absolute continuity of GTS processes with respect to the underlying stable process is studied. This should be compared with the results of Rosinski [16] and Caballero, Pardo, and Pérez [2].

Theorem 2.5.1. *Let P and Q be two probability measures such that under P the canonical process is a GTS process,$\{X_t^{GTS}\}_{t\ge 0} \sim GTS_\alpha(\sigma, q, a)$, with spectral measure σ and drift a*

and under Q it is an α-stable process, $\{X_t^\alpha\}_{t\geq 0} \sim S_\alpha(\sigma_1, b)$*, with spectral measure* σ_1 *and drift b. Let* $\{\mathcal{F}_t\}$ *be the canonical filtration. Then, we have the absolute continuity iff both of the following conditions hold:*

i)

$$\int_{S^{d-1}} \int_0^1 \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)\right)^2 \; r^{-\alpha-1}\, dr\, \sigma(d\xi) < \infty \tag{2.2}$$

ii)

$$a - b = \begin{cases} \displaystyle\int_{\|x\|\leq 1} x \; \nu^{GTS}(dx) & if \quad \alpha \in (0,1) \\ \displaystyle\int_{\|x\|\leq 1} x \; (\nu^{GTS} - \nu^1_{\sigma_1})(dx) & if \quad \alpha = 1 \\ \displaystyle\int_{\|x\|\leq 1} x \; (\nu^{GTS} - \nu^\alpha_{\sigma_1})(dx) - \int_{\|x\|>1} x \; \nu^\alpha_{\sigma_1}(dx) & if \quad \alpha \in (1,2) \end{cases}$$

Proof. Since, for every Borel set B, $\nu^{GTS}(B)$ is equal to

$$\int_{S^{d-1}} \int_0^\infty 1_B(r\xi)\, q(r,\xi)\, r^{-\alpha-1}\, dr\, \sigma(d\xi) = \int_{S^{d-1}} \int_0^\infty 1_B(r\xi)\, \frac{q(r,\xi)}{c_1(\xi)}\, r^{-\alpha-1}\, dr\, c_1(\xi)\sigma(d\xi)$$

$$= \int_{S^{d-1}} \int_0^\infty 1_B(r\xi)\, \frac{q(r,\xi)}{c_1(\xi)}\, r^{-\alpha-1}\, dr\, \sigma_1(d\xi) = \nu^\alpha_{\sigma_1}(A),$$

so it follows that the Radon Nikodym derivative is

$$\frac{d\nu^{GTS}}{d\nu^\alpha_{\sigma_1}}(x) = \frac{q\left(\|x\|, \frac{x}{\|x\|}\right)}{c_1\left(\frac{x}{\|x\|}\right)}.$$

And, by Theorem 33.1 in Sato [19], it is necessary to show:

$$\int_{\mathbb{R}^d} \left(e^{\phi(x)/2} - 1\right)^2 \nu^\alpha_{\sigma_1}(dx) < \infty$$

where

$$\phi(x) := \log \frac{q\left(\|x\|, \frac{x}{\|x\|}\right)}{c_1\left(\frac{x}{\|x\|}\right)}.$$

Well, notice

$$\int_{\mathbb{R}^d} \left(e^{\phi(x)/2} - 1\right)^2 \nu^\alpha_{\sigma_1}(dx) = \int_{S^{d-1}} \int_0^\infty \left(\left(\frac{q(r,\xi)}{c_1(\xi)}\right)^{\frac{1}{2}} - 1\right)^2 r^{-\alpha-1}\, dr\, \sigma_1(d\xi)$$

$$= \int_{S^{d-1}} \int_0^\infty \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)\right)^2 r^{-\alpha-1}\, dr\, \sigma(d\xi)$$

$$= \int_{S^{d-1}} \int_0^1 \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)\right)^2 r^{-\alpha-1}\, dr\, \sigma(d\xi)$$

$$+ \int_{S^{d-1}} \int_1^\infty \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)\right)^2 r^{-\alpha-1}\, dr\, \sigma(d\xi).$$

And the second integral is finite, since

$$\int_{S^{d-1}} \int_1^\infty \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)\right)^2 r^{-\alpha-1}\, dr\, \sigma(d\xi)$$

$$\leq \int_{S^{d-1}} \int_1^\infty q(r,\xi)\, r^{-\alpha-1}\, dr\, \sigma(d\xi) + \int_{S^{d-1}} \int_1^\infty c_1(\xi)\, r^{-\alpha-1}\, dr\, \sigma(d\xi)$$

$$= \int_{\|x\|>1} \nu^{GTS}(dx) + \int_{\|x\|>1} \nu^\alpha_{\sigma_1}(dx) < \infty$$

as ν^{GTS} and $\nu^\alpha_{\sigma_1}$ are Lévy measures. Thus,

$$\int_{\mathbb{R}^d} \left(e^{\phi(x)/2} - 1\right)^2 \nu^\alpha_{\sigma_1}(dx) < \infty$$

if and only if

$$\int_{S^{d-1}} \int_0^1 \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)\right)^2 r^{-\alpha-1}\, dr\, \sigma(d\xi) < \infty,$$

which completes the proof of Part (i).

For part (ii), notice X_1^{GTS} has cumulant function given by

$$\int_{\mathbb{R}^d_0} \left(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_{\|x\|\leq 1}(x)\right) \nu^{GTS}(dx) + i\langle u,\, a\rangle\,.$$

And, X_1^α has cumulant function given by

$$\int_{\mathbb{R}_0^d} \left(e^{i\langle u,x\rangle} - 1 - i\left\langle u, x\right\rangle 1_{\|x\|\le 1}(x)\right)\, \nu_{\sigma_1}^\alpha(dx) + i\left\langle u,\, b + n_\alpha\right\rangle$$

where

$$n_\alpha = \begin{cases} \displaystyle\int_{\|x\|\le 1} x\ \nu_{\sigma_1}^\alpha(dx) & if \quad \alpha\in(0,1) \\ 0 & if \quad \alpha = 1 \\ \displaystyle -\int_{\|x\|>1} x\ \nu_{\sigma_1}^\alpha(dx) & if \quad \alpha\in(1,2) \end{cases}$$

Now, by Theorem 33.1 of Sato [19], it is necessary to show

$$a - (b + n_\alpha) - \int_{\|x\|\le 1} x\ \left(\nu^{GTS} - \nu_{\sigma_1}^\alpha\right)(dx) = 0.$$

Well,

$$a - (b + n_\alpha) - \int_{\|x\|\le 1} x\ \left(\nu^{GTS} - \nu_{\sigma_1}^\alpha\right)(dx)$$

$$= \begin{cases} \displaystyle a - b - \int_{\|x\|\le 1} x\ \nu_{\sigma_1}^\alpha(dx) - \int_{\|x\|\le 1} x\ \left(\nu^{GTS} - \nu_{\sigma_1}^\alpha\right)(dx) & if \quad \alpha\in(0,1) \\ a - b - \int_{\|x\|\le 1} x\ \left(\nu^{GTS} - \nu_{\sigma_1}^\alpha\right)(dx) & if \quad \alpha = 1 \\ \displaystyle a - b + \int_{\|x\|>1} x\ \nu_{\sigma_1}^\alpha(dx) - \int_{\|x\|\le 1} x\ \left(\nu^{GTS} - \nu_{\sigma_1}^\alpha\right)(dx) & if \quad \alpha\in(1,2) \end{cases}$$

$$= \begin{cases} \displaystyle a - b - \int_{\|x\|\le 1} x\ \nu^{GTS}(dx) & if \quad \alpha\in(0,1) \\ \displaystyle a - b - \int_{\|x\|\le 1} x\ (\nu^{GTS} - \nu_{\sigma_1}^1)(dx) & if \quad \alpha = 1 \\ \displaystyle a - b - \int_{\|x\|\le 1} x\ (\nu^{GTS} - \nu_{\sigma_1}^\alpha)(dx) + \int_{\|x\|>1} x\ \nu_{\sigma_1}^\alpha(dx) & if \quad \alpha\in(1,2) \end{cases}$$

which, of course, equals 0 if and only if condition (ii) is satisfied. So, this concludes the proof of the theorem. $\square$

Remark 2.5.2. Rosinski [16] mentioned that the rate of convergence directly impacts whether or not a tempered stable process is absolutely continuous with respect to the underlying stable process. Here, an attempt is made to quantify this fact. The goal is to find some easier condition to check in lieu of condition (i) (equation 2.2) in the previous theorem by looking at the rate of convergence with respect to $r^{\frac{\alpha}{2}}$. The claim is that it would be enough to calculate:

$$\lim_{r\to 0} \frac{|q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)|}{r^{\frac{\alpha}{2}+\epsilon}} := C \qquad \text{for each} \quad \xi \in S^{d-1} \quad \text{and} \quad \epsilon > 0.$$

Then, if $C < \infty$ for some $\epsilon > 0$ condition (i) is satisfied, that is,

$$\int_{S^{d-1}} \int_0^1 \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi) \right)^2 r^{-\alpha-1}\, dr\, \sigma(d\xi) < \infty$$

and if $C = \infty$ for all $\epsilon > 0$ condition (i) is not satisfied since

$$\int_{S^{d-1}} \int_0^1 \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi) \right)^2 r^{-\alpha-1}\, dr\, \sigma(d\xi) = \infty.$$

Proof. If $C < \infty$, then there exists an r_0 and a K such that

$$\sup_{r\le r_0} \frac{|q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)|}{r^{\frac{\alpha}{2}+\epsilon}} \le K < \infty \qquad \text{for each} \quad \xi \in S^{d-1}.$$

Now,

$$\int_{S^{d-1}} \int_0^1 \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi) \right)^2 r^{-\alpha-1}\, dr\, \sigma(d\xi)$$

equals

$$\int_{S^{d-1}} \int_0^1 \left(\frac{q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)}{r^{\frac{\alpha}{2}+\epsilon}} \right)^2 r^{2\epsilon-1}\, dr\, \sigma(d\xi)$$

which is less than or equal to

$$\int_{S^{d-1}} \int_0^{r_0} K^2\, r^{2\epsilon-1}\, dr\, \sigma(d\xi) + \int_{S^{d-1}} \int_{r_0}^1 \left(\frac{q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)}{r^{\frac{\alpha}{2}+\epsilon}} \right)^2 r^{2\epsilon-1}\, dr\, \sigma(d\xi)$$

$$= K^2\sigma(S^{d-1})\int_0^{r_0} r^{2\epsilon-1}\,dr + \int_{S^{d-1}}\int_{r_0}^1 \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)\right)^2 r^{-\alpha-1}\,dr\,\sigma(d\xi)$$

which equals

$$K^2\sigma(S^{d-1})\frac{(r_0)^{2\epsilon}}{2\epsilon} + \int_{S^{d-1}}\int_{r_0}^1 q(r,\xi)\ r^{-\alpha-1}\,dr\,\sigma(d\xi) + \int_{S^{d-1}}\int_{r_0}^1 c_1(\xi)\ r^{-\alpha-1}\,dr\,\sigma(d\xi)$$

$$\leq \frac{K^2\sigma(S^{d-1})}{2\epsilon} + \int_{\|x\|>r_0} \nu^{GTS}(dx) + \int_{\|x\|>r_0} \nu_{\sigma_1}^{\alpha}(dx)$$

which is finite since ν^{GTS} and $\nu_{\sigma_1}^{\alpha}$ are Lévy measures and σ is a finite measure on S^{d-1}. Thus, condition (i) is satisfied.

If $C = \infty$, then there exists an $K > 0$ and an r_0 such that

$$\sup_{r\leq r_0} \frac{|q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)|}{r^{\frac{\alpha}{2}+\epsilon}} > K \qquad \text{for each} \quad \xi \in S^{d-1}$$

which implies

$$\sup_{r\leq r_0} \frac{|q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)|}{r^{\frac{\alpha}{2}}} > K \qquad \text{for each} \quad \xi \in S^{d-1}$$

since $r \leq 1$. And,

$$\int_{S^{d-1}}\int_0^1 \left(q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)\right)^2 r^{-\alpha-1}\,dr\,\sigma(d\xi)$$

$$= \int_{S^{d-1}}\int_0^1 \left(\frac{q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)}{r^{\frac{\alpha}{2}}}\right)^2 r^{-1}\,dr\,\sigma(d\xi)$$

which is greater than or equal to

$$K^2\int_{S^{d-1}} \sigma(d\xi)\int_0^{r_0} r^{-1}\,dr$$

$$= K^2\ \sigma(S^{d-1})\ \left[\ln\, r_0 - \lim_{r\to 0}\ \ln\, r\right] = \infty.$$

Thus, condition (i) is not satisfied. □

Remark 2.5.3. Of course this limit may not be easily computed. Actually, all that is really needed for absolute continuity is to have an r_0 and a positive integrable function k on S^{d-1}

such that

$$\sup_{r\leq r_0} \frac{|q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)|}{r^{\frac{\alpha}{2}+\epsilon}} \leq k(\xi)$$

for each $\xi \in S^{d-1}$ and some $\epsilon > 0$.

And, to show it is not absolutely continuous, it would be enough to show that there exists an r_0 and a positive integrable function k on S^{d-1} such that

$$\sup_{r\leq r_0} \frac{|q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)|}{r^{\frac{\alpha}{2}}} > k(\xi) \qquad \text{for each} \quad \xi \in S^{d-1}.$$

This is clear by inspection of the proof of the previous remark.

Example 2.5.1. (*A Tempered Stable Process*) Let $q(r,\xi) = e^{-r}$, $c_1 = 1$, and σ be any finite measure on S^{d-1}. Then,

$$\lim_{r\to 0} \frac{q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)}{r^{\frac{\alpha}{2}+\epsilon}} = \lim_{r\to 0} \frac{\sqrt{e^{-r}} - 1}{r^{\frac{\alpha}{2}+\epsilon}}$$
$$= \lim_{r\to 0} \frac{-1}{\alpha + 2\epsilon}(e^{-r})^{\frac{1}{2}} r^{1-(\frac{\alpha}{2}+\epsilon)} = 0$$

for any $\epsilon < 1 - \frac{\alpha}{2}$. So, there is absolute continuity with respect to the underlying stable process.

Example 2.5.2. (*Another Tempered Stable Process*) Let $q(r,\xi) = e^{-r^\beta}$, $c_1 = 1$, and σ be any finite measure on S^{d-1} where $0 < \beta \leq \frac{\alpha}{2}$. Then,

$$\lim_{r\to 0} \frac{q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)}{r^{\frac{\alpha}{2}+\epsilon}} = \lim_{r\to 0} \frac{\sqrt{e^{-r^\beta}} - 1}{r^{\frac{\alpha}{2}+\epsilon}}$$
$$= \lim_{r\to 0} \frac{-\beta}{\alpha + 2\epsilon}(e^{-r^\beta})^{\frac{1}{2}} r^{\beta-(\frac{\alpha}{2}+\epsilon)} = \infty$$

for all $\epsilon > 0$. So, there is no absolute continuity in this case.

Example 2.5.3. (*Lamperti Stable Processes*) Let

$$q(r,\xi) = \frac{e^{rf(\xi)}}{(e^r - 1)^{\alpha+1}} r^{\alpha+1}$$

where $f : S^{d-1} \to \mathbb{R}$ is such that $\sup_{\xi\in S^{d-1}} f(\xi) < \alpha + 1$. Then, it can be shown that

$$\lim_{r\to 0} \frac{q^{\frac{1}{2}}(r,\xi) - c_1^{\frac{1}{2}}(\xi)}{r^{\frac{\alpha}{2}+\epsilon}} = 0$$

for some $\epsilon > 0$ where $c_1 = 1$. So, there is always absolute continuity of Lamperti stable processes with respect to the underlying stable process.

2.6 Series representation

First, a series representation using LePage's Method [15] is examined. It involves looking at a decomposition of the Lévy measure of the form:

$$\nu^{GTS}(A) = \int_{S^{d-1}} \int_0^\infty 1_A(r\xi)\ \rho_*(dr, \xi)\, \sigma_*(d\xi)$$

where $\sigma_*(d\xi) := \sigma(d\xi)/\sigma(S^{d-1}) := \sigma(d\xi)/\|\sigma\|$ is a probability measure on S^{d-1} and ρ_* is a measure on $\mathbb{R}$ such that

$$\rho_*([x, \infty), \xi) = \|\sigma\|\, \rho([x, \infty), \xi)$$

where

$$\rho([x, \infty), \xi) := \int_x^\infty q(r, \xi)\ r^{-\alpha-1}\, dr. \tag{2.3}$$

LePage's Method requires that ρ_* be a Lévy measure on $\mathbb{R}$ for each $\xi \in S^{d-1}$, and it is, since:

$$\int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\nu^{GTS}(dx) = \int_{S^{d-1}} \int_0^\infty (1 \wedge r^2)\ \rho_*(dr, \xi)\, \sigma_*(d\xi) < \infty$$

which implies that

$$\int_0^\infty (1 \wedge r^2)\rho_*(dr, \xi) < \infty.$$

The inverse is defined as:

$$\rho_*^{-1}(u, \xi) := \inf\{x > 0 : \rho_*([x, \infty), \xi) < u\}.$$

$$= \inf\{x > 0 : \rho([x, \infty), \xi) < \frac{u}{\|\sigma\|}\}$$

$$= \rho^{-1}(u/\|\sigma\|, \xi).$$

Theorem 2.6.1. *Let $\{\Gamma_i\}_{i\geq 1}$ be a sequence of partial sums of iid exponential random variables* (1)*. Let $\{U_i\}_{i\geq 1}$ be a sequence of iid uniform random variables on $[0, T]$ and let $\{V_i\}_{i\geq 1}$ be a sequence of random variables in S^{d-1} with common distribution $\sigma(d\xi)/\|\sigma\|$. Then,*

$$\sum_{i=1}^\infty \left\{ \rho^{-1}\left(\frac{\Gamma_i}{\|\sigma\|T}, V_i\right) V_i\, 1_{\{U_i \leq t\}} - c_i \frac{t}{T} \right\} \tag{2.4}$$

converges uniformly a.s. to a process, $\{X_t^{GTS}\}_{t\geq 0} \sim GTS_\alpha(\sigma, q, 0)$ *where*

$$c_i = \int_{i-1}^{i} E\left(\rho^{-1}\left(\frac{s}{\|\sigma\|T}, V_1\right) V_1 1_{\rho^{-1}\left(\frac{s}{\|\sigma\|T}, V_1\right)\leq 1}\right) ds.$$

In particular, if there exists a function g *on* $(0, \infty)$ *such that* $q(r, \xi) = g(r)$*, then* ρ^{-1} *will not depend on* ξ *and the above series representation is simplified to the following:*

$$\sum_{i=1}^{\infty} \left\{\rho^{-1}\left(\frac{\Gamma_i}{\|\sigma\|T}\right) V_i 1_{\{U_i\leq t\}} - c_i \frac{t}{T}\right\} \tag{2.5}$$

where

$$c_i = E(V_1) \int_{i-1}^{i} \left(\rho^{-1}\left(\frac{s}{\|\sigma\|T}\right) 1_{\rho^{-1}\left(\frac{s}{\|\sigma\|T}\right)\leq 1}\right) ds.$$

Proof. A proof of this theorem is omitted, as it is merely a reformulation of the results presented by Rosinski in [15]. The representation is generated by LePage's method and the series convergences by Rosinski.

□

Example 2.6.1. *(Lamperti Stable Processes)* It is possible to find the inverse in some cases. For example, consider a Lamperti stable process with

$$q(r, \xi) = \frac{e^{rf(\xi)}}{(e^r - 1)^{\alpha+1}} r^{\alpha+1}$$

and $f(\xi) := 1$. Then,

$$\rho([x, \infty), \xi) = \int_x^{\infty} \frac{e^{rf(\xi)}}{(e^r - 1)^{\alpha+1}} r^{\alpha+1}\, r^{-\alpha-1}\, dr$$

$$= \int_x^{\infty} \frac{e^{rf(\xi)}}{(e^r - 1)^{\alpha+1}}\, dr = \alpha^{-1} (e^x - 1)^{-\alpha}$$

which implies that

$$\rho^{-1}(u, \xi) = \ln\left(1 + (\alpha u)^{-\frac{1}{\alpha}}\right).$$

Thus,

$$\sum_{i=1}^{\infty} \left\{\ln\left(1 + \left(\frac{\alpha\Gamma_i}{\|\sigma\|T}\right)^{-\frac{1}{\alpha}}\right) V_i 1_{\{U_i\leq t\}} - c_i \frac{t}{T}\right\} \tag{2.6}$$

converges to a Lamperti stable process with $f(\xi) := 1$.

Example 2.6.2. *(Stable Processes)* Consider a stable process with

$$q(r, \xi) = 1.$$

Then,

$$\rho([x, \infty), \xi) = \int_x^\infty r^{-\alpha-1}\, dr$$
$$= \frac{x^{-\alpha}}{\alpha}$$

which implies that

$$\rho^{-1}(u, \xi) = (\alpha u)^{-\frac{1}{\alpha}}.$$

Thus,

$$\sum_{i=1}^{\infty} \left\{ \left(\frac{\alpha \Gamma_i}{\|\sigma\| T} \right)^{-\frac{1}{\alpha}} V_i\, 1_{\{U_i \le t\}} - c_i \frac{t}{T} \right\} \tag{2.7}$$

where

$$c_i = E(V_1) \int_{i-1}^{i} \left(\left(\frac{\alpha s}{\|\sigma\| T} \right)^{-\frac{1}{\alpha}} 1_{\frac{\alpha s}{\|\sigma\| T} \le 1} \right) ds$$

converges to an α-stable process.

Example 2.6.3. *(A tempered stable process)* Consider a simple tempered stable process with

$$q(r, \xi) = e^{-r}.$$

Then,

$$\rho([x, \infty), \xi) = \int_x^\infty e^{-r}\, r^{-\alpha-1}\, dr$$

which is not invertible, so we cannot get a closed form of the formula for ρ^{-1} in this case.

Remark 2.6.2. As Rosinski mentioned in [16], it is not always easy to find the inverse! In the case when the q function is bounded, it may be advantageous to use the rejection method to generate series representations.

Theorem 2.6.3. *Let $\{\Gamma_i\}_{i \ge 1}$ be a sequence of partial sums of iid exponential random variables* (1)*. Let $\{U_i\}_{i \ge 1}$ be a sequence of iid uniform random variables on $[0, T]$ and let $\{V_i\}_{i \ge 1}$ be a sequence of random variables in S^{d-1} with common distribution $\sigma(d\xi)/\|\sigma\|$. Also, let $\{W_i\}_{i \ge 1}$ be a sequence of iid uniform random variables that is independent of the rest.*

Now, suppose there exists a function g on $(0,\infty)$ such that

$$(*) \qquad sup_{\xi\in S^{d-1}}\, q(r,\xi) \le g(r) \qquad \textit{for each} \quad r\in(0,\infty)$$

And, let

$$\sum_{i=1}^{\infty}\left\{\rho^{-1}\left(\frac{\Gamma_i}{\|\sigma\|T}\right)V_i\,1_{\{U_i\le t\}} - c_i\frac{t}{T}\right\}$$

where

$$c_i = E(V_1)\int_{i-1}^{i} E\left(\rho^{-1}\left(\frac{s}{\|\sigma\|T}\right)1_{\rho^{-1}(\frac{s}{\|\sigma\|T})\le 1}\right)ds$$

be the series representation of the process with Lévy measure

$$\nu^0(B) = \int_{S^{d-1}}\int_0^{\infty} 1_B(r\xi)\,g(r)\,r^{-\alpha-1}\,dr\,\sigma(d\xi)$$

for each Borel set B in $\mathbb{R}^d_0$.

Now, let $J_i^0 := \rho^{-1}\left(\frac{\Gamma_i}{\|\sigma\|T}\right)$ and

$$\beta_i := \begin{cases} 1 & \textit{if} \quad f\left(J_i^0\right) \ge W_i \\ 0 & \textit{else} \end{cases}$$

where $f(x) = q(\|x\|, \frac{x}{\|x\|})/g(\|x\|)$. Then,

$$\sum_{i=1}^{\infty}\left\{\beta_i\,\rho^{-1}\left(\frac{\Gamma_i}{\|\sigma\|T}\right)V_i\,1_{\{U_i\le t\}} - c_i\frac{t}{T}\right\}_{t\in[0,T]} \tag{2.8}$$

converges uniformly a.s. to a process, $\{X_t^{GTS}\}_{t\ge 0}\sim GTS_\alpha(\sigma,q,0)$ where

$$c_i = E(V_1)\int_{i-1}^{i} E\left(\rho^{-1}\left(\frac{s}{\|\sigma\|T}\right)1_{\rho^{-1}(\frac{s}{\|\sigma\|T})\le 1}\right)ds.$$

Proof. A proof of this theorem is also omitted, as it is merely a reformulation of the results presented by Rosinski in [15]. It is based on the rejection method from [15] and relies on the fact that if

$$\sup_{\xi\in S^{d-1}} q(r,\xi) \le g(r) \qquad \text{for each} \quad r\in(0,\infty)$$

then

$$\frac{d\nu^{GTS}}{d\nu^0}(r,\xi) = \frac{q(r,\xi)}{g(r)} \le 1.$$

□

Remark 2.6.4. This theorem is useful when ρ is not invertible, but the q function is bounded by a g function that does have an invertible ρ.

Example 2.6.4. *(Tempered Stable Processes)* Consider any proper tempered stable process. Then we have that

$$\sup_{\xi \in S^{d-1}} q(r,\xi) \leq 1$$

so we can use the series representation in Theorem 2.6.3 with $g(r) = 1$. Thus,

$$\sum_{i=1}^{\infty} \left\{ \beta_i \left(\frac{\alpha \Gamma_i}{\|\sigma\| T} \right)^{-\frac{1}{\alpha}} V_i \, 1_{\{U_i \leq t\}} - c_i \frac{t}{T} \right\} \tag{2.9}$$

where

$$c_i = E(V_1) \int_{i-1}^{i} \left(\left(\frac{\alpha s}{\|\sigma\| T} \right)^{-\frac{1}{\alpha}} 1_{\frac{\alpha s}{\|\sigma\| T} \leq 1} \right) ds$$

converges to a tempered stable process where the β_i's are given by

$$\beta_i := \begin{cases} 1 & \text{if} \quad f\left(J_i^0\right) \geq W_i \\ \\ 0 & \text{else} \end{cases}$$

where $f(x) = q(\|x\|, \frac{x}{\|x\|})/g(\|x\|) = q(\|x\|, \frac{x}{\|x\|})$. So, they are given by

$$\beta_i := \begin{cases} 1 & \text{if} \quad q(\|y_i\|, \frac{y_i}{\|y_i\|}) \geq W_i \\ \\ 0 & \text{else} \end{cases}$$

where

$$y_i := J_i^0 = \rho^{-1} \left(\frac{\Gamma_i}{\|\sigma\| T} \right) = \left(\frac{\alpha \Gamma_i}{\|\sigma\| T} \right)^{-\frac{1}{\alpha}}.$$

Note: this is analogous to rejecting terms from a stable process.

Example 2.6.5. *(Lamperti Stable Processes)* Again, consider a Lamperti stable process with

$$q(r,\xi) = \frac{e^{rf(\xi)}}{(e^r - 1)^{\alpha+1}} r^{\alpha+1}$$

with $f(\xi)=1$. Then,

$$\rho^{-1}(u,\xi)=\rho^{-1}(u)=\ln\left(1+(\alpha u)^{-\frac{1}{\alpha}}\right).$$

Now, assume only that $f(\xi)\leq 1$. Since

$$q(r,\xi)=\frac{e^{rf(\xi)}}{(e^r-1)^{\alpha+1}}r^{\alpha+1}\leq\frac{e^r}{(e^r-1)^{\alpha+1}}r^{\alpha+1}$$

we can use

$$g(r)=\frac{e^r}{(e^r-1)^{\alpha+1}}r^{\alpha+1}$$

and get that

$$\sum_{i=1}^{\infty}\left\{\beta_i\ \ln\left(1+\left(\frac{\alpha\Gamma_i}{\|\sigma\|T}\right)^{-\frac{1}{\alpha}}\right)V_i\,1_{\{U_i\leq t\}}-c_i\frac{t}{T}\right\} \tag{2.10}$$

converges to a Lamperti stable process with $f(\xi)\leq 1$ where the β_i's are given by

$$\beta_i:=\begin{cases}1 & \text{if}\quad f_*\left(J_i^0\right)\geq W_i\\ \\ 0 & \text{else}\end{cases}$$

where $f_*(x)=q(\|x\|,\frac{x}{\|x\|})/g(\|x\|)$. Now,

$$q(r,\xi)=\frac{e^{rf(\xi)}}{(e^r-1)^{\alpha+1}}r^{\alpha+1}\qquad\text{and}\qquad g(r)=\frac{e^r}{(e^r-1)^{\alpha+1}}r^{\alpha+1}$$

imply that

$$\frac{q}{g}(r,\xi)=\frac{e^{rf(\xi)}}{e^r}=e^{r(f(\xi)-1)}.$$

So,

$$\frac{q}{g}(x)=e^{\|x\|\left(f\left(\frac{x}{\|x\|}\right)-1\right)}.$$

And,

$$\rho^{-1}(u)=\ln\left(1+(\alpha u)^{-\frac{1}{\alpha}}\right)$$

implies that

$$J_i^0:=\rho^{-1}\left(\frac{\Gamma_i}{\|\sigma\|T}\right)=\ln\left(1+\left(\frac{\alpha\Gamma_i}{\|\sigma\|T}\right)^{-\frac{1}{\alpha}}\right).$$

So,

$$f_*(J_i^0)=e^{\|\ln y_i\|\left(f\left(\frac{\ln y_i}{\|\ln y_i\|}\right)-1\right)}$$

where $y_i := 1 + \left(\frac{\alpha\Gamma_i}{\|\sigma\|T}\right)^{-\frac{1}{\alpha}}$. Thus,

$$\beta_i := \begin{cases} 1 & \text{if} \quad \|\ln y_i\| \left(f\left(\frac{\ln y_i}{\|\ln y_i\|}\right) - 1\right) \geq \ln W_i \\ \\ 0 & \text{else} \end{cases}$$

where $y_i := 1 + \left(\frac{\alpha\Gamma_i}{\|\sigma\|T}\right)^{-\frac{1}{\alpha}}$.

Note that in [2], an explicit representation was only given for the case $f(\xi) = 1$. Here, an explicit representation is given when $f(\xi) \leq 1$. But, in general, the f function in Lamperti stable processes satisfies $sup_{\xi \in S^{d-1}} f(\xi) := \gamma < \alpha + 1$.

Example 2.6.6. *(Modified Tempered Stable Distribution (MTS distribution))* Consider an MTS distribution with

$$q(r,\xi) = \begin{cases} 2^{\frac{1-\alpha}{2}} \left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1} (\lambda_+ r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ r) & \text{if} \quad \xi = 1 \\ \\ 2^{\frac{1-\alpha}{2}} \left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1} (\lambda_- r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- r) & \text{if} \quad \xi = -1 \end{cases}$$

where $\lambda_+, \lambda_- > 0$, $\alpha < 2$, and K_p is the modified Bessel function of the second kind which is defined by

$$K_p(x) = \frac{\pi}{2\sin(p\pi)} \left(\sum_{k=0}^{\infty} \frac{(x/2)^{2k-p}}{k!\,\Gamma(k-p+1)} - \sum_{k=0}^{\infty} \frac{(x/2)^{2k+p}}{k!\,\Gamma(k+p+1)} \right)$$

and has an integral representation given by

$$K_p(x) = \frac{1}{2}\left(\frac{x}{2}\right)^p \int_0^{\infty} e^{-t-\frac{x^2}{4t}} t^{-p-1} dt.$$

This Bessel function satisfies

$$K_p(x) \sim \frac{\Gamma(p)}{2}\left(\frac{2}{x}\right)^p \quad \text{as} \quad x \to 0^+ \quad \text{and} \quad p > 0$$

and

$$K_p(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}} \quad \text{as} \quad x \to \infty \quad \text{and} \quad p \geq 0.$$

It also satisfies the property that

$$K_p(x) = K_{-p}(x)$$

since

$$\begin{aligned}
K_p(x) &= \frac{\pi}{2\sin(p\pi)}\left(\sum_{k=0}^{\infty}\frac{(x/2)^{2k-p}}{k!\,\Gamma(k-p+1)} - \sum_{k=0}^{\infty}\frac{(x/2)^{2k+p}}{k!\,\Gamma(k+p+1)}\right)\\
&= \frac{\pi}{2(-\sin(p\pi))}\left(\sum_{k=0}^{\infty}\frac{(x/2)^{2k+p}}{k!\,\Gamma(k+p+1)} - \sum_{k=0}^{\infty}\frac{(x/2)^{2k-p}}{k!\,\Gamma(k-p+1)}\right)\\
&= \frac{\pi}{2\sin((-p)\pi)}\left(\sum_{k=0}^{\infty}\frac{(x/2)^{2k-(-p)}}{k!\,\Gamma(k-(-p)+1)} - \sum_{k=0}^{\infty}\frac{(x/2)^{2k+(-p)}}{k!\,\Gamma(k+(-p)+1)}\right)\\
&= K_{-p}(x)
\end{aligned}$$

since $\sin(x)$ is an odd function. And, notice that

$$K_p(x) \leq \frac{\Gamma(p)}{2}\left(\frac{2}{x}\right)^p$$

since

$$\begin{aligned}
K_p(x) &= K_{-p}(x)\\
&= \frac{1}{2}\left(\frac{x}{2}\right)^{-p}\int_0^{\infty} e^{-t-\frac{x^2}{4t}}t^{p-1}dt\\
&= \frac{1}{2}\left(\frac{x}{2}\right)^{-p}\int_0^{\infty} e^{-\frac{x^2}{4t}}e^{-t}t^{p-1}dt\\
&\leq \frac{1}{2}\left(\frac{2}{x}\right)^{p}\int_0^{\infty} e^{-t}t^{p-1}dt\\
&= \frac{\Gamma(p)}{2}\left(\frac{2}{x}\right)^p.
\end{aligned}$$

Thus,

$$K_{\frac{\alpha+1}{2}}(\lambda r) \leq \frac{\Gamma(\frac{\alpha+1}{2})}{2}\left(\frac{2}{\lambda r}\right)^{\frac{\alpha+1}{2}}.$$

Now, let $\lambda := \max\{\lambda_+, \lambda_-\}$ and $\lambda^* := \min\{\lambda_+, \lambda_-\}$. Then, we have

$$\begin{aligned} sup_{\xi\in S^{d-1}}\, q(r,\xi) &\leq 2^{\frac{1-\alpha}{2}} \left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1} (\lambda r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda^* r) \\ &\leq 2^{\frac{1-\alpha}{2}} \left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1} (\lambda r)^{\frac{\alpha+1}{2}} \frac{\Gamma(\frac{\alpha+1}{2})}{2} \left(\frac{2}{\lambda^* r}\right)^{\frac{\alpha+1}{2}} \\ &= 2^{\frac{1-\alpha}{2}} (\lambda)^{\frac{\alpha+1}{2}} \frac{1}{2} \left(\frac{2}{\lambda^*}\right)^{\frac{\alpha+1}{2}} \\ &= \left(\frac{\lambda}{\lambda^*}\right)^{\frac{\alpha+1}{2}} \end{aligned}$$

and we set this equal to $g(r)$ (although it does not depend on r). Then, we look at the series representation associated with g and then reject appropriate terms from it. Now,

$$\rho([x,\infty),\xi) = \int_x^\infty g(r)\; r^{-\alpha-1}\, dr$$

$$= \left(\frac{\lambda}{\lambda^*}\right)^{\frac{\alpha+1}{2}} \frac{x^{-\alpha}}{\alpha}$$

which implies that

$$\rho^{-1}(u,\xi) = \rho^{-1}(u) = \left(\left(\frac{\lambda^*}{\lambda}\right)^{\frac{\alpha+1}{2}} \alpha u\right)^{-\frac{1}{\alpha}}.$$

Then,

$$\sum_{i=1}^{\infty} \left\{ \beta_i \left(\left(\frac{\lambda^*}{\lambda}\right)^{\frac{\alpha+1}{2}} \left(\frac{\alpha\Gamma_i}{\|\sigma\| T}\right)\right)^{-\frac{1}{\alpha}} V_i\, 1_{\{U_i\leq t\}} - c_i \frac{t}{T} \right\}_{t\in[0,T]} \tag{2.11}$$

converges uniformly a.s. to an MTS process where

$$\beta_i := \begin{cases} 1 & \text{if} \quad f\,(J_i^0) \geq W_i \\ \\ 0 & \text{else} \end{cases}$$

and $f(x) = q(\|x\|, \frac{x}{\|x\|})/g(\|x\|)$. And, lastly, the β_i's are computed explicitly below.

$$f(r,\xi) = \frac{q(r,\xi)}{g(r)} = \frac{q}{g}(r,\xi) = \left(\frac{\lambda^*}{\lambda}\right)^{\frac{\alpha+1}{2}} q(r,\xi)$$

$$= \left(\frac{\lambda^*}{\lambda}\right)^{\frac{\alpha+1}{2}} \begin{cases} 2^{\frac{1-\alpha}{2}}\left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1}(\lambda_+ r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ r) & \text{if} \quad \xi = 1 \\ \\ 2^{\frac{1-\alpha}{2}}\left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1}(\lambda_- r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- r) & \text{if} \quad \xi = -1 \end{cases}$$

$$= \begin{cases} 2^{\frac{1-\alpha}{2}}\left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1}(\lambda^{-1}\lambda^*\lambda_+ r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ r) & \text{if} \quad \xi = 1 \\ \\ 2^{\frac{1-\alpha}{2}}\left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1}(\lambda^{-1}\lambda^*\lambda_- r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- r) & \text{if} \quad \xi = -1 \end{cases}$$

So,

$$\begin{aligned} f\left(J_i^0\right) &= f\left(\rho^{-1}\left(\frac{\Gamma_i}{\|\sigma\| T}\right)\right) \\ &= f\left(\left(\frac{\lambda^*}{\lambda}\right)^{\frac{\alpha+1}{-2\alpha}}\left(\frac{\alpha\Gamma_i}{\|\sigma\| T}\right)^{-\frac{1}{\alpha}}\right) \\ &= 2^{\frac{1-\alpha}{2}}\left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1}(\lambda^{-1}\lambda^*\lambda_+ y_i)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ y_i) \end{aligned}$$

where

$$y_i := \left(\frac{\lambda^*}{\lambda}\right)^{\frac{\alpha+1}{-2\alpha}}\left(\frac{\alpha\Gamma_i}{\|\sigma\| T}\right)^{-\frac{1}{\alpha}}.$$

Thus,

$$\beta_i := \begin{cases} 1 \quad \text{if} \quad (\lambda^{-1}\lambda^*\lambda_+ y_i)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ y_i) \geq 2^{\frac{2}{1-\alpha}}\left(\Gamma\left(\frac{\alpha+1}{2}\right)\right) W_i \\ \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{else} \end{cases}$$

and thus we have calculated the correct centers for the series representation of modified tempered stable.

This chapter introduces GTS processes. As was mentioned before, the definition is constructed to describe the minimal necessary conditions for the short and long term behavior to hold. The definition is a new result, to the best knowledge of the author, however the short term behavior is similar to that of "layered stable processes" [8]. But, due to the error in their definition (described in a remark following the result) a slight modification using convergence in L^1 as opposed to pointwise convergence was necessary. After the definition of GTS processes is stated, numerous example obtained from the literature in this area are noted.

After establishing the definition of GTS processes, the main result of this section, the short and long term behavior is stated. The main result is a new advancement in this area of research, as it displays that the short and long term behavior of these processes (which has been investigated separately in the previous papers on variations of TS processes) still holds under the minimal assumptions made by the definition of GTS processes. The result is proven for the marginals at time 1 and then easily expands to convergence in the space of continuous functions, as GTS processes are Lévy processes.

Next, the absolute continuity is studied, following the work of Jan Rosinksi in [16]. The absolute continuity condition is more complex in this case, as GTS processes are less quantifiable than TS processes. In order to assist in this manner, a limit condition is stated as an alternative to the integrability condition 2.2 and examples are presented to represent its ease of use.

The final topic of investigation is series representation. The first series representation given by equation 2.4 is based on LePage's method and relies on the condition that the ρ-function (2.3) is invertible. Of course, since there is not always an invertible ρ, so a different series representation (equation 2.8) using the rejection method is offered and examples of generating series representations are given, which can be used for simulation of GTS processes. The examples presented exemplify how to generate series representations of GTS processes, which are unique to each case and must be specified separately.

Chapter 3

Fractional generalized tempered stable motion

In 2006, Houdré and Kawai in [7] introduced fractional tempered stable motion by taking a stochastic integral of a Volterra kernel, a deterministic function, with respect to a tempered stable process which are described by Rosinski in [16]. They observed that the process behaves as fractional Brownian motion in large scales and behaves as fractional stable motion in small scales. In this chapter, fractional generalized tempered stable motion (fGTSm) is defined and studied and the short and long term behavior is investigated and established, following the work of Houdré and Kawai in [7].

3.1 Preliminaries

First, a review of GTS processes is given. Please note that this is the same as the definition of GTS processes introduced in the previous chapter, except for the fact that we will always assume square integrability in this chapter.

Definition 3.1.1. A Lévy process is called a generalized tempered stable process if its Lévy measure at time 1 is given in polar coordinates as equation 2.1 where B is a Borel set in $\mathbb{R}_0^d$, $\alpha \in (0,2)$, σ is a finite measure on S^{d-1}, and q is a measurable function from $(0,\infty) \times S^{d-1}$ to $(0,\infty)$ such that:

$$\lim_{r \to 0} q(r, \cdot) = c_1(\cdot) \qquad \text{in} \quad L^1(S^{d-1}, \sigma)$$

and

$$\int_{\{\|x\|>1\}} \|x\|^2 \, \nu^{GTS}(dx) < \infty.$$

Also, assume that $E(X_t^{GTS}) = 0$ for any $t \geq 0$.

Also, recall that the characteristic function of X_1^{GTS} is given by

$$E \exp\{i \langle u, X_1^{GTS} \rangle\} = \exp\left\{ \int_{\mathbb{R}_0^d} \left(e^{i\langle u,x\rangle} - 1 - i \langle u, x\rangle 1_{\|x\|\leq 1}(x) \right) \nu^{GTS}(dx) \right\}.$$

3.2 Definition of fractional GTS motion

Define the following Volterra Kernel, $K : (\mathbb{R}^+, \mathbb{R}^+) \to \mathbb{R}^+$ as

$$K_{H,\alpha}(t,s) := c_{H,\alpha} \Bigg\{ \left(\frac{t}{s}\right)^{H-1/\alpha} (t-s)^{H-1/\alpha}$$

$$- \left(H - \frac{1}{\alpha}\right) s^{1/\alpha - H} \int_s^t u^{H-1/\alpha-1} (u-s)^{H-1/\alpha} \, du \Bigg\} 1_{[0.t]}(s)$$

where $H \in (1/\alpha - 1/2, 1/\alpha + 1/2)$ and

$$c_{H,\alpha} := \left(\frac{G(1-2G)\Gamma(1/2-G)}{\Gamma(2-2G)\Gamma(1/2+G)} \right)^{\frac{1}{2}}$$

where $G := H - 1/\alpha + 1/2$.

Below are some known properties of $K_{H,\alpha}$ (see Houdré and Kawai [7]).

1. It is only defined on $[0, t]$
2. $K_{1/\alpha,\alpha}(t,s) = 1_{[0,t]}(s)$
3. If $H \in (1/\alpha, 1/\alpha + 1/2)$, then it may be simplified to

$$K_{H,\alpha}(t,s) = c_{H,\alpha} \left(H - \frac{1}{\alpha}\right) s^{1/\alpha - H} \int_s^t (u-s)^{H-1/\alpha-1} u^{H-1/\alpha} \, du \; 1_{[0,t]}(s)$$

4. *(Square integrability)* $K_{H,\alpha}(t, \cdot) \in L^2([0,t])$
5. *(Scaling property)* For each $h > 0$, $K_{H,\alpha}(ht, s) = h^{H-1/\alpha} K_{H,\alpha}(t, s/h)$

6. For $t, s > 0$,

$$\int_0^{t\wedge s} K_{H,\alpha}(t,u)K_{H,\alpha}(s,u)du = \frac{1}{2}\left(t^{2G} + s^{2G} - |t-s|^{2G}\right).$$

Definition 3.2.1. Define fractional GTS motion (fGTSm) in $\mathbb{R}$ as $\{Y_t^H : t \geq 0\}$ where

$$Y_t^H := \int_0^t K_{H,\alpha}(t,s)dX_s^{GTS} \qquad \text{for} \quad t \geq 0.$$

Let $t_1, \ldots, t_k$ be a finite nondecreasing sequence of times and let $a_1, \ldots, a_k$ be real numbers. Then, the finite dimensional distributions of fGTSm are given by:

$$\begin{aligned} E[\exp\{iu\sum_{i=1}^k a_i Y_{t_i}^H\}] =& E[\exp\{iu\sum_{i=1}^k a_i \left(\int_0^{t_i} K_{H,\alpha}(t_i,s)dX_s^{GTS}\right)\}] \\ =& E[\exp\{iu\left(\int_0^{t_k}\sum_{i=1}^k a_i K_{H,\alpha}(t_i,s)dX_s^{GTS}\right)\}] \\ =& \exp\left\{\int_0^{t_k}\psi_0(u\sum_{i=1}^k a_i K_{H,\alpha}(t_i,s))ds\right\} \end{aligned}$$

where

$$\psi_0(u) := \int_{\mathbb{R}_0}\left(e^{iux} - 1 - iux1_{|x|\leq 1}(x)\right)\nu^{GTS}(dx).$$

$$= \exp\int_0^{t_k}\int_{\mathbb{R}}\left(\exp\left\{iu\sum_{i=1}^k a_i K_{H,\alpha}(t_i,s)x\right\} - 1 - iu\sum_{i=1}^k a_i K_{H,\alpha}(t_i,s)x1_{|x|\leq 1}(x)\right)\nu^{GTS}(dx)\,ds$$

Remark 3.2.2. Note that when $H = 1/\alpha$, fGTSm is simply a GTS process since

$$K_{1/\alpha,\alpha}(t,s) = 1_{[0,t]}(s)$$

implies that

$$Y_t^{1/\alpha} = \int_0^t 1_{[0,t]}(s)dX_s^{GTS} = X_t^{GTS} \qquad \text{for} \quad t \geq 0.$$

Remark 3.2.3. As is mentioned in Houdré and Kawai [7], an advantage of using this Volterra kernel as opposed to other kernels, such as the moving average kernel, is that it is only defined on $[0,t]$. So, while the moving average kernel has a more simple form, that is:

$$(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}$$

it has the disadvantage that it is defined on the whole real line.

3.3 Short and long term behavior

Some preliminaries are necessary before the main theorem of this section is stated. First, we begin with a proposition that has been adapted from Houdré and Kawai [7].

Lemma 3.3.1. *Let* $\{Y_t^H : t \geq 0\}$ *be fGTSm in* $\mathbb{R}$. *Then,*

(a) $\{Y_t^H : t \geq 0\}$ *has covariance structure given by*

$$Cov(Y_t^H, Y_s^H) = \frac{1}{2}\left(t^{2G} + s^{2G} - |t-s|^{2G}\right) E(X_1^{GTS})^2$$

where $t, s \in \mathbb{R}^+$.

(b) *For each* $t > 0$ *and* $h > 0$, *fGTSm satisfies what Houdré and Kawai refer to as "second-order self-similarity." That is,*

$$E(Y_{ht}^H)^2 = h^{2G} E(Y_t^H)^2$$

(c) *For each* $t > 0$ *and* $s > 0$, *fGTSm satisfies what Houdré and Kawai refer to as "second-order stationary increments." That is,*

$$E(Y_t^H - Y_s^H)^2 = |t-s|^{2G} E(X_1^{GTS})^2 = E(Y_{|t-s|}^H)^2$$

Proof. (b) and (c) follow directly from (a), so it is only necessary to prove (a). Now, notice that since $E(Y_t^H) = 0$ for each $t > 0$, we have

$$\begin{aligned} Cov(Y_t^H, Y_s^H) =& E(Y_t^H Y_s^H) \\ =& E\left(\int_0^t K_{H,\alpha}(t,u) dX_u^{GTS} \int_0^s K_{H,\alpha}(s,u) dX_u^{GTS}\right) \\ =& E(X_1^{GTS})^2 E\left(\int_0^{t\wedge s} K_{H,\alpha}(t,u) K_{H,\alpha}(s,u) dX_u^{GTS}\right) \end{aligned}$$

by the Wiener-Ito isometry. And, by the properties of the Volterra kernel, the above is equal to

$$\frac{1}{2}\left(t^{2G} + s^{2G} - |t-s|^{2G}\right) E(X_1^{GTS})^2.$$

Thus, the lemma is proven. □

Lemma 3.3.2. *If $H \in (1/\alpha, 1/\alpha + 1/2)$, then there exists a continuous version of $\{Y_t^H : t \geq 0\}$. Moreover, for any $c \in (0, H - 1/\alpha)$, this continuous version is a.s. locally Hölder continuous with exponent c.*

Proof. Since

$$E|Y_t^H - Y_s^H|^2 = |t-s|^{2G} E(X_1^{GTS})^2,$$

there is a continuous version by Kolmogorov-Chentsov (see Theorem 3.23 in Kallenberg [9]). And, the continuous version is a.s. locally Hölder continuous with exponent

$$c \in \left(0, \frac{2G-1}{2}\right) = \left(0, \frac{2(H - 1/\alpha + 1/2) - 1}{2}\right) = (0, H - 1/\alpha).$$

□

Now we define fractional Brownian motion and fractional stable motion since they arise as limiting processes of fGTSm.

Definition 3.3.3. Fractional Brownian motion (fBm), $\{B_t^G : t \in \mathbb{R}\}$ with $G \in (0, 1]$ is a centered Gaussian process with continuous paths and covariance given by:

$$Cov(B_t^G, B_s^G) = \frac{1}{2}\left(t^{2G} + s^{2G} - |t-s|^{2G}\right)$$

where $t, s \in \mathbb{R}$.

Definition 3.3.4. Fractional stable motion (fSm) in $\mathbb{R}$ is defined as

$$Y_t^{H,\alpha} := \int_0^t K_{H,\alpha}(t,s) dX_s^\alpha \qquad \text{for} \quad t \geq 0$$

where $\{X_t^\alpha : t \geq 0\}$ is a stable process with characteristic function at time 1 given by:

$$E\exp\{iuX_1^\alpha\} = \begin{cases} \exp\left\{\int_{\mathbb{R}_0} \left(e^{iux} - 1\right) \nu_{\sigma_1}^\alpha(dx)\right\} & if \quad \alpha \in (0,1) \\ \\ \exp\left\{\int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux1_{|x|\leq 1}(x)\right) \nu_{\sigma_1}^1(dx)\right\} & if \quad \alpha = 1 \\ \\ \exp\left\{\int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux\right) \nu_{\sigma_1}^\alpha(dx)\right\} & if \quad \alpha \in (1,2). \end{cases}$$

Let $t_1, \ldots, t_k$ be a finite nondecreasing sequence of times and let $a_1, \ldots, a_k$ be real numbers. Then, the finite dimensional distributions of fractional stable motion are given by:

$$\begin{aligned} E[\exp\{iu\sum_{i=1}^k a_i Y_{t_i}^{H,\alpha}\}] =& E[\exp\{iu\sum_{i=1}^k a_i \left(\int_0^{t_i} K_{H,\alpha}(t_i,s)dX_s^\alpha\right)\}] \\ =& E[\exp\{iu\left(\int_0^{t_k} \sum_{i=1}^k a_i K_{H,\alpha}(t_i,s)dX_s^\alpha\right)\}] \\ =& \exp\left\{\int_0^{t_k} \psi_\alpha(u\sum_{i=1}^k a_i K_{H,\alpha}(t_i,s))ds\right\} \end{aligned}$$

where

$$\psi_\alpha(u) := \begin{cases} \int_{\mathbb{R}_0} \left(e^{iux} - 1\right) \nu^{GTS}(dx) \quad if \quad \alpha \in (0,1) \\ \\ \int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux1_{|x|\leq 1}(x)\right) \nu^{GTS}(dx) \quad if \quad \alpha = 1 \\ \\ \int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux\right) \nu^{GTS}(dx) \qquad if \quad \alpha \in (1,2). \end{cases}$$

Remark 3.3.5. In [7], Houdré and Kawai claim that fSm has stationary increments. That is, for any $t > s$,

$$L_t^{H,\alpha} - L_s^{H,\alpha} \stackrel{d}{=} L_{t-s}^{H,\alpha}.$$

But, as is mentioned by Pipiras and Taqqu in [13], this is not true.

Proof. To see this, let $t := 2$ and $s := 1$. Then,

$$E[\exp\{i\theta(L_t^{H,\alpha} - L_s^{H,\alpha})\}] = E[\exp\{i\theta\left(\int_0^2 K_{H,\alpha}(2,u)dX_u^\alpha - \int_0^1 K_{H,\alpha}(1,u)dX_u^\alpha\right)\}]$$

$$= E[\exp\{i\theta\left(\int_0^2 (K_{H,\alpha}(2,u) - K_{H,\alpha}(1,u))\, dX_u^\alpha\right)\}],$$

which is not equal to

$$E[\exp\{i\theta\left(\int_0^1 K_{H,\alpha}(1,u) dX_u^\alpha\right)\}],$$

which is the same as

$$E[\exp\{i\theta(L_{t-s}^{H,\alpha})\}].$$

This is due to the fact (letting $\alpha := 1$) that

$$\int_0^2 |K_{H,1}(2,u) - K_{H,1}(1,u)|^\alpha\, du \neq \int_0^1 K_{H,1}(1,u) du.$$

To see this, consider $H \in (1/\alpha, 1/\alpha + 1/2)$. Then the kernel may be simplified to

$$K_{H,\alpha}(t,u) = c_{H,\alpha}\left(H - \frac{1}{\alpha}\right) u^{1/\alpha - H} \int_u^t (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[0,t]}(u),$$

which is positive (since $H > 1/\alpha$) and increasing in t. Thus, $K_{H,1}(2,u) - K_{H,1}(1,u) \geq 0$ and

$$\int_0^2 |K_{H,1}(2,u) - K_{H,1}(1,u)|^\alpha\, du = \int_0^2 (K_{H,1}(2,u) - K_{H,1}(1,u))\, du.$$

Now,

$$K_{H,\alpha}(2,u) - K_{H,\alpha}(1,u)$$

$$= K\, u^{1/\alpha - H}\Big\{ \int_u^2 (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[0,2]}(u)$$

$$- \int_u^1 (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[0,1]}(u)\Big\}$$

for some constant K. And, this is equal to

$$K\, u^{1/\alpha - H}\Big\{ \int_1^2 (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[0,1)}(u)$$

$$+\int_u^2 (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[1,2]}(u)\Big\}$$

$$= K\, u^{1/\alpha-H} \int_1^2 (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[0,1)}(u)$$

$$+K\, u^{1/\alpha-H} \int_u^2 (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[1,2]}(u)$$

$$:= A + B.$$

Now, consider the expression involving A. We have,

$$K\, u^{1/\alpha-H} \int_0^2 A\, du$$

$$= K \int_0^2 u^{1/\alpha-H} \int_1^2 (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[0,1)}(u)\, du$$

$$= K \int_0^1 \int_1^2 \left(u^{1/\alpha-H} (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\right) dw\, du := I.$$

And, for the B component, notice

$$K\, u^{1/\alpha-H} \int_0^2 B\, du$$

$$= K \int_0^2 \int_u^2 u^{1/\alpha-H} (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[1,2]}(u)\, du$$

$$= K \int_1^2 \int_u^2 \left(u^{1/\alpha-H} (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\right) dw\, du := II.$$

And,

$$K_{H,\alpha}(1,u) = K\, u^{1/\alpha-H} \int_u^1 (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[0,1]}(u) := C.$$

Now,

$$\int_0^1 C\, du = K \int_0^1 u^{1/\alpha-H} \int_u^1 (w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\, dw\ 1_{[0,1]}(u)\, du$$

$$= K \int_0^1 \int_u^1 \left(u^{1/\alpha - H}(w-u)^{H-1/\alpha-1} w^{H-1/\alpha}\right) dw\, du := III.$$

And, $I - II \neq III$, thus,

$$E[\exp\{i\theta(L_t^{H,\alpha} - L_s^{H,\alpha})\}] \neq E[\exp\{i\theta(L_{t-s}^{H,\alpha})\}],$$

so fSm does not have stationary increments.

□

Now, we are in a position to state the main theorem of this section.

Theorem 3.3.6. *Consider a fractional GTS motion with no drift term. Let*

$$k_t := \int_0^t K_{H,\alpha}(t,s)ds$$

and

$$n_\alpha := \begin{cases} -\int_{\{|x|\leq 1\}} x\, \nu^{GTS}(dx) & if \quad \alpha \in (0,1) \\ 0 & if \quad \alpha = 1 \\ \int_{\{|x|>1\}} x\, \nu^{GTS}(dx) & if \quad \alpha \in (1,2). \end{cases}$$

(a) *Then,*

Case(i): (Small Scales) *as $h \to 0$,*

$$\{h^{-H} Y_{ht}^H - h^{1-1/\alpha}\, n_\alpha k_t\}_{t\geq 0} \xrightarrow{fdd} \{Y_t^{H,\alpha}\}_{t\geq 0}$$

where $\{Y_t^{H,\alpha}\}_{t\geq 0}$ is a fractional stable motion of index α.

Case(ii): (Large Scales) *as $h \to \infty$,*

$$\{h^{-G} Y_{ht}^H\} \xrightarrow{fdd} \{B_t^G\}_{t\geq 0}$$

where $\{B_t^G\}_{t\geq 0}$ a centered fractional Brownian motion with $G := H - 1/\alpha + 1/2$.

(b) *Moreover, for $H \in (1/\alpha, 1/\alpha + 1/2)$, the convergence results in case (ii) can be strengthened to weak convergence in the space $C([0,\infty), \mathbb{R})$.*

Proof. **Part (a) Case(i): (Small Scales)** Let

$$Z_t^h := h^{-H}(Y_{ht}^H) - h^{1-1/\alpha}\, n_\alpha k_t.$$

Then, we need to show that when $t_1, \ldots, t_k$ is a finite nondecreasing sequence of times and $a_1, \ldots, a_k$ is a sequence of real numbers, we have that $\sum_{i=1}^k a_i Z_{t_i}^h$ converges in law to $\sum_{i=1}^k a_i Y_{t_i}^{H,\alpha}$ as $h \to 0$. So, first notice that

$$Z_t^h = h^{-H}(Y_{ht}^H) - h^{1-1/\alpha}\, n_\alpha k_t = \int_0^t \left(h^{-H} K_{H,\alpha}(ht, s)\right)\, dX_s^{GTS} - h^{1-1/\alpha}\, n_\alpha k_t,$$

which, by the scaling property equals

$$\int_0^t \left(h^{-H} h^{H-1/\alpha} K_{H,\alpha}(t, s/h)\right)\, dX_s^{GTS} - h^{1-1/\alpha}\, n_\alpha k_t$$

$$= \int_0^t h^{-1/\alpha} K_{H,\alpha}(t, s/h) dX_s^{GTS} -$$

$$\begin{cases} -h^{1-1/\alpha} \int_0^t \int_{\{|x|\le 1\}} x K_{H,\alpha}(t,s)\, \nu^{GTS}(dx)\, ds & if \quad \alpha \in (0,1) \\ 0 & if \quad \alpha = 1 \\ h^{1-1/\alpha} \int_0^t \int_{\{|x|>1\}} x K_{H,\alpha}(t,s)\, \nu^{GTS}(dx)\, ds & if \quad \alpha \in (1,2). \end{cases}$$

Thus, the finite dimensional characteristic function of $Z_t^h = h^{-H}(Y_{ht}^H) - h^{1-1/\alpha} t\, n_\alpha k_t$ is given by

$$\exp\left\{ \int_0^{t_k} \psi_0(u h^{-1/\alpha} \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s/h)) ds - h^{1-1/\alpha} u \sum_{i=1}^k a_i\, n_\alpha k_t \right\}$$

$$= \exp\left\{ \int_0^{t_k} h\psi_0(u h^{-1/\alpha} \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s)) ds - h^{1-1/\alpha} u \sum_{i=1}^k a_i\, n_\alpha k_t \right\}$$

which equals

$$\exp\left\{ \int_0^{t_k} h\psi(u h^{-1/\alpha} \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s)) ds \right\}$$

where

$$\psi(u) := \begin{cases} \int_{\mathbb{R}_0} \left(e^{iux} - 1\right) \nu^{GTS}(dx) \quad if \quad \alpha \in (0,1) \\ \\ \int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux1_{|x|\leq 1}(x)\right) \nu^{GTS}(dx) \quad if \quad \alpha = 1 \\ \\ \int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux\right) \nu^{GTS}(dx) \qquad if \quad \alpha \in (1,2). \end{cases}$$

For case (i), we need to show

$$\exp\left\{\int_0^{t_k} h\psi(uh^{-1/\alpha}\sum_{i=1}^{k} a_i K_{H,\alpha}(t_i,s))ds\right\} \to \exp\left\{\int_0^{t_k} \psi_\alpha(u\sum_{i=1}^{k} a_i K_{H,\alpha}(t_i,s))ds\right\}$$

as $h \to 0$ where

$$\psi_\alpha := \begin{cases} \int_{\mathbb{R}_0} \left(e^{iux} - 1\right) \nu_{\sigma_1}^{\alpha}(dx) \quad if \quad \alpha \in (0,1) \\ \\ \int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux1_{|x|\leq 1}(x)\right) \nu_{\sigma_1}^{\alpha}(dx) \quad if \quad \alpha = 1 \\ \\ \int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux\right) \nu_{\sigma_1}^{\alpha}(dx) \qquad if \quad \alpha \in (1,2). \end{cases}$$

Observe that

$$h\psi(uh^{-1/\alpha}\sum_{i=1}^{k} a_i K_{H,\alpha}(t_i,s)) \to \psi_\alpha(u\sum_{i=1}^{k} a_i K_{H,\alpha}(t_i,s))$$

as $h \to 0$. To see this, consider $\alpha = 1$ and, for simplicity, let $K := \sum_{i=1}^{k} a_i K_{H,\alpha}(t_i,s)$. Then,

$$h\psi(uh^{-1/\alpha}\sum_{i=1}^{k} a_i K_{H,\alpha}(t_i,s))$$

$$= \int_{\mathbb{R}_0} \left(e^{iuh^{-1/\alpha}Kx} - 1 - iuh^{-1/\alpha}Kx1_{|x|\leq 1}(x)\right) h\nu^{GTS}(dx)$$

which equals

$$\int_{\mathbb{R}_0} \left(e^{iuKh^{-1/\alpha}x} - 1 - iuKh^{-1/\alpha}x1_{|h^{-1/\alpha}x|\leq 1}(h^{-1/\alpha}x)\right) h\nu^{GTS}(dx)$$

$$= \int_{\mathbb{R}_0} \left(e^{iuKx} - 1 - iuKx1_{|x|\le 1}(x)\right) h(\nu^{GTS} \circ \phi_h)(dx)$$

where

$$\phi_h(x) := h^{1/\alpha} x.$$

So,

$$h(\nu^{GTS} \circ \phi_h)(dx) = h\, \nu^{GTS}(d(h^{1/\alpha} r), d\xi)$$

equals

$$h\; q(h^{1/\alpha} r)\, (h^{1/\alpha} r)^{-\alpha-1}\, d(h^{1/\alpha} r)\, \sigma(d\xi)$$

$$= q(h^{1/\alpha} r)\, r^{-\alpha-1}\, dr\, \sigma(d\xi).$$

Now, since

$$lim_{h\to 0} q(h^{1/\alpha} r, \cdot) = c_1(\cdot) \qquad \text{in} \quad L^1(S^{d-1}, \sigma),$$

and since $c_1(\xi)\sigma(d\xi) := \sigma_1(d\xi)$, we have

$$\int_{\mathbb{R}_0} \left(e^{iuKx} - 1 - iuKx1_{|x|\le 1}(x)\right) h(\nu \circ \phi_h)^{GTS}(dx)$$

$$\to \int_{\mathbb{R}_0} \left(e^{iuKx} - 1 - iuKx1_{|x|\le 1}(x)\right) \nu^{\alpha}_{\sigma_1}(dx)$$

as $h \to 0$. And, the Lebesgue Dominated Convergence Theorem applies by the known inequality

$$\left|e^{it} - 1 - it1_{|t|\le 1}(t)\right| \le 2\{1 \wedge |t|^2\}$$

since

$$\int_0^{t_k} \int_{\mathbb{R}_0} \left|e^{iuKx} - 1 - iuKx1_{|x|\le 1}(x)\right| h(\nu^{GTS} \circ \phi_h)(dx) ds$$

$$= \int_0^{t_k} \int_S \int_0^{\infty} \left|e^{iuKr\xi} - 1 - iuKr\xi 1_{|r|\le 1}(r\xi)\right| q(h^{1/\alpha} r, \xi) r^{-\alpha-1} dr \sigma(d\xi) ds$$

is less than or equal to

$$\int_0^{t_k} \int_S \int_0^{\infty} 2\{1 \wedge |uKr\xi|^2\} q(h^{1/\alpha} r, \xi) r^{-\alpha-1} dr \sigma(d\xi) ds$$

$$\le 2 \int_0^{t_k} 1 \vee |uK|^2\, ds \int_S \int_0^{\infty} \{1 \wedge r^2\} q(h^{1/\alpha} r, \xi) r^{-\alpha-1} dr \sigma(d\xi)$$

$$= 2\int_0^{t_k} 1 \vee |uK|^2\, ds \int_0^\infty \{1\wedge r^2\} \left[\int_S q(h^{1/\alpha} r,\xi)\sigma(d\xi)\right] r^{-\alpha-1} dr$$

And, notice that since

$$\int_{S^{d-1}} q(r,\xi)\,\sigma(d\xi) \to \int_{S^{d-1}} c_1(\xi)\,\sigma(d\xi) \qquad \text{as} \quad r\to 0,$$

there exists an r_0 such that:

$$\sup_{r\le r_0} \int_{S^{d-1}} q(r,\xi)\,\sigma(d\xi) < \infty.$$

So, we can choose h small enough such that

$$\int_S q(h^{1/\alpha} r,\xi)\sigma(d\xi) = C < \infty.$$

Thus,

$$2\int_0^{t_k} 1 \vee |uK|^2\, ds \int_0^\infty \{1\wedge r^2\} \left[\int_S q(h^{1/\alpha} r,\xi)\sigma(d\xi)\right] r^{-\alpha-1} dr$$

is less than or equal to

$$2C\int_0^{t_k} 1 \vee |uK|^2\, ds \int_0^\infty \{1\wedge r^2\} r^{-\alpha-1} dr < \infty$$

$$= 2C\int_0^{t_k} 1 \vee \left|u\sum_{i=1}^k a_i K_{H,\alpha}(t_i,s)\right|^2 ds \int_0^\infty \{1\wedge r^2\} r^{-\alpha-1} dr < \infty$$

since $K_{H,\alpha}(t_i,\cdot)\in L^2([0,t_k])$ and $\left|\sum_{i=1}^k a_i K_{H,\alpha}(t_i,s)\right|^2 \le k\sum_{i=1}^k (a_i K_{H,\alpha}(t_i,s))^2$.

Part (a) Case(ii): (Large Scales) Let

$$Z_t^h := h^{-G} Y_{ht}^H - h^{1-1/2} n_\alpha k_t.$$

Let $t_1,\ldots,t_k$ be a finite nondecreasing sequence of times and $a_1,\ldots,a_k$ be a sequence of real numbers. Now, notice that

$$Z_t^h = h^{-G} Y_{ht}^H = \int_0^t \left(h^{-G} K_{H,\alpha}(ht,s)\right)\, dX_s^{GTS}$$

which, by the scaling property equals

$$\begin{aligned}\int_0^t \left(h^{-G}h^{H-1/\alpha}K_{H,\alpha}(t,s/h)\right)\, dX_s^{GTS} &= \int_0^t \left(h^{-(H-1/\alpha+1/2)}h^{H-1/\alpha}K_{H,\alpha}(t,s/h)\right)\, dX_s^{GTS} \\ &= \int_0^t \left(h^{-1/2}K_{H,\alpha}(t,s/h)\right)\, dX_s^{GTS} \\ &= \int_0^t h^{-1/2}K_{H,\alpha}(t,s/h)dX_s^{GTS}\end{aligned}$$

Thus, the finite dimensional characteristic function of $Z_t^h = h^{-G}Y_{ht}^H$ is given by

$$\exp\left\{\int_0^{t_k} \psi_0(uh^{-1/2}\sum_{i=1}^k a_iK_{H,\alpha}(t_i,s/h))ds\right\}$$

$$= \exp\left\{\int_0^{t_k} h\psi_0(uh^{-1/2}\sum_{i=1}^k a_iK_{H,\alpha}(t_i,s))ds\right\}$$

where

$$\psi_0(u) := \int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux1_{|x|\le 1}(x)\right)\nu^{GTS}(dx).$$

Now, it is known that there exists a constant C such that

$$\left|e^{it} - 1 - it1_{|t|\le 1}(t) + \frac{t^2}{2}\right| \le C|t|^3$$

Now, letting $K := \sum_{i=1}^k a_iK_{H,\alpha}(t_i,s)$ we see that

$$\left|e^{iuh^{-1/2}Kx} - 1 - iuh^{-1/2}Kx1_{|x|\le 1}(x) - \frac{(uh^{-1/2}Kx)^2}{2}\right|$$

is less than or equal to

$$Ch|uh^{-1/2}Kx|^3$$

$$= h^{-1/2}\cdot C|uK_{H,\alpha}(t_i,s)x|^3 \to 0 \quad \text{as} \quad h\to\infty$$

Thus,

$$hf(uh^{-1/2}\sum_{i=1}^k a_iK_{H,\alpha}(t_i,s)) \to -\frac{1}{2}\left(u\sum_{i=1}^k a_iK_{H,\alpha}(t_i,s)x\right)^2 \quad \text{as} \quad h\to\infty.$$

where

$$f(x) := e^{iux} - 1 - iux1_{|x|\leq 1}(x).$$

And, the Lebesgue Dominated Convergence Theorem applies since

$$\left|e^{it} - 1 - it1_{|t|\leq 1}(t)\right| \leq 2|t|^2$$

implies

$$\int_0^{t_k}\int_{\mathbb{R}_0}\left|h\psi_0(uh^{-1/2}\sum_{i=1}^{k}a_iK_{H,\alpha}(t_i,s))\right|\nu^{GTS}(dx)ds$$

$$= \int_0^{t_k}\int_{\mathbb{R}_0}\left|e^{iuh^{-1/2}Kx} - 1 - iuh^{-1/2}Kx1_{|x|\leq 1}(x)\right| h\nu^{GTS}(dx)ds$$

is less than or equal to

$$\int_0^{t_k}\int_{\mathbb{R}_0} 2\left|uh^{-1/2}Kx\right|^2 h\nu^{GTS}(dx)ds$$

which equals

$$\int_0^{t_k}\int_{\mathbb{R}_0} 2\left|u\sum_{i=1}^{k}a_iK_{H,\alpha}(t_i,s)x\right|^2 \nu^{GTS}(dx)ds$$

$$= 2u^2\int_0^{t_k}\left(\sum_{i=1}^{k}a_iK_{H,\alpha}(t_i,s)\right)^2 ds\int_{\mathbb{R}_0}|x|^2\,\nu^{GTS}(dx)$$

which is finite since $\int_{\mathbb{R}_0}|x|^2\,\nu^{GTS}(dx) < \infty$ and since $K_{H,\alpha}(t_i,\cdot) \in L^2([0,t_k])$.

Now, the finite dimensional characteristic function of the limiting process is:

$$\int_0^{t_k}\int_{\mathbb{R}_0} -\frac{1}{2}\left(u\sum_{i=1}^{k}a_iK_{H,\alpha}(t_i,s)x\right)^2 \nu^{GTS}(dx)ds.$$

And, the marginal characteristic function of the limiting process is:

$$\int_0^{t}\int_{\mathbb{R}_0} -\frac{1}{2}\left(uK_{H,\alpha}(t,s)x\right)^2 \nu^{GTS}(dx)ds$$

which equals

$$-\frac{1}{2}u^2\int_{\mathbb{R}_0}x^2\nu^{GTS}(dx)\int_0^t\left(K_{H,\alpha}(t,s)\right)^2 ds$$

$$= -\frac{1}{2}t^{2G}cu^2$$

using property (6) of the Volterra kernel and letting $c := \int_{\mathbb{R}_0}x^2\nu^{GTS}(dx) < \infty$.

Part (b) Case(ii) We will show the tightness of the sequence $\{h^{-G}Y_{ht}^{H}\}$ using Corollary 16.9 of Kallenberg [9]. Since,

$$E(Y_t^H - Y_s^H)^2 = |t-s|^{2G} E(X_1^{GTS})^2$$

we have that

$$\begin{aligned} E|h^{-G}Y_{ht}^{H} - h^{-G}Y_{hs}^{H}|^2 &= h^{-2G}\,|ht-hs|^{2G}E(X_1^{GTS})^2 \\ &= |t-s|^{2G}E(X_1^{GTS})^2. \end{aligned}$$

Thus, the sequence is tight in $C([0,\infty),\mathbb{R})$ and convergence in the space $C([0,\infty),\mathbb{R})$ follows immediately.

□

Remark 3.3.7. Beware of the typo in the convergence result in Houdré and Kawai [7]. They claim that the limit in the Gaussian case is fBm with exponent $H - 1/2 + 1/\alpha$, which is incorrect. The correct exponent is $H - 1/\alpha + 1/2$, as is stated in this theorem.

Remark 3.3.8. Recall that when $H = 1/\alpha$, fGTSm is a GTS process, and the above results are analogous to the short and long term behavior of GTS processes. Notice that the short time limiting process is a stable process and the long time limiting process is actually a Brownian motion when $H = 1/\alpha$.

The purpose of this chapter is to examine processes driven by GTS processes, just as Houdré and Kawai studied processes driven by TS processes in "On fractional tempered stable motion" in 2006. Houdré and Kawai introduced fractional tempered stable motion (fTSm) as an analog of fractional Brownian motion (fBm). As is discussed in [7], the light tails of fBm are often inadequate for modeling phenomena with higher variability. On the other hand, stable generalizations have infinite second moments. So they proposed fTSm as an alternative, since it possesses properties of both fBm and stable generalizations. They found that fTSm has the same covariance structure as fBm. Also, in the long term it behaves as a fBm while behaving like fractional stable motion in the short term. This chapter establishes the appropriate analog definition of fGTSm and establishes the corresponding short and long term behavior. And, please note that the convergence results obtained in this chapter required more work to prove than the convergence results for GTS processes. This is due to the fact that GTS processes are Lévy processes, but fGTSm is not necessarily a Lévy process.

Chapter 4

Multifractal infinitely divisible processes

In their 2007 paper "Small and large scale asymptotics of some Lévy stochastic integrals," Vladas Pipiras of University of North Carolina at Chapel Hill and Murad S. Taqqu of Boston University described general conditions for normalized, time-scaled stochastic integrals of independently scattered Levy random measures to converge to a limit [13]. The idea is to provide general conditions to bypass the use of characteristic functions, which can sometimes have tedious calculations, to simplify and shorten the proofs of convergence of infinitely divisible processes, which have previously been done on a case by case basis. It is of particular interest to study both small and large scale asymptotics, since there are many applications, such as modeling internet traffic. The purpose of this chapter is to generalize these results to the greater class of all stochastically continuous (or, more generally, separable in probability) infinitely divisible processes with no Gaussian part and to expand the results to the multi-dimensional case. There are some interesting examples where rescaling towards a small time scale yields an infinite variance limit and rescaling towards a large time scale yields a Gaussian limit. These examples are presented at the end of the chapter.

4.1 Preliminaries

Let X be a d-dimensional stochastically continuous infinitely divisible processes, without Gaussian component and drift. All such processes have an integral representation given by

$$X_t = \int_S f_t(s) \left(N(ds) - \frac{\nu(ds)}{1 \vee \|f_t(s)\|} \right) \tag{4.1}$$

where (S, S) is a measurable space, $\{f_t(s)\}$ are measurable deterministic functions in $\mathbb{R}^d$ and $N(ds)$ is a Poisson random measure on (S, S) with intensity measure $\nu(ds)$.

It is known by Theorem 2.7 of Rajput and Rosinski [14] that the integrals exist if and only if

$$\int_S 1 \wedge \|f_t(s)\|^2 \nu(ds) < \infty. \tag{4.2}$$

The process X has finite-dimensional characteristic function given by

$$E \exp\{i \sum_{j=1}^{n} \langle \theta_j, X_{t_j} \rangle\} = \exp\left\{ \int_S \left(\exp\{i \sum_{j=1}^{n} \langle \theta_j, f_{t_j}(s) \rangle\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, f_{t_j}(s) \rangle}{1 \vee \|f_{t_j}(s)\|} \right) \nu(ds) \right\}.$$

for $\theta_j \in \mathbb{R}^d, t_j \in \mathbb{R}$.

4.2 Lemmas

We will need the following technical lemmas in the proof of the main theorem. Please understand that we are not making any assumptions about integrability yet, we merely need these inequalities to hold true so they may be used later.

Lemma 4.2.1. *Let*

$$E := \left| \exp\{i \sum_{j=1}^{n} \langle \theta_j, x_j \rangle\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, x_j \rangle}{1 \vee \|x_j\|} + \frac{1}{2} \left(\sum_{j=1}^{n} \langle \theta_j, x_j \rangle \right)^2 \right|.$$

where $x_j, \theta_j \in \mathbb{R}^d$. Then,

$$E \leq C \left(\sum_{j=1}^{n} \|x_j\| \right)^3$$

for some constant C depending only on n and $\theta_1, \ldots, \theta_n$.

Proof. First, notice that E is equal to

$$\left| \exp\{i \sum_{j=1}^{n} \langle \theta_j, x_j \rangle\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, x_j \rangle}{1 \vee \|x_j\|} + \frac{1}{2} \left(\sum_{j=1}^{n} \langle \theta_j, x_j \rangle \right)^2 \right|$$

$$\leq \left| \exp\{i \sum_{j=1}^{n} \langle \theta_j, x_j \rangle\} - 1 - i \sum_{j=1}^{n} \langle \theta_j, x_j \rangle + \frac{1}{2} \left(\sum_{j=1}^{n} \langle \theta_j, x_j \rangle \right)^2 \right|$$

$$+\left|\sum_{j=1}^{n}\langle\theta_j,x_j\rangle-\sum_{j=1}^{n}\frac{\langle\theta_j,x_j\rangle}{1\vee\|x_j\|}\right|$$

$$:= E_1+E_2.$$

Now, for E_1, we will use the inequality:

$$\left|\exp\{it\}-1-it+\frac{t^2}{2}\right|\leq\frac{1}{6}|t|^3,$$

which is true for any value $t\in\mathbb{R}$ (not to be confused, of course, with the time variable). So, we have,

$$E_1=\left|\exp\{i\sum_{j=1}^{n}\langle\theta_j,x_j\rangle\}-1-i\sum_{j=1}^{n}\langle\theta_j,x_j\rangle+\frac{1}{2}\left(\sum_{j=1}^{n}\langle\theta_j,x_j\rangle\right)^2\right|$$

$$\leq\frac{1}{6}\left|\sum_{j=1}^{n}\langle\theta_j,x_j\rangle\right|^3$$

$$\leq\left(\sum_{j=1}^{n}|\langle\theta_j,x_j\rangle|\right)^3$$

by the triangle inequality and the fact that $f(x):=x^3$ is increasing for $x\geq 0$ (and, of course the fact that $\frac{1}{6}\leq 1$). Now,

$$\left(\sum_{j=1}^{n}|\langle\theta_j,x_j\rangle|\right)^3\leq\left(\sum_{j=1}^{n}\|\theta_j\|\cdot\|x_j\|\right)^3$$

by the Cauchy-Schwarz inequality. And, if we let $\theta^*:=\max\{\|\theta_1\|,\ldots,\|\theta_n\|\}$, then

$$\left(\sum_{j=1}^{n}\|\theta_j\|\cdot\|x_j\|\right)^3\leq(\theta^*)^3\left(\sum_{j=1}^{n}\|x_j\|\right)^3.$$

So, we have an upper bound for E_1. Now, we will turn our attention to E_2. Observe that if $\|x_j\|\leq 1$ for *all* of the $x_j's$ then E_2 would equal zero, since E_2 would be equal to

$$\left|\sum_{j=1}^{n}\langle\theta_j,x_j\rangle-\sum_{j=1}^{n}\frac{\langle\theta_j,x_j\rangle}{1}\right|,$$

which indeed equals zero. So, in order for E_2 to be nonzero, we must have $\|x_j\| > 1$ for at least one x_j, or equivalently, $\max_{j\leq n} \|x_j\| > 1$.

Now,

$$E_2 = \left| \sum_{j=1}^{n} \langle \theta_j, x_j \rangle - \sum_{j=1}^{n} \frac{\langle \theta_j, x_j \rangle}{1 \vee \|x_j\|} \right|$$

$$= \left| \sum_{j=1}^{n} \langle \theta_j, x_j \rangle \left(1 - \frac{1}{1 \vee \|x_j\|} \right) \right|$$

$$\leq \sum_{j=1}^{n} | \langle \theta_j, x_j \rangle |$$

by the triangle inequality and the fact that

$$\left| 1 - \frac{1}{1 \vee \|x_j\|} \right| \leq 1$$

for any x_j. And,

$$\sum_{j=1}^{n} | \langle \theta_j, x_j \rangle | \leq \sum_{j=1}^{n} \|\theta_j\| \cdot \|x_j\|$$

by the Cauchy-Schwarz inequality. And, if we again let $\theta^* := \max\{\|\theta_1\|, \ldots, \|\theta_n\|\}$, then

$$\sum_{j=1}^{n} | \langle \theta_j, x_j \rangle | \leq \sum_{j=1}^{n} \|\theta_j\| \cdot \|x_j\| \leq \theta^* \sum_{j=1}^{n} \|x_j\|.$$

Now, since $\max_{j\leq n} \|x_j\| > 1$, it must be that $\sum_{j=1}^{n} \|x_j\| > 1$, so

$$\theta^* \sum_{j=1}^{n} \|x_j\| \leq \theta^* \left(\sum_{j=1}^{n} \|x_j\| \right)^3$$

since $x \leq x^3$ for any real number,$x \geq 1$. So, we have an upper bound for E_2.

Now, let $C := 2 \cdot \max\{\theta^*, (\theta^*)^3\}$. Then, we have,

$$E \leq E_1 + E_2 \leq C \left(\sum_{j=1}^{n} \|x_j\| \right)^3$$

which is the desired inequality.

□

Lemma 4.2.2. *Let*

$$E := \left| \exp\{i \sum_{j=1}^{n} \langle \theta_j, x_j \rangle\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, x_j \rangle}{1 \vee \|x_j\|} \right|$$

where $x_j, \theta_j \in \mathbb{R}^d$. Then,

$$E \leq C \cdot \sum_{j=1}^{n} \{1 \wedge \|x_j\|^2\}$$

for some constant C (possibly different) depending only on n and $\theta_1, \ldots, \theta_n$.

Moreover, since

$$\{1 \wedge \|x_j\|^2\} \leq \|x_j\|^2,$$

for each x_j we can also conclude that

$$E \leq C \cdot \sum_{j=1}^{n} \|x_j\|^2.$$

Proof. Notice

$$E \leq \left| \exp\{i \sum_{j=1}^{n} \langle \theta_j, x_j \rangle\} - 1 - i \frac{\sum_{j=1}^{n} \langle \theta_j, x_j \rangle}{1 \vee |\sum_{j=1}^{n} \langle \theta_j, x_j \rangle|} \right|$$

$$+ \left| \frac{\sum_{j=1}^{n} \langle \theta_j, x_j \rangle}{1 \vee |\sum_{j=1}^{n} \langle \theta_j, x_j \rangle|} - \sum_{j=1}^{n} \frac{\langle \theta_j, x_j \rangle}{1 \vee \|x_j\|} \right|$$

$$:= E_1 + E_2.$$

First, we will find an upper bound for E_1. To do this, we will need the following inequality, which is true for any $t \in \mathbb{R}$:

$$\left| \exp\{it\} - 1 - i \frac{t}{1 \vee |t|} \right| \leq 3 \cdot \{|t|^2 \wedge 1\}.$$

To verify the inequality, we will consider 2 cases: $|t| \leq 1$ and $|t| > 1$. If we assume first that $|t| \leq 1$, we get that the RHS of the inequality is equal to $3|t|^2$ and the LHS is equal to

$$|\exp\{it\} - 1 - it| \leq \frac{|t|^2}{2}$$

$$\leq 3|t|^2$$

which is equal to the RHS of the inequality. And, for the case $|t| > 1$, notice that the RHS of the inequality equals 3, while the LHS equals

$$\left|\exp\{it\} - 1 - i\frac{t}{|t|}\right| \leq |\exp\{it\}| + |1| + \left|i\frac{t}{|t|}\right|$$

by the triangle inequality, and this is less than or equal to $1 + 1 + 1 = 3$, which is the value of the RHS of the inequality, so the inequality is verified.

Now,

$$E_1 = \left|\exp\{i\sum_{j=1}^{n}\langle\theta_j, x_j\rangle\} - 1 - i\frac{\sum_{j=1}^{n}\langle\theta_j, x_j\rangle}{1 \vee |\sum_{j=1}^{n}\langle\theta_j, x_j\rangle|}\right|$$

$$\leq 3 \cdot \left\{\left|\sum_{j=1}^{n}\langle\theta_j, x_j\rangle\right|^2 \wedge 1\right\}$$

$$\leq 3 \cdot \left\{\left(\sum_{j=1}^{n}|\langle\theta_j, x_j\rangle|\right)^2 \wedge 1\right\}$$

by the triangle inequality and the fact that $f(x) := x \wedge 1$ and $g(x) := x^2$ are both nondecreasing functions for $x \geq 0$. Then, by Cauchy-Schwarz, we get that

$$3 \cdot \left\{\left(\sum_{j=1}^{n}|\langle\theta_j, x_j\rangle|\right)^2 \wedge 1\right\} \leq 3 \cdot \left\{\left(\sum_{j=1}^{n}\|\theta_j\| \cdot \|x_j\|\right)^2 \wedge 1\right\}.$$

Now, once again letting $\theta^* := \max\{\|\theta_1\|, \ldots, \|\theta_n\|\}$, we obtain

$$3 \cdot \left\{\left(\sum_{j=1}^{n}\|\theta_j\| \cdot \|x_j\|\right)^2 \wedge 1\right\} \leq 3 \cdot \left\{\left\{(\theta^*)^2\left(\sum_{j=1}^{n}\|x_j\|\right)^2\right\} \wedge 1\right\}$$

$$\leq 3 \cdot \left\{\left\{(\theta^*)^2 n\sum_{j=1}^{n}\|x_j\|^2\right\} \wedge 1\right\}$$

using the fact that $(\sum_{j=1}^{m} a_j)^2 \leq m\sum_{j=1}^{m} a_j^2$ for any a_j. Now, notice that since $f(x) := x \wedge 1$ is a nondecreasing function for $x \geq 0$,

$$3 \cdot \left\{\left\{(\theta^*)^2 n\sum_{j=1}^{n}\|x_j\|^2\right\} \wedge 1\right\} \leq 3 \cdot \left\{(\theta^*)^2 n \vee 1\right\} \cdot \left\{\sum_{j=1}^{n}\|x_j\|^2 \wedge 1\right\}\Bigg\}$$

$$\leq 3 \cdot \left\{ (\theta^*)^2 n \vee 1\} \cdot \sum_{j=1}^{n} \{\|x_j\|^2 \wedge 1\} \right\}$$

$$:= C^1 \cdot \sum_{j=1}^{n} \{\|x_j\|^2 \wedge 1\}$$

where C^1 depends only on n and $\theta_1, \dots, \theta_n$. So, we have proven that

$$E_1 \leq C^1 \cdot \sum_{j=1}^{n} \{\|x_j\|^2 \wedge 1\}.$$

Now, we will shift our focus to E_2, which you will recall is given by

$$E_2 = \left| \frac{\sum_{j=1}^{n} \langle \theta_j, x_j \rangle}{1 \vee \left| \sum_{j=1}^{n} \langle \theta_j, x_j \rangle \right|} - \sum_{j=1}^{n} \frac{\langle \theta_j, x_j \rangle}{1 \vee \|x_j\|} \right|.$$

First, it is necessary to observe that, under certain conditions, $E_2 = 0$, so it is certainly bounded. To see this, notice that if $\left| \sum_{j=1}^{n} \langle \theta_j, x_j \rangle \right| \leq 1$and $\|x_j\| \leq 1$, then

$$E_2 = \left| \frac{\sum_{j=1}^{n} \langle \theta_j, x_j \rangle}{1} - \sum_{j=1}^{n} \frac{\langle \theta_j, x_j \rangle}{1} \right|$$

which indeed equals zero.

Now, we must alter these conditions somewhat, to suit our purposes. So, let $C_0 := \{1 \wedge \frac{1}{n\theta^*}\}$, which is another constant from our point of view, since it only depends on n and $\theta_1, \dots, \theta_n$. Then, if we suppose that

$$\max_{j \leq n} \|x_j\| \leq C_0,$$

this will imply that $\left| \sum_{j=1}^{n} \langle \theta_j, x_j \rangle \right| \leq 1$, since

$$\begin{aligned}\left|\sum_{j=1}^{n}\langle\theta_j,x_j\rangle\right| &\leq \sum_{j=1}^{n}|\langle\theta_j,x_j\rangle| \\ &\leq \sum_{j=1}^{n}\|\theta_j\|\cdot\|x_j\| \\ &\leq \theta^*\sum_{j=1}^{n}\|x_j\| \\ &\leq \theta^*\cdot n\cdot C_0 \\ &= 1\end{aligned}$$

by the triangle inequality and Cauchy-Schwarz inequality. Also, this condition implies that $\|x_j\| \leq 1$ for each j since $\|x_j\| \leq \max_{j\leq n}\|x_j\| \leq C_0 \leq 1$. Thus, E_2 can only be nonzero when $\max_{j\leq n}\|x_j\| > C_0$. So, for the rest of the proof that E_2 is bounded, we may assume $\max_{j\leq n}\|x_j\| > C_0$.

Now,

$$\begin{aligned}E_2 &= \left|\frac{\sum_{j=1}^{n}\langle\theta_j,x_j\rangle}{1\vee\left|\sum_{j=1}^{n}\langle\theta_j,x_j\rangle\right|} - \sum_{j=1}^{n}\frac{\langle\theta_j,x_j\rangle}{1\vee\|x_j\|}\right| \\ &\leq \sum_{j=1}^{n}|\langle\theta_j,x_j\rangle|\cdot\left|\frac{1}{1\vee|\sum_{j=1}^{n}\langle\theta_j,x_j\rangle|} - \frac{1}{1\vee\|x_j\|}\right| \\ &\leq \sum_{j=1}^{n}|\langle\theta_j,x_j\rangle| \\ &\leq \sum_{j=1}^{n}\|\theta_j\|\cdot\|x_j\| \\ &\leq \sqrt{\sum_{j=1}^{n}\|\theta_j\|^2}\cdot\sqrt{\sum_{j=1}^{n}\|x_j\|^2}\end{aligned}$$

by the Cauchy-Schwarz inequality. And,

$$\sqrt{\sum_{j=1}^{n}\|\theta_j\|^2}\cdot\sqrt{\sum_{j=1}^{n}\|x_j\|^2} \leq \sqrt{n}\theta^*\cdot\sqrt{\sum_{j=1}^{n}\|x_j\|^2}$$

$$= \sqrt{n}\theta^* C_0 \cdot \sqrt{\frac{1}{C_0^2}\sum_{j=1}^n \|x_j\|^2}.$$

Now, notice that $\max_{j\le n}\|x_j\| > C_0$ implies that $\frac{1}{C_0^2}\sum_{j=1}^n \|x_j\|^2 > 1$, so if we use the fact that $\sqrt{x} \le x$ for $x \ge 1$ then we get that

$$\sqrt{n}\theta^* C_0 \cdot \sqrt{\frac{1}{C_0^2}\sum_{j=1}^n \|x_j\|^2} \le \sqrt{n}\theta^* \frac{1}{C_0} \cdot \sum_{j=1}^n \|x_j\|^2$$
$$:= C^* \sum_{j=1}^n \|x_j\|^2$$

Now, if we could show that $E_2 \le C^{**}$ for some constant C^{**}, then we could let $C_2 := \{\max\{C^*, C^{**}\} \vee 1\}$ and we would have $E_2 \le C_2 \sum_{j=1}^n \|x_j\|^2$ and we would also have $E_2 \le C_2$ which would imply:

$$E_2 \le \left\{C_2 \wedge C_2 \sum_{j=1}^n \|x_j\|^2\right\}$$
$$\le C_2 \cdot \left\{1 \wedge \sum_{j=1}^n \|x_j\|^2\right\}.$$

So, we need to show that $E_2 \le C^{**}$ for some constant C^{**}.

To see this, notice

$$\begin{aligned}
E_2 =& \left| \frac{\sum_{j=1}^n \langle \theta_j, x_j\rangle}{1 \vee \left|\sum_{j=1}^n \langle \theta_j, x_j\rangle\right|} - \sum_{j=1}^n \frac{\langle \theta_j, x_j\rangle}{1 \vee \|x_j\|}\right| \\
\le& \frac{|\sum_{j=1}^n \langle \theta_j, x_j\rangle|}{1 \vee \left|\sum_{j=1}^n \langle \theta_j, x_j\rangle\right|} + \left|\sum_{j=1}^n \frac{\langle \theta_j, x_j\rangle}{1 \vee \|x_j\|}\right| \\
\le& 1 + \sum_{j=1}^n \left|\left\langle \theta_j, \frac{x_j}{1 \vee \|x_j\|}\right\rangle\right| \\
\le& 1 + \sum_{j=1}^n \|\theta_j\| \cdot \frac{\|x_j\|}{1 \vee \|x_j\|} \\
\le& 1 + n\theta^* := C^{**}.
\end{aligned}$$

Thus, we have

$$E \leq E_1 + E_2 \leq C_1 \cdot \{1 \wedge \sum_{j=1}^{n} \|x_j\|^2\} + C_2 \cdot \{1 \wedge \sum_{j=1}^{n} \|x_j\|^2\}$$

$$\leq C \cdot \{1 \wedge \sum_{j=1}^{n} \|x_j\|^2\}$$

where $C := 2 \cdot \max\{C_1, C_2\}$. And,

$$C \cdot \{1 \wedge \sum_{j=1}^{n} \|x_j\|^2\} \leq C \cdot \sum_{j=1}^{n} \{1 \wedge \|x_j\|^2\}.$$

So, the lemma is proven.

□

4.3 Theorem

We now present the main theorem of this chapter, the short and long term behavior of infinitely divisible processes.

Theorem 4.3.1. *Suppose β is a positive parameter tending to some value β_0. Also suppose there are one to one invertible measurable maps ϕ_β from S into S and normalizing sequences n_β and m_β such that for any $t \in \mathbb{R}$,*

$$m_\beta f_{\beta t}(\phi_\beta(s)) \to h_t(s) \qquad a.e. - \nu$$

$$\|m_\beta f_{\beta t}(\phi_\beta(s))\| \leq k_t(s) \qquad a.e. - \nu$$

$$\frac{d(\nu \circ \phi_\beta)}{d\nu}(s) \left(\frac{n_\beta}{m_\beta}\right)^2 \to g(s) \qquad a.e. - \nu$$

$$\frac{d(\nu \circ \phi_\beta)}{d\nu}(s) \left(\frac{n_\beta}{m_\beta}\right)^2 \leq l(s) \qquad a.e. - \nu$$

as $\beta \to \beta_0$ for some functions k_t, h_t, g and l on the space S. Let

$$c_{\beta,t} = \int_S \left(\frac{n_\beta f_{\beta t}(s)}{1 \vee \|n_\beta f_{\beta t}(s)\|} - \frac{n_\beta f_{\beta t}(s)}{1 \vee \|f_{\beta t}(s)\|} \right) \nu(ds).$$

Then,

$$n_\beta X_{\beta t} - c_{\beta,t} \to Y_t$$

in the sense of finite dimensional distributions as $\beta \to \beta_0$ where Y_t is characterized in the following cases:

(i) If

$$\frac{n_\beta}{m_\beta} \to 0$$

and

$$l \cdot k_t^2 \quad \in \quad L_\nu(S)$$

then,

$$Y_t = \int_S h_t(s) Z(ds)$$

where Z is an independently scattered, Gaussian random measure with control measure $g \cdot \nu$.

(ii) If

$$m_\beta = n_\beta$$

and

$$l \cdot \{1 \wedge k_t^2\} \quad \in \quad L_\nu(S)$$

then,

$$Y_t = \int_S h_t(s) \left(M(ds) - \frac{g(s)\nu(ds)}{1 \vee \|h_t(s)\|} \right)$$

where M is a Poisson random measure with intensity measure $g \cdot \nu$.

Proof. Notice that the ln of the finite dimensional characteristic function of $n_\beta X_{\beta t} - c_{\beta,t}$ is given by

$$\ln E \exp \left\{ i \sum_{j=1}^{n} \left\langle \theta_j, n_\beta X_{\beta t_j} - c_{\beta, t_j} \right\rangle \right\}$$

$$= \ln E \exp\left\{ i \sum_{j=1}^{n} (\langle \theta_j, n_\beta X_{\beta t_j} \rangle - \langle \theta_j, c_{\beta, t_j} \rangle) \right\}$$

which is equal to

$$\ln E \exp\left\{ i \sum_{j=1}^{n} \langle \theta_j, n_\beta X_{\beta t_j} \rangle - i \sum_{j=1}^{n} \langle \theta_j, c_{\beta, t_j} \rangle \right\}$$

and this equals

$$\int_S \left(\exp\left\{ i \sum_{j=1}^{n} \langle \theta_j, n_\beta f_{\beta t_j}(s) \rangle \right\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, n_\beta f_{\beta t_j}(s) \rangle}{1 \vee \| f_{\beta t_j}(s) \|} \right) \nu(ds)$$

$$-i \sum_{j=1}^{n} \left\langle \theta_j, \int_S \left(\frac{n_\beta f_{\beta t_j}(s)}{1 \vee \| n_\beta f_{\beta t_j}(s) \|} - \frac{n_\beta f_{\beta t_j}(s)}{1 \vee \| f_{\beta t_j}(s) \|} \right) \nu(ds) \right\rangle$$

$$= \int_S \left(\exp\left\{ i \sum_{j=1}^{n} \langle \theta_j, n_\beta f_{\beta t_j}(s) \rangle \right\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, n_\beta f_{\beta t_j}(s) \rangle}{1 \vee \| f_{\beta t_j}(s) \|} \right) \nu(ds)$$

$$-i \sum_{j=1}^{n} \int_S \left\langle \theta_j, \left(\frac{n_\beta f_{\beta t_j}(s)}{1 \vee \| n_\beta f_{\beta t_j}(s) \|} - \frac{n_\beta f_{\beta t_j}(s)}{1 \vee \| f_{\beta t_j}(s) \|} \right) \right\rangle \nu(ds)$$

$$= \int_S \left(\exp\left\{ i \sum_{j=1}^{n} \langle \theta_j, n_\beta f_{\beta t_j}(s) \rangle \right\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, n_\beta f_{\beta t_j}(s) \rangle}{1 \vee \| f_{\beta t_j}(s) \|} \right) \nu(ds)$$

$$-i \sum_{j=1}^{n} \int_S \left(\frac{\langle \theta_j, n_\beta f_{\beta t_j}(s) \rangle}{1 \vee \| n_\beta f_{\beta t_j}(s) \|} - \frac{\langle \theta_j, n_\beta f_{\beta t_j}(s) \rangle}{1 \vee \| f_{\beta t_j}(s) \|} \right) \nu(ds)$$

$$= \int_S \left(\exp\left\{ i \sum_{j=1}^{n} \langle \theta_j, n_\beta f_{\beta t_j}(s) \rangle \right\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, n_\beta f_{\beta t_j}(s) \rangle}{1 \vee \| n_\beta f_{\beta t_j}(s) \|} \right) \nu(ds)$$

which is the same as

$$\int_S \left(\exp\left\{ i \sum_{j=1}^{n} \langle \theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s)) \rangle \right\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s)) \rangle}{1 \vee \| n_\beta f_{\beta t_j}(\phi_\beta(s)) \|} \right) \frac{d\nu \circ \phi_\beta}{d\nu}(s) \nu(ds)$$

$$:= \int_S F_\beta(s) \nu(ds)$$

For case (i), observe that

$$F_\beta(s) \to -\frac{g(s)}{2}\left(\sum_{j=1}^{n}\left\langle \theta_j, h_{t_j}(s)\right\rangle\right)^2 \qquad a.e. - \nu$$

as $\beta \to \beta_0$ since

$$\left|F_\beta(s) + \frac{1}{2}\frac{d\nu\circ\phi_\beta}{d\nu}(s)\left(\frac{n_\beta}{m_\beta}\right)^2\left(\sum_{j=1}^{n}\left\langle \theta_j, m_\beta f_{\beta t_j}(\phi_\beta(s))\right\rangle\right)^2\right|$$

$$= \left|\left(\exp\left\{i\sum_{j=1}^{n}\left\langle \theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s))\right\rangle\right\} - 1 - i\sum_{j=1}^{n}\frac{\left\langle \theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s))\right\rangle}{1 \vee \|n_\beta f_{\beta t_j}(\phi_\beta(s))\|}\right)\frac{d\nu\circ\phi_\beta}{d\nu}(s)\right.$$

$$\left. + \frac{1}{2}\frac{d\nu\circ\phi_\beta}{d\nu}(s)\left(\sum_{j=1}^{n}\left\langle \theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s))\right\rangle\right)^2\right|$$

which is less than or equal to

$$\frac{d\nu\circ\phi_\beta}{d\nu}(s)\left|\exp\left\{i\sum_{j=1}^{n}\left\langle \theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s))\right\rangle\right\} - 1 - i\sum_{j=1}^{n}\frac{\left\langle \theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s))\right\rangle}{1 \vee \|n_\beta f_{\beta t_j}(\phi_\beta(s))\|}\right.$$

$$\left. + \frac{1}{2}\left(\sum_{j=1}^{n}\left\langle \theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s))\right\rangle\right)^2\right|$$

$$\leq C \cdot \frac{d\nu\circ\phi_\beta}{d\nu}(s)\left(\sum_{j=1}^{n}\|n_\beta f_{\beta t_j}(\phi_\beta(s))\|\right)^3$$

by Lemma 4.2.1. And this is equal to

$$C \cdot \frac{d\nu\circ\phi_\beta}{d\nu}(s)\left(\frac{n_\beta}{m_\beta}\right)^2\left(\frac{n_\beta}{m_\beta}\right)\left(\sum_{j=1}^{n}\|m_\beta f_{\beta t_j}(\phi_\beta(s))\|\right)^3$$

$$\to C \cdot g(s) \cdot 0 \cdot \left(\sum_{j=1}^{n}\|h_{t_j}(s)\|\right)^3$$

$$= 0$$

as $\beta \to \beta_0$. So, we have proven that

$$F_\beta(s) \to -\frac{g(s)}{2}\left(\sum_{j=1}^{n}\langle\theta_j, h_{t_j}(s)\rangle\right)^2 \qquad a.e. - \nu$$

as $\beta \to \beta_0$. Now, to finish the proof, we would like to apply the Lebesgue Dominated Convergence Theorem to

$$\int_S F_\beta(s)\nu(ds).$$

So, it is necessary to prove that $F_\beta(s)$ is bounded by a function in $L_\nu(S)$. To do this, notice,

$$|F_\beta(s)| = \left|\exp\left\{i\sum_{j=1}^{n}\langle\theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s))\rangle\right\} - 1 - i\sum_{j=1}^{n}\frac{\langle\theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s))\rangle}{1 \vee \|n_\beta f_{\beta t_j}(\phi_\beta(s))\|}\right| \frac{d\nu\circ\phi_\beta}{d\nu}(s)$$

$$\leq C\cdot\frac{d(\nu\circ\phi_\beta)}{d\nu}(s)\cdot\sum_{j=1}^{n}\|n_\beta f_{\beta t_j}(\phi_\beta(s))\|^2$$

by Lemma 4.2.2, and this is equal to

$$C\cdot\frac{d(\nu\circ\phi_\beta)}{d\nu}(s)\left(\frac{n_\beta}{m_\beta}\right)^2\cdot\sum_{j=1}^{n}\|m_\beta f_{\beta t_j}(\phi_\beta(s))\|^2$$

which is

$$\leq C\cdot\sum_{j=1}^{n} l(s)\cdot k_{t_j}(s)^2.$$

which is $\in L_\nu(S)$ since $l\cdot k_{t_j}^2 \in L_\nu(S)$ for each t_j by the assumptions of the theorem, and we are taking a finite sum of these functions in L_ν and then multiplying them by a constant. So, thus, $F_\beta(s)$ is in $L_\nu(S)$.

So, we can apply the Lebesgue Dominated Convergence Theorem to get

$$\int_S F_\beta(s)\nu(ds) \to \int_S -\frac{g(s)}{2}\left(\sum_{j=1}^{n}\langle\theta_j, h_{t_j}(s)\rangle\right)^2\nu(ds)$$

as $\beta \to \beta_0$. So, case (i) is proven.

For case (ii) observe that

$$F_\beta(s) \to \left(\exp\left\{ i \sum_{j=1}^{n} \langle \theta_j, h_{t_j}(s) \rangle \right\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, h_{t_j}(s) \rangle}{1 \vee \|h_{t_j}(s)\|} \right) g(s) \qquad a.e. - \nu$$

as $\beta \to \beta_0$.

Now, we would like to have $F_\beta(s) \in L_\nu(S)$ so we could apply the Lebesgue Dominated Convergence Theorem to

$$\int_S F_\beta(s)\nu(ds)$$

to get

$$\int_S F_\beta(s)\nu(ds) \to \int_S \left(\exp\{ i \sum_{j=1}^{n} \langle \theta_j, h_{t_j}(s) \rangle - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, h_{t_j}(s) \rangle}{1 \vee \|h_{t_j}(s)\|} \right) g(s)\nu(ds)$$

as $\beta \to \beta_0$ and the result would be proven. So, it remains to be shown that $F_\beta(s) \in L_\nu(S)$.

To do this, notice

$$|F_\beta(s)| = \left| \exp\left\{ i \sum_{j=1}^{n} \langle \theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s)) \rangle \right\} - 1 - i \sum_{j=1}^{n} \frac{\langle \theta_j, n_\beta f_{\beta t_j}(\phi_\beta(s)) \rangle}{1 \vee \|n_\beta f_{\beta t_j}(\phi_\beta(s))\|} \right| \frac{d\nu \circ \phi_\beta}{d\nu}(s)$$

$$\leq C \cdot \frac{d\nu \circ \phi_\beta}{d\nu}(s) \sum_{j=1}^{n} \{1 \wedge \|n_\beta f_{\beta t_j}(\phi_{\beta t_j}(s))\|^2\}$$

by Lemma 2. And, this is

$$\leq C \cdot l(s) \sum_{j=1}^{n} \{1 \wedge \|k_{t_j}\|^2\}$$

from the observation that $\|n_\beta f_{\beta t_j}(\phi_{\beta t_j}(s))\| \leq k_{t_j}(s)$ and $\frac{d\nu \circ \phi_\beta}{d\nu}(s) \leq l(s)$ by assumption (and, again, using the fact that $f(x) := x \wedge 1$ is nondecreasing for $x \geq 0$).

And, the above is equal to

$$C \cdot \sum_{j=1}^{n} l(s) \cdot \{1 \wedge \|k_{t_j}\|^2\}$$

which is a constant times a finite sum of functions that are in $L_\nu(S)$ by assumption. So, we have proven $F_\beta(s) \in L_\nu(S)$, so we have proven case (ii) and, hence, the theorem.

□

4.4 Corollaries

Corollary 4.4.1. *Let X be a d-dimensional stochastic process given by*

$$X_t = \int_S f_t(s) \left(N(ds) - \nu(ds) \right). \tag{4.3}$$

It is known by Theorem 2.7 of Rajput and Rosinski [14] that the integrals exist if and only if

$$\int_S \|f_t(s)\| \wedge \|f_t(s)\|^2 \nu(ds) < \infty. \tag{4.4}$$

(i) Then, assuming the conditions of Theorem 4.3.1 case (i), we get

$$n_\beta X_{\beta t} \to Y_t = \int_S h_t(s) Z(ds)$$

in the sense of finite-dimensional distributions as $\beta \to \beta_0$ where Z is an independently scattered, Gaussian random measure with control measure $g \cdot \nu$.

(ii) Then, assuming the conditions of Theorem 4.3.1 case (ii) and assuming further that $l \cdot \{k_t \wedge k_t^2\} \in L_\nu(S)$, we get

$$n_\beta X_{\beta t} \to Y_t = \int_S h_t(s)(M(ds) - g(s)\nu(ds))$$

in the sense of finite-dimensional distributions as $\beta \to \beta_0$ where M is a Poisson random measure with intensity measure $g \cdot \nu$.

Proof. First, write

$$X_t = \int_S f_t(s) \left(N(ds) - \frac{\nu(ds)}{1 \vee \|f_t(s)\|} \right) - \int_S f_t(s) \left(1 - \frac{1}{1 \vee \|f_t(s)\|} \right) \nu(ds)$$

$$:= X_t^* - a_t$$

and note that a_t exists under the condition $\int_S \|f_t(s)\| \wedge \|f_t(s)\|^2 \nu(ds) < \infty$ since

$$\int_S \left\| f_t(s) \left(1 - \frac{1}{1 \vee \|f_t(s)\|} \right) \right\| \nu(ds)$$

$$= \int_S 1_{\|f_t(s)\|>1}(s)\|f_t(s)\| \left\| \left(1 - \frac{1}{\|f_t(s)\|}\right) \right\| \nu(ds)$$

which is less than or equal to

$$\int_S 1_{\|f_t(s)\|>1}(s)\|f_t(s)\|\nu(ds)$$

$$\leq \int_S \|f_t(s)\| \wedge \|f_t(s)\|^2 \nu(ds)$$

since $\{1_{\|f_t(s)\|>1}\|f_t(s)\|\} \leq \{\|f_t(s)\| \wedge \|f_t(s)\|^2\}$.

Now,

$$n_\beta X_{\beta t} = n_\beta X^*_{\beta t} - n_\beta a_{\beta t}$$

$$= (n_\beta X^*_{\beta t} - c_{\beta,t}) + (c_{\beta,t} - n_\beta a_{\beta t})$$

and $n_\beta X^*_{\beta t} - c_{\beta,t}$ converges in the sense of finite dimensional distributions to $Y^*_t \quad a.e. - \nu$ as $\beta \to \beta_0$ by Theorem 4.3.1. So, now we will characterize the limit in two cases.

In case (i),

$$Y^*_t = \int_S h_t(s) Z(ds).$$

And, since we hope to prove that $n_\beta X_{\beta t}$ converges to

$$\int_S h_t(s) Z(ds),$$

we need $c_{\beta,t} - n_\beta a_{\beta t}$ to converge to zero. To see this, notice $c_{\beta,t} - n_\beta a_{\beta t}$ is equal to

$$\int_S \frac{n_\beta f_{\beta t}(s)}{1 \vee \|n_\beta f_{\beta t}(s)\|} - \frac{n_\beta f_{\beta t}(s)}{1 \vee \|f_{\beta t}(s)\|}\nu(ds) - \int_S n_\beta f_{\beta t}(s) \left(1 - \frac{1}{1 \vee \|f_{\beta t}(s)\|}\right) \nu(ds)$$

and since the last two integrands are in $L_\nu(S)$, they can be combined under one integral to get

$$\int_S \left(\frac{n_\beta f_{\beta t}(s)}{1 \vee \|n_\beta f_{\beta t}(s)\|} - \frac{n_\beta f_{\beta t}(s)}{1 \vee \|f_{\beta t}(s)\|} - n_\beta f_{\beta t}(s) + \frac{n_\beta f_{\beta t}(s)}{1 \vee \|f_{\beta t}(s)\|} \right) \nu(ds)$$

which is equivalent to

$$\int_S \left(\frac{n_\beta f_{\beta t}(s)}{1 \vee \|n_\beta f_{\beta t}(s)\|} - n_\beta f_{\beta t}(s) \right) \nu(ds)$$

$$= \int_S \left(\frac{n_\beta f_{\beta t}(\phi_\beta(s))}{1 \vee \|n_\beta f_{\beta t}(\phi_\beta(s))\|} - n_\beta f_{\beta t}(\phi_\beta(s)) \right) \frac{d(\nu \circ \phi_\beta)}{d\nu}(s)\nu(ds)$$

And, we claim that this converges to zero. In order to prove this, we will use the Lebesgue Dominated Convergence Theorem. So, first we will prove that the application of the Lebesgue Dominated Convergence Theorem is valid. To see this, first notice that if $\|n_\beta f_{\beta t}(\phi_\beta(s))\| \leq 1$, then this integrand is zero, and hence integrable. Now, if $\|n_\beta f_{\beta t}(\phi_\beta(s))\| > 1$, then the norm of the above integrand is equal to

$$\left\| \left(\frac{n_\beta f_{\beta t}(\phi_\beta(s))}{\|n_\beta f_{\beta t}(\phi_\beta(s))\|} - n_\beta f_{\beta t}(\phi_\beta(s)) \right) \right\| \frac{d(\nu \circ \phi_\beta)}{d\nu}(s)$$

$$\leq \|n_\beta f_{\beta t}(\phi_\beta(s))\| \left\| \frac{1}{\|n_\beta f_{\beta t}(\phi_\beta(s)\|} - 1 \right\| \frac{d(\nu \circ \phi_\beta)}{d\nu}(s)$$

which is less than or equal to

$$\|n_\beta f_{\beta t}(\phi_\beta(s))\| \frac{d(\nu \circ \phi_\beta)}{d\nu}(s)$$

$$\leq \|n_\beta f_{\beta t}(\phi_\beta(s))\|^2 \frac{d(\nu \circ \phi_\beta)}{d\nu}(s)$$

since $\|n_\beta f_{\beta t}(\phi_\beta(s))\| > 1$. And this is equal to

$$\|m_\beta f_{\beta t}(\phi_\beta(s))\|^2 \left(\frac{n_\beta}{m_\beta} \right)^2 \frac{d(\nu \circ \phi_\beta)}{d\nu}(s)$$

$$\leq k_t(s)^2 \cdot l(s)$$

which is integrable with respect to ν by the assumptions of the theorem. So, an application of the Lebesgue Dominated Convergence Theorem is valid.

Now, notice that the norm of the integrand is equal to

$$\left\| \frac{n_\beta f_{\beta t}(\phi_\beta(s))}{1 \vee \|n_\beta f_{\beta t}(\phi_\beta(s))\|} - n_\beta f_{\beta t}(\phi_\beta(s)) \right\| \frac{d(\nu \circ \phi_\beta)}{d\nu}(s)$$

and, as in the proof that this is in $L_\nu(S)$, notice that the above is only nonzero when $\|n_\beta f_{\beta t}(\phi_\beta(s))\| > 1$. And, from the proof that it is in $L_\nu(S)$, we know that

$$\left\| \frac{n_\beta f_{\beta t}(\phi_\beta(s))}{1 \vee \|n_\beta f_{\beta t}(\phi_\beta(s))\|} - n_\beta f_{\beta t}(\phi_\beta(s)) \right\| \frac{d(\nu \circ \phi_\beta)}{d\nu}(s)$$

$$\leq \|n_\beta f_{\beta t}(\phi_\beta(s))\| \frac{d(\nu \circ \phi_\beta)}{d\nu}(s).$$

And since $\|n_\beta f_{\beta t}(\phi_\beta(s))\| > 1$, we get that $\|n_\beta f_{\beta t}(\phi_\beta(s))\| \leq \|n_\beta f_{\beta t}(\phi_\beta(s))\|^3$, so

$$\|n_\beta f_{\beta t}(\phi_\beta(s))\| \frac{d(\nu \circ \phi_\beta)}{d\nu}(s) \leq \|n_\beta f_{\beta t}(\phi_\beta(s))\|^3 \frac{d(\nu \circ \phi_\beta)}{d\nu}(s).$$

And, this is equal to

$$\frac{n_\beta}{m_\beta} \|m_\beta f_{\beta t}(\phi_\beta(s))\|^3 \left(\frac{n_\beta}{m_\beta}\right)^2 \frac{d(\nu \circ \phi_\beta)}{d\nu}(s)$$

$$\to 0 \cdot h_t(s)^3 \cdot g(s) = 0$$

as $\beta \to \beta_0$. Then, we apply the Lebesgue Dominated Convergence Theorem to get that $c_{\beta,t} - n_\beta a_{\beta t}$ converges to zero.

Now, we have

$$n_\beta X_{\beta t} = (n_\beta X^*_{\beta t} - c_{\beta,t}) + (c_{\beta,t} - n_\beta a_{\beta t})$$

$$\to Y^*_t + 0 \qquad a.e. - \nu$$

as $\beta \to \beta_0$ by Slutsky's Theorem. And this equals

$$\int_S h_t(s) Z(ds).$$

So, we have proven case (i).

For case (ii), we know that the limit of $n_\beta X^*_{\beta t} - c_{\beta,t}$ is

$$Y^*_t = \int_S h_t(s) \left(M(ds) - \frac{g(s)\mu(ds)}{1 \vee \|h_t(s)\|} \right) \qquad a.e. - \mu$$

by Theorem 4.3.1 case (ii).

So, we need to find a limit for $c_{\beta,t} - n_\beta a_{\beta t}$. Notice $c_{\beta,t} - n_\beta a_{\beta t}$ is equal to

$$= \int_S \left(\frac{n_\beta f_{\beta t}(\phi_\beta(s))}{1 \vee \|n_\beta f_{\beta t}(\phi_\beta(s))\|} - n_\beta f_{\beta t}(\phi_\beta(s)) \right) \frac{d(\nu \circ \phi_\beta)}{d\mu}(s)\mu(ds)$$

as in case (i). And this converges as $\beta \to \beta_0$ via the Lebesgue Dominated Convergence Theorem to

$$b_t := \int_S \left(\frac{h_t(s)}{1 \vee \|h_t(s)\|} - h_t(s) \right) g(s)\mu(ds) \qquad a.e. - \mu.$$

Note that the application of the Lebesgue Dominated Convergence Theorem is valid for

$$\int_S \left(\frac{n_\beta f_{\beta t}(\phi_\beta(s))}{1 \vee \|n_\beta f_{\beta t}(\phi_\beta(s))\|} - n_\beta f_{\beta t}(\phi_\beta(s)) \right) \frac{d(\nu \circ \phi_\beta)}{d\mu}(s)\mu(ds)$$

since

$$\left\| \left(\frac{n_\beta f_{\beta t}(\phi_\beta(s))}{1 \vee \|n_\beta f_{\beta t}(\phi_\beta(s))\|} - n_\beta f_{\beta t}(\phi_\beta(s)) \right) \right\| \frac{d(\nu \circ \phi_\beta)}{d\mu}(s)$$

$$= \left\| 1_{|n_\beta f_{\beta t}(\phi_\beta(s))\| > 1}(s) n_\beta f_{\beta t}(\phi_\beta(s)) \left(\frac{1}{\|n_\beta f_{\beta t}(\phi_\beta(s))\|} - 1 \right) \right\| \frac{d(\nu \circ \phi_\beta)}{d\mu}(s)$$

which is less than or equal to

$$\left\| 1_{\|n_\beta f_{\beta t}(\phi_\beta(s))\| > 1}(s) n_\beta f_{\beta t}(\phi_\beta(s)) \right\| \frac{d(\nu \circ \phi_\beta)}{d\mu}(s)$$

$$\leq 1_{k_t(s) > 1}(s) k_t(s) \cdot l(s)$$

$$\leq \{k_t(s) \wedge k_t^2(s)\} \cdot l(s)$$

and $l \cdot \{k_t \wedge k_t^2\} \in L_\mu(S)$ by assumption.

Now,

$$n_\beta X_{\beta t} = (n_\beta X^*_{\beta t} - c_{\beta,t}) + (c_{\beta,t} - n_\beta a_{\beta t})$$

$$\to Y^*_t + b_t \qquad a.e. - \mu$$

as $\beta \to \beta_0$ in the sense of finite dimensional distributions by an application of Slutsky's Theorem. And, note that this limit is equal to

$$\int_S h_t(s)\left(M(ds) - \frac{g(s)\mu(ds)}{1 \vee \|h_t(s)\|}\right) + \int_S h_t(s)\left(\frac{g(s)}{1 \vee \|h_t(s)\|} - g(s)\right)\mu(ds)$$

$$= \int_S h_t(s)\left(M(ds) - \frac{g(s)\mu(ds)}{1 \vee \|h_t(s)\|} + \frac{g(s)\mu(ds)}{1 \vee \|h_t(s)\|} - h_t(s)g(s)\mu(ds)\right)$$

$$= \int_S h_t(s)\left(M(ds) - g(s)\mu(ds)\right).$$

Thus, the result is proven using Theorem 4.3.1. □

Corollary 4.4.2. *Let X be a d-dimensional stochastic process given by*

$$X_t = \int_S f_t(s)N(ds). \tag{4.5}$$

It is known by Theorem 2.7 of Rajput and Rosinski [14] that the integrals exist if and only if

$$\int_S \left(\|f_t(s)\| \wedge 1\right)\nu(ds) < \infty. \tag{4.6}$$

(i) Then, assuming the conditions of Theorem 4.3.1 case (i), we get

$$n_\beta X_{\beta t} - b_{\beta,t} \to Y_t = \int_S h_t(s)Z(ds)$$

in the sense of finite-dimensional distributions as $\beta \to \beta_0$ where Z is an independently scattered, Gaussian random measure with control measure $g \cdot \mu$ and

$$b_{\beta,t} := \int_S \frac{n_\beta f_{\beta t}(s)}{1 \vee \|n_\beta f_{\beta t}(s)\|}\nu(ds).$$

(ii) Then, assuming the conditions of Theorem 4.3.1 case (ii) and assuming further that $l \cdot \{k_t \wedge 1\} \in L_\mu(S)$, we get

$$n_\beta X_{\beta t} \to Y_t = \int_S h_t(s)M(ds)$$

in the sense of finite-dimensional distributions as $\beta \to \beta_0$ where M is a Poisson random measure with intensity measure $g \cdot \mu$.

Proof. First, write

$$X_t = \int_S f_t(s)\left(N(ds) - \frac{\nu(ds)}{1 \vee \|f_t(s)\|}\right) + \int_S \frac{f_t(s)}{1 \vee \|f_t(s)\|}\nu(ds)$$

$$:= X_t^* + a_t$$

and note that a_t exists under the condition $\int_S (\|f_t(s)\| \wedge 1)\, \nu(ds) < \infty$ since

$$\int_S \left\| \frac{f_t(s)}{1 \vee \|f_t(s)\|} \right\| \nu(ds)$$

$$= \int_S \left(\|f_t(s)\| 1_{\|f_t(s)\| \leq 1} + 1 \cdot 1_{\|f_t(s)\| > 1} \right) \nu(ds)$$

$$= \int_S (\|f_t(s)\| \wedge 1)\, \nu(ds)$$

which is finite by assumption.

Now for case (i), notice

$$n_\beta X_{\beta t} - b_{\beta,t} = n_\beta X_{\beta t}^* + n_\beta a_{\beta t} - b_{\beta,t}$$

$$= (n_\beta X_{\beta t}^* - c_{\beta,t}) + (c_{\beta,t} + n_\beta a_{\beta t} - b_{\beta,t})$$

and, notice $c_{\beta,t} + n_\beta a_{\beta t} - b_{\beta,t}$ equals zero. To see this, observe that $c_{\beta,t} + n_\beta a_{\beta t} - b_{\beta,t}$ is equal to

$$= \int_S \left(\frac{n_\beta f_{\beta t}(s)}{1 \vee \|n_\beta f_{\beta t}(s)\|} - \frac{n_\beta f_{\beta t}(s)}{1 \vee \|f_{\beta t}(s)\|} \right) \nu(ds) + \int_S \frac{n_\beta f_{\beta t}(s)}{1 \vee \|f_{\beta t}(s)\|} \nu(ds)$$

$$- \int_S \frac{n_\beta f_{\beta t}(s)}{1 \vee \|n_\beta f_{\beta t}(s)\|} \nu(ds)$$

and the three integrands can be combined into one since they are all in $L_\nu(S)$ and there are cancellations and the resulting integral equals zero. Thus, $c_{\beta,t} + n_\beta a_{\beta t} - b_{\beta,t}$ equals zero.

So,

$$n_\beta X_{\beta t} - b_{\beta,t} = n_\beta X_{\beta t}^* - c_{\beta,t}$$

and by Theorem 1, $n_\beta X_{\beta t}^* - c_{\beta,t}$ converges in the sense of finite dimensional distributions as $\beta \to \beta_0$ to Y_t where

$$Y_t = \int_S h_t(s) Z(ds).$$

So, we get

$$n_\beta X_{\beta t} - b_{\beta,t} \xrightarrow{fdd} \int_S h_t(s) Z(ds),$$

as $\beta \to \beta_0$ as desired.

For case (ii),

$$n_\beta X_{\beta t} = n_\beta X^*_{\beta t} + n_\beta a_{\beta t}$$

$$= (n_\beta X^*_{\beta t} - c_{\beta,t}) + (c_{\beta,t} + n_\beta a_{\beta t})$$

and $n_\beta X_{\beta t} - c_{\beta,t}$ converges in the sense of finite dimensional distributions to Y^*_t where

$$Y^*_t = \int_S h_t(s) \left(M(ds) - \frac{g(s)\mu(ds)}{1 \vee \|h_t(s)\|} \right) \qquad a.e. - \mu$$

as $\beta \to \beta_0$ by Theorem 4.3.1. Now, it is necessary to determine the convergence of $c_{\beta,t}+n_\beta a_{\beta t}$.

Well, $c_{\beta,t} + n_\beta a_{\beta t}$

$$= \int_S \left(\frac{n_\beta f_{\beta t}(s)}{1 \vee |n_\beta f_{\beta t}(s)|} - \frac{n_\beta f_{\beta t}(s)}{1 \vee \|f_{\beta t}(s)\|} \right) \nu(ds) + \int_S \frac{n_\beta f_{\beta t}(s)}{1 \vee \|f_{\beta t}(s)\|} \nu(ds)$$

and the two integrands on the right can be combined into one since they are both in $L_\nu(S)$. Then, there is a cancellation and we arrive at

$$\int_S \frac{n_\beta f_{\beta t}(s)}{1 \vee \|n_\beta f_{\beta t}(s)\|} \nu(ds)$$

$$= \int_S \frac{n_\beta f_{\beta t}(\phi_\beta(s))}{1 \vee \|n_\beta f_{\beta t}(\phi_\beta(s))\|} \frac{d\nu \circ \phi_\beta}{d\mu}(s)\mu(ds)$$

and this converges to

$$d_t := \int_S \frac{h_t(s)}{1 \vee \|h_t(s)\|} g(s)\mu(ds) \qquad a.e. - \mu$$

as $\beta \to \beta_0$ by the Lebesgue Dominated Convergence Theorem since

$$\left\| \frac{n_\beta f_{\beta t}(\phi_\beta(s))}{1 \vee \|n_\beta f_{\beta t}(\phi_\beta(s))\|} \right\| \frac{d(\nu \circ \phi_\beta)}{d\mu}(s)$$

$$= \{\|n_\beta f_{\beta t}(\phi_\beta(s))\| \wedge 1\} \cdot \frac{d(\nu \circ \phi_\beta)}{d\mu}(s)$$

$$\leq \{k_t \wedge 1\} \cdot l(s)$$

and $\{k_t \wedge 1\} \cdot l(s) \in L_\mu(S)$ by assumption.

So,

$$n_\beta X_{\beta t} = (n_\beta X^*_{\beta t} - c_{\beta,t}) + (c_{\beta,t} + n_\beta a_{\beta t})$$

$$\to Y^*_t + d_t$$

in the sense of finite dimensional distributions as $\beta \to \beta_0$ by Slutsky's Theorem. And, note that this limit is equal to

$$\int_S h_t(s)\left(M(ds) - \frac{g(s)\mu(ds)}{1 \vee \|h_t(s)\|}\right) + \int_S \frac{h_t(s)}{1 \vee \|h_t(s)\|} g(s)\mu(ds)$$

$$= \int_S h_t(s) M(ds)$$

and this concludes the proof. □

4.5 Examples

Remark 4.5.1. Please recall that for the special case when the process is a Lévy process, it has a version given by the integral representation:

$$X_t = \int_{\mathbb{R}^+ \times \mathbb{R}^d} v 1_{(0,t]}(u)\left(N(du, dv) - \frac{du\, \eta(dv)}{1 \vee \|v 1_{(0,t]}(u)\|}\right). \tag{4.7}$$

Example 4.5.1 (Stable Lévy motions). Consider a $S\alpha S$ Lévy motion X_t,

$$X_t = \int_S f_t(s)\left(N(ds) - \frac{\nu(ds)}{1 \vee \|f_t(s)\|}\right),$$

where,

$$S = \mathbb{R} \times \mathbb{R}, \quad s = (u, w), \quad \nu(du, dw) = du c|w|^{-\alpha-1} dw,$$

and

$$f_t(u, w) = 1_{[0,t)}(u) w$$

such that $c > 0$ is a constant.

Take $\phi_\beta(u,w) := (\beta u, \beta^{\frac{1}{\alpha}} w)$ and $m_\beta = n_\beta := \beta^{-\frac{1}{\alpha}}$. Then, notice:

$$m_\beta f_{\beta t}(\phi_\beta(u,w)) = \beta^{-\frac{1}{\alpha}} 1_{[0,\beta t)}(\beta u) \beta^{\frac{1}{\alpha}} w = f_t(u,w).$$

So, $h_t(s) = f_t(s)$ and

$$\frac{d\nu \circ \phi_\beta}{d\nu} = \frac{\beta du c |\beta^{\frac{1}{\alpha}} w|^{-\alpha-1} \beta^{\frac{1}{\alpha}} dw}{du c |w|^{-\alpha-1} dw} = 1.$$

So, $g(s) = 1$.

Thus, by Theorem 4.3.1,

$$\beta^{-\frac{1}{\alpha}} X_{\beta t} - c_{\beta,t} \xrightarrow{fdd} Y_t$$

where

$$Y_t = \int_S f_t(s) \left(M(ds) - \frac{\nu(ds)}{1 \vee \|f_t(s)\|} \right)$$

and M is a Poisson random measure with intensity measure ν.

The purpose of the previous example is to illustrate how to use the main Theorem. There is actually no convergence here, only equality. It is merely a reformulation of a scaling property and holds for all β. Pipiras and Taqqu considered this example in the case $\alpha \in (1,2)$ only. Theorem 4.3.1 expands this result to all α such that $\alpha \in (0,2)$.

The purpose of the next example is to illustrate the power of Theorem 4.3.1. The short and long term behavior of tempered stable processes were considered by Rosinski in 2007. The goal here is to prove Theorem 3.1 of [16] without the use of characteristic functions and other calculations. This demonstrates the fact that we can use our Theorem to bypass the use of characteristic functions to prove convergence. Also, it provides an example where short and long term behavior yield different limits. Moreover, it provides another extension of Pipiras and Taqqu from the case $\alpha \in (1,2)$ to the case $\alpha \in (0,2)$

Example 4.5.2 (Tempered stable Lévy motions). Consider a tempered stable Lévy motion X_t,

$$X_t = \int_S f_t(s) \left(N(ds) - \frac{\nu(ds)}{1 \vee |f_t(s)|} \right),$$

where,

$$S = \mathbb{R}^+ \times \mathbb{R}, \quad s = (u,w), \quad \nu(du,dw) = du\ \alpha q|w||w|^{-\alpha-1}\, dw,$$

and

$$f_t(u,w) = 1_{[0,t)}(u) w$$

where $q(w)$ is a positive, bounded function satisfying $q(0+) = 1$.

First, consider small scales, that is $\beta \to 0$. Take $\phi_\beta(u,w) := (\beta u, \beta^{\frac{1}{\alpha}} w)$ and $m_\beta = n_\beta := \beta^{-\frac{1}{\alpha}}$. Then, notice:

$$m_\beta f_{\beta t}(\phi_\beta(u,w)) = \beta^{-\frac{1}{\alpha}} 1_{[0,\beta t)}(\beta u)\beta^{\frac{1}{\alpha}} w = f_t(u,w).$$

So, $h_t(s) = f_t(s)$ and

$$\begin{aligned}
\frac{d\nu \circ \phi_\beta}{d\nu} &= \frac{\beta du\ \alpha q|\beta^{\frac{1}{\alpha}} w||\beta^{\frac{1}{\alpha}} w|^{-\alpha-1}\beta^{\frac{1}{\alpha}}\, dw}{du\ \alpha q|w||w|^{-\alpha-1}\, dw} \\
&= \frac{du\ \alpha q|\beta^{\frac{1}{\alpha}} w||w|^{-\alpha-1}\, dw}{du\ \alpha q|w||w|^{-\alpha-1}\, dw} \\
&= \frac{q|\beta^{\frac{1}{\alpha}} w|}{q|w|} \to \frac{1}{q|w|} \quad as \quad \beta \to 0.
\end{aligned}$$

So, $g(s) = \frac{1}{q|w|}$.

Thus, by Theorem 4.3.1 case (ii),

$$\beta^{-\frac{1}{\alpha}} X_{\beta t} - c_{\beta,t} \xrightarrow{fdd} Y_t$$

as $\beta \to 0$ where

$$Y_t = \int_S f_t(s)\left(M(ds) - \frac{\nu(ds)}{1 \vee |f_t(s)|}\right)$$

and M is a Poisson random measure with intensity measure $g(s)\nu(ds)$ where

$$g(s)\nu(ds) = \frac{1}{q|w|}\nu(du,dw) = \frac{1}{q|w|} du\ \alpha q|w||w|^{-\alpha-1}\, dw = du\ \alpha|w|^{-\alpha-1}\, dw$$

So, the limit is a stable process at small scales.

Now, consider large scales: $\beta \to \infty$. Take $\phi_\beta(u,w) := (\beta u, w)$ and $m_\beta := 1$ and $n_\beta := \beta^{-\frac{1}{2}}$. Then, we have

$$\frac{n_\beta}{m_\beta} = \frac{1}{\beta^{\frac{1}{2}}} \to 0$$

as $\beta \to \infty$. Also,

$$m_\beta f_{\beta t}(\phi_\beta(u,w)) = 1_{[0,\beta t)}(\beta u) w = f_t(u,w)$$

so, $h_t(s) = f_t(s)$. And,

$$\frac{d\nu \circ \phi_\beta}{d\nu} = \frac{\beta du\ \alpha q(-w)|w|^{-\alpha-1}\, dw}{du\ \alpha q(-w)|w|^{-\alpha-1}\, dw} = \beta.$$

So,

$$\frac{d(\nu \circ \phi_\beta)}{d\nu}(s)\left(\frac{n_\beta}{m_\beta}\right)^2 = \beta(\beta^{-\frac{1}{2}})^2 = 1.$$

Thus, $g(s) = 1$.

So, if we assume that

$$\int_{\mathbb{R}} \|w\|^{1-\alpha} q(w)dw < \infty$$

then, by Theorem 4.3.1 case (i),

$$\beta^{-\frac{1}{2}} X_{\beta t} - c_{\beta,t} \xrightarrow{fdd} Y_t = \int_S f_t(s) Z(ds)$$

as $\beta \to \infty$ where Z is an independently scattered, Gaussian random measure with control measure ν.

Example 4.5.3 (Standard Poissonized Telecom process). This example is inspired by the standard Poissonized telecom process studied by Pipiras and Taqqu in [13]. They consider a Poissonized telecom process X_t, as originally described in [4], given by,

$$X_t = \int_S f_t(s)\left(N(ds) - \nu(ds)\right),$$

where,

$$S = (0,\infty) \times \mathbb{R} \times \mathbb{R} \quad s = (x,u,w) \quad \nu(dx,du,dw) = dx\,du\,\frac{1_{\{|w|<1\}}dw}{|w|^{\alpha+1}}$$

and

$$f_t(x,u,w) = \Big([(t+u)\wedge 0) + x]_+ - [(u \wedge 0) + x]_+\Big) x^{-(1-k)-\frac{1}{\alpha}} w$$

for $\alpha \in (1,2)$ and $k \in (0, 1-\frac{1}{\alpha})$.

We wish to prove that as $\beta \to 0$,

$$\beta^{-(k+\frac{1}{\alpha})} X_{\beta t} \xrightarrow{fdd} Y_t$$

where Y_t is given by

$$\int_S f_t(s)\left(M(ds) - \mu(ds)\right)$$

and

$$\mu(ds) = \mu(dx,du,dw) = dx\,du\,\frac{dw}{|w|^{\alpha+1}}.$$

Here, we will take $\phi_\beta(x,u,w) := (\beta x, \beta u, \beta^{\frac{2}{\alpha}} w)$ and notice that this ϕ_β satisfies the conditions of the Theorem. It is necessary to choose the appropriate n_β to apply the Theorem. It is obvious to choose

$$n_\beta := \beta^{-(k+\frac{1}{\alpha})}.$$

We know that

$$n_\beta f_{\beta t}(x,u,w) = \beta^{-(k+\frac{1}{\alpha})} f_{\beta t}(x,u,w)$$

and we wish that

$$n_\beta f_{\beta t}(\phi_\beta(x,u,w)) \to f_t(x,u,w)$$

as $\beta \to 0$.

so we calculate

$$\begin{aligned} f_{\beta t}(\phi_\beta(x,u,w)) &= f_{\beta t}(\beta x, \beta u, \beta^{\frac{2}{\alpha}} w) \\ &= \Big([(\beta t + \beta u) \wedge 0) + \beta x]_+ - [(\beta u \wedge 0) + \beta x]_+ \Big) (\beta x)^{-(1-k)-\frac{1}{\alpha}} \beta^{\frac{2}{\alpha}} w \\ &= \beta^{k+\frac{1}{\alpha}} \Big([(t+u) \wedge 0) + x]_+ - [(u \wedge 0) + x]_+ \Big) x^{-(1-k)-\frac{1}{\alpha}} w \\ &= \beta^{k+\frac{1}{\alpha}} f_t(x,u,w). \end{aligned}$$

Thus,

$$n_\beta f_{\beta t}(\phi_\beta(x,u,w)) = \beta^{-(k+\frac{1}{\alpha})} \beta^{k+\frac{1}{\alpha}} f_t(x,u,w) = f_t(x,u,w).$$

Now to check the conditions:

$$(*) \qquad n_\beta f_{\beta t}(\phi_\beta(x,u,w)) \to f_t(x,u,w)$$

as $\beta \to 0$ as before, and

$$(**) \qquad \frac{d\nu \circ \phi_\beta}{d\mu} = \frac{\beta dx\, \beta du\, 1_{\{|\beta^{\frac{2}{\alpha}} w|<1\}} \beta^{\frac{2}{\alpha}} |\beta^{\frac{2}{\alpha}} w|^{-\alpha-1} dw}{dx\, du\, |w|^{-\alpha-1} dw}$$

$$= \beta\beta\beta^{\frac{2}{\alpha}}\beta^{-2}\beta^{-\frac{2}{\alpha}} \frac{dx\, du\, 1_{\{|\beta^{\frac{2}{\alpha}} w|<1\}} |w|^{-\alpha-1} dw}{dx\, du\, |w|^{-\alpha-1} dw}$$

$$= 1_{\{|w|<\beta^{-\frac{2}{\alpha}}\}} \to 1 \quad as \quad \beta \to 0.$$

So,

$$g(s) = 1,\ h_t(s) = f_t(s),\ \mu(ds) = \mu(dx, du, dw) = dx\, du\, |w|^{-\alpha-1} dw.$$

Thus, by Corollary 4.4.1,

$$\beta^{-(k+\frac{1}{\alpha})} X_{\beta t} \xrightarrow{fdd} Y_t$$

as $\beta \to 0$ where

$$Y_t = \int_S f_t(s)\, (M(ds) - \mu(ds))$$

and M is a Poisson random measure with intensity measure μ where $\mu(ds) = \mu(dx, du, dw) = dx\, du\, \frac{dw}{|w|^{\alpha+1}}$.

Now, for large scales ($\beta \to \infty$), we can get a result where the limit is a Gaussian-type ID process. So, we want to use the second part of the Theorem to prove:

$$\beta^{\frac{1}{\alpha}-k-1} X_{\beta t} \xrightarrow{fdd} Y_t$$

as $\beta \to \infty$ where

$$Y_t = \int_S f_t(s) Z(ds)$$

and Z is an independently scattered, Gaussian random measure with control measure ν.

Here, we will take $\phi_\beta(x, u, w) := (\beta x, \beta u, w)$ and notice that this ϕ_β satisfies the conditions of the Theorem (and is a multiplicative flow). Again, it is necessary to choose the appropriate n_β and m_β to apply the Theorem. Recall that for the second part of the Theorem, it is necessary for

$$\frac{n_\beta}{m_\beta} \to 0 \qquad \text{as } \beta \to \infty$$

so this suggests setting

$$n_\beta := \beta^{\frac{1}{\alpha}-k-1} \quad \text{and} \quad m_\beta := \beta^{\frac{1}{\alpha}-k}.$$

So, we calculate that $m_\beta f_{\beta t}(\phi_\beta(x, u, w))$ equals

$$\beta^{\frac{1}{\alpha}-k}\bigg([(\beta t+\beta u)\wedge 0)+\beta x]_+ - [(\beta u\wedge 0)+\beta x]_+\bigg)(\beta x)^{-(1-k)-\frac{1}{\alpha}}w$$

$$=\beta^{\frac{1}{\alpha}-k}\beta^{k-\frac{1}{\alpha}}\bigg([(t+u)\wedge 0)+x]_+ - [(u\wedge 0)+x]_+\bigg)x^{-(1-k)-\frac{1}{\alpha}}w$$

thus,

$$m_\beta f_{\beta t}(\phi_\beta(x,u,w)) = f_t(x,u,w).$$

Now to check the conditions:

$$(*)\qquad m_\beta f_{\beta t}(\phi_\beta(x,u,w)) \to f_t(x,u,w)$$

as $\beta\to\infty$ as before, and

$$\frac{d\nu\circ\phi_\beta}{d\nu} = \frac{\beta dx\,\beta du\,\frac{1_{\{|w|<1\}}dw}{|w|^{\alpha+1}}}{dx\,du\,\frac{1_{\{|w|<1\}}dw}{|w|^{\alpha+1}}} = \beta^2$$

So,

$$(**)\qquad \frac{d(\nu\circ\phi_\beta)}{d\nu}(s)\left(\frac{n_\beta}{m_\beta}\right)^2 = \beta^2(\beta^{-1})^2 = 1$$

and we get

$$g(s)=1,\; h_t(s)=f_t(s),\; \mu(ds)=\mu(dx,du,dw)=dx\,du\,\frac{1_{\{|w|<1\}}dw}{|w|^{\alpha+1}}.$$

Then, by Corollary 4.4.1,

$$\beta^{\frac{1}{\alpha}-k-1}X_{\beta t}\xrightarrow{fdd} Y_t=\int_S f_t(s)Z(ds)$$

as $\beta\to\infty$ where Z is an independently scattered, Gaussian random measure with control measure ν. So, the limit is a Gaussian-type ID process at large scales.

Example 4.5.4 (Layered Stable Processes).

Definition 4.5.2. A Lévy process is called a layered stable process if its Lévy measure at time 1 is given by

$$\nu^{LS}(B)=\int_{S^{d-1}}\int_0^\infty 1_B(r\xi)q(r,\xi)dr\sigma(d\xi)$$

where B is a Borel set in $\mathbb{R}_0^d$ and σ is a finite positive measure on S^{d-1} and q is a locally integrable function from $(0,\infty)\times S^{d-1}$ to $(0,\infty)$ such that:

$$q(r,\xi)\sim c_1(\xi)r^{-\alpha-1} \qquad as\ r\to 0$$

and

$$q(r,\xi)\sim c_2(\xi)r^{-\alpha_1-1} \qquad as\ r\to \infty$$

for σ-almost every $\xi\in S^{d-1}$ where c_1 and c_2 are positive integrable with respect to σ functions on S^{d-1} and $\alpha\in(0,2)$ and $\alpha_1\in(0,\infty)$.

Remark 4.5.3. As was mentioned in Chapter 2, the assumptions on the behavior of the q function near zero are incorrect. There is a slight error in the assumptions made in the layered stable paper [8] and a counterexample is described in Chapter 2. The correct assumption is the following:

$$\lim_{r\to 0}\frac{q(r,\cdot)}{r^{-\alpha-1}}=c_1(\cdot) \qquad \text{in} \quad L^1(S^{d-1},\sigma).$$

Notation: Notice that if $q(r,\xi)=r^{-\alpha-1}$, then we have an α-stable Lévy measure, so Layered Stable is a generalization of α-stable. We write ν_σ^α to denote an α-stable Lévy measure with spectral measure σ. Also, let

$$\sigma_1(d\xi):=c_1(\xi)\sigma(d\xi)$$

$$\sigma_2(d\xi):=c_2(\xi)\sigma(d\xi)$$

and let $X_t^{LS}\sim LS_{\alpha,\alpha_1}(\sigma,q,\eta)$ denote a Layered Stable Process with Lévy measure ν^{LS} at time 1 and drift η.

Consider a Layered Stable Process with the integral representation:

$$X_t=\int_S f_t(s)\left(N(ds)-\frac{\nu(ds)}{1\vee\|f_t(s)\|}\right),$$

where

$$S=\mathbb{R}^+\times\mathbb{R}^d,\quad s=(u,w),\quad \nu(du,dw)=du\,\nu^{LS}(dw),$$

and

$$f_t(u,w)=1_{[0,t)}(u)w.$$

Case(i): (Small Scales). As $\beta \to 0$,

$$\beta^{-\frac{1}{\alpha}} X_{\beta t} - c_{\beta,t} \xrightarrow{fdd} Y_t$$

where

$$Y_t = \int_S f_t(s) \left(M(ds) - \frac{g(s)\nu(ds)}{1 \vee \|f_t(s)\|} \right)$$
$$= \int_{\mathbb{R}^+ \times \mathbb{R}^d} w 1_{(0,t]}(u) \left(M(du, dw) - \frac{du\, \nu_{\sigma_1}^{\alpha}(dw)}{1 \vee \|w 1_{(0,t]}(u)\|} \right)$$

Case(ii): (Large Scales).

(a) If $\beta \in (0, 2)$, then as $\beta \to \infty$,

$$\beta^{-\frac{1}{\alpha_1}} X_{\beta t} - c_{\beta,t} \xrightarrow{fdd} Y_t$$

where

$$Y_t = \int_S f_t(s) \left(M(ds) - \frac{g(s)\nu(ds)}{1 \vee \|f_t(s)\|} \right)$$
$$= \int_{\mathbb{R}^+ \times \mathbb{R}^d} w 1_{(0,t]}(u) \left(M(du, dw) - \frac{du\, \nu_{\sigma_2}^{\alpha_1}(dw)}{1 \vee \|w 1_{(0,t]}(u)\|} \right)$$

(b) If $\beta \in (2, \infty)$, then as $\beta \to \infty$,

$$\beta^{-\frac{1}{2}} X_{\beta t} - c_{\beta,t} \xrightarrow{fdd} Y_t$$

where

$$Y_t = \int_S f_t(s) Z(ds)$$

and Z is an independently scattered Gaussian random measure with control measure ν^{LS}.

Case (i). Take $\phi_\beta(u, w) := (\beta u, \beta^{\frac{1}{\alpha}} w)$ and $m_\beta = n_\beta := \beta^{-\frac{1}{\alpha}}$. Then, notice:

$$m_\beta f_{\beta t}(\phi_\beta(u, w)) = \beta^{-\frac{1}{\alpha}} 1_{[0,\beta t)}(\beta u) \beta^{\frac{1}{\alpha}} w = f_t(u, w).$$

So, $h_t(s) = f_t(s)$. And, $\|m_\beta f_{\beta t}(\phi_\beta(s))\| \leq k_t(s)$ where $k_t(s) := \|f_t(s)\|$.

And,

$$\frac{d\nu \circ \phi_\beta}{d\nu}(s) \left(\frac{n_\beta}{m_\beta} \right)^2 = \frac{d\nu \circ \phi_\beta}{d\nu}(u, w) = \frac{d(\beta u)\, \nu^{LS}(d(\beta^{\frac{1}{\alpha}} w))}{du\, \nu^{LS}(dw)}$$

and letting $r := \|w\|$ and $\xi := \frac{w}{\|w\|}$ we get that the above is equal to

$$\beta^{1+\frac{1}{\alpha}} \frac{du\ q(\beta^{\frac{1}{\alpha}} r, \xi)\, dr\, \sigma(d\xi)}{du\, \nu^{LS}(dw)}$$

$$= \beta^{1+\frac{1}{\alpha}} \frac{du\ q(\beta^{\frac{1}{\alpha}} r, \xi)\, dr\, \sigma(d\xi)}{du\ |r|^{-\alpha-1}\, dr\ c_1(\xi)\, \sigma(d\xi)} \left(\frac{du\ |r|^{-\alpha-1}\, dr\ c_1(\xi)\, \sigma(d\xi)}{du\, \nu^{LS}(dw)} \right)$$

which equals

$$\left(\frac{du\ dr\, \sigma(d\xi)}{du\ dr\, \sigma(d\xi)} \right) \left(\frac{q(\beta^{\frac{1}{\alpha}} r, \xi)}{|\beta^{\frac{1}{\alpha}} r|^{-\alpha-1}\, c_1(\xi)} \right) \left(\frac{du\ |r|^{-\alpha-1}\, dr\ c_1(\xi)\, \sigma(d\xi)}{du\, \nu^{LS}(dw)} \right)$$

$$\to \left(\frac{du\ |r|^{-\alpha-1}\, dr\ c_1(\xi)\, \sigma(d\xi)}{du\, \nu^{LS}(dw)} \right)$$

as $\beta \to 0$, so

$$g(s) = g(u, r, \xi) = \left(\frac{du\ |r|^{-\alpha-1}\, dr\ \sigma_1(d\xi)}{du\, \nu^{LS}(dw)} \right).$$

Which implies

$$g(s)\nu(ds) = du\ |r|^{-\alpha-1}\, dr\ \sigma_1(d\xi)$$

And,

$$\frac{d\nu \circ \phi_\beta}{d\nu}(s) \left(\frac{n_\beta}{m_\beta} \right)^2 \le l(s)$$

where

$$l(s) = l(u, r, \xi) := \beta^{1+\frac{1}{\alpha}} \left(\frac{q(\beta^{\frac{1}{\alpha}} r, \xi)}{|r|^{-\alpha-1}\, c_1(\xi)} \right) \left(\frac{du\ |r|^{-\alpha-1}\, dr\ c_1(\xi)\, \sigma(d\xi)}{du\, \nu^{LS}(dw)} \right)$$

$$= \beta^{1+\frac{1}{\alpha}} \frac{q(\beta^{\frac{1}{\alpha}} r, \xi)\ dr\ \sigma(d\xi)}{\nu^{LS}(dw)}$$

Now, to have the Poissonian-type limit, we need to have

$$l \cdot \{1 \wedge k_t^2\} \quad \in \quad L_\mu(S)$$

Well,

$$\left\| \int_{\mathbb{R}^+ \times \mathbb{R}^d} l \cdot \{1\ \wedge\ k_t^2\} \mu(du, dw) \right\|$$

$$\le \int_{\mathbb{R}^+} \int_{S^{d-1} \times (0,\infty)} \beta^{1+\frac{1}{\alpha}} \frac{q(\beta^{\frac{1}{\alpha}} r, \xi)\ dr\ \sigma(d\xi)}{\nu^{LS}(dw)} \{1\ \wedge\ \{r^2 1_{[0,t)}(u)\}\} du\, \nu^{LS}(dw)$$

which equals

$$\int_0^t \int_{S^{d-1}\times(0,\infty)} \beta^{1+\frac{1}{\alpha}}\{1 \ \wedge \ r^2\}\ du\ q(\beta^{\frac{1}{\alpha}}r,\xi)\,dr\ \sigma(d\xi)$$

$$= t\int_{S^{d-1}\times(0,\infty)} \beta^{1+\frac{1}{\alpha}}\{1 \ \wedge \ r^2\}\ q(\beta^{\frac{1}{\alpha}}r,\xi)\,dr\ \sigma(d\xi)$$

and is equivalent to

$$t\int_0^\infty \beta^{1+\frac{1}{\alpha}}\{1 \ \wedge \ r^2\}\ \left\{\int_{S^{d-1}} q(\beta^{\frac{1}{\alpha}}r,\xi)\ \sigma(d\xi)\right\}\ dr$$

$$= t\int_0^\infty \{1 \ \wedge \ r^2\}\ \left\{\int_{S^{d-1}} \frac{q(\beta^{\frac{1}{\alpha}}r,\xi)}{(\beta^{\frac{1}{\alpha}}r)^{-\alpha-1}}\ \sigma(d\xi)\right\}\ r^{-\alpha-1}dr.$$

And, for β sufficiently small, this is

$$\leq K\,t\int_0^\infty \{1 \ \wedge \ r^2\}\ |r|^{-1-\alpha}\,dr < \infty.$$

Thus, by Theorem 4.3.1, as $\beta \to 0$,

$$\beta^{-\frac{1}{\alpha}}X_{\beta t} - c_{\beta,t} \xrightarrow{fdd} Y_t$$

where

$$Y_t = \int_S f_t(s)\left(M(ds) - \frac{g(s)\nu(ds)}{1\vee\|f_t(s)\|}\right)$$

$$= \int_{\mathbb{R}^+\times\mathbb{R}^d} w1_{(0,t]}(u)\left(M(du,dw) - \frac{du\,\nu^{\alpha}_{\sigma_1}(dw)}{1\vee\|w1_{(0,t]}(u)\|}\right)$$

which is a representation of an α-stable process.

□

Case (ii): Note that for part (a) since $\alpha_1 \in (0,2)$ we get a result similar to the result in case (i) since we can take $\alpha_1 = \alpha$ and use the fact that

$$q(r,\xi) \sim c_2(\xi)r^{-\alpha_1-1} \qquad as\ r\to\infty$$

to see that as $\beta \to \infty$,

$$\beta^{-\frac{1}{\alpha_1}}X_{\beta t} - c_{\beta,t} \xrightarrow{fdd} Y_t$$

where

$$Y_t = \int_S f_t(s)\left(M(ds) - \frac{g(s)\nu(ds)}{1 \vee \|f_t(s)\|}\right)$$

$$= \int_{\mathbb{R}^+ \times \mathbb{R}^d} w1_{(0,t]}(u)\left(M(du, dw) - \frac{du\, \nu_{\sigma_2}^{\alpha_1}(dw)}{1 \vee \|w1_{(0,t]}(u)\|}\right)$$

which is a representation of an α_1-stable process since $\alpha_1 \in (0,2)$. Just look at the proof for case (i) and substitute $\beta \to \infty$ for $\beta \to 0$ and α_1 for α.

But, for part (b) since $\alpha_1 \in (2, \infty)$, we have a different convergence result. To see this, let $n_\beta := \beta^{-\frac{1}{2}}$, $m_\beta := 1$, and $\phi_\beta(s) = \phi_\beta(u, w) := (\beta u, w)$. Then, clearly

$$m_\beta f_{\beta t}(\phi_\beta(u, w)) = 1_{[0,\beta t)}(\beta u)w = f_t(u, w).$$

So, $h_t(s) = f_t(s)$. And,

$$\|f_t(s)\| = \|f_t(u, w)\| \leq \|w\|1_{[0,t)}(u) := k_t(u, w) = k_t(s)$$

And,

$$\frac{d(\nu \circ \phi_\beta)}{d\nu}(s)\left(\frac{n_\beta}{m_\beta}\right)^2 = \frac{d(\nu \circ \phi_\beta)}{d\nu}(u, w)\left(\frac{\beta^{-\frac{1}{2}}}{1}\right)^2$$

$$= \beta^{-1}\frac{d(\beta u)\, \nu^{LS}(dw)}{du\, \nu^{LS}(dw)} = 1.$$

So, $g(s) = 1$ and

$$\frac{d(\nu \circ \phi_\beta)}{d\nu}(s)\left(\frac{n_\beta}{m_\beta}\right)^2 \leq 1 := l(s).$$

Now, to apply Theorem 4.3.1 case (i), we need to show that

$$\frac{n_\beta}{m_\beta} \to 0$$

as $\beta \to \infty$ and

$$l \cdot k_t^2 \quad \in \quad L_\nu(S).$$

Well,

$$\frac{n_\beta}{m_\beta} = \beta^{-\frac{1}{2}}$$

which clearly goes to zero as $\beta \to \infty$ and

$$\int_S l \cdot k_t^2 \ \mu(ds) = \int_{\mathbb{R}^+ \times \mathbb{R}^d} \|w\|^2 1_{(0,t]}(u) \ du \, \nu^{LS}(dw)$$

$$= t \int_{\mathbb{R}^d} \|w\|^2 \ \nu^{LS}(dw)$$

$$= t \int_{\|w\| \le 1} \|w\|^2 \ \nu^{LS}(dw) + t \int_{\|w\|>1} \|w\|^2 \ \nu^{LS}(dw).$$

The first integral is obviously finite since ν^{LS} is a Lévy measure. So, we need only show that the second integral is finite. Well, the second integral is equal to

$$t \int_{\|w\|>1} \|w\|^2 \ \nu^{LS}(dw) = t \int_{S^{d-1}} \int_1^\infty r^2 \ q(r,\xi) \, dr \, \sigma(d\xi)$$

which equals

$$t \int_{S^{d-1}} \int_{\beta^{-\frac{1}{\alpha_1}}}^\infty (\beta^{\frac{1}{\alpha_1}} r)^2 \ q(\beta^{\frac{1}{\alpha_1}} r, \xi) \, d(\beta^{\frac{1}{\alpha_1}} r) \, \sigma(d\xi)$$

And, for sufficiently large β, this is

$$\le K_\beta t \int_{S^{d-1}} \int_{\beta^{-\frac{1}{\alpha_1}}}^\infty (\beta^{\frac{1}{\alpha_1}} r)^2 \ |\beta^{\frac{1}{\alpha_1}} r|^{-\alpha_1 - 1} c_2(\xi) \, d(\beta^{\frac{1}{\alpha_1}} r) \, \sigma(d\xi)$$

$$\le t \int_{S^{d-1}} \int_{\beta^{-\frac{1}{\alpha_1}}}^\infty \ (\beta^{\frac{1}{\alpha_1}} r)^{1-\alpha_1} c_2(\xi) \, d(\beta^{\frac{1}{\alpha_1}} r) \, \sigma(d\xi)$$

which equals

$$t \int_{S^{d-1}} \int_1^\infty \ r^{1-\alpha_1} \ dr \ c_2(\xi) \, \sigma(d\xi)$$

$$= \frac{1}{\alpha - 2} \int_{S^{d-1}} c_2(\xi) \, \sigma(d\xi)$$

which is finite since σ_2 is integrable with respect to σ by assumption. Thus, by Theorem 4.3.1, as $\beta \to \infty$,

$$\beta^{-\frac{1}{2}} X_{\beta t} - c_{\beta,t} \xrightarrow{fdd} Y_t$$

where

$$Y_t = \int_S f_t(s) Z(ds)$$

and Z is an independently scattered Gaussian random measure with control measure ν^{LS}.

And, the result is proven. □

As was mentioned previously, the work presented in this chapter is based on the work of Pipiras and Taqqu in [13]. The short and long term behavior theorem, Theorem 4.3.1, is the main result and is new to the best of the author's knowledge. The most important advancement made in this work is the removal of the moment condition imposed by the main theorem of Pipiras and Taqqu. Their main theorem arises as a corollary to our main theorem. However, Corollary 4.4.1 is still more general than their result, as it holds in multiple dimensions and Corollary 4.4.2 is a new result. And, most of the examples investigated in this chapter are obtained from the original paper, but are studied in more generality.

Chapter 5

Summary

The purpose of this work is to unify the variations of multifractal processes and make general statements in regards to their short and long term behavior. It is of particular interest to study the small and large scale asymptotics of multifractal processes because of the many modeling applications. This work addresses many known papers in this field and expands, corrects, and exemplifies many previous results.

Since real life occurrences are not always describable by a normal distribution, stable distributions were introduced as an alternative, since they allow for more extreme outliers to occur because of their fat tails and since they have a desirable scaling property. But, stable distributions are not practical in applications because of their infinite variance. As a result, tempered stable distributions were introduced to tilt the tails of the stable distribution, making it have finite variance, thus making it useful in applications, such as physics formulas. Since then, many modifications of tempered stable distributions and their corresponding Lévy processes have been introduced.

In an attempt to unify the many modifications of tempered stable processes, generalized tempered stable processes are defined and studied in chapter 2. It is proven that the processes display multifractal properties at different scales in the space of cadlag functions equipped with the Skorokhod topology. Conditions under which the process is absolutely continuous with respect to the underlying stable process are given. Also, the series representations that lead to simulations of the process are discussed. This work unifies the area of tempered stable processes by identifying the minimal conditions necessary for the short and long term behavior.

In chapter 3, we take the integral of a Volterra kernel, $K_{H,\alpha}(t,s)$, with respect to a generalized tempered stable process:

$$X_t := \int_0^t K_{H,\alpha}(t,s)dX_s^{GTS}, \quad t \geq 0.$$

In short time, it is close fractional stable Levy motion, but in the long term is similar to fractional Brownian motion. The short and long term behavior of processes driven by a generalized tempered stable process are considered so that it is not necessary to study processes driven by the different variations of tempered stable processes separately. Also, errors in the work of Houdré and Kawai in "Fractional tempered stable motion" [7] are identified.

In chapter 4, a much more general class of infinitely divisible processes is considered. This involves the generalization of a work by Pipiras and Taqqu [13] in 2006. As was mentioned earlier, in their paper the authors examine processes with finite first moment given by the integral representations:

$$X_t := \int_S f_t(s)N(ds), \quad t \in \mathbb{R},$$

where N is a compensated Poisson random measure on a measurable space S. They describe general conditions for the normalized and time-scaled process to converge to a limit. The main advancement made in this chapter is the removal of the moment condition imposed on the original work of Pipiras and Taqqu, thus expanding the results to include every infinitely divisible process without Gaussian component, which can be represented as a stochastic integral of this type. Also, the multidimensional case is considered, as the earlier result only held in dimension one.

Please note that generalized tempered stable processes and fractional tempered stable motion could be thought of as special cases of the infinitely divisible processes studied in chapter 4. However, the short and long term behavior results obtained in chapter 4 involve convergence of finite dimensional distributions only. In the case of Lévy processes, convergence of finite dimensional distributions is enough to obtain convergence in the space of cadlag functions equipped with the Skorokhod topology. However, fractional tempered stable motion, which is considered in chapter 3, is not a Lévy process in general. So, more work is needed to obtain convergence results in the space of cadlag functions. The large scales case result is given and proven in this paper and the author is currently investigating the short scales case. It is proving to be quite a challenge due to the fact that the limiting processes do not actually have stationary increments as was earlier claimed by Houdré and Kawai [7]. But, due to preliminary calculations, it appears that the short scales case will hold in the space of cadlag functions.

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Appendices

Appendix A

Simulations of standard Lamperti stable processes

Some series representations using Example 2.6.1 are implemented in MATLAB in dimension one and are shown here. The representations are adapted from a code given by Jan Rosinski for stable processes. Recall that the series representation using LePage's Method [15] was modified to the case of GTS processes. It involves looking at a decomposition of the Lévy measure of the form:

$$\nu^{GTS}(A) = \int_{S^{d-1}} \int_0^\infty 1_A(r\xi)\ \rho_*(dr,\xi)\,\sigma_*(d\xi)$$

where $\sigma_*(d\xi) := \sigma(d\xi)/\sigma(S^{d-1}) := \sigma(d\xi)/\|\sigma\|$ is a probability measure on S^{d-1} and ρ_* is a measure on $\mathbb{R}$ such that

$$\rho_*([x,\infty),\xi) = \|\sigma\|\,\rho([x,\infty),\xi)$$

where

$$\rho([x,\infty),\xi) := \int_x^\infty q(r,\xi)\ r^{-\alpha-1}\,dr.$$

The inverse is defined as:

$$\begin{aligned}\rho_*^{-1}(u,\xi) &:= \inf\{x>0 : \rho_*([x,\infty),\xi) < u\}.\\ &= \inf\{x>0 : \rho([x,\infty),\xi) < \frac{u}{\|\sigma\|}\}\\ &= \rho^{-1}(u/\|\sigma\|,\xi).\end{aligned}$$

But, a problem is that ρ is not always invertible. As is mentioned in Example 2.6.1, in the special case of Lamperti stable processes when $f = 1$, this ρ-function is actually invertible. So, we are able to present some sample simulations directly. Below is a sample code, which produces Figure A.1. It considers the case $\alpha := .25$, although other stable indices can be obtained by simply changing this number. And, for convenience, the Lamperti stable processes are referred to as "standard" when $f = 1$.

```
a = .25;     % stable index

T = 1;     % time length

n = 2000;     % number of simulations

m = 1000;     % number of partitions of [0,T]

t = 0 : (T/m) : T;     % time goes from 0 to T with interval T/m

G =cumsum(-log(rand(1,n)));

    % Gamma random variable: Sum of n exponential(1) random variables

U = T*rand(n,1);     % times of jumps

V = 2*(rand(1,n)<= 0.5) -ones(1,n);     % random signs m

    % rand(1,n) <= 0.5 gives 1 x n matrix of zeros or ones with probability 1/2

J = V.*(log(repmat(1,1,n) +G. ∧ (−1/a)));

    % Gives the jumps: Γ^(−1/α) with random signs

clear G V;

X = J*(repmat(U, 1, m + 1) <= repmat(t, n, 1));

    % repmat(U, 1, m + 1) <= repmat(t, n, 1) produces 1_{u_j≤t}
```

figure

plot($t, X,' .r'$, 'markersize', 6); % graphs X

Various ranges of the parameter α are represented in the simulations. Notice that as α increases, the frequency and size of large jumps decreases.

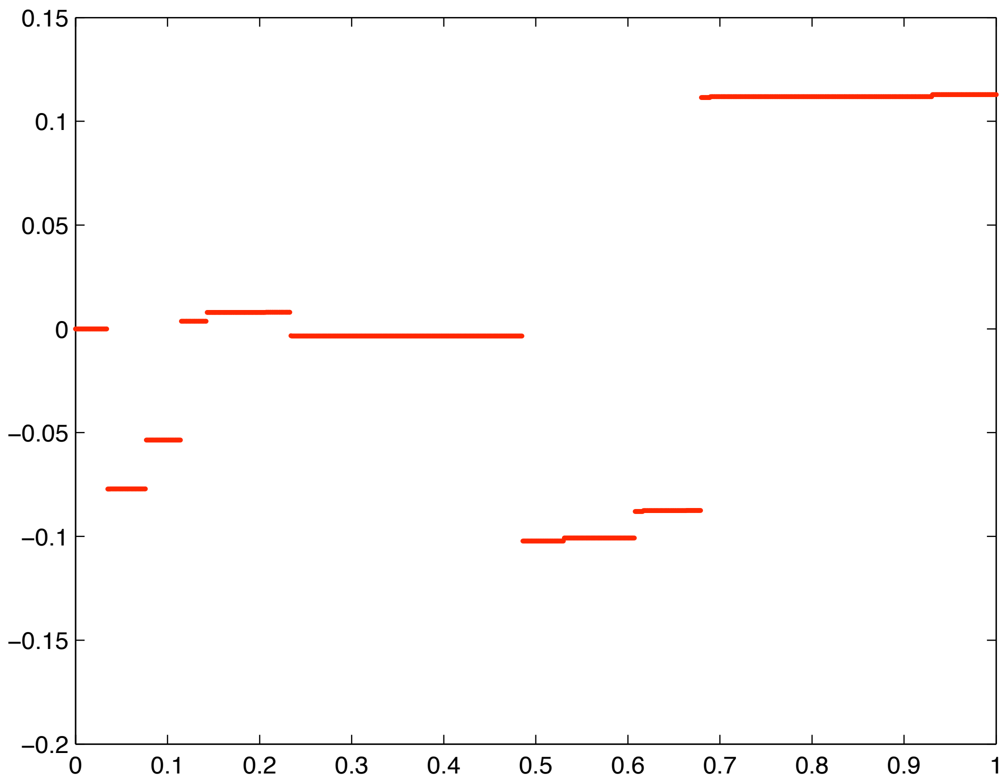

Figure A.1: Lamperti Stable Process, $\alpha = .25$, $f \equiv 1$

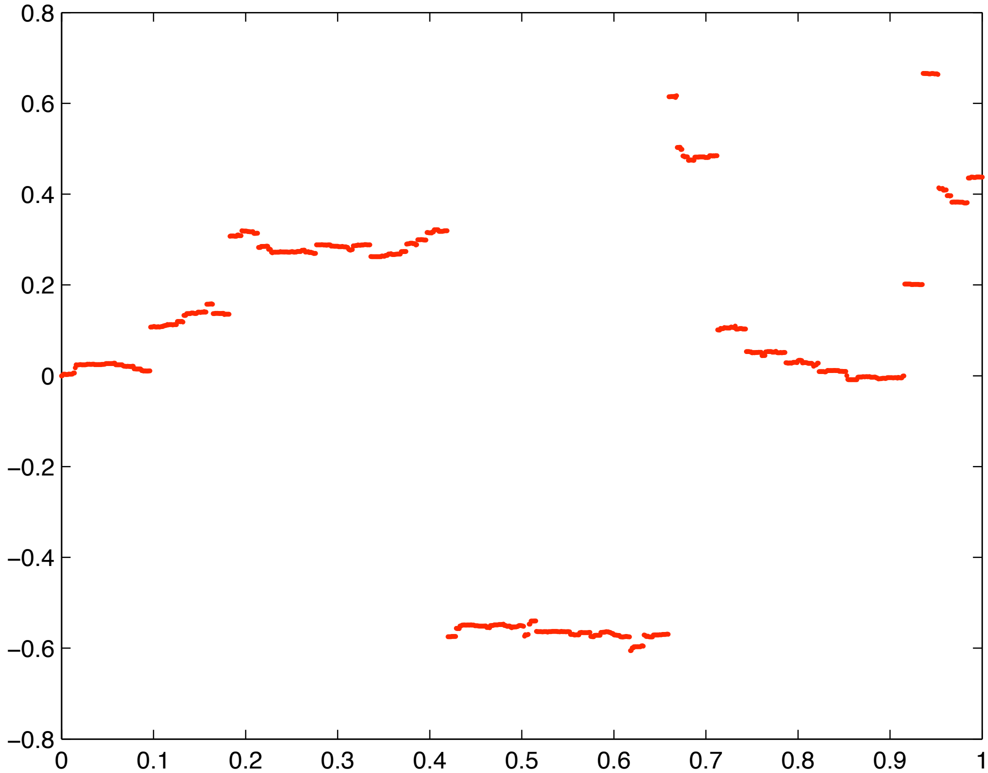

Figure A.2: Lamperti Stable Process, $\alpha = .75$, $f \equiv 1$

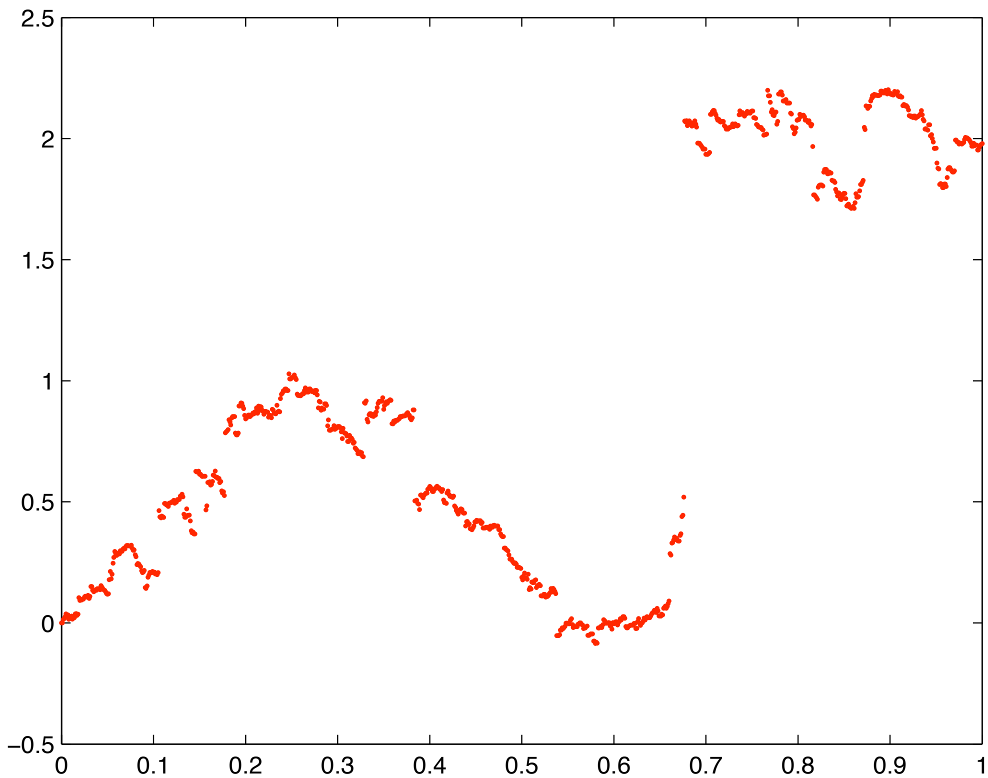

Figure A.3: Lamperti Stable Process, $\alpha = 1.25$, $f \equiv 1$

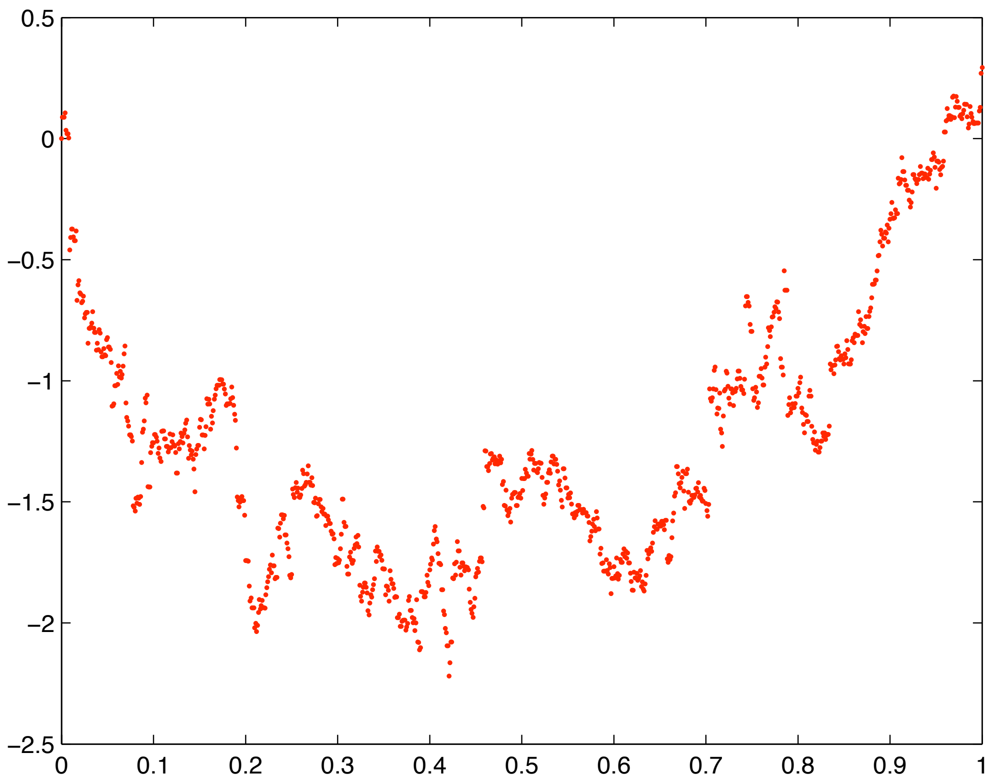

Figure A.4: Lamperti Stable Process, $\alpha = 1.75$, $f \equiv 1$

Appendix B

Simulations of non-standard Lamperti stable processes

Recall that in chapter 2 section 6, it was mentioned that Caballero, et. al. in [2] only conduct simulations of Lamperti stable processes in the case where $f(\xi) \equiv 1$. Here, simulations where $f \not\equiv 1$ are given using the series representation derived in Example 2.6.5 for various ranges of α. This code works for any values of the function, as long as $f \leq 1$.

$a = .25;$ % stable index

$T = 1;$ % time length

$b = .5;$ % $f(1) := .5$

$c = -.75;$ % $f(-1) := -.75$

$n = 2000;$ % number of simulations

$m = 1000;$ % number of partitions of [0,T]

$t = 0 : (T/m) : T;$ % time goes from 0 to T with interval T/m

G =cumsum(-log(rand(1,n)));

% Gamma random variable: Sum of n exponential(1) random variables

$U = T^*$rand(n,1);   % times of jumps

% gives an n x 1 matrix of random values from (0,1)

$W = T^*$rand(1,n);   % new uniform random variable

$V = 2^*$(rand(1,n)<= 0.5) -ones(1,n);   % random signs m

% rand(1,n)<= 0.5 gives $1 \times n$ matrix of zeros or ones with probability 1/2

% produces one or negative one with probability 1/2

L =log(repmat(1,1,n) $+ G. \wedge (-1/a)$)

B =abs(log(L)).*(((b)*(sign(L) >= 0) +(c)*(sign(L) <= 0)) -1)

% produces the V_i's.

$J = V.^*(L).^*(B >=$ log(W));   % jumps $\Gamma^{-1/\alpha}$ with random signs and rejection term

% $J = \sum_{j=1}^{n} \epsilon_j \Gamma_j^{-1/\alpha}$

clear G V;

$X = J^*$(repmat$(U, 1, m+1) <=$ repmat$(t, n, 1)$);

% repmat$(U, 1, m+1)$ <=repmat$(t, n, 1)$ produces $1_{u_j \leq t}$

figure

plot$(t, X,' .r'$, 'markersize', 6);   % graphs X

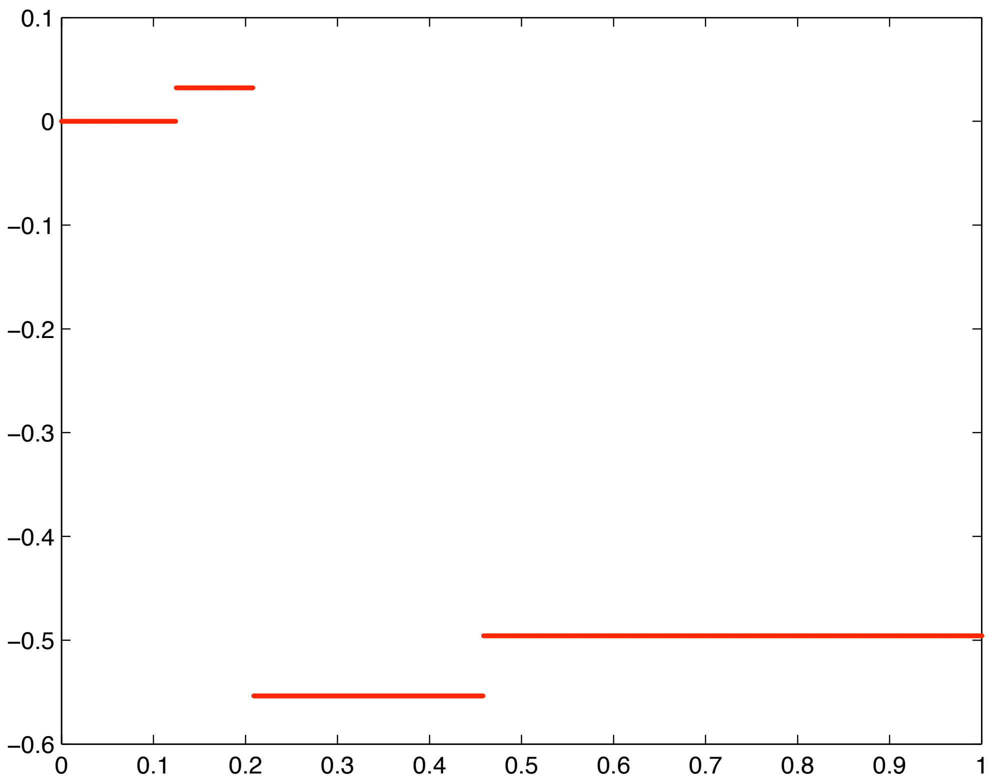

Figure B.1: Lamperti Stable Process, $\alpha = .25$, $f \not\equiv 1$

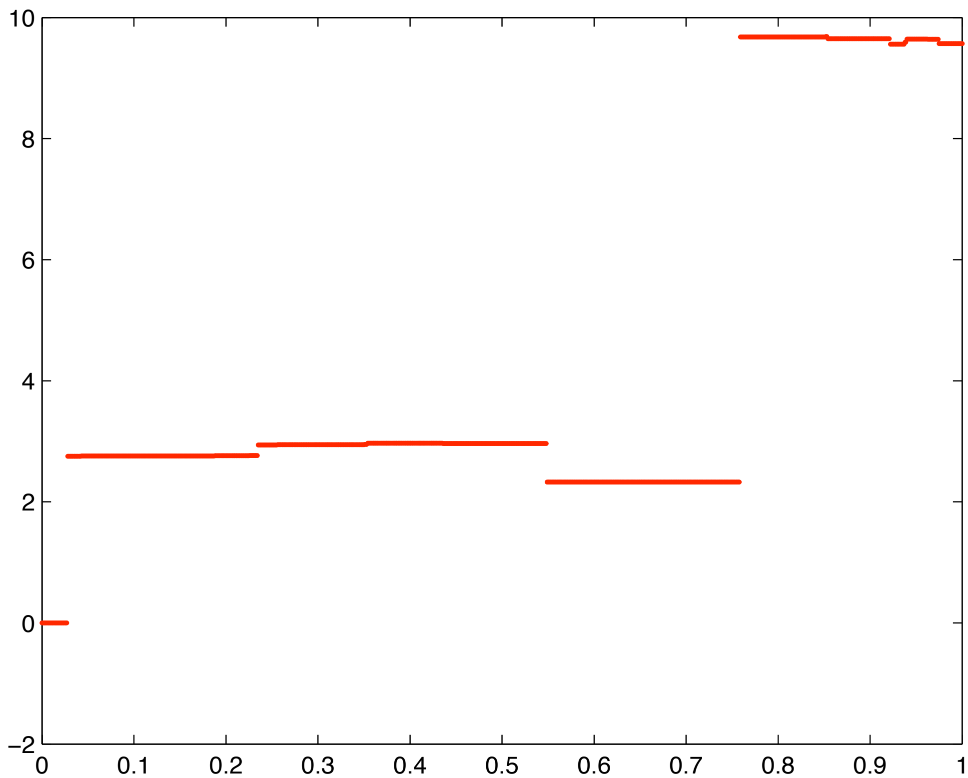

Figure B.2: Lamperti Stable Process, $\alpha = .75$, $f \not\equiv 1$

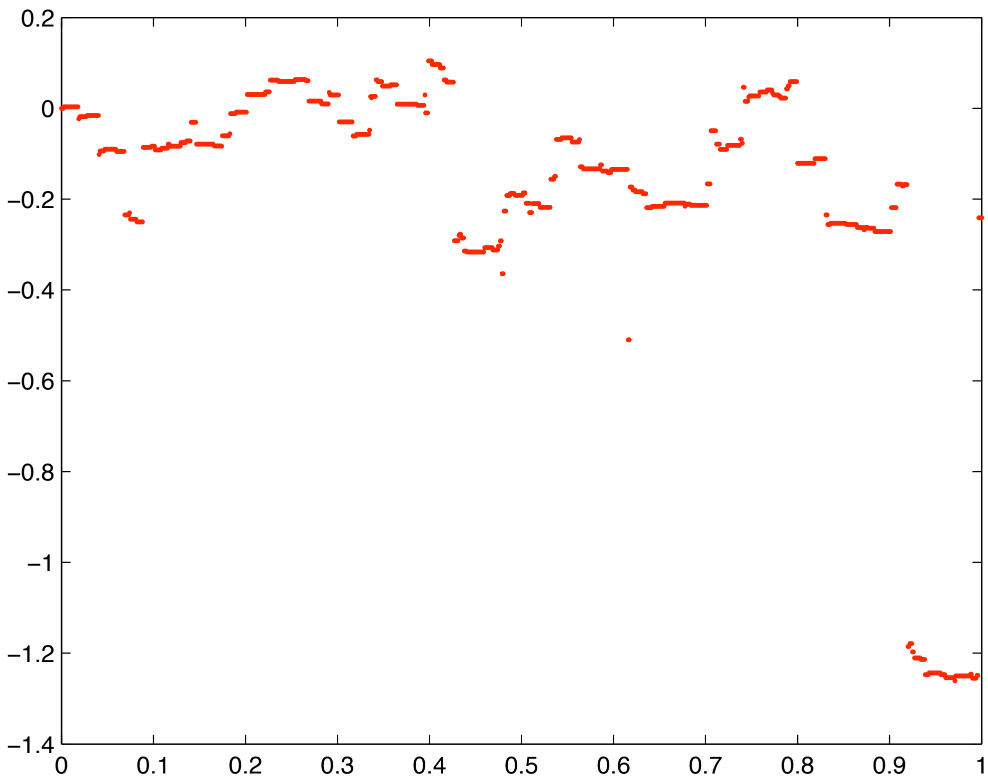

Figure B.3: Lamperti Stable Process, $\alpha = 1.25$, $f \not\equiv 1$

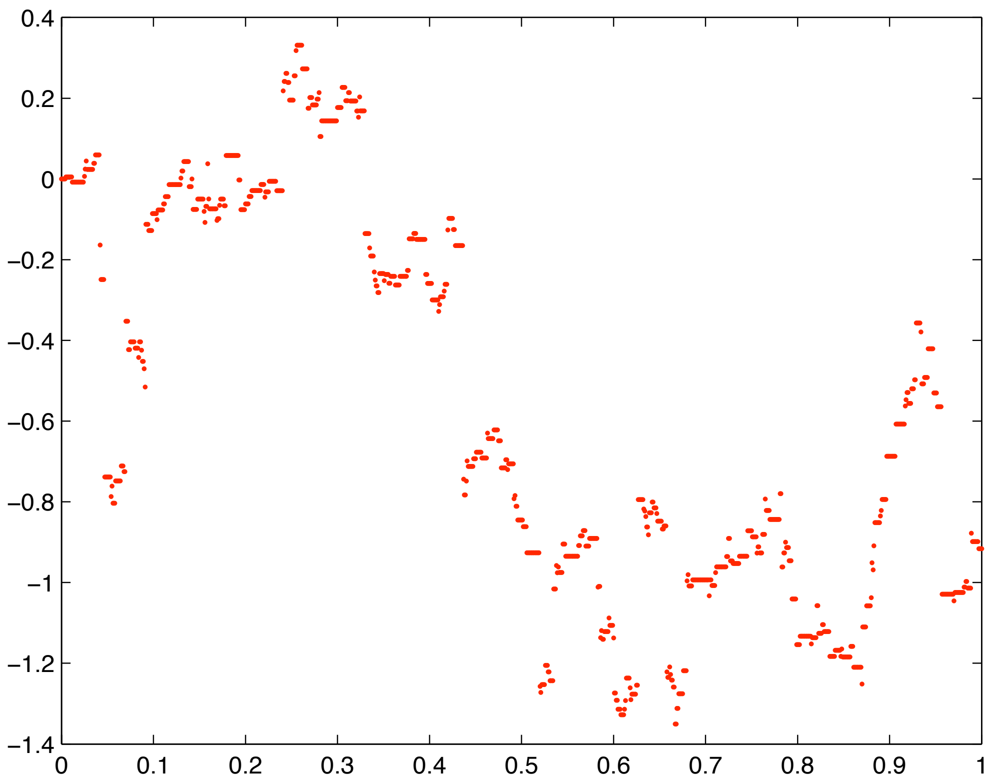

Figure B.4: Lamperti Stable Process, $\alpha = 1.75$, $f \not\equiv 1$

Vita

Jennifer Sinclair was raised in the metro Atlanta area of Georgia. She graduated from Norcross High School in May 1999 and attended the University of the South in Sewanee, Tennessee in August. She graduated from the University of the South in May 2003 with a Bachelor of Science in Mathematics. The next Fall, she began her time as a doctoral student of Mathematics at the University of Tennessee, Knoxville, working as a Graduate Teaching Assistant. The next year, she was promoted to Graduate Teaching Associate. Eventually, she received a partial fellowship provided by the Science Alliance of the University of Tennessee, which continued through 2009. She successfully defended her dissertation in July 2009.