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On the Number of Tilings of a Square by Rectangles

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ON THE NUMBER OF TILINGS OF A SQUARE BY RECTANGLES

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Abstract. How many ways are there to partition a square into n rectangles? In [2], Reading exhibits a bijection between a subset of permutations called 2-clumped permutations and those rectangulations with no 4 valent vertices, i.e., occurrences of $+$ We develop a recursion for generating the number of all rectangulations (also called tilings) of a square by n rectangles. This formula specializes to agree with Readings calculations. This results in an interesting periodicity modulo 2 which we verify for small values of n, but the general result remains elusive, hinting at some unseen structure of the rectangulations analogous to Readings bijection. Then, considering the topological space of all 0 to n-rectangulations, we use discrete Morse theory to prove homotopy equivalence of this space to wedges of n-1 dimensional spheres. The Euler characteristics obtained from the recursion allow us to compute the exact homotopy types.

1. Introduction

We are considering all rectangular tilings of the unit square. A tiling is any partition of the square into a finite number of rectangles such that the edges of the rectangles are parallel to the edges of the square. We define $\mathbb{T}_n$ to be the topological space of tilings by $\leq n$ rectangles. The topology is straight forward. We equip $\mathbb{T}_n$ with the Hausdorff metric on the edges of the rectangles. Thus perturbation of an edge in a tiling yields another tiling close to the first. We are concerned with the homotopy types of this space. Our main result uses discrete Morse theory to establish homotopy equivalence between $\mathbb{T}_n$ and a wedge of n-1 dimensional spheres.

Theorem 1. *There is a homotopy equivalence* $\mathbb{T}_n \simeq \vee_{i=1}^{k_n} S^{n-1}$*, for some nonnegative integer* k_n*.*

To compute the homotopy type, we must compute the k_n, and this motivates our treatment of the tilings as combinatorial equivalence classes. Two tilings are said to be combinatorially equivalent if there exists a homeomorphism between them which fixes the corners of the square. Equivalently, two tilings are equivalent if all corners which end on the same line segment, or wall, have the same relative positions in each tiling Now $\mathbb{T}_n$ can be seen as a cell complex, and a rectangulation with m rectangles and e 4-valent vertices, i.e. occurrences of $+$ Has dimension $m - e - 1$. Each combinatorial equivalence class corresponds bijectively to a cell in $\mathbb{T}_n$. The numbers k_n are determined by the number of cells in each dimension using the Euler characteristic. Thus combinatorially, we need to count the number of rectangulations with m rectangles and e 4-valent vertices which we denote $t_{m,e}$.

We develop a recursion to generate more complex rectangulations from simpler ones. This two step process involves sliding in vertical lines from the right side of the square and then adding horizontal lines to the right of the vertical lines just added. Thus we index over horizontal lines in a given rectangulation which hit the right edge of the square and arrive at our second main result. Let $t_{m,r,e}$ be the number of rectangulations of m rectangles, r horizontal lines hitting the right edge, and e 4-valent vertices.

Theorem 2.

$$t_{m,r,e} = \sum_{s=0}^{n-1}\sum_{n=1}^{m-1}\sum_{f=0}^{e}\sum_{c=1}^{\lceil (s+1)/2\rceil} (-1)^{c+1}\binom{\ell-1}{c-1}\binom{s+2-\ell}{c}\binom{\ell-c}{\nabla}\binom{\Delta-c-\nabla+\ell-1}{\ell-1}t_{n,s,f}$$

where $\nabla = e-f$, $\Delta = m-n$, and $\ell = s+m-n-r$. The base of the recursion is given by $t_{k,k-1,0} = 1$ for $k \geq 1$.

Clearly the quantity we need, $t_{m,e} = \sum_{r=0}^{m-1} t_{m,r,e}$. A closed form for either $t_{m,e}$ or $t_m := \sum_{e=0}^{m-1} t_{m,e}$ = the number of total rectangulations of m rectangles would be desirable and is still an open problem. In [2], the numbers $t_{m,0}$ are shown to count a certain class of permutations of $\{1,\dots,m\}$, called 2-clumped permutations. In particular, since the first 2-clumped permutation does not occur until $m=5$, the interesting pattern $t_{1,0}=1, t_{2,0}=2, t_{3,0}=6$, and $t_{4,0}=24$ arises. Reading's beautiful result, however, does not easily give a formula for $t_{m,0}$ or extend to $t_{m,e}$ for $e \geq 1$. From the recursion, a simpler expression for t_m can be formulated which only indexes over the number of rectangles and the lines hitting the right edge. However, since it does not keep track of the number of 4-valent vertices, it is less interesting topologically. Now for this corollary only, let $t_{m,r}$ denote the number of rectangulations with m rectangles and r horizontal lines hitting the right edge of the square.

Corollary 3.

$$t_{m,r} = \sum_{s=0}^{n-1}\sum_{n=1}^{m-1}\sum_{c=1}^{\lceil (s+1)/2\rceil}\sum_{i=0}^{\ell-c}(-1)^{i+c+1}\binom{\ell-1}{c-1}\binom{s+2-\ell}{c}\binom{\ell-c}{i}\binom{\Delta+2(\ell-c-1-i)}{2\ell-c-2}t_{n,s}$$

where $\Delta = m-n$, and $\ell = s+m-n-r$. The base of the recursion is given by $t_{k,k-1} = 1$ for $k \geq 1$.

The above recursion yields the following data, where $t_{m,e}$ now counts the number of 4-valent vertices again.

m	1	2	3	4	5	6	7	8	9	10	11	12
$t_{m,0}$	1	2	6	24	116	642	3938	26194	186042	1395008	10948768	89346128
$t_{m,1}$	0	0	0	1	12	114	1028	9220	83540	768916	7200852	68611560
$t_{m,2}$	0	0	0	0	0	2	48	770	10502	132210	1593934	18755516
$t_{m,3}$	0	0	0	0	0	0	0	10	348	7680	137940	2206972
$t_{m,4}$	0	0	0	0	0	0	0	0	1	104	4020	106338
$t_{m,5}$	0	0	0	0	0	0	0	0	0	0	20	1571
$t_{m,6}$	0	0	0	0	0	0	0	0	0	0	0	2
t_m	1	2	6	25	128	758	5014	36194	280433	2303918	19885534	179028087

Although t_m does not have topological significance, it is an interesting combinatorial quantity.

The sequence t_m continues

$$
\begin{aligned}
&1,2,6,25,128,758,5014,36194,280433,2303918,19885534,179028087,1671644720,\\
&16114138846,159761516110,1623972412726,16880442523007,179026930243822,\\
&1933537655138482,21231023519199575,236674460790503286,2675162663681345170,\\
&30625903703241927542,354767977792683552908,4154708768196322925749,\\
&49152046198035152483150,587011110939295781585102,7072674305834582713614923
\end{aligned}
$$

This sequence motivates the following conjecture about the 8 fold periodicity of the parity of t_m.

Conjecture 4. *$t_n \equiv 1 \bmod 2$ if $n = 8k+1$ or $n = 8k+3$. Otherwise $t_n \equiv 0 \bmod 2$.*

Returning to our main result, we can use the calculations of $t_{m,e}$ above to calculate k_n:

Proposition 5. *The sequence k_n referred to in Theorem 1 is given by:*

$$0, 2, 4, 19, 85, 445, 2513, 15221, 97436, 653290, 4554620, 32833261, \ldots$$

for $n \geq 1$

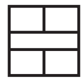
2. Recursive formula for the number of tilings

We first enumerate the combinatorially distinct rectangulations

Theorem 6. *Let $t_{m,r,e}$ be the number of distinct tilings with m tiles, r edges that meet the right-hand side of the square and e 4-valent vertices.*

$$t_{m,r,e} = \sum_{s=0}^{n-1} \sum_{n=1}^{m-1} \sum_{f=0}^{e} \sum_{c=1}^{\lceil (s+1)/2 \rceil} (-1)^{c+1} \binom{\ell-1}{c-1} \binom{s+2-\ell}{c} \binom{\ell-c}{\nabla} \binom{\Delta - c - \nabla + \ell - 1}{\ell - 1} t_{n,s,f}$$

where $\nabla = e - f$, $\Delta = m - n$, and $\ell = s + m - n - r$. The base of the recursion is given by $t_{k,k-1,0} = 1$ for $k \geq 1$.

Proof. Every rectangular tiling, except ones with only vertical edges can be generated from a simpler tiling by the process in Figure 1, where $c = 1$. The simpler tiling is pictured in (A). Then one pushes an edge of length ℓ in from the right, blocking $\ell - 1$ horizontal edges from hitting the right edge, as in (B). One then adds horizontal edges in the newly created box, some of which create 4-valent vertices as in (C), and some of which do not as in (D). However, some tilings may be generated in more than one way from this move. For example, the tiling  comes from two different simpler tilings. To take care of this we use an inclusion-exclusion argument and write

$$t_{m,r,e} = \sum_{c\geq 1}(-1)^{c+1}(\#\text{ of ways to push in } c \text{ edges from the right from a simpler tiling})$$

First we count the ways to push in c lines from the right with total length ℓ, as in Figure 1 (B). (Note that $\ell = s - r + m - n$, because ΔBoxes $= \Delta$(Right edges) $- \ell$.) Since there are $s+1$ available slots on the right, this is the count of the number of c-component subsets of $s+1$ with a total length of ℓ, which by Lemma 7, is $\binom{\ell-1}{c-1}\binom{s+2-\ell}{c}$. Next, we need to create $e-f$ 4-valent vertices, and the only way to do this is to put a horizontal line at one of the existing pushed in horizontal lines, as in (C). There are $\ell - c$ pushed in lines, so there are $\binom{\ell-c}{e-f}$ choices available. Finally, we need to figure out how to distribute the remaining horizontal edges to get an m-tile configuration with s right-hitting edges. The number of bins these new horizontal lines can go to is ℓ. Each pushed in component creates a new box making c, and each 4-valent vertex also creates a new box, making $c+e-f$. So we need to create $m-n-(c+e-f)$ new boxes. Hence we need to count the number of ways to distribute $m-n-c-e+f$ edges into the ℓ distinct slots they can go, as in (D). By Lemma 8, this is $\binom{m-n-c-e+f+\ell-1}{\ell-1}$. Thus we have accounted for all four factors of the coefficient in the formula.

The limits of the summations are explained as follows. Given a tiling where s edges hit the right edge, one can push in at most $\lceil (s+1)/2 \rceil$ edges. The number of tiles in the simpler tiling must be smaller, so n ranges to $m-1$. The number of edges meeting the right may not be smaller in the simpler tiling, but we can at least say it has to be less than the number of tiles n. Finally the number of 4-valent vertices must indeed be less than or equal to the number in the more complex tiling. □

Lemma 7. *The number of c-component subsets of $\{1,\ldots,s+1\}$ of total size ℓ is given by the formula*

$$\binom{\ell-1}{c-1}\binom{s+2-\ell}{c}.$$

Proof. First we count the number of ways to break ℓ into c nonzero pieces, which is $\binom{\ell-1}{c-1}$. Then we count the ways of inserting those c pieces into the rest of the slots. There are $s+1-\ell$ slots remaining, and there are $s+2-\ell$ interstices available, accounting for the $\binom{s+2-\ell}{c}$ term. □

The following lemma is well-known and can be found, for example, in [1].

Lemma 8. *The number of ordered nonnegative integer partitions of n with k parts is* $\binom{n+k-1}{k-1}$.

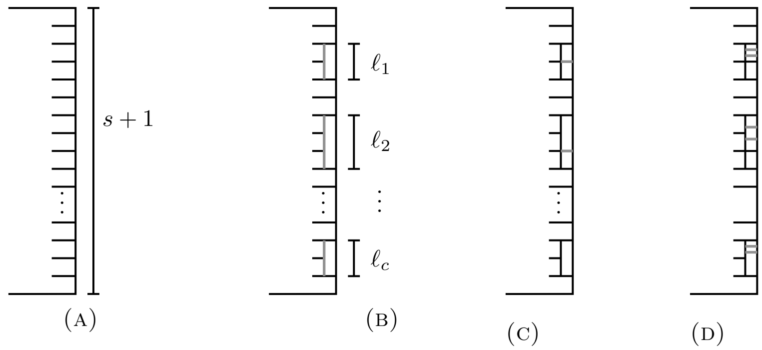


FIGURE 1. (A): The right side of the square with s edges hitting it. (B): Pushing in c vertical edges, of total length $\ell_1 + \cdots + \ell_c = \ell$. (C): Adding $e-f$ horizontal line segments to create $e-f$ 4-valent vertices. (D): Adding edges to the ℓ available bins.

Using the recursive strategy in the proof of Theorem 6 we obtain a different recursion, simpler because it does not index over the number of 4-valent vertices, but less topologically interesting for this same reason.

Corollary 9. *Let $t_{m,r}$ denote the number of rectangulations with m rectangles and r horizontal lines hitting the right edge of the square. Then*

$$t_{m,r} = \sum_{s=0}^{n-1}\sum_{n=1}^{m-1}\sum_{c=1}^{\lceil (s+1)/2\rceil}\sum_{i=0}^{\ell-c}(-1)^{i+c+1}\binom{\ell-1}{c-1}\binom{s+2-\ell}{c}\binom{\ell-c}{i}\binom{\Delta+2(\ell-c-1-i)}{2\ell-c-2}t_{n,s}$$

where $\Delta = m-n$, and $\ell = s+m-n-r$. The base of the recursion is given by $t_{k,k-1} = 1$ for $k \geq 1$.

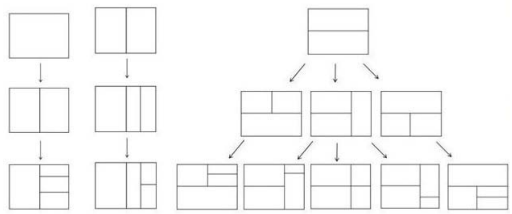
Proof. The proof replicates that of theorem 6 until steps (C) and (D). We distribute the remaining $m-n-c$ horizontal lines after sliding in the c vertical lines at once. There are $(\ell-c)+(\ell-1)$ possible slots for a line and the $\ell-c$ slots corresponding to the creation of a 4-valent vertex can hold at most one line. Thus by Lemma 10 the number of ways to do this is $\binom{(m-n-c)+(\ell-c)+(\ell-1)}{(\ell-c)+(\ell-1)-1}$, and the result follows by moving the summation sign and simplifying. $\square$

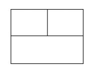
Lemma 10. *The number of ordered nonnegative integer partitions of n of size $k+j$ such that the first k entries are either 0 or 1 is*

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{n+k+j-1-2i}{k+j-1}$$

Proof. Let $A :=$ the total number of partitions of n and $A_i =$ the number of partitions of n such that the i^{th} term is ≥ 2 for $i = 1, \ldots k$. We need $|A - \bigcup A_i|$ which by the principle of inclusion and exclusion $= \sum_{I\subset\{1,\ldots,k\}}(-1)^{|I|}|\bigcap_{i\in I} A_i| = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{n+k+j-1-2i}{k+j-1}$ $\square$

The recursive strategy of the proof of Theorem 6 is illustrated in the following diagram. Here we generate rectangulations of 4 rectangles, starting with the empty tiling and the two tilings of two rectangles. The first step is sliding in the vertical line followed by adding the remaining horizontal lines to the right of it.



Similarly one can construct new tilings starting with  and so on.

2.1. **Calculations.** Now, let $t_{m,e}$ denote the number of tilings which have m rectangles and e 4-valent vertices. So $t_{m,e} = \sum_{r=0}^{m-1} t_{m,r,e}$. The above recursion yields the following data.

m	1	2	3	4	5	6	7	8	9	10	11	12
$t_{m,0}$	1	2	6	24	116	642	3938	26194	186042	1395008	10948768	89346128
$t_{m,1}$	0	0	0	1	12	114	1028	9220	83540	768916	7200852	68611560
$t_{m,2}$	0	0	0	0	0	2	48	770	10502	132210	1593934	18755516
$t_{m,3}$	0	0	0	0	0	0	0	10	348	7680	137940	2206972
$t_{m,4}$	0	0	0	0	0	0	0	0	1	104	4020	106338
$t_{m,5}$	0	0	0	0	0	0	0	0	0	0	20	1571
$t_{m,6}$	0	0	0	0	0	0	0	0	0	0	0	2
t_m	1	2	6	25	128	758	5014	36194	280433	2303918	19885534	179028087

Indeed the sequence t_m continues

$$1,2,6,25,128,758,5014,36194,280433,2303918,19885534,179028087,1671644720,\\ 16114138846,159761516110,1623972412726,16880442523007,179026930243822,\\ 1933537655138482,21231023519199575,236674460790503286,2675162663681345170,\\ 30625903703241927542,354767977792683552908,4154708768196322925749,\\ 49152046198035152483150,587011110939295781585102,7072674305834582713614923$$

3. Symmetric tiles and a mod 2 counting conjecture

From the sequence t_n, we notice an 8 fold periodicity of parity. Here is t_n mod 2, for $1 \leq n \leq 28$.

$$1,0,0,1,0,0,0,0,1,0,0,1,0,0,0,0,1,0,0,1,0,0,0,0,1,0,0,1,\ldots$$

This sequence appears to satisfy the surprising property that the number of rectangulations is even unless $n = 8k + 1$ or $8k + 4$ in which case it is odd.

Conjecture 11. $t_n \equiv 1 \bmod 2$ *if* $n = 8k + 1$ *or* $n = 8k + 4$. *Otherwise* $t_n \equiv 0 \bmod 2$.

We consider the action of the dihedral group of 8 elements D_8 on T_n. Let s_n be the number of tilings fixed by this action. That is, s_n counts the totally symmetric tilings.

Lemma 12. $s_n \equiv t_n \bmod 2$

Proof. The orbits of the D_8 action on T_n have an even number of elements except for the singleton orbits. □

Thus to prove Conjecture 11 we need only to count the totally symmetric rectangulations of T_n. Furthermore, again considering the action of D_8 we have the following:

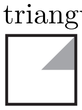
Lemma 13. *A totally symmetric rectangulation has either* $4k$ *tiles or* $4k + 1$ *tiles.*

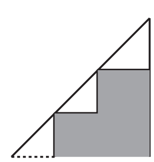
Proof. Given a totally symmetric rectangulation, D_8 acts on the individual rectangles within it. The orbit of a rectangle under the D_8 action has either 1, 4, or 8 elements. It has 1 element if and only if the rectangle contains the square's center in its interior. □

This gives the immediate result

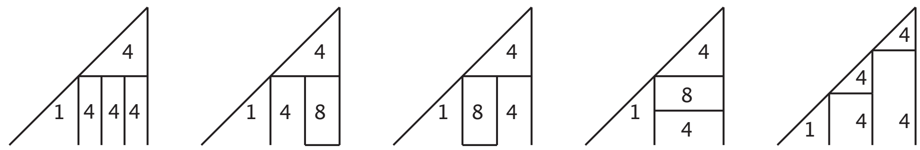
Proposition 14. $s_n = 0$ *unless* $n = 4k$ *or* $n = 4k + 1$. *Furthermore* $s_{4k+1} = s_{4k+4}$.

Proof. The first statement follows from Lemma 13. The bijection corresponding to $s_{4k+1} = s_{4k+4}$ is given by subdividing the central square into 4 squares. □

Conjecture 11 can be independently verified for small n by directly counting symmetric configurations. Every symmetric rectangulation is determined by what it looks like in a triangular fundamental domain for the D_8 action, depicted in grey in the following picture:
. So we study the possible configurations when restricted to this triangle. It is clear that they must look as follows



where the grey region is a rectangular tiling, and there are some number of "sawteeth" that hit the diagonal. The dashed edge may or may not be there, and accounts for the equality $s_{4k+1} = s_{4k+4}$. So for example, here is a count of the symmetric tilings by 17 rectangles.



Here the numbers refer to the number of rectangles in the orbit of a given region, and must add up to 17. We see that $s_{17} = 5$, which is consistent with our calculation that $t_{17} \equiv 1 \mod 2$.

4. Topological remarks

Let $\mathbb{T}_n$ be the topological space of tilings of the unit square by $\leq n$ rectangles. The topology is straightforward: if you move a vertex slightly the new tiling is near the old tiling. To make this precise is to consider the Hausdorff metric on the edges of the rectangles. This makes $\mathbb{T}_n$ into a metric space.

The space $\mathbb{T}_n$ is a cell complex, where the cells correspond to combinatorially distinct tilings. The dimension of a cell with m tiles and e 4-valent vertices is $m - e - 1$.

Define the reduced Euler characteristic $\tilde{\chi}(K)$ of a complex K to be $\chi(K) - 1$ where $\chi(K)$ is the classical Euler characteristic. Let $x_n = \tilde{\chi}(\mathbb{T}_n)$. We can use our calculations of $t_{m,e}$ to calculate reduced Euler characteristics.

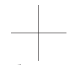
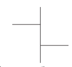
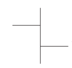
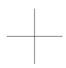
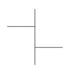
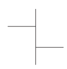
Proposition 15. *The sequence x_n of reduced Euler characteristics is given by:*

$$0, -2, 4, -19, 85, -445, 2513, -15221, 97436, -653290, 4554620, -32833261, \ldots$$

for $n \geq 1$

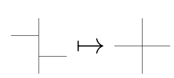
4.1. **Proof of Theorem 1.** We define a discrete vector field in the sense of Forman [3] on the complex $\mathbb{T}_n$. This is a collection of pairs of cells (α, β), called *vectors*, where α is a codimension 1 face of β, in the sense that the degree of the attaching map is ± 1. Every cell of $\mathbb{T}_n$ is allowed to appear in at most 1 pair. Furthermore, we need the vector field to be a gradient field, which means that no chain $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \ldots$ can loop back on itself, where each (α_i, β_i) is a pair from the vector field, and α_{i+1} is a cell in the boundary of β_i distinct from α_i, with nonzero multiplicity. The critical cells are defined to be those that don't appear in any pair in the vector field. Forman's theorem implies that $\mathbb{T}_n$ is homotopy equivalent to a complex which has cells in $1 - 1$ correspondence with the critical cells.

We define the vector field as follows. Given a rectangulation R, define $\llbracket R$ to be the rectangulation with a new long thin box added on the left of the ambient square. Every nontrivial rectangulation R can be uniquely written $\llbracket^k S$ where $S \neq \llbracket T$ for any T. By convention we think of the trivial rectangulation with 1 tile as $\llbracket\emptyset$, although $\emptyset$ does not correspond to a rectangulation in $\mathbb{T}_n$. Create a vector field by forming all possible pairs $(\llbracket^{2i} S, \llbracket^{2i+1} S)$. Depending on orientation conventions $\llbracket^{2i} S$ appears with coefficient ± 1 in $\partial \llbracket^{2i+1} S$ because there are $2i + 1$ different terms in the boundary that correspond to $\llbracket^{2i} S$ which mostly cancel. (On the other hand notice that $\llbracket^{2i-1} S$ appears with coefficient 0 in $\partial \llbracket^{2i} S$.) By design these pairs do not overlap at all. To see there are no closed gradient loops, notice that in a chain $\alpha_1, \beta_1, \alpha_2, \beta_2$, we must have $\alpha_1 = \llbracket^{2i} S$, $\beta_1 = \llbracket^{2i+1} S$ and $\alpha_2 = \llbracket^{2i+1} S'$ for some S'. Since α_2 is the first in a pair, $S' = \llbracket^{2\ell+1} T$ for some ℓ and T. In particular α_2 has more leading rectangles than α_1. Since the number of leading rectangles strictly increases along gradient paths, there can be no closed loops.

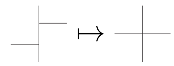
The critical cells for this vector field consist of the trivial rectangulation, which is the unique vertex, as well as cells corresponding to rectangulations with n tiles that are not of the form $\llbracket^{2i+1} S$. These are of dimension $n-1$ minus the number of 4-valent vertices. Our next task is to extend the previous vector field to a vector field that includes all singular rectangulations (i.e. rectangulations with at least one 4-valent vertex.) Among critical rectangulations from the previous vector field, we define a map Δ as follows. Given a rectangulation t, find all positions in the tiling of the form  or . Locate the position which is furthest right, and if there is more than one that is furthest right, take the one that is closest to the top. If this position is of the form , then $\Delta(t) = 0$. If it is of the form  then define $\Delta(t)$ to be the rectangulation where this position is changed to . Now the vector field consists of all pairs $(t, \Delta(t))$ where $\Delta(t) \neq 0$, and t is critical for the previous vector field. (Which implies that $\Delta(t)$ is also.) All of these pairs are disjoint since if the upper right instance is , then it is the first coordinate of a pair, and if it is , then it is the second coordinate of a pair. All singular rectangulations appear either as the first or second coordinate of a pair, so the critical cells are either the unique 0-cell or are $(n-1)$-dimensional. Finally we argue there are no closed gradient loops in the combined vector field. We claim that a gradient loop cannot contain any pairs $(\llbracket^{2i} S, \llbracket^{2i+1} S)$. If we have $\alpha_2 = \llbracket^{2j} S'$, then $j > i$ and $\beta_2 = \llbracket^{2j+1} S''$. If we have $\alpha_2 = \llbracket^{2j+1} S'$, then this is a contradiction since such rectangulations are always the second coordinate of a vector. So if a gradient path contains a pair $(\llbracket^{2i} S, \llbracket^{2i+1} S)$, then all subsequent pairs are of this form, and so by the previous argument, there is no closed loop. So now we can concentrate on pairs $(t, \Delta(t))$ only.

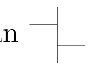
Notice that Δ preserves the number of vertical walls, and ∂ cannot increase the number. [Discussion of all codimension 1 events.] Thus a closed gradient loop must have a constant number of vertical walls for every α_i and β_i. Also, once Δ operates on a given vertical wall it can never operate on one below it or to the left. So in a loop, it must operate on a single vertical wall. Similarly, the number of edges meeting the wall from the left and right must be constant since Δ preserves this number and ∂ cannot increase it. Thus it makes sense to label the edges meeting the wall from the left and right by numbers which are constant throughout the purported gradient loop. We are then reduced to the following question. Consider the following moves:

(Y1):



(Y2):



These are the two possible moves $\beta_i \to \alpha_{i+1}$ consisting of taking a term in the boundary where the number of walls and attaching edges from the left and right are preserved. However the move Y2 can never be part of a loop, since the left edge starts out below the right edge, and Δ cannot reverse their order. Hence only $Y1$ moves are available. So consider $\alpha_1, \beta_1, \alpha_2$ where α_2 is obtained from β_1 by a $Y1$ move. Then it must have operated on a site below the one that changed from α_1 to β_1, so that α_2 still has an  as its top instance. Thus $\Delta(\alpha_2) = 0$ and it cannot be the first cell in a pair.

References

[1] **J.H. van Lint** and **R.M. Wilson**. *A course in combinatorics.* Cambridge University Press, Cambridge, 1992. xii+530 pp. ISBN: 0-521-41057-6; 0-521-42260-4
[2] **N. Reading**, *Generic Rectangulations*, arxiv:1105.3090.v1 [math.CO]
[3] **Forman**, *Morse Theory for Cell Complexes*, Advances in Mathematics 134, 90-145 (1998)