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A Random Walk Production-Inventory Policy: Rationale and Implementation

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A RANDOM WALK PRODUCTION-INVENTORY POLICY: RATIONALE AND IMPLEMENTATION*

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The smoothing of fluctuations in production and inventory has been handled successfully as a problem in servomechanisms: an optimal linear filter (production scheduling policy) balances costs of inventory fluctuation and costs of production fluctuation. This approach requires the system to respond (by making production rate changes) either continuously or at regular intervals: the usual assumption is that the cost of responding is proportional to the size of the response. This paper explores some reasons why the frequency of production changes may have to be controlled, and offers a new class of policies, called random walk policies, to accomplish this. One of these, the (a, b, c) policy, is investigated for stationary operating characteristics, and a specific example is provided to illustrate the technique of finding optimal values for policy parameters. Some of the characteristics of systems in which random walk inventory policies may be useful are discussed; and more general techniques for identifying the form of the optimal control policy are mentioned.

1. Introduction

In this paper we deal with the production smoothing problem. Our model specifies these conditions: (a) one homogeneous product is manufactured; (b) time enters the model as a continuous parameter (but we will use the "discrete equal intervals" formulation where appropriate for discussing other analyses of the smoothing problem); (c) the rate of production is the only decision variable; (d) the objective is minimization of total cost associated with inventory and production. In symbolic form, letting p_t, $\underline{s}_t$ and $\underline{I}_t$ represent the dated rates of production and demand, and the "on hand" level of inventory, letting $\underline{z}$ represent the number of production rate changes in a time interval $(t, t + \Delta t)$ and distinguishing all random variables by underbars, the problem is to minimize the total stationary expected cost at some future time t:

$$(1) \qquad E[L(t)] = E[\lambda_1(\underline{I}_t \mid \underline{I}_t > 0) + \lambda_2(\underline{I}_t \mid \underline{I}_t \leqq 0) + \lambda_3(p_t) + \lambda_4(\underline{z})]$$

subject to the inventory balance identity

$$(2) \qquad \underline{I}_{t+\Delta t} = \underline{I}_t + \Delta t(p_t - \underline{s}_t)$$

(where Δt is a very short interval) and the demand probability rule

$$(3) \qquad \text{Prob}\ (\underline{s}_t \leq \underline{s}) = \Phi(\underline{s}).$$

The cost components of (1) are respectively the inventory holding cost, the inventory shortage cost, the production cost, and the production change cost.

For convenience we assume that the demand distribution is stationary, time-

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independent and random, i.e., demands during $(t, t + \Delta t)$ depend only upon the length of the interval, and not upon its beginning point in time.

In the following, we contend that properties of production associated costs, which may be sufficiently important to warrant the modification of production policies, have heretofore been ignored or glossed over. Our view of the cost encountered in changing production rates is compared to the production-associated costs found in the existing inventory-production literature.

2. Descriptions of Production-Associated Costs

There seems to be no consensus regarding a "best" representation of production-associated costs. This is particularly true of the fourth term of (1)—the cost of changing the rate of production. The divergence of opinion on this point emerges clearly from two analyses which attempt to incorporate these costs explicitly into their respective loss functions. Under "total cost of manufacture," Simon [13] includes an element he calls "... sticky costs, proportional to the rate of manufacture when that is constant, but not capable of being reduced immediately when that declines." (p. 265) On the other hand, Hoffman and Jacobs [7] see production change costs as dependent upon the magnitude of increase in production: reductions in output are costless.

Arrow, Karlin and Scarf ([2], Chapter 2) present a static model similar to (1–3). However, they do not develop explicit forms for the costs in this model: most of their analytic work is with dynamic models[1] which take costs of production change to be zero. In describing production change costs, they state that

> ... an increase in production may require a rapid increase in some variable factors for a period in which other factors, such as equipment, cannot be increased, and therefore a temporary increase in total costs will occur which will be reduced as relatively immobile factors become adapted to the new production level. There are other elements in the cost of changing the rate of production; the hiring of inexperienced personnel, the need for learning new organizational methods appropriate to a higher production rate, the breaking-in of new equipment. There may even be costs to reducing the rate of production, such as those involved in separation of personnel (intensified by guaranteed wage plans) or in making special provisions for the care of inactive equipment.[2]

A group at the Carnegie Institute of Technology has developed an extensive literature on production and inventory control, as reported in a recent volume [8]. Their work is largely based upon a result presented in [12]: if all cost components of a loss function are quadratic, chance variables (e.g., demands) may be represented by single-valued unbiased estimates, instead of by probability distributions. Accordingly, in an effort to capitalize on this "certainty equivalence" property, the group has, in its applied work, used quadratic functions to approximate a wide variety of cost components. For example, in a study of the

[1] We use a one-period representation, in lieu of the more usual recursive functional equation, in anticipation of determining the stationary operating characteristics of the particular scheduling policy we are investigating.

[2] [2], p. 22.

operations of a paint factory,[3] they used the following to represent the cost of changing the rate of production:

$$k_1(w_t - w_{t-1} - k_2)^2 + k_3(p_t - cw_t)^2 + k_4p_t - k_5w_t + k_6p_tw_t$$

where w is the size of the work force, measured in the number of man-periods (e.g., man weeks), p is the production rate, and c and the k's are constants. For simplicity, these costs could be approximated by the term

$$k(p_t - p_{t-1})^2, \tag{4}$$

i.e., the cost of production adjustment is a smooth function of the size of the adjustment.

An interesting variation on this smoothing analysis is offered by Mills [11]. He shows that a *servo policy* of the form

$$p_t = \alpha p_{t-1} + (1 - \alpha)\underset{\sim}{s}_{t-1} \tag{5}$$

is optimal, provided all pertinent costs are functions of the first two moments of the series $\{\underset{\sim}{I}\}$ and $\{p\}$, and provided identical probability distributions generate demands independently in each discrete time period.[4] By increasing α, $\text{var}(p)$ is reduced and $\text{var}(\underset{\sim}{I})$ is increased. This result is weakened by the necessity to relate production change costs to the variance of the series $\{p\}$; in most systems, costs of production change are more aptly represented by successive differences, i.e., (4). But the policy (5) assures that the first autocovariance term $E(p_t - p_{t-1})^2$ will fluctuate with the variance; as α approaches 0, both $\text{var}(p)$ and $E(p_t - p_{t-1})^2$ approach their maximum values, respectively $\text{var}(\underset{\sim}{s})$ and $E(\underset{\sim}{s}_t - \underset{\sim}{s}_{t-1})^2$; and as α approaches 1, both $\text{var}(p)$ and $E(p_t - p_{t-1})^2$ approach 0.

Perhaps the most serious drawback of the "servo" approach suggested in the above work is that it is incapable of accommodating "lump sum" costs, which are associated with an action *per se*, rather than with any dimension of the action. Even in the absence of "lump-sum" costs of this type, however, it is not always safe to use quadratic approximations. As Beckmann [4] has shown, a piecewise linear production change cost function such as

$$\lambda_4(p_{t+1} - p_t) = \begin{cases} C_1(p_{t+1} - p_t) & p_{t+1} > p_t \\ C_2(p_t - p_{t+1}) & p_{t+1} < p_t \\ 0 & p_{t+1} = p_t \end{cases} \tag{6}$$

implies a far different form for the optimal policy than does a quadratic approximation thereto, the alternative suggested in [8], Ch. 2. Beckmann shows that because of the vanishing first derivative at the zero value of the argument of (6), the optimal policy specifies boundaries $a(\underset{\sim}{I}_t)$ and $b({}_t\underset{\sim}{I})$ such that if

$$a(\underset{\sim}{I}_t) \geq p_t \geq b(\underset{\sim}{I}_t)$$

[3] [8], Ch. 2.

[4] John F. Muth has indicated that this can be inferred from Simon's result [12]; if we take a loss function of the form $L = \lambda_1(p_t - p^*)^2 + \lambda_2(I_{+t1} - I^*)^2$, and set $p^* = E(\underset{\sim}{s}) = \mu$, i.e., set "target" production equal to mean demands, then the policy (5) follows from differentiating the loss function with respect to p_t.

p_{t+1} is optimally set equal to p_t, while if one of the inequalities is violated, the smallest change is made which suffices to make the inequalities hold when p_{t+1} is substituted for p_t.

In the next section, the costs incurred by altering production rates too frequently are discussed.

3. Short-Run Production Change Costs

The distinction drawn between the long run and the short run in the theory of the firm is based upon the premise that capital investment once made, is not profitably varied in response to temporary changes in critical variables such as the demand function confronting the firm; changes in these variables must give reasonable promise of permanence before alteration of capital can be considered. All productive inputs other than capital are classed as variable, though they may not be available in the desired quality and amount as soon as a short-run output decision has been made. In the traditional comparative statics the time between successive short-run equilibria is negligible, since the equilibrium positions and not the process of transition are of interest.

Inventory and production scheduling, however, cannot ignore differences in the ease with which "variable" inputs can be altered. With some inputs (electric power), a variation in the rate of use may be effected instantly, and no cost is attached to changing the rate: if anything, increased consumption is economical. Other inputs (some raw materials) can be varied at no out-of-pocket cost, but with a time lag; the attention Holt, Modigliani, Muth and Simon [8] devote to scheduling the work force is evidence of the difficulty of varying this input.

It would seem that trouble caused by differing degrees of stickiness in the "variable" inputs may be overcome by lengthening the scheduling interval. For example, if the size of the labor force is the input most difficult to vary in the short run (which seems likely), and ten days are required to adjust it to any but very small output changes, then we cannot schedule major production changes requiring labor force changes every week. In practice, large output changes would be permitted monthly, or perhaps quarterly,[5] with smaller adjustments, such as can be effected with no change in the labor force, called for weekly or even daily.[6]

[5] Clearly, arbitrarily to select a month or a quarter as the scheduling interval may be unwise: some costs (e.g., those of hiring and firing) vary inversely with the length of the scheduling interval, while others (e.g., inventory-associated costs) will vary directly: hence there will be at least one interval of optimal length, which should be identifiable by ordinary analytic means. We will refer to this as the *optimal short-run rescheduling interval*, implying that all inputs usually regarded as variable in the short run may be varied economically at intervals of this length.

[6] A standard postulate is: efficient operation over a wide range of outputs in the short run, obtained by use of flexible capital equipment, can be traded for extremely low unit cost of operation, which is obtained by use of highly specialized capital equipment. (This idea, which is evidently of long standing, is spelled out in [3], Appendix to Chapter 5.) It would seem that the same point holds more strongly when the labor force, as well as the capital structure, is not variable: a substantial degree of output flexibility in the shorter

Since the size of the labor force cannot be varied freely, we will regard employment scheduling as a short-run analog to the long-run problem of capital planning. Following this analogy, we define a "shorter run" and contrast it with the short run: in the shorter run both the firm's capital and the size of its labor force are fixed, while in the short run the former remains fixed while the latter is variable. To increase output in the shorter run, one or more of the following alternatives may be available: overtime operation; temporary reduction of maintenance, cleanup, or other parts of the normal operating routine; or increased operating rates with no change in the direct labor hour input. (See footnote 6 concerning this last possibility.) Shorter-run reductions in output lead to costs of undertime, the temporary diversion of labor into activities other than production.

The size of the work force is viewed as inflexible because of costs of hiring (paperwork and costs of interviewing), costs of training (time spent by experienced personnel in instruction, and wages of the new workers during the nonproductive initial period); or in the case of output decreases, costs of firing (terminal pay, reassignment of the remaining workers, further paperwork, and possible ill will of the labor force).[7] These costs may be largely independent of the size of the desired change in the labor force: they will depend upon the available labor pool, the skill of available workers compared to the requirements of new jobs, and the acumen of placement officers.

Another input usually assumed perfectly flexible in making production rate changes in managerial skill. If an extra shift or stage of the production process is added, a new foreman may be necessary, but except in such cases, expenditures on management overhead are not usually regarded as variable when planning changes in the rate of production. If the line manager's function is trouble-shooting, then production rate changes add to his difficulties, but no charge need be made for the time he spends coping with them. The increased routinization of operation, which attends stable production rates, may enable an increase in the span of managerial control. The firm then must choose between increased participation by line management in cost reduction activity, or decreased management overhead; in either case, the firm stands to gain.

4. A Random Walk Policy

It is seen in [8] that servo policies can be adapted for making production adjustments which involve labor force changes. However, it is necessary to make sure the time interval between reschedulings is of sufficient length. In effect, the length of the time period index t enters the model as another parameter for which an optimal value must be selected. If this course is adopted, and the optimal value of t (the optimal rescheduling interval) is long, difficulty may be encountered in establishing a criterion of adequate control against stockouts.

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run (i.e., a wide range of output rates for which the unit cost is near the minimum level attainable with the given capital and labor force) is obtained at the expense of a higher minimum unit cost of output.

[7] These costs are discussed at greater length in [8].

Periodic control is best analyzed in a discrete-time model, and such models usually offer period-ending inventory as a measure of performance against stockouts; this may be satisfactory for lot-size ordering control, but in a production-inventory complex, there is a possibility of long stockouts during an interval, which go undetected because cumulative production overtakes cumulative demand before the period is over.[8]

It is possible that shorter-run output changes can take care of intra-period demand variations and reduce the danger of intra-period stockouts, which are undetected (in the model). Such a pattern of major schedule changes at long intervals and more frequent minor adjustments at shorter intervals is believed to be widely followed by manufacturers.[9] However, the goal is not to make major schedule changes at regular intervals; rather it is to avoid rescheduling too frequently. A class of policies affords this objective by letting inventory absorb all fluctuations in demand until it becomes too high (too low) and then adjusting production rates downward (upward) until inventory moves back within the desired range. For example, the following rules (which specify a continuous review of stock levels) embody this idea: identify three inventory levels, $a > c > b$, and three production rates $h > n > l$,[10] with the operating instructions:

$$(7) \qquad p_t = \begin{cases} h \text{ if } I_t \text{ passes } b \text{ from above} \\ n \text{ if } I_t \text{ passes } c \\ l \text{ if } I_t \text{ passes } a \text{ from below} \\ p_{t-\tau}(\tau \to 0) \text{ if none of these passages occur during } (t - \tau, t). \end{cases}$$

This policy, which we call the (a, b, c) policy, applies the same rationale to production problems as underlies the s,S [1] or two-bin [15] policies for dealing with ordering problems. These earlier policies allow for fixed costs to a retailer each time an order is placed; these costs of ordering are controlled by regulation of the frequency of ordering; at the same time, both policies specify that orders must be placed when inventory passes a certain critically low level. Our contention is that costs of changing production rates in the short run require the frequency of these changes to be controlled; uncorrected inventory variation in the range (a, b) provides the desired respite from major production rate changes. While inventory remains within this range, production is held constant at some efficient rate.

[8] By taking account of the danger of intra-period stockouts, the analytic routine for optimizing the value of t can be forced to produce a scheduling interval which does not lead to disproportionately high costs in this area. However, this could lead to substantially higher overall cost, incurred just for the sake of using a special kind of scheduling policy, i.e., a servo policy.

[9] The writer recently spent two years at the Procter and Gamble Company, working on production and inventory scheduling. Such a system of short and shorter run adjustments was used by Procter and Gamble until a random walk scheduling policy was installed.

[10] Mnemonically, a, b, c = above, below, center; l, n, h = low, normal, high.

Another feature of the policy (7) is that it calls for only three short-run production rates: limitation to very few output rates may result in more rapid and efficient transitions from one rate to another, and thus in lower total cost through time. After a transition from, e.g., n to h has been made several times, the cost of management control may be reduced; repetition may lead to discovery of improved methods and routines, and to lower costs of change. Similarly, with few allowable rates, the permanent members of the labor force may become versed in the process of transition, mastering efficient patterns of adjustment, and reducing the disruption during the change. We provide no quantitative analysis of the virtues or flaws of few rates in the following; however this feature may be significant in certain specific situations.

The policy (7) is one of a class which we will call "random walk" policies. The rule that bounded inventory fluctuations be permitted without signalling a production change causes inventory to perform the generalized one dimensional random walk, which has been analyzed by Wald and others.

Further simplifying assumptions enable us to treat the inventory random walk generated by the (a, b, c) policy as a diffusion process; these assumptions do not seem overly restrictive. This greatly simplifies the problem of finding the stationary inventory distribution which is addressed next. In the following analysis, we assume no shorter-run production changes are made.

5. The Steady-State Inventory Distribution

The diffusion process provides an extremely simple and tractable model of inventory behavior; these attributes stem from the fact that it is a limiting case of the simplest random walk. This process, called the gambler's ruin, has drawbacks as a representation of inventory behavior; however, by passing to limits in a manner we will describe, objections can be successfully overcome. Since a lucid elementary exposition of the gambler's ruin and diffusion random walks is provided by Feller ([6], ch. XIV), we will not deal with them descriptively. Rather, we will summarize the results on diffusion processes which will be found useful, and the assumptions which must necessarily be satisfied by the inventory process if the diffusion representation is to be admissible.

1. If the inventory level is zero at time zero, and if demand in a time period of length t is a normally distributed random variable with mean μt and variance t, and if total production is kt during $(0, t)$, then inventory on hand at time t will be a normally distributed random variable, with mean $(k - \mu)t = \gamma t$ and variance t.

2. If the stochastic inventory process is homogeneous in time, i.e., in successive short intervals Δt, the probability of selling a small amount ΔI out of inventory is $(1 - \gamma)/2 = q$ while the probability of no sale is $(1 + \gamma)/2 = p$; then during these intervals Δt, the behavior of inventory is governed by a binomial distribution.[11]

[11] Feller shows that ΔI and Δt must be allowed to pass to zero in such a way that $(\Delta I)^2/\Delta t$ approaches a finite limit: this limit (the diffusion coefficient) we use to define the unit of our I-axis.

Given these two properties, that demands are Gaussian, and sufficiently regular in time to permit the inventory process to be treated as binomial in very these intervals Δt, Feller shows that the differential equation

$$\partial f(I, t)/\partial t = -2\gamma \partial f(I, t)/\partial I + \partial^2 f(I, t)/\partial I^2 \tag{8}$$

describes the behavior of inventory through time. The stationarity condition is $\partial f/\partial t = 0$; hence, (8) becomes

$$f''(I) - 2\gamma f'(I) = 0 \tag{9}$$

a second order differential equation.

This equation has the parameter γ, which is the *drift* of the diffusion process; $\gamma = k - \mu$. The three values assigned to k in the (a, b, c) policy, n, h, and l will each lead to a different γ, and hence the stationary inventory distribution is described by three equations of the form (9).

In Section 7 we will justify a simplifying assumption that is useful now: two of our decision parameters, n and c, take the values $n = \mu$ and $c = (a + b)/2 = b + r = a - r$. The first of these conditions implies that (9) becomes $f''(I) = 0$ for $p_t = n$, since then $\gamma = 0$. This equation has solutions of the form

$$f_n(I) = \alpha_1 + \alpha_6 I; \tag{10}$$

the other two solutions are of the form

$$f_h(I) = \alpha_5 + \alpha_2 e^{\gamma'(I-c)} \tag{11}$$

where $\gamma' = h - \mu$, and

$$f_l(I) = \alpha_4 + \alpha_3 e^{\gamma''(I-c)} \tag{12}$$

where $\gamma'' = l - \mu$; $\alpha_1 - \alpha_6$ are six arbitrary constants in (10–12).

The three solutions $f_n(I)$, $f_h(I)$ and $f_l(I)$ are the three steady-state inventory densities which correspond to the three production states n, l and h.

To obtain values for the six α constants we use the following boundary conditions. First, our assumptions $n = \mu$ and $c = (a + b)/2$ imply that passage to a is as likely as passage to b when production is in the n-state. This implies $f_n(a) = f_n(b)$; hence $\alpha_6 = 0$.

Second, the integral of either tail of $f(I)$ must be finite, hence α_4 and $\alpha_5 = 0$. This leaves us with the arbitrary constants α_1, α_2 and α_3 to be evaluated. To obtain these values, we use the conditions that $f(I)$ is continuous at a and b; and the density condition,

$$\int_{-\infty}^{\infty} f(I)\, dI = 1.$$

The necessity for the continuity conditions is best visualized by means of an alternative mechanism for obtaining the stationary distribution $F(I)$: this is to take the limiting value of equation (8) as t grows large without bound. If the time-independent distribution thus obtained is to be stationary, it is necessary that for all states I, the probability of moving to some adjacent higher state

$I + \Delta I$ from I be equal to the probability of returning from $I + \Delta I$ to I, and similar conditions must obtain regarding movement between I and the adjacent lower state $I - \Delta I$. Otherwise I is a transient state, and its stationary probability of occupancy is 0. In particular, for the boundary state a we obtain the two equations

$$p''f_l(a) = q''f_l(a + \Delta I) \tag{13-1}$$

$$q''f_l(a) = p''f_l(a - \Delta I) + \tfrac{1}{2}f_n(a - \Delta I) \tag{13-2}$$

where p'' and q'' are respectively $\frac{1}{2} + \gamma''\Delta I$ and $\frac{1}{2} - \gamma''\Delta I$. A similar pair of equations can be obtained at the barrier b.

Because of the continuity of the physical inventory process at a and b, which permits passage between adjacent states only, we have from the equations (13) the relation

$$f(a) = f_l(a) + f_n(a)$$

as ΔI approaches 0. Thus, the stationary occupancy probability for the boundary states a and b is obtained as an unweighted sum of the occupancy probabilities associated with the two possible production states. These continuity conditions on $f(I)$ at a and b thus govern the relative sizes of α_1, α_2 and α_3 : their absolute sizes are governed by the condition that $f(I)$ is a density.

We have

$$\begin{aligned} 1 = {} & \alpha_2 \int_{-\infty}^{c-r} e^{\gamma'(I-c)}\, dI + \left[\int_{c-r}^{c} (\alpha_2 e^{\gamma'(I-c)} + \alpha_1)\, dI\right] \\ & + \left[\int_{c}^{c+r} (\alpha_3 e^{\gamma''(I-c)} + \alpha_1)\, dI\right] + \alpha_3 \int_{c+r}^{\infty} e^{\gamma''(I-c)}\, dI \end{aligned} \tag{14}$$

with the conditions (from equation (13-2) and the analogous equation for b):

$$\alpha_1 = -2\gamma''\alpha_3 e^{\gamma'' r}, \qquad \alpha_1 = 2\,\gamma'\alpha_2 e^{\gamma'(-r)} \tag{15}$$

where r takes the sign of the direction of deviations from c. From these conditions we see that

$$\alpha_2 = -(\gamma''/\gamma')e^{(\gamma'+\gamma'')}\alpha_3 . \tag{16}$$

Substituting (15) and (16) into (14) and solving, we obtain

$$\begin{aligned} \alpha_3 = \Bigg[& -(\gamma''/\gamma')\, e^{(\gamma'+\gamma'')} \int_{-\infty}^{c} e^{\gamma'(I-c)}\, dI \\ & + \int_{c}^{\infty} e^{\gamma''(I-c)}\, dI - \int_{b}^{a} 2\gamma''\, e^{\gamma'' r}\, dI \Bigg]^{-1} \end{aligned} \tag{17}$$

or

$$\alpha_3 = \{-[\gamma''/(\gamma')^2]e^{(\gamma'+\gamma'')} - 4\,\gamma''\, re^{\gamma'' r} - 1/\gamma''\}^{-1}$$

Thus, the stationary inventory distribution is

$$F(I) = \frac{(\gamma''/\gamma')e^{(\gamma'+\gamma'')} \int_{-\infty}^{c} e^{\gamma'(I-c)}\, dI - \int_{c}^{\infty} e^{\gamma''(I-c)}\, dI + 2\gamma'' e^{\gamma'' r} \int_{c-r}^{c+r} dI}{\{[\gamma''/(\gamma')^2]e^{(\gamma''+\gamma')} + 4\gamma'' re^{\gamma'' r} + 1/\gamma''\}} \tag{18}$$

In (18) our original policy parameters have been modified as follows:

$$h = \mu + \gamma'$$
$$l = \mu + \gamma''$$
$$n = \mu$$
$$a = c + r$$
$$b = c - r.$$

6. An Example

We turn to the problem of optimizing the six decision parameters of the (a, b, c) policy. The loss function with which we will deal has the following form:

λ_1 : The inventory holding cost function is given by

$$E[\lambda_1(I_t \mid I_t > 0)] = \eta \int_0^\infty I f(I)\, dI \tag{19}$$

where $f(I)$ is the steady-state inventory density, and η is a constant.

λ_2 : The runout cost is given by

$$E[\lambda_2(I_t \mid I_t \leqq 0)] = \rho \int_{-\infty}^0 f(I)\, dI \tag{20}$$

where ρ is a constant.

λ_3 : The production cost is given by

$$E[\lambda_3(p_t)] = \phi[(k - p^*)^2 \text{ Prob } (p_j = k),\ k = h, n, l]$$

where p^* is the minimum cost production rate, and ϕ is a constant.

λ_4 : The production change cost is given by

$$E[\lambda_4(z)] = Kz$$

where K is a constant, and z is the frequency of production rate changes.

The objective is to minimize expected cost at some future time t, where t is sufficiently far in the future to assure that all transient effects (effects of present inventory levels and production rates upon future cost) shall have died out. In accord with this objective, the inventory-associated costs are adequately specified, but it will be necessary to reformulate λ_3 and λ_4. Two properties of the inventory random walks are useful for this purpose.

The *passage times* of the inventory random walks, denoted $\theta(i)$, are random variables representing the times required for inventory to move from its origin to the barrier (or one of the barriers) when production is in state i. For example, $\theta(h)$ is the time required for inventory to move from b to c, given that the production rate is h, $\theta(n)$ is the time required by inventory to move from c to a or b, given production is n, etc. The *passage probability* of the inventory random walk, denoted $\psi_n(x, y)$ is the probability that inventory reaches x before it reaches y, given that the production rate is n. Of particular interest is the passage probability $\psi_n(a, b)$; this gives the probability that the production state n will be followed by the production state l. From a lemma of Wald [14], we know that if the variance of the demand distribution $\neq 0$, $\psi_n(a, b) + \psi_n(b, a) = 1$, and $E[\theta(n)]$ is finite.

Wald has provided analytic representations of these passage times and passage probabilities for the case where time is a discrete parameter. Subsequent research has shown that the several significant passage times and passage probabilities of the (a, b, c) policy random walks take the following values, under the assumption that the inventory process is a diffusion process, $c = (a + b)/2$, and $r = (a - b)/2$:

$$E[\theta(h)] = r/(h - \mu) \tag{21}$$

$$E[\theta(l)] = r/(\mu - l) \tag{22}$$

and assuming $n = \mu = p^*$,

$$E[\theta(n)] = r^2/2 \tag{23}$$

where μ and 1 are respectively the rates at which the mean and variance of demands increases as the time interval increases; also

$$\psi_n(a, b) = (a - c)/(a - b) = \tfrac{1}{2}. \tag{24}$$

The four properties given in (21-24) are useful in explicitly relating production associated costs to the decision parameters of the (a, b, c) policy. Production cost is given by

$$E[\lambda_3(p_t)] = \frac{\phi(h - p^*)^2\psi_n(b, a)E[\theta(h)] + \phi(p^* - l)^2\psi_n(a, b)E[\theta(l)]}{E[\theta(n)] + \psi_n(b, a)E[\theta(h)] + \psi_n(a, b)E[\theta(l)]} \tag{25}$$

This is a function of the form

$$\phi(h - p^*)^2 \text{ Prob } (p_t = h) + \phi(p^* - l)^2 \text{ Prob } (p_t = l).$$

The denominator of the expression (25) we call the *mean renewal time* of the (a, b, c) process; this is the expected time required to pass from c to a (with probability $\psi_n(a, b)$) or to b (with probability $\psi_n(b, a)$) *plus* the expected time required to return from a or b to c. An assumption which we continue to use, and justify in Section 7, is that $n = \mu = p^*$.

Production change cost is reformulated

$$E[\lambda_4(z)] = \frac{2K}{E[\theta(n)] + \psi_n(b, a)E[\theta(h)] + \psi_n(a, b)E[\theta(l)]}$$

The cost K is incurred upon making a production rate change; two changes are made during each renewal cycle. Hence, total production-associated costs may be written

$$\begin{aligned} &E[\lambda_3(p_t) + \lambda_4(z)] \\ &= \{2K + \phi(h - p^*)^2\psi_n(b, a)E[\theta(h)] + \phi(p^* - l)^2\psi_n(a, b)E[\theta(l)]\}/E(R) \end{aligned} \tag{26}$$

where $E(R)$ is the mean renewal time from (25).

The loss function composed of (19-20), (26) is expressed completely in terms of three properties of the (a, b, c) inventory process: the expected passage times, the passage probabilities, and the stationary distribution of occupancy probabilities for the various inventory levels. In the next section, we explore the conditions under which the useful simplifications $c = (a + b)/2$ and $n = \mu = p^*$ can be employed.

7. Optimal Values for Policy Parameters

The arguments which establish the values $c = (a + b)/2$ and $n = p^* = \mu$ are presented in this section. First, $p^* = \mu$ is an assumption; one which is susceptible to rationalization, however. It is equivalent to the assumptions we originally made, that demands are stationary and random, and that the objective is to minimize total production- and inventory-associated costs. A firm confronted by a stationary demand series will find it possible to adjust its capital structure in the long run, to bring costs of production into line with the scale of demands. Alternatively, by suitable pricing policy, a market can be found in the short run such that the quantity sold can be efficiently produced with present capital. Thus, either the production cost schedule or the demand schedule can be adjusted so that $p^* = \mu$; our contention is that it will be profitable to make such an adjustment.[12]

Next, we establish the conditions necessary to the proposition: for given a, b, l and h, the values of c and n which maximize $E(R)$ are respectively $(a + b)/2$ and μ.

In the diffusion model, we assume the inventory variate takes steps of ΔI at intervals of Δt: let the probability of a step in the positive direction be P, and in the negative direction Q. ΔI and Δt are permitted to pass to zero in such a way that $(\Delta I)^2/\Delta t$ is always bounded. (cf. Feller [6], pp. 324–5). We define $(P - Q) = \gamma$, the drift of the process. If the production rate n imposes the drift γ, the rate of change of the expected value of inventory is $\gamma\Delta I/\Delta t$ while $p_t = n$. Suppose γ is positive: then the maximal expected renewal time is obtained by setting c as close to b as possible; the expected renewal time then is

$$\frac{2r}{\gamma\Delta I/\Delta t} + \frac{2r}{\gamma^*\Delta I/\Delta t} \tag{27}$$

where γ^* is the negative drift imposed when inventory passes the barrier a. (A similar result is obtained for $\gamma < 0$ by setting c as close to a as possible.) In

[12] This construction seemingly directly contradicts the equilibrium result for non-perfectly competitive firms in microeconomic theory: there it is seen that the optimal output rate will be to the left of the average cost function's minimum point. This contradiction is only apparent: our function $\lambda_2(p_t - p^*)^2$ is a *total* variable cost function, which takes into account production costs only. If this function is increasing less than linearly immediately to the right of its minimum point p^*, it will, other costs not considered, be necessary that the minimum of the ATC curve lie to the right of the minimum of the λ_2 function.

order for the requisite boundedness of $(\Delta I)^2/\Delta t$ to obtain, it is necessary that γ and γ^* be of the order of ΔI.

Our proposition asserts that the expression (27) is always less than

$$\frac{r^2}{(\Delta I)^2/\Delta t} + \frac{\frac{1}{2}r}{\gamma \Delta I/\Delta t} + \frac{\frac{1}{2}r}{\gamma^* \Delta I/\Delta t}, \tag{28}$$

the expected renewal time of the (a, b, c) policy when $c = (a + b)/2$, $n = \mu$, $h = \mu + \gamma$ and $l = \mu - \gamma^*$.

Now, (27) and (28) can by our proposition be rewritten

$$r^2/(\Delta I)^2 \geqq 3r/(\Delta I)^2$$

which must always hold for $r \gg 3$, even allowing for the approximation $\gamma \sim \gamma^* \sim \Delta I$.

In order to completely establish the optimality of these values for n and c, we must be assured that their effects upon inventory cost are either positive or neutral. It is true that if all other decision parameters are fixed, the value of $\psi_n(a, b)$ can be increased by setting $c > (a + b)/2$ or $n > \mu$; and thus holding cost can be increased and runout cost decreased. But notice that if h, l, n, and r are held fixed, the same effect can be achieved by increasing the value of b (and thus c and a).

The marginal cost of this latter alternative is the increase in holding cost η; it has no effect upon production-associated costs. If runout costs are reduced by manipulation of n and/or c, to establish a comparison it is necessary to choose a value of n and c such that

$$\psi_n(b, a)\theta(n) = \psi_h(b, c + \Delta)\psi_\mu(b + \Delta, a + \Delta)\theta(\mu)$$

where the right side represents the frequency of reaching the level b after the entire inventory range (a, b) has been moved Δ units farther from zero. Now, if we choose an $n > \mu$ or $c > b + r$, the effect upon the stationary inventory distribution is to skew it toward a; a term $e^{\gamma'''(a-I)}$ (where γ''' is the drift coefficient introduced into the n- state) will appear in $f_n(I)$ if $n \neq \mu$; or the constant $\alpha_6 \neq 0$ in equation (10) if $n = \mu$ but $c > b + r$. Thus it is entirely possible that holding cost will be increased more than they would by the straightforward increase of b, leaving $c = b + r$ and $a = b + 2r$. We assume production-associated costs are sufficient to offset the holding cost advantage accruing from manipulation of c and n rather than b, if any; similarly, we assume it is more economical to reduce b and a if holding costs are excessive vis-a-vis runout costs.

With these two policy parameters thus established, our loss function (19–20), (26) becomes

$$\begin{aligned} E[L(t)] = \Big[&\eta(\gamma''/\gamma')e^{\gamma''+\gamma'} \int_0^c Ie^{\gamma'(I-c)}\,dI - \eta \int_c^\infty Ie^{\gamma''(I-c)}\,dI \\ &+ 2\eta\gamma'' e^{\gamma'' r} r \int_{c-r}^{c+r} dI + \rho(\gamma''/\gamma')e^{(\gamma''+\gamma')} \int_{-\infty}^{0} e^{\gamma'(I-c)}\,dI \Big] \\ &\div \{[\gamma''/(\gamma')^2]e^{(\gamma''+\gamma')} + 4\gamma'' re^{\gamma'' r} + 1/\gamma''\} \\ &+ (2k - \phi r\gamma''/2 + \phi r\gamma'/2)/(r^2 - r/2\gamma'' + r/2\gamma'). \end{aligned} \tag{29}$$

Probably the simplest way to obtain minimizing values of γ'', γ', r and c for a given set of costs η, ρ, K, ϕ, is by a gradient routine on a high-speed computer. However, for the sake of a numerical illustration, the following specific numerical values were used, and (29) was solved on a desk caclulator by successive approximations: $\eta = .1, \phi = 1, K = 10, \rho = 100$. The solution routine considered only multiples of $\frac{1}{4}$ as values for γ' and γ'', and multiples of 2 as values for c and r. The resulting solution was: $\gamma' = \frac{1}{2}, \gamma'' = -\frac{1}{2}, c = 16, r = 14$. The loss function was found to be quite flat in this neighborhood.

8. Alternative Solution Procedures

At least two computationally feasible computation routines have been developed during the past five years, which would enable direct minimization of the loss function (1); the specification that an (a, b, c) policy be used is unnecessary. One of these approaches is the discrete dynamic programming technique exposited by Howard [9], the other is the method of sequential stochastic linear programming, due to Manne [10] and d'Epenoux [5]. The computational feasibility of both routines depends upon the stationarity, randomness and independence of demands. These discrete-argument algorithms may lead to a policy[13] which yields a lower value of the loss function than is obtainable from the (a, b, c) policy. However, unless the computational routine is constrained (e.g., by adding a term to the loss function which has cost proportional to the number of production rates used), the benefits of a limited number of production rates will not be realized.

9. Summary

Random walk inventory policies, characterized by uncorrected inventory fluctuation between prespecified barriers, arise naturally out of a variety of cost characteristics and computational methods. If costs of output variation are significant, several alternatives are open to the firm: the rescheduling interval can be lengthened (which may be difficult because of increased probability of intra-period stockout); one of the recently developed discrete dynamic programming algorithms may be employed (a particularly useful approach when the loss function does not contain too many special terms); or a particular policy of simple form, e.g., the (a, b, c) policy analyzed in this paper, may be used; optimal policy parameters can then be located in any of a number of standard ways.

A "lump sum" cost of production rate changes is not a necessary condition on the desirability of random walk policies, as is seen in Beckmann's analysis of a case in which the production-associated cost function is continuous [4]. He shows that a discontinuity in the first derivative of this function suffices as an indicator of random walk policies.

[13] A *policy* is any rule which maps the current production rate and inventory level into the new production rate; it need not be of simple form, like (5) or (7).

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References

1. Arrow, Kenneth J., Theodore E. Harris, and Jacob Marschak, "Optimal Inventory Policy," *Econometrica*, Vol. 19, (1951).
2. ——, Samuel Karlin, and Herbert Scarf, *Studies in the Mathematical Theory of Inventory and Production*, Stanford, 1958.
3. Baumol, William J., *Economic Dynamics*, New York, Macmillan, 1951.
4. Beckmann, Martin, "Production Smoothing and Inventory Control," *Operations Research*, Vol. 9, (1961).
5. d'Epenoux, F., "Sur un Problème de Production et de Stockage dans l'Aléatoire," *Révue Française de Récherche Opérationelle*, XIV, 1960.
6. Feller, William, *An Introduction to Probability Theory and Its Applications*, Vol. 1, New York, John Wiley & Sons, Inc., 1957, 2nd ed.
7. Hoffman, Alan and Walter Jacobs, "Smooth Patterns of Production," *Management Science*, Vol. 1, No. 1 (October 1954), pp. 86–91.
8. Holt, Charles C., Franco Modigliani, John F. Muth, and Herbert Simon, *Planning Production, Inventories, and Work Force*, Englewood Cliffs, N. J., Prentice-Hall, Inc., 1960.
9. Howard, Ronald, *Dynamic Programming and Markov Processes*, New York, John Wiley & Sons, Inc., 1960.
10. Manne, Alan S., "Linear Programming and Sequential Decision Models," *Management Science*, Vol. 6, No. 3 (April 1960), pp. 259–267.
11. Mills, Harlan D., "Smoothing in Inventory Processes," *Navy Supply System Research, Study I*, Mathematica, Princeton, N. J., 1960.
12. Simon, Herbert, "Dynamic Programming under Uncertainty with a Quadratic Criterion Function," *Econometrica*, Vol. 24, (1956).
13. ——, "On the Application of Servomechanism Theory in the Study of Production Control," *Econometrica*, Vol. 20, (1951).
14. Wald, Abraham, "On Cumulative Sums of Random Variables," *Annals of Mathematical Statistics*, Vol. 15, (1944).
15. Whitin, Thomson M., *Theory of Inventory Management*, Princeton, 1957, 2nd Ed.