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Volume Comparison with Integral Bounds in Lorentz Manifolds

Nikhyl Bryon Aragam
University of Tennessee - Knoxville

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BRYON ARAGAM

ABSTRACT. Ten years ago, Ehrlich and Sanchez produced a pointwise statement of the classical Bishop volume comparison theorem for so-called SCLV subsets of the causal future in a Lorentz manifold, while Petersen and Wei developed and proved an integral version for Riemannian manifolds. We apply Peterson and Wei's method to the SCLV sets, and verify that two essential differential equations from the Riemannian proof extend to the Lorentz setting. As a result, we obtain a volume comparison theorem for Lorentz manifolds with integral, rather than pointwise, bounds. We also briefly discuss the history of the problem, starting with Bishop's original theorem from 1963.

1. INTRODUCTION

Over 40 years ago, Bishop and Gromov proved a series of fundamental results concerning the relationship between the volume of regions in a space and the elementary structure of the space itself. Quite simply, they showed that the more curved a space is, the less volume small regions of space consume. For instance, one may compare the volume contained inside a circle on a globe compared to the volume contained in a circle of the same radius on a flat sheet of paper. A moment's thought should be enough to convince one that the volume on the globe will be smaller. Bishop's result maintains that this remains the case in any dimension on any Riemannian manifold.

The past decade has seen a flurry of work to generalize these results. The first positive step forward came in 1997, when Petersen and Wei showed that we may relax the hypotheses of Bishop's theorem considerably. Instead of assuming a lower bound on the curvature at every point, Petersen and Wei averaged the curvature using an integral over small balls to put a bound on the volume ratio [5]. The estimates used to establish this bound have been successfully recycled to prove several other comparison and finiteness theorems [4].

Another natural generalization was to consider the case when the manifold is Lorentzian. This proved to be a delicate problem, requiring several new concepts and ideas to properly formulate the correct generalization. In 1998, Ehrlich and Sanchez managed to complete this program, proving a theorem similar to Bishop's classical theorem for Lorentz manifolds [2].

Given Ehrlich and Sanchez's result, the obvious question arose: can we combine the approaches of Petersen-Wei and Ehrlich-Sanchez to produce a general volume comparison theorem for Lorentz manifolds? This is the question we studied under the guidance of Dr. Justin Corvino with the support of the 2008 Lafayette College REU. It turns out to have an affirmative answer. The proof follows the lines of Petersen and Wei's original proof rather closely, and offers a new way to study geometric properties of Lorentz manifolds without being overly bothered by the complex structure which arises from having a nondegenerate metric.

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2. The Riemannian Case

Before stating our theorem, it will be a worthwhile digression to discuss the three aforementioned results which motivated it.

First, let us establish some notation. (M, g) will always denote an n-dimensional semi-Riemannian manifold M with metric tensor g. The manifold may be Riemannian or Lorentz depending on the context. Given a semi-Riemannian manifold, we may define the Ricci tensor Ric on $T_xM \times T_xM$, where T_xM is the tangent space at x. Finally, let $v(n, \lambda, r)$ denote the volume of a ball of radius r in the n-dimensional simply connected manifold of constant curvature λ. Since these manifolds are homogeneous, the centre of the ball is irrelevant.

With these preliminaries, we can now state Bishop's volume comparison theorem [1]:

Theorem 2.1 (Bishop). *Let M be a Riemannian manifold and suppose* $\mathrm{Ric}(v, v) \geq (n-1)\lambda$ *for all $v \in TM$. Then*

$$\operatorname{vol} B(x, r) \leq v(n, \lambda, r)$$

for any $x \in M$ and $r > 0$.

In simple terms, more curvature implies less volume. Note that the comparison is not between two arbitrary Riemannian manifolds, however. Instead, given a Riemannian manifold, we compare it to a standard constant curvature space. Note also that the hypotheses require a pointwise inequality for all tangent vectors in the tangent bundle–that is, at every point. This is a fairly strong hypothesis, which we are interested in relaxing.

To generalize Bishop's theorem, we turn to the approach made by Petersen and Wei [5]. For each point $x \in M$, define $g(x)$ to be the smallest eigenvalue of the Ricci tensor of M at x (viewed as an operator on T_xM). Now define

$$k(\lambda, p) = \int_M \max\{-g + (n-1)\lambda, 0\}^p \, dv.$$

This function averages how much the Ricci curvature dips below λ. Note that in the case $\mathrm{Ric}(v, v) \geq (n-1)\lambda$ as in Bishop's theorem, we have $k \equiv 0$. The function k is thus is some sense a measure of the deviation from the rigid hypothesis of Bishop's theorem, and is the correct tool for generalizing it. More specifically we have:

Theorem 2.2 (Petersen-Wei). *Suppose M is a Riemannian manifold. Let $x \in M$, $\lambda \leq 0$, $p > \frac{n}{2}$ be given. For any $R > 0$, there exists a constant $C(n,p,\lambda,R)$ which is nondecreasing in R such that for $r < R$, we have*

$$\left(\frac{\operatorname{vol} B(x,R)}{v(n,\lambda,R)}\right)^{1/2p} - \left(\frac{\operatorname{vol} B(x,r)}{v(n,\lambda,r)}\right)^{1/2p} \leq C(n,p,\lambda,R)\cdot k(\lambda,p)^{1/2p}.$$

To better understand what this theorem says, it will be instructive to consider what happens when the hypotheses of Bishop's theorem are satisfied. In this case, we have already noted that $k \equiv 0$, so

$$\left(\frac{\operatorname{vol} B(x,R)}{v(n,\lambda,R)}\right)^{1/2p} \leq \left(\frac{\operatorname{vol} B(x,r)}{v(n,\lambda,r)}\right)^{1/2p}.$$

Taking $r \to 0$, it is easy to see that the right side tends to unity. Thus for any $R > 0$,

$$\operatorname{vol} B(x,R) \leq v(n,\lambda,R)$$

as in Bishop's theorem. Hence Theorem 2.2 is truly a generalization of the classical result.

The conclusion of Petersen and Wei's theorem appears much more complicated than Bishop's. What it says is aside from the constant C, the function k quantifies to what extent the volume comparison in Theorem 2.1 fails to hold. When $k \equiv 0$, the same comparison holds. As k gets larger, the comparison becomes weaker and weaker.

These two results summarize the bulk of the work done on Riemannian volume comparisons.

3. The Lorentzian Case

We now turn to the case when our manifold is Lorentzian. Recall that this means that the metric $\mathbf{g}$ on M is not positive definite, but instead nondegenerate with index one. This implies, among things, that nonzero tangent vectors can have zero length. In the Riemannian setting, the positive definite metric gives rise to a genuine distance function. This is not the case in the Lorentz setting. Thus, unlike in the previous two theorems where the volumes of metric balls were being compared, when we are in a Lorentz manifold we do not have such luxury. We must compare more general sets.

To resolve this issue, Ehrlich and Sanchez made the following definition [2]:

Definition 3.1. *A normal neighbourhood $U \subset M$ of a point x is called* standard for comparison of Lorentz volumes *if $U = \exp_x(\overline{U})$, where $\overline{U}$ is a subset of the causal future of the origin and $\operatorname{cl}\overline{U}$ is compact.*

It is common to call these sets *SCLV sets* for short. The compactness requirement is simply to ensure $\operatorname{vol} U < \infty$.

Once we are given an SCLV set U, we must find a suitable set in a constant curvature space to compare it with. Denote the Lorentz space form with constant curvature λ by M_λ. Fix a point $p \in M_\lambda$ and a linear isometry $\iota : T_xM \to T_pM_\lambda$. Thus we have the following diagram:

$$\begin{array}{ccc} T_xM & \xrightarrow{\iota} & T_pM_\lambda \\ \exp_x \big\downarrow & & \big\downarrow \exp_p \\ M & & M_\lambda \end{array}$$

Define the *transplantation* of U in M_λ, denoted by U_λ, by following the sequence of maps suggested by this diagram: $U_\lambda = \exp_p(\iota(\exp_x^{-1}(U))) = \exp_p(\iota(\overline{U}))$. We thus have four related sets in four different spaces:

$$\begin{aligned} U &\subset M \text{ is SCLV}, \\ \overline{U} &= \exp_x^{-1}(U) \subset T_xM, \\ U_\lambda &= \exp_p(\iota(\overline{U})) \subset M_\lambda, \\ \overline{U}_\lambda &= \exp_p^{-1}(U_\lambda) \subset T_pM_\lambda. \end{aligned}$$

We are interested in comparing $\operatorname{vol} U$ and $\operatorname{vol} U_\lambda$. This is precisely the result due to Ehrlich and Sanchez:

Theorem 3.1 (Ehrlich-Sanchez). *Let $(M, \mathbf{g})$ be a time-oriented Lorentz manifold and suppose $\operatorname{Ric}(v,v) \geq (n-1)\lambda \cdot \mathbf{g}(v,v)$ for all timelike and radial vectors tangent to an SCLV subset U. Then*

$$\operatorname{vol}(U) \leq \operatorname{vol}(U_\lambda).$$

This is essentially Bishop's theorem for Lorentz manifolds minus a few technical difficulties. Recall that a time-oriented Lorentz manifold M is one for which there is a continuous nonvanishing timelike future-directed vector field defined on all of M.

Thus far we have three comparison theorems:

- Riemannian with pointwise curvature bounds [Bishop]
- Riemannian with integral curvature bounds [Petersen-Wei]
- Lorentzian with pointwise curvature bounds [Ehrlich-Sanchez]

All that is missing is a Lorentzian version with integral bounds. This was accomplished in 2007 by the Lafayette College REU geometry group consisting of Dr. Justin Corvino, Bryon Aragam, Andrew Karl, and Austin Rochford. Inspired by the work of Petersen and Wei, we proved the following theorem:

Theorem 3.2 (Aragam-Corvino-Karl-Rochford). *Suppose M is a time-oriented Lorentz manifold. Let $x \in M$, $\lambda \leq 0$, $p > \frac{n}{2}$ be given. For any $R > 0$ there is a constant $C(n,p,\lambda,R)$ which is nondecreasing in R such that for $r < R$ we have*

$$\left(\frac{\operatorname{vol}\overline{U}_R}{v(n,\lambda,R)}\right)^{1/2p} - \left(\frac{\operatorname{vol}\overline{U}_r}{v(n,\lambda,r)}\right)^{1/2p} \leq C(n,p,\lambda,R)\cdot k(\lambda,p,R)^{1/2p}$$

for any comparable set U in T_xM.

It is worth stressing that the constant $C(n,p,\lambda,R)$ in our theorem does not depend on the comparable set or the point chosen. The remainder of this paper will focus on the proof of Theorem 3.2

4. Preliminaries

Let $(M, \mathbf{g})$ be a time-oriented Lorentz manifold. For $x \in M$, let $P_{x,r}$ be the hypersurface consisting of all points in M that can be connected to x by a future timelike geodesic of length r. For $m \in P_{x,r}$, let γ_m be such a unit-speed geodesic, parametrized by arc length, so for some $t > 0$, $\gamma_m(t) = m$. It follows from the generalized Gauss lemma [3], $\dot{\gamma}_m(t)$ is the outward pointing unit normal to $P_{x,r}$ at m.

The formulas documented in the following two lemmas figure heavily in the proof of the Riemannian theorem. They will prove just as important for us.

Lemma 4.1. *For all $x \in M$, $r > 0$, the shape operator S_r of $P_{x,r}$ obeys a Ricatti differential equation. That is,*

$$S_r' - S_r^2 + R_{\dot{\gamma}_m} = 0$$

where $R_X Y = R(X, Y, X)$.

Proof. For $v \in T_x M$,

$$\begin{aligned} S_t'(v) &= (\nabla_{\dot{\gamma}_m} S_t)(v) \\ &= -\nabla_{\dot{\gamma}_m} \nabla_v \dot{\gamma}_m - S_t(\nabla_{\dot{\gamma}_m} v) \\ &= -R_{\dot{\gamma}_m} v - \nabla_{[\dot{\gamma}_m, v]} \dot{\gamma}_m - S_t(\nabla_{\dot{\gamma}_m} v) \\ &= -R_{\dot{\gamma}_m} v + S_t(\nabla_{\dot{\gamma}_m} v - \nabla_v \dot{\gamma}_m) - S_t(\nabla_{\dot{\gamma}_m} v) \\ &= -R_{\dot{\gamma}_m} v + S_t^2(v). \end{aligned}$$

Since $R_{\dot{\gamma}_m} v = \nabla_{\dot{\gamma}_m} \nabla_v \dot{\gamma}_m - \nabla_v \nabla_{\dot{\gamma}_m} \dot{\gamma}_m - \nabla_{[\dot{\gamma}_m, v]} \dot{\gamma}_m = \nabla_{\dot{\gamma}_m} \nabla_v \dot{\gamma}_m - \nabla_{[\dot{\gamma}_m, v]} \dot{\gamma}_m$ because γ_m is a geodesic. □

Corollary. $h' + \frac{h^2}{n-1} \leq \mathrm{Ric}(\dot{\gamma}_m, \dot{\gamma}_m)$, *where h is the mean curvature of $P_{x,r}$.*

This corollary is obtained by tracing the previous theorem and applying the Cauchy-Schwarz inequality.

The next lemma is a simple formula involving the volume form on M. If dV is the volume form on M, take $dV = \omega dt \wedge d\theta_{n-1}$ where $d\theta_{n-1}$ is the volume form on $\mathbb{H}^{n-1}$. Similarly, take $dV_\lambda = \omega_\lambda dt \wedge d\theta_{n-1}^\lambda$ where dV_λ is the volume form on the n-dimensional simply-connected constant curvature space form of curvature λ. Thus ω and ω_λ can be viewed as real-valued functions of the parameter t.

Lemma 4.2. $\omega' = h\omega$, *where h is the mean curvature of the hypersurfaces $P_{x,r}$. Also, $\omega_\lambda' = h_\lambda \omega_\lambda$.*

Proof. The future timecone in $T_x M$ can be idenitifed with $(0, \infty) \times \mathbb{H}^{n-1}$ with the appropriate product metric. Choose a frame for the future timecone such that the pullback of E_1 corresponds to the interval in the above product and $\{E_i\}_{i=2,\dots,n}$ gets pulled back to a frame on $\mathbb{H}^{n-1}$, so we have that $[E_1, E_i] = 0$ for all of the frame vectors. Now, since $\{E_i\}$ is a frame, $\omega = dV(E_1, \dots, E_n)$, so

$$\begin{aligned} \omega' &= (\nabla_{E_1} dV)(E_1, \dots, E_n) \\ &= \nabla_{E_1}(dV(E_1, \dots, E_n)) - (dV(\nabla_{E_1} E_1, \dots, E_n) + \dots + dV(E_1, \dots, \nabla_{E_1} E_n)) \\ &= dV(SE_1, \dots, E_n) + \dots + dV(E_1, \dots, SE_n) - (dV([E_1, E_1], \dots, E_n) + \dots \\ &\qquad + dV(E_1, \dots, [E_1, E_n])) \\ &= \mathrm{tr} S \, dV(E_1, \dots, E_n) \\ &= h\omega. \end{aligned}$$

□

5. The Main Result

With these preliminaries out of the way, we are ready to begin discussing the proof of the main theorem. First we must define precisely what is meant by a 'comparable' set in Theorem 3.2.

Definition 5.1. *For $x \in M$, let $U_x \subset \mathbb{H}_x^{n-1} = \{v \in T_xM | \langle v, v\rangle = -1\}$ be open with compact closure, such that $exp_x|_U$ is a diffeomorphism onto its image. Such a set will be called* comparable. *For $r > 0$, $U_r = \{sv \mid v \in U, s \leq r\}$. Furthermore, take* $\overline{U}_r = \exp_x U_r$

The sets $\overline{U}_r$ are the sets whose volume will be compared. These sets are completely specified by $x \in M$, $r > 0$, and $U \subset \mathbb{H}_x^{n-1}$. Once U is fixed, we denote by $v(n, \lambda, r)$ the volume of the transplantation of U in the Lorentz space form of constant curvature λ (see Section 3). This will be our standard for comparisons.

Given a local coordinate chart, we define $\rho(t, \cdot) = \max\{(n-1)\lambda - \mathrm{Ric}(\partial_t, \partial_t), 0\}$ for $\lambda \leq 0$. Outside this chart we use the default that ρ is zero. Now define

$$k(p, \lambda, r) = \int_{\overline{U}_r} \rho^p \, dV = \int_U \int_0^r \rho(t, \cdot)^p \omega dt \wedge d\theta_{n-1}.$$

Evidently the function k measures how much curvature falls below the value $(n-1)\lambda$ inside in the set $\overline{U}_r$. We will also need another auxiliary function to measure differences in mean curvature between our comparison spaces: define $\psi = \psi(t, \cdot) = \max\{0, h(t, \cdot) - h_\lambda(t, \cdot)\}$ and declare ψ to be zero outside the coordinate chart.

We are now ready to prove two important lemmas which bridge the gap between the Ricatti equation, mean curvature, and our function k. To this end, consider a point $x \in M$ and a fixed comparable set $U \subset \mathbb{H}_x^{n-1}$. We will assume $\lambda \leq 0$ for the remainder of this paper.

Lemma 5.1. *For the volume ratio $\frac{\mathrm{vol}\,\overline{U}_r}{v(n,\lambda,r)}$ we have that*

$$\frac{d}{dr}\frac{\mathrm{vol}\,\overline{U}_r}{v(n, \lambda, r)} \leq C_1(n, \lambda, r)\left(\frac{\mathrm{vol}\,\overline{U}_r}{v(n, \lambda, r)}\right)^{1-\frac{1}{2p}} \left(\int_{\overline{U}_r} \psi^{2p} dV\right)^{\frac{1}{2p}} (v(n, \lambda, r))^{-\frac{1}{2p}}$$

where

$$C_1(n, \lambda, r) = \max_{t\in[0,r]} \frac{t \cdot \omega_\lambda(t)}{\int_0^t \omega_\lambda(s) ds}$$

$$C_1(n, \lambda, 0) = n.$$

Proof. Begin by noting that the fraction ω/ω_λ satisfies

$$\frac{d}{dt}\frac{\omega}{\omega_\lambda} \leq (h - h_\lambda)\frac{\omega}{\omega_\lambda} \leq \psi\frac{\omega}{\omega_\lambda}. \tag{5.1}$$

Note that away from the cut locus the first inequality is actually an equality. At the cut locus the singular part of the derivative of ω has negative measure, and so when the derivative is interpreted correctly we get inequality. This implies

$$\frac{d}{dr}\frac{\int_{U_r} \omega(r, \cdot) d\theta_{n-1}}{\int_{U_r} \omega_\lambda(r) d\theta_{n-1}} = \frac{1}{\mathrm{vol}\, U_r}\int_{U_r} \frac{d}{dr}\frac{\omega(r, \cdot)}{\omega_\lambda(r)} d\theta_{n-1} \leq \frac{1}{\mathrm{vol}\, U_r}\int_{U_r} \psi\frac{\omega}{\omega_\lambda} d\theta_{n-1},$$

where the equality follows from pulling out the term $\mathrm{vol}\, U_r = \int_{U_r} d\theta_{n-1}$ in the denominator and noting that $\omega_\lambda(r)$ is constant with respect to $d\theta_{n-1}$, and the inequality follows from Eq. 5.1. So for $t \leq r$ we have

$$\frac{\int_{U_r} \omega(r, \cdot) d\theta_{n-1}}{\int_{U_r} \omega_\lambda(r, \cdot) d\theta_{n-1}} - \frac{\int_{U_r} \omega(t, \cdot) d\theta_{n-1}}{\int_{U_r} \omega_\lambda(t, \cdot) d\theta_{n-1}} \leq \frac{1}{\mathrm{vol}\, U_r}\int_t^r \int_{U_r} \psi\frac{\omega}{\omega_\lambda} d\theta_{n-1} \wedge ds$$

which, by cross-multiplication, gives us

$$\begin{aligned}&\int_{U_r}\omega(r,\cdot)d\theta_{n-1}\cdot\int_{U_r}\omega_\lambda(t)d\theta_{n-1}-\int_{U_r}\omega_\lambda(r)d\theta_{n-1}\cdot\int_{U_r}\omega(t,\cdot)d\theta_{n-1}\\&\leq\frac{1}{\operatorname{vol}U_r}\left(\int_{U_r}\omega_\lambda(r)d\theta_{n-1}\right)\left(\int_{U_r}\omega_\lambda(t)d\theta_{n-1}\right)\cdot\int_t^r\int_{U_r}\psi\frac{\omega}{\omega_\lambda}d\theta_{n-1}\wedge ds\\&\leq\left(\int_{U_r}\omega_\lambda(r)d\theta_{n-1}\right)\int_t^r\int_{U_r}\psi\omega d\theta_{n-1}\wedge ds,\end{aligned}\tag{5.2}$$

where the final inequality stems from the relation $\int_{U_r} d\theta_{n-1} = \operatorname{vol} U_r$ and the fact that ω_λ is non-decreasing in r, meaning $\frac{\omega_\lambda(t)}{\omega_\lambda(\Xi)} \leq 1$ for all $\Xi \in [t,r]$. Since $\psi\omega$ is nonnegative and, again, $\int_{U_r} d\theta_{n-1} = \operatorname{vol} U_r$ we obtain

$$(\text{RHS of Eq. 5.2}) \leq \omega_\lambda(r)\operatorname{vol}U_r\int_0^r\int_{U_r}\psi\omega d\theta_{n-1}\wedge ds.\tag{5.3}$$

Noting again that $\psi\omega$ is non-negative, we use Hölder's inequality to obtain

$$\begin{aligned}(\text{RHS of Eq. 5.3}) &\leq \omega_\lambda(r)\operatorname{vol}U_r\left(\int_0^r\int_{U_r}\psi^{2p}\omega d\theta_{n-1}\wedge ds\right)^{1/2p}\left(\int_0^r\int_{U_r}\omega d\theta_{n-1}\wedge ds\right)^{1-\frac{1}{2p}}\\&\leq \operatorname{vol}U_r\,\omega_\lambda(r)(\operatorname{vol}\overline{U}_r)^{1-\frac{1}{2p}}\left(\int_{\overline{U}_r}\psi^{2p}dV\right)^{1/2p},\end{aligned}$$

where the last inequality follows from the definition $\operatorname{vol}\overline{U}_r = \int_0^r\int_{U_r}\omega d\theta_{n-1}\wedge dt$. Now using the volume elements from above, we have

$$\frac{\operatorname{vol}\overline{U}_r}{v(n,\lambda,r)} = \frac{\int_0^r\int_{U_r}\omega d\theta_{n-1}\wedge dt}{\int_0^r\int_{U_r}\omega_\lambda d\theta_{n-1}\wedge dt}.$$

Thus, by the chain rule, we have

$$\begin{aligned}\frac{d}{dr}\frac{\operatorname{vol}\overline{U}_r}{v(n,\lambda,r)} &= \frac{\left(\int_{U_r}\omega(r,\cdot)d\theta_{n-1})\right)\cdot\left(\int_0^r\int_{U_r}\omega_\lambda(t)d\theta_{n-1}\wedge dt\right)}{(v(n,\lambda,r))^2}\\&\quad-\frac{(\int_{U_r}\omega_\lambda(r)d\theta_{n-1})\cdot\left(\int_0^r\int_{U_r}\omega(t,\cdot)d\theta_{n-1}\wedge dt\right)}{(v(n,\lambda,r))^2}.\end{aligned}$$

As $\omega(r,\cdot)$ is fixed (with respect to r), we can write the numerator as

$$\begin{aligned}&\int_0^r\left(\int_{U_r}\omega(r,\cdot)d\theta_{n-1}\cdot\int_{U_r}\omega_\lambda(t)d\theta_{n-1}\right)dt-\int_0^r\left(\int_{U_r}\omega_\lambda(r)d\theta_{n-1}\cdot\int_{U_r}\omega(t,\cdot)d\theta_{n-1}\right)dt\\&\leq\int_0^r\left(\operatorname{vol}U_r\cdot\omega_\lambda(r)\cdot(\operatorname{vol}\overline{U}_r)^{1-\frac{1}{2p}}\cdot\left(\int_{\overline{U}_r}\psi^{2p}dV\right)^{\frac{1}{2p}}\right)dt\\&=\operatorname{vol}U_r\cdot r\cdot\omega_\lambda(r)\cdot(\operatorname{vol}\overline{U}_r)^{1-\frac{1}{2p}}\cdot\left(\int_{\overline{U}_r}\psi^{2p}dV\right)^{1/2p},\end{aligned}$$

where the inequality is the same one derived earlier in this lemma and the equality is true because the integrand is constant in r. Thus we have

$$\frac{d}{dr}\frac{\operatorname{vol}\overline{U}_r}{v(n,\lambda,r)} \leq \frac{\operatorname{vol}U_r\cdot r\cdot\omega_\lambda(r)\cdot(\operatorname{vol}\overline{U}_r)^{1-\frac{1}{2p}}\cdot\left(\int_{\overline{U}_r}\psi^{2p}dV\right)^{\frac{1}{2p}}}{(v(n,\lambda,r))^2}$$

$$\leq C_1(n,\lambda,r)\cdot\left(\frac{\operatorname{vol}\overline{U}_r}{v(n,\lambda,r)}\right)^{1-\frac{1}{2p}}\left(\int_{\overline{U}_r}\psi^{2p}dV\right)^{\frac{1}{2p}}(v(n,\lambda,r))^{-\frac{1}{2p}}$$

because

$$\frac{\operatorname{vol}U_r\cdot r\cdot\omega_\lambda(r)}{v(n,\lambda,r)}=\frac{r\cdot\omega_\lambda(r)}{\int_0^r\omega_\lambda(s)ds}$$

converges to n as $r\to 0$ and can therefore be estimated by its maximum value on $[0,r]$. Recall from the statement of this lemma that $C_1(n,\lambda,r)$ is defined to be this maximum. Note that when $\lambda<0$ and either if $r\to\infty$ or if $\lambda\to-\infty$, then the constant $C_1\to\infty$. However, when $\lambda=0$ we can use $C_1=n$ for all r. (It can be seen that $C_1\to\infty$ when $\lambda<0$ and $r\to\infty$ by using l'Hôpital's rule on C_1 and noting that $h_\lambda\to 1$ as $r\to\infty$. Extend this to show also that $C_1\to\infty$ when $\lambda\to-\infty$ by using the identity $\omega_\lambda(r)=\omega_1(r\sqrt{\lambda})$.) □

We must now estimate $\int_{\overline{U}_r}\psi^{2p}\,dV$ in terms of $k(p,\lambda,r)$. We can reduce the problem by writing

$$\int_{\overline{U}_r}\psi^{2p}\,dV=\int_U\int_0^r\psi^{2p}\omega dt\wedge d\theta_{n-1}.$$

It thus suffices to estimate $\int_0^r\psi^{2p}\omega dt$.

Lemma 5.2. *There is a constant $C_2(n,p)$ such that when $p>n/2$ we have*

$$\int_0^r\psi^{2p}\omega dt\leq C_2(n,p)\int_0^r\rho^p\omega dt.$$

Proof. By the estimates mentioned in the introduction, we have

$$\psi'+\frac{\psi^2}{n-1}+2\frac{\psi\cdot h_\lambda}{n-1}\leq\rho.$$

Since ψ is absolutely continuous, we can multiply this through by $\psi^{2p-2}\omega$ and integrate to get

$$\int_0^r\psi'\psi^{2p-2}\omega dt+\frac{1}{n-1}\int_0^r\psi^{2p}\omega dt+\frac{2}{n-1}\int_0^r h_\lambda\psi^{2p-1}\omega dt\leq\int_0^r\rho\cdot\psi^{2p-2}\omega dt.$$

Integrating by parts,

$$\begin{aligned}\int_0^r\psi'\psi^{2p-2}\omega dt&=\frac{1}{2p-1}\psi^{2p-1}\omega\Big|_0^r-\frac{1}{2p-1}\int_0^r\psi^{2p-1}h\omega dt\\&\geq-\frac{1}{2p-1}\int_0^r\psi^{2p-1}h\omega dt\\&\geq-\frac{1}{2p-1}\int_0^r\psi^{2p}\omega dt-\frac{1}{2p-1}\int_0^r\psi^{2p-1}h_\lambda\omega dt.\end{aligned}$$

In the last line we have used that $h-h_\lambda\leq\psi$. Inserting this into the previous inequality we obtain

$$\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)\int_0^r\psi^{2p}\omega dt+\left(\frac{2}{n-1}-\frac{1}{2p-1}\right)\int_0^r h_\lambda\psi^{2p-1}\omega dt$$
$$\leq\int_0^r\rho\cdot\psi^{2p-2}\omega dt.$$

Thus when $p > n/2$

$$\left(\frac{1}{n-1} - \frac{1}{2p-1}\right) \int_0^r \psi^{2p} \omega dt \le \int_0^r \rho \cdot \psi^{2p-2} \omega dt$$
$$\le \left(\int_0^r \rho^p \omega dt\right)^{\frac{1}{p}} \cdot \left(\int_0^r \psi^{2p} \omega dt\right)^{1-\frac{1}{p}}.$$

Rearranging, we have

$$\int_0^r \psi^{2p} \omega dt \le \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-p} \cdot \int_0^r \rho^p \omega dt,$$

which is the desired inequality with $C_2(n,p) = \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-p}$. □

Combining the previous two lemmas, we have

$$\frac{d}{dr} \frac{\operatorname{vol} \overline{U}_r}{v(n,\lambda,r)} \le C_3(n,p,\lambda,r) \cdot \left(\frac{\operatorname{vol} \overline{U}_r}{v(n,\lambda,r)}\right)^{1-\frac{1}{2p}} k(p,\lambda,r)^{\frac{1}{2p}}\, v(n,\lambda,r)^{-\frac{1}{2p}},$$

where $C_3(n,p,\lambda,r) = C_1(n,\lambda,r) \cdot C_2(n,p)^{\frac{1}{2p}}$. We are now ready to prove the main theorem.

Theorem 5.1. *There is a constant $C(n,p,\lambda,R)$ which is nondecreasing in R such that when $r < R$ we have*

$$\left(\frac{\operatorname{vol} \overline{U}_R}{v(n,\lambda,R)}\right)^{1/2p} - \left(\frac{\operatorname{vol} \overline{U}_r}{v(n,\lambda,r)}\right)^{1/2p} \le C(n,p,\lambda,R) \cdot k(\lambda,p,R)^{1/2p}.$$

Proof. By the above observation, we have a differential inequality of the type

$$y' \le \alpha\, y^{1-\frac{1}{2p}} \cdot f(x)$$
$$y(0) = 1 \text{ and } y > 0$$

where

$$y(x) = \frac{\operatorname{vol} \overline{U}_x}{v(n,\lambda,x)},$$
$$f(x) = v(n,\lambda,x)^{-\frac{1}{2p}}, \text{ and}$$
$$\alpha = C_3(n,p,\lambda,R)\, k(p,\lambda,R)^{\frac{1}{2p}}.$$

Separating variables and integrating yields

$$2p \cdot y^{1/2p}(R) - 2p \cdot y^{1/2p}(r) \le \alpha \int_r^R f(x) dx.$$

Thus we can simply use

$$C = \frac{1}{2p} C_3(n,p,\lambda,R) \int_0^R v(n,\lambda,t)^{-\frac{1}{2p}}\, dt.$$

The final observation to be made is that the integral

$$\int_0^R v(n,\lambda,t)^{-\frac{1}{2p}}\, dt$$

in fact converges. □

We observed previously that Bishop's theorem is a special case of Theorem 2.2. Is it true that Ehrlich and Sanchez's result can be obtained from our result? Of course. We have $k \equiv 0$ if $\mathrm{Ric}(v,v) \geq (n-1)\lambda \cdot \mathbf{g}(v,v)$, just as before. Letting $r \to 0$, we obtain

Theorem 5.2. *Let $(M, \mathbf{g})$ be a Lorentz manifold, $x \in M$, $\lambda \leq 0$. Suppose U is a comparable set in T_xM and $\mathrm{Ric}(v,v) \geq (n-1)\lambda \cdot \mathbf{g}(v,v)$ for all timelike vectors tangent to $\overline{U}_r$, $r > 0$. Then*

$$\mathrm{vol}\,\overline{U}_r \leq v(n, \lambda, r).$$

The results presented here, while in some sense completing one circle of ideas in the machinery of volume comparisons, also open up a new way of studying the geometry of Lorentz spaces. Instead of getting bogged down in the complexities of more classical tools such as Jacobi fields and connections, we reduced the problem to some hard analysis of simple functions defined in terms of integrals. The ease with which this was accomplished suggests that these ideas can be pushed further.

We were quite surprised to find Petersen and Wei's paper almost completely analytic in nature. This made the transition from Riemannian manifolds to Lorentz manifolds almost unnoticeable. Indeed, our proofs follow the lines of [5] almost exactly. The only real difficulty for us was verifying that Lemmas 4.1 and 4.2—which are more geometric in spirit—indeed carried over to the Lorentzian setting. Once we had these squared away, the rest was easy.

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