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A study of
the Generalized Catenary Problem

by

Hai Xuan Nguyen

A Senior Thesis submitted to the Mathematics Department of
The University of Tennessee - Knoxville
in partial fulfillment of the requirements for the Honors degree of

Bachelor of Science

The University of Tennessee - Knoxville
Knoxville, April 2007

THE UNIVERSITY OF TENNESSEE - KNOXVILLE

RESEARCH ADVISER APPROVAL

of a thesis submitted by

Hai Xuan Nguyen

This thesis has been read by the following adviser and has been found to be satisfactory:

Apr 25, 2007 — Date

Jochen Denzler

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FINAL READING APPROVAL

To the Honors Program:

I have read the thesis of Hai Xuan Nguyen in its final form and have found that (1) its format, citations, and bibliographic style are consistent and acceptable; (2) its illustrative materials including figures, are in place; and (3) the final manuscript is satisfactory and is ready for submission to The Honors Program.

Apr 25 2007

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Jochen Denzler

Approved for the Honors Program:

Conrad Plaut

Director

Abstract

Consider an ideal homogeneous chain of length $2l$, suspended at two pre-chosen points. This paper discusses the qualitative behaviors of the shape of the curve drawn by the chain in different potentials. We first solve the problem under the constant gravity near the surface of the earth, then construct and study a general model with rotationally symmetric gravitational fields. Our primary considerations are Newton potentials in $n+2$ dimensions, for $n > 1$. At the end, we also extend the discussion to the two-dimensional Newton potential $\ln r$.

Acknowledgment

First, I reserve the most special thank for my adviser, Dr. Jochen Denzler. Over the course of my studying here at the University and pursuing this research, he has always been there to listen and to give advice. His patience is the greatest grace I have known. Moreover, he was the one who introduced me to the concept of Catenary, who inspired me to conduct the research, and who guided me throughout the entire work. My paper closely associates with his 1999 publication; and all the figures are drawn primarily by him.

Additionally, I would like to thank Dr. Conrad Plaut for the encouragement, the Chancellor's Honors Program for the internship funding during the summer of 2006 that made this paper possible, and especially my family and friends for their enormous and never ending supports.

Contents

1 Introduction

During his life time (1564 -1642), Galilei conjectured that the shape of a hanging chain under the Earth's gravity was the parabola, ([2], pg. 149). He expressed this idea in his book "Dialogues Concerning Two New Sciences" in 1638 and was bolstered among the scientific community. Although in 1646, Huygens successfully disproved Galilei's conjecture, it was forty-five years later that Huygens himself, Bernoulli and Leibniz, independently became able to provide the exact answer, ([3],pg. 236-237). Nowadays, it is well-known under the name the Catenary Problem (which originated from "catena" in Latin for chain); and the answer includes variants of the graph of the hyperbolic function $cosh$, and can be obtained through various methods.

In 1998, the paper "Catenaria Vera - The True Catenary" by Jochen Denzler and Andreas M. Hinz once again considered the Catenary problem, but in a different gravitational field — namely, the rotational symmetric $-\frac{1}{r}$ potential. The problem was then solved explicitly by means of the Euler-Lagrange equation, and then further discussed with a number of methods for an existence proof.

Based on the ground of the classical Catenary and inspired by the "True Catenary", this paper extends the idea and henceforth studies the following question: "What general qualitative behaviors does a hanging chain of prescribed length have in a rotationally symmetric potential?" Although the research will not provide a complete answer, it will examine some aspects of it, and leave the rest open for further study.

This paper consists of five major sections, including the Introduction. In section 2, we will first consider and solve the classical Catenary with the modern techniques of Calculus of Variations, thereby constructing a general model for the problem. Section 3 will derive qualitative behaviors of the chains in gravitational fields that satisfy our assumptions. A discussion on the chain's behaviors in $\ln r$ potential is available in section 4. Finally, the last section, Conclusion and Outlook, will summarize the result and suggest possible directions for further investigations of the Catenary.

2 The Catenary

2.1 The Classical Case of Constant Gravity

Let two points A and B be given in a vertical plane and a perfectly flexible (and unstretchable) chain of prescribed length $2l$. If $2l$ is smaller than the distance between A and B, there cannot be a physical solution; if equal, we have the uninterestingly straight line. Hence, without lost of generality, we assume that $2l$ is strictly greater than the distance between A and B. The rectifiable curve from A to B created by the chain under the sole influence of gravity is the one that minimizes the potential energy of the chain. For any other curve, the exceeding energy will be converted to kinetic energy, therefore lead to a deformation of its shape, (see [1]). In this section, we will solve the classical problem using the standard techniques from Calculus of Variations.

Since the chain is solely under gravity, we expect that the graph is a smooth curve and can be represented by a continously differentiable function. Let $x \mapsto r(x)$ be the function in cartesian coordinates describing the curve, whose graph is in study. Then let ρ be the density. The two points A and B have the coordinations (x_1, r_1) and (x_2, r_2), respectively. Then $\sqrt{dx^2 + dr^2}$ is the element of arclength of the curve. The potential energy of an unit length near the surface of the earth is given by:

$$\rho g r \sqrt{dx^2 + dr^2} = \rho g r \sqrt{1 + \dot{r}^2} dx$$

(Here, we use the notion $\dot{r}$ to mean the derivative of r with respect to x.) Therefore, the potential energy of the chain is:

$$F[r] = \int_{x_1}^{x_2} \rho g r \sqrt{1 + \dot{r}^2} dx \tag{1}$$

We have to minimize (1) among all the sufficiently regular functions $r : [x_1, x_2] \to]-\infty, \infty[$, $r(x_1) = r_1$ and $r(x_2) = r_2$, satisfying the length condition:

$$L[r] := \int_{x_1}^{x_2} \sqrt{1 + \dot{r}^2} dx = 2l \tag{2}$$

Because this is a variational problem with a constraint, we will use the method of Lagrange multipliers. From (1) and (2), we are now interested in optimizing:

$$F^*[r] = \int_{x_1}^{x_2} (\rho g r + \lambda) \sqrt{1 + \dot{r}^2} dx \tag{3}$$

where λ denotes the Lagrange multiplier.

Let $I = (\rho gr + \lambda)\sqrt{1+\dot{r}^2}$; then the Euler-Lagrange equation for this problem will read:

$$\exists \lambda \in \mathbb{R} : I_r(r, \dot{r}) = \frac{d}{dx} I_{\dot{r}}(r, \dot{r}) \tag{4}$$

Since I is independent of x, we can also use the "energy integral" of the Euler-Lagrange equation:

$$I - \dot{r}\frac{\partial I}{\partial \dot{r}} = c_1 \tag{5}$$

$$\Rightarrow (\rho gr + \lambda)\sqrt{1+\dot{r}^2} - \dot{r}\frac{\partial}{\partial \dot{r}}[(\rho gr + \lambda)\sqrt{1+\dot{r}^2}] = c_1$$

$$\Rightarrow (\rho gr + \lambda)\sqrt{1+\dot{r}^2} - \dot{r}\frac{(\rho gr + \lambda)\dot{r}}{\sqrt{1+\dot{r}^2}} = c_1$$

$$\Rightarrow (\rho gr + \lambda)(\frac{1}{\sqrt{1+\dot{r}^2}}) = c_1 \tag{6}$$

Solve (6) for $\dot{r}$:

$$\dot{r} = \pm\sqrt{(\frac{\rho gr + \lambda}{c_1})^2 - 1}$$

$$\Rightarrow dx = \pm\frac{1}{\sqrt{(\frac{\rho gr+\lambda}{c_1})^2 - 1}}dr$$

$$\Rightarrow x = \pm\int \frac{1}{\sqrt{(\frac{\rho gr+\lambda}{c_1})^2 - 1}}dr$$

$$\Rightarrow x = \pm\frac{c_1}{\rho g}\ln\left| \frac{\rho gr + \lambda}{c_1} + \sqrt{(\frac{\rho gr + \lambda}{c_1})^2 - 1} \right| + c_2$$

$$^1 \Rightarrow x = \pm\frac{c_1}{\rho g}\cosh^{-1}\left(\frac{\rho gr + \lambda}{c_1}\right) + c_2$$

Therefore[2],

$$r = \frac{c_1}{\rho g}\cosh\left(\frac{\rho g(x - c_2)}{c_1}\right) - \frac{\lambda}{\rho g} \tag{7}$$

The graph of $r(x)$ is called a catenary.

As it turns out, using the boundary conditions $r(x_1) = r_1$, $r(x_2) = r_2$, and the prescribed length $2l$, one can certainly determine the constants λ, c_1 and c_2. There are exactly two

[1] One can verify this quite straightforwardly:
$\cosh\left(\ln(y + \sqrt{y^2+1})\right) = \frac{e^{\ln(y+\sqrt{y^2+1})} + e^{-\ln(y+\sqrt{y^2+1})}}{2} = \frac{y+\sqrt{y^2+1}+\frac{1}{y+\sqrt{y^2+1}}}{2} = y$

[2] The "±" sign drops out because cosh is an even function.

solutions, one of which ($c_1 > 0$) is the catenary. The other ($c_1 < 0$) gives an upsidedown catenary that maximizes the potential energy.

At this point, let us consider (4) again.

$$(4) \Leftrightarrow \rho g\sqrt{1+\dot{r}^2} = \frac{d}{dx}[(\rho g r+\lambda)\frac{\dot{r}}{\sqrt{1+\dot{r}^2}}]$$

$$\Rightarrow \rho g\sqrt{1+\dot{r}^2} = \rho g\frac{\dot{r}^2}{\sqrt{1+\dot{r}^2}} + (\rho g r+\lambda)\frac{\ddot{r}}{\sqrt{1+\dot{r}^2}^3}$$

$$\Rightarrow \rho g\frac{1}{\sqrt{1+\dot{r}^2}} = \frac{(\rho g r+\lambda)\ddot{r}}{\sqrt{1+\dot{r}^2}^3} \tag{8}$$

Note that a minimizing curve $r(x)$ must be convex, because a point reflection of a graph segment would otherwise decrease the potential energy, (see Figure 1). Thus, $\ddot{r}$ is always non-negative. Since the left hand side of (8) is positive, we deduce that both $\ddot{r}$ and $\rho g r + \lambda$ must be greater than 0. We shall see later (in Theorem I) that the fact $\rho g r + \lambda > 0$ means there is only one type of solution.

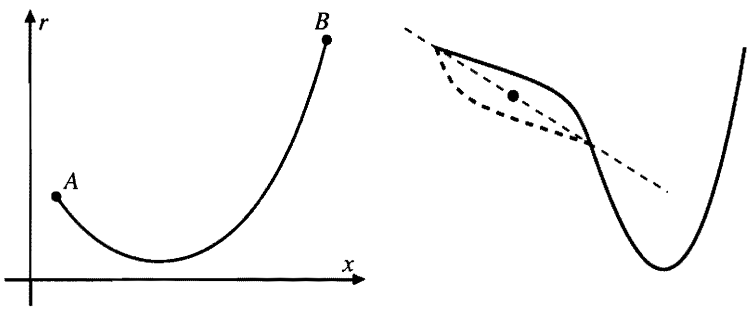

Figure 1: Classical Catenary (left); and Point reflection reduces potential energy (right)

2.2 The Case of General Rotationally Symmetric Potential

We will now consider the model in a general gravitational field, i.e. $V :]0, \infty[\to \mathbb{R}$, centered at 0. We further assume that $V'(r)$ is strictly greater than 0. All assumptions about the chain and its position (i.e. suspended at A and B) hold as previously discussed.

Let $\varphi \mapsto r(\varphi)$ be the function that describes the chain. Note that r is now a function of the angle φ, centered at 0. Then a length element of the curve is: $\sqrt{(rd\varphi)^2 + (dr)^2} = \sqrt{r^2 + \dot{r}^2}d\varphi$. Accordingly, the potential of a length element is: $V(r(\varphi))\sqrt{r^2 + \dot{r}^2}d\varphi$

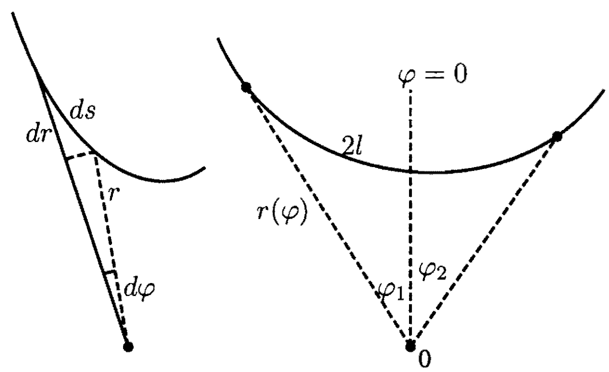

Figure 2: Length Element (left) and Chain in polar coordinates (right)

Our ultimate goal is to optimize the potential of the chain. Hence, the problem reads:

Minimize the functional:

$$F[r] := \int_{\varphi_1}^{\varphi_2} V(r(\varphi))\sqrt{r(\varphi)^2 + \dot{r}(\varphi)^2}d\varphi \tag{9}$$

among all the sufficiently regular functions $r : [\varphi_1, \varphi_2] \to]0, \infty[$ with $r(\varphi_1) = r_1$ and $r(\varphi_2) = r_2$, satisfying the length condition:

$$L[r] := \int_{\varphi_1}^{\varphi_2} \sqrt{r(\varphi)^2 + \dot{r}(\varphi)^2}d\varphi = 2l \tag{10}$$

Using the Lagrange multiplier method as in the previous chapter, the Euler-Lagrange equation reads:

$$\exists \lambda \in \mathbb{R} : \frac{\partial}{\partial r}(\lambda + V(r))\sqrt{r^2 + \dot{r}^2} = \frac{d}{d\varphi}\left[\frac{\partial}{\partial \dot{r}}(\lambda + V(r))\sqrt{r^2 + \dot{r}^2}\right] \tag{11}$$

which, by multiplying with an integrating factor $\dot{r}$ and then integrating both sides, leads to the Integral Euler-Lagrange Equation:

$$\exists \lambda \in \mathbb{R}\ \exists \alpha \in \mathbb{R}\backslash\{0\} : \ \frac{1}{\alpha} = (\lambda + V(r))\frac{r^2}{\sqrt{r^2 + \dot{r}^2}} \tag{12}$$

where λ is the Lagrange multiplier, and α is a constant. Equation (12) is also known as the Energy Equation. In calling the constant of integral $\frac{1}{\alpha}$, we have anticipated that this constant could not be 0, because $\lambda + V(r)$ cannot vanish.

From (11), we can also deduce that:

$$\frac{rV'(r)}{\sqrt{r^2 + \dot{r}^2}} = \kappa(\lambda + V(r)) \tag{13}$$

where κ is the curvature:

$$\kappa = \frac{r\ddot{r} - r^2 - 2\dot{r}^2}{\sqrt{r^2 + \dot{r}^2}^3} \tag{14}$$

From now on, we will refer to equation (13) as the Curvature Equation. Since the left hand side of (13) is strictly positive (remember that we assume $V'(r) > 0$), it is clear that $\lambda + V(r)$ can never vanish, as claimed above.

Here, we have completed building our general model.

3 Behaviors of *chains* and *arches* for various potentials

From the Curvature Equation, it follows that the κ and $\lambda + V(r)$ have the same sign. The reason is that r is positive, and $V'(r)$ is assumed to be strictly positive. From the Energy Equation, it is trivial to see that α and $\lambda + V$ also have the same sign. The sign of α will therefore determine that of the curvature. For instant, if α is positive, we get a positive κ. We will call solutions that correspond to positive α *chains*. Accordingly, we will call those with negative curvature *arches*. It will turn out that in the limit where α goes to 0, the Energy Equation does not produce a physical solution because it poses an infinite tension on the chain.

In the case of *chains*, the curve must be strictly convex because $\alpha > 0$. Its maximal domain is an interval of length less than π, and $r(\varphi)$ will go to infinity as φ approaches the boundary of this interval. Therefore, the function $\varphi \mapsto r(\varphi)$ has exactly one local extremum, which is an absolute minimum.

On the other hand, *arches* can have a maximum, as well as a minimum, either one or both at the same time. If there is only a minimum, we will call the curve a *flat arch*. If we have only a maximum, then it is a *steep arch*. Later, we will show that an *arch* can wrap around the center many times before diverging to infinity (or converging to the center). Furthermore, if the solution $r(\varphi)$ has neither maximum nor minimum, it will be a *spiral arch*. Finally, if there are both maximum and minimum, we can have either a *circular*, or an *exotic arch*. In many interesting potentials, *exotic arches* can be ruled out. We will elaborate more on each case in the next sections.

For convenience, let φ_0 be the (or: an) angle at which $r(\varphi)$ reaches its extremum, and $r_0 = r(\varphi_0)$ (if an extremum exists at all). It follows that $\dot{r}(\varphi_0) = 0$

Together with the Energy Equation, we have:

$$\alpha r_0(\lambda + V(r_0)) = 1 \tag{15}$$

Now, take the Energy Equation again and solve for $\dot{r}$. We obtain:

$$\dot{r} = \pm r\sqrt{\alpha^2 r^2(\lambda + V(r))^2 - 1} \tag{16}$$

(The "plus" sign in the above expression corresponds to one side of the chain, and "minus" to the other.)

$$\Rightarrow d\varphi = \pm \frac{1}{r\sqrt{\alpha^2 r^2(\lambda + V(r))^2 - 1}} dr \tag{17}$$

From here on, we will refer to equation (15) as the Extremum Equation, and equation (17) as the Delta Phi Equation.

We will now state the main theorem.

Theorem I:

Consider an ideal homogeneous chain under the sole influence of a potential V. Assume:

A-I: *V is rotationally symmetric around the origin 0*

A-II: *V can be described as a continuously differentible function of r, i.e. $V : r \mapsto V(r)$*

A-III: *$V'(r) > 0$, in particular $V(r)$ is strictly increasing*

A-IV: *$rV(r)$ satisfies: $\lim_{r\to 0} rV(r) = -\infty$ and $\lim_{r\to\infty} rV(r) = 0$*

A-V: *$V(r) + rV'(r)$ is strictly decreasing, and never equals to 0 $\forall r$*

Then there exist λ and $\alpha \neq 0$ in $\mathbb{R}$ such that the Euler Lagrange Equation and the Energy Equation hold. Furthermore,

1. *If $\alpha > 0$, then $\lambda > 0$. Solutions must be chains.*
2. *If $\alpha < 0$ and*
   (a) *$\lambda \geq 0$, then solutions must be steep arches*
   (b) *$\lambda < 0$, then there exists a threshold α_0 such that:*
      i. *$\alpha < \alpha_0$, solutions are spiral arches*
      ii. *$\alpha = \alpha_0$, solutions are circular arches*
      iii. *$\alpha > \alpha_0$, solutions can be steep arches, flat arches, or exotic arches*

Note: The five assumptions we have made above in particular allow the potential in consideration to belong to the family $-\frac{1}{r^n}$, for $n > 1$.

We will now consider the problem in $-\frac{1}{r}$, $-\frac{1}{r^2}$ potentials, before studying the general case.

3.1 The potential $-\frac{1}{r}$

The potential $-\frac{1}{r}$ does not satisfy assumptions **A-IV** and **A-V**; however it directly relates to the $-\frac{1}{r^n}$ family. In fact, we shall see that the behaviors of the chain under this potential and under a general one differ only marginally.

Take the Energy Equation, and let $V(r) = -\frac{1}{r}$, we have:

$$\frac{1}{\alpha} = (\lambda - \frac{1}{r})\frac{r^2}{\sqrt{r^2 + \dot{r}^2}}$$

$$\Rightarrow \lambda r - 1 = \frac{1}{\alpha}\frac{\sqrt{r^2 + \dot{r}^2}}{r} \tag{18}$$

Correspondingly, the Delta Phi Equation in this case reads:

$$d\varphi = \pm\frac{1}{r\sqrt{\alpha^2(\lambda r - 1)^2 - 1}}dr \tag{19}$$

And the Extremum Equation reads:

$$\alpha(\lambda r_0 - 1) = 1 \tag{20}$$

$$\Rightarrow r_0 = \frac{1}{\lambda}(1 + \frac{1}{\alpha}) \tag{21}$$

We will consider different cases of α as in the theorem.

Case 1: The case of *chains*, or $\alpha > 0$

The right hand side of equation (18) is positive when $\alpha > 0$. Hence, λ must be strictly positive.

Now, let $\lambda r = \frac{1}{s}$, then $\lambda \dot{r} = -\frac{\dot{s}}{s^2}$. Substitute in (18) to have:

$$\alpha(1 - s) = sign(\lambda)\sqrt{s^2 + \dot{s}^2}$$

With some algebraic calculations, i.e. squaring and taking the derivative, then canceling $\dot{s}$ from both side, the resulting equation is:

$$\ddot{s} + (1 - \lambda^2)s = -\alpha^2 \tag{22}$$

Solving (22) and expressing the solutions in term of *cosh*, we then have:

$$s(\varphi) = \frac{\alpha\Big(\alpha - \cosh[\sqrt{\alpha^2 - 1}(\varphi - \varphi_0)]\Big)}{\alpha^2 - 1}$$

where φ_0 is as previously discussed.

Hence, the solution to the proposed problem is:

$$r(\varphi) = \frac{\alpha^2 - 1}{\lambda\alpha\Big(\alpha - \cosh[\sqrt{\alpha^2 - 1}(\varphi - \varphi_0)]\Big)} \tag{23}$$

The function $r(\varphi)$ describes the shape of the catenary, with yet undetermined constants λ, α, and φ_0. Fortunately, with boundary conditions $r(\varphi_1) = r_1$, $r(\varphi_2) = r_2$, the length constraint and the additional condition $\dot{r}(\varphi_0) = 0$, we can determine [uniquely] the values of those mentioned unknowns. For discussion of the existence and uniqueness of the solution, one can refer to section 3 of the paper by Denzler and Hinz, (see [1]).

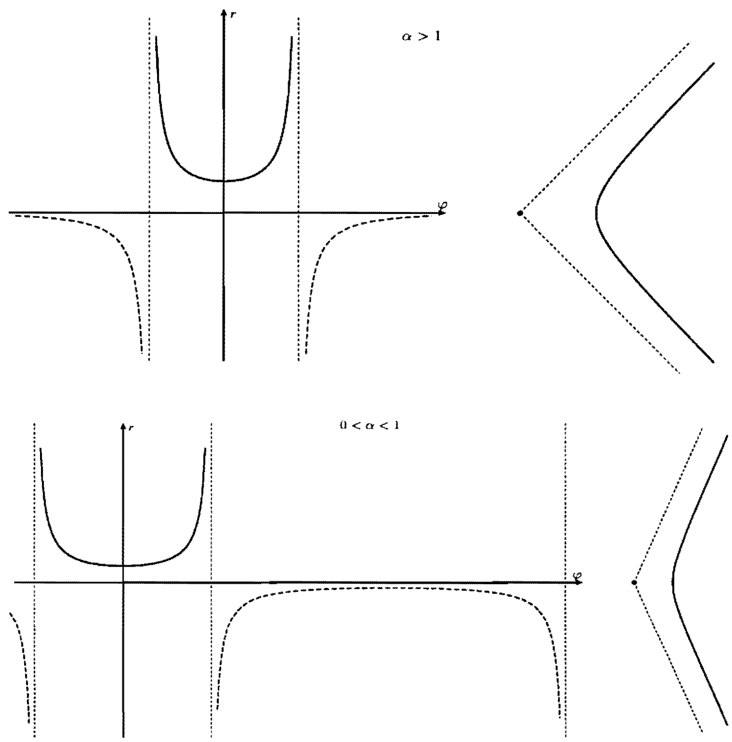

Figure 3: Graph of solutions in (φ, r) plane (left) and as seen in the potential (right)

Notice that in equation (23), $\alpha^2 - 1$ is under a square root. Thus, we consider two sub-cases: $0 < \alpha < 1$ and $\alpha > 1$, which give us respective qualitative pictures (see Figure 3). (The case $\alpha = 1$ is neglected, but one can study it as a limiting case $\alpha \longrightarrow 1$ of (23).)

Case 2: The case of *arches*, or $\alpha < 0$

Case 2.1: $\lambda = 0$

Substitute $\lambda = 0$ to equation (20), we have $\alpha = -1$. Now solve for (18), we get $\dot{r} = 0$. Hence, the solution $r(\varphi)$ depicts a circle, centered at 0.

Case 2.2: $\lambda > 0$

We require the quantity under the square root in equation (16) to be non-negative.

$$\alpha^2(\lambda r - 1)^2 \geq 1$$

$$\Rightarrow \; or \begin{cases} r \geq \frac{1}{\lambda}(1 - \frac{1}{\alpha}) \\ 0 < r \leq \frac{1}{\lambda}(1 + \frac{1}{\alpha}) \end{cases}$$

We ignore the possibility of the first inequality (r has only a minimum), because $\frac{1}{\lambda}(1 - \frac{1}{\alpha})$ does not satisfy (21). We are left with the second inequality, which implies that the solution is a *steep arch*. Furthermore, we have:

$$0 < r \leq \frac{1}{\lambda}(1 + \frac{1}{\alpha})$$

$$\Rightarrow 1 + \frac{1}{\alpha} > 0$$

$$\Rightarrow \alpha < -1$$

Thus, in this case of *arches*, where $\lambda > 0$, the solution exists only when $\alpha < -1$. Moreover, the solution will be a *steep arch*.

Case 2.3: $\lambda < 0$

Again, we require the quantity under the square root in equation (16) to be non-negative.

$$\alpha^2(\lambda r - 1)^2 \geq 1$$

$$\Rightarrow \; or \begin{cases} r \geq \frac{1}{\lambda}(1 + \frac{1}{\alpha}) \\ 0 < r \leq \frac{1}{\lambda}(1 - \frac{1}{\alpha}) \end{cases}$$

Here, we omit the possibility of the second inequality (r has a minimum), because $\frac{1}{\lambda}(1 - \frac{1}{\alpha})$ does not satisfy equation (21). We are left with the first inequality:

$$r \geq \frac{1}{\lambda}(1 + \frac{1}{\alpha})$$

Notice that the right hand side of the equation is positive only when α is between -1 and 0. In such a case, solutions will be *flat arches*. On the other hand, when $\alpha \leq -1$, the lower bound condition is essentially ineffective. Hence, solutions are *spiral arches*.

Summary of the results in $-\frac{1}{r}$:

1. If $\alpha > 0$, solutions are *chains*.
2. If $\alpha < 0$ and
   (a) $\lambda = 0$, solutions are circles, centered at 0, which can also be viewed as *steep arches*.
   (b) $\lambda > 0$, solutions are *steep arches*
   (c) $\lambda < 0$, then the threshold is $\alpha_0 = -1$ such that:
      i. $\alpha < \alpha_0$, solutions are *spiral arches*
      ii. $\alpha_0 \leq \alpha < 0$, solutions are *flat arches*

The results match closely with the theorem.

3.2 The potential $-\frac{1}{r^2}$

Take the Energy Equation and let $V(r) = -\frac{1}{r^2}$:

$$\frac{1}{\alpha} = (\lambda - \frac{1}{r^2})\frac{r^2}{\sqrt{r^2 + \dot{r}^2}}$$

$$\Rightarrow \frac{1}{\alpha}\frac{\sqrt{r^2 + \dot{r}^2}}{r} = \lambda r - \frac{1}{r} \tag{24}$$

Correspondingly, equation (16) reads:

$$\Rightarrow \alpha^2(\lambda r - \frac{1}{r})^2 \geq 1 \tag{25}$$

Additionally, the Extremum Equation reads:

$$\alpha(\lambda r_0 - \frac{1}{r_0}) = 1$$

$$\Rightarrow r_0 = \frac{1 \pm \sqrt{1 + 4\alpha^2\lambda}}{2\alpha\lambda}$$

Let $r_+ = \frac{1+\sqrt{1+4\alpha^2\lambda}}{2\alpha\lambda}$, and $r_- = \frac{1-\sqrt{1+4\alpha^2\lambda}}{2\alpha\lambda}$

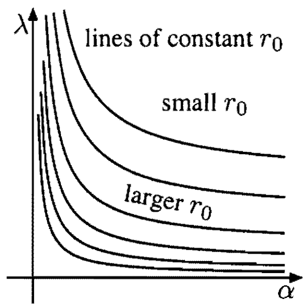

Figure 4: Iso-r_0-lines when both α and λ are positive

Case 1: The case of *chains*, or $\alpha > 0$

Because the left hand side of (24) is now positive, we deduce that λ must be greater than 0.

Also, since we are in the case of *chains*, we know that $r(\varphi)$ will have a minimum and cannot have a maximum. From (25), we can therefore deduce that:

$$r \geq \frac{1 + \sqrt{1 + 4\alpha^2\lambda}}{2\alpha\lambda}$$

And the extremum is: $r_0 = r_+$

On the other hand, we observe that:

$$r_+ = \frac{1 + \sqrt{1 + 4\alpha^2\lambda}}{2\alpha\lambda}$$

$$\Rightarrow \lambda = \frac{1}{r_+^2} + \frac{1}{\alpha r_+}$$

As a result, in the first quadrant of the (α, λ) plane, we have a *iso-r_0-line*, for every given level of r_+ (see Figure 4). In other words, if we know the minimum of the function $r(\varphi)$, we can conclude that the pair (α, λ) that solves the problem must be on this line.

Case 2: The case of *arches*, or $\alpha < 0$

Case 2.1: $\lambda = 0$

Take the Energy Equation and substitute in $V(r) = -\frac{1}{r^2}$, and $\lambda = 0$, we get:

$$\frac{1}{\alpha} = -\frac{1}{\sqrt{r^2 + \dot{r}^2}}$$

Squaring both side, solving for $\dot{r}^2$, and then taking the square root again gives:

$$\dot{r} = \pm\sqrt{\alpha^2 - r^2}$$

$$\Rightarrow d\varphi = \pm\frac{1}{\sqrt{\alpha^2 - r^2}}dr$$
$$\Rightarrow \int d\varphi = \pm\int\frac{1}{\sqrt{\alpha^2 - r^2}}dr$$
$$\Rightarrow \varphi = \pm\arcsin(\frac{r}{|\alpha|}) + \phi$$
$$r(\varphi) = \pm\alpha\sin(\varphi - \phi) \tag{26}$$

We, thus, have obtained an explicit solution, which depicts a circle through the center 0.

<u>Case 2.2:</u> $\lambda > 0$

From (25), we have:

$$or\begin{cases} \lambda r - \frac{1}{r} \le \frac{1}{\alpha} \\ \lambda r - \frac{1}{r} \ge -\frac{1}{\alpha} \end{cases}$$
$$\Rightarrow\ or\begin{cases} 0 < r \le \frac{1-\sqrt{1+4\alpha^2\lambda}}{2\alpha\lambda} \\ r \ge -\frac{1+\sqrt{1+4\alpha^2\lambda}}{2\alpha\lambda} \end{cases}$$

Although the second inequality suggests a *flat arch*, the minimum $-\frac{1+\sqrt{1+4\alpha^2\lambda}}{2\alpha\lambda}$ does not satisfy the Extremum Equation. We, therefore, omit it.

The first inequality gives us *steep arches*, which have a maximum at $r_0 = r_-$. With this in mind, we now integrate the Delta Phi Equation from 0 to r_-. Since we are in a rotationally symmetric potential, we first consider the "plus" sign only, and then multiply by 2 in order to get the maximum angle φ.

$$\Delta\varphi_{max} = 2\int_0^{r_-}\frac{1}{r\sqrt{\alpha^2(\lambda r - \frac{1}{r})^2 - 1}}dr \tag{27}$$

Let $s = \alpha(\lambda r - \frac{1}{r}) > 0$, then:

$$r = \frac{s \pm \sqrt{s^2 + 4\alpha^2\lambda}}{2\alpha\lambda}$$

Since $\alpha < 0$ and $r > 0$,

$$\Rightarrow r = \frac{s - \sqrt{s^2 + 4\alpha^2\lambda}}{2\alpha\lambda}$$
$$\Rightarrow \frac{dr}{ds} = -\frac{r}{\sqrt{s^2 + 4\alpha^2\lambda}}$$
$$\Rightarrow \frac{dr}{r} = -\frac{ds}{\sqrt{s^2 + 4\alpha^2\lambda}}$$

Substitute s in (27). Since s goes to ∞ as r goes to 0, and at r_0, $s_0 = 1$, we have:

$$\Delta\varphi_{max} = -2\int_{\infty}^{1} \frac{1}{\sqrt{s^2+4\alpha^2\lambda}\sqrt{s^2-1}}ds$$

Let $s = \cosh t$:

$$\Rightarrow \Delta\varphi_{max} = 2\int_{0}^{\infty} \frac{1}{\sqrt{\cosh^2 t+4\alpha^2\lambda}}dt$$

$$\Rightarrow \Delta\varphi_{max} < 2\int_{0}^{\infty} \frac{1}{\cosh t}dt$$

$$\Rightarrow \Delta\varphi_{max} < \pi \tag{28}$$

As we can see, the integral expressing $\Delta\varphi_{max}$ converges nicely at the lower limit $r = 0$. Furthermore, the magnitude of $\Delta\varphi_{max}$ cannot exceed π, which means we have a *steep arch* that converges directly into the center, (see Figure 5).

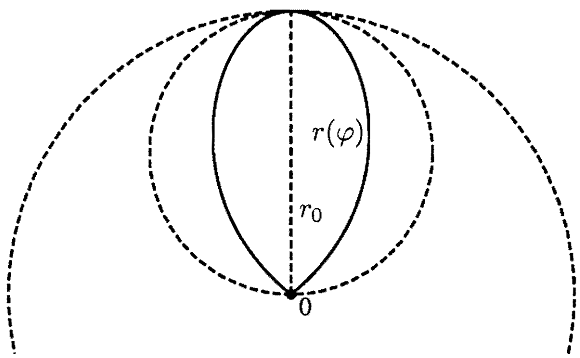

Figure 5: Steep Arch

<u>Case 2.3:</u> $\lambda < 0$

Before proceeding with our analysis, let us note that if we know beforehand that λ is negative, then as a result of the Euler Lagrange Equation, α will surely be negative. We can immediately expect *arches* as solutions.

Recall that $r_+ = \frac{1+\sqrt{1+4\alpha^2\lambda}}{2\alpha\lambda}$, and $r_- = \frac{1-\sqrt{1+4\alpha^2\lambda}}{2\alpha\lambda}$. Previously, the expressions under the square roots were always positive. In this case, however, we need to require λ such that $1+4\alpha^2\lambda \geq 0$. Thus, we have three sub-cases:

- If $\alpha < -\frac{1}{2\sqrt{-\lambda}}$, then the Extremum Equation cannot be satisfied. On the other hand, $\alpha^2(\lambda r - \frac{1}{r})^2 > \frac{1}{-4\lambda}(\lambda r - \frac{1}{r})^2 = \frac{1}{4}(-\lambda r^2 + \frac{1}{-\lambda r^2} + \frac{1}{2}) \geq 1$. Equation (25) is always satisfied. Therefore, we will have *spiral arches*.

- If $\alpha = -\frac{1}{2\sqrt{-\lambda}}$, then $r_0 = \frac{1}{\sqrt{-\lambda}}$. Both maximum and minimum of solutions equal to r_0. Therefore, we will have circles, centered at 0.

- If $-\frac{1}{2\sqrt{|\lambda|}} < \alpha < 0$, then we will have two cases: *flat* and *steep arches*.

Since λ is negative, we deduce from (25) that:

$$or \begin{cases} r \geq r_+ \\ 0 < r \leq r_- \end{cases} \tag{29}$$

The first inequality gives *flat arches*, whereas the second inequality gives *steep arches*. We then estimate the maximum angle for each case as done previously.

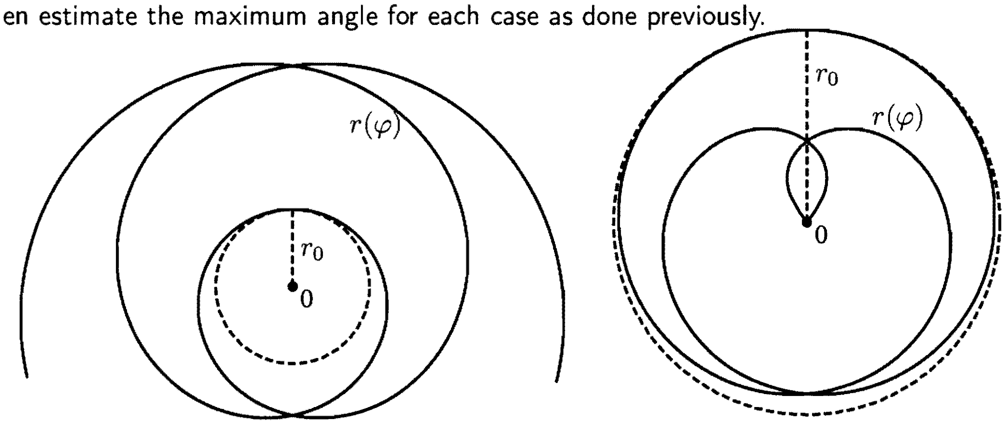

Figure 6: *Flat arch* (left) - and *Steep Arch* (right)

$$or \begin{cases} \Delta\varphi_{max} = 2\int_{r_+}^{\infty} \frac{1}{r\sqrt{\alpha^2(\lambda r - \frac{1}{r})^2 - 1}} dr \\ \Delta\varphi_{max} = 2\int_{0}^{r_-} \frac{1}{r\sqrt{\alpha^2(\lambda r - \frac{1}{r})^2 - 1}} dr \end{cases} \tag{30}$$

Repeat the substitution for s, and then t, we get the following equation for both cases:

$$\Delta\varphi_{max} = 2\int_0^{\infty} \frac{1}{\sqrt{\cosh^2 t + 4\alpha^2\lambda}} dt < \infty \tag{31}$$

(Note: During the substitution, we have to solve for r in term of s, which results in a choice between $r = \frac{s \pm \sqrt{s^2 + 4\alpha\lambda}}{2\alpha\lambda}$. As r goes to r_+ (r_-, respectively), s goes to 1. Hence, we end up with the "plus" sign in the expression of r.)

Now, by letting $\Delta\varphi_{max} =: 2k\pi$, we can see that solutions wrap around the center k times; and k can be made arbitrarily large by making $4\alpha^2\lambda$ close to -1

In the case of *flat arches*, each solution lies outside a circle of radius r_+; and in the case of *steep arches*, each solution lies inside a circle of radius r_- (see Figure 6).

Summary of the results in $-\frac{1}{r^2}$:

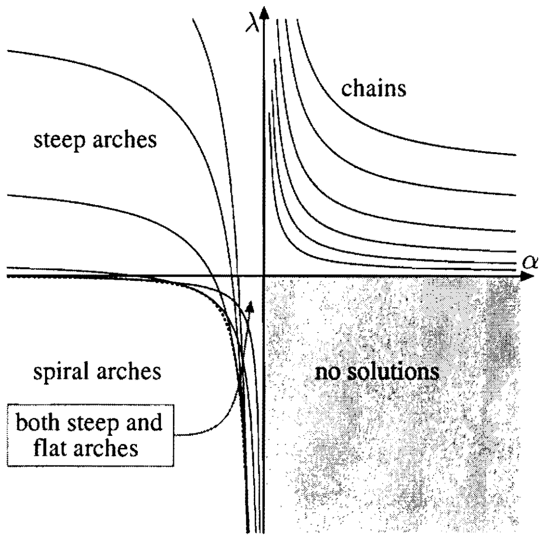

Figure 7: Case Diagram

1. If $\alpha > 0$, solutions are *chains*

2. If $\alpha < 0$ and

   (a) $\lambda = 0$, solutions are circles, going through the center 0

   (b) $\lambda > 0$, solutions are *steep arches*

   (c) $\lambda < 0$, then the threshold is $\alpha_0 = -\frac{1}{2\sqrt{-\lambda}}$ such that:

      i. $\alpha < \alpha_0$, solutions are *spiral arches*

      ii. $\alpha = \alpha_0$, solutions are circles, centered at 0

      iii. $\alpha_0 \leq \alpha < 0$, solutions are either *flat arches* or *steep arches*

The results, once again, match with the stated theorem.

3.3 The general field

Recall the model we build in section 2.2, and other essential equations that will help analyzing the problems. The Energy Equation gives us a necessary condition, which we derived from the Euler-Lagrange equation. Equation (14) expresses the curvature of the curve; a positive curvature means a *chain*, while a negative one gives an *arch*. The Extremum Equation is an additional constraint that we have for r_0 - the extremum of the curve. Finally, the quantity under the square root in equation (16) needs to be non-negative:

$$\Rightarrow \alpha^2 r^2 (\lambda + V(r))^2 \geq 1$$
$$\Rightarrow |\lambda r + rV(r)| \geq \frac{1}{|\lambda|} \tag{32}$$

On the other hand, $\frac{d}{dr}(rV(r)) = V(r) + rV'(r)$ never crosses 0 (because of assumption **A-V**), while $rV(r)$ itself approaches $-\infty$ and 0 as r goes to 0 and ∞, respectively. As a result, $V(r) + rV'(r)$ must be positive, and in particular $rV(r)$ is strictly increasing. Furthermore, by assumption **A-V** that $V(r) + rV'(r)$ is strictly decreasing, we deduce that for any constant λ, if λ is greater than or equal 0, $\lambda + V(r) + rV'(r)$ remains positive on $[0, \infty[$; and if λ is less than 0, $\lambda + V(r) + rV'(r)$ has exactly one zero. Hence, $\lambda r + rV(r)$ is strictly increasing if λ is non-negative, and has exactly one extremum otherwise.

Additionally, assumption **A-IV** ensures that the function $\lambda r + rV(r)$ approaches $-\infty$ as r goes to 0, and is asymptotic to λr as r goes to ∞.

With these observations in mind (which will become very essential in future discussions), let us now consider different cases of α and λ as in the previous section.

Case 1: The case of *chains*, or $\alpha > 0$

Since $V(r)$ is always negative, we deduce from the Energy Equation that λ must be positive. Therefore, $\lambda r + rV(r)$ is strictly increasing as discussed above. In the plane $(r, \lambda r + rV(r))$, the function $\lambda r + rV(r)$ then has two asymptotes: the vertical axis and the line λr, (see Figure 8).

Inequality (32) translates into two inequalities:

$$\Rightarrow \ or \begin{cases} \lambda r + rV(r) \geq \frac{1}{\alpha} \\ \lambda r + rV(r) \leq -\frac{1}{\alpha} \end{cases}$$

By the monotonicity property of $\lambda r + rV(r)$, the first inequality will give a minimum for r, whereas the second will give a maximum (which we ignore because we are in the case of

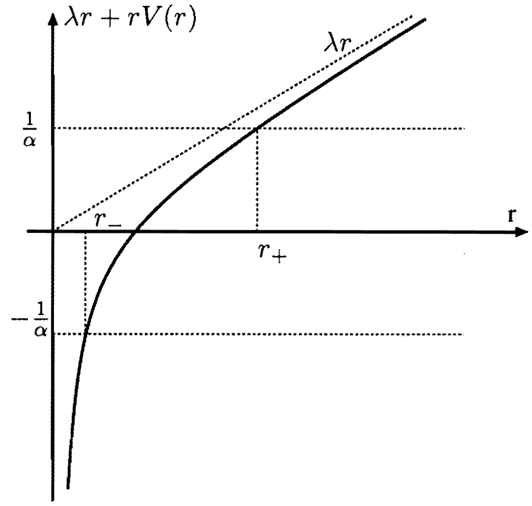

Figure 8: In general potential, $\lambda > 0$

chains). Furthermore, there will be exactly one value of $r = r_+$, which solves the Extremum Equation. We have $\lambda r_+ + r_+ V(r_+) = \frac{1}{\alpha}$. For all $r \geq r_+$, the first inequality is satisfied. $r_0 = r_+$ is the extremum of the curve.

Additionally, λ can be expressed as:

$$\lambda = -V(r_+) + \frac{1}{\alpha r_+}$$

Consequently in the first quadrant of the (α, λ) plane, we have an *iso-r_0-line*, for every given level of r_+, (see Figure 4).

Case 2: The case of *arches*, or $\alpha < 0$

Case 2.1: $\lambda = 0$

With a similar discussion as in the previous case, we also obtain a qualitative picture for $\lambda r + rV(r)$. However, since $\lambda = 0$, we are left with $rV(r)$, which is strictly increasing and is asymptotic to both axes.

Inequality (32) then reads:

$$\Rightarrow \ or \begin{cases} rV(r) \leq \frac{1}{\alpha} \\ rV(r) \geq -\frac{1}{\alpha} \end{cases}$$

Since $rV(r)$ is less than 0, the second inequalities can be omitted immediately. The first inequality, by the monotonicity property of $rV(r)$, gives a maximum for r. Moreover, we will expect exactly one solution, r_-, for $rV(r) = \frac{1}{\alpha}$. Fortunately, r_- satisfies the Extremum Equation. Thus, we have a *steep arch* in this case, with r_- being the maximum.

Now, consider the Delta Phi Equation:

$$d\varphi = \pm\frac{1}{r\sqrt{\alpha^2 r^2 V(r)^2 - 1}}dr$$

The maximum angle φ is:

$$\Rightarrow \Delta\varphi_{max} = 2\int_0^{r_-} \frac{1}{r\sqrt{\alpha^2 r^2 V(r)^2 - 1}}dr \tag{33}$$

Let $s = \alpha r V(r)$, then $\frac{ds}{dr} = \alpha(V(r) + rV'(r))$. Because $V(r) + rV'(r)$ is positive and α is negative, $\frac{ds}{dr} < 0$. Hence, we have monotonicity, and can view r as a function of s, which is differentiable by the Implicit Function Theorem.

Therefore, there exists a function $g(s)$, such that $r = g(s)$. Then, $r = \frac{s}{\alpha V(g(s))}$, and $\frac{ds}{dr} = \alpha(V(r) + rV'(r))$ implies $dr = \frac{1}{\alpha[V(g(s))+g(s)V'(g(s))]}ds$. Substitute in (33), we get:

$$\Delta\varphi_{max} = 2\int_{\infty}^{1} \frac{V(g(s))}{s\sqrt{s^2-1}\,[V(g(s)) + g(s)V'(g(s))]}ds$$

Substitute $s = \cosh t$, we finally have:

$$\Delta\varphi_{max} = 2\int_0^{\infty} \frac{-V(g(\cosh t))}{\cosh(t)[V(g(\cosh t)) + g(\cosh t)V'(g(\cosh t))]}dt$$

Let $2k =: \sup\limits_{r\in[0,\infty[} \frac{-V(r)}{V(r)+rV'(r)}$, then:

$$\Delta\varphi_{max} \leq 2\int_0^{\infty} \frac{2k}{\cosh(t)}dt$$

$$\Rightarrow \Delta\varphi_{max} \leq 2k\pi$$

Our *steep arch*, therefore, will wrap around the center at most k times. The entire curve will lie within a circle of radius r_-.

<u>Case 2.2</u>: $\lambda > 0$

Once again, we will employ the argument regarding asymptotic behavior of $\lambda r + rV(r)$. Since $\lambda > 0$, $\lambda r + rV(r)$ is strictly increasing from 0 to ∞, and is asymptotic to the vertical axis and the line λr. Therefore, we have a similar picture as Figure 8.

Inequality (32) is now equivalent to:

$$\Rightarrow \ or \begin{cases} \lambda r + rV(r) \leq \frac{1}{\alpha} \\ \lambda r + rV(r) \geq -\frac{1}{\alpha} \end{cases}$$

By the monotonicity of $\lambda r + rV(r)$, the first inequality gives a maximum for r, whereas the second gives a minimum. There is only one solution r_- to the Extremum Equation, which also is the maximum of r for the first inequality. The second inequality is omitted, for its minimum allowable value of r does not satisfy the Extremum Equation. Henceforth, we get a *steep arch*, with r_- being the maximum.

Repeating the discussion which concerns the maximum angle φ, we will then obtain a similar result. Solutions to this case are *steep arches*, each of which wraps around the center at most k times and lies inside a circle of radius r_-.

<u>Case 2.3</u>: $\lambda < 0$

Behaviors of $\lambda r + rV(r)$ are significantly different in this case than all previous cases, because λ is negative. First of all, the graph of $\lambda r + rV(r)$ lies entirely in the second quadrant, and is bounded by the vertical axis and the downward-sloping line λr. $\lambda r + rV(r)$ goes to $-\infty$ both as r goes to 0 and ∞. Secondly, we have insisted earlier that $\lambda + V(r) + rV'(r)$ will have exactly one zero, which means the graph of $\lambda r + rV(r)$ will change its direction (from increasing to decreasing) exactly one time. Hence, $\lambda r + rV(r)$ attains a unique maximum at some r^*.

Let α_0 be a real, negative number such that $\frac{1}{\alpha_0} = \lambda r^* + r^* V(r^*)$.

Now, let us consider inequality (32):

$$\Rightarrow \ or \begin{cases} \lambda r + rV(r) \leq \frac{1}{\alpha} \\ \lambda r + rV(r) \geq -\frac{1}{\alpha} \end{cases}$$

Since $\lambda r + rV(r)$ is negative for all r in this case, the second inequality can be omitted immediately. We are left with only the first inequality. From now on, we will refer to it as the Necessary Inequality.

- If $\alpha_0 < \alpha < 0$, or $\frac{1}{\alpha_0} < \frac{1}{\alpha}$, then the Necessary Inequality is satisfied for all r. The Extremum Equation does not have solution. There is neither a maximum nor a minimum for r. Hence, we have *spirals*.

- If $\alpha_0 = \alpha$, or $\frac{1}{\alpha_0} = \frac{1}{\alpha}$, then the Necessary Inequality is satisfied for all $r \geq r^*$ and all

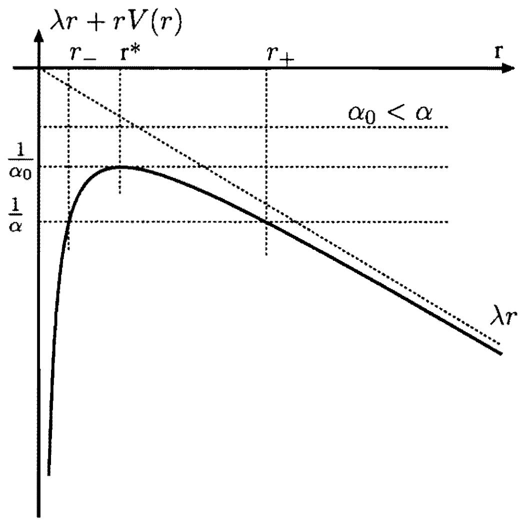

Figure 9: In general potential, $\lambda < 0$

$r \leq r^*$. The Extremum Equation is satisfied at $r = r^*$. Consequently, a solution must attain both its maximum and minimum at $r = r^*$. Hence, a solution is a *circular arch* of radius r^*.

- If $\alpha < \alpha_0$, or $\frac{1}{\alpha} < \frac{1}{\alpha_0}$, then we consider the equation $\lambda r + rV(r) = \frac{1}{\alpha}$ in two separate intervals $]0, r^*]$ and $[r^*, \infty[$. In the first interval, $\lambda r + rV(r)$ increases continuously from $-\infty$ to $\frac{1}{\alpha_0}$. Thus, there must exist uniquely r_- such that $\lambda r_- + r_- V(r_-) = \frac{1}{\alpha}$. Similarly in the second interval where the function decreases continuously from $\frac{1}{\alpha_0}$ to $-\infty$, there must exist uniquely r_+ such that $\lambda r_+ + r_+ V(r_+) = \frac{1}{\alpha}$.

The Necessary Inequalities, therefore, becomes equivalent to:

$$\Rightarrow \; or \begin{cases} 0 \leq r \leq r_- \\ r \geq r_+ \end{cases}$$

The first inequality gives *steep arches*, whereas the second gives *flat arches*. Both r_- and r_+ satisfy the Extremum Equation, and hence are the maximum and minimum of r, respectively.

The maximum angle φ (for both cases) is estimated through a similar calculation as it has been done when $\lambda = 0$. The expressions of $\Delta\varphi_{max}$ are:

$$\begin{cases} \Delta\varphi_{max} = 2\int_0^{r_-} \frac{1}{r\sqrt{\alpha^2(\lambda r + rV(r))^2 - 1}} dr \\ \Delta\varphi_{max} = 2\int_{r_+}^{\infty} \frac{1}{r\sqrt{\alpha^2(\lambda r + rV(r))^2 - 1}} dr \end{cases}$$

After substitutions, we get:

$$\Delta\varphi_{max} = 2\int_0^\infty \frac{-[\lambda + V(g(\cosh t))]}{\cosh(t)[\lambda + V(g(\cosh t)) + g(\cosh t)V'(g(\cosh t))]}dt$$

Let $2k =: \sup_{r\in[0,\infty[} \frac{-[\lambda+V(r)]}{\lambda+V(r)+rV'(r)}$, then:

$$\Delta\varphi_{max} \leq 2\int_0^\infty \frac{2k}{\cosh(t)}dt$$

$$\Rightarrow \Delta\varphi_{max} \leq 2k\pi$$

Our $arches$, therefore, will wrap around the center at most k times. A $flat\ arch$ will lie outside a circle of radius r_+, whereas a $steep\ arch$ will lie inside a circle or radius r_-.

Overall, the results in different cases of α give us exactly what Theorem I has stated. We, therefore, have completed proving the theorem.

4 *Exotic Arches*

At the beginning of section 3 where we distinguished between cases according to the signs of α, we mentioned the possibility of *exotic arches*. This type of solutions, however, has not appeared in our analysis so far. In this section, we will examine the potential $V(r) = \ln r$, and see that it can produce *exotic arches*. It is clear that $\ln r$ does not satisfy assumptions **A-IV** and **A-V**; the potential however relates closely to the family $-\frac{1}{r^n}$. Indeed, $\ln r$ and $-\frac{1}{r^n}$ are both Newton potentials (in 2 and $n+2$ dimensions, respectively). The analysis can be done similarly.

The Energy Equation when $V(r) = \ln r$ is:

$$\frac{1}{\alpha} = (\lambda + \ln r)\frac{r^2}{\sqrt{r^2 + \dot{r}^2}}$$

and the Extremum Equation is:

$$\alpha r_0(\lambda + \ln r_0) = 1$$

The necessary condition (which comes from equation (16)) is:

$$|r(\lambda + \ln r)| \geq \frac{1}{|\alpha|}$$

- If α is positive, we have *chains*.

- If α is negative, we have following inequalities:

$$\Rightarrow \; or \begin{cases} r(\lambda + \ln r) \geq \frac{1}{-\alpha} \\ r(\lambda + \ln r) \leq \frac{1}{\alpha} \end{cases}$$

A further study will show that the first inequality gives *flat arches*. However, we are interested in the second inequality at the moment.

Consider the function $r(\lambda + \ln r)$, which equals to 0 at $r = 0$, and which diverges to ∞ as $r \to \infty$. Solving for r^* (where $r(\lambda + \ln r)$ reaches its extremum) we have:

$$\frac{d}{dr} r(\lambda + \ln r) = \lambda + \ln r + 1 = 0$$

$$\Rightarrow r^* = e^{-\lambda - 1}$$

Therefore, $r(\lambda + \ln r)$ has a unique extremum at r^* (see Figure 10).

$$r^*(\lambda + \ln r^*) = -e^{-\lambda-1} = -\frac{1}{e^{\lambda+1}}$$

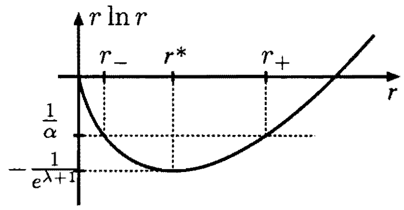

Figure 10: In the potential $V(r) = \ln r$

Apply the second inequality above, we have:

- If $-e^{\lambda+1} < \alpha$, there is no solution.

- If $-e^{\lambda+1} = \alpha$, a solution has both maximum and minimum at $r = r^*$. Hence, solutions are circle of radius r^*, centered at 0.

- If $\alpha < -e^{\lambda+1}$, then a solution has a maximum at some r_+, and a minimum at some r_-. Hence, we obtain an *exotic arch.*

Consider the last case, when $\alpha < -e^{\lambda+1}$. The angle that a solution covers when going from a minimum to the closest maximum reads:

$$\Delta\varphi = \int_{r_-}^{r_+} \frac{1}{r\sqrt{\alpha^2 r^2 (\lambda + \ln r)^2 - 1}} dr \tag{34}$$

Before continuing to analyze the above expression, we consider the following lemma.

Lemma I:

Consider the problem as described in Theorem I. If $r : \varphi \mapsto r(\varphi)$ is a solution of the problem, corresponding to the pair (α, λ), then $\forall\ k \neq 0$, there exists a pair $(\tilde{\alpha}, \tilde{\lambda})$ such that $kr(\varphi)$ satisfies the Euler-Lagrange Equation (11), as well as the Energy Equation (12).

Proof: Let $\tilde{\alpha} = \frac{\alpha}{k}$ and $\tilde{\lambda} = \lambda - \ln k$. □

With Lemma I, it is now without loss of generality that we can assume $\lambda = 0$ in equation (34). The equation then reads:

$$\Delta\varphi = \int_{r_-}^{r_+} \frac{1}{r\sqrt{\alpha^2 r^2 \ln^2 r - 1}} dr \tag{35}$$

The integrand is finite everywhere, except at the limits r_- and r_+. Thus, whether or not the integral is finite depends solely on the behaviors of the integrand near the two limits. We will now prove that the integral always remains finite.

Let $h(r) = \alpha^2 r^2 \ln^2 r - 1$. The Taylor Series for $h(r)$ around r_- is:

$$h(r) = h(r_-) + h'(r)(r - r_-) + O((r - r_-)^2)$$

$$= 0 + \alpha^2 r_- \ln r_-(1 + \ln r_-)(r - r_-) + O((r - r_-)^2)$$

$$= \alpha(1 + \ln r_-)(r - r_-) + O((r - r_-)^2)$$

Since $r_- < r*$ and $\alpha < 0$, the coefficient of the first order term is actually positive. So, when we get sufficiently close to r_-, it will dominate the rest of the expression. Hence, there exists $\epsilon > 0$, such that $\forall r \in (r_-, r_- + \epsilon)$, we have:

$$\frac{1}{2}\alpha(1 + \ln r_-)(r - r_-) < h(r) < 2\alpha(1 + \ln r_-)(r - r_-)$$

$$\Rightarrow \int_{r_-}^{r_-+\epsilon} \frac{1}{r\sqrt{2\alpha(1 + \ln r_-)(r - r_-)}} dr < \int_{r_-}^{r_-+\epsilon} \frac{1}{r\sqrt{\alpha^2 r^2 \ln^2 r - 1}} dr$$

$$< \int_{r_-}^{r_-+\epsilon} \frac{1}{r\sqrt{\frac{1}{2}\alpha(1 + \ln r_-)(r - r_-)}} dr$$

On the other hand,

$$\int_{r_-}^{r_-+\epsilon} \frac{1}{r\sqrt{\alpha(1 + \ln r_-)(r - r_-)}} dr$$

$$= \frac{1}{r_-\sqrt{\alpha(1 + \ln r_-)}} \left[\int_{r_-}^{r_-+\epsilon} \frac{1}{\sqrt{r - r_-}} dr - \int_{r_-}^{r_-+\epsilon} \frac{\sqrt{r - r_-}}{r} dr \right]$$

which is finite. Hence, $\int_{r_-}^{r_-+\epsilon} \frac{1}{r\sqrt{\alpha^2 r^2 \ln^2 r - 1}} dr$ is finite.

Similarly, we can write down the Taylor Series around r_+ for $h(r)$ and show that

$$\int_{r_+-\epsilon'}^{r_+} \frac{1}{r\sqrt{\alpha^2 r^2 \ln^2 r - 1}} dr$$

is finite. Consequently, $\Delta\varphi$ is always finite, regardless of α (and λ).

Furthermore, with the assistance of the software package *Mathematica*, we have estimated equation (35) numerically, with a wide range of α. The preliminary result indicates that $\Delta\varphi$ is bounded between $\frac{\pi}{2}$ and π. Then:

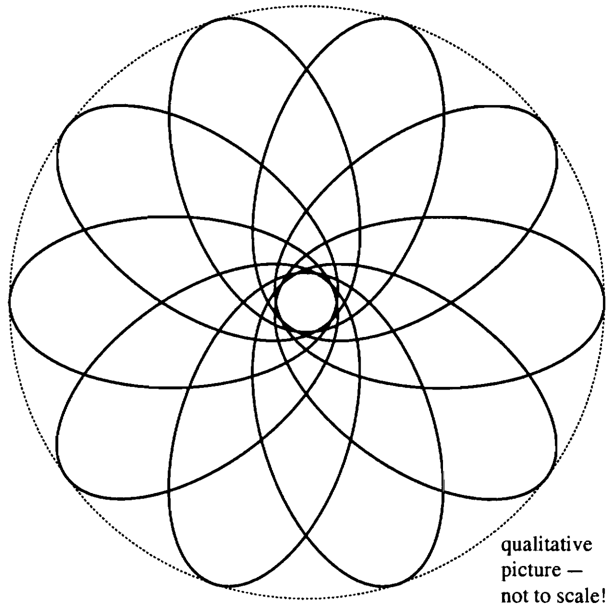

Figure 11: A ten-petal *exotic arch*

- For a value of $\Delta\varphi$ that is an irrational multiple of π, a solution, which lies outside a circle of radius r_- and inside a circle of radius r_+, will wrap around the center infinitely many times.

- Otherwise, let $\Delta\varphi = \frac{p}{q}\pi$, with $(p, q) = 1$. Tracing a solution in this case from its minimum to the next minimum, we get an angle of $2\Delta\varphi$, because we are still in a rotationally symmetric potential. (When going from minimum to minimum, the solution will draw a shape, which will be called a *petal*.) Now, starting at a minimum and following the solution in one direction, either clockwise or counter-clockwise, we then come back to the starting point after covering the smallest angle that is a multiple of 2π. Consequently, a solution will go from minimum to minimum q times, and create exactly q *petals*. The angle it covers is then $2p\pi$, which means it wrap around the center p times. An example is $\Delta\varphi = \frac{9}{10}\pi$, which is depicted in Figure 11.

To sum up, by studying the potential $V(r) = \ln r$, we have showed a possible behavior of solutions, which we did not see in earlier study. Accordingly, we obtain *exotic arches*, which will be bounded from both above and below. Each solution of this type will lie inside a circle of radius r_+, and outside a circle of radius r_-; furthermore has multiple extremums (as opposed to a unique extremum in previous cases).

5 Conclusion and Outlook

In this paper, our main objective was to find qualitative behaviors of a hanging chain in a general rotationally symmetric potential. By first discussing the problem in the regular gravity, $-\frac{1}{r}$ and $-\frac{1}{r^2}$ potentials, we have stated and proved Theorem I, which answers our initial question.

Furthermore, by basing our research on the paper by Denzler and Hinz, we have generalized parts of the method by studying qualitative behaviors of solutions to the Euler-Lagrange equation. The question of uniqueness for the boundary problem with a constrained length, however, was done explicitly for the $\frac{1}{r}$ potential in [1], and the argument there hinged on the explicit calculation. A generalization of this uniqueness problem to more general potentials, using qualitative methods, inspired the present work. In future research, one can study this question, using our paper as a starting base. The understanding of the case diagram should help in looking for any kind of monotonicity results that could be used in such a pursuit.

The assumption **A-V** of Theorem I, in particular, was designed to give us the monotonicity of the expression $\lambda r + rV(r)$, which then ruled out the possibility of *exotic arches*. As a consequence, the case diagram was simple and clear, and was filled with iso-r_0-lines. In the absence of assumption **A-V**, such as when $V(r) = \ln r$, chapter 4 shows that *exotic arches* enter. Mixed monotonicity behaviors of $\lambda r + rV(r)$, as in $V(r) = \ln r - \frac{1}{r^2}$, would make our case distinctions much more complicated.

Now, with a general potential that satisfies all five assumptions of Theorem I, we immediately have a case diagram as in Figure 7. By specifying, e.g., the boundary conditions, one can shift along any corresponding iso-r_0-line and examine the changes in the length of the chain. From there, one can hope to derive a conclusion about the uniqueness for the boundary problem.

We, nonetheless, rest our study at this point.

References

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[3] Stillwell, John. Mathematics and Its History. 2nd. New York: Springer, 2005.