Spring 5-2002

Higher Mathematical Concepts Using the Rubik's Cube

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Appendix E - **UNIVERSITY HONORS PROGRAM**
SENIOR PROJECT - APPROVAL

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College: Arts & Sciences Department: Mathematics

Faculty Mentor: Morwen Thistlethwaite

PROJECT TITLE: Higher Mathematical Concepts Using the Rubik's Cube

I have reviewed this completed senior honors thesis with this student and certify that it is a project commensurate with honors level undergraduate research in this field.

Signed: Morwen Thistlethwaite, Faculty Mentor

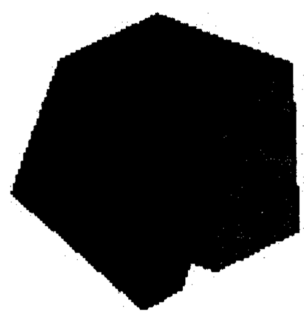
Date: 5/1/02

General Assessment - please provide a short paragraph that highlights the most significant features of the project.

Comments (Optional):

University of Tennessee Honors Department

Senior Project: Higher Mathematical Concepts using the Rubik's Cube



By Pawel Nazarewicz

Spring Semester 2002

With the special help of Dr. Morwen Thistlethwaite

Abstract

Special thanks needs to go to my mentor for all his help:
Senior Project Mentor: Dr. Morwen Thistlethwaite

Purpose: The main intent behind this is to show high school students a preview of higher mathematics in ways that they can relate to and understand. The focus will be examples instead of proofs, to give the students a conceptual idea of the subjects covered. Proofs will be minimized at this level due to the restrictions of time and the understanding of students.

Format: A PDF formatted syllabus that is divided into a five day, 50 minutes-per-day teaching aides for high school teachers.

Focus: Many higher mathematical concepts can be shown using the Rubik's Cube and the lesson outline will be as follows:

Lesson 1: The Basics. *This section introduces the students with they layout of the cube, structures how to move around, and hopefully starts to dispel some of the mystery associated with the Cube. This is a "get you feet wet" section.*

Lessons 2 and 3: Permutations. *This lesson will focus on defining Permutations, Identities, Inverses, and the Order of an Element. The focus will be understanding permutations and why they are important.*

Lesson 4 and 5: Groups. *Here we will define four laws of a group and show that the Rubik's Cube is indeed a group that satisfies the definition. We will also look at other examples of groups and subgroups. Other concepts that will be covered will be Generators, Orders of a Group, and Caley Tables.*

Chapter Outline

Terminology

The first thing we have to establish is a common notation so we can standardize everything we are talking about.. This is the standard way of holding the cube, showing the maximum number of faces.

1. Basic Layout of the Cube

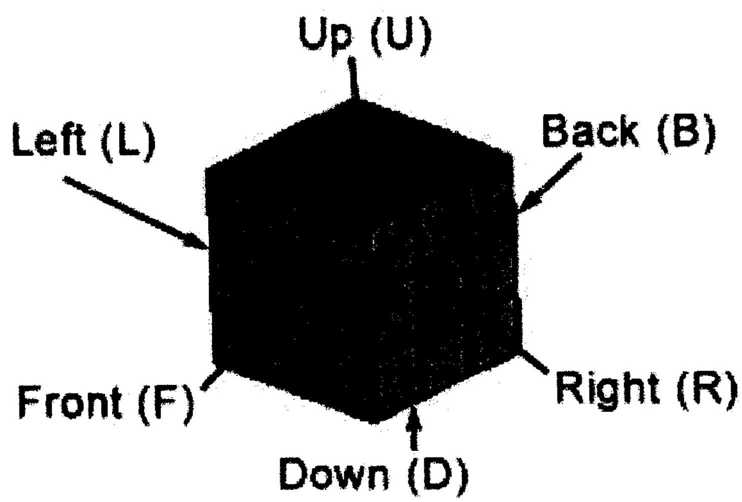


Figure 1.1

- There are six faces to the cube labeled by their position - Up, Down, Left, Right, Front, Back. We use that notation instead of colors to allow for different colored cubes. The default position for the Cube is to show the Front, Up, and Right faces.
- Each face is divided into nine facellets, so there are 6 x 9 = 54 facellets.
- Each facellet is either part of an edge piece, corner piece, or middle piece.
- There are 12 edge pieces - a piece with two facellets.

  - Each edge piece can be labeled using it's position. For example, 'rf' means the edge piece is the RIGHT-FRONT piece - as seen in Figure 1.1.
- There are 8 corner pieces - a piece with three facellets.

  - As with edge pieces, each corner piece can be labeled as well. The RIGHT-FRONT-UP piece can also be seen in Figure 1.1.
- There are 6 middle pieces - a piece with one facellet.

2. Moving around

Now that we have the layout taken care of, we have to worry about how to make sure our moves are standardized. Here, we will use the following notations:

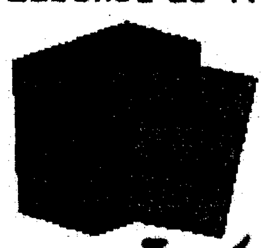
- R, F, U, D, L, B - these represent the face that you want to turn <u>clockwise</u>. As seen in Figure 2.1, if one was given the instruction to perform 'R' to the cube, the result would be a clockwise rotation of the Right Face by 90°.

- Since we done 'R' as a move clockwise, 'R^{-1}' is a logical notation for a counterclockwise move of the Right Face. This is seen in Figure 2.2.

- If we want to rotate a face of the cube by 180°, this requires two rotations of 90° each, so we call this 'R^2' - or R * R. Notice that 'R^2' = 'R^{-2}'

- Finally, we see that 'R^3' = 'R^{-1}', so it would be redundant to have it. Also, 'R^4' is simply 1 (which we call the ***identity*** - but more on that in the chapter on "Orders of an Element")

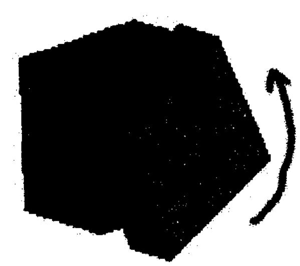
A Right Twist
Labelled as 'R'



The move is done clockwise.

Figure 2.1

A Right Twist
Labelled as 'R^{-1}'



The move is done counter-clockwise.

Figure 2.2

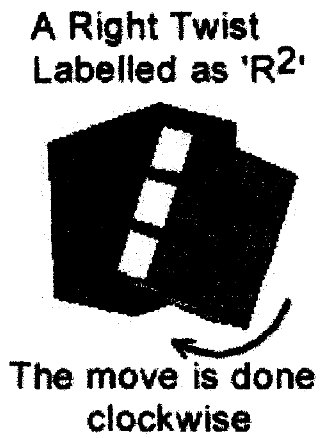


Figure 2.3

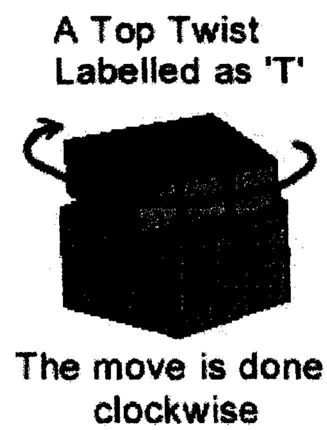


Figure 2.4

So now let's apply all this in the following examples:

Example 2.1: Perform the following operation, and remember to be very careful - it's very easy to mess up the cube. See if your output looks like ours:

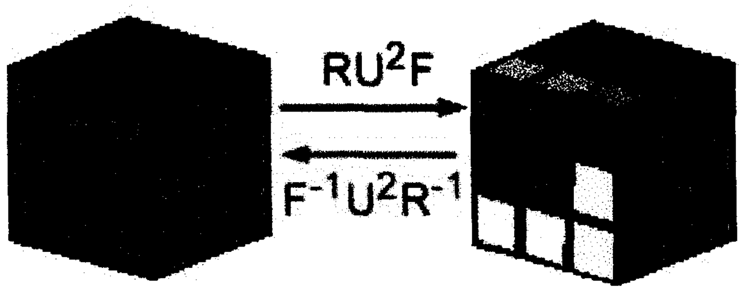


Figure 2.5

On average, any cube can be restored back to it's original state in 19 moves - in optimal scenarios - usually solved by computer. One computer program that will spit out a sequence for you is Herbert Kociemba's The Cube Explorer.

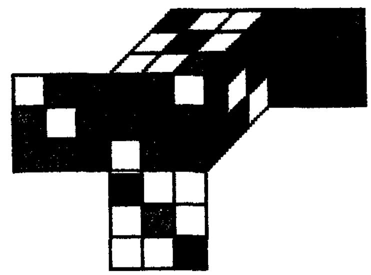


Figure 2.6

Once you program the layout of your cube into it (Figure 2.6), it will spit out a solution that looks a bit like this:

G1 = <F2 D' R2 D' L' U' L' R B D' U B L F2 L U2>

G1 is a ***generator*** of that sequence - more on generators later. For now, if you have a clean cube, try to perform G1 to it. Keep in mind that in Kociemba's notation, $D^{-1} = D'$.

3. Wrapping Up, Looking Ahead.

Now that you have the basics in place, the tools in your hand, we are ready to move forward and try to understand the Cube better. While it's certainly a very mysterious object to the majority of people, it will serve us to understand some higher mathematical concepts, and hopefully further your own understanding and appreciation for the power that mathematics can provide you with.

We will figuratively dissect the cube and try to control it's movements - nothing we do will be random - that would only get us in trouble. Attached, there is a **Definitions** chapter that should be used as a reference throughout.

You're now ready to move on and learn about permutations - our **Insight # 1** to the workings of the Cube.

Insight # 1: Permutations

There is a difference is being able to solve a cube fast and being able to solve it using the fewest moves possible. In order to solve it with few moves, one needs to understand what process moves certain pieces in certain positions. Any move that rearranges the **facellets** is called a ***permutation***.

4. The Basics

The best way to illustrate this is by an example:

Example 4.1: *What is the Permutation for 'R'?*

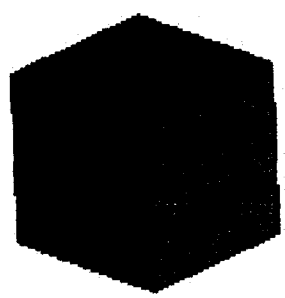


Figure 4.1

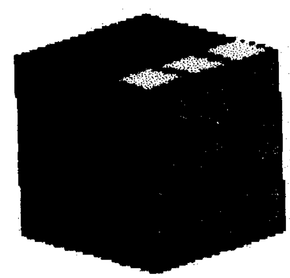


Figure 4.2

Here we apply a 'R^{-1}' move to our home position. Notice that eight pieces are moved out of place. The rotation of the edge pieces can be denoted in the following manner:

(ru) → (rf)
(rf) → (rd)
(rd) → (rb)
(rb) → (ru)

... and the corner pieces are shifted as follows:

(ruf) → (rdf)
(rdf) → (rdb)
(rdb) → (rub)
(rub) → (ruf)

But that's a lot of writing. As you can see, one element points to another, which points to yet another, and eventually it returns to the first element. We can write this in a more concise form.

The shorthand notation of the permutation (for the edges) would look like this:

(ru, rf, rd, rb)

This is read as such: *'ru' goes to 'rf' which goes to 'rd' which goes to 'rb' which goes back to 'ru' and we are done.*

It doesn't matter where you start out with in writing your permutation. Since we have a finite number of facellets in the cube, all our permutations are ***cyclic*** - the cycle return to the original piece to its original position.

So let's take this a step further with the next example:

Example 4.2 - *What is the permutation of the 'RF^{-1}' move?*

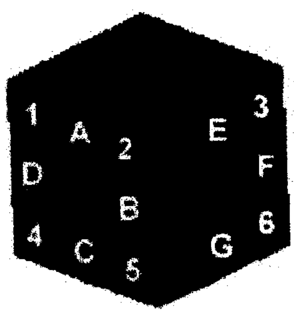


Figure 4.3

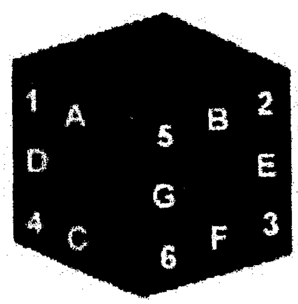


Figure 4.4

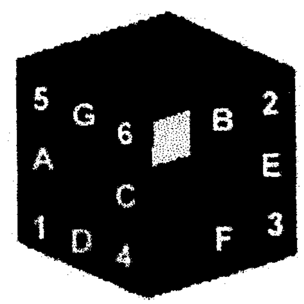


Figure 4.5

Let's start with the edge pieces, and it doesn't matter which one we start with (so we will go alphabetically). Also, let's simplify our notation as shown in Figure 4.3 by using numbers to indicate our corner pieces and letters to indicate our edge pieces. Go ahead and write out the permutation for the 'R' move:

- Edge Permutation for 'R': (B, E, F, G)

- Edge Permutation for 'F^{-1}': (A, D, C, B)

And notice what happens during the overall move:

- Edge Permutation for 'RF^{-1}': (A, D, C, B, E, F, G)

- Corner Permutation for 'RF^{-1}': (1, 4, 5), (2, 3, 6) - and here we have two disjoint sets of corner permutations.

5. Identities and Inverses

As we have shown, getting from one stage of the cube to another (a permutation) can be accomplished by a series of moves. The hard part is getting back to the original position. That's where ***inverses*** come into play.

An **inverse** undoes a move - it restores the current permutation of the cube to the original - the ***identity***. For our purposes, the ***identity*** is the solved cube. Let's look at a few examples:

Example 5.1: *What is the inverse of 'F'? 'FR'? 'F^2R^2'?*

- This one is fairly simple - let's start with 'F'. Once we perform 'F' to the cube, all we need to do is apply 'F^{-1}' to it and we are back to our original position.
- Now let's look at 'FR'. The obvious answer would be to do '$F^{-1}R^{-1}$' ... but look at figure 5.1 and notice that it's not what we want. '$FRF^{-1}R^{-1}$' does not give us the identity back.

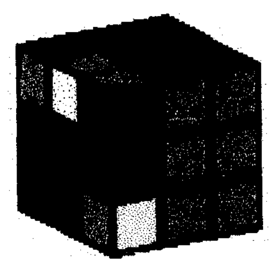


Figure 5.1

The question now becomes why that is so. Let us look at that concept in a little more detail. Here is the natural assumption:

$$(FR)^{-1} = F^{-1} R^{-1} \text{ (This is \underline{wrong}!)}$$

In his book about the cube, Alexander Frey provides a clever analogy to inverses - they work like socks and shoes. Think of 'F' as putting on your socks and 'R' as putting on your shoes.

- 'FR' would be putting on your socks and then putting on your shoes.
- '$F^{-1}R^{-1}$' would be taking off your socks and then taking off your shoes. It's evident that you can't take your socks off while your shoes are still on - try it!

Therefore, the inverse of 'Socks on, Shoes on' would have to be 'Shoes off, Socks off' - or '$R^{-1} F^{-1}$'. Mathematically, this makes sense:

$$(FR) * (R^{-1}F^{-1}) = F (R * R^{-1}) * F^{-1} = F * (1) * F^{-1} = F * F^{-1} = 1$$

So, you have to work backwards when does inverses - something that might not seem so obvious initially, since when dealing with arithmetic and most of algebra, we aren't taught to work with inverses like that.

- So let's look at the last one - F^2R^2. Working backwards, we need to take care of the R^2 first - its inverse is R^2 (since $R^2 * R^2 = R^4 = 1$). Next, we get rid of the F^2 with another F^2, and the final inverse is R^2F^2. As you can see, inverses are pretty easy.

So are inverses unique? Can I apply two different moves to 'R' to get the identity back? The answer are no, inverses are not unique. Name two different moves one can apply to 'R' to get the identity back.

6. Orders of a Permutation

When you apply the 'R' move to the cube four times, you get the identity. Can the same be true for multiple moves? If you apply them enough times, will you get the identity back? Try to see how many times you need to do 'F^2R^2' to get the identity again.

Let's apply some shorthand and let 'F^2R^2' = X.

Observe that X^3 gives changes the position of four pieces - names it gives us the following permutation (as illustrated in Figure 6.1):

(uf, df), (ur, dr)

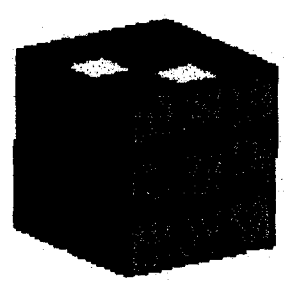


Figure 6.1

So this left most of the cube unchanged - it is a very powerful technique to be able to change a select few things when dealing with the cube.

But getting back to the problem of X. Notice that $X^6 = 1$. This is called the ***order of X***. The **order** of an element is the number of times it must be repeated before you arrive back at the identity.

Example 6.1: *What is the order of '$FRF^{-1}R^{-1}$'?*

One way to do this would be to manually repeat the sequence until we arrive at the identity and count the number of times we made the move. This is lengthy and silly, and can be done with much greater ease using permutations.

For simplicity, let's call $'FRF^{-1}R^{-1}' = Y$, We will use the labels from Figure 4.3. Figure 5.1 shows the permutation. Let's write it out:

- First, the <u>Edge Permutation</u>: (A, B, G) (C) (D) (E) (F)

Notice that C, D, E, and F all remain unmoved and untwisted. We can simplify this notation to (A, B, G) - the rest is understood.

- Now, let's look at the <u>Corner Permutation</u>: (1, 2) (3) (4) (5, 6)

So the <u>Overall Permutation</u> is (A, B, G) (1, 2) (5, 6).

Before we mathematically blow this problem out of the water, let's notice what happens at subsequent powers of Y.

$\mathbf{Y^2}$	The corners are in their original positions, but they are twisted
$\mathbf{Y^3}$	The edge pieces are back to their original position - and are untwisted.
$\mathbf{Y^4}$	Corners back (twisted), edges just like in Y.
$\mathbf{Y^5}$	Corners <u>AND</u> edges are out of position.
$\mathbf{Y^6}$	Perfecto! We are back to the original.

So what is happening here? The corners are on a 6-cycle (due to the twist), and the edges on a 3-cycle. We are lucky that six is divisible by three and it all works out. Other problems aren't that nice.

Example 6.2: *What is the permutation and order of 'FR'?*

Now we are going to do this strictly mathematically. First, let's look at the permutation of 'FR' as seen in Figure 6.2:

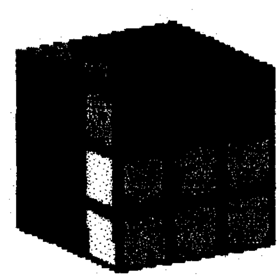


Figure 6.2

Now, let's look at the resulting permutation:

9. <u>Edges</u>: (A, E, F, G, B, C, D)

- Corners: (1, 3, 6, 5, 4)

So it is going to take 7 applications of 'FR' to get back to our identity with the edges, but 5 to get back with the edges. The **Lowest Common Multiple** of 5 and 7 is 35. At 35, we are all back in place ... but twisted!

So it's going to take 105 applications of 'FR' to get back to the identity.

7. A few words about Transpositions ...

A ***transposition*** is simply a permutation that changes the location of two elements. Any permutation can be written as a sum of transpositions.

- If a permutation ends up being a product of an even number of transpositions, it is called an ***even*** permutation.
- If it's a product of an odd number of transpositions, it's called an ***odd*** permutation.

Any move with the cube results in an **even** permutation. We're not going to spend a lot of time on this, but it is important to note it for computing the size of the Cube Group later on.

8. Conjugations

Now we can look at what conjugations do and what makes them so powerful. Perform the following permutation on the cube:

$P = < B^{-1}\, U^2\, B^2\, U\, B^{-1}\, U^{-1}\, B^{-1}\, U^2\, F\, R\, B\, R^{-1}\, F\text{-}1 >$

Notice that you get the following permutation:

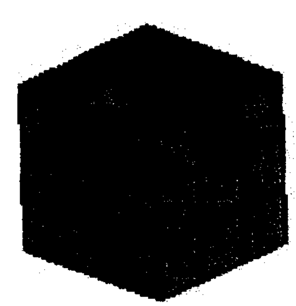


Figure 8.1

As you can see, the sequence changes the position of two adjacent edge pieces (with a small rotation). The beauty of ***conjugations*** is that once you know how to do that, you can change the position (and rotation) of just about any two edge pieces. Can you see how that can be done?

We will use Figure 4.3 for reference. Suppose you want to flip the edges labeled 'A' and 'G'. We know that our **generator** <P> (from above) changes the position of 'A' and 'E'. Therefore, if we place 'G' in 'E's spot, we're set. It's going to take a 'R^2' move to get that done though ... and another 'R^2' move at the end to restore everything back to normal. Try it:

$$Q = < F^2\ P\ F^2 > = < \mathbf{F^2}\ B^{-1}\ U^2\ B^2\ U\ B^{-1}\ U^{-1}\ B^{-1}\ U^2\ F\ R\ B\ R^{-1}\ F\text{-}1\ \mathbf{F^2} >$$

Can you see how you could change the location of any two given edge pieces using the method? Also, if you find a **generator** that changes any other pieces (and <u>only</u> those pieces), how that would prove useful in controlling the cube?

9. Wrapping up, Looking ahead

Knowledge of permutations gives us incredible power - as illustrated in Example 6.2. Understanding their behavior allows us to predict what is going to happen as we perform certain moves - it gives us foresight and structure - something not commonly associated with the Cube.

Insight # 2 to the cube will be groups. We already started hinted at what a group might be, but we will outline a formal definition in the next section. Groups will put even more structure on how the cube behaves, and it will become even less mysterious.