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Morphisms and Order Ideals of Toric Posets

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Abstract. Toric posets are in some sense a natural "cyclic" version of finite posets in that they capture the fundamental features of a partial order but without the notion of minimal or maximal elements. They can be thought of combinatorially as equivalence classes of acyclic orientations under the equivalence relation generated by converting sources into sinks, or geometrically as chambers of toric graphic hyperplane arrangements. In this paper, we define toric intervals and toric order-preserving maps, which lead to toric analogues of poset morphisms and order ideals. We develop this theory, discuss some fundamental differences between the toric and ordinary cases, and outline some areas for future research. Additionally, we provide a connection to cyclic reducibility and conjugacy in Coxeter groups.

1. Introduction

A finite poset can be described by at least one directed acyclic graph where the elements are vertices and directed edges encode relations. We say "at least one" because edges implied by transitivity may be present or absent. The operation of converting a source vertex into a sink generates an equivalence relation on finite posets over a fixed graph. Equivalence classes are called *toric posets*. These objects have arisen in a variety of contexts in the literature, including but not limited to chip-firing games [Eri94], Coxeter groups [EE09, Shi01, Spe09], graph polynomials [Che10], lattices [ES09, Pro93], and quiver representations [MRZ03]. These equivalence classes were first formalized as toric posets in [DMR15], where the effort was made to develop a theory of these objects in conjunction with the existing theory of ordinary posets. The name "toric poset" is motivated by a bijection between toric posets over a fixed (undirected) graph and chambers of the toric graphic hyperplane arrangement of that graph. This is an analogue to the well-known bijection between ordinary posets over a fixed graph G and chambers of the graphic hyperplane arrangement of G, first observed by Greene [Gre77] and later extended to signed graphs by Zaslavsky [Zas91].

Combinatorially, a poset over a graph G is determined by an *acyclic orientation* ω of G. We denote the resulting poset by $P(G,\omega)$. A toric poset over G is determined by an acyclic orientation, up to the *equivalence* generated by converting sources into sinks. We denote this by $P(G,[\omega])$. Though most standard features of posets have elegant geometric interpretations, this viewpoint is usually unnecessary. In contrast, for most features of toric posets, i.e., the toric analogues of standard posets features, the geometric viewpoint is needed to see the natural proper definitions and to prove structure theorems. Once this is done, the definitions and characterizations frequently have simple combinatorial (non-geometric) interpretations.

To motivate our affinity for the geometric approach, consider one of the fundamental hallmarks of an ordinary poset P: its binary relation, $<_P$. Most of the classic features of posets (chains, transitivity, morphisms, order ideals, etc.) are defined in terms of this relation. Toric posets have no such binary relation, and so this is why we *need* to go to the geometric setting to define the basic features. Perhaps surprisingly, much of the theory of posets carries over to the toric setting despite the absence of a relation, and current toric poset research strives to understand just how much and what does carry over. As an analogy from a different area of mathematics, topology can be thought of as "analysis without the metric." A fundamental hallmark of a metric space is its distance function. Many of the classic features of metric spaces, such as open, closed, and

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compact sets, and continuous functions, are defined using the distance function. However, once one establishes an equivalent characterization of continuity in terms of the inverse image of open sets, many results can be proven in two distinct ways: via epsilon-delta proofs, or topologically. When one moves from metric spaces to topological spaces, one loses the distance function and all of the tools associated with it, so this first approach goes out the window. Remarkably, much of the theory of real analysis carries over from metric spaces to topological spaces. Back to the poset world, one can prove theorems of ordinary posets using either the binary relation or the geometric definitions. However, upon passing to toric posets, the binary relation and all of the tools associated with it are lost, so one is forced to go to the geometric setting. Remarkably, much of the theory of ordinary posets still carries over to toric posets. This analogy is not perfect, because toric posets are not a generalization of ordinary posets like how topological spaces extend metric spaces. However, it should motivate the reliance on geometric methods throughout this paper.

An emerging theme of toric poset structure theorems, from both the original paper [DMR15] and this one, is that characterizations of toric analogues, when they exist, usually have one of two forms. In one, the feature of a toric poset $P(G,[\omega])$ is characterized by it being the analogous feature of the ordinary poset $P(G,\omega')$, *for all* $\omega' \in [\omega]$. In the other, the feature of $P(G,[\omega])$ is characterized by it being the analogous feature of $P(G,\omega')$ *for some* $\omega' \in [\omega]$. Several examples of this are given below. It is not obvious why this should happen or which type of characterization a given toric analogue should have *a priori*. Most of these are results from this paper, and so this list provides a good overview for what is to come.

The "*For All*" structure theorems:

- A set $C \subseteq V$ is a toric chain of $P(G,[\omega])$ iff C is a chain of $P(G,\omega')$ for all $\omega' \in [\omega]$. (Proposition 5.3)
- The edge $\{i,j\}$ is in the toric transitive closure of $P(G,[\omega])$ iff $\{i,j\}$ is in the transitive closure of $P(G,\omega')$ for all $\omega' \in [\omega]$. (Proposition 5.15)

The "*For Some*" structure theorems:

- A partition $\pi \in \Pi_V$ is a closed toric face partition of $P(G,[\omega])$ iff π is a closed face partition of $P(G,\omega')$ for some $\omega' \in [\omega]$. (Theorem 4.7)
- A set $A \subseteq V$ is a (geometric) toric antichain of $P(G,[\omega])$ iff A is an antichain of $P(G,\omega')$ for some $\omega' \in [\omega]$. (Proposition 5.17)
- If a set $I \subseteq V$ is a toric interval of $P(G,[\omega])$, then I is an interval of $P(G,\omega')$ for some $\omega' \in [\omega]$. (Proposition 5.14)
- A set $J \subseteq V$ is a toric order ideal of $P(G,[\omega])$ iff J is an order ideal of $P(G,\omega')$ for some $\omega' \in [\omega]$. (Proposition 7.3)
- Collapsing $P(G,[\omega])$ by a partition $\pi \in \Pi_V$ is a morphism of toric posets iff collapsing $P(G,\omega')$ by π is a poset morphism for some $\omega' \in [\omega]$. (Corollary 6.2)
- If an edge $\{i,j\}$ is in the Hasse diagram of $P(G,\omega')$ for some $\omega' \in [\omega]$, then it is in the toric Hasse diagram of $P(G,[\omega])$. (Proposition 5.15)

This paper is organized as follows. In the next section, we formally define posets and preposets and review how to view them geometrically in terms of faces of chambers of graphic hyperplane arrangements. In Section 3, we translate well-known properties of poset morphisms to this geometric setting. In Section 4, we define toric posets and preposets geometrically in terms of faces of chambers of toric hyperplane arrangements, and we study the corresponding "toric face partitions" and the bijection between toric preposets and lower-dimensional faces. In Section 5, we define the notion of a toric interval and review some features of toric posets needed for toric order-preserving maps, or morphisms, which are finally presented in Section 6. In Section 7, we introduce toric order ideals and filters, which are essentially the preimage of one element upon mapping into a two-element toric poset. The toric order ideals and filters of a toric poset turn out to coincide. They form a graded poset $J_{\mathrm{tor}}(P)$, but unlike the ordinary case, this need not be a lattice. In Section 8, we provide a connection of this theory to Coxeter groups, and then we conclude with a summary and discussion of current and future research in Section 9.

2. Posets geometrically

2.1. **Posets and preposets.** A *binary relation* R on a set V is a subset $R \subseteq V \times V$. A *preorder* or *preposet* is a binary relation that is reflexive and transitive. This means that $(x,x) \in R$ for all $x \in V$, and if $(x,y),(y,z) \in R$, then $(x,z) \in R$. We will use the notation $x \preceq_R y$ instead of (x,y), and say that $x \prec_R y$ if $x \preceq_R y$ and $y \not\preceq_R x$. Much of the basics on preposets can be found in [PRW08].

An *equivalence relation* is a preposet whose binary relation is symmetric. For any preposet P, we can define an equivalence relation $\sim_P$ on P by saying $x \sim_P y$ if and only if $x \preceq_P y$ and $y \preceq_P x$ both hold. A *partially ordered set*, or *poset*, is a preposet P such that every $\sim_P$-class has size 1. We say that a preposet is *acyclic* if it is also a poset.

Every preorder P over V determines a directed graph $\omega(P)$ over V that contains an edge $i \to j$ if and only if $i \preceq_P j$ and $i \neq j$. Not every directed graph arises from such a preorder, since edge transitivity is required. That is, if $i \to j$ and $j \to k$ are edges, then $i \to k$ must also be an edge. However, any directed graph can be completed to a transitive graph via transitive closure, which adds in all "missing" edges. Note that the graph $\omega(P)$ is acyclic if and only if P is a poset. The strongly connected components are the $\sim_P$-classes, and so the quotient $\omega(P)/\!\sim_P$ is acyclic and $P/\!\sim_P$ inherits a natural poset structure from P.

If R_1 and R_2 are preorders on V, then we can define their union $R_1 \cup R_2$ as the union of the subsets R_1 and R_2 of $V \times V$. This need not be a preorder, but its *transitive closure* $\overline{R_1 \cup R_2}$ will be.

Another way we can create a new preorder from an old one is by an operation called contraction. Given a binary relation $R \subseteq V \times V$, let R^{op} denote the *opposite* binary relation, meaning that $(x,y) \in R^{op}$ if and only if $(y,x) \in R$. If P and Q are preposets on V, then Q is a *contraction* of P if there is a binary relation $R \subseteq P$ such that $Q = \overline{P \cup R^{op}}$. Intuitively, each added edge $(x,y) \in R^{op}$ forces $x \sim_Q y$ because $(y,x) \in P \subseteq Q$ by construction. Note that in this context, contraction is a *different* concept than what it often means in graph theory – modding out by a subset of vertices, or "collapsing" a set of vertices into a single vertex.

2.2. **Chambers of hyperplane arrangements.** It is well known that every finite poset corresponds to a chamber of a graphic hyperplane arrangement [Wac07]. This correspondence will be reviewed here. Let P be a poset over a finite set $V = [n] := \{1,\dots,n\}$. This poset can be identified with the following open polyhedral cone in $\mathbb{R}^V$:

$$c = c(P) := \{x \in \mathbb{R}^V : x_i < x_j \text{ if } i <_P j\}. \tag{1}$$

It is easy to see how the cone c determines the poset $P = P(c)$: one has $i <_P j$ if and only if $x_i < x_j$ for all x in c. Each such cone c is a connected component of the complement of the graphic hyperplane arrangement for at least one graph $G = (V,E)$. In this case, we say that P is a *poset over* G (or "on G"; both are used interchangeably). Given distinct vertices i,j of a simple graph G, the hyperplane H_{ij} is the set

$$H_{ij} := \{x \in \mathbb{R}^V : x_i = x_j\}.$$

The *graphic arrangement* of G is the set $\mathcal{A}(G)$ of all hyperplanes H_{ij} in $\mathbb{R}^V$ where $\{i,j\}$ is in E. Under a slight abuse of notation, at times it is convenient to refer to $\mathcal{A}(G)$ as the set of points in $\mathbb{R}^V$ on the hyperplanes, as opposed to the actual finite set of hyperplanes themselves. It should always be clear from the context which is which.

Each point $x = (x_1,\dots,x_n)$ in the *complement* $\mathbb{R}^V - \mathcal{A}(G)$ determines an *acyclic orientation* $\omega(x)$ of the edge set E: direct the edge $\{i,j\}$ in E as $i \to j$ if and only if $x_i < x_j$. Clearly, the fibers of the mapping $\alpha_G : x \longmapsto \omega(x)$ are the *chambers* of the hyperplane arrangement $\mathcal{A}(G)$. Thus, α_G induces a bijection between the set $\mathrm{Acyc}(G)$ of acyclic orientations of G and the set

$\mathrm{Cham}\,\mathcal{A}(G)$ of chambers of $\mathcal{A}(G)$:

$$\begin{tikzcd} \mathbb{R}^V - \mathcal{A}(G) \arrow[rr, "\alpha_G"] \arrow[dr] & & \mathrm{Acyc}(G) \\ & \mathrm{Cham}\,\mathcal{A}(G) \arrow[ur, dotted] & \end{tikzcd} \tag{2}$$

We denote the poset arising from an acyclic orientation $\omega \in \mathrm{Acyc}(G)$ by $P = P(G, \omega)$. An open cone $c = c(P)$ may be a chamber in several graphic arrangements, because adding or removing edges implied by transitivity does not change the poset. Geometrically, the hyperplanes corresponding to these edges do not cut c, though they intersect its boundary. Thus, there are, in general, many pairs (G, ω) of a graph G and acyclic orientation ω that lead to the same poset $P = P(G, \omega) = P(c)$. Fortunately, this ambiguity is not too bad, in that with respect to inclusion of edge sets, there is a unique minimal graph $\hat{G}^{\mathrm{Hasse}}(P)$ called the *Hasse diagram* of P and a unique maximal graph $\bar{G}(P)$, where $(G, \omega) \longmapsto (\bar{G}(P), \bar{\omega})$ is *transitive closure*.

Given two posets P, P' on a set V, one says that P' is an *extension* of P when $i <_P j$ implies $i <_{P'} j$. Geometrically, P' is an extension of P if and only if $c(P') \subseteq c(P)$. Moreover, P' is a *linear extension* if $c(P')$ is a chamber of $\mathcal{A}(K_V)$, where K_V is the complete graph.

2.3. **Face structure of chambers.** Let $\pi = \{B_1, \ldots, B_r\}$ be a partition of V into nonempty blocks. The set Π_V of all such partitions has a natural poset structure: $\pi \leq_V \pi'$ if every block in π is contained in some block in π'. When this happens, we say that π is *finer* than π', or that π' is *coarser* than π.

Intersections of hyperplanes in $\mathcal{A}(G)$ are called *flats*, and the set of flats is a lattice, denoted $L(\mathcal{A}(G))$. Flats are partially ordered by *reverse inclusion*: If $X_1 \subseteq X_2$, then $X_2 \leq_L X_1$. Every flat of $\mathcal{A}(G)$ has the form

$$D_\pi := \{x \in \mathbb{R}^V : \ x_i = x_j \text{ for every pair } i, j \text{ in the same block } B_k \text{ of } \pi\},$$

for at least one partition π of V. Note that $D_\pi \leq_L D_{\pi'}$ if and only if $\pi \leq_V \pi'$; this should motivate the convention of partially ordering $L(\mathcal{A}(G))$ by reverse inclusion.

Given a poset $P = P(G, \omega)$ over $G = (V, E)$, a partition π of V defines a preposet $P_\pi := (\pi, \leq_P)$ on the blocks, where $B_i \leq_P B_j$ for $x \leq_P y$ for some $x \in B_i$ and $y \in B_j$ (and taking the transitive closure). This defines a directed graph $\omega/{\sim_\pi}$, formed by collapsing out each block B_i into a single vertex. Depending on the context, we may use P_π or $\omega/{\sim_\pi}$ interchangeably. If this preposet is acyclic (i.e., if P_π is a poset, or equivalently, the directed graph $\omega/{\sim_\pi}$ is acyclic), then we say that π is *compatible* with P. In this case, there is a canonical surjective poset morphism $P \to P_\pi$. We call such a morphism a *quotient*, as to distinguish it from inclusions and extensions which are inherently different.

Compatibility of partitions with respect to a poset can be characterized by a closure operator on Π_V. If $P_\pi = (\pi, \leq_P)$ is a preposet that is not acyclic, then there is a unique minimal coarsening $\mathrm{cl}_P(\pi)$ of π such that the contraction $(\mathrm{cl}_P(\pi), \leq_P)$ is acyclic. This is the partition achieved by merging all pairs of blocks B_i and B_j such that $B_i \sim_P B_j$, and we call it the *closure* of π with respect to P. If P is understood, then we may write this as simply $\bar{\pi} := \mathrm{cl}_P(\pi)$. A partition π is *closed* (with respect to P) if $\bar{\pi} = \pi$, which is equivalent to being compatible with respect to P. Geometrically, it means that for any $i \neq j$, there is some $x \in \overline{c(P)} \cap D_\pi$ such that $x_{b_i} \neq x_{b_j}$ for some $b_i \in B_i$ and $b_j \in B_j$. If π is not closed, then the polyhedral face $\overline{c(P)} \cap D_\pi$ has strictly lower dimension than D_π. In this case, $\bar{\pi}$ is the unique coarsening that is closed with respect to P and satisfies $\overline{c(P)} \cap D_{\bar{\pi}} = \overline{c(P)} \cap D_\pi$.

Still assuming that P is a poset over $G = (V, E)$, and π is a partition of V, define

$$\bar{F}_\pi(P) := \overline{c(P)} \cap D_\pi \,. \tag{3}$$

If D_π is a flat of $\mathcal{A}(\bar{G}(P))$, then $\bar{F}_\pi(P)$ is a face of the (topologically) closed polyhedral cone $\overline{c(P)}$. In this case, we say that π is a *face partition* of P. Since it is almost always clear what P is, we will usually write $\bar{F}_\pi$ instead of $\bar{F}_\pi(P)$. If D_π is not a flat of $\mathcal{A}(G)$, then the subspace D_π still

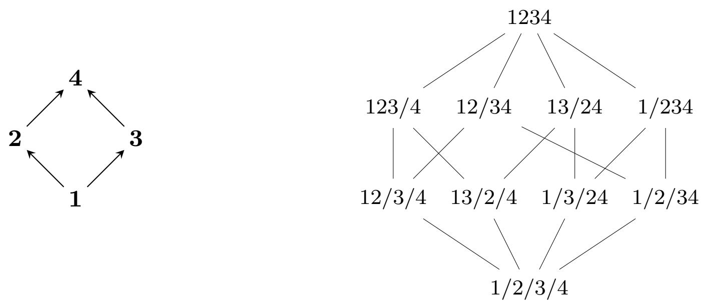


FIGURE 1. A poset P and its lattice of closed face partitions.

intersects $\overline{c(P)}$ in at least the line $x_1 = \cdots = x_n$. Though this may intersect in the interior of $c(P')$, it is a face of $\overline{c(P')} \subseteq \overline{c(P)}$, for at least one extension P' of P.

To characterize the facial structure of the cone $\overline{c(P)}$, it suffices to characterize the closed face partitions. This is well known – it was first described by Geissinger [Gei81], and also done by Stanley [Sta86] in the characterization of the face structure of the *order polytope* of a poset, defined by

$$\mathcal{O}(P) = \{x \in [0,1]^V : x_i \leq x_j \text{ if } i \leq_P j\}\,. \tag{4}$$

Clearly, if π is a closed face partition of P, then the subposets induced by the individual blocks are connected (that is, their Hasse diagrams are connected). We call such a partition *connected*, with respect to P.

Theorem 2.1. *([Sta86], Theorem 1.2) Let P be a poset over $G = (V, E)$. A partition π of V is a closed face partition of P if and only if it is connected and compatible with P.*

To summarize Theorem 2.1, characterizing the faces of $\mathcal{O}(P)$ amounts to characterizing which face partitions π are closed. If π is not compatible with P, then it is not closed. On the other hand, if π not connected, then it is either not closed, or the flat D_π cuts through the interior of $\mathcal{O}(P)$ (and hence of $c(P)$), in which case π is not a face partition.

Example 2.2. Let P be the poset shown at left in Figure 1; its lattice of closed face partitions is shown at right. In this and in later examples, we denote the blocks of a partition using dividers rather than set braces, e.g., $\pi = B_1/B_2/\cdots/B_r$.

The partition $\sigma = 1/23/4$ is closed but not connected; it is not a face partition because $D_\sigma = H_{23}$ intersects the interior of $\overline{c(P)}$. The partition $\pi = 124/3$ is connected but not closed. Finally, the partition $\pi' = 14/23$ is neither connected nor closed. However, both π and π' are face partitions because the subspaces D_π and $D_{\pi'}$ intersect $\overline{c(P)}$ in the line $x_1 = x_2 = x_3 = x_4$, which is the flat D_V. Therefore, both of these partitions have the same closure: $\mathrm{cl}_P(\pi) = \mathrm{cl}_P(\pi') = 1234$.

If $\bar{\pi} = \pi = \{B_1, \ldots, B_r\}$ is a closed face partition of P, then D_π is an r-dimensional flat of $\mathcal{A}(G)$, and the closed face $\bar{F}_\pi$ is an r-dimensional subset of $D_\pi \subseteq \mathbb{R}^V$. The interior of $\bar{F}_\pi$ with respect to the subspace topology of D_π will be called an open face. So as to avoid confusion between open and closed faces, and open and closed chambers, we will speak of faces as being features of the actual poset, not of the chambers. It should be easy to relate these definitions back to the chambers if one so desires.

Definition 2.3. A set $F \subseteq \mathbb{R}^V$ is a *closed face* of the poset P if $F = \bar{F}_\pi = \overline{c(P)} \cap D_\pi$ for some closed face partition $\pi = \mathrm{cl}_P(\pi)$ of V. The interior of $\bar{F}_\pi$ with respect to the subspace topology of D_π is called an *open face* of P, and denoted F_π. Let $\mathrm{Face}(P)$ and $\overline{\mathrm{Face}}(P)$ denote the set of open and closed faces of P, respectively. Finally, define the faces of the graphic arrangement $\mathcal{A}(G)$ to be the faces of the posets over G:

$$\mathrm{Face}\,\mathcal{A}(G) = \bigcup_{\omega \in \mathrm{Acyc}(G)} \mathrm{Face}(P(G,\omega))\,, \qquad \overline{\mathrm{Face}}\,\mathcal{A}(G) = \bigcup_{\omega \in \mathrm{Acyc}(G)} \overline{\mathrm{Face}}(P(G,\omega))\,.$$

Faces of co-dimension 1 are called *facets*.

Remark 2.4. The dimension of the face $F_\pi(P(G,\omega))$ is the number of strongly connected components of $\omega/{\sim_\pi}$. As long as G is connected, there is a unique 1-dimensional face of $\mathcal{A}(G)$, which is the line $x_1 = \cdots = x_n$ and is contained in the closure of every chamber. There are no 0-dimensional faces of $\mathcal{A}(G)$. The n-dimensional faces of $\mathcal{A}(G)$ are its chambers. Additionally, $\mathbb{R}^V$ is a disjoint union of open faces of $\mathcal{A}(G)$:

$$\mathbb{R}^V = \dot{\bigcup_{F \in \mathrm{Face}\,\mathcal{A}(G)}} F\,.$$

If P is a fixed poset over G, then there is a canonical isomorphism between the lattice of closed face partitions and the lattice of faces of P, given by the mapping $\pi \mapsto F_\pi$. Recall that since π is closed, $P_\pi = (\pi, \leq_P)$ is an acyclic preposet (i.e., poset) of size $|\pi| = r$. This induces an additional preposet over V (i.e., of size $|V| = n$), which is $\leq_P$ with the additional relations that $x \sim_\pi y$ for all $x, y \in B_i$. We will say that this is a *preposet over* G, because it can be described by an (not necessarily acyclic) orientation ω_π of G. The notation reflects the fact that this orientation can be constructed by starting with some $\omega \in \mathrm{Acyc}(G)$ and then making each edge bidirected if both endpoints are contained in the same block of π. Specifically, ω_π orients edge $\{i,j\}$ as $i \to j$ if $i \leq_P j$ and as $i \leftrightarrow j$ if additionally $i \sim_\pi j$. Let $\mathrm{Pre}(G)$ be the set of all such orientations of G that arise in this manner. That is,

$$\mathrm{Pre}(G) = \{\omega_\pi \mid \omega \in \mathrm{Acyc}(G),\ \pi \text{ closed face partition of } P(G,\omega)\}\,.$$

When working with preposets over G, sometimes it is more convenient to quotient out by the strongly connected components and get an acyclic graph $\omega_\pi/{\sim_\pi}$. Note that this quotient is the same as $\omega'/{\sim_\pi}$ for at least one $\omega' \in \mathrm{Acyc}(G)$. In particular, $\omega' = \omega$ will always do. In summary, a preposet over G can be expressed several ways:

(i) as a unique orientation ω_π of G, where π is the partition into the strongly connected components;

(ii) as a unique acyclic quotient $\omega/{\sim_\pi}$ of an acyclic orientation $\omega \in \mathrm{Acyc}(G)$.

Note that while the orientation ω_π and acyclic quotient $\omega/{\sim_\pi}$ are both unique to the preposet, the choice of representative ω is not. Regardless of how an element in $\mathrm{Pre}(G)$ is written, it induces a canonical partial order $(\pi, \leq)$ on the blocks of π. However, information is lost by writing it this way; in particular, the original graph G cannot necessarily be determined from just $(\pi, \leq)$.

The mapping $\mathbb{R}^V - \mathcal{A}(G) \xrightarrow{\alpha_G} \mathrm{Acyc}(G)$ in Eq. (2) can be extended to all of $\mathbb{R}^V$ by adding both edges $i \to j$ and $j \to i$ if $x_i = x_j$. This induces a bijection between the set $\mathrm{Pre}(G)$ of all preposets on G and the set of faces of the graphic hyperplane arrangement:

$$\begin{tikzcd} \mathbb{R}^V \arrow[rr, "\alpha_G"] \arrow[dr] & & \mathrm{Pre}(G) \\ & \mathrm{Face}\,\mathcal{A}(G) \arrow[ur, dotted, "c^{-1}"'] & \end{tikzcd} \tag{5}$$

Consequently, for any preposet ω_π over G, we can let $c(\omega_\pi)$ denote the open face of $\mathcal{A}(G)$ containing any (equivalently, all) $x \in \mathbb{R}^V$ such that $\alpha_G(x) = \omega_\pi$.

Moreover, if we restrict to the preposets on exactly r strongly connected components, then the α_G-fibers are the r-dimensional open faces of $\mathcal{A}(G)$. If x lies on a face $F_\pi(P) = F_{\bar\pi}(P)$ for some poset P and closed face partition $\bar\pi = \mathrm{cl}_P(\pi)$, then the preposet $P_\pi = (\pi, \leq_P)$ has vertex set $\pi = \{B_1, \ldots B_r\}$; these are the strongly connected components of the orientation $\alpha_G(x)$.

3. Morphisms of ordinary posets

Poset isomorphisms are easy to describe both combinatorially and geometrically. An *isomorphism* between two finite posets P and P' on vertex sets V and V' is a bijection $\phi \colon V \to V'$ characterized

- *combinatorially* by the condition that $i <_P j$ is equivalent to $\phi(i) <_{P'} \phi(j)$ for all $i, j \in V$;

- *geometrically* by the equivalent condition that the induced isomorphism Φ on $\mathbb{R}^V \to \mathbb{R}^{V'}$ maps $c(P)$ to $c(P')$ bijectively.

By "induced isomorphism," we mean that Φ permutes the coordinates of $\mathbb{R}^V$ in the same way that ϕ permutes the vertices of V:

$$(x_1, x_2, \ldots, x_n) \overset{\Phi}{\longmapsto} (x_{\phi^{-1}(1)}, x_{\phi^{-1}(2)}, \ldots, x_{\phi^{-1}(n)})\,. \tag{6}$$

Morphisms of ordinary posets are also well understood. The "combinatorial" definition is easiest to modify. If P and P' are as above, then a morphism, or *order-preserving map*, is a function $\phi\colon V \to V'$ such that $i <_P j$ implies $\phi(i) <_{P'} \phi(j)$ for all $i, j \in V$. The geometric characterization is trickier because quotients, injections, and extensions are inherently different. These three types of order-preserving maps generate all poset morphisms, up to isomorphism. Below we will review this and give a geometric interpretation of each, which will motivate their toric analogues.

3.1. **Quotient.**

3.1.1. *Contracting partitions.* Roughly speaking, a quotient morphism of a poset $P(G,\omega)$ is described combinatorially by contracting ω by the blocks of a partition $\pi = \{B_1, \ldots, B_r\}$ while preserving acyclicity. Geometrically, the chamber $c(P)$ is orthogonally projected to $\bar{F}_{\bar\pi} := \overline{c(P)} \cap D_{\bar\pi}$, where $\bar\pi = \mathrm{cl}_P(\pi)$. This is the mapping

$$d_\pi\colon c(P) \longrightarrow D_\pi\,, \qquad d_\pi(x) = (x + D_{\bar\pi}^{\perp}) \cap D_{\bar\pi}\,. \tag{7}$$

By construction, the image of this map is $F_{\bar\pi}$, which is a face of P if π is a face partition. Though the map d_π extends to the closure $\overline{c(P)}$, it does *not* do so in a well-defined manner; the image $d_\pi(x)$ for some x on a hyperplane depends on the choice of P, and each hyperplane intersects two (closed) chambers along facets, and intersects the boundary of every chamber.

Example 3.1. Let $G = K_3$, the complete graph on 3 vertices. There are six acyclic orientations of G, and three of them are shown in Figure 2. The curved arrows point to the chamber $c(P_i)$ of $\mathcal{A}(G)$ for each $P_i := P(G,\omega_i)$, $i = 1,2,3$. The intersection of each closed chamber $\overline{c(P_i)}$ with $[0,1]^3$ is the order polytope, $\mathcal{O}(P_i)$.

Contracting ω_1 and ω_2 by the partition $\pi = \{B_1 = \{1\}, B_2 = \{2,3\}\}$ yields a poset over $\{B_1, B_2\}$; these are shown directly below the orientations in Figure 2. Therefore, π is closed with respect to P_1 and P_2. Geometrically, the flat D_π intersects the closed chambers $\overline{c(P_i)}$ for $i = 1,2$ in two-dimensional faces.

In contrast, contracting P_3 by π yields a preposet that is not a poset. Therefore, π is not closed with respect to P_3. Indeed, $\bar\pi := \mathrm{cl}_P(\pi) = 123 := \{\{1,2,3\}\}$, and the flat $D_{\bar\pi}$ intersects the closed chamber $\overline{c(P_3)}$ in a line. Modding out the preposet $P_\pi := (\pi, \leq_{P_3})$ by its strongly connected components yields a one-element poset. Geometrically, the chamber $c(P_3)$ projects onto the one-dimensional face $\bar{F}_{\bar\pi}$.

To see why the map d_π from Eq. (7) does not extend to the closure of the chambers in a well-defined manner, consider the point y shown in Figure 2 that lies on the hyperplane $x_1 = x_2$, and the same partition, $\pi = 1/23$. The orthogonal projection $d_\pi\colon c(P_3) \to D_\pi$ as defined in Eq. (7) and extended continuously to the closed chamber maps $\overline{c(P_3)}$ onto the line $x_1 = x_2 = x_3$. However, if $\overline{c(P_4)}$ is the other closed chamber containing y (that is, the one for which $x_3 \leq x_2 \leq x_1$), then $d_\pi\colon c(P_4) \to D_\pi$ extended to the closure maps $\overline{c(P_4)}$ onto a 2-dimensional closed face $\bar{F}_\pi(P_4) = \overline{c(P_4)} \cap D_\pi$. The point y is projected orthogonally onto the plane D_π, and does not end up on the line $x_1 = x_2 = x_3$.

Despite this, there is a natural way to extend d_π to all of $\mathbb{R}^V$, though not continuously. To do this, we first have to extend the notion of the closure of a partition π with respect to a poset, to a preposet P over G. This is easy, since the original definition did not specifically require P to actually be a poset. Specifically, the closure of π with respect to a preposet P is the unique minimal coarsening $\bar\pi := \mathrm{cl}_P(\pi)$ of π such that $(\bar\pi, \leq_P)$ is acyclic. The map d_π can now be extended to all of $\mathbb{R}^V$, as

$$d_\pi\colon \mathbb{R}^V \longrightarrow D_\pi\,, \qquad d_\pi(x) = (x + D_{\bar\pi}^{\perp}) \cap D_{\bar\pi}\,, \quad \text{where } \bar\pi = \overline{\pi(x)} := \mathrm{cl}_{\alpha_G(x)}(\pi)\,, \tag{8}$$

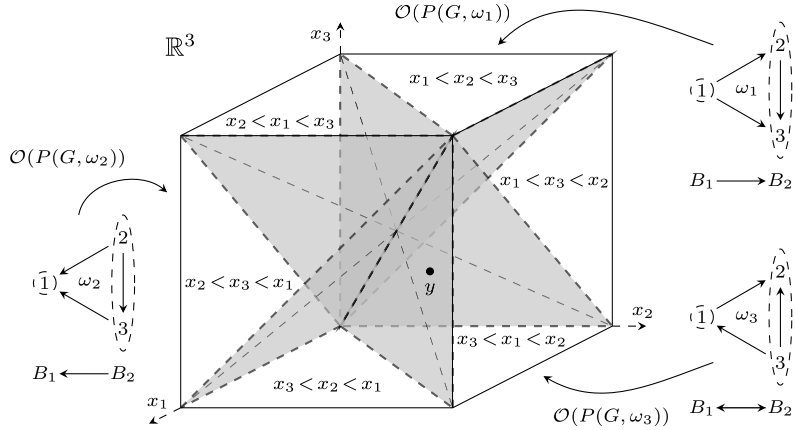


FIGURE 2. The hyperplane arrangement $\mathcal{A}(G)$ for $G = K_3$. Three orientations in $\mathrm{Acyc}(G)$ are shown, along with the corresponding chambers of $\mathcal{A}(G)$, and the preposet that results when contracting $P_i = P(G, \omega_i)$ by the partition $\pi = \{B_1 = \{1\}, B_2 = \{2,3\}\}$ of V. The intersection of each (closed) chamber $\overline{c(P_i)}$ with $[0,1]^3$ is the order polytope $\mathcal{O}(P_i)$ of P_i. The point y is supposed to lie on the hyperplane $x_1 = x_2$.

where α_G is the map from Eq. (5) sending a point to the unique open face (i.e., preposet over G) containing it.

Let us return to the case where P is a poset over G, and examine the case when π is not a face partition of P. Indeed, for an arbitrary partition π of V with $\bar{\pi} = \mathrm{cl}_P(\pi)$, the subset $\bar{F}_\pi := \overline{c(P)} \cap D_{\bar{\pi}}$ need not be a face of P; it could cut through the interior of the chamber. In this case, it is the face of at least one extension of P. Specifically, let G'_π be the graph formed by making each block B_i a clique, and let $G/{\sim_\pi}$ be the graph formed by contracting these cliques into vertices, with loops and multiedges removed. Clearly, D_π is a flat of the graphic arrangement $\mathcal{A}(G'_\pi)$ (this choice is not unique, but it is a canonical one that works). Thus, the set $\bar{F}_{\bar{\pi}} = \bar{F}_\pi = \overline{c(P)} \cap D_\pi$, for $\bar{\pi} = \mathrm{cl}_P(\pi)$, is a closed face of $\mathcal{A}(G'_\pi)$, and hence a face of some poset P' over G'_π for which $\bar{\pi} = \mathrm{cl}_{P'}(\pi)$.

Whether or not π is a face partition of a particular poset P over G, the map d_π in Eq. (8) projects a chamber $c(P)$ onto a flat $D_{\bar{\pi}}$ of $\mathcal{A}(G'_\pi)$, where $\bar{\pi} = \mathrm{cl}_P(\pi)$. From here, we need to project it homeomorphically onto a coordinate subspace of $\mathbb{R}^V$ so it is a chamber of a lower-dimensional arrangement. Specifically, for a partition $\pi = \{B_1, \dots, B_r\}$, let $W \subsetneq V$ be any subset formed by removing all but 1 coordinate from each B_i, and let $r_\pi \colon \mathbb{R}^V \longrightarrow \mathbb{R}^W$ be the induced projection. The r_π-image of $\mathcal{A}(G)$ is the graphic arrangement of $G/{\sim_\pi}$. The following ensures that r_π is a homeomorphism, and that the choice of W does not matter. We omit the elementary proof.

Lemma 3.2. *Let $\pi = \{B_1, \dots, B_r\}$ be a partition of V, and $W = \{b_1, \dots, b_r\} \subseteq V$ with $b_i \in B_i$. The restriction $r_\pi|_{D_\pi} \colon D_\pi \longrightarrow \mathbb{R}^W$ is a homeomorphism.*

Moreover, all such projection maps for a fixed π are topologically conjugate in the following sense: If $W' = \{b'_1, \dots, b'_r\} \subseteq V$ with $b'_i \in B_i$ and projection map $r'_\pi|_{D_\pi} \colon D_\pi \longrightarrow \mathbb{R}^{W'}$, and σ is the permutation of V that transposes each b_i with b'_i, then the following diagram commutes:

$$\begin{tikzcd} D_\pi \arrow[r, hook] & \mathbb{R}^V \arrow[r, "r_\pi"] \arrow[d, "\Sigma"] & \mathbb{R}^W \arrow[d, "\Sigma'"] \\ D_\pi \arrow[r, hook] & \mathbb{R}^V \arrow[r, "r'_\pi"] & \mathbb{R}^{W'} \end{tikzcd}$$

Here, $\sigma' \colon W \to W'$ is the map $b_i \mapsto b'_i$, with Σ and Σ' being the induced linear maps as defined in Eq. (6).

By convexity, d_π induces a well-defined map $\delta_\pi \colon \operatorname{Face}\mathcal{A}(G) \longrightarrow \operatorname{Face}\mathcal{A}(G'_\pi)$ making the following diagram commute:

$$\begin{tikzcd} \mathbb{R}^V \arrow[r, "d_\pi"] \arrow[d] & D_\pi \arrow[d] \\ \operatorname{Face}\mathcal{A}(G) \arrow[r, dotted, "\delta_\pi"] & \operatorname{Face}\mathcal{A}(G'_\pi) \end{tikzcd} \tag{9}$$

The map δ_π is best understood by looking at a related map $\bar{\delta}_\pi$ on closed faces. Let $\bar{F} = \overline{c(P)} \cap D_\sigma$ be a closed face of $\mathcal{A}(G)$, for some closed face partition $\sigma \in \Pi_V$. Then the map $\bar{\delta}_\pi$ is defined by

$$\bar{\delta}_\pi \colon \overline{\operatorname{Face}}\,\mathcal{A}(G) \longrightarrow \overline{\operatorname{Face}}\,\mathcal{A}(G'_\pi)\,, \qquad \bar{\delta}_\pi \colon \overline{c(P)} \cap D_\sigma \longmapsto \overline{c(P)} \cap D_\sigma \cap D_\pi\,.$$

The map δ_π is between the corresponding open faces. These faces are then mapped to faces of the arrangement $\mathcal{A}(G/{\sim_\pi})$ under the projection $r_\pi|_{D_\pi} \colon D_\pi \longrightarrow \mathbb{R}^W$. [Alternatively, we could simply identity the quotient space $\mathbb{R}^V/D_\pi^\perp$ with $\mathbb{R}^W$.]

To summarize, the open faces of $\mathcal{A}(G)$ arise from preposets $\omega = \omega_\sigma$ in $\operatorname{Pre}(G)$, where without loss of generality, the blocks for $\sigma \in \Pi_V$ are the strongly connected components. The contraction of this preposet formed by adding all relations (edges) of the form (v, w) for $v, w \in B_i$ yields a preposet ω'_π over G'_π. Then, modding out by the strongly connected components yields an acyclic preposet, i.e., a poset. This two-step process is a composition of maps

$$\operatorname{Pre}(G) \xrightarrow{q_\pi} \operatorname{Pre}(G'_\pi) \xrightarrow{p_\pi} \operatorname{Pre}(G/{\sim_\pi})\,, \qquad \omega_\sigma = \omega \overset{q_\pi}{\longmapsto} \omega'_\pi \overset{p_\pi}{\longmapsto} \omega'_\pi/{\sim_{\bar{\pi}}} = \omega/{\sim_{\bar{\pi}}}\,.$$

Here $\bar{\pi} = \operatorname{cl}_{P_\sigma}(\pi)$, the closure of π with respect to the *preposet* P_σ, which we have been denoting by ω_σ under a slight abuse of notation.

Putting this all together gives a commutative diagram that illustrates the relationship between the points in $\mathbb{R}^V$, the open faces of the graphic arrangement $\mathcal{A}(G)$, and the preposets over G. The left column depicts the acyclic preposets – those that are also posets.

$$\begin{tikzcd} \mathbb{R}^V - \mathcal{A}(G) \arrow[r, hook] \arrow[d] & \mathbb{R}^V \arrow[r, "d_\pi"] \arrow[d] & D_\pi \arrow[r, "r_\pi"] \arrow[d] & \mathbb{R}^W \arrow[d] \\ \operatorname{Cham}\mathcal{A}(G) \arrow[r, hook] \arrow[d, leftrightarrow] & \operatorname{Face}\mathcal{A}(G) \arrow[r, "\delta_\pi"] \arrow[d, leftrightarrow] & \operatorname{Face}\mathcal{A}(G'_\pi) \arrow[r, "\rho_\pi"] \arrow[d, leftrightarrow] & \operatorname{Face}\mathcal{A}(G/{\sim_\pi}) \arrow[d, leftrightarrow] \\ \operatorname{Acyc}(G) \arrow[r, hook] & \operatorname{Pre}(G) \arrow[r, "q_\pi"] & \operatorname{Pre}(G'_\pi) \arrow[r, "p_\pi"] & \operatorname{Pre}(G/{\sim_\pi}) \end{tikzcd}$$

3.1.2. *Intervals and antichains.* Poset morphisms that are quotients are characterized geometrically by projecting the chamber $c(P)$ onto a flat $D_{\bar{\pi}}$ of $\mathcal{A}(G'_\pi)$ for some partition $\bar{\pi} = \operatorname{cl}_P(\pi)$, and then homeomorphically mapping this down to a chamber of a lower-dimensional graphic arrangement $\mathcal{A}(G/{\sim_\pi})$. Equivalently, contracting $P(G, \omega)$ by $\bar{\pi}$ yields an acyclic preposet $P_{\bar{\pi}} = (\bar{\pi}, \leq_P)$. It is well known that contracting a poset by an interval or an antichain yields an acyclic preposet. Verification of this is elementary, but first recall how these are defined.

Definition 3.3. Let P be a poset over V. An *interval* of P is a subset $I \subseteq V$, sometimes denoted $[i, j]$, such that

$$I = \{x \in P : i \leq_P x \leq_P j\}, \quad \text{for some fixed } i, j \in P\,.$$

An *antichain* of P is a subset $A \subseteq V$ such that any two elements are incomparable.

We will take a moment to understand how contracting an interval or antichain fits in the partition framework described above, which will help us understand the toric analogue. Given a nonempty subset $S \subseteq V$, define the partition π_S of V by

$$\pi_S = \{B_1 = S, B_2, \ldots, B_r\}, \qquad \text{where } |B_i| = 1 \text{ for } i = 2, \ldots, r. \tag{10}$$

Contracting an interval $I \subseteq V$ in a poset P yields the poset $P_{\pi_I} = (\pi_I, \leq_P)$. In this case, $\pi_I = \operatorname{cl}_P(\pi_I)$ is a face partition and F_π is a $(|V| - |I| + 1)$-dimensional face of P. Similarly, collapsing an antichain $A \subseteq V$ yields the poset $P_{\pi_A} = (\pi_A, \leq_P)$. Note that D_{π_I} is a flat of $\mathcal{A}(G)$ and lies on the boundary of $c(P)$, but the $(|V| - |A| + 1)$-dimensional subspace D_{π_A} cuts through the interior of $c(P)$. For both of these cases, $S = I$ and $S = A$, the subspace D_{π_S} is trivially a flat of $\mathcal{A}(G'_{\pi_S})$.

3.2. **Extension.** Given two posets P, P' on a set V, one says that P' *is an extension of* P when $i <_P j$ implies $i <_{P'} j$. In this case, the identity map $P \longrightarrow P'$ is a poset morphism. Geometrically, P' is an extension of P if and only if one has an inclusion of their open polyhedral cones $c(P') \subseteq c(P)$. Each added relation $i <_{P'} j$ amounts to intersecting $c(P)$ with the half-space $H_{i<j} := \{x \in \mathbb{R}^V : x_i < x_j\}$.

3.3. **Inclusion.** The last operation that yields a poset morphism is an injection $\phi \colon P \hookrightarrow P'$. This induces a canonical inclusion $\Phi \colon \mathbb{R}^V \hookrightarrow \mathbb{R}^{V'}$. Note that $i <_P j$ implies $i <_{P'} j$, but not necessarily vice-versa. Thus, up to isomorphism, an inclusion can be decomposed into the composition $P \hookrightarrow P'' \to P'$, where the first map adds the elements $\{n+1, \ldots, m\}$ to P but no extra relations, and then the map $P'' \to P'$ is an extension. This gives an inclusion of polyhedral cones:

$$\begin{aligned} c(P') := \{x \in \mathbb{R}^{V'} : x_i < x_j \text{ for } i <_{P'} j\} \subseteq c(P'') &= \{x \in \mathbb{R}^{V'} : x_i < x_j \text{ for } i <_P j\} \\ &\cong c(P) \times \mathbb{R}^{V'-V}. \end{aligned}$$

3.4. **Summary.** Up to isomorphism, every morphism of a poset $P = P(G, \omega)$ can be decomposed into a sequence of three steps:

(i) *quotient*: Collapsing G by a partition π that preserves acyclicity of ω (projecting $c(P)$ to a flat D_π of $\mathcal{A}(G'_\pi)$ for some closed partition $\pi = \mathrm{cl}_P(\pi)$).
(ii) *inclusion*: Adding vertices (adding dimensions).
(iii) *extension*: Adding relations (intersecting with half-spaces).

In the special case of the morphism $P \longrightarrow P'$ being surjective, the inclusion step is eliminated and the entire process can be described geometrically by projecting $\overline{c(P)}$ to a flat D_π and then intersecting with a collection of half-spaces.

4. Toric posets and preposets

4.1. **Toric chambers and posets.** Toric posets, introduced in [DMR15], arise from ordinary (finite) posets defined by acyclic orientations under the equivalence relation generated by converting maximal elements into minimal elements, or sources into sinks. Whereas an ordinary poset corresponds to a chamber of a graphic arrangement $\mathcal{A}(G)$, a toric poset corresponds to a chamber of a *toric graphic arrangement* $\mathcal{A}_{\mathrm{tor}}(G) = q(\mathcal{A}(G))$, which is the image of $\mathcal{A}(G)$ under the quotient map $\mathbb{R}^V \xrightarrow{q} \mathbb{R}^V/\mathbb{Z}^V$. Elements of $\mathcal{A}_{\mathrm{tor}}(G)$ are *toric hyperplanes*

$$H_{ij}^{\mathrm{tor}} := \{x \in \mathbb{R}^V/\mathbb{Z}^V : x_i \bmod 1 = x_j \bmod 1\} = q(H_{ij}).$$

Definition 4.1. A connected component c of the complement $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G)$ is called a *toric chamber* for G, or simply a chamber of $\mathcal{A}_{\mathrm{tor}}(G)$. Let $\operatorname{Cham} \mathcal{A}_{\mathrm{tor}}(G)$ denote the set of all chambers of $\mathcal{A}_{\mathrm{tor}}(G)$.

A *toric poset* P is a set c that arises as a toric chamber for at least one graph G. We may write $P = P(c)$ or $c = c(P)$, depending upon the context.

If we fix a graph $G = (V, E)$ and consider the arrangement $\mathcal{A}_{\mathrm{tor}}(G)$, then each point in $\mathbb{R}^V/\mathbb{Z}^V$ naturally determines a preposet on G via a map $\bar{\alpha}_G \colon x \mapsto \omega(x)$. Explicitly, for $x = (x_1 \bmod 1, \ldots, x_n \bmod 1)$ in $\mathbb{R}^V/\mathbb{Z}^V$, the directed graph $\omega(x)$ is constructed by doing the following for each edge $\{i, j\}$ in E:

- If $x_i \bmod 1 \leq x_j \bmod 1$, then include edge $i \to j$;
- If $x_j \bmod 1 \leq x_i \bmod 1$, then include edge $j \to i$.

The mapping $\bar{\alpha}_G$ is essentially the same as α_G from Eq. (5) except done modulo 1, so many of its properties are predictably analogous. For example, the undirected version of $\omega(x)$ is G. The edge $\{i, j\}$ is bidirected in $\omega(x)$ if and only if $x_i \bmod 1 = x_j \bmod 1$. Therefore, $\omega(x)$ is acyclic if and only if x lies in $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G)$; in this case $\omega(x)$ describes a poset. Otherwise it describes a preposet (that is not a poset). Modding out by the strongly connected components yields an acyclic graph $\omega(x)/{\sim_x}$ that describes a poset.

Definition 4.2. When two preposets $\omega(x)$ and $\omega(y)$ are such that the directed graphs $\omega(x)/{\sim_x}$ and $\omega(y)/{\sim_y}$ differ only by converting a source vertex (equivalence class) into a sink, or vice-versa, we say they differ by a *flip*. The transitive closure of the flip operation generates an equivalence relation on $\mathrm{Pre}(G)$, denoted by $\equiv$.

In the special case of restricting to preposets that are acyclic, we get $\mathrm{Acyc}(G) \subseteq \mathrm{Pre}(G)$ and a bijective correspondence between toric posets and chambers of toric graphic arrangements. This is Theorem 1.4 in [DMR15]. A generalization of this to a bijection between toric preposets and faces of the toric graphic arrangement appears later in this section (Proposition 4.11).

Theorem 4.3. *([DMR15], Theorem 1.4) The map* $\bar{\alpha}_G$ *induces a bijection between* $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$ *and* $\mathrm{Acyc}(G)/{\equiv}$ *as follows:*

$$\begin{tikzcd} \mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G) \arrow[r, "\bar{\alpha}_G"] \arrow[d] & \mathrm{Acyc}(G) \arrow[d] \\ \mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G) \arrow[r, dotted, "\bar{\alpha}_G"'] & \mathrm{Acyc}(G)/{\equiv} \end{tikzcd} \tag{11}$$

In other words, two points x, x' *in* $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G)$ *have* $\bar{\alpha}_G(x) \equiv \bar{\alpha}_G(x')$ *if and only if* x, x' *lie in the same toric chamber of* $\mathcal{A}_{\mathrm{tor}}(G)$.

By Theorem 4.3, every pair $(G, [\omega])$ of a graph G and $\omega \in \mathrm{Acyc}(G)$ determines a toric poset, and we denote this by $P(G, [\omega])$. *Specifically,* $P(G, [\omega])$ *is the toric poset* $P(c)$ *such that* $\bar{\alpha}_G(c) = [\omega]$. If the graph G is understood, then we may denote the corresponding toric chamber by $c_{[\omega]} := c(P(G, [\omega]))$.

If $G = (V, E)$ is fixed, then the unit cube $[0,1]^V$ in $\mathbb{R}^V$ is the union of order polytopes $\mathcal{O}(P(G,\omega))$, any two of which only intersect in a subset of a flat of $\mathcal{A}(G)$:

$$[0,1]^V = \bigcup_{\omega \in \mathrm{Acyc}(G)} \mathcal{O}(P(G,\omega))\,. \tag{12}$$

When G is understood, we will say that the order polytopes $\mathcal{O}(P(G,\omega))$ and $\mathcal{O}(P(G,\omega'))$ are torically equivalent whenever $\omega' \in [\omega]$. Under the natural quotient $q\colon [0,1]^V \to \mathbb{R}^V/\mathbb{Z}^V$, each order polytope $\mathcal{O}(P(G,\omega))$ is mapped into the closed toric chamber $\bar{c}_{[\omega]}$. Moreover, by Theorem 4.3, the closed chambers of $\mathcal{A}_{\mathrm{tor}}(G)$ are unions of q-images of torically equivalent order polytopes.

Corollary 4.4. *Let* $P = P(G, [\omega])$ *be a toric poset, and* $q\colon [0,1]^V \to \mathbb{R}^V/\mathbb{Z}^V$ *the natural quotient. The closure of the chamber* $c(P)$ *is*

$$\overline{c(P)} = \bigcup_{\omega' \in [\omega]} q(\mathcal{O}(P(G,\omega')))\,.$$

4.2. **Toric faces and preposets.** Let $P = P(c)$ be a toric poset over $G = (V, E)$. To define objects like a face of P or its dimension, it helps to first lift c up to a chamber of the *affine graphic arrangement* which lies in $\mathbb{R}^V$:

$$\mathcal{A}_{\mathrm{aff}}(G) := q^{-1}(\mathcal{A}_{\mathrm{tor}}(G)) = q^{-1}(q(\mathcal{A}(G))) = \bigcup_{\substack{\{i,j\} \in E \\ k \in \mathbb{Z}}} \{x \in \mathbb{R}^V : x_i = x_j + k\}. \tag{13}$$

The affine chambers are open unbounded convex polyhedral regions in $\mathbb{R}^V$, the universal cover of $\mathbb{R}^V/\mathbb{Z}^V$. The path lifting property guarantees that two points x and y in $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G)$ are in the same toric chamber if and only if they have lifts $\hat{x}$ and $\hat{y}$ that are in the same affine chamber. Moreover, since Corollary 4.4 characterizes the closed toric chamber $\overline{c(P)}$ as a union of torically equivalent order polytopes under a universal covering map, each closed affine chamber is a union of translated copies of torically equivalent order polytopes in $\mathbb{R}^V$.

We usually denote an affine chamber by $\hat{c}$ or c^{aff}. Each hyperplane H_{ij}^{tor} has a unique preimage containing the origin in $\mathbb{R}^V$ called its *central preimage*; this is the ordinary hyperplane $H_{ij} = \{x \in \mathbb{R}^V : x_i = x_j\}$. Thus, the set of central preimages of $\mathcal{A}_{\text{tor}}(G)$ is precisely the graphic arrangement $\mathcal{A}(G)$ in $\mathbb{R}^V$. Each closed affine chamber $\bar{c}^{\text{aff}}$ contains at most one order polytope $\mathcal{O}(P(G,\omega))$ for $\omega \in \text{Acyc}(G)$. Affine chambers whose closures contain precisely one order polytope $\mathcal{O}(P(G,\omega))$ are *central affine chambers*.

We will call nonempty sets that arise as intersections of hyperplanes in $\mathcal{A}_{\text{aff}}(G)$ *affine flats* and nonempty sets that are intersections of hyperplanes in $\mathcal{A}_{\text{tor}}(G)$ *toric flats*. Since the toric flats have a nonempty intersection, they form a lattice that is denoted $L(\mathcal{A}_{\text{tor}}(G))$, and partially ordered by reverse inclusion.

Since a toric flat of $\mathcal{A}_{\text{tor}}(G)$ is the image of a flat of $\mathcal{A}(G)$, it too is determined by a partition π of V, and so it is of the form

$$D_\pi^{\text{tor}} = \{x \in \mathbb{R}^V/\mathbb{Z}^V :\ x_i = x_j \text{ for every pair } i,j \text{ in the same block } B_k \text{ of } \pi\} = q(D_\pi)\,. \tag{14}$$

Since $\mathbb{R}^V \xrightarrow{q} \mathbb{R}^V/\mathbb{Z}^V$ is a covering map, it is well-founded to declare the dimension of a toric flat D_π^{tor} in $\mathbb{R}^V/\mathbb{Z}^V$ to be the same as the dimension of its central preimage D_π in $\mathbb{R}^V$.

Recall that a partition $\pi = \{B_1, \dots, B_r\}$ is compatible with an ordinary poset P if contracting the blocks of π yields a preposet $P_\pi = (\pi, \le_P)$ that is acyclic (also a poset). The notion of compatible partitions does not carry over well to toric posets, because compatibility is not preserved by toric equivalence. Figure 2 shows an example of this: the preposets $(\pi, \le_{P_1})$ and $(\pi, \le_{P_2})$ are acyclic but $(\pi, \le_{P_3})$ is not. Despite this, every set D_π^{tor}, whether or not it is a toric flat of $\mathcal{A}_{\text{tor}}(G)$, intersects the closed toric chamber $\overline{c(P)}$ in at least the line $x_1 = \cdots = x_n$. We denote this intersection by

$$\bar{F}_\pi^{\text{tor}}(P) := \overline{c(P)} \cap D_\pi^{\text{tor}}\,. \tag{15}$$

If D_π^{tor} does not intersect $c(P)$, then we say that π is a *toric face partition*, since it intersects the closed toric chamber along its boundary. Compare this to the definition of face partitions of an ordinary poset $P(G,\omega)$, which are those $\pi \in \Pi_V$ characterized by D_π being a flat of the graphic arrangement of the transitive closure, or equivalently, by $D_\pi \cap c(P(G,\omega)) = \varnothing$. The transitive closure $\bar{G}(P(G,\omega))$ is formed from G by adding all additional edges $\{i,j\}$ such that $H_{i,j} \cap c(P(G,\omega)) = \varnothing$. Similarly, we can define the *toric transitive closure* of $P(G,[\omega])$ as the graph G along with the extra edges $\{i,j\}$ such that $H_{i,j}^{\text{tor}} \cap c(P) = \varnothing$. This was done in [DMR15], and we will return to it in the Section 5.3 when we discuss toric Hasse diagrams.

Now, let $\pi \in \Pi_V$ be an arbitrary partition. Since flats of $\mathcal{A}_{\text{tor}}(K_V)$ are closed under intersections, there is a unique maximal toric subspace $D_{\bar{\pi}}^{\text{tor}}$ (that is, of minimal dimension) for which $\bar{F}_\pi^{\text{tor}} = \overline{c(P)} \cap D_{\bar{\pi}}^{\text{tor}}$. The partition $\bar{\pi}$ is the unique minimal coarsening of π for which $\bar{F}_\pi^{\text{tor}} = \bar{F}_{\bar{\pi}}^{\text{tor}}$, and it is the lattice-join of all such partitions. We call it the *closure* of π with respect to the toric poset P, denoted $\text{cl}_P^{\text{tor}}(\pi)$, and we define $\dim(\bar{F}_\pi^{\text{tor}}) := \dim(D_{\bar{\pi}}^{\text{tor}})$. A partition π is *closed* with respect to the toric poset P if $\pi = \text{cl}_P^{\text{tor}}(\pi)$. Note that the closure is defined for all partitions, not just toric face partitions.

Definition 4.5. A set $F \subseteq \mathbb{R}^V/\mathbb{Z}^V$ is a *closed face* of the toric poset P if $F = \bar{F}_\pi^{\text{tor}} = \overline{c(P)} \cap D_\pi^{\text{tor}}$ for some closed toric face partition $\pi = \text{cl}_P^{\text{tor}}(\pi)$. The interior of $\bar{F}_\pi^{\text{tor}}$ with respect to the subspace topology of D_π^{tor} is called an *open face* of P, and denoted F_π^{tor}. Let $\text{Face}(P)$ and $\overline{\text{Face}}(P)$ denote the set of open and closed faces of P, respectively. Finally, define the faces of the toric graphic arrangement $\mathcal{A}_{\text{tor}}(G)$ to be the faces of the toric posets over G:

$$\text{Face}\,\mathcal{A}_{\text{tor}}(G) = \bigcup_{\omega \in \text{Acyc}(G)} \text{Face}(P(G,[\omega]))\,, \qquad \overline{\text{Face}}\,\mathcal{A}_{\text{tor}}(G) = \bigcup_{\omega \in \text{Acyc}(G)} \overline{\text{Face}}(P(G,[\omega]))\,.$$

Toric faces of co-dimension 1 are called *facets*.

The following remark is the toric analogue of Remark 2.4.

Remark 4.6. Let $P = P(G,[\omega])$ be a toric poset. The dimension of $\bar{F}^{\mathrm{tor}}_\pi(P) := \overline{c(P)} \cap D^{\mathrm{tor}}_\pi$ is simply the maximum dimension of $\overline{c^{\mathrm{aff}}(P)} \cap D_\pi$ taken over all affine chambers that descend down to $\overline{c(P)}$. Since closed affine chambers are unions of translations of order polytopes, this is the maximum dimension of $\overline{c(P(G,\omega'))} \cap D_\pi$ taken over all $\omega' \in [\omega]$. In other words,

$$\dim F^{\mathrm{tor}}_\pi(P(G,[\omega])) = \max_{\omega' \in [\omega]} \dim F_\pi(P(G,\omega')) .$$

On the level of graphs, this is the maximum number of strongly connected components that $\omega'/\!\sim_\pi$ can have for some $\omega' \in [\omega]$. In particular, a partition π is closed with respect to $P(G,[\omega])$ if and only if $\omega'/\!\sim_\pi$ is acyclic for *some* $\omega' \in [\omega]$.

As long as G is connected, there is a unique 1-dimensional face of $\mathcal{A}_{\mathrm{tor}}(G)$, which is the line $x_1 = \cdots = x_n$ and is contained in the closure of every chamber. There are no 0-dimensional faces of $\mathcal{A}_{\mathrm{tor}}(G)$. The n-dimensional faces of $\mathcal{A}_{\mathrm{tor}}(G)$ are its chambers. Additionally, $\mathbb{R}^V/\mathbb{Z}^V$ is a disjoint union of open faces of $\mathcal{A}_{\mathrm{tor}}(G)$:

$$\mathbb{R}^V/\mathbb{Z}^V = \dot{\bigcup_{F \in \mathrm{Face}\,\mathcal{A}_{\mathrm{tor}}(G)}} F . \tag{16}$$

As in the case of ordinary posets, there is a canonical bijection between the closed toric face partitions of P and open faces (or closed faces) of P, via $\pi \mapsto F^{\mathrm{tor}}_\pi$. To classify the faces of a toric poset, it suffices to classify the closed toric face partitions.

Theorem 4.7. *Let $P = P(G,[\omega])$ be a toric poset over $G = (V,E)$. A partition π of V is a closed toric face partition of P if and only if it is connected and compatible with $P(G,\omega')$, for some $\omega' \in [\omega]$.*

The proof of Theorem 4.7 will be done later in this section, after the following lemma, which establishes that $\mathrm{cl}^{\mathrm{tor}}_P$ is a closure operator [War42] on the partition lattice Π_V and compares it with cl_P.

Lemma 4.8. *Let ω be an acyclic orientation of a graph $G = (V,E)$, and π a partition of V.*
(a) If π is closed with respect to $P(G,\omega)$, then π is closed with respect to $P(G,[\omega])$.
(b) Closure is monotone: if $\pi \leq_V \pi'$, then $\mathrm{cl}^{\mathrm{tor}}_P(\pi) \leq_V \mathrm{cl}^{\mathrm{tor}}_P(\pi')$.
(c) If $\pi \leq_V \pi' \leq_V \mathrm{cl}^{\mathrm{tor}}_P(\pi)$, then $\mathrm{cl}^{\mathrm{tor}}_P(\pi') = \mathrm{cl}^{\mathrm{tor}}_P(\pi)$.
(d) $\mathrm{cl}^{\mathrm{tor}}_{P(G,[\omega])}(\pi) \leq_V \mathrm{cl}_{P(G,\omega)}(\pi)$.

Proof. If π is closed with respect to $P(G,\omega)$, then the preposet $\omega/\!\sim_\pi$ is acyclic. By Remark 4.6, this means that π is closed with respect to $P(G,[\omega])$, which establishes (a).

Part (b) is obvious. Part (c) follows from taking the closure of each term in the chain of inequalities $\pi \leq_V \pi' \leq_V \mathrm{cl}^{\mathrm{tor}}_P(\pi)$:

$$\mathrm{cl}^{\mathrm{tor}}_P(\pi) \leq_V \mathrm{cl}^{\mathrm{tor}}_P(\pi') \leq_V \mathrm{cl}^{\mathrm{tor}}_P\big(\mathrm{cl}^{\mathrm{tor}}_P(\pi)\big) = \mathrm{cl}^{\mathrm{tor}}_P(\pi) .$$

To prove (d), let $\bar{\pi} = \mathrm{cl}_{P(G,\omega)}(\pi)$, which is closed with respect to $P(G,\omega)$. By (a), $\bar{\pi}$ is closed with respect to $P(G,[\omega])$. Using this, along with (b) applied to $\pi \leq_V \bar{\pi}$, yields

$$\mathrm{cl}^{\mathrm{tor}}_{P(G,[\omega])}(\pi) \leq_V \mathrm{cl}^{\mathrm{tor}}_{P(G,[\omega])}(\bar{\pi}) = \bar{\pi} = \mathrm{cl}_{P(G,\omega)}(\pi) ,$$

whence the theorem. $\square$

Example 4.9. For an example both of where the converse to Lemma 4.8(a) fails, and where $\mathrm{cl}^{\mathrm{tor}}_{P(G,[\omega])}(\pi) \lneq_V \mathrm{cl}_{P(G,\omega)}(\pi)$, consider $G = K_3$, and the partition $\pi = 1/23$. Using the same notation as in Example 3.1 and Figure 2, we see that $\mathrm{cl}_{P(G,\omega_3)}(\pi) = 123$, because the intersection of $\overline{c(P(G,\omega_3))}$ and D_π is one-dimensional. Equivalently, the preposet $\omega_3/\!\sim_\pi$ has one strongly connected component. In contrast, the intersection of $\overline{c(P(G,[\omega_3]))}$ with D^{tor}_π is two-dimensional. Indeed, the preposets $\omega_1/\!\sim_\pi$ and $\omega_2/\!\sim_\pi$ both have two strongly connected components, and $\omega_1 \equiv \omega_2 \equiv \omega_3$. Therefore,

$$1/23 = \pi = \mathrm{cl}^{\mathrm{tor}}_{P(G,[\omega_3])}(\pi) \lneq_V \mathrm{cl}_{P(G,\omega_3)}(\pi) = 123 ,$$

and so π is closed with respect to $P(G,[\omega_3])$ but not with respect to $P(G,\omega_3)$.

Proof of Theorem 4.7. Suppose π is a closed toric face partition of $P = P(G, [\omega])$, and that $\bar{F}_\pi^{\mathrm{tor}} := \overline{c(P)} \cap D_\pi^{\mathrm{tor}}$ has dimension k. Then for some $\omega' \in [\omega]$, the hyperplane D_π must intersect the order polytope $\mathcal{O}(P(G, \omega'))$ in a k-dimensional face. Thus, $\bar{F}_\pi := \overline{c(P(G, \omega'))} \cap D_\pi$ has dimension k for some $\omega' \in [\omega]$, and so π is a face partition of $P(G, \omega')$. To see why π is closed with respect to $P(G, \omega')$, suppose there were a coarsening $\pi' \geq_V \pi$ such that

$$\bar{F}_{\pi'} = \overline{c(P(G, \omega'))} \cap D_{\pi'} = \overline{c(P(G, \omega'))} \cap D_\pi = \bar{F}_\pi\,. \tag{17}$$

It suffices to show that $\pi' = \pi$. Descending down to the torus, the intersection $\bar{F}_{\pi'}^{\mathrm{tor}} := \overline{c(P)} \cap D_{\pi'}^{\mathrm{tor}}$ must have dimension at least k by Eq. (17), but no more than k because $\bar{F}_{\pi'}^{\mathrm{tor}} \subseteq \bar{F}_\pi^{\mathrm{tor}}$. Thus, we have equality $\bar{F}_{\pi'}^{\mathrm{tor}} = \bar{F}_\pi^{\mathrm{tor}}$, which means $\pi' \leq_V \mathrm{cl}_P^{\mathrm{tor}}(\pi) = \pi$, whence $\pi' = \pi$. Since π is a closed face partition with respect to $P(G, \omega')$, it is connected and compatible with $P(G, \omega')$ by Theorem 2.1.

Conversely, suppose that π is connected and compatible with respect to $P(G, \omega')$ for some $\omega' \in [\omega]$. By Theorem 2.1, π is a closed face partition of $P(G, \omega')$. Since π is connected, D_π is a flat of $\mathcal{A}(G)$. Therefore, D_π^{tor} is a toric flat of $\mathcal{A}_{\mathrm{tor}}(G)$, and so $\bar{F}_\pi^{\mathrm{tor}} := \overline{c(P)} \cap D_\pi^{\mathrm{tor}}$ is a face of the toric poset $P(G, [\omega']) = P(G, [\omega]) = P$. Therefore, π is a toric face partition. Closure of π with respect to $P = P(G, [\omega]) = P(G, [\omega'])$ follows immediately from Lemma 4.8(a) applied to the fact that π is closed with respect to $P(G, \omega')$. □

Unlike the ordinary case, where faces of posets are literally faces of a convex polyhedral cone, it is not quite so "geometrically obvious" what subsets can be toric faces. The following example illustrates this.

Example 4.10. There are only two simple graphs $G = (V, E)$ over $V = \{1, 2\}$: The edgeless graph $G_0 = (V, \varnothing)$, and the complete graph $K_2 = (V, \{\{1, 2\}\})$. For both graphs, the complement $\mathbb{R}^2/\mathbb{Z}^2 - \mathcal{A}(G)$ is connected. The respective chambers are

$$c_0 := \mathbb{R}^2/\mathbb{Z}^2\,, \qquad \text{and} \qquad c := \mathbb{R}^2/\mathbb{Z}^2 - H_{12}^{\mathrm{tor}}\,,$$

and so they represent different toric posets, $P(c_0)$ and $P(c)$. Despite this, these chambers have the same topological closures: $\bar{c_0} = \bar{c} = \mathbb{R}^2/\mathbb{Z}^2$. The lattice of flats of G_0 contains one element: $\mathcal{A}(G_0) = \{D_{1/2}^{\mathrm{tor}}\}$, and this flat arises from the permutation $\pi = 1/2$. The lattice of flats of K_2 has two elements: $\mathcal{A}(K_2) = \{D_{1/2}^{\mathrm{tor}}, D_{12}^{\mathrm{tor}}\}$, where $D_{1/2}^{\mathrm{tor}} = \mathbb{R}^V/\mathbb{Z}^V$, and $D_{12}^{\mathrm{tor}} = H_{12}^{\mathrm{tor}} = \{x_1 = x_2\}$. Thus, the closed faces of the corresponding toric posets are

$$\overline{\mathrm{Face}}(P(c_0)) = \{\mathbb{R}^V/\mathbb{Z}^V\}\,, \qquad \text{and} \qquad \overline{\mathrm{Face}}(P(c)) = \{\mathbb{R}^V/\mathbb{Z}^V,\ H_{12}^{\mathrm{tor}}\}.$$

The subtlety in Example 4.10 does not arise for ordinary posets, because distinct ordinary posets never have chambers with the same topological closure. In contrast, if $G = (V, E)$ and $G' = (V, E')$ are both forests, then $\mathcal{A}_{\mathrm{tor}}(G)$ and $\mathcal{A}_{\mathrm{tor}}(G')$ both have a single toric chamber. This is because the number of chambers is counted by the Tutte polynomial evaluation $T_G(1, 0)$, which is always 1 for a forest; see [DMR15]. In this case, the closures of both chambers will be all of $\mathbb{R}^V/\mathbb{Z}^V$. A more complicated example involving a toric poset over a graph with three vertices, will appear soon in Example 4.13.

Recall the map $\bar{\alpha}_G$ from Eq. (11) that sends a point x in $\mathbb{R}^V/\mathbb{Z}^V$ to a preposet $\omega(x)$. By Theorem 4.3, when restricted to the points in $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G)$, this map induces a bijection between toric posets and toric chambers. Toric faces F_π^{tor} that are open in D_π^{tor} are chambers in lower-dimensional arrangements that are contractions of $\mathcal{A}_{\mathrm{tor}}(G)$, namely by the subspace D_π^{tor}. Thus, the bijection between toric equivalence classes of $\mathrm{Acyc}(G)$ (n-element preposets) and toric chambers (n-dimensional faces) extends naturally to a bijection between toric preposets over G and open faces of $\mathcal{A}_{\mathrm{tor}}(G)$.

Proposition 4.11. *The map* $\bar{\alpha}_G$ *induces a bijection between* $\mathrm{Face}\,\mathcal{A}_{\mathrm{tor}}(G)$ *and* $\mathrm{Pre}(G)/\equiv$ *as follows:*

$$\begin{tikzcd} \mathbb{R}^V/\mathbb{Z}^V \arrow[r, "\bar{\alpha}_G"] \arrow[d] & \mathrm{Pre}(G) \arrow[d] \\ \mathrm{Face}\,\mathcal{A}_{\mathrm{tor}}(G) \arrow[r, dotted, "\bar{\alpha}_G"'] & \mathrm{Pre}(G)/\equiv \end{tikzcd} \tag{18}$$

In other words, given two points x, x' *in* $\mathbb{R}^V/\mathbb{Z}^V$ *the equivalence* $\bar{\alpha}_G(x) \equiv \bar{\alpha}_G(x')$ *holds if and only if* x, x' *lie on the same open face of* $\mathcal{A}_{\mathrm{tor}}(G)$.

Definition 4.12. A *toric preposet* is a set that arises as an open face of a toric poset $P = P(c)$ for at least one graph G.

If x lies on a toric face F_π^{tor} of P, where (without loss of generality) $\pi = \mathrm{cl}_P^{\mathrm{tor}}(\pi)$, then the strongly connected components of the preposet $\bar{\alpha}_G(x)$ are $\pi = \{B_1, \dots B_r\}$.

Example 4.13. Let $G = K_3$, as in Example 3.1. The six acyclic orientations of G fall into two toric equivalence classes. The three orientations shown in Figure 3.1 comprise one class, and so the corresponding toric poset is $P = P(G, [\omega_i])$ for any $i = 1, 2, 3$. Equivalently, the closed toric chamber is a union of order polytopes under the natural quotient map:

$$\overline{c(P)} = \bar{c}_{[\omega_i]} = \bigcup_{i=1}^{3} q(\mathcal{O}(P(G, \omega_i)))\,.$$

This should be visually clear from Figure 3.1. The two chambers in $\mathcal{A}_{\mathrm{tor}}(G)$ are the three-dimensional faces of P. Each of the three toric hyperplanes in $\mathcal{A}_{\mathrm{tor}}(G) = \{H_{12}^{\mathrm{tor}}, H_{13}^{\mathrm{tor}}, H_{23}^{\mathrm{tor}}\}$ are two-dimensional faces of P, and these (toric preposets) correspond to the following toric equivalence classes of size-2 preposets $\omega_\pi/\!\sim_\pi$ over K_3:

$$\left\{ \{3\} \longrightarrow \{1,2\}\,,\ \{3\} \longleftarrow \{1,2\} \right\} \qquad \left\{ \{2\} \longrightarrow \{1,3\}\,,\ \{2\} \longleftarrow \{1,3\} \right\} \qquad \left\{ \{1\} \longrightarrow \{2,3\}\,,\ \{1\} \longleftarrow \{2,3\} \right\}$$

The toric flat $D_{\{V\}}$ is the unique one-dimensional face of P, and this corresponds to the unique size-1 preposet over K_3; when $x_1 = x_2 = x_3$, which is trivially in its own toric equivalence class.

5. Toric intervals and antichains

Collapsing an interval or antichain of an ordinary poset defines a poset morphism. This remains true in the toric case, as will be shown in Section 6, though the toric analogues of these concepts are trickier to define. Toric antichains were introduced in [DMR15], but toric intervals are new to this paper. First, we need to review some terminology and results about toric total orders, chains, transitivity, and Hasse diagrams. This will also be needed to study toric order ideals and filters in Section 7. Much of the content in Sections 5.1–5.3 can be found in [DMR15]. Throughout, $G = (V, E)$ is a fixed undirected graph with $|V| = n$, and coordinates x_i of points $x = (x_1, \dots, x_n)$ in a toric chamber $c(P)$ are assumed to be reduced modulo 1, i.e., $x_i \in [0, 1)$.

5.1. **Toric total orders.** A toric poset P' is a *total toric order* if $c(P')$ is a chamber of $\mathcal{A}_{\mathrm{tor}}(K_V)$. If $P(G, [\omega])$ is a total toric order, then $P(G, \omega')$ is a total order for each $\omega' \in [\omega]$, and thus $[\omega]$ has precisely $|V|$ elements. Since each $P(G, \omega')$ has exactly one linear extension, total toric orders are indexed by the $(n-1)!$ cyclic equivalence classes of permutations of V:

$$[w] = [(w_1, \dots, w_n)] := \big\{(w_1, \dots, w_n), (w_2, \dots, w_n, w_1), \dots, (w_n, w_1, \dots, w_{n-1})\big\}\,.$$

Recall that if P and P' are toric posets over G, then P' is an extension of P if $c(P') \subseteq c(P)$. Moreover, P' is a *total toric extension* if P' is a total toric order. Analogous to how a poset is determined by its linear extensions, a toric poset P is determined by its set of total toric extensions, denoted $\mathcal{L}_{\mathrm{tor}}(P)$.

Theorem 5.1. *([DMR15], Proposition 1.7) Any toric poset P is completely determined by its total toric extensions in the following sense:*

$$\overline{c(P)} = \bigcup_{P' \in \mathcal{L}_{\mathrm{tor}}(P)} \overline{c(P')}\,.$$

5.2. **Toric directed paths, chains, and transitivity.** A *chain* in a poset $P(G,\omega)$ is a totally ordered subset $C \subseteq V$. Equivalently, this means that the elements in C all lie on a common *directed path* $i_1 \to i_2 \to \cdots \to i_m$ in ω. Transitivity can be characterized in this language: if i and j lie on a common chain, then i and j are comparable in $P(G,\omega)$. Geometrically, i and j being comparable means the hyperplane $H_{i,j}$ does not cut (i.e., is disjoint from) the chamber $c(P(G,\omega))$.

The toric analogue of a chain is "essentially" a totally cyclically ordered set, but care must be taken in the case when $|C| = 2$ because every size-two subset $C \subseteq V$ is trivially totally cyclically ordered. Define a *toric directed path* in ω, to be a directed path $i_1 \to i_2 \to \cdots \to i_m$ such that the edge $i_1 \to i_m$ is also present. We denote such a path by $i_1 \to_{\mathrm{tor}} i_m$. Toric directed paths of size 2 are simply edges, and every singleton set is a toric directed path of size 1. A fundamental property of toric directed paths is that up to cyclic shifts, they are invariants of toric-equivalence classes. That is, $i_1 \to_{\mathrm{tor}} i_m$ is a toric directed path of ω if and only if each $\omega' \in [\omega]$ has a toric directed path $j_1 \to_{\mathrm{tor}} j_m$, for some cyclic shift $(j_1,\dots,j_m)$ in $[(i_1,\dots,i_m)]$. This is Proposition 4.2 of [DMR15], and it leads to the notion of a toric chain, which is a totally cyclically ordered subset.

Definition 5.2. Let $P = P(G,[\omega])$ be a toric poset. A subset $C = \{i_1,\dots,i_m\} \subseteq V$ is a *toric chain* of P if there exists a cyclic equivalence class $[(i_1,\dots,i_m)]$ of linear orderings of C with the following property: for every $x \in c(P)$ there exists some $(j_1,\dots,j_m)$ in $[(i_1,\dots,i_m)]$ for which

$$0 \le x_{j_1} < x_{j_2} < \cdots < x_{j_m} < 1\,.$$

In this situation, we will say that $P|_C = [(i_1,\dots,i_m)]$.

The following is a reformulation of Proposition 6.3 of [DMR15] using the language of this paper, where notation such as $P(G,\omega)$ and $P(G,[\omega])$ is new.

Proposition 5.3. *Fix a toric poset $P = P(G,[\omega])$, and $C = \{i_1,\dots,i_m\} \subseteq V$. The first three of the following four conditions are equivalent, and when $m = |C| \neq 2$, they are also equivalent to the fourth.*

(a) C is a toric chain in P, with $P|_C = [(i_1,\dots,i_m)]$.
(b) For every $\omega' \in [\omega]$, the set C is a chain of $P(G,\omega')$, ordered in some cyclic shift of $(i_1,\dots,i_m)$.
(c) For every $\omega' \in [\omega]$, the set C occurs as a subsequence of a toric directed path in ω', in some cyclic shift of the order $(i_i,\dots,i_m)$.
(d) Every total toric extension $[w]$ in $\mathcal{L}_{\mathrm{tor}}(P)$ has the same restriction $[w|_C] = [(i_1,\dots,i_m)]$.

For ordinary posets, all subsets of chains are chains. The same holds in the toric case.

Proposition 5.4. *Subsets of toric chains are toric chains.*

Having the concept of a toric chain leads to the notion of *toric transitivity*, which is completely analogous to ordinary transitivity when stated geometrically.

Proposition 5.5. *Let $i,j \in V$ be distinct. Then the hyperplane $H^{\mathrm{tor}}_{i,j}$ does not cut the chamber $c(P(G,[\omega]))$ if and only if i and j lie on a common toric chain.*

5.3. **Toric Hasse diagrams.** One of the major drawbacks to studying toric posets combinatorially, as equivalences of acyclic orientations (rather than geometrically, as toric chambers), is that a toric poset P or chamber $c = c(P)$ generally arises in multiple toric graphic arrangements $\mathcal{A}_{\mathrm{tor}}(G)$ over the same vertex set. That is, one can have $P(G,[\omega]) = P(G',[\omega'])$ for different graphs, leading to ambiguity in labeling a toric poset P with a pair $(G,[\omega])$ consisting of a graph G and equivalence class $[\omega]$ in $\mathrm{Acyc}(G)/\!\equiv$.

Toric transitivity resolves this issue. As with ordinary posets, there is a well-defined notion for toric posets of what it means for an edge to be "implied by transitivity." The *toric Hasse* *diagram* is the graph $\hat{G}^{\text{torHasse}}$ with all such edges removed. In Section 5.3, we encountered the *toric transitive closure*, which is the graph $\bar{G}^{\text{tor}}$ with all such edges included. In other words, given any toric poset $P = P(G, [\omega])$, there is always a unique minimal pair $(\hat{G}^{\text{torHasse}}(P), [\omega^{\text{torHasse}}(P)])$ and maximal pair $(\bar{G}^{\text{tor}}(P), [\bar{\omega}^{\text{tor}}(P)])$ with the property that the set $c(P)$ is in $\text{Cham}\,\mathcal{A}_{\text{tor}}(G)$ iff

$$\hat{G}^{\text{torHasse}}(P) \subseteq G \subseteq \bar{G}^{\text{tor}}(P)$$

where $\subseteq$ is inclusion of edges. In this case, ω can be taken to be the restriction to G of any orientation in $[\bar{\omega}^{\text{tor}}(P)]$.

Geometrically, the existence of a unique toric Hasse diagram is intuitive; it corresponds to the minimal set of toric hyperplanes that bound the chamber $c(P)$, and the edges implied by transitivity correspond to the additional hyperplanes that do not cut $c(P)$. The technical combinatorial reason for the existence of a unique Hasse diagram (respectively, toric Hasse diagram) follows because the transitive closure (respectively, toric transitive closure) $A \longmapsto \bar{A}$ is a *convex closure*, meaning it satisfies the following anti-exchange condition; also see [EJ85]:

$$\text{for } a \neq b \text{ with } a, b \notin \bar{A} \text{ and } a \in \overline{A \cup \{b\}}, \text{ one has } b \notin \overline{A \cup \{a\}}\,.$$

Edges $\{i, j\}$ in the Hasse diagram (respectively, toric Hasse diagram) are precisely those whose removal "change" the poset (respectively, toric poset), and the geometric definitions make this precise. Though the ordinary and toric cases are analogous, there are a few subtle differences. For example, consider the following "folk theorem."

Proposition 5.6. *Let P be a poset over $G = (V, E)$ and $\{i, j\} \in E$. Then the following are equivalent:*

(i) The edge $\{i, j\}$ is in the Hasse diagram, $\hat{G}^{\text{Hasse}}(P)$.
(ii) Removing $H_{i,j}$ enlarges the chamber $c(P)$.
(iii) $\overline{c(P)} \cap H_{i,j}$ is a (closed) facet of P.
(iv) The interval $[i, j]$ is precisely $\{i, j\}$.
(v) Removing $H_{i,j}$ from $\mathcal{A}(G)$ strictly increases the number of chambers.

Since toric posets are defined geometrically as subsets of $\mathbb{R}^V/\mathbb{Z}^V$ that are chambers of a graphic hyperplane arrangement, the equivalence (i)$\Leftrightarrow$(ii) is immediate for toric posets. Condition (iii) says that the edges $\{i, j\}$ of the Hasse diagram are precisely the size-2 intervals, and Condition (iv) says these are the closed face partitions having two blocks of the form $\pi = \{\{i, j\}, V - \{i, j\}\}$. Finally, note that the implication (i)$\Rightarrow$(v) in Proposition 5.6 trivially fails for toric posets. For a simple counterexample, take any tree G with at least one edge. Since $\mathcal{A}_{\text{tor}}(G)$ has only one chamber, removing any $H^{\text{tor}}_{i,j}$ will never increase the number of chambers.

Since adding or removing edges implied by toric transitivity does not change the toric poset, it does not change which sets are toric chains. Thus, to characterize the toric chains of $P(G, [\omega])$, it suffices to characterize the toric chains of $P(\bar{G}^{\text{tor}}(P), [\bar{\omega}^{\text{tor}}(P)])$. The following is immediate.

Remark 5.7. Let P be a toric poset. A size-2 subset $C = \{i, j\}$ of V is a toric chain of P if and only if $\{i, j\}$ is an edge of $\bar{G}^{\text{tor}}(P)$. In particular, if C is a maximal toric chain, then $\{i, j\}$ is an edge in $\hat{G}^{\text{torHasse}}(P)$.

5.4. **Toric intervals.** To motivate the definition of a toric interval, it helps to first interpret the classical definition in several different ways.

Definition 5.8. Let i and j be elements of a poset $P = P(G, \omega)$. The *interval* $[i, j]$ is the set I characterized by one of the following equivalent conditions:

(i) $I = \{k \in V : x_i \leq x_k \leq x_j, \text{ for all } x \in c(P)\}$;
(ii) $I = \{k \in V : k \text{ appears between } i \text{ and } j \text{ (inclusive) in any linear extension of } P\}$;
(iii) $I = \{k \in V : k \text{ lies on a directed path from } i \text{ to } j \text{ in } \omega\}$.

Note that if $i = j$, then $[i, j] = \{i\}$, and if $i \not\leq j$, then $[i, j] = \varnothing$.

We will define toric intervals geometrically, motivated by Condition (i), and show how it is equivalent to the toric version of Condition (ii). In contrast, Condition (iii) has a small wrinkle –

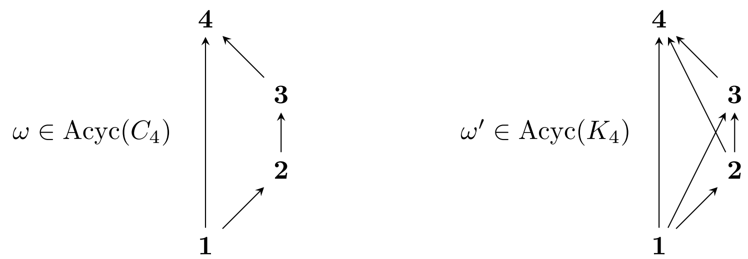


FIGURE 3. Despite the equality $P(C_4, [\omega]) = P(K_4, [\omega'])$, the set $I = \{1, 3\}$ lies on a toric directed path $1 \to_{\text{tor}} 3$ in ω' but not for any representative of $[\omega]$.

the property of lying on a directed path from i to j does not depend on the choice of (G, ω) for P. Specifically, if $P(G, \omega) = P(G', \omega')$ and k lies on an ω-directed path from i to j, then k lies on an ω'-directed path from i to j. This is not the case for toric directed paths in toric posets, as the following example illustrates. As a result, we will formulate and prove a modified version of Condition (iii) for toric intervals.

Example 5.9. Consider the circular graph $G = C_4$, and $\omega \in \mathrm{Acyc}(C_4)$ as shown at left in Figure 3. Let $P = P(C_4, [\omega])$, which is a total toric order. Therefore, the toric transitive closure of $(C_4, [\omega])$ is the pair $(K_4, [\omega'])$, where ω' is shown in Figure 3 on the right. Therefore, $P(C_4, [\omega]) = P(K_4, [\omega'])$.

Now, let $i = 1$ and $j = 3$. The set $\{i, j\}$ lies on a toric directed path from 1 to 3 in ω' (which also contains 2). However, none of the 4 representatives in $[\omega]$ contain a toric directed path from 1 to 3.

Another obstacle to formulating the correct toric analogue of an interval is how to characterize which size-2 subsets should be toric intervals. This ambiguity arises from the aforementioned "size-2 chain problem" of all size-2 subsets being totally cyclically ordered. Since ordinary intervals are unions of chains, we will require this to be a feature of toric intervals.

Definition 5.10. Let i and j be elements of a toric poset $P = P(G, [\omega])$. The *toric interval* $I = [i, j]^{\text{tor}}$ is the empty set if i, j do not lie on a common toric chain, and otherwise is the set

$$I = [i, j]^{\text{tor}} := \{i, j\} \cup \{k \in V : P|_{\{i,j,k\}} = [(i, k, j)]\}\,. \tag{19}$$

If there is no $k \notin \{i, j\}$ satisfying Eq. (19), then $[i, j]^{\text{tor}} = \{i, j\}$. If $i = j$, then $[i, j]^{\text{tor}} = \{i\}$.

Remark 5.11. If i, j, k are distinct elements of the toric interval $I = [i, j]^{\text{tor}}$ of $P = P(G, [\omega])$, then for each x in $c(P)$, exactly one of the following must hold:

$$0 \le x_i < x_k < x_j < 1\,, \qquad 0 \le x_j < x_i < x_k < 1\,, \qquad 0 \le x_k < x_j < x_i < 1\,. \tag{20}$$

By Theorem 5.1, we can rephrase Remark 5.11 as the toric analogue of Definition 5.8(ii): the toric interval $[i, j]^{\text{tor}}$ in P is the set of elements between i and j in the cyclic order of any total toric extension of P.

Corollary 5.12. *Suppose $I = [i, j]^{\text{tor}}$ is a toric interval of $P = P(G, [\omega])$ of size $|I| \ge 3$. Then*

$$[i, j]^{\text{tor}} = \{i, j\} \cup \{k : [w|_{\{i,j,k\}}] = [(i, k, j)], \ \text{for all } [w] \in \mathcal{L}_{\text{tor}}(P)\}\,.$$

Finally, the toric analogue of Definition 5.8(iii) can be obtained by first passing to the toric transitive closure.

Proposition 5.13. *Fix a toric poset $P = P(G, [\omega])$. An element k is in $[i, j]^{\text{tor}}$ if and only if k lies on a toric directed path $i \to_{\text{tor}} j$ in $\bar{\omega}'$, for some $\bar{\omega}' \in [\bar{\omega}^{\text{tor}}(P)]$.*

Proof. Throughout, let $C = \{i, j, k\}$. Assume that $|C| \ge 3$; the result is trivial otherwise. Suppose k is in $[i, j]^{\text{tor}}$, which means that $P|_C = [(i, k, j)]$. Take any $\omega' \in [\omega]$ for which i is a source. By Proposition 5.3, the elements of C occur as a subsequence of a toric directed path in ω', ordered (i, k, j). Since this is a toric chain, the edges $\{i, k\}$, $\{k, j\}$, and $\{i, j\}$ are all implied by toric transitivity. Thus, k lies on a toric directed path $i \to_{\rm tor} j$ in $\bar{\omega}'$, the unique orientation of $[\bar{\omega}^{\rm tor}(P)]$ whose restriction to G is ω'.

Conversely, suppose that k lies on a toric directed path $i \to_{\rm tor} j$ in $\bar{\omega}'$, for some $\bar{\omega}' \in [\bar{\omega}^{\rm tor}(P)]$. Then C is a toric chain, ordered $P|_C = [(i,k,j)]$, hence k is in $[i,j]^{\rm tor}$. □

Proposition 5.14. *Let $P = P(G,[\omega])$ be a toric poset. If a set $I \subseteq V$ is a toric interval $I = [i,j]^{\rm tor}$, then there is some $\omega' \in [\omega]$ for which the set I is the interval $[i,j]$ of $P(G,\omega')$. The converse need not hold.*

Proof. Without loss of generality, assume that $G = \bar{G}^{\rm tor}(P)$. The statement is trivial if $|I| < 2$. We need to consider the cases $|I| = 2$ and $|I| \geq 3$ separately. In both cases, we will show that one can take ω' to be any orientation that has i as a source.

First, suppose $|I| = 2$, which means that $I = \{i,j\}$ is an edge of G. Take any $\omega' \in [\omega]$ for which i is a source. Since $|I| = 2$, there is no other $k \notin \{i,j\}$ on a directed path from i to j in ω', as this would form a toric directed path. Therefore, the interval $[i,j]$ in $P(G,\omega')$ is simply $\{i,j\}$.

Next, suppose $|I| \geq 3$. As before, take any $\omega' \in [\omega]$ such that i is a source in ω'. Since $G = \bar{G}^{\rm tor}(P)$, the directed edge $i \to j$ is present, and so by Proposition 5.13, $[i,j]^{\rm tor}$ consists of all $k \in V$ that lie on a directed path from i to j. This is precisely the definition of the interval $[i,j]$ in $P(G,\omega')$.

To see how the converse can fail, take G to be the line graph on 3 vertices, and ω to be the orientation $1 \to 2 \to 3$. In $P(G,\omega')$, the interval $[1,3]$ is $\{1,2,3\}$ but since 1 and 3 do not lie on a common toric chain, $[1,3]^{\rm tor} = \varnothing$ in $P(G,[\omega])$. □

Proposition 5.15. *For any toric poset $P = P(G,[\omega])$,*

$$E(\hat{G}^{\rm Hasse}(P(G,\omega))) \subseteq E(\hat{G}^{\rm torHasse}(P(G,[\omega]))) \subseteq E(\bar{G}^{\rm tor}(P)) = \bigcap_{\omega' \in [\omega]} E(\bar{G}(P(G,\omega'))) \,. \tag{21}$$

Proof. Given the toric Hasse diagram of $P(G,[\omega])$, the ordinary Hasse diagram of $P(G,\omega)$ is obtained by removing the edge $\{i_1,i_m\}$ for each toric directed path $i_1 \to_{\rm tor} i_m$ in $P(G,\omega)$ of size at least $m \geq 3$. This establishes the first inequality in Eq. (21).

The second inequality is obvious. Loosely speaking, the final equality holds because edges in the toric transitive closure are precisely the size-2 toric chains, which are precisely the subsets that are size-2 chains in every representative poset. We will prove each containment explicitly. For "$\subseteq$", take an edge $\{i,j\}$ of $\bar{G}^{\rm tor}(P)$, which is a size-2 toric chain. By Proposition 5.3, $\{i,j\}$ is a toric chain of $P(G,\omega')$ for all $\omega' \in [\omega]$, which means that it is an edge of the transitive closure $\bar{G}(P(G,\omega'))$. The "$\supseteq$" containment is analogous: suppose $\{i,j\}$ is an edge of $\bar{G}(P(G,\omega'))$ for each $\omega' \in [\omega]$. Then by Proposition 5.3, it is a toric chain of P, and hence an edge of $\bar{G}^{\rm tor}(P)$. □

5.5. **Toric antichains.** An *antichain* of an ordinary poset P is a subset $A \subseteq V$ characterized

- *combinatorially* by the condition that no pair $\{i,j\} \subseteq A$ with $i \neq j$ are comparable, that is, they lie on no common chain of P, or
- *geometrically* by the equivalent condition that the $(|V| - |A| + 1)$-dimensional subspace D_{π_A} intersects the open polyhedral cone $c(P)$ in $\mathbb{R}^V$.

As shown in [DMR15], these two conditions in the toric setting lead to different notions of toric antichains which are both easy to formulate. Unlike the case of ordinary posets, these two definitions are non-equivalent; leading to two distinct versions of a toric antichains, combinatorial and geometric. The following is the geometric one which we will use in this paper. Its appearance in Proposition 5.17, which is one of the "*For Some*" structure theorems listed in the Introduction, suggests that it is the more natural toric analogue of the two.

Definition 5.16. Given a toric poset P on V, say that $A \subseteq V$ is a *geometric toric antichain* if $D^{\rm tor}_{\pi_A}$ intersects the open toric chamber $c(P)$ in $\mathbb{R}^V/\mathbb{Z}^V$.

The following characterization of toric antichains was established in [DMR15]. It follows because if $D^{\rm tor}_{\pi_S}$ intersects the open toric chamber $c = c(P)$ in $\operatorname{Cham}\mathcal{A}_{\rm tor}(G)$, then D_{π_S} intersects the open chamber upstairs in $\operatorname{Cham}\mathcal{A}(G)$.

Proposition 5.17. *Let $P = P(G, [\omega])$ be a toric poset. Then a set $A \subseteq V$ is a geometric toric antichain of P if and only if A is an antichain of $P(G, \omega')$ for some $\omega' \in [\omega]$.*

6. Morphisms of toric posets

Morphisms of ordinary posets have equivalent combinatorial and geometric characterizations. In contrast, while there seems to be no simple or obvious combinatorial description for morphisms of toric posets, the geometric version has a natural toric analogue.

Firstly, it is clear how to define a *toric isomorphism* between two toric posets P and P' on vertex sets V and V': a bijection $\phi\colon V \to V'$ such that the induced isomorphism on $\mathbb{R}^V/\mathbb{Z}^V \to \mathbb{R}^{V'}/\mathbb{Z}^{V'}$ maps $c(P)$ to $c(P')$ bijectively. The other types of ordinary poset morphisms have the following toric analogues:

- *quotients* that correspond to projecting the toric chamber onto a flat of $\mathcal{A}_{\mathrm{tor}}(G'_\pi)$ for some closed toric face partition $\pi = \mathrm{cl}_P^{\mathrm{tor}}(\pi)$;
- *inclusions* that correspond to embedding a toric chamber into a higher-dimensional chamber;
- *extensions* that add relations (toric hyperplanes).

Since every poset morphism can be expressed as the composition of a quotient, an inclusion, and an extension, it is well-founded to define a *toric poset morphism* to be the composition of the toric analogues of these maps. In the remainder of this section, we will describe toric morphisms in detail. Most of the difficulties have already been done in Section 3, when interpreting the well-known concept of an ordinary poset morphism geometrically. In contrast, this section is simply an adaptation of this geometric framework from $\mathbb{R}^V$ to $\mathbb{R}^V/\mathbb{Z}^V$, though there are some noticeable differences. For example, there is no toric analogue of intersecting a chamber with a half-space, because the torus minus a hyperplane is connected.

6.1. Quotient.

In the ordinary poset case, a quotient is performed by contracting $P(G, \omega)$ by a partition $\pi = \{B_1, \dots, B_r\}$. Each B_i gets collapsed into a single vertex, and the resulting acyclic graph is denoted by $\omega/{\sim_\pi}$, which is an element of $\mathrm{Acyc}(G/{\sim_\pi})$. This does not carry over to the toric case, because in general, contracting a partition will make some representatives acyclic and others not. However, the geometric definition has a natural analogue.

Now, let $P = P(G, [\omega])$ be a toric poset, and π be any partition of V closed with respect to P, i.e., $\pi = \mathrm{cl}_P^{\mathrm{tor}}(\pi)$. By construction, D_π^{tor} is a flat of $\mathcal{A}_{\mathrm{tor}}(G'_\pi)$, and so the subset $F_\pi^{\mathrm{tor}}(P)$ is a face of $\mathcal{A}_{\mathrm{tor}}(G'_\pi)$. First, we need a map that projects a point x in $c(P)$ onto this face, which is relatively open in the subspace topology of D_π^{tor}. This can be extended to the entire torus, by taking the unique map $\bar{d}_\pi$ that makes the following diagram commute, where d_π is the mapping from Eq. (9):

$$\begin{tikzcd} \mathbb{R}^V \arrow[r, "d_\pi"] \arrow[d, "q"'] & D_\pi \arrow[d, "q"] \\ \mathbb{R}^V/\mathbb{Z}^V \arrow[r, dotted, "\bar{d}_\pi"] & D_\pi^{\mathrm{tor}} \end{tikzcd}$$

Explicitly, the map $\bar{d}_\pi$ takes a point $x \in \mathbb{R}^V/\mathbb{Z}^V$, lifts it to a point $\hat{x}$ in an order polytope in $\mathbb{R}^V$, projects it onto the flat D_π as in Eq. (9), and then maps that point down to the toric flat D_π^{tor}. In light of this, we will say that the map $\bar{d}_\pi$ is a *projection* onto the toric flat D_π^{tor}.

After projecting a chamber $c(P)$ onto a flat D_π^{tor} of $\mathcal{A}(G'_\pi)$, we need to project it homeomorphically onto a coordinate subspace of $\mathbb{R}^V/\mathbb{Z}^V$ so it is a chamber of a lower-dimensional toric arrangement. As in the ordinary case, let $W \subseteq V$ be any subset formed by removing all but 1 coordinate from each B_i, and let $\bar{r}_\pi\colon \mathbb{R}^V/\mathbb{Z}^V \longrightarrow \mathbb{R}^W/\mathbb{Z}^W$ be the induced projection. The $\bar{r}_\pi$-image of $\mathcal{A}_{\mathrm{tor}}(G)$ will be the toric arrangement $\mathcal{A}_{\mathrm{tor}}(G/{\sim_\pi})$. As before, the following easily verifiable lemma ensures that our choice of $W \subseteq V$ does not matter.

Lemma 6.1. *Let $\pi = \{B_1, \dots, B_r\}$ be a partition of V, and $W = \{b_1, \dots, b_r\} \subseteq V$ with $b_i \in B_i$. The restriction $\bar{r}_\pi|_{D_\pi^{\mathrm{tor}}}\colon D_\pi^{\mathrm{tor}} \longrightarrow \mathbb{R}^W/\mathbb{Z}^W$ is a homeomorphism.*

Moreover, all such projection maps for a fixed π *are topologically conjugate in the following sense: If* $W' = \{b'_1, \dots, b'_r\} \subseteq V$ *with* $b'_i \in B_i$*, and projection map* $\bar{r}'_\pi|_{D_\pi} : D_\pi^{\mathrm{tor}} \longrightarrow \mathbb{R}^{W'}/\mathbb{Z}^{W'}$*, and* σ *is the permutation of* V *that transposes each* b_i *with* b'_i*, then the following diagram commutes:*

$$\begin{tikzcd}
D_\pi^{\mathrm{tor}} \arrow[r, hook] & \mathbb{R}^V/\mathbb{Z}^V \arrow[r, "\bar{r}_\pi"] \arrow[d, "\bar{\Sigma}"'] & \mathbb{R}^W/\mathbb{Z}^W \arrow[d, "\bar{\Sigma}'"] \\
D_\pi^{\mathrm{tor}} \arrow[r, hook] & \mathbb{R}^V/\mathbb{Z}^V \arrow[r, "\bar{r}'_\pi"] & \mathbb{R}^{W'}/\mathbb{Z}^{W'}
\end{tikzcd}$$

Here, $\sigma' : W \to W'$ *is the map* $b_i \mapsto b'_i$*, with* $\bar{\Sigma}$ *and* $\bar{\Sigma}'$ *being the induced linear maps as defined in Eq.* (6)*, but done modulo* 1*.*

By convexity (in the fundamental affine chambers), two points in the same face of $\mathcal{A}_{\mathrm{tor}}(G)$ get mapped to the same face in $\mathcal{A}_{\mathrm{tor}}(G/{\sim_\pi})$. In other words, $\bar{d}_\pi$ induces a well-defined map $\bar{\delta}_\pi$ from $\mathrm{Face}\,\mathcal{A}_{\mathrm{tor}}(G)$ to $\mathrm{Face}\,\mathcal{A}_{\mathrm{tor}}(G'_\pi)$ making the following diagram commute:

$$\begin{tikzcd}
\mathbb{R}^V/\mathbb{Z}^V \arrow[r, "\bar{d}_\pi"] \arrow[d] & D_\pi^{\mathrm{tor}} \arrow[d] \\
\mathrm{Face}\,\mathcal{A}_{\mathrm{tor}}(G) \arrow[r, dotted, "\bar{\delta}_\pi"] & \mathrm{Face}\,\mathcal{A}_{\mathrm{tor}}(G/{\sim_\pi})
\end{tikzcd}$$

Explicitly, the map $\bar{\delta}_\pi$ is easiest to defined by the analogous map on closed faces:

$$\bar{\delta}_\pi : \overline{\mathrm{Face}}\,\mathcal{A}_{\mathrm{tor}}(G) \longrightarrow \overline{\mathrm{Face}}\,\mathcal{A}_{\mathrm{tor}}(G'_\pi)\,, \qquad \bar{\delta}_\pi : \overline{c(P)} \cap D_\sigma^{\mathrm{tor}} \longmapsto \overline{c(P)} \cap D_\sigma^{\mathrm{tor}} \cap D_\pi^{\mathrm{tor}}\,.$$

The open faces of $\mathcal{A}_{\mathrm{tor}}(G'_\pi)$ are then mapped to faces of the arrangement $\mathcal{A}_{\mathrm{tor}}(G/{\sim_\pi})$ under the projection $\bar{r}_\pi|_{D_\pi} : D_\pi^{\mathrm{tor}} \longrightarrow \mathbb{R}^W/\mathbb{Z}^W$. Combinatorially, the open faces of $\mathcal{A}_{\mathrm{tor}}(G)$ are toric preposets $[\omega]$ over G (i.e., in $\mathrm{Pre}(G)/{\equiv}$). These are mapped to toric preposets over $G/{\sim_\pi}$ via the composition

$$\mathrm{Pre}(G) \xrightarrow{\bar{q}_\pi} \mathrm{Pre}(G'_\pi) \xrightarrow{\bar{p}_\pi} \mathrm{Pre}(G/{\sim_\pi})\,, \qquad [\omega] \overset{\bar{q}_\pi}{\longmapsto} [\omega'_\pi] \overset{\bar{p}_\pi}{\longmapsto} [\omega'_\pi/{\sim_\pi}] = [\omega/{\sim_\pi}]\,.$$

The following commutative diagram illustrates the relationship between the points in $\mathbb{R}^V/\mathbb{Z}^V$, the faces of the toric graphic arrangement $\mathcal{A}_{\mathrm{tor}}(G)$, and the toric preposets over G. The left column depicts the toric preposets over G that are also toric posets.

$$\begin{tikzcd}
\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G) \arrow[r, hook] \arrow[d] & \mathbb{R}^V/\mathbb{Z}^V \arrow[r, "\bar{d}_\pi"] \arrow[d] & D_\pi^{\mathrm{tor}} \arrow[r, "\bar{r}_\pi"] \arrow[d] & \mathbb{R}^W/\mathbb{Z}^W \arrow[d] \\
\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G) \arrow[r, hook] \arrow[d, leftrightarrow] & \mathrm{Face}\,\mathcal{A}_{\mathrm{tor}}(G) \arrow[r, "\bar{\delta}_\pi"] \arrow[d, leftrightarrow] & \mathrm{Face}\,\mathcal{A}_{\mathrm{tor}}(G'_\pi) \arrow[r, "\bar{\rho}_\pi"] \arrow[d, leftrightarrow] & \mathrm{Face}\,\mathcal{A}_{\mathrm{tor}}(G/{\sim_\pi}) \arrow[d, leftrightarrow] \\
\mathrm{Acyc}(G)/{\equiv} \arrow[r, hook] & \mathrm{Pre}(G)/{\equiv} \arrow[r, "\bar{q}_\pi"] & \mathrm{Pre}(G'_\pi)/{\equiv} \arrow[r, "\bar{p}_\pi"] & \mathrm{Pre}(G/{\sim_\pi})/{\equiv}
\end{tikzcd}$$

To summarize, toric poset morphisms that are quotients are characterized geometrically by projecting the toric chamber $c(P)$ onto a flat of $\mathcal{A}_{\mathrm{tor}}(G'_\pi)$, for some closed toric face partition $\pi = \mathrm{cl}_P^{\mathrm{tor}}(\pi)$. Applying Theorem 4.7 gives a combinatorial interpretation of this, which was not *a priori* obvious.

Corollary 6.2. *Let* $P = P(G, [\omega])$ *be a toric poset. Contracting* G *by a partition* $\pi \in \Pi_V$ *yields a morphism to a toric poset over* $G/{\sim_\pi}$ *if and only if* $\omega'/{\sim_\pi}$ *is acyclic for some orientation* $\omega' \in [\omega]$*.* □

The following is now immediate from Propositions 5.14 and 5.17.

Corollary 6.3. *Let* P *be a toric poset over* V*. Then contracting a toric interval* $I \subseteq V$ *or a geometric toric antichain* $A \subseteq V$ *defines a toric morphism.* □

6.2. **Inclusion.** Just like for ordinary posets, a toric poset can be included in larger one. Let P be a poset over V and let $V \subsetneq V'$. The simplest injection adds vertices (dimension) but no edges (extra relations). In this case, the inclusion $\phi\colon P \hookrightarrow P'$ defines a canonical inclusion $\Phi\colon \mathbb{R}^V/\mathbb{Z}^V \hookrightarrow \mathbb{R}^{V'}/\mathbb{Z}^{V'}$. This sends the arrangement $\mathcal{A}_{\mathrm{tor}}(G)$ in $\mathbb{R}^V$, where $G=(V,E)$, to the same higher-dimensional arrangement:

$$\mathcal{A}_{\mathrm{tor}}(G \cup (V'-V)) := \{H_{ij}^{\mathrm{tor}} \text{ in } \mathbb{R}^{V'} : \{i,j\} \in E\}\,.$$

The toric chamber $c = c(P)$ is sent to the chamber

$$c(P') = \{x \in \mathbb{R}^{V'} : x_i < x_j \text{ for } \{i,j\} \in E\}\,.$$

More generally, an injection $P \to P'$ can have added relations in P' either among the vertices in P or those in $V'-V$. Such a map is simply the composition of an inclusion described above and a toric extension, described below.

6.3. **Extension.** Extensions of ordinary posets were discussed in Section 3.2. A poset P' is an extension of P (both assumed to be over the same set V) if any of the three equivalent conditions holds:

- $i \leq_P j$ implies $i \leq_{P'} j$;
- $\hat{G}^{\mathrm{Hasse}}(P) \subseteq \hat{G}^{\mathrm{Hasse}}(P')$, where $\subseteq$ is inclusion of edge sets;
- $c(P') \subseteq c(P)$.

The first of these conditions does not carry over nicely to the toric setting, but the second two do. A toric poset P' is a *toric extension* of P if and only one has an inclusion of their open polyhedral cones $c(P') \subseteq c(P)$ in $\mathbb{R}^V/\mathbb{Z}^V$, which is equivalent to $\hat{G}^{\mathrm{torHasse}}(P) \subseteq \hat{G}^{\mathrm{torHasse}}(P')$.

6.4. **Summary.** Up to isomorphism, every toric poset morphism $P = P(G,[\omega])$ can be decomposed into a sequence of three steps:

(i) *quotient*: Collapsing G by a partition π that preserves acyclicity of some $\omega' \in [\omega]$ (projecting to a flat D_π^{tor} of $\mathcal{A}(G'_\pi)$ for some partition $\pi = \mathrm{cl}_P^{\mathrm{tor}}(\pi)$).
(ii) *inclusion*: Adding vertices (adding dimensions).
(iii) *extension*: Adding relations (cutting the chamber with toric hyperplanes).

Note that in the special case of the morphism $P \longrightarrow P'$ being surjective, the inclusion step is eliminated and the entire process can be described geometrically by projecting $\overline{c(P)}$ to a toric flat D_π^{tor} and a then adding toric hyperplanes.

7. Toric order ideals and filters

Let P be a poset over a set V of size at least 2, and suppose $\phi\colon P \to P'$ is a morphism to a poset over a size-2 subset $V' \subseteq V$. This is achieved by projecting $c(P)$ onto a flat D_π of $\mathcal{A}(G'_\pi)$ such that $\pi = \mathrm{cl}_P(\pi)$ has at most two blocks, and hence $\bar{F}_\pi = \overline{c(P)} \cap D_\pi$ is at most 2-dimensional. A point $x = (x_1, \dots, x_n)$ on F_π has at most two distinct entries. Thus, the partition $\pi = \{I, J\}$ of V satisfies

- $x_{i_k} = x_{i_\ell}$ for all i_k, i_ℓ in I;
- $x_{j_k} = x_{j_\ell}$ for all j_k, j_ℓ in J;
- $x_i \leq x_j$ for all $i \in I$ and $j \in J$.

The set I is called an *order ideal* or just an *ideal* of P and J is called a *filter*.

Ideal/filter pairs are thus characterized by closed partitions π of V such that D_π intersects $\overline{c(P)}$ in at most two dimensions. The set of ideals has a natural poset structure by subset inclusion. Allowing I or J to be empty, this poset has a unique maximal element $I = V$ (corresponding to $J = \varnothing$) and minimal element $I = \varnothing$ (corresponding to $J = V$). Moreover, the order ideal poset is a lattice; this is well-known [Sta01]. Similarly, the set of filters is a lattice as well.

Toric order ideals and filters can be defined similarly.

Definition 7.1. Let P be a toric poset over V, and suppose $\phi\colon P \to P'$ is a morphism to a toric poset over a size-2 subset $V' \subseteq V$. This projects $c(P)$ onto a toric flat D_π^{tor} of $\mathcal{A}_{\mathrm{tor}}(G'_\pi)$ for some $\pi = \mathrm{cl}_P^{\mathrm{tor}}(\pi)$ such that $\bar{F}_\pi^{\mathrm{tor}} = \overline{c(P)} \cap D_\pi^{\mathrm{tor}}$ is at most 2-dimensional. For the partition $\pi = \{I, J\}$ of V, each point $x = (x_1, \dots, x_n)$ on F_π^{tor} satisfies

- $x_{i_k} \bmod 1 = x_{i_\ell} \bmod 1$ for all i_k, i_ℓ in I;
- $x_{j_k} \bmod 1 = x_{j_\ell} \bmod 1$ for all j_k, j_ℓ in J.

The set I is called a *toric order ideal* of P.

Remark 7.2. By symmetry, if I is a toric order ideal, then so is $J := V - I$. A *toric filter* can be defined analogously, and it is clear that these two concepts are identical. Henceforth, we will stick with the term "toric filter" to avoid ambiguity with the well-established but unrelated notion of a toric ideal from commutative algebra and algebraic geometry [Stu96].

By construction, toric filters are characterized by closed toric partitions π of V such that D_π^{tor} intersects $\overline{c(P)}$ in at most two dimensions – either a two-dimensional face of P or of an extension P' over G'_π.

Proposition 7.3. *Let $P(G, [\omega])$ be a toric poset. The following are equivalent for a subset $I \subseteq V$.*

(i) I is a toric filter of $P(G, [\omega])$;
(ii) I is an ideal of $P(G, \omega')$ for some $\omega' \in [\omega]$;
(iii) I is a filter of $P(G, \omega'')$ for some $\omega'' \in [\omega]$;
(iv) In at least one total toric extension of $P(G, [\omega])$, the elements in I appear in consecutive cyclic order.

Proof. The result is obvious if $I = \varnothing$ or $I = V$, so assume that $\varnothing \subsetneq I \subsetneq V$, and $\pi = \{I, V - I\}$. This forces D_π^{tor} to be two-dimensional (rather than one-dimensional).

(i)⇒(ii): If I is a toric filter of $P(G, [\omega])$, then D_π^{tor} intersects $c(P(G, [\omega]))$ in two-dimensions, and so D_π intersects an order polytope $\mathcal{O}(P(G, \omega'))$ in two-dimensions, for some $\omega' \in [\omega]$. Therefore, D_π intersects the chamber $c(P(G, \omega'))$ in two-dimensions, and hence I is an ideal of $P(G, \omega')$.

(ii)⇒(i): Suppose that I is an ideal of $P(G, \omega')$ for $\omega' \in [\omega]$. Then $\bar{F}_\pi = \overline{c(P(G, \omega'))} \cap D_\pi$ is two-dimensional, and it descends to a two-dimensional face $\bar{F}_\pi^{\mathrm{tor}}$ of the toric poset $P(G, [\omega']) = P(G, [\omega])$ or of some extension (if $\bar{F}^{\mathrm{tor}}$ intersects the interior). Therefore, I is a toric filter of $P(G, [\omega])$.

(ii)⇔(iii): Immediate by Remark 7.2 upon reversing the roles of I and $V - I$.

(ii)⇒(iv): If I is a size-k ideal of $P(G, \omega')$, then by a well-known property of posets, there is a linear extension of the form $(i_1, \dots, i_k, v_{k+1}, \dots, v_n)$, where each $i_j \in I$. The cyclic equivalence class $[(i_1, \dots, i_k, v_{k+1}, \dots, v_n)]$ is a total toric extension of $P(G, [\omega']) = P(G, [\omega])$ in which the elements of I appear in consecutive cyclic order.

(iv)⇒(ii): Suppose $[(i_1, \dots, i_k, v_{k+1}, \dots, v_n)]$ is a toric total extension of $P(G, [\omega])$. This means that for some $x \in \mathbb{R}^V / \mathbb{Z}^V$,

$$0 \le x_{i_1} < \cdots < x_{i_k} < x_{v_{k+1}} < \cdots < x_{v_n} < 1\,. \tag{22}$$

The unique preimage $\hat{x}$ of this point in $[0,1)^V$ under the quotient map $q\colon \mathbb{R}^V \to \mathbb{R}^V/\mathbb{Z}^V$ is in the order polytope of some $P(G, \omega')$ that maps into $\overline{c(P(G, [\omega']))} = \overline{c(P(G, [\omega]))}$. Since the coordinates of $\hat{x}$ are totally ordered as in Eq. (22), $(i_1, \dots, i_k, v_{k+1}, \dots, v_n)$ is a linear extension of $P(G, \omega')$. Moreover, the coordinates in I form an initial segment of this linear extension, hence I is an ideal of $P(G, \omega')$.

(iii)⇔(iv): Immediate by Remark 7.2 upon reversing the roles of I and $V - I$. □

Proposition 7.3 along with a result of Stanley gives a nice characterization of the toric filters in terms of vertices of order polytopes. Every filter I of a poset P has a *characteristic function*

$$\chi_I\colon P \longrightarrow \mathbb{R}\,, \qquad \chi_I(k) = \begin{cases} 1, & k \in I \\ 0, & k \notin I. \end{cases}$$

We identify $\chi_I(P)$ with the corresponding vector in $\{0,1\}^V \subseteq \mathbb{R}^V$.

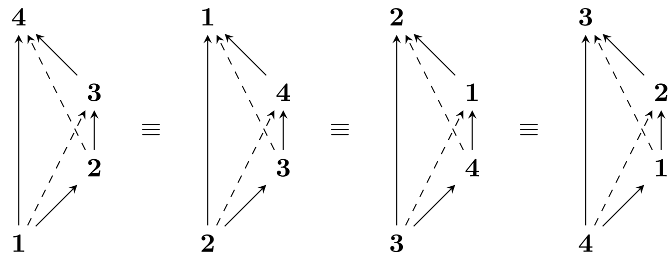
$\omega \in \mathrm{Acyc}(C_4)$  $\equiv$ $\equiv$ $\equiv$

FIGURE 4. The four torically equivalent orientations to $\omega \in \mathrm{Acyc}(C_4)$, shown at left. The edges implied by toric transitivity are dashed.

Proposition 7.4. *([Sta86], Corollary 1.3) Let P be a poset. The vertices of the order polytope $\mathcal{O}(P)$ are the characteristic functions χ_I of filters of P.*

Given a toric poset, we can define the characteristic function χ_I of a toric filter similarly. However, one must be careful because under the canonical quotient to the torus, the vertices of every order polytope get identified to $(0,\dots,0)$. Therefore, we will still identify χ_I with a point in $\mathbb{R}^V$, not $\mathbb{R}^V/\mathbb{Z}^V$.

Corollary 7.5. *Let $P = P(G,[\omega])$ be a toric poset and $I \subseteq V$. Then χ_I is the characteristic function of a toric filter of P if and only if χ_I is the vertex of $\mathcal{O}(P(G,\omega'))$ for some $\omega' \in [\omega]$.*

Let $J_{\mathrm{tor}}(P)$ denote the set of toric filters of P. This has a natural poset structure by subset inclusion. Once again, there is a unique maximal element $I = V$ and minimal element $I = \varnothing$.

Proposition 7.6. *With respect to subset inclusion and cardinality rank function, $J_{\mathrm{tor}}(P)$ is a graded poset.*

Proof. Let $P = P(G,[\omega])$ be a toric poset over $G = (V,E)$. It suffices to show that every nonempty toric filter J contains a toric filter J' of cardinality $|J'| = |J| - 1$.

By Proposition 7.3, the set J is an order ideal of $P' = P(G,\omega')$ for some $\omega' \in [\omega]$. Choose any minimal element $v \in V$ of P', which is a source of ω'. Let ω'' be the orientation obtained by flipping v into a sink. The set $J' := J - \{v\}$ is an ideal of $P(G,\omega'')$, and so by Proposition 7.3, it is a toric filter of $P(G,[\omega'']) = P(G,[\omega])$. □

Example 7.7. Let $G = C_4$, the circle graph on 4 vertices, and let $\omega \in \mathrm{Acyc}(G)$ be the orientation shown at left in Figure 4. The Hasse diagram of the poset $P = P(G,\omega)$ is a line graph $\hat{G}^{\mathrm{Hasse}}(P) = L_4$, and the transitive closure is $\bar{G}(P) = K_4$. Since V is a size-4 toric chain, it is totally cyclically ordered in every $\omega' \in [\omega]$, and the dashed edges are additionally implied by toric transitivity. Thus,

$$L_4 = \hat{G}^{\mathrm{Hasse}}(P(G,\omega)) \subsetneq \hat{G}^{\mathrm{torHasse}}(P(G,[\omega])) = C_4\,, \qquad \bar{G}(P(G,\omega)) = \bar{G}^{\mathrm{tor}}(P(G,[\omega])) = K_4\,.$$

The 4 torically equivalent orientations are shown in Figure 4. The only total toric extension of $P(G,[\omega])$ is

$$[(1,2,3,4)] = \{(1,2,3,4)\,,\ (2,3,4,1)\,,\ (3,4,1,2)\,,\ (4,1,2,3)]\,,$$

and this is shown at right in Figure 5. The toric filters are all subsets of V that appear as an initial segment in one of these four total orders. The poset $J_{\mathrm{tor}}(P(C_4,[\omega]))$ is shown at left in Figure 5. Note that unlike the ordinary poset case, it is not a lattice.

Example 7.8. Let $G = C_4$, as in Example 7.7, but now let $\omega' \in \mathrm{Acyc}(G)$ be the orientation shown at left in Figure 6. The only nonempty toric chains are the four vertices (size 1) and the four edges (size 2). Since ω' has no toric chains of size greater than 2, the Hasse diagram and the transitive closure of the toric poset $P(G,[\omega'])$ are both C_4. Note that the transitive closure of the (ordinary) poset $P(G,\omega')$ contains the edge $\{1,3\}$, and so as graphs, $\bar{G}(P(G,\omega')) \neq \bar{G}^{\mathrm{tor}}(P(G,[\omega']))$. The 6 torically equivalent orientations of ω' are shown in Figure 6. There are four total toric extensions of $P(G,[\omega'])$ which are shown on the right in Figure 7, as cyclic words. The toric filters are all

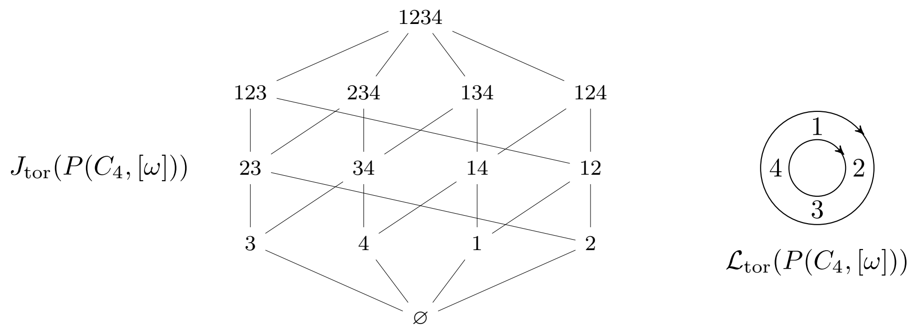


FIGURE 5. The toric filters of the toric poset $P(C_4, [\omega])$ form a poset that is not a lattice. The vertices should be thought of as subsets of $\{1, 2, 3, 4\}$; order does not matter. That is, 134 represents $\{1, 3, 4\}$.

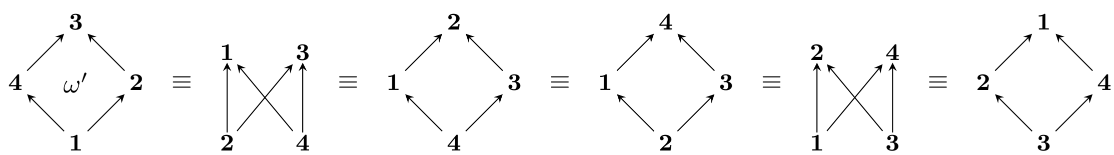


FIGURE 6. The six torically equivalent orientations to $\omega' \in \mathrm{Acyc}(C_4)$, shown at left.

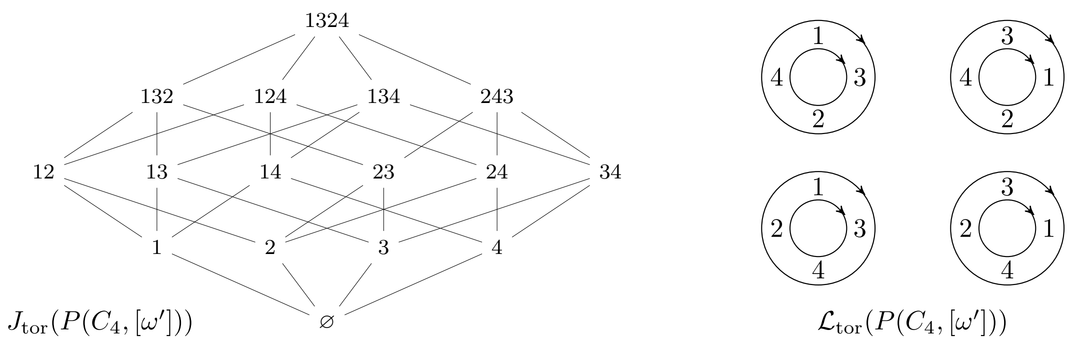


FIGURE 7. The toric filters of the toric poset $P(C_4, [\omega'])$ form a poset that happens to be a lattice. Each toric filter appears as a consecutive sequence in one of the total toric extensions, shown at right.

subsets of V that appear as a consecutive segment in one of these four total orders. The poset $J_{\mathrm{tor}}(P(C_4, [\omega']))$ of toric filters is shown at left in Figure 7. In this particular case, the poset of toric filters is a lattice. In fact, it is isomorphic to a Boolean lattice, because every subset of $\{1, 2, 3, 4\}$ appears consecutively (ignoring relative order) in one of the four cyclic words in Figure 7.

8. Application to Coxeter groups

A Coxeter system is a pair (W, S) consisting of a *Coxeter group* W generated by a finite set of involutions $S = \{s_1, \ldots, s_n\}$ with presentation

$$W = \langle S \mid s_i^2 = 1,\ (s_i s_j)^{m_{i,j}} = 1 \rangle$$

where $2 \leq m_{i,j} \leq \infty$ for $i \neq j$. The corresponding Coxeter graph Γ has vertex set $V = S$ and edges $\{i, j\}$ for each $m_{i,j} \geq 3$ labeled with $m_{i,j}$ (label usually omitted if $m_{i,j} = 3$). A *Coxeter element* is the product of the generators in some order, and every Coxeter element $c \in W$ defines a partial ordering on S via an acyclic orientation $\omega(c) \in \mathrm{Acyc}(\Gamma)$: Orient $s_i \to s_j$ iff s_i precedes

s_j in some (equivalently, every) reduced expression for c. Conjugating a Coxeter element by an initial generator (note that $s_i = s_i^{-1}$) cyclically shifts it:

$$s_{x_1}(s_{x_1}s_{x_2}\cdots s_{x_n})s_{x_1} = s_{x_2}\cdots s_{x_n}s_{x_1}\,,$$

and the corresponding acyclic orientation differs by reversing the orientations of all edges incident to s_{x_1}, thereby converting it from a source to a sink vertex. In 2009, H. and K. Eriksson showed [EE09] that two Coxeter elements c and c' are conjugate if and only if $\omega(c) \equiv \omega(c')$. Thus, there are bijections between the set $\mathrm{C}(W)$ of Coxeter elements and $\mathrm{Acyc}(\Gamma)$, as well as between the corresponding conjugacy classes and the toric equivalence classes:

$$\begin{array}{ccc} \mathrm{C}(W) \longrightarrow \mathrm{Acyc}(\Gamma) & \qquad & \mathrm{Conj}(\mathrm{C}(W)) \longrightarrow \mathrm{Acyc}(\Gamma)/\!\equiv \\ c \longmapsto \omega(c) & & \mathrm{cl}_W(c) \longmapsto [\omega(c)] \end{array}$$

As an example, suppose $W = W(\widetilde{A}_3)$, the affine Coxeter group of type A, and let $c = s_1s_2s_3s_4$ (using s_4 instead of the usual s_0), as shown below:

$$c = s_1s_2s_3s_4 \in W(\widetilde{A}_3) \qquad \begin{tikzcd} s_1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\Gamma"] & s_2 \arrow[d, dash] \\ s_4 \arrow[r, dash] & s_3 \end{tikzcd} \qquad \begin{tikzcd} s_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\omega(c)"] & s_2 \arrow[d] \\ s_4 & s_3 \arrow[l] \end{tikzcd}$$

The conjugate Coxeter elements to $c \in W$ are thus $s_1s_2s_3s_4$, $s_2s_3s_4s_1$, $s_3s_4s_1s_2$, and $s_4s_1s_2s_3$. The toric filters of $P(G,[\omega(c)])$ describe which subwords can appear in an initial segment of some reduced expression of one of these conjugate Coxeter elements. The poset of these toric filters was shown in Figure 5 (replace k with s_k).

Now, consider the element $c' = s_1s_3s_2s_4$ in $W(\widetilde{A}_3)$, as shown below:

$$c' = s_1s_3s_2s_4 \in W(\widetilde{A}_3) \qquad \begin{tikzcd} s_1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\Gamma"] & s_2 \arrow[d, dash] \\ s_4 \arrow[r, dash] & s_3 \end{tikzcd} \qquad \begin{tikzcd} s_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\omega(c')"] & s_2 \\ s_4 & s_3 \arrow[l] \arrow[u] \end{tikzcd}$$

The toric equivalence class containing $\omega(c')$ has six orientations, which were shown in Figure 6. Each of these describes a unique conjugate Coxeter element:

$$\begin{array}{cccccc} s_1s_2s_4s_3 & s_2s_4s_1s_3 & s_4s_1s_3s_2 & s_2s_1s_3s_4 & s_1s_3s_2s_4 & s_3s_2s_4s_1 \\ = s_1s_4s_2s_3 & = s_2s_4s_3s_1 & = s_4s_3s_1s_2 & = s_2s_3s_1s_4 & = s_1s_3s_4s_2 & = s_3s_4s_2s_1 \\ & = s_4s_2s_3s_1 & & & = s_3s_1s_4s_2 & \\ & = s_4s_2s_1s_3 & & & = s_3s_1s_2s_4 & \end{array}$$

These are listed above so that the Coxeter element in the i^{th} column corresponds to the i^{th} orientation in Figure 6. The linear extensions of each orientation describe the reduced expressions of the corresponding Coxeter element, which are listed in the same column above.

The toric poset $P(G,[\omega(c')])$ has four total toric extensions, and these were shown on the right in Figure 7 (replace k with s_k). The toric filters of $P(G,[\omega(c')])$ correspond to the subsets that appear consecutively in one of these cyclic words. The poset $J_{\mathrm{tor}}(P(C_4,[\omega']))$ of toric filter appears on the left in Figure 7.

9. Concluding remarks

In this paper, we further developed the theory of toric posets by formalizing the notion of toric intervals, morphisms, and order ideals. In some regards, much of the theory is fairly analogous to that of ordinary posets, though there are some noticeable differences. Generally speaking, the one recurring theme was the characterization of the toric analogue of a feature in $P(G,[\omega])$ by the characterization of the ordinary version of that feature in $P(G,\omega')$ either for some $\omega' \in [\omega]$, or for all $\omega' \in [\omega]$.

One question that arises immediately is whether there is a toric order complex. While there may exist such an object, there are some difficulties unique to the toric case. For example, a

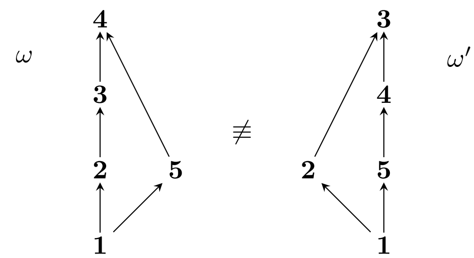


FIGURE 8. Two non-torically equivalent orientations $\omega \not\sim \omega'$ in $\mathrm{Acyc}(C_5)$ for which $P(C_5, [\omega])$ and $P(C_5, [\omega'])$ have the same set of toric chains.

poset is completely determined by its chains, in that if one specifies which subsets of V are the chains of P, and then the toric order of the elements within each chain, the entire poset can be reconstructed. This is not the case for toric posets, as shown in Figure 8. Here, two torically non-equivalent orientations of C_5 are given, but the toric posets $P(C_5, [\omega])$ and $P(C_5, [\omega'])$ have the same sets of toric chains: the 5 vertices and the 5 edges.

The fact that an ordinary poset is determined by its chains just means that once one specifies the total order between every chain of size $k \geq 2$, then the entire partial order is determined. The problem for toric posets, which we encountered in this paper, is that *every* size-2 subset is trivially cyclically ordered, whether it lies on a toric chain or not. In other words, a total order can be defined on two elements, but a cyclic order needs three. The analogous statement for toric posets would be that specifying the total cyclic order between every toric chain of size $k \geq 3$ specifies the entire toric order. Such a statement would establish the intuitive idea that knowing all total cyclic orders should determine the toric partial order "modulo the size-2 toric chains." Current works suggests that there is an analogue of the aforementioned properties for toric posets, but it requires a new generalization of the concept of a chain. The details are too preliminary and complicated to describe here, and it is not clear whether it will lead to a combinatorial object such as a toric order complex. Without this, there might not be a natural way to study toric posets topologically.

Another important feature of ordinary posets that does not seem to have any obvious toric analogue are Möbius functions, and this is vital to much of the theory of ordinary posets. Recall the analogy from the Introduction about how topology is like "analysis without the metric." Similarly, many of the basic features of ordinary posets have toric analogues, despite the fact that toric posets have no binary relation. However, much of the more advanced theory is likely to fail because one also seems to lose valuable tools such as an order complex and a Möbius function. Even the theory that does carry over has its shortcomings. For example, morphisms have a simple combinatorial characterization using the binary relation: $i <_P j$ implies $\phi(i) <_{P'} \phi(j)$. The geometric definition requires a patchwork of quotients, extensions, and inclusions. It would be desirable to have a more "holistic" characterization of toric poset morphisms, though it is not clear that that such a description should exist.

Finally, the connection of toric posets to Coxeter groups is the subject of a paper nearing completion on cyclic reducibility and conjugacy in Coxeter groups. Loosely speaking, reduced expressions can be formalized as labeled posets called heaps. This was formalized by Stembridge [Ste96, Ste98] in the 1990s. The *fully commutative* (FC) elements are those such that "long braid relations" (e.g., $sts \mapsto tst$) do not arise. Equivalently, they have a unique heap. The cyclic version of the FC elements are the *cyclically fully commutative* CFC elements, introduced by the author and collaborators in [BBE$^+$12]. In 2013, T. Marquis showed that two CFC elements are conjugate if and only if their heaps are torically equivalent [Mar14]. These elements were further studied by M. Pétréolle [Pét14]. In our forthcoming paper, we will formalize the notion of a toric heap, which will essentially be a labeled toric poset. This allows us to formalize objects such as cyclic words, cyclic commutativity classes, and develop a theory of cyclic reducibility in Coxeter groups using the toric heap framework.

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