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What Moser Could Have Asked: Counting Hamilton Cycles in Tournaments

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Abstract

Moser asked for a construction of explicit tournaments on n vertices having at least $(\frac{n}{3e})^n$ Hamilton cycles. We show that he could have asked for rather more.

1 Introduction

> ...the cycle has taken us up through forests.
>
> *Robert M. Pirsig*

In his classic book on tournaments, Moon [4, Section 10] discusses the question of exhibiting tournaments with a large number of Hamilton cycles. He poses the question (Exercise 4, attributed to Moser), of constructing a tournament on n vertices having at least $(\frac{n}{3e})^n$ Hamilton cycles. Presumably, the intended construction is to take three tournaments, T_1, T_2, T_3, on $\frac{n}{3}$ vertices, and construct a new tournament $C_3(T_1, T_2, T_3)$ by orienting all edges from T_1 to T_2, T_2 to T_3, and T_3 to T_1 (See Figure 1). The number of Hamilton cycles

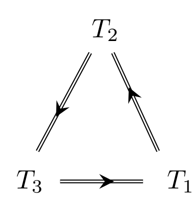

Figure 1: $C_3(T_1, T_2, T_3)$

AMS Subject Classification: 05C20, 05C30, 05A16 in $C_3(T_1, T_2, T_3)$ which do not use any edges internal to T_1, T_2, or T_3 is

$$\frac{(\frac{n}{3})!^3}{\frac{n}{3}} \sim \sqrt{\frac{8\pi^3 n}{3}} \left(\frac{n}{3e}\right)^n > \left(\frac{n}{3e}\right)^n.$$

In this note, we show that this construction has many more Hamilton cycles. Indeed, if T_1, T_2, and T_3 are all transitive, we show that the number of Hamilton cycles is asymptotic to $\frac{1}{(1-\log 2)} \frac{(n-1)!}{(3 \log 2)^n}$.

2 Background and Definitions

A *tournament* is an oriented, complete graph. A *Hamilton cycle* or *path* in a tournament T, is a spanning directed cycle or directed path in T. A tournament with no directed cycles is called *transitive*

Counting Hamilton paths and cycles in tournaments is a very old problem, dating back to the 1940's: in one of the first applications of the probabilistic method, Szele [6] showed that the expected number of Hamilton paths in a random tournament is $\frac{n!}{2^{n-1}}$, therefore showing that there exists a tournament on n vertices with at least this many Hamilton paths. The same argument shows that there exists a tournament with at least $\frac{(n-1)!}{2^n}$ Hamilton cycles. Moon observed that it seems difficult to give explicit tournaments with at least this many Hamilton cycles.

Deep results of Cuckler [3] show that every regular tournament on n vertices has at least $\frac{n!}{(2+o(1))^n}$ Hamilton cycles.

Given tournaments T_1, T_2, T_3, we can construct a tournament $C_3(T_1, T_2, T_3)$ by orienting all edges from T_1 to T_2, T_2 to T_3, and T_3 to T_1. We will call such tournaments *triangular*. Wormald [8] showed that if T_1, T_2, T_3 are random tournaments, then the expected number of Hamilton cycles is $2\frac{(n-1)!}{2^n}$.

We show that *all* triangular tournaments have a relatively large number of Hamilton cycles, even in the extreme case when they constructed from transitive tournaments.

Let $S(m, k)$ denote the Stirling number of the second kind, that is, $S(m, k)$ is the number of set partitions of $\{1, 2, \ldots m\}$ into exactly k parts.

3 Main Result

Theorem 1. *Let T_1, T_2, and T_3 be any tournaments on m_1, m_2, and m_3 vertices respectively. Then the number H of Hamilton cycles in $C_3(T_1, T_2, T_3)$ is at least*

$$H \geq \sum_{k=1}^{\min\{m_1, m_2, m_3\}} S(m_1, k) S(m_2, k) S(m_3, k) \frac{k!^3}{k}, \tag{1}$$

with equality when T_1, T_2, and T_3 are transitive.

Corollary 2. *If T_1, T_2, and T_3 are transitive tournaments on $\frac{n}{3}$ vertices, the number of Hamilton cycles in $C_3(T_1, T_2, T_3)$ is asymptotic to*

$$\frac{1}{(1-\log 2)}\frac{(n-1)!}{(3\log 2)^n} \simeq 3.25889\frac{(n-1)!}{(2.07944)^n}. \tag{2}$$

Proof of Theorem 1. Take any Hamilton cycle C, of $C_3(T_1, T_2, T_3)$, and consider C restricted to T_1, T_2, and T_3. Since a Hamilton cycle meets every vertex in T_1, T_2, and T_3 exactly once, C visits each subtournament the same number of times, say k. Hence for each T_i, C will induce a collection of k disjoint paths that cover the vertices of T_i. We will refer to such a collection of k paths as a k-path cover. Similarly, given k-path covers for T_1, T_2, T_3, we can construct a Hamilton cycle by joining these k-path covers together. The number of ways of doing this is $k!^3/k$. Thus, if $P(T_i, k)$ denotes the number of k-path covers of T_i, then the number of Hamilton cycles of $C_3(T_1, T_2, T_3)$ which induce k-path covers in T_1, T_2, and T_3 is

$$P(T_1,k)P(T_2,k)P(T_3,k)\frac{k!^3}{k}. \tag{3}$$

It follows that the number of Hamilton cycles in $C_3(T_1, T_2, T_3)$ is

$$\sum_{k=1}^{\min\{m_1,m_2,m_3\}} P(T_1,k)P(T_2,k)P(T_3,k)\frac{k!^3}{k}.$$

For any set partition of the vertex set of T_i into k nonempty sets, each part will induce a subtournament of T_i. Rédei [5] showed that every tournament has a Hamilton path, thus each partition into k sets will induce at least one k-path cover of T_i. Therefore the number of Hamilton cycles in $C_3(T_1, T_2, T_3)$ is at least

$$\sum_{k=1}^{\min\{m_1,m_2,m_3\}} S(m_1,k)S(m_2,k)S(m_3,k)\frac{k!^3}{k}$$

as claimed.

In the case that each T_i is transitive, each subtournament will have exactly one Hamilton path, hence we have equality in (1). □

Proof of Corollary 2. Suppose now that each T_i is a transitive tournament on m vertices, then the number of Hamilton cycles in $C_3(T_1, T_2, T_3)$ is equal to

$$\sum_{k=1}^{m} S(m,k)^3\frac{k!^3}{k}. \tag{4}$$

As with many combinatorial sums, the summands in (4) are approximated rather well by a normal distribution. Indeed, if we let

$$\mu = \frac{1}{2\log 2} \quad \text{and} \quad \sigma = \frac{\sqrt{1-\log 2}}{2\log 2},$$

define $f(m) = \sum_{k=1}^{m} S(m,k)k!$, and write $p(m,k) = \frac{S(m,k)k!}{f(m)}$, then Bender [2] shows that $p(m,k)$ is asymptotically normal with mean μm and variance $\sigma^2 m$. Hence, $p(m,k)^3$ is also proportional to a normal distribution, at least in a range of k close to μm. This allows us to approximate the sum $\sum_{k=1}^{m} S(m,k)^3 \frac{k!^3}{k}$ by an integral, showing that

$$\sum_{k=1}^{m} S(m,k)^3 \frac{k!^3}{k} \sim f(m)^3 \frac{3^{\frac{1}{2}} 2^{\frac{1}{2}} \pi^{\frac{1}{2}}}{3\mu\sigma^2 (2\pi)^{\frac{3}{2}} m^2}.$$

From Wilf [7, p. 176], we know that

$$f(m) \sim \frac{m!}{2(\log 2)^{m+1}},$$

Therefore with $n = 3m$, two applications of Stirling's approximation for $n!$ yields

$$\begin{aligned}
\sum_{k=1}^{m} S(m,k)^3 \frac{k!^3}{k} &\sim \left(\frac{m!}{2(\log 2)^{m+1}}\right)^3 \frac{3^{\frac{1}{2}} 2^{\frac{1}{2}} \pi^{\frac{1}{2}}}{3\mu\sigma^2 (2\pi)^{\frac{3}{2}} m^2} \\
&\sim \frac{\sqrt{2\pi n}}{n(1-\log 2)} \left(\frac{n}{3e\log 2}\right)^n \\
&\sim \frac{1}{(1-\log 2)} \frac{(n-1)!}{(3\log 2)^n}
\end{aligned}$$

□

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