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An Introduction to Homological Algebra and its Applications

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An Introduction to Homological Algebra and its Applications

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Chapter 1

Preliminaries

It will be assumed that the reader is already familiar with introductory-level abstract algebra as well as the following definitions and results. Assume R is a commutative ring with identity throughout.

1.1 Exact Sequences and Projective Modules

Some basic properties of modules and (short) exact sequences will be essential in this document, so we present a number of them here.

Definition 1.1.1. Let M_1, M_2, M_3 be R-modules. Then a sequence of R-module homomorphisms

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

is *exact* if $\mathrm{Im}\,(f) = \mathrm{Ker}\,(g)$. More generally, a sequence of R-module homomorphisms

$$\dots \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \xrightarrow{d_{i-1}} \dots$$

is exact if $\mathrm{Im}\,(d_{i+1}) = \mathrm{Ker}\,(d_i)$, for all relevant i.

Fact 1.1.2. *Let U, V, W be R-modules.*

(a) The following sequence is exact if and only if α is injective.

$$0 \xrightarrow{\epsilon} U \xrightarrow{\alpha} V$$

(b) The following sequence is exact if and only if β is surjective.

$$V \xrightarrow{\beta} W \xrightarrow{\rho} 0$$

(c) The following sequence is exact if and only if α is injective, β in surjective, and $\mathrm{Im}\,(\alpha) = \mathrm{Ker}\,(\beta)$.

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0$$

Proof. By Definition 1.1.1, the sequence is exact if and only if $\mathrm{Ker}(\alpha) = \mathrm{Im}\,(\epsilon) = \{0\}$, which proves (a). Part (b) also holds by definition of exactness, since the sequence is exact if and only if $\mathrm{Im}\,(\beta) = \mathrm{Ker}(\rho) = W$. Part (c) is a corollary of parts (a) and (b). □

Definition 1.1.3. When the sequence in part (c) above is exact, it is a *short exact sequence.*

Example 1.1.4. If M and N are R-modules, then so is $M \oplus N$ and we claim the sequence

$$0 \longrightarrow M \xrightarrow{\epsilon} M \oplus N \xrightarrow{\pi} N \longrightarrow 0$$

is a short exact sequence, where ϵ and π are the natural injection and surjection, respectively. Let $(m, n) \in M \oplus N$. Then $\pi(m, n) = 0$ if and only if $n = 0$, which holds if and and only if $(m, n) \in \mathrm{Im}\,(\epsilon)$. Therefore the sequence is exact in the center by Fact 1.1.2 (c), since ϵ and π are injective and surjective, respectively.

Fact 1.1.5. *Let A, B, C be R-modules.*

(a) The sequence below is exact if and only if α is an isomorphism.

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

(b) The sequence below is exact if and only if C is the zero module.

$$0 \longrightarrow C \longrightarrow 0$$

Proof. Both parts follow from Fact 1.1.2. For part (a), note α is injective if and only if the sequence is exact and α is surjective if and only if the sequence is exact. For part (b), note the sequence is exact if and only if the map $0 \longrightarrow C$ is surjective, i.e., if and only if $C = 0$. □

Definition 1.1.6. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

$$0 \longrightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \longrightarrow 0$$

be two short exact sequences. A *homomorphism of short exact sequences* is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & \curvearrowright & \downarrow{\scriptstyle \beta} & \curvearrowright & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \longrightarrow & 0
\end{array}$$

where α, β, and γ are R-module homomorphisms. The homomorphism is an *isomorphism* if α, β, and γ are isomorphisms. This is an equivalence if $A = A'$, $C = C'$, $\alpha = id_A$, and $\gamma = id_C$. That is, we have equivalence if our diagram can be written

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & {\scriptstyle id_A}\downarrow & \curvearrowright & \downarrow{\scriptstyle \beta} & \curvearrowright & \downarrow{\scriptstyle id_C} & & \\
0 & \longrightarrow & A & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C & \longrightarrow & 0
\end{array}$$

Note in this case β is necessarily an isomorphism (see Fact 1.1.9).

Fact 1.1.7. *Given any R-module homomorphism $g : A \longrightarrow B$, there exists an exact sequence*

$$0 \longrightarrow \mathrm{Ker}\,(g) \xrightarrow{\epsilon} A \xrightarrow{g} B \xrightarrow{\tau} \mathrm{Coker}\,(g) \longrightarrow 0$$

where ϵ is the natural injection, τ is the natural surjection, and

$$\mathrm{Coker}\,(g) := \frac{B}{\mathrm{Im}\,(g)}$$

Definition 1.1.8. Given R-modules A, B, and C, the short exact sequence

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$$

is said to be *split* if there is an R-module complement to $\psi(A)$ in B. In this case $B \cong A \oplus C$, or to be precise, $B = \psi(A) \oplus C'$ where $C' \subseteq B$ is a submodule and $\phi(C') \cong C$. The module B is said to be a *split extension* of C by A.

An equivalent definition is to say that the above short exact sequence splits if there exists an equivalence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\epsilon} & A \oplus C & \xrightarrow{\rho} & C & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}_A}\downarrow{\scriptstyle \cong} & \curvearrowright & {\scriptstyle \Gamma}\downarrow{\scriptstyle \cong} & \curvearrowright & {\scriptstyle \mathrm{id}_C}\downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & A & \xrightarrow[\psi]{} & B & \xrightarrow[\phi]{} & C & \longrightarrow & 0 \end{array}$$

where ϵ and ρ are the natural injection and surjection, respectively.

Fact 1.1.9. *In the following commutative diagram with exact rows, the R-module homomorphism β must be an isomorphism.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\downarrow{\scriptstyle \cong} & \curvearrowright & {\scriptstyle \beta}\downarrow & \curvearrowright & {\scriptstyle \gamma}\downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \longrightarrow & 0 \end{array}$$

Proof. To show β is injective, let $b \in \mathrm{Ker}(\beta)$ be given and we want to show $b = 0$. By the commutivity of the diagram, $0 = g'(\beta(b)) = \gamma(g(b))$, so $g(b) \in \mathrm{Ker}(\gamma) = \{0\}$. Since $b \in \mathrm{Ker}(g) = \mathrm{Im}\,(f)$, let $a \in A$ such that $f(a) = b$. By the commutivity of the diagram, $f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$, so $\alpha(a) \in \mathrm{Ker}(f') = \{0\}$. Since α is injective, $a = 0$ and therefore $b = f(a) = 0$.

To show β is surjective, let $b' \in B'$ be given and we want to find a lift of this element in B. Since both γ and g are surjective, let $b \in B$ such that $(\gamma \circ g)(b) = g'(b')$. By the commutivity of the diagram it also holds that $(g' \circ \beta)(b) = g'(b')$, so the element $b' - \beta(b) \in \mathrm{Ker}(g')$. Since the rows are exact and α an isomorphism, we may lift to some $a \in A$ such that $(f' \circ \alpha)(a) = b' - \beta(b)$ and the commutivity of the diagram implies $(\beta \circ f)(a) = b' - \beta(b)$, whereby we conclude

$$\beta(f(a) + b) = (\beta \circ f)(a) + \beta(b) = b'$$

as desired. □

Fact 1.1.10. *A short exact sequence as in 1.1.8 splits if and only if there exists an R-module homomorphism $\mu : C \longrightarrow B$ such that $\phi \circ \mu = \mathrm{id}_C$. In this case μ is called a* splitting homomorphism *for the sequence.*

Proof. First assume an equivalence of short exact sequences exists as in Definition 1.1.8 and define

$$\mu : C \longrightarrow B$$
$$c \longmapsto \Gamma(0, c)$$

This is a well-defined R-module homomorphism, because Γ is a well-defined R-module homomorphism. For an arbitrary element $c \in C$ the commutivity of the diagram gives

$$(\phi \circ \mu)(c) = (\phi \circ \Gamma)(0, c) = (\mathrm{id}_C \circ \rho)(0, c) = c$$

Second, assume instead there exists a homomorphism $\mu : C \longrightarrow B$ such that $\phi \circ \mu = \mathrm{id}_C$ (cf. 1.1.8). Define the following map.

$$\begin{aligned} \Gamma : A \oplus C &\longrightarrow B \\ (a, c) &\longmapsto \psi(a) + \mu(c) \end{aligned}$$

Since both ψ and μ are well-defined R-module homomorphisms, so is Γ. Moreover for any $a \in A$ we have

$$(\Gamma \circ \epsilon)(a) = \Gamma(a, 0) = \psi(a)$$

and for any $(a, c) \in A \oplus C$ we have

$$(\phi \circ \Gamma)(a, c) = \phi(\psi(a) + \mu(c)) = (\phi \circ \psi)(a) + (\phi \circ \mu)(c) = 0 + c = (\mathrm{id}_C \circ \rho)(a, c).$$

Therefore the diagram commutes. By Fact 1.1.9, Γ is also an isomorphism, so the bottom row is split. □

Definition 1.1.11. Let R be a ring, let C be an R-module, and let $A \subseteq C$ be a submodule. We say A *is a direct summand of* C if there exists some R-submodule $B \subseteq C$ such that $C = A \oplus B$.

Definition 1.1.12. A *category* consists of a collection of objects, a collection of morphisms for each pair of objects, and a binary operation on pairs of morphisms called composition (provided the morphisms have compatible domain and codomain). A *functor* is a map between categories that respects compositions and identity morphisms. A functor F is *covariant* if a morphism $\phi : A \longrightarrow B$ becomes

$$F(\phi) : F(A) \longrightarrow F(B)\ .$$

A functor G is *contravariant* if the morphism becomes

$$G(\phi) : G(B) \longrightarrow G(A)\ .$$

Remark 1.1.13. Contravariant and covariant functors respect compositions differently. Let γ, ρ, and φ be morphisms in the same category such that $\varphi = \gamma \circ \rho$ and let F and G be covariant and contravariant functors, respectively, on this category. Then $F(\varphi) = F(\gamma) \circ F(\rho)$ and $G(\varphi) = G(\rho) \circ G(\gamma)$. In particular, if $\gamma \circ \rho = 0$, then $F(\gamma) \circ F(\rho) = 0$ and $G(\rho) \circ G(\gamma) = 0$.

Definition 1.1.14. An R-module P is *projective* if it satisfies any one (and therefore all) of the following equivalent conditions.

(a) The covariant functor $\mathrm{Hom}_R(P, -)$ is exact. That is, for any R-modules L, M, and N, the exactness of the sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0$$

implies the following sequence is also exact.

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Hom}_R(P, L) & \xrightarrow{\psi'} & \mathrm{Hom}_R(P, M) & \xrightarrow{\phi'} & \mathrm{Hom}_R(P, N) & \longrightarrow 0 \\ & \rho & \longmapsto & \psi \circ \rho \qquad \gamma & \longmapsto & \phi \circ \gamma & \end{array}$$

(b) For any R-modules M and N, if $M \xrightarrow{\phi} N \longrightarrow 0$ is exact, then every R-module homomorphism from P into N lifts to an R-module homomorphism into M. In other words, given $f \in \mathrm{Hom}_R(P, N)$ there is a lift $F \in \mathrm{Hom}_R(P, M)$ making the following diagram commute.

$$\begin{array}{ccccc} & & P & & \\ & \overset{F}{\swarrow} & \downarrow f & & \\ M & \underset{\phi}{\longrightarrow} & N & \longrightarrow & 0 \end{array}$$

(c) For any R-module M, if P is isomorphic to a quotient of M (i.e., $P \cong M/M'$ for some submodule $M' \subseteq M$), then P is isomorphic to a direct summand of M.

(d) Every short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0$ splits.

(e) P is a direct summand of a free R-module.

Definition 1.1.15. An R-module I is *injective* if it satisfies any one (and therefore all) of the following equivalent conditions.

(a) The contravariant functor $\mathrm{Hom}_R(-, I)$ is exact. That is, for any short exact sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0$$

the following sequence is exact as well.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(L, I) & \xrightarrow{\psi'} & \mathrm{Hom}_R(M, I) & \xrightarrow{\phi'} & \mathrm{Hom}_R(N, I) & \longrightarrow & 0 \\ & & \rho \longmapsto \psi \circ \rho & & & \gamma \longmapsto \phi \circ \gamma & & & \end{array}$$

(b) For any R-modules X, Y and any R-module homomorphisms $X \hookrightarrow Y$ and $X \longrightarrow I$, there exists an R-module homomorphism h such that the following diagram commutes.

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \overset{f}{\hookrightarrow} & Y \\ & & g \downarrow & \overset{\text{s.t.}}{\curvearrowright} \quad \swarrow \exists h & \\ & & I & & \end{array}$$

(c) For any R-module M, if I is isomorphic to a submodule $I' \subseteq M$, then I' is a direct summand of M.

(d) Every short exact sequence $0 \longrightarrow I \longrightarrow M \longrightarrow K \longrightarrow 0$ splits.

1.2 Localization

We briefly look at the construction of localized rings and modules and their properties. Of particular usefulness throughout this document will be the correspondence of prime ideals under localization given in Fact 1.2.11. To conclude the section we also introduce the terms covariant and contravariant. Assume M and N are R-modules throughout.

Definition 1.2.1. A subset $U \subseteq R$ is *multiplicatively closed* if $1 \in U$ and the product $uv \in U$ for all $u, v \in U$.

Example 1.2.2. For any element $s \in R$, the subset $S = \{\, s^{\epsilon} \mid \epsilon \in \mathbb{N}_0\}$ is multiplicatively closed. If $\mathfrak{p} \lneq R$ is a prime ideal, then $R \setminus \mathfrak{p}$ is multiplicatively closed as well.

Definition 1.2.3. Let $U \subseteq R$ be multiplicatively closed and we may define a relation on $M \times U$: let $(m, u) \sim (n, v)$ if there exists $w \in U$ such that $w(vm - un) = 0$. One can show that this is an equivalence relation. We therefore define

$$U^{-1}M := \{\text{equivalence classes from } M \times U \text{ under } \sim\}$$

and denote the equivalence class (m, u) as $\dfrac{m}{u}$ or m/u.

Fact 1.2.4. *In general $U^{-1}M$ is an abelian group by the operations*

$$\frac{m}{u} + \frac{n}{v} := \frac{vm + um}{uv} \qquad\qquad 0_{U^{-1}M} := \frac{0_M}{1_R} = \frac{0_M}{u}$$

an R-module by the operation

$$r \cdot \frac{m}{u} := \frac{rm}{u}$$

and a $U^{-1}R$-module by the operation

$$\frac{r}{v} \cdot \frac{m}{u} := \frac{rm}{vu}.$$

The special case when $M = R$ gives a commutative ring $U^{-1}R$ with the following operations and identities.

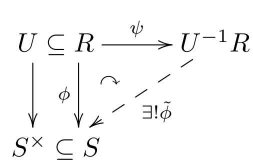
$$\frac{m}{u} + \frac{n}{v} := \frac{vm + un}{uv} \qquad\qquad \frac{m}{u} \cdot \frac{n}{v} := \frac{mn}{uv}$$
$$0_{U^{-1}R} := \frac{0_R}{u} = \frac{0_R}{1_R} \qquad\qquad 1_{U^{-1}R} := \frac{u}{u} = \frac{1_R}{1_R}$$

Moreover there exists a ring homomorphism

$$\begin{aligned} \psi : & R \longrightarrow U^{-1}R \\ & r \longmapsto \frac{r}{1} = \frac{ur}{u}. \end{aligned}$$

Notation 1.2.5. Let $R^{\times}$ denote the collection of all units in R.

Theorem 1.2.6 (Universal Mapping Property)**.** *Let R and S be commutative rings with identity. Given any ring homomorphism $\phi : R \longrightarrow S$ such that $\phi(U) \subseteq S^{\times}$, there exists a unique ring homomorphism $\tilde{\phi} : U^{-1}R \longrightarrow S$ such that $\tilde{\phi} \circ \psi = \phi$. This is summed up by a commutative diagram.*



Example 1.2.7. If R is an integral domain, then $0 \lneq R$ is a prime ideal and $R \setminus 0$ is multiplicatively closed, so we call $(R \setminus 0)^{-1}R$ the *field of fractions* of R, denoted $Q(R)$, and $(R \setminus 0)^{-1}M$ is a vector space over the field of fractions.

Notation 1.2.8. Recall Example 1.2.2. If $s \in R$, then $M_s := S^{-1}M$. If $\mathfrak{p} \lneq R$ is prime then $M_\mathfrak{p} := (R \setminus \mathfrak{p})^{-1}M$. Notice that in the M_s case, the multiplicatively closed subset *does contain* the element s, but in the $M_\mathfrak{p}$ case, the multiplicatively closed subset *does not* contain $\mathfrak{p}$.

Fact 1.2.9. *There is a one-to-one correspondence of prime ideals under this localization process. Explicitly, if $U \subseteq R$ is a multiplicatively closed subset and ψ is the ring homomorphism from Theorem 1.2.6, then we have*

$$\{\textit{prime ideals of } U^{-1}R\} \rightleftarrows \{\textit{prime ideals } \mathfrak{q} \lneq R \mid \mathfrak{q} \cap U = \emptyset\}$$

$$Q \longmapsto \psi^{-1}(Q) = \{x \in R \mid \psi(x) \in Q\}$$

$$(x/1 \mid x \in \mathfrak{q})\, U^{-1}R = \mathfrak{q}(U^{-1}R) \longleftarrow\!\!\shortmid \mathfrak{q}$$

and the isomorphic relations

$$\frac{U^{-1}R}{\mathfrak{q}(U^{-1}R)} \cong U^{-1}(R/\mathfrak{q}) \qquad\qquad (U^{-1}R)_{\mathfrak{q}(U^{-1}R)} \xleftarrow{\cong} R_\mathfrak{q}\ .$$

$$\frac{r/1}{z/1} \longleftarrow\!\!\shortmid \frac{r}{z}$$

Example 1.2.10. Let $\mathfrak{p}$ be a prime ideal. The correspondence for $R_\mathfrak{p}$ under the description in Fact 1.2.9 is

$$\{\text{prime ideals of } R_\mathfrak{p}\} \leftrightharpoons \{\text{prime ideals } \mathfrak{q} \lneq R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$$

$$\frac{R_\mathfrak{p}}{\mathfrak{q}R_\mathfrak{p}} \cong (R/\mathfrak{q})_\mathfrak{p}$$

$$(R_\mathfrak{p})_{\mathfrak{q}R_\mathfrak{p}} \cong R_\mathfrak{q}$$

Considering the special case $\mathfrak{q} = \mathfrak{p}$ we have two ways of thinking about a field.

$$\frac{R_\mathfrak{p}}{\mathfrak{p}R_\mathfrak{p}} \cong (R/\mathfrak{p})_\mathfrak{p} \cong Q(R/\mathfrak{p})$$

Under the correspondence we know $\mathfrak{p}R_\mathfrak{p}$ is the unique maximal ideal of $\mathfrak{p}R_\mathfrak{p}$, so on the left-hand side we have a local ring modulo the unique maximal ideal, which must be a field. On the right-hand side, we have the field of fractions on the integral domain $R/\mathfrak{p}$.

Fact 1.2.11. *Given any R-module homomorphism $f : M \longrightarrow N$, this induces the following well-defined $U^{-1}R$-module homomorphism.*

$$U^{-1}f : U^{-1}M \longrightarrow U^{-1}N$$

$$\frac{m}{u} \longmapsto \frac{f(m)}{u}$$

Proof. We need to check well-definedness and $U^{-1}R$-linearity. If $m/u, m'/u' \in U^{-1}M$ such that $m/u = m'/u'$, then there exists some $v \in U$ such that $vu'm = vum'$. Therefore

$$v \cdot u'f(m) = f(vu'm) = f(vum') = v \cdot uf(m')$$

which implies $f(m)/u = f(m')/u'$, so $U^{-1}f$ preserves equality. Since it also lands well by construction, it is well-defined. Letting $m/u, x/w \in U^{-1}M$ and $r/u \in U^{-1}R$, we verify linearity as follows.

$$
\begin{aligned}
(U^{-1}f)\left(\frac{m}{u}+\frac{x}{w}\right) &= (U^{-1}f)\left(\frac{wm+ux}{uw}\right)\\
&= \frac{f(wm+ux)}{uw}\\
&= \frac{w\cdot f(m)+u\cdot f(x)}{uw}\\
&= \frac{w\cdot f(m)}{uw}+\frac{u\cdot f(x)}{uw}\\
&= \frac{f(m)}{u}+\frac{f(x)}{w}\\
&= (U^{-1}f)\left(\frac{m}{u}\right)+(U^{-1}f)\left(\frac{x}{w}\right)
\end{aligned}
$$

$$
\begin{aligned}
(U^{-1}f)\left(\frac{r}{u}\cdot\frac{m}{v}\right) &= (U^{-1}f)\left(\frac{rm}{uv}\right)\\
&= \frac{f(rm)}{uv}\\
&= \frac{r\cdot f(m)}{uv}\\
&= \frac{r}{u}\cdot\frac{f(m)}{v}\\
&= \frac{r}{u}\cdot(U^{-1}f)\left(\frac{m}{v}\right)
\end{aligned}
$$

□

Fact 1.2.12. *The operation $U^{-1}(-)$ is a covariant functor. Therefore it respects function composition and $U^{-1}(\mathrm{id}_M)=\mathrm{id}_{U^{-1}M}$.*

Chapter 2

Motivating Ext

In this chapter we motivate our study of Ext modules by discussing three applications in abstract algebra. We also give a few major results that will be explored more fully in later chapters.

2.1 Application 1: Long Exact Sequence

Given a short exact sequence of R-modules and R-module homomorphisms

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$$

and given an R-module N, the induced sequence

$$0 \longrightarrow \operatorname{Hom}_R(N, M_1) \xrightarrow{f_{1*}} \operatorname{Hom}_R(N, M_2) \xrightarrow{f_{2*}} \operatorname{Hom}_R(N, M_3) \tag{2.1.0.1}$$

is exact, where f_{i*} denotes $\operatorname{Hom}_R(N, f_i)$ and is defined as follows.

$$\begin{aligned} f_{i*} : \operatorname{Hom}_R(N, M_i) &\longrightarrow \operatorname{Hom}_R(N, M_{i+1}) \\ \phi &\longmapsto f_i \circ \phi \end{aligned}$$

A similar sequence was seen previously in Definition 1.1.10.

Here is demonstrated why we say Hom is *left-exact*. Writing the zero on the left in (2.1.0.1) maintains the exactness of the sequence. The contravariant sequence below is exact as well.

$$0 \longrightarrow \operatorname{Hom}_R(M_3, N) \xrightarrow{f_2^*} \operatorname{Hom}_R(M_2, N) \xrightarrow{f_1^*} \operatorname{Hom}_R(M_1, N)$$

Here f_i^* functions analogously to f_{i*} above.

$$\begin{aligned} f_i^* : \operatorname{Hom}_R(M_{i+1}, N) &\longrightarrow \operatorname{Hom}_R(M_i, N) \\ \psi &\longmapsto \psi \circ f_i \end{aligned}$$

If we were to put the zero module on the right of either the covariant sequence or the contravariant sequence, the exactness would fail in general at that point of the sequence. We can, however, compute something else on the right for a longer exact sequence. This is one of the first great achievements of homological algebra and the application from the title of this section. We will prove this in Section 6.2 (see Theorem 6.2.1).

Theorem 2.1.1 (Long Exact Sequences)**.** *Given the short exact sequence*

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$$

and an R-module N as above, there exist exact sequences

$$0 \longrightarrow \operatorname{Hom}_R(N, M_1) \longrightarrow \operatorname{Hom}_R(N, M_2) \longrightarrow \operatorname{Hom}_R(N, M_3)$$
$$\longrightarrow \operatorname{Ext}^1_R(N, M_1) \longrightarrow \operatorname{Ext}^1_R(N, M_2) \longrightarrow \operatorname{Ext}^1_R(N, M_3)$$
$$\longrightarrow \operatorname{Ext}^2_R(N, M_1) \longrightarrow \cdots$$

and

$$0 \longrightarrow \operatorname{Hom}_R(M_3, N) \longrightarrow \operatorname{Hom}_R(M_2, N) \longrightarrow \operatorname{Hom}_R(M_1, N)$$
$$\longrightarrow \operatorname{Ext}^1_R(M_3, N) \longrightarrow \operatorname{Ext}^1_R(M_2, N) \longrightarrow \operatorname{Ext}^1_R(M_1, N)$$
$$\longrightarrow \operatorname{Ext}^2_R(M_3, N) \longrightarrow \cdots$$

where $\operatorname{Ext}^i_R(-,-)$ *will be defined after some discussion. We will simply say colloquially here that* Ext^1_R *measures the lack of right exactness of* Hom.

Discussion 2.1.2. Given an R-module M, there exists a projective R-module P_0 and a surjective homomorphism $P_0 \xrightarrow{\tau} M$, because every R-module is a homomorphic image of a projective R-module. The sequence

$$P_0 \xrightarrow{\tau} M \longrightarrow 0$$

can be thought of as approximating M by the projective module P_0 where the error of the approximation is $\operatorname{Ker}(\tau)$. The sequence can be lengthened into the short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\tau) \xrightarrow{\subseteq} P_0 \xrightarrow{\tau} N \longrightarrow 0\ .$$

The R-module $\operatorname{Ker}(\tau)$ can likewise be approximated by a projective R-module. That is there exists a sequence

$$P_1 \xrightarrow{\tau_1} \operatorname{Ker}(\tau) \longrightarrow 0$$

and the short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\tau_1) \xrightarrow{\subseteq} P_1 \xrightarrow{\tau_1} \operatorname{Ker}(\tau) \longrightarrow 0\ .$$

Inductively there exists a short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\tau_i) \xrightarrow{\subseteq} P_i \xrightarrow{\tau_i} \operatorname{Ker}(\tau_{i-1}) \longrightarrow 0$$

for any $i \geq 2$, giving us diagram (2.1.5.1). Moreover, a standard diagram chase shows the infinite sequence

$$\ldots \longrightarrow P_4 \xrightarrow{\partial^P_4} P_3 \xrightarrow{\partial^P_3} P_2 \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

is exact by virtue of the exactness of the short exact sequences that compose it, as we show next.

Proof. For any $i \geq 1$ we want to show $\text{Im}\left(\partial_{i+1}^P\right) = \text{Ker}\left(\partial_i^P\right)$ by mutual containment. For any $b \in \text{Im}\left(\partial_{i+1}^P\right)$, there exists some $a \in P_{i+1}$ such that $\partial_{i+1}^P(a) = b$ and by the commutivity of diagram (2.1.5.1), $b = \tau_{i+1}(a) \in \text{Ker}(\tau_i)$, so $\tau_i(b) = 0$. Again by the commutivity of the diagram $\partial_i^P(b) = 0$, so $b \in \text{Ker}\left(\partial_i^P\right)$ and thus $\text{Im}\left(\partial_{i+1}^P\right) \subseteq \text{Ker}\left(\partial_i^P\right)$.

For any $d \in \text{Ker}\left(\partial_i^P\right)$, the commutivity of the diagram implies $d \in \text{Ker}(\tau_i)$. Since τ_{i+1} surjective we let $c \in P_{i+1}$ such that $\tau_{i+1}(c) = d$ and by the commutivity of the diagram $\partial_{i+1}^P(c) = d$, so $d \in \text{Im}\left(\partial_{i+1}^P\right)$ and therefore $\text{Ker}\left(\partial_i^P\right) \subseteq \text{Im}\left(\partial_{i+1}^P\right)$, which establishes equality. The proof at the $i = 0$ step using τ is just as straightforward. □

From this construction we define some new notation.

Definition 2.1.3. Every R-module M has an associated exact sequence, called an *augmented projective resolution*,

$$P_\bullet^+ \;=\; \cdots \longrightarrow P_4 \xrightarrow{\partial_4^P} P_3 \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

where each module P_i is projective and τ is a surjection, an associated *(truncated) projective resolution* (not typically exact),

$$P_\bullet \;=\; \cdots \longrightarrow P_4 \xrightarrow{\partial_4^P} P_3 \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \longrightarrow 0$$

and an associated Hom sequence

$$P_\bullet^* \;=\; \text{Hom}_R(P_\bullet, N) \;=\; 0 \longrightarrow P_0^* \xrightarrow{\left(\partial_1^P\right)^*} P_1^* \xrightarrow{\left(\partial_2^P\right)^*} P_2^* \xrightarrow{\left(\partial_3^P\right)^*} P_3^* \xrightarrow{\left(\partial_4^P\right)^*} P_4^* \longrightarrow \cdots.$$

The maps ∂_i^P are the *differentials* of the resolution.

Fact 2.1.4. *In the notation of 2.1.3 we have*

$$\left(\partial_{n+1}^P\right)^* \circ \left(\partial_n^P\right)^* = \left(\partial_n^P \circ \partial_{n+1}^P\right)^* = 0^* = 0.$$

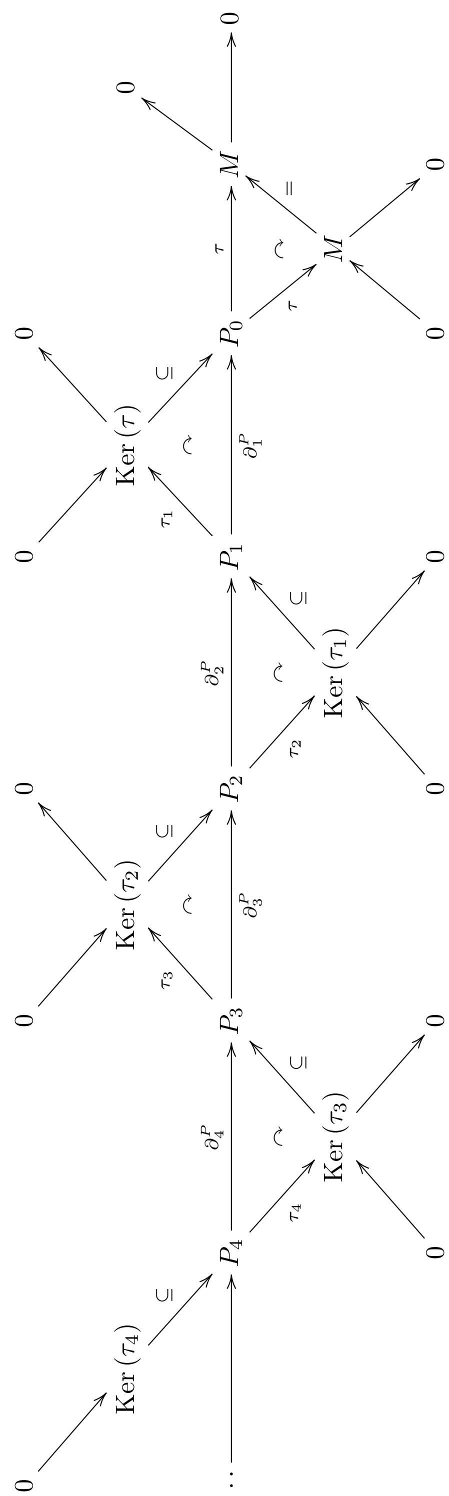
In other words, since $\text{Hom}_R(-, N)$ *is a functor we have* $\text{Im}\left(\partial_i^{P*}\right) \subseteq \text{Ker}\left(\partial_{i+1}^{P*}\right)$ *by Remark 1.1.13.*

Definition 2.1.5. Given a projective resolution of an R-module M in the notation of Definition 2.1.3 and given an arbitrary R-module N, we define

$$\text{Ext}_R^i(M, N) = \frac{\text{Ker}\left(\left(\partial_{i+1}^P\right)^*\right)}{\text{Im}\left(\left(\partial_i^P\right)^*\right)}.$$

Colloquially

$$\text{Ext}_R^i(M, N) = \frac{\text{Ker}\left(\text{outgoing from } i^{th} \text{ position}\right)}{\text{Im}\left(\text{incoming to } i^{th} \text{ position}\right)}.$$



(2.1.5.1)

Example 2.1.6. Let N be an R-module. Then

$$\operatorname{Ext}^i_R(R,N) \cong \begin{cases} N & i=0 \\ 0 & i \neq 0 \end{cases}.$$

Indeed, since R is projective (consider part (e) of Definition 1.1.14), we have the augmented projective resolution of R

$$P^+_\bullet \ = \qquad 0 \longrightarrow R \xrightarrow{\text{id}} R \longrightarrow 0$$

which is exact by Fact 1.1.5. The corresponding projective resolution is therefore

$$P_\bullet \ = \qquad 0 \longrightarrow R \longrightarrow 0\ .$$

To compute Ext we need the sequence $\operatorname{Hom}_R(P_\bullet, N)$.

$$0 \xrightarrow{f} \operatorname{Hom}_R(R,N) \xrightarrow{g} 0$$

From position $i=0$ we have

$$\operatorname{Ext}^0_R(R,N) = \frac{\operatorname{Ker}(g)}{\operatorname{Im}(f)} = \frac{\operatorname{Hom}_R(R,N)}{0} \cong \operatorname{Hom}_R(R,N) \cong N$$

and for any $i \neq 0$ we have

$$\operatorname{Ext}^i_R(R,N) = \frac{0}{0} \cong 0.$$

Note 2.1.7. In general, if $P_i = 0$, then $\operatorname{Hom}_R(P_i,N) = 0$ and therefore $\operatorname{Ext}^i_R(M,N) = 0$.

Notation 2.1.8. For an R-module M and any $r \in R$, the multiplication map

$$\begin{aligned} \mu_r \colon M &\longrightarrow M \\ m &\longmapsto rm \end{aligned}$$

is a well-defined R-module homomorphism by the axioms for R- modules. Unless otherwise noted, we will let μ_x denote a multiplication map by the element x.

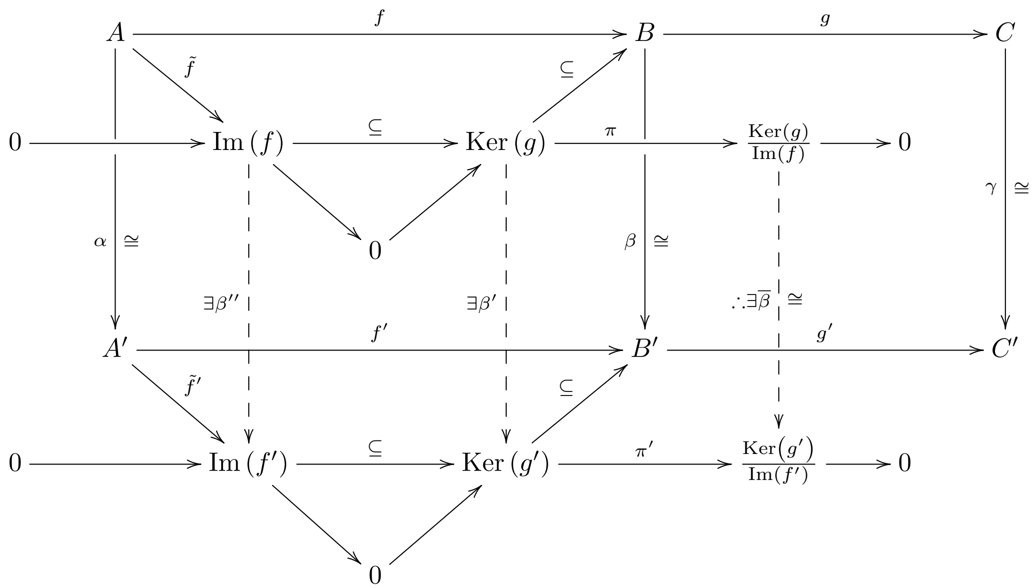
Lemma 2.1.9. *Consider a commutative diagram of R-modules and R-module homomorphisms*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \alpha \downarrow \cong & \curvearrowright & \beta \downarrow \cong & \curvearrowright & \gamma \downarrow \cong \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C \end{array}$$

and assume $g \circ f = 0$ (and consequently $g' \circ f' = 0$). Then there is a well-defined R-module isomorphism

$$\begin{aligned} \overline{\beta} : \frac{\operatorname{Ker}(g)}{\operatorname{Im}(f)} &\longrightarrow \frac{\operatorname{Ker}(g')}{\operatorname{Im}(f')} \\ b + \operatorname{Im}(f) &\longmapsto \beta(b) + \operatorname{Im}(f')\,. \end{aligned}$$

Proof. We give here only a sketch via a commutative diagram.

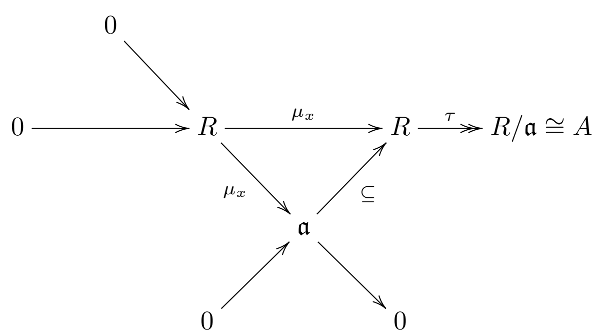


□

Example 2.1.10. Let A be a non-zero commutative ring with identity and set $R = A[x]$, $\mathfrak{a} = (x)R$. Note $R/\mathfrak{a} \cong A$ and therefore A is an R-module. Then we will show

$$\mathrm{Ext}^i_R(A, R) \cong \begin{cases} A & i = 1 \\ 0 & i \neq 1 \end{cases} \qquad \mathrm{Ext}^i_R(A, A) \cong \begin{cases} A & i = 0, 1 \\ 0 & \text{else.} \end{cases}$$

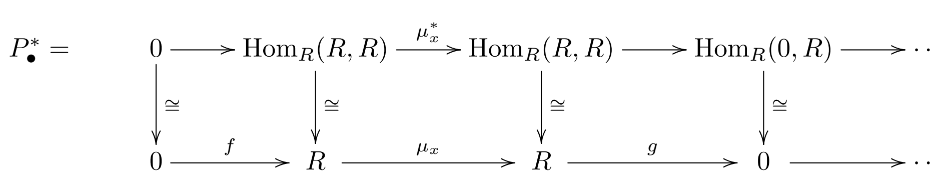
We begin with an augmented projective resolution of A from the diagram



where μ_x is multiplication by x and τ is the natural surjection. Define $P^+_\bullet$ to be the row from the above diagram. Hence

$$P_\bullet = \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow R \xrightarrow{\mu_x} R \longrightarrow 0$$

and



where the vertical isomorphisms are by Hom-cancellation. We can now calculate $\mathrm{Ext}^i_R(A,R)$ from the bottom row of this diagram, because of Lemma 2.1.9.

$$\mathrm{Ext}^i_R(A,R) = \begin{cases} 0/0 = 0 & i \neq 0,\ i \neq 1 \\ \mathrm{Ker}\,(\mu_x)\,/\,\mathrm{Im}\,(f) = 0/0 & i = 0 \\ \mathrm{Ker}\,(g)\,/\,\mathrm{Im}\,(\mu_x) = R/\mathfrak{a} \cong A & i = 1 \end{cases}$$

We calculate $\mathrm{Ext}^i_R(A,A)$ similarly.

$$\mathrm{Hom}_R(P_\bullet, A) \cong \qquad 0 \xrightarrow{h} A \xrightarrow[0]{\mu_x^A} A \xrightarrow{k} 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\implies \mathrm{Ext}^i_R(A,A) = \begin{cases} 0/0 = 0 & i \neq 0, i \neq 1 \\ \mathrm{Ker}\left(\mu_x^A\right)/\,\mathrm{Im}\,(h) = A/0 \cong A & i = 0 \\ \mathrm{Ker}\,(k)\,/\,\mathrm{Im}\left(\mu_x^A\right) = A/0 \cong A & i = 1 \end{cases}$$

Note in this case μ_x^A is the zero map since $\mathrm{Im}\left(\mu_x^A\right) = \mathfrak{a}/\mathfrak{a} = \{0\} \subset R/\mathfrak{a} \cong A$.

One might wonder why we did not write $\mathrm{Ext}^0_R(N, M_\ell)$ in Example 2.1.1, so here we give a reason in the form of a proposition.

Proposition 2.1.11. *For any two R-modules M and N*

$$\mathrm{Ext}^0_R(M,N) \cong \mathrm{Hom}_R(M,N).$$

Proof. Let M and N be two R-modules and let

$$P_\bullet^+ \; = \qquad \ldots \longrightarrow P_4 \xrightarrow{\partial_4^P} P_3 \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

be an augmented projective resolution of M. Since Hom is left-exact, the following piece of the sequence $\mathrm{Hom}_R(P_\bullet^+, N)$ is exact as well.

$$0 \longrightarrow \mathrm{Hom}_R(M,N) \xrightarrow{\tau^*} \mathrm{Hom}_R(P_0,N) \xrightarrow{(\partial_1^P)^*} \mathrm{Hom}_R(P_1,N)$$

This exactness yields

$$\mathrm{Ker}\left(\partial_1^{P*}\right) = \mathrm{Im}\,(\tau^*) \overset{(1)}{\cong} \frac{\mathrm{Hom}_R(M,N)}{\mathrm{Ker}\,(\tau^*)} \overset{(2)}{=} \frac{\mathrm{Hom}_R(M,N)}{\{0\}} \cong \mathrm{Hom}_R(M,N)$$

where (1) holds by the First Isomorphism Theorem and (2) holds since

$$\mathrm{Ker}\,(\tau^*) = \mathrm{Im}\Big(\, 0 \longrightarrow \mathrm{Hom}_R(M,N)\,\Big) = \{0\}.$$

On the other hand, from the definition of Ext we have

$$\mathrm{Ext}^0_R(M,N) = \frac{\mathrm{Ker}\left(\left(\partial_1^P\right)^*\right)}{\mathrm{Im}\,(0 \to \mathrm{Hom}_R(P_0,N))} = \frac{\mathrm{Ker}\left(\left(\partial_1^P\right)^*\right)}{\{0\}} \cong \mathrm{Ker}\left(\left(\partial_1^P\right)^*\right).$$

□

Proposition 2.1.12. *Given R-modules and a projective resolution as in the above discussion, we have the following.*

(a) $\mathrm{Ext}^i_R(M,N) = 0$ *for all* $i < 0$

(b) $\mathrm{Ext}^i_R(M,0) = 0$ *for all* $i \in \mathbb{Z}$

(c) $\mathrm{Ext}^i_R(0,N) = 0$ *for all* $i \in \mathbb{Z}$

Proof. (a). We have $(P^*_\bullet)_i = 0$ for all $i < 0$. Therefore $(\partial^P_i)^* \colon 0 \longrightarrow 0$ for all $i < 0$ and $(\partial^P_0)^* \colon 0 \longrightarrow P^*_0$. It follows that

$$\mathrm{Ext}^i_R(M,N) = \frac{\mathrm{Ker}\left((\partial^P_{i+1})^*\right)}{\mathrm{Im}\left((\partial^P_i)^*\right)} = \frac{0}{0} = 0$$

for all $i < 0$.

(b). For any $i \in \mathbb{Z}$ we have

$$\mathrm{Hom}_R(P_\bullet, 0)_{-i} = \mathrm{Hom}_R(P_i, 0) = 0.$$

Then $(\partial^P_i)^* \colon 0 \longrightarrow 0$ and therefore

$$\mathrm{Ext}^i_R(M,0) = \frac{0}{0} = 0$$

for all $i \in \mathbb{Z}$.

(c). We can define a projective resolution of the R-module $M = 0$.

$$P^+_\bullet = P_\bullet = \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Therefore

$$\mathrm{Hom}_R(P_\bullet, N)_{-i} = \mathrm{Hom}_R(0, N) = 0$$

for all $i \in \mathbb{Z}$ and hence

$$\mathrm{Ext}^i_R(0,N) = \frac{0}{0} = 0$$

again for all $i \in \mathbb{Z}$. □

Fact 2.1.13. *Ext is well-defined. That is, the calculation of* $\mathrm{Ext}^i_R(M,N)$ *is independent (up to isomorphism) of our choice of projective resolution of* M.

Establishing the Fact 2.1.13 is the main point of Chapter 6. See Theorem 6.5.2.

2.2 Application 2: Depth

Depth is a nice tool on which to perform induction arguments. One thing that makes it so versatile is that it has strong ties to Ext modules.

Definition 2.2.1. Let M be an R-module. An element $x \in R$ is a *non-zero-divisor* on M if the sequence $0 \longrightarrow M \xrightarrow{\mu_x} M$ is exact (i.e., for all $m \in M$, $xm = 0$ implies $m = 0$). We say x is *M-regular* if x is a non-zero-divisor on M and $xM \neq M$ (i.e., $M/xM \neq 0$). A sequence $\underline{x} = x_1, \ldots, x_n \in R$ is *M-regular* if x_1 is M-regular and x_i is $M/(x_1, \ldots, x_{i-1})M$-regular for all $i = 2, \ldots, n$.

Fact/Definition 2.2.2. *Let R be noetherian and $\mathfrak{a} \leq R$ an ideal such that $\mathfrak{a}M \neq M$. Then there exists a* maximal *M-regular sequence in $\mathfrak{a}$. That is, there exists an M-regular sequence $\underline{x} = x_1, \ldots, x_n \in \mathfrak{a}$ such that for all $y \in \mathfrak{a}$, the sequence $x_1, \ldots, x_n, y$ is not M-regular. The longest length n of an M-regular sequence in $\mathfrak{a}$ is called the* depth *of $\mathfrak{a}$ on M, denoted*

$$n = \operatorname{depth}_R(\mathfrak{a}; M).$$

Fact 2.2.3. *Depth is independent of our choice of maximal M-regular sequence as long as M is finitely generated. The proof of this fact requires Ext. One proves there exists some $n \in \mathbb{N}_0$ such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M) = 0$ whenever $0 \leq i \leq n-1$ and $\operatorname{Ext}^n_R(R/\mathfrak{a}, M) \neq 0$, in order to conclude*

$$\operatorname{depth}_R(\mathfrak{a}; M) = \inf \left\{ i \geq 0 \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0 \right\}.$$

Establishing Fact 2.2.3 is the goal of Chapter 3. See Theorem 3.5.16.

2.3 Application 3: Localization Problem for Regular Local Rings

Here we introduce regular rings. The question of whether regularity is preserved under localization (Question 2.3.7) was one of the great open questions solved using homological methods. We give an answer immediately in Theorem 2.3.8, which is seen again later (Theorem 7.4.11). Throughout the section assume $(R, \mathfrak{m}, \mathfrak{K})$ is a local, noetherian ring. That is, assume $\mathfrak{m}$ is the unique maximal ideal and $\mathfrak{K} \cong R/\mathfrak{m}$.

Definition 2.3.1. The *Krull dimension*, or just *dimension*, of R can be said to measure the size of R and is defined

$$\dim(R) = \sup \{ n \geq 0 \mid \exists\, \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq R \text{ s.t. } \mathfrak{p}_i \text{ prime}, \forall\, i = 1, \ldots, n \}.$$

Under our local and noetherian assumptions, Krull dimension is finite.

Definition 2.3.2. The *embedding dimension* is defined as the dimension of a particular R-module as a $\mathfrak{K}$-vector space.

$$\operatorname{edim}(R) = \dim_{\mathfrak{K}}(\mathfrak{m}/\mathfrak{m}^2)$$

Since $\mathfrak{m}/\mathfrak{m}^2$ is an R-module satisfying $\mathfrak{m} \cdot (\mathfrak{m}/\mathfrak{m}^2) = 0$, it is also an $R/\mathfrak{m}$-module. That is, it is a $\mathfrak{K}$-vector space (since $\mathfrak{K}$ a field) and moreover since R is noetherian, $\mathfrak{m}/\mathfrak{m}^2$ is finitely generated over R and is consequently a finite dimensional vector space over $\mathfrak{K}$. In summary, the noetherian assumption on R again guarantees a finite dimension.

Theorem 2.3.3. *One has*

$$\operatorname{depth}_R(\mathfrak{m}; R) \overset{(1)}{\leq} \dim(R) \overset{(2)}{\leq} \operatorname{edim}(R).$$

Definition 2.3.4. R is *Cohen-Macauley* if (1) is an equality and R is *regular* if (2) is an equality.

Fact 2.3.5. *Every regular ring is Cohen-Macaulay.*

Example 2.3.6. For the localization ring

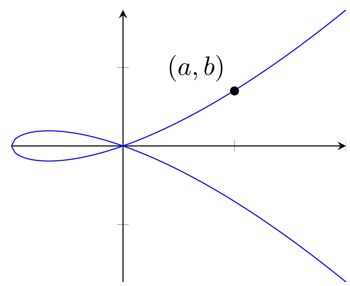
$$R = \mathbb{K}[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$$

with unique maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)R$, we have $\dim(R) = n$ and $\operatorname{edim}(R) = n$, so the ring is regular and one can think of R as the geometric object $\mathbb{K}^n$ (e.g., $\mathbb{R}^n$ or $\mathbb{C}^n$). There is more on the construction of localization rings in the preliminaries.

In many ways the quotient ring

$$R_0 = \frac{\mathbb{R}[x,y]}{(y^2 - x^2(x+1))}$$

represents the curve $y^2 = x^2(x+1)$, which we plot in the Cartesian coordinate plane below. This plot tells us a number of things about the ring R_0, though none of them are necessarily obvious.



- Points $p = (a,b)$ on the curve correspond to maximal ideals $\mathfrak{m}_p = (x-a, y-b)R_0$ and the local ring $(R_0)_{\mathfrak{m}_p}$ has the maximal ideal $(x-a, y-b)(R_0)_{\mathfrak{m}_p}$.
- All rings $(R_0)_{\mathfrak{m}_p}$ have Krull dimension 1, because the curve is 1-dimensional.
- If p is a smooth point of the curve, then the ring $(R_0)_{\mathfrak{m}_p}$ is regular.
- $\operatorname{edim}((R_0)_{\mathfrak{m}_p}) = \dim_{\mathbb{R}}(\text{tangent space at } p)$.
- At the origin $p = (0,0)$, $\operatorname{edim}((R_0)_{\mathfrak{m}_p}) = 2$ and therefore $(R_0)_{\mathfrak{m}_p}$ is not regular.
- The localization in this example can be thought of as zooming in on some neighborhood of your point, so it should at least not make the singularity worse.

An important question from the early 1900's asked if regularity is preserved under localization, which is the thrust of this section.

Question 2.3.7. If R is regular and $\mathfrak{p} \lneq R$ is prime, must $R_{\mathfrak{p}}$ necessarily be regular as well?

It turns out that the answer is 'yes'. This is highly nontrivial because while one can exert some control from $\dim(R)$ to $\dim(R_{\mathfrak{p}})$, controlling $\operatorname{edim}(R_{\mathfrak{p}})$ is harder and requires homological algebra. The essential point is in the following theorem.

Theorem 2.3.8 (Auslander, Buchsbaum, Serre)**.** *The following are equivalent.*

(i) *R is regular.*

(ii) *For any two finitely generated modules M and N, $\operatorname{Ext}^i_R(M,N) = 0$ for all $i > \dim(R)$.*

(iii) $\operatorname{Ext}_R^{\dim(R)+1}(\mathfrak{K}, \mathfrak{K}) = 0$.

(iv) *There exists some $d \geq 0$ such that $\operatorname{Ext}^d_R(\mathfrak{K}, \mathfrak{K}) = 0$.*

The proof of this result, unfortunately, is outside the scope of the present document.

Chapter 3

Depth by Ext

In this chapter we build the tools we need to characterize depth in terms of Ext (Fact 2.2.3), which is given with proof as Theorem 3.5.16 at the end of the chapter.

3.1 Hom and Direct Sums of Modules

In this section we observe that direct sums of modules interact very intuitively with functors like $\mathrm{Hom}_R(-,N)$ and $U^{-1}(-)$. We conclude the section by proving in Proposition 3.1.8 that with a few assumptions, the two functors interact with one another exactly as one might like them to.

Fact 3.1.1. *If M and M' are two R-modules, then there is a split short exact sequence*

$$0 \longrightarrow M \underset{\tau}{\overset{\epsilon}{\rightleftarrows}} M \oplus M' \underset{\epsilon'}{\overset{\tau'}{\rightleftarrows}} M' \longrightarrow 0$$

where $\tau \circ \epsilon = \mathrm{id}_M$ and $\tau' \circ \epsilon' = \mathrm{id}_{M'}$, and we have

$$\begin{aligned} \mathrm{Hom}_R(M \oplus M', N) &\xrightarrow[\cong]{\omega} \mathrm{Hom}_R(M,N) \oplus \mathrm{Hom}_R(M',N) \\ \psi &\longmapsto (\psi \circ \epsilon, \psi \circ \epsilon') = (\epsilon^*(\psi), \epsilon'^*(\psi)) \end{aligned}$$

Proof. Applying $\mathrm{Hom}_R(-,N)$ to the split exact sequence above we get

$$0 \longrightarrow \mathrm{Hom}_R(M',N) \xrightarrow{\tau'^*} \mathrm{Hom}_R(M \oplus M', N) \underset{\tau^*}{\overset{\epsilon^*}{\rightleftarrows}} \mathrm{Hom}_R(M,N) \dashrightarrow 0 \qquad (3.1.1.1)$$

tacking on a zero on the right-hand side. We claim this is a short exact sequence. Indeed since Hom is left-exact and

$$\epsilon^* \circ \tau^* = (\tau \circ \epsilon)^* = (\mathrm{id}_M)^* = \mathrm{id}_{\mathrm{Hom}_R(M,N)}$$

we know ϵ^* is surjective and therefore (3.1.1.1) is a short exact sequence.

From here we can take one of two approaches to reach the desired conclusion. On the one hand, note that we now have a split exact sequence in (3.1.1.1), so the desired isomorphism follows immediately from the definition of a split sequence (1.1.8). On the other hand, we can also prove directly that the map ω is an isomorphism as follows.

We claim the following is a homomorphism of short exact sequences, for which it suffices to show ω is a well-defined R-module homomorphism and the proposed maps make the diagram commute. This will actually complete the proof by the Short-5 Lemma.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(M',N) & \xrightarrow{\tau'^*} & \mathrm{Hom}_R(M\oplus M',N) & \xrightarrow{\epsilon^*} & \mathrm{Hom}_R(M,N) & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & \circlearrowleft ? & \downarrow{\scriptstyle \omega} & \circlearrowleft ? & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
0 & \longrightarrow & \mathrm{Hom}_R(M',N) & \xrightarrow[E']{} & \mathrm{Hom}_R(M,N)\oplus \mathrm{Hom}_R(M',N) & \xrightarrow[T]{} & \mathrm{Hom}_R(M,N) & \longrightarrow & 0
\end{array}$$

$$\alpha \overset{E'}{\longmapsto} (0,\alpha)$$

$$(\beta,\gamma) \overset{T}{\longmapsto} \beta$$

For an arbitrary pair of elements $\psi_1,\psi_2 \in \mathrm{Hom}_R(M\oplus M',N)$ and for any $r\in R$ we have

$$\begin{aligned}
\omega(r\psi_1+\psi_2) &= ((r\psi_1+\psi_2)\circ\epsilon\,,\,(r\psi_1+\psi_2)\circ\epsilon') \\
&= ((r\psi_1)\circ\epsilon+\psi_2\circ\epsilon\,,\,(r\psi_1)\circ\epsilon'+\psi_2\circ\epsilon') \\
&= (r(\psi_1\circ\epsilon)+\psi_2\circ\epsilon\,,\,r(\psi_1\circ\epsilon')+\psi_2\circ\epsilon') \\
&= (r(\psi_1\circ\epsilon)\,,\,r(\psi_1\circ\epsilon'))+(\psi_2\circ\epsilon\,,\,\psi_2\circ\epsilon') \\
&= r(\psi_1\circ\epsilon\,,\,\psi_1\circ\epsilon')+(\psi_2\circ\epsilon\,,\,\psi_2\circ\epsilon') \\
&= r\cdot\omega(\psi_1)+\omega(\psi_2)
\end{aligned}$$

Thus ω is a well-defined R-module homomorphism. Consider an arbitrary $\alpha\in\mathrm{Hom}_R(M',N)$ and we have

$$(\omega\circ\tau'^*)(\alpha)=\omega(\alpha\circ\tau')=(\alpha\circ\tau'\circ\epsilon,\alpha\circ\tau'\circ\epsilon')\overset{(1)}{=}(\alpha\circ 0,\alpha\circ\mathrm{id}_{M'})=(0,\alpha)=E'(\alpha)$$

where (1) holds since $\mathrm{Im}\,(\epsilon)=\mathrm{Ker}(\tau')$. Now taking an arbitrary $\psi\in\mathrm{Hom}_R(M\oplus M',N)$ we have

$$(T\circ\omega)(\psi)=T(\psi\circ\epsilon,\psi\circ\epsilon')=\psi\circ\epsilon=\epsilon^*(\psi)$$

So the diagram commutes and ω must be an isomorphism by the Short-Five Lemma. ☐

Example 3.1.2. Using 3.1.1 as a base case, one can prove inductively that

$$\mathrm{Hom}_R\left(\bigoplus_{i=1}^n M_i, N\right)\longrightarrow \bigoplus_{i=1}^n \mathrm{Hom}_R(M_i,N)$$

$$\psi\longmapsto\begin{pmatrix}\epsilon_1^*(\psi)\\ \vdots \\ \epsilon_n^*(\psi)\end{pmatrix}$$

is an isomorphism, where $\epsilon_j\colon M_j \longrightarrow \bigoplus_{i=1}^n M_i$ is the standard injection. In particular the map

$$\omega_n:\mathrm{Hom}_R(R^n,R)\longrightarrow R^n$$

$$\psi\longmapsto\begin{pmatrix}\psi(e_1)\\ \vdots \\ \psi(e_n)\end{pmatrix}$$

is an isomorphism, where $e_1,\ldots,e_n\in R^n$ is the standard basis for R^n. Note in this case the base case is simply Hom-cancellation $\mathrm{Hom}_R(R,R)\cong R$. Moreover, if we let $v_1,\ldots,v_m\in R^m$ be the standard basis vectors of R^m, let $e_1^*, \dots, e_n^*$ and $v_1^*, \dots, v_m^*$ be the respective dual basis vectors, and let $\phi : R^m \longrightarrow R^n$ be an R-module homomorphism represented by a matrix A, where the jth column of A is $\phi(v_j)$, then $\text{Hom}_R(-, R)$ yields

$$\begin{array}{ccc} \{e_1^*, \dots, e_n^*\} \subset \text{Hom}_R(R^n, R) & \xrightarrow{\phi^*} & \text{Hom}_R(R^m, N) \supset \{v_1^*, \dots, v_m^*\} \\ \cong \big\downarrow \omega_n & \curvearrowright & \omega_m \big\downarrow \cong \\ \{e_1, \dots, e_n\} \subset R^n & \xrightarrow[A^T]{} & R^m \supset \{v_1, \dots, v_m\}. \end{array}$$

One can prove the diagram commutes using the basis vectors and the dual basis vectors, which in conjunction with linearity, proves the entire diagram commutes.

$$(\omega_m \circ \phi^*)(e_i^*) = \omega_m(e_i^* \circ \phi) = \begin{pmatrix} (e_i^* \circ \phi)(v_1) \\ \vdots \\ (e_i^* \circ \phi)(v_m) \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} a_{i1} \\ \vdots \\ a_{im} \end{pmatrix} = \left(i^{th} \text{ row of } A\right)^T$$

where (2) holds since $e_i^*(\phi(v_j))$ is simply e_i^* applied to the j^{th} column of A, which is a_{ij}.

$$(A^T \circ \omega_n)(e_i^*) = A^T \cdot \begin{pmatrix} e_i^*(e_1) \\ \vdots \\ e_i^*(e_i) \\ \vdots \\ e_i^*(e_n) \end{pmatrix} = i^{th} \text{ column of } (A^T)$$

We now state an even more general version of Fact 3.1.1 without proof.

Fact 3.1.3. *For a direct sum of an arbitrary collection of R-modules, denoted* $\bigoplus_{\lambda \in \Lambda} M_\lambda$*, we have*

$$\text{Hom}_R\left(\bigoplus_{\lambda \in \Lambda} M_\lambda, N\right) \cong \prod_{\lambda \in \Lambda} \text{Hom}_R(M_\lambda, N).$$

Theorem 3.1.4. $U^{-1}(-)$ *is exact, i.e.,* $U^{-1}(-)$ *respects short exact sequences (and therefore exact sequences).*

Proof. Let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

be a short exact sequence and consider

$$0 = U^{-1}0 \longrightarrow U^{-1}M \xrightarrow{U^{-1}f} U^{-1}N \xrightarrow{U^{-1}g} U^{-1}P \longrightarrow U^{-1}0 = 0 \,.$$

First, and most straightforward to show, is the containment $\text{Im}\left(U^{-1}f\right) \subseteq \text{Ker}\left(U^{-1}g\right)$.

$$(U^{-1}g) \circ (U^{-1}f) = U^{-1}(g \circ f) = U^{-1}0 = 0$$

Second, to verify the reverse containment we let $n/u \in \text{Ker}\left(U^{-1}g\right)$ and show it has a preimage under $U^{-1}f$. Residing in the kernel implies $g(u)/n = 0$, i.e., there exists some $v \in U$ such that $0 = v \cdot g(n) = g(vn)$. Since $\text{Ker}\,(g) \subseteq \text{Im}\,(f)$, we have $f(m) = vn$ for some $m \in M$ and we consider the element $m/uv \in U^{-1}M$.

$$(U^{-1}f)\left(\frac{m}{uv}\right) = \frac{f(m)}{uv} = \frac{vn}{uv} = \frac{n}{u}$$

Third, we want to show $U^{-1}f$ is injective. Let $m/u \in \operatorname{Ker}\left(U^{-1}f\right)$ and similar to the previous part this implies there exists some $v \in U$ such that $v \cdot f(m) = 0$. This also implies $f(vm) = v \cdot f(m) = 0$ and since f is injective, $vm = 0$. Therefore we have

$$\frac{m}{u} = \frac{vm}{vu} = \frac{0}{vu} = 0.$$

So $U^{-1}f$ has trivial kernel and is therefore injective.

Finally, let $p/u \in U^{-1}P$ and note $p = g(n)$ for some $n \in N$ since g is surjective. The immediate implication is

$$\frac{p}{u} = \frac{g(n)}{u} = U^{-1}g\left(\frac{n}{u}\right) \in \operatorname{Im}\left(U^{-1}g\right)$$

Hence $U^{-1}(-)$ preserves short exact sequences. To expand to the arbitrary sequence suppose

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$$

is exact. Around this sequence we build four short exact sequences as in diagram (3.1.6.1). The point in this construction is applying $U^{-1}(-)$ to it will preserve commutivity of the diagram and exactness of the diagonals. Then a standard diagram chase (omitted) shows the exactness of the row in which we are interested is also preserved. □

Fact 3.1.5. *We have results similar to those in 3.1.1 and 3.1.2 for localizations. For $U \subseteq R$ a multiplicatively closed set and for R-modules M and M' we have the following isomorphism.*

$$\begin{array}{ccc} U^{-1}(M \oplus M') & \xrightarrow{\cong} & U^{-1}(M) \oplus U^{-1}(M') \\ \dfrac{(m,m')}{u} & \longmapsto & \left(\dfrac{m}{u}, \dfrac{m'}{u}\right) \\ \dfrac{(u'm, um')}{uu'} & \longleftarrow\!\!\mid & \left(\dfrac{m}{u}, \dfrac{m'}{u'}\right) = \left(\dfrac{u'm}{uu'}, \dfrac{um'}{uu'}\right) \end{array}$$

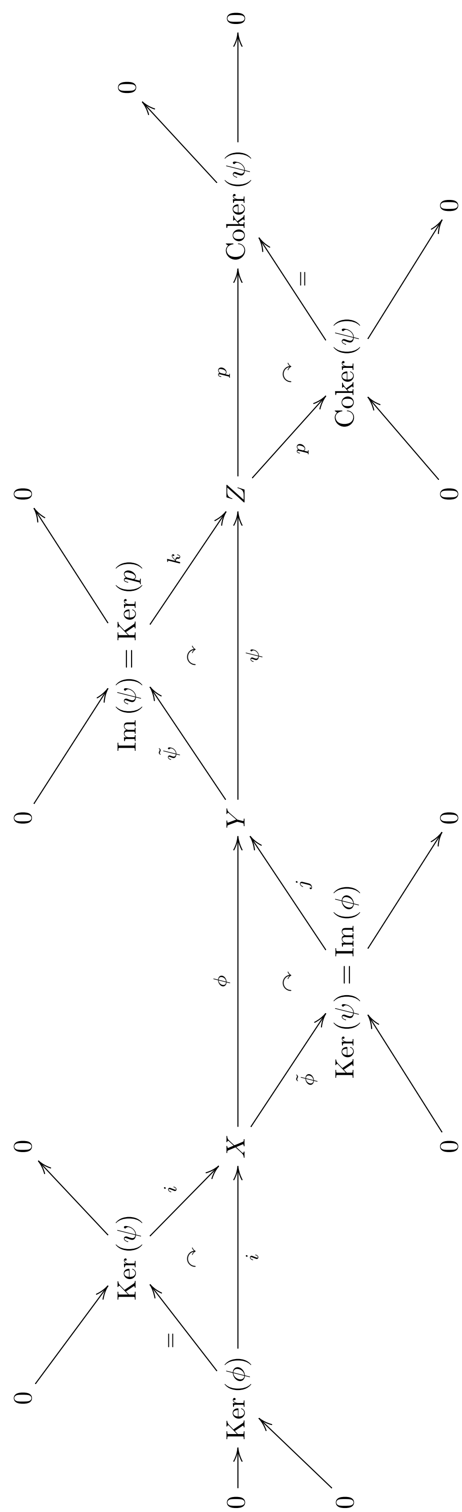
More generally we write

$$U^{-1}\left(\bigoplus_{i=1}^{n} M_i\right) \xrightarrow{\cong} \bigoplus_{i=1}^{n} U^{-1}M_i$$

$$\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}/u \longmapsto \begin{pmatrix} m_1/u \\ \vdots \\ m_n/u \end{pmatrix}$$

Remark 3.1.6. Replacing M_i above with copies of R shows that the notation $U^{-1}R^n$ is not ambiguous, because $U^{-1}(R^n)$ is isomorphic to $(U^{-1}R)^n$. Thus homomorphisms between modules in the form of the former induce homomorphisms between modules in the form of the latter. We summarize this relationship in the following commutative diagram, where (a_{ij}) is an $n \times m$ matrix over R.

$$\begin{array}{ccc} U^{-1}(R^m) & \xrightarrow{U^{-1}(a_{ij})} & U^{-1}(R^n) \\ {\scriptstyle\cong}\downarrow & \curvearrowright & \downarrow{\scriptstyle\cong} \\ (U^{-1}R)^m & \xrightarrow[(a_{ij}/1)]{} & (U^{-1}R)^n \end{array}$$



(3.1.6.1)

Definition 3.1.7. An R-module M is *finitely presented* if there exists an exact sequence

$$R^m \xrightarrow{f} R^n \xrightarrow{g} M \longrightarrow 0 .$$

Proposition 3.1.8. *Let R be a non-zero commutative ring with identity, let M and N be R-modules, and let $U \subseteq R$ be a multiplicatively closed subset.*

(a) For all $\phi/u \in U^{-1}\operatorname{Hom}_R(M,N)$, the map ϕ_u below is a well-defined $U^{-1}R$-module homomorphism.

$$\phi_u : U^{-1}M \longrightarrow U^{-1}N$$

$$m/v \longmapsto \phi(m)/(uv)$$

(b) The function $\Theta_{U,M,N}$ below is a well-defined $U^{-1}R$-module homomorphism.

$$\Theta_{U,M,N} : U^{-1}\operatorname{Hom}_R(M,N) \longrightarrow \operatorname{Hom}_{U^{-1}R}\big(U^{-1}M, U^{-1}N\big)$$

$$\phi/u \longmapsto \phi_u$$

(c) If M is finitely presented, then $\Theta_{U,M,N}$ is an isomorphism.

(d) If R is noetherian and M is finitely generated, then

$$\operatorname{Hom}_{U^{-1}R}\big(U^{-1}M, U^{-1}N\big) \cong U^{-1}\operatorname{Hom}_R(M,N)$$

as $U^{-1}R$-modules (via $\Theta_{U,M,N}$).

Proof. (a). We prove this part in two steps. First let $u \in U$ and $\phi \in \operatorname{Hom}_R(M,N)$. For any $m/v \in U^{-1}M$ we have

$$\phi_u\left(\frac{m}{v}\right) = \frac{\phi(m)}{uv} = \frac{1}{u}\cdot(U^{-1}\phi)\left(\frac{m}{v}\right) = (\mu_{\frac{1}{u}} \circ U^{-1}\phi)\left(\frac{m}{v}\right)$$

where $\mu_{1/u}$ is the standard product map (see Example 2.1.10). Thus ϕ_u is the composition of two well-defined $U^{-1}R$-module homomorphisms, so it is itself a well-defined $U^{-1}R$-module homomorphism. The second question of well-definedness has to do with our choice of representative from $U^{-1}\operatorname{Hom}_R(M,N)$, so let $\phi/u = \phi'/u'$. This means there exists some $u'' \in U$ such that $uu''\phi' = u'u''\phi$, so for any $m/v \in U^{-1}M$ we have

$$uu''\cdot\phi'(m) = (uu''\phi')(m) = (u'u''\phi)(m) = u'u''\cdot\phi(m)$$

and therefore

$$\phi_u\left(\frac{m}{v}\right) = \frac{\phi(m)}{uv} = \frac{u'u''\phi(m)}{u'u''uv} = \frac{uu''\phi'(m)}{u'u''uv} = \frac{\phi'(m)}{u'v} = \phi'_{u'}\left(\frac{m}{v}\right).$$

(b). The well-definedness of $\Theta_{U,M,N}$ is a consequence of part (a), so we need only show it is $U^{-1}R$-linear. Let $\phi/u, \phi'/u' \in U^{-1}\operatorname{Hom}_R(M,N)$ be given and note that showing Θ respects sums is equivalent to showing

$$(u'\phi + u\phi')_{uu'} = \phi_u + \phi'_{u'}$$

since

$$\frac{\phi}{u} + \frac{\phi'}{u'} = \frac{u'\phi + u\phi'}{uu'}.$$

To this end, for any $m/v \in U^{-1}M$ we have

$$\begin{aligned}(u'\phi + u\phi')_{uu'}\left(\frac{m}{v}\right) &= \frac{(u'\phi + u\phi')(m)}{uu'v} \\ &= \frac{(u'\phi)(m) + (u\phi')(m)}{uu'v} \\ &= \frac{u' \cdot \phi(m)}{uu'v} + \frac{u \cdot \phi'(m)}{uu'v} \\ &= \frac{\phi(m)}{uv} + \frac{\phi'(m)}{u'v} \\ &= \phi_u\left(\frac{m}{v}\right) + \phi'_{u'}\left(\frac{m}{v}\right).\end{aligned}$$

To complete the proof of part (b) let $r/t \in U^{-1}R$ be given and we observe for any $m/v \in U^{-1}M$

$$\begin{aligned}\Theta_{U,M,N}\left(\frac{r\phi}{tu}\right)\left(\frac{m}{v}\right) &= (r\phi)_{tu}\left(\frac{m}{v}\right) \\ &= \frac{(r\phi)(m)}{tuv} \\ &= \frac{r}{t} \cdot \frac{\phi(m)}{uv} \\ &= \frac{r}{t} \cdot \phi_u\left(\frac{m}{v}\right) \\ &= \frac{r}{t} \cdot \Theta_{U,M,N}\left(\frac{\phi}{u}\right)\left(\frac{m}{v}\right).\end{aligned}$$

(c). We complete this part in four steps. First we claim $\Theta_{U,M\oplus M',N}$ is an isomorphism if and only if both $\Theta_{U,M,N}$ and $\Theta_{U,M',N}$ are isomorphisms. We prove this by showing diagram (3.1.8.3) of $U^{-1}R$-modules and homomorphisms commutes. To make clear some of our notation we define the following homomorphisms.

$$\gamma : U^{-1}(M \oplus M') \longrightarrow (U^{-1}M) \oplus (U^{-1}M') \qquad \gamma^{-1} : (U^{-1}M) \oplus (U^{-1}M') \longrightarrow U^{-1}(M \oplus M')$$

$$\frac{(m, m')}{v} \longmapsto \left(\frac{m}{v}, \frac{m'}{v}\right) \qquad \left(\frac{m}{v}, \frac{m'}{v'}\right) \longmapsto \frac{(v'm, vm')}{vv'}$$

Consider the map ω as defined in Fact 3.1.1 and from the same fact, consider the standard injections ϵ and ϵ' along with the standard projections τ and τ', all of which we reproduce below.

$$\epsilon : U^{-1}M \longrightarrow (U^{-1}M) \oplus (U^{-1}M') \qquad \epsilon' : U^{-1}M' \longrightarrow (U^{-1}M) \oplus (U^{-1}M')$$

$$\frac{m}{u} \longmapsto \left(\frac{m}{u}, 0\right) \qquad \frac{m'}{u} \longmapsto \left(0, \frac{m'}{u}\right)$$

$$\tau : (U^{-1}M) \oplus (U^{-1}M') \longrightarrow U^{-1}M \qquad \tau' : (U^{-1}M) \oplus (U^{-1}M') \longrightarrow U^{-1}M'$$

$$\left(\frac{m}{u}, \frac{m'}{u'}\right) \longmapsto \frac{m}{u} \qquad \left(\frac{m}{u}, \frac{m'}{u'}\right) \longmapsto \frac{m'}{u'}$$

The maps Γ and Ω will be defined implicitly in the diagram chase.

For any $\psi/u \in U^{-1}\operatorname{Hom}_R(M \oplus M', N)$ we have

$$\frac{\psi}{u} \overset{U^{-1}\omega}{\longmapsto} \frac{\omega(\psi)}{u} = \frac{(\psi\circ\tau, \psi\circ\tau')}{u} \overset{\Gamma}{\longmapsto} \left(\frac{\psi\circ\tau}{u}, \frac{\psi\circ\tau'}{u}\right) \xmapsto{\Theta_{U,M,N}\oplus\Theta_{U,M',N}} ((\psi\circ\tau)_u, (\psi\circ\tau')_u)\ . \tag{3.1.8.1}$$

Tracking along the other half of the diagram we find

$$\frac{\psi}{u} \xmapsto{\Theta_{U,M\oplus M',N}} \psi_u \overset{(\gamma^{-1})^*}{\longmapsto} \psi_u\circ\gamma^{-1} \overset{\Omega}{\longmapsto} (\psi_u\circ\gamma^{-1}\circ\epsilon, \psi_u\circ\gamma^{-1}\circ\epsilon')\ . \tag{3.1.8.2}$$

Now it is a matter of showing the resulting maps in (3.1.8.1) and (3.1.8.2) are equivalent.

For any $m/v \in U^{-1}M$ and any $m'/v' \in U^{-1}M'$, (3.1.8.1) produces

$$\left((\psi\circ\tau)_u\left(\frac{m}{v}\right), (\psi\circ\tau')_u\left(\frac{m'}{v'}\right)\right) = \left(\frac{(\psi\circ\tau)(m)}{uv}, \frac{(\psi\circ\tau')(m')}{uv'}\right) = \left(\frac{\psi(m,0)}{uv}, \frac{\psi(0,m')}{uv'}\right).$$

and likewise (3.1.8.2) produces

$$\begin{aligned}\left((\psi_u\circ\gamma^{-1}\circ\epsilon)\left(\frac{m}{v}\right), (\psi_u\circ\gamma^{-1}\circ\epsilon')\left(\frac{m'}{v'}\right)\right) &= \left(\psi_u\left(\gamma^{-1}\left(\frac{m}{v},0\right)\right), \psi_u\left(\gamma^{-1}\left(0,\frac{m'}{v'}\right)\right)\right)\\ &= \left(\psi_u\left(\gamma^{-1}\left(\frac{m}{v},\frac{0}{v}\right)\right), \psi_u\left(\gamma^{-1}\left(\frac{0}{v'},\frac{m'}{v'}\right)\right)\right)\\ &= \left(\psi_u\left(\frac{(m,0)}{v}\right), \psi_u\left(\frac{(0,m')}{v'}\right)\right)\\ &= \left(\frac{\psi(m,0)}{uv}, \frac{\psi(0,m')}{uv'}\right)\end{aligned}$$

Hence the diagram commutes and our first claim follows from a standard diagram chase.

Next we claim $\Theta_{U,\oplus_{i=1}^n M_i,N}$ is an isomorphism if and only if $\Theta_{U,M_i,N}$ is an isomorphism for every $i = 1,\ldots,n$. The base case is our first claim, so assume our second claim holds for R-modules $M_1,\ldots,M_{n-1}$ and let M_n be another R-module. By our first claim we have $\Theta_{U,\oplus_{i=1}^n M_i,N}$ is an isomorphism if and only if both $\Theta_{U,\oplus_{i=1}^{n-1} M_i,N}$ and $\Theta_{U,M_n,N}$ are isomorphisms, so our second claim follows from our induction hypothesis.

Third we claim $\Theta_{U,R^n,N}$ is an isomorphism, for which it suffices to show $\Theta_{U,R,N}$ is an isomorphism (by our second claim). Consider the diagram

$$\begin{array}{ccc} U^{-1}\operatorname{Hom}_R(R,N) & \xrightarrow{\Theta_{U,R,N}} & \operatorname{Hom}_{U^{-1}R}\big(U^{-1}R, U^{-1}N\big) \\ {\scriptstyle U^{-1}f}\big\downarrow{\scriptstyle \cong} & \swarrow_{F}^{\cong} & \\ U^{-1}N & & \end{array}$$

where f and F are the evaluation maps at 1_R and $1_{U^{-1}R}$, respectively. The diagram commutes since for any $\frac{\psi}{u} \in U^{-1}\operatorname{Hom}_R(R,N)$ we have the following.

$$\begin{aligned}(U^{-1}f)\left(\frac{\psi}{u}\right) &= \frac{f(\psi)}{u} = \frac{\psi(1)}{u}\\ (F\circ\Theta_{U,R,N})\left(\frac{\psi}{u}\right) &= F(\psi_u) = \psi_u(1) = \psi_u\left(\frac{1}{1}\right) = \frac{\psi(1)}{1\cdot u}\end{aligned}$$

Since the evaluation maps $U^{-1}f$ and F are known isomorphisms, a standard diagram chase shows $\Theta_{U,R,N}$ is an isomorphism also.

To finish the proof of part (c), assume the sequence

$$R^m \xrightarrow{f} R^n \xrightarrow{g} M \longrightarrow 0$$

is exact (f no longer an evaluation map). Since Hom is left-exact the sequence

$$0 \longrightarrow \operatorname{Hom}_R(M,N) \xrightarrow{g^*} \operatorname{Hom}_R(R^n,N) \xrightarrow{f^*} \operatorname{Hom}_R(R^m,N)$$

is exact as well, where $*$ is defined as

$$(-)^* := \operatorname{Hom}_R(-,N).$$

Localization is also exact, so

$$\begin{array}{ccccccc}
0 \longrightarrow & U^{-1}\operatorname{Hom}_R(M,N) & \xrightarrow{U^{-1}(g^*)} & U^{-1}\operatorname{Hom}_R(R^n,N) & \xrightarrow{U^{-1}(f^*)} & U^{-1}\operatorname{Hom}_R(R^m,N) \\
& \downarrow \Theta_{U,M,N} & \curvearrowright & \cong \downarrow \Theta_{U,R^n,N} & \curvearrowright & \cong \downarrow \Theta_{U,R^m,N} \\
0 \longrightarrow & \operatorname{Hom}_{U^{-1}R}\big(U^{-1}M, U^{-1}N\big) & \xrightarrow[(U^{-1}g)^\star]{} & \operatorname{Hom}_{U^{-1}R}\big(U^{-1}R^n, U^{-1}N\big) & \xrightarrow[(U^{-1}f)^\star]{} & \operatorname{Hom}_{U^{-1}R}\big(U^{-1}R^m, U^{-1}N\big)
\end{array}$$

is a homomorphism of exact sequences, where the commutivity of the diagram is verified as above, the isomorphisms therein follow from our third claim, and $\star$ is defined as

$$(-)^\star := \operatorname{Hom}_{U^{-1}R}\big(-, U^{-1}N\big)\,.$$

Another diagram chase allows us to conclude that $\Theta_{U,M,N}$ is an isomorphism as desired, completing the proof of (c).

(d). Since R noetherian and M finitely generated, there exists an exact sequence

$$\ldots \longrightarrow R^b \longrightarrow R^a \longrightarrow R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0\,.$$

Therefore

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

is exact and M is finitely presented. Part (d) then follows from part (c). $\square$

$$
\begin{array}{ccc}
U^{-1}\operatorname{Hom}_R(M \oplus M', N) & \xrightarrow{\Theta_{U,M\oplus M',N}} & \operatorname{Hom}_{U^{-1}R}(U^{-1}(M \oplus M'), U^{-1}N) \\
{\scriptstyle U^{-1}\omega}\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle (\gamma^{-1})^*} \\
U^{-1}(\operatorname{Hom}_R(M,N) \oplus \operatorname{Hom}_R(M',N)) & & \operatorname{Hom}_{U^{-1}R}((U^{-1}M) \oplus (U^{-1}M'), U^{-1}N) \\
{\scriptstyle \Gamma}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle \Omega} \\
U^{-1}\operatorname{Hom}_R(M,N) \oplus U^{-1}\operatorname{Hom}_R(M',N) & \xrightarrow{\Theta_{U,M,N}\oplus\Theta_{U,M',N}} & \operatorname{Hom}_{U^{-1}R}(U^{-1}M, U^{-1}N) \oplus \operatorname{Hom}_{U^{-1}R}(U^{-1}M', U^{-1}N)
\end{array}
\tag{3.1.8.3}
$$

3.2 Modules and Prime Spectra

The prime spectrum of a ring and related constructs are used heavily throughout the remainder of the chapter and we introduce them here. Remark 3.2.11 in particular will get a lot of use and will be used directly in the proof of Theorem 3.5.16, the ultimate goal of the chapter.

Notation 3.2.1. For any natural number $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Definition 3.2.2. Let I be an ideal of the ring R. The *prime spectrum* of R is

$$\operatorname{Spec}(R) = \{\, \mathfrak{p} \leq R \mid \mathfrak{p} \text{ prime}\}$$

the *variety* of I is

$$V(I) = \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$$

and the *radical* of I is

$$\operatorname{rad}(I) = \{\, x \in R \mid \exists n \in \mathbb{N} \text{ s.t. } x^n \in I\}$$

also denoted $r(I)$ or $\sqrt{I}$.

Remark 3.2.3. We have the following properties of the radical ideal and the variety of an ideal.

1. $\operatorname{rad}(I) \leq R$
2. $I \subseteq \operatorname{rad}(I)$
3. $I \supseteq J \leq R \implies \operatorname{rad}(J) \subseteq \operatorname{rad}(I)$
4. $\operatorname{rad}(\operatorname{rad}(I)) = \operatorname{rad}(I)$
5. $I \subseteq J \implies V(I) \supseteq V(J)$
6. $I = R \iff \operatorname{rad}(I) = R$

Example 3.2.4. Let R be a principal ideal domain. For any $x \in R\setminus\{0\}$ there exists a unit $u \in R^\times$, prime elements $p_1, \ldots, p_n \in R$, and positive $e_1, \ldots, e_n \in \mathbb{N}$ such that

$$x = up_1^{e_1} \cdots p_n^{e_n}$$

and $p_iR \neq p_jR$ whenever $i \neq j$. If we define $I = xR$, then $V(I) = \{p_1R, \ldots, p_nR\}$. This is because $qR \in \operatorname{Spec}(R)$ is such that qR contains xR if and only if $q|x = up_1^{e_1} \cdots p_n^{e_n}$. That is, $qR \in V(I)$ if and only if $q \sim p_i$ for some i.

We can also show $\operatorname{rad}(I) = p_1 \cdots p_nR$. Note by Remark 3.2.3.6 above, we may assume without loss of generality that $n \geq 1$ (i.e., x is not a unit). Define $e = \max_i(e_i)$ and we have

$$(p_1 \cdots p_n)^e = p_1^e \cdots p_n^e \in p_1^{e_1} \cdots p_n^{e_n}R = xR.$$

So the product $p_1 \cdots p_n \in \operatorname{rad}(I)$ and hence $p_1 \cdots p_nR \subseteq \operatorname{rad}(I)$, because $\operatorname{rad}(I)$ is an ideal. For the reverse containment let $y \in \operatorname{rad}(I)$ and let $m \in \mathbb{N}$ such that $y^m \in I$. This implies $up_1^{e_1} \cdots p_n^{e_n}|y^m$. For each $i \in [n]$, $e_i \geq 1$ so $p_i|y^m$ and $p_i|y$. Moreover $p_i \not\sim p_j$ whenever $i \neq j$ implies $p_1 \cdots p_n|y$ and therefore $y \in p_1 \cdots p_nR$. Hence $\operatorname{rad}(I) \subseteq p_1 \cdots p_nR$, concluding the proof.

To give a more explicit example, consider $x = 2^5 3^{17} 19 \in \mathbb{Z}$. By what we have shown above $V(x\mathbb{Z}) = \{2\mathbb{Z}, 13\mathbb{Z}, 19\mathbb{Z}\}$ and $\operatorname{rad}(x\mathbb{Z}) = 2 \cdot 13 \cdot 19\mathbb{Z}$.

Fact 3.2.5. *If $I \leq R$, then $V(\operatorname{rad}(I)) = V(I)$.*

Proof. The forward containment follows from parts 2 and 5 in Remark 3.2.3 above. For the reverse containment, let $\mathfrak{p} \in V(I)$. For any $x \in \operatorname{rad}(I)$ with $x^n \in I \subseteq \mathfrak{p}$, we know $x \in \mathfrak{p}$, implying $\operatorname{rad}(I) \subseteq \mathfrak{p}$. Having shown an arbitrary prime ideal containing I must also contain $\operatorname{rad}(I)$, we conclude $V(\operatorname{rad}(I)) \supseteq V(I)$. □

Proposition 3.2.6. *If $I \leq R$, then*

$$\operatorname{rad}(I) = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

Proof. First we deal with a special case. If $V(I) = \emptyset$, then we have the empty intersection on the right, which is defined to be all of R. Moreover in this case I must actually be the entire ring, since if $I \lneq R$, then I must be contained in some maximal (and therefore prime) ideal, violating the emptiness of $V(I)$. This gives $\operatorname{rad}(I) = \operatorname{rad}(R) = R$, so the proposition holds in this case.

Now assume without loss of generality that $V(I)$ is nonempty and therefore $I \neq R$. For any $x \in \operatorname{rad}(I)$ with $x^n \in I$ for some $n \in \mathbb{N}$, if I lies in some prime ideal $\mathfrak{p}$, then $x^n \in I \subseteq \mathfrak{p}$ and therefore $x \in \mathfrak{p}$. Having shown that $\operatorname{rad}(I)$ is contained in an arbitrary element of $V(I)$, we conclude

$$\operatorname{rad}(I) \subseteq \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

For the other containment, we use a clever application of localization. Let $x \in R \setminus \operatorname{rad}(I)$ and define the multiplicatively closed subset $S = \{1, x, x^2, x^3, \dots\} \subseteq R$. Since $\operatorname{rad}(I) \cap S = \emptyset$, it follows that $S^{-1}\operatorname{rad}(I)$ contains no units of $S^{-1}R$ and therefore is a proper ideal of $S^{-1}R$ (see Fact 3.2.10). Then we may let $S^{-1}\mathfrak{q} \lneq S^{-1}R$ be a maximal ideal containing $S^{-1}\operatorname{rad}(I)$. By this we know $\mathfrak{q} \in \operatorname{Spec}(R)$ satisfies $\mathfrak{q} \cap S = \emptyset$ and $\operatorname{rad}(I) \subseteq \mathfrak{q}$, so $\mathfrak{q} \in V(\operatorname{rad}(I)) = V(I)$ by Fact 3.2.5. Since $\mathfrak{q} \cap S = \emptyset$, we know $x^n \notin \mathfrak{q}$ for any integer $n \geq 0$ and in particular

$$x \notin \mathfrak{q} \supseteq \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

Hence we have proven the reverse containment by contraposition. □

Lemma 3.2.7. *If $I, J \leq R$ and $V(J) \subseteq V(I)$, then $I \subseteq \operatorname{rad}(J)$. If I is also finitely generated over R, then $I^n \subseteq J$, for all sufficiently large $n > 0$.*

Proof. The first implication is a corollary of Remark 3.2.3 and Proposition 3.2.6:

$$I \subseteq \operatorname{rad}(I) = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in V(J)} \mathfrak{p} = \operatorname{rad}(J).$$

For the second part, let $x_1, \dots, x_n \in R$ be such that $(x_1, \dots, x_n)R = I \subseteq \operatorname{rad}(J)$. By definition of the radical there exist $e_1, \dots, e_n \in \mathbb{N}$ such that $x_1^{e_1}, \dots, x_n^{e_n} \in J$ and we define $e = \sum_{i=1}^n e_i$. We then have

$$I^e = \left\langle x_1^{f_1} \cdots x_n^{f_n} \mid \sum_{i=1}^n f_i = e \right\rangle.$$

Since for any generator of I^e above, the f_i's and e_i's both sum to e, we know $f_i \geq e_i$ for some i and therefore

$$x_1^{f_1} \cdots x_i^{f_i} \cdots x_n^{f_n} \in (x_i^{f_i})R \subseteq (x^{e_i})R \subseteq J.$$

Hence $I^e \subseteq J$ and therefore $I^t \subseteq I^e \subseteq J$ for all $t \geq e$. □

Definition 3.2.8. For all $m \in M$, the *annihilator* of m is

$$\operatorname{Ann}_R(m) = \{r \in R \mid rm = 0\}.$$

Similarly we may define the *annihilator* of M as

$$\operatorname{Ann}_R(M) = \{r \in R \mid rM = 0\} = \{r \in R \mid rm = 0, \forall m \in M\} = \bigcap_{m \in M} \operatorname{Ann}_R(m).$$

The *support* of M is the set of all prime ideals for which M "survives the localization process"; formally we write

$$\operatorname{Supp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_\mathfrak{p} \neq 0\}.$$

Example 3.2.9. Let $U \subseteq R$ be multiplicatively closed.

(a) For any $m \in M$ the following are equivalent.

  (i) $m/1 = 0 \in U^{-1}M$.

  (ii) There exists some $u \in U$ such that $um = 0$.

  (iii) $U \cap \operatorname{Ann}_R(m) \neq \emptyset$.

(b) If M is finitely generated, then the following are equivalent.

  (i) $U^{-1}M = 0$.

  (ii) There exists some $u \in U$ such that $uM = 0$.

  (iii) $U \cap \operatorname{Ann}_R(M) \neq \emptyset$.

The majority of the above implications are simply restatements of definitions, so we will only prove (i) implies (ii) in part (b).

Proof. If there exists $u \in U$ such that $uM = 0$, then for any $m/v \in U^{-1}M$ we have

$$\frac{m}{v} = \frac{um}{uv} = \frac{0}{uv} = 0.$$

Therefore $U^{-1}M = 0$. This proves one direction and we point out here that we did not need the finitely generated assumption. If $M = (m_1, \ldots, m_n)R$ and $U^{-1}M = 0$, then notice $m_i/1 = 0$ for each $i = 1, \ldots, n$. By part (a) this means there exist $u_1, \ldots, u_n \in U$ such that $u_i m_i = 0$ for $i = 1, \ldots, n$. Define $u = \prod_{i=1}^n u_i$ and let $m = \sum_{i=1}^n r_i m_i$ be given. It follows that

$$um = u\left(\sum_{i=1}^n r_i m_i\right) = \sum_{i=1}^n \left(r_i \prod_{j \neq i} u_j\right)(u_i m_i) = 0$$

implying $uM = 0$. □

In order to be explicit in our reasoning in Remark 3.2.11, we prove a fact about ideals under localization.

Fact 3.2.10. *Let $I \leq R$ be an ideal and let $U \subseteq R$ be a multiplicatively closed subset. Then $U^{-1}I = U^{-1}R$ if and only if $I \cap U \neq \emptyset$.*

Proof. If we first assume there exists an element $u \in I \cap U$, then we write

$$1_{U^{-1}R} = \frac{u}{u} \in U^{-1}I$$

and therefore $U^{-1}I = U^{-1}R$. On the other hand if we assume $U^{-1}I = U^{-1}R$, then $1_{U^{-1}R} \in U^{-1}I$ and we have

$$1_{U^{-1}R} = \frac{1}{1} = \frac{a}{u}$$

for some $a \in I$ and some $u \in U$. By the definition of equality in $U^{-1}R$, there exists an element $v \in U$ such that

$$\underbrace{va}_{\in I} = \underbrace{vu}_{\in U}$$

and we conclude $I \cap U \neq \emptyset$. □

Remark 3.2.11. We have the following relationships between annihilators, supports, and prime spectra.

1. $\mathrm{Ann}_R(m), \mathrm{Ann}_R(M) \leq R$
2. $\mathrm{Supp}_R(R) = \mathrm{Spec}(R)$
3. $\mathrm{Supp}_R(0) = \emptyset$
4. $\mathrm{Supp}_R(R/I) = V(I)$
5. M finitely generated $\implies$ $\mathrm{Supp}_R(M) = V(\mathrm{Ann}_R(M))$

Proof. 1. The annihilators are non-empty since they each contain 0. They are closed under addition and subtraction as a result of the distributive property. They contain additive inverses, because

$$rm = 0 \implies (-r)m = (-1)rm = 0.$$

Finally, they absorb multiplication from R as a result of the associative property.

2. Supports are special sets of prime ideals and spectra contain all prime ideals of the particular ring, so $\mathrm{Supp}_R(R) \subseteq \mathrm{Spec}(R)$ from the definitions. On the other hand, for any $\mathfrak{p} \in \mathrm{Spec}(R)$, $1 \notin \mathfrak{p}$ so 1 is an allowable denominator and we write

$$0 \neq \frac{1}{1} \in R_\mathfrak{p}$$

implying $R_\mathfrak{p} \neq 0$. Thus $\mathfrak{p} \in \mathrm{Supp}_R(R)$.

3. This holds simply because there is no localization under which zero is 'resurrected' to something non-zero. That is, $0_\mathfrak{p} = 0$ for any $\mathfrak{p} \in \mathrm{Spec}(R)$.

4. For any $\mathfrak{p} \in \mathrm{Spec}(R)$, by Fact 3.2.10 above we have $I_\mathfrak{p} = R_\mathfrak{p}$ if and only if $I \cap (R \setminus \mathfrak{p}) \neq \emptyset$. This is equivalent to $I \not\subseteq \mathfrak{p}$ which is equivalent to $\mathfrak{p} \notin V(I)$. Therefore

$$I_\mathfrak{p} \subsetneq R_\mathfrak{p} \iff \mathfrak{p} \in V(I). \tag{3.2.11.1}$$

Consider the short exact sequence

$$0 \longrightarrow I \xrightarrow{\subseteq} R \xrightarrow{\pi} R/I \longrightarrow 0$$

where π is the canonical surjection. Since localization is exact by Theorem 3.1.4, the sequence

$$0 \longrightarrow I_\mathfrak{p} \xrightarrow{i} R_\mathfrak{p} \xrightarrow{\pi_\mathfrak{p}} (R/I)_\mathfrak{p} \longrightarrow 0$$

is also exact. The First Isomorphism Theorem for modules applied to $\pi_\mathfrak{p}$ yields

$$(R/I)_\mathfrak{p} = \mathrm{Im}\,(\pi_\mathfrak{p}) \cong \frac{R_\mathfrak{p}}{\mathrm{Ker}(\pi_\mathfrak{p})} = \frac{R_\mathfrak{p}}{\mathrm{Im}\,(i)} = R_\mathfrak{p}/I_\mathfrak{p}. \tag{3.2.11.2}$$

Our application of short exact sequences shortens the proof immensely.

$$\begin{aligned}
\mathfrak{p} \in \mathrm{Supp}_R(R/I) &\iff (R/I)_\mathfrak{p} \neq 0 && \text{definition of support} \\
&\iff R_\mathfrak{p}/I_\mathfrak{p} \neq 0 && (3.2.11.2) \\
&\iff I_\mathfrak{p} \subsetneq R_\mathfrak{p} && \\
&\iff \mathfrak{p} \in V(I) && (3.2.11.1)
\end{aligned}$$

5. This requires only definitions and Example 3.2.9.

$$\begin{aligned}
\mathfrak{p} \in \operatorname{Supp}_R(M) &\iff M_\mathfrak{p} \neq 0 && \text{definition of support}\\
&\iff (R \setminus \mathfrak{p}) \cap \operatorname{Ann}_R(M) = \emptyset && 3.2.9\\
&\iff \operatorname{Ann}_R(M) \subseteq \mathfrak{p} &&\\
&\iff \mathfrak{p} \in V(\operatorname{Ann}_R(M)) && \text{definition of variety}
\end{aligned}$$

□

Example 3.2.12. Let $\mathbb{K}$ be a field and define the ring $R = \mathbb{K}[x, y]$.

(a) For every polynomial $f \in R$

$$\operatorname{Supp}_R(R/fR) = V(fR) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid f \in \mathfrak{p}\}.$$

(b) For every $m, n \in \mathbb{N}$ we have

$$\operatorname{Supp}_R\left(\frac{R}{(x^m, y^n)R}\right) = \{(x, y)R\} = \operatorname{Supp}_R\left(\frac{R}{((x, y)R)^m}\right).$$

(c) For the ideal $L = (x^2, xy)R \leq R$ we have

$$\operatorname{Supp}_R(R/L) = V(xR) = \operatorname{Supp}_R(R/xR)$$

and

$$\operatorname{rad}(L) = xR.$$

Proof. Here we will only justify part (b) of the example, as the other parts follow more or less similarly. By point 4 in Remark 3.2.11, to prove (b) it suffices to show

$$V((x^m, y^n)R) \overset{(1)}{=} \{(x, y)R\} \overset{(2)}{=} V(((x, y)R)^m).$$

If $\mathfrak{p} \in \operatorname{Spec}(R)$ and $(x^m, y^n)R \subseteq \mathfrak{p} \subsetneq R$, then $x^m, y^n \in \mathfrak{p}$ and thus $x, y \in \mathfrak{p}$ since $\mathfrak{p}$ prime. It follows that $(x, y)R \subseteq \mathfrak{p}$ and the strictness of $\mathfrak{p}$ implies $(x, y)R = \mathfrak{p}$, because $(x, y)R$ is maximal. Therefore

$$V((x^m, y^n)R) \subseteq \{(x, y)R\}.$$

On the other hand, $(x, y)R \in \operatorname{Spec}(R)$ and $x^m, y^n \in (x, y)R$, so $(x^m, y^n)R \subseteq (x, y)R$. Hence equality (1) holds by mutual containment.

Now for equality (2). Since $(x, y)R \in \operatorname{Spec}(R)$ and $x^a y^b \in (x, y)R$ for any $a, b \geq 0$, we know

$$((x, y)R)^m = \left\{ x^a y^b \;\middle|\; a, b \geq 0,\ a + b = m \right\} \subseteq (x, y)R$$

implying $(x, y)R \in V(((x, y)R)^m)$, so we have containment in one direction. For the reverse, let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $((x, y)R)^m \subseteq \mathfrak{p}$. Then $x^m, y^m \in \mathfrak{p}$ a prime ideal, so $x, y \in \mathfrak{p}$ and therefore $(x, y)R \subseteq \mathfrak{p}$. It follows that

$$V(((x, y)R)^m) \subseteq \{(x, y)R\}$$

so equality (2) holds by mutual containment. □

Definition 3.2.13. A prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is *associated to M* if there exists $m \in M$ such that $\mathfrak{p} = \operatorname{Ann}_R(m)$. The set of all such ideals is the set of *associated primes*, denoted as follows.

$$\begin{aligned}
\operatorname{Ass}_R(M) &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \text{ is associated to } M\}\\
&= \{\operatorname{Ann}_R(m) \leq R \mid m \in M\} \cap \operatorname{Spec}(R)\\
&= \{\operatorname{Ann}_R(m) \leq R \mid m \in M,\ \operatorname{Ann}_R(m) \text{ is a prime ideal}\}
\end{aligned}$$

In other words, $\operatorname{Ass}_R(M)$ is the set of prime ideals of R that are also the annihilator of some element of M.

Example 3.2.14. Let $\mathfrak{p} \in \operatorname{Spec}(R)$, $r \in R$, and $r + \mathfrak{p} \in R/\mathfrak{p}$.

$$\operatorname{Ann}_R(r+\mathfrak{p}) = \begin{cases} R & r \in \mathfrak{p} \qquad (\because r + \mathfrak{p} = 0_{R/\mathfrak{p}}) \\ \mathfrak{p} & r \notin \mathfrak{p} \qquad (\because r + \mathfrak{p} \neq 0_{R/\mathfrak{p}},\ R/\mathfrak{p} \text{ a domain}) \end{cases}$$

In general, determining the set of associated primes of R/I is difficult, but in this case we have just shown that

$$\operatorname{Ass}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}.$$

Example 3.2.15. Assume R is a principal ideal domain and let $I = xR$. If $x \in R^\times$, then $xR = R$ and $R/xR = 0$, implying

$$\operatorname{Ass}_R(R/xR) = \operatorname{Ass}_R(0) = \{\operatorname{Ann}_R(0)\} \cap \operatorname{Spec}(R) = \{R\} \cap \operatorname{Spec}(R) = \emptyset.$$

If $x = 0$, then xR is the (prime) zero ideal and therefore by Example 3.2.14 we have

$$\operatorname{Ass}_R(R/xR) = \{xR\} = \{0\}.$$

So let $x \in R \setminus (R^\times \cup \{0\})$ and $p_1, \ldots, p_n \in R$ primes (not necessarily distinct) such that $x = p_1 \cdots p_n$. We claim

$$\operatorname{Ass}_R(R/xR) = \{p_1R, \ldots, p_nR\}.$$

Proof. For the reverse containment, first define $x' = p_2 \cdots p_n$. Since prime factorizations are unique in R, this implies $\{\, r \in R \mid rx' \in xR\} = p_1R$. We can also write

$$\{\, r \in R \mid rx' \in xR\} = \{\, r \in R \mid r(x' + xR) = 0 \text{ in } R/xR\} = \operatorname{Ann}_R(x' + xR).$$

We have therefore shown

$$p_1R \in \operatorname{Spec}(R/xR) \cap \{\operatorname{Ann}_R(y + xR) \leq R \mid y + xR \in R/xR\} = \operatorname{Ass}_R(R/xR).$$

Since multiplication in R is commutative, we conclude $p_iR \in \operatorname{Ass}_R(R/xR)$ for all $i = 1, \ldots, n$, proving the reverse containment.

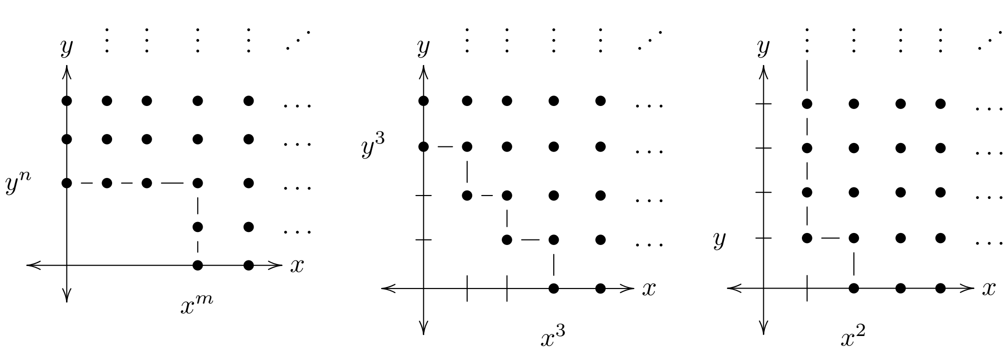
Let $y \in R$ such that $\operatorname{Ann}_R(y + xR) \in \operatorname{Spec}(R)$ and set $\operatorname{Ann}_R(y + xR) = pR$ for some prime $p \in R$. Then $p(y + xR) = \overline{0}$ or in other words $py = xr = p_1 \cdots p_n r$ for some $r \in R$. This implies $xr \in pR$ a prime ideal, so either $r \in pR$ or $p_i \in pR$ for some $i \in [n]$. Suppose $r \in pR$, so $r = pz$ for some $z \in R$. Therefore

$$py = p_1 \cdots p_n r = p_1 \cdots p_n pz$$

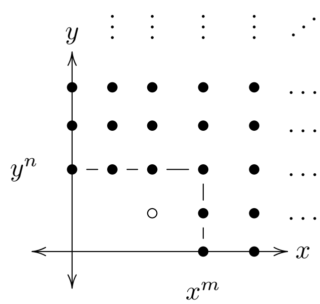
and since R is commutative $y = p_1 \cdots p_n z$ by cancellation. However, this implies $y \in xR$ and thus $pR = \operatorname{Ann}_R(\overline{0}) = R$, a contradiction. Therefore $p_i \in pR$ for some $i \in [n]$ and it follows that $pR = p_iR$ for some $i \in [n]$. ☐

Note 3.2.16. Given the polynomial ring $R = \mathbb{K}[x, y]$ as in Example 3.2.12, we can plot graphic representations of ideals generated by monomials in the ring.

$(x^m, y^n)R$ $\qquad$ $((x,y)R)^3$ $\qquad$ $(x^2, xy)R$



Example 3.2.17. Here we make use of lattice diagrams representing monomial ideals in the polynomial ring $R = \mathbb{K}[x, y]$, where $\mathbb{K}$ is a field. Given the lattice representation of an ideal from Note 3.2.16, certain corners in the lattice give us information about the associated primes of the residual ring. Specifically, first consider $I = (x^m, y^n)R$ with lattice representation



where we have denoted the element $(x^{m-1}y^{n-1}) \in R$ with '∘'. For the element $\overline{x^{m-1}y^{n-1}} \in R/I$, since $x, y \in \operatorname{Ann}_R\left(\overline{x^{m-1}y^{n-1}}\right)$ and $1 \notin \operatorname{Ann}_R\left(\overline{x^{m-1}y^{n-1}}\right)$, we have

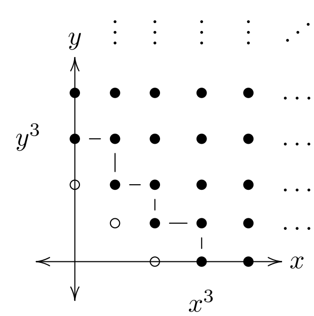
$$\operatorname{Ann}_R\left(\overline{x^{m-1}y^{n-1}}\right) = (x, y)R.$$

This implies

$$(x, y)R \in \operatorname{Ass}_R(R/I)$$

and in fact we will later show $\operatorname{Ass}_R(R/I) = \{(x, y)R\}$.

Next consider the ideal $J = ((x, y)R)^3 = (x^3, x^2y, xy^2, y^3)R$ with lattice representation



where $y^2, xy, x^2 \in R$ have been marked. Since $(x, y)R \subseteq \operatorname{Ann}_R(\overline{xy}) \subsetneq R$ for $\overline{xy} \in R/J$ and $(x, y)R$ is maximal, we have $(x, y)R = \operatorname{Ann}_R(\overline{xy})$ and an identical argument shows $\operatorname{Ann}_R(\overline{y^2}) = (x, y)R = \operatorname{Ann}_R(\overline{x^2})$. Hence

$$(x, y)R \in \operatorname{Ass}_R(R/J).$$

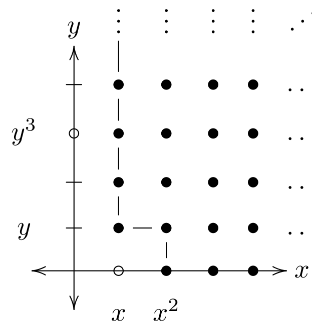
Lastly let $L = (x^2, xy)R$ and consider the elements $\overline{y^3}, \overline{x} \in R/L$ for which we have

$$(x)R = \operatorname{Ann}_R\left(\overline{y^3}\right)$$
$$(x, y)R = \operatorname{Ann}_R(\overline{x})$$

and therefore

$$\{(x)R, (x, y)R\} \subseteq \operatorname{Ass}_R(R/L).$$

As with the previous two ideals, the element $\overline{x}$ that get annihilated resides near the corner in our lattice diagram below.



Contrary to our first two examples however, notice $\overline{y^3}$ instead lies in the 'corridor' along the vertical axis and in fact could have been $\overline{y^t}$ for any $t \geq 1$. So when looking for associated primes, we look near corners and in the corridors of the corresponding lattice diagram.

Remark 3.2.18. For any $m \in M$ there exists a well-defined R-module homomorphism

$$\lambda_m \colon R \longrightarrow M$$
$$r \longmapsto rm.$$

If we let $I = \operatorname{Ann}_R(m)$, then $\operatorname{Ker}(\lambda_m) = I$, $\operatorname{Im}(\lambda_m) = Rm$, and the First Isomorphism Theorem gives $R/I \cong Rm \subseteq M$. Moreover there exists an injective R-module homomorphism

$$\overline{\lambda}_m \colon R/I \hookrightarrow M$$
$$r + I \longmapsto rm.$$

In particular, if $\mathfrak{p} \in \operatorname{Ass}_R(M)$, then there exists an injective R-module homomorphism from $R/\mathfrak{p}$ into M.

Conversely, if $\varphi \colon R/\mathfrak{p} \hookrightarrow M$ is an injective R-module homomorphism, then for the element $m = \varphi\left(\overline{1}\right)$, we have $\operatorname{Ann}_R(m) = \mathfrak{p}$ and so $\mathfrak{p} \in \operatorname{Ass}_R(M)$. Hence $\mathfrak{p} \in \operatorname{Ass}_R(M)$ if and only if there exists an injection $\varphi \colon R/\mathfrak{p} \hookrightarrow M$.

Proposition 3.2.19. *Assume R is noetherian and let M be a non-zero R-module.*

(a) The set

$$A_R(M) = \{ \operatorname{Ann}_R(m) \mid m \in M \setminus \{0\}\}$$

has maximal elements, and every maximal element is prime. Therefore $\operatorname{Ass}_R(M) \neq \emptyset$.

(b) If we define the set

$$\operatorname{ZD}_R(M) = \{\textit{zero divisors on } M \textit{ in } R\},$$

then we have

$$\{0\} \cup \operatorname{ZD}_R(M) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p}.$$

(c) Independent of the noetherian assumption we have $\operatorname{Ass}_R(M) \subseteq \operatorname{Supp}_R(M)$.

Proof. (a). Since $A_R(M)$ is a nonempty set of ideals of R, the maximum condition for noetherian rings guarantees $A_R(M)$ has a maximal element $I = \mathrm{Ann}_R(m)$ for some $m \in M \setminus \{0\}$. Note $m \neq 0$ implies $I \neq R$. To show I is prime, let $a, b \in R$ such that $ab \in I$ and $a \notin I$. Since $a \notin I$, it follows that $am \neq 0$ and we have

$$I = \mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(am) \in A_R(M)$$

where the set containment holds by the commutivity of R. Moreover, since $\mathrm{Ann}_R(am) \neq R$, the maximality of I implies

$$I = \mathrm{Ann}_R(m) = \mathrm{Ann}_R(am).$$

In particular, $b \in \mathrm{Ann}_R(am) = I$.

(b). By part (a) every element of $A_R(M)$ is contained in an associated prime of M. That is, for every $m \in M \setminus \{0\}$ there exists $\mathfrak{p} \in \mathrm{Ass}_R(M)$ such that $\mathrm{Ann}_R(m) \subseteq \mathfrak{p}$. Hence

$$\{0\} \cup \mathrm{ZD}_R(M) \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}.$$

On the other hand, $\mathfrak{p} \in \mathrm{Ass}_R(M)$ means precisely that $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some $m \in M$, so $(\mathfrak{p} \setminus \{0\}) \subseteq \mathrm{ZD}_R(M)$.

(c). Let $\mathfrak{p} = \mathrm{Ann}_R(\hat{m}) \in \mathrm{Ass}_R(M)$, for some $\hat{m} \in M$. Then we can define the following injective R-module homomorphism.

$$\pi_{\hat{m}} : R/\mathfrak{p} \hookrightarrow M$$

$$r + \mathfrak{p} \longmapsto r\hat{m}$$

Localizing at $\mathfrak{p}$ (which preserves injectivity) we have

$$0 \neq Q(R/\mathfrak{p}) = (R/\mathfrak{p})_{\mathfrak{p}} \xhookrightarrow{(\pi_{\hat{m}})_{\mathfrak{p}}} M_{\mathfrak{p}}$$

where $Q(R/\mathfrak{p})$ denotes the field of fractions (recall Example 1.2.7). So $M_{\mathfrak{p}}$ contains a non-zero submodule and therefore $M_{\mathfrak{p}} \neq 0$. Hence $\mathfrak{p} \in \mathrm{Supp}_R(M)$. □

Remark 3.2.20. Recall Examples 3.2.12 and 3.2.17. Applying Proposition 3.2.19(c), we can justify the equalities in the next display.

$$\{(x,y)R\} = \mathrm{Ass}_R(R/I)$$
$$\{(x,y)R\} = \mathrm{Ass}_R(R/J)$$
$$\{(x)R,\ (x,y)R\} \subseteq \mathrm{Ass}_R(R/L)$$

Proof. Since $\mathrm{rad}(I) = (x,y)R$, justifying the first equality is done as follows and the second is proven almost identically.

$$\{(x,y)R\} \overset{3.2.17}{\subseteq} \mathrm{Ass}_R(R/I) \overset{3.2.19(c)}{\subseteq} \mathrm{Supp}_R(R/I) \overset{3.2.11.4}{=} V(I) \overset{3.2.5}{=} V((x,y)R) = \{(x,y)R\}$$

Later in Example 3.3.10 we will see we have equality in the third case as well. Right now we have

$$\{(x)R, (x,y)R\} \subseteq \mathrm{Ass}_R(R/L) \subseteq \mathrm{Supp}_R(R/L) = V(xR)$$

but this does not give the desired equality. What we will later see is that we can greatly refine the list of primes to consider on the far right-hand side of this containment. □

Since $M = \{0\}$ implies $\mathrm{Ass}_R(M) = \emptyset$, we may strengthen the conclusion of Proposition 3.2.19(a).

Corollary 3.2.21. *If R is noetherian, then an R-module M is non-zero if and only if* $\mathrm{Ass}_R(M) \neq \emptyset$.

Example 3.2.22. Let $\mathbb{K}$ be a field. The ring defined as

$$R = \prod_{i=1}^{\infty} \mathbb{K} = \{ (a_1, a_2, \dots) \mid a_i \in \mathbb{K}, \ \forall i \in \mathbb{N} \}$$

is a commutative ring with identity under component-wise operations, but it is not noetherian. Indeed, consider the proper ideal

$$I = \bigoplus_{i=1}^{\infty} \mathbb{K} = \{ (a_1, a_2, \dots) \in R \mid a_i = 0, \ \forall i \gg 0 \} \not\ni (1, 1, \dots).$$

Some examples of maximal ideals of R include those of the form

$$\mathfrak{m}_i = \{ (a_1, a_2, \dots) \in R \mid a_i = 0 \} = \mathrm{Ker}\,(\tau_i)$$

where $\tau_i : R \twoheadrightarrow \mathbb{K}$ maps sequences from R to their i^{th} entry and the maximality of $\mathfrak{m}$ follows from the First Isomorphism Theorem for rings (i.e., $R/\mathfrak{m} \cong$ *field* implies $\mathfrak{m}$ maximal). Since no $\mathfrak{m}_i$ contains I, there must be some other maximal ideal $\mathfrak{m} \lneq R$ such that $I \subseteq \mathfrak{m}$. It is actually quite difficult to write down $\mathfrak{m}$ explicitly. In addition, it can be shown that $\mathrm{Ass}_R(R/I) = \emptyset$, even though $R/I \neq 0$, thereby demonstrating the necessity of the noetherian assumption in Corollary 3.2.21.

Fact 3.2.23. *If R is not noetherian, but M is a noetherian module over R, then the conclusion of Corollary 3.2.21 still holds.*

Proof. The details are omitted here, but the crux of the proof is M noetherian lets us conclude after some work that $R/\mathrm{Ann}_R(M)$ is noetherian. ☐

We give a fact that will be used to prove the proposition that follows.

Fact 3.2.24. *Given a short exact sequence* $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ *of R-module homomorphisms, $A = 0$ if and only if $A' = 0 = A''$.*

Proof. Assume $A = 0$. Since f is injective, A' must be zero and since g is surjective, A'' must be zero as well. Conversely if we assume $A' = 0 = A''$, by the exactness of the sequence we have $0 = \mathrm{Im}\,(f) = \mathrm{Ker}(g) = A$ as desired. ☐

Proposition 3.2.25. *Consider a short exact sequence of R-module homomorphisms.*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

(a) $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(M') \cup \mathrm{Supp}_R(M'')$

(b) $\mathrm{Ass}_R(M') \subseteq \mathrm{Ass}_R(M) \subseteq \mathrm{Ass}_R(M') \cup \mathrm{Ass}_R(M'')$

Proof. (a). First note by Fact 3.2.24 that $A \neq 0$ if and only if either $A' \neq 0$ or $A'' \neq 0$. Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and since localization is exact we have the short exact sequence

$$0 \longrightarrow M'_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M''_{\mathfrak{p}} \longrightarrow 0$$

implying the following string of equivalent conditions.

$$\begin{aligned} \mathfrak{p} \in \mathrm{Supp}_R(M) &\iff M_{\mathfrak{p}} \neq 0 \\ &\iff M'_{\mathfrak{p}} \neq 0 \text{ or } M''_{\mathfrak{p}} \neq 0 \\ &\iff \mathfrak{p} \in \mathrm{Supp}_R(M') \text{ or } \mathfrak{p} \in \mathrm{Supp}_R(M'') \\ &\iff \mathfrak{p} \in \mathrm{Supp}_R(M') \cup \mathrm{Supp}_R(M'') \end{aligned}$$

This completes the proof of this part.

(b). For the first containment let $\mathfrak{p} \in \operatorname{Ass}_R(M')$. By Remark 3.2.18 there exists an injective R-module homomorphism $\pi : R/\mathfrak{p} \hookrightarrow M'$. Composing with f we conclude there exists another injective R-module homomorphism $f \circ \pi : R/\mathfrak{p} \hookrightarrow M$ and by the same remark $\mathfrak{p} \in \operatorname{Ass}_R(M)$.

For the second containment let $\mathfrak{q} \in \operatorname{Ass}_R(M)$. Then there exists a submodule $N \subseteq M$ such that $N \cong R/\mathfrak{q}$ by Remark 3.2.18. For any $\overline{r} \in R/\mathfrak{q} \cong N$ such that $\overline{r} \neq 0$ (i.e., $r \notin \mathfrak{q}$), notice

$$\operatorname{Ann}_R(\overline{r}) = \{ s \in R \mid s\overline{r} = 0 \in R/\mathfrak{q} \} = \mathfrak{q}.$$

We now consider two possibilities regarding the intersection of the image of f and the submodule N.

In the first case when $N \cap \operatorname{Im}(f) \neq \{0\}$, there is a non-zero element $\alpha \in N \cap \operatorname{Im}(f)$ with $\alpha = f(\beta)$ for some $\beta \in M'$. Since f is injective, $\operatorname{Ann}_R(\beta) = \operatorname{Ann}_R(\alpha) = \mathfrak{q}$ and hence $\mathfrak{q} \in \operatorname{Ass}_R(M')$.

In the second case we have

$$\{0\} = N \cap \operatorname{Im}(f) = N \cap \operatorname{Ker}(g) = \operatorname{Ker}(g|_N : N \longrightarrow M'')$$

so the restriction $g|_N$ is injective. This yields

$$R/\mathfrak{q} \cong N \cong g(N) \subseteq M''$$

implying $\mathfrak{q} \in \operatorname{Ass}_R(M'')$. □

Remark 3.2.26. For any R-modules A and B, if there exists an injective R-module homomorphism $\phi\colon A \hookrightarrow B$, then by Proposition 3.2.25 we have $\operatorname{Ass}_R(A) \subseteq \operatorname{Ass}_R(B)$.

Lemma 3.2.27. *Let M be an R-module and assume there exists a finite filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M.$$

(a) $\operatorname{Supp}_R(M) = \bigcup_{i=1}^{n} \operatorname{Supp}_R\left(\frac{M_i}{M_{i-1}}\right)$

(b) $\operatorname{Ass}_R(M_i) \subseteq \operatorname{Ass}_R(M) \subseteq \bigcup_{i=1}^{n} \operatorname{Ass}_R\left(\frac{M_i}{M_{i-1}}\right)$

Proof. (a). We will induct on n. The base case $n = 0$ holds trivially. So we assume $n \geq 1$ and the result holds for all modules with filtrations of length $n-1$. Given a filtration of length n above, we know M_{n-1} has a filtration of length $n-1$. Therefore under our induction hypothesis

$$\operatorname{Supp}_R(M_{n-1}) = \bigcup_{i=1}^{n-1} \left(\frac{M_i}{M_{i-1}}\right).$$

To the short exact sequence

$$0 \longrightarrow M_{n-1} \xrightarrow{\subseteq} M_n \xrightarrow{\pi} M_n/M_{n-1} \longrightarrow 0$$

with canonical epimorphism π, we apply Proposition 3.2.25 to get

$$\begin{aligned}
\operatorname{Supp}_R(M) = \operatorname{Supp}_R(M_n) &= \operatorname{Supp}_R(M_{n-1}) \cup \operatorname{Supp}_R(M_n/M_{n-1}) \\
&= \left(\bigcup_{i=1}^{n-1} \operatorname{Supp}_R\left(\frac{M_i}{M_{i-1}}\right)\right) \cup \operatorname{Supp}_R(M_n/M_{n-1}) \\
&= \bigcup_{i=1}^{n} \operatorname{Supp}_R\left(\frac{M_i}{M_{i-1}}\right).
\end{aligned}$$

(b). The first containment follows from Remark 3.2.26 and the inclusion maps $M_i \xrightarrow{\epsilon_i} M$. For the second containment we will again induct on n, skipping the trivial base cases when $n = 0$ or 1. Assume $n \geq 2$ and the result holds for modules with filtrations of length $n - 1$. We again use the short exact sequence

$$0 \longrightarrow M_{n-1} \xrightarrow{\subseteq} M_n \longrightarrow M_n/M_{n-1} \longrightarrow 0$$

which yields

$$\begin{aligned} \mathrm{Ass}_R(M_n) &\subseteq \mathrm{Ass}_R(M_{n-1}) \cup \mathrm{Ass}_R(M_n/M_{n-1}) && \text{3.2.25} \\ &\subseteq \bigcup_{i=1}^{n-1} \mathrm{Ass}_R(M_i/M_{i-1}) \cup \mathrm{Ass}_R(M_n/M_{n-1}) && \text{induction hypothesis} \\ &= \bigcup_{i=1}^{n} \mathrm{Ass}_R(M_i/M_{i-1}). \end{aligned}$$

□

Lemma 3.2.28. *Let $M_1, \ldots, M_n$ be R-modules and set $M = \bigoplus_{i=1}^{n} M_i$.*

(a) $\mathrm{Supp}_R(M) = \bigcup_{i=1}^{n} \mathrm{Supp}_R(M_i)$

(b) $\mathrm{Ass}_R(M) = \bigcup_{i=1}^{n} \mathrm{Ass}_R(M_i)$

Proof. (a). We can explicitly build the following finite filtration of M.

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq \bigoplus_{i=1}^{n-2} M_i \subseteq \bigoplus_{i=1}^{n-1} M_i \subseteq \bigoplus_{i=1}^{n} M_i = M$$

For any $j \in [n]$, applying the First Isomorphism Theorem for modules to the canonical projection $\pi_j : \bigoplus_{i=1}^{j} M_i \to M_j$ gives the following isomorphism.

$$\frac{\bigoplus_{i=1}^{j} M_i}{\bigoplus_{i=1}^{j-1} M_i} \cong M_j$$

This, along with Lemma 3.2.27, allows us to write

$$\mathrm{Supp}_R(M) = \bigcup_{j=1}^{n} \mathrm{Supp}_R\left(\frac{\bigoplus_{i=1}^{j} M_i}{\bigoplus_{i=1}^{j-1} M_i}\right) = \bigcup_{j=1}^{n} \mathrm{Supp}_R(M_j)$$

so part (a) holds.

(b). By Lemma 3.2.27 we have

$$\mathrm{Ass}_R(M) \subseteq \bigcup_{j=1}^{n} \mathrm{Ass}_R\left(\frac{\bigoplus_{i=1}^{j} M_i}{\bigoplus_{i=1}^{j-1} M_i}\right) = \bigcup_{j=1}^{n} \mathrm{Ass}_R(M_j).$$

For the reverse containment, notice for any $j \in [n]$ we have canonical injection and surjection ϵ_j and π_j, respectively, by which we construct the exact sequence

$$0 \longrightarrow M_j \xrightarrow{\epsilon_j} M \xrightarrow{\pi_j} M/M_j \longrightarrow 0$$

Applying 3.2.25, we conclude $\mathrm{Ass}_R(M_j) \subseteq \mathrm{Ass}_R(M)$ for every $j \in [n]$ and hence so is the union of all such $\mathrm{Ass}_R(M_j)$. □

3.3 Prime Filtrations

We will see in Theorem 3.3.3 (and Corollary 3.3.4) that in the noetherian setting, finite prime filtrations of modules guarantee the sets of associated primes are finite as well. Moreover, Theorem 3.3.3 will be used either directly or indirectly in a number of future results (e.g., Corollary 3.4.2 and Lemma 3.4.19). Proposition 3.3.13 is another significant result. Part (a) in particular leverages prime filtrations to give equality between three noteworthy sets of primes.

Theorem 3.3.1. *Assume R is noetherian and let M be a finitely generated R-module. There exists a finite filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$$

such that for all $i \in [n]$, there exists an ideal $\mathfrak{p}_i \in \mathrm{Spec}(R)$ for which $M_i/M_{i-1} \cong R/\mathfrak{p}_i$.

Proof. If $M = 0$, then the empty filtration will suffice with $n = 0$, so assume $M \neq 0$. Set $M_0 = 0$ and since R is noetherian, there exists an ideal $\mathfrak{p} \in \mathrm{Ass}_R(M)$ by Proposition 3.2.19. By the First Isomorphism Theorem there exists a submodule $M_1 \subseteq M$ such that $M_1/0 \cong M_1 \cong R/\mathfrak{p}$. If $M_1 = M$, then stop here with a finite filtration of length one. Otherwise $M/M_1 \neq 0$ and there exists $\mathfrak{p}_2 \in \mathrm{Ass}_R(M/M_1)$ by the same proposition. Hence by the Fourth Isomorphism Theorem there exists a submodule $M_2 \subseteq M$ with $M_1 \subseteq M_2$ and $M_2/M_1 \cong R/\mathfrak{p}_2$. If $M_2 = M$, then stop here with a finite filtration of length two. Otherwise continue the process, which must terminate after finitely many steps since M is noetherian (finitely generated modules over a noetherian ring are themselves noetherian). □

Fact 3.3.2. *The conclusion of Theorem 3.3.1 holds if we replace the noetherian assumption on the ring R with a noetherian assumption on the module M.*

Theorem 3.3.3. *Assume M has a filtration as in Theorem 3.3.1.*

(a) $\mathrm{Ass}_R(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \mathrm{Supp}_R(M)$ *and therefore* $|\mathrm{Ass}_R(M)| < \infty$.

(b) For any ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, $\mathfrak{p} \in \mathrm{Supp}_R(M)$ if and only if $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some $i \in [n]$. In other words

$$\mathrm{Supp}_R(M) = \bigcup_{i=1}^{n} V(\mathfrak{p}_i).$$

Proof. (a). From Lemma 3.2.27 and Example 3.2.14 we have

$$\mathrm{Ass}_R(M) \subseteq \bigcup_{i=1}^{n} \mathrm{Ass}_R(M_i/M_{i-1}) = \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/\mathfrak{p}_i) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$$

thereby proving the first containment. For any $i \in [n]$ we have

$$0 \neq Q(R/\mathfrak{p}_i) \cong \left(\frac{R}{\mathfrak{p}_i}\right)_{\mathfrak{p}_i} \cong \left(\frac{M_i}{M_{i-1}}\right)_{\mathfrak{p}_i} \cong \frac{(M_i)_{\mathfrak{p}_i}}{(M_{i-1})_{\mathfrak{p}_i}}$$

where the second isomorphism is a consequence of Theorem 3.1.4. Therefore $(M_i)_{\mathfrak{p}_i}$ is a non-zero submodule of $M_{\mathfrak{p}_i}$. Hence $M_{\mathfrak{p}_i}$ is non-zero, i.e., $\mathfrak{p}_i \in \operatorname{Supp}_R(M)$.

(b). This part is a corollary of previous results.

$$\begin{aligned} \operatorname{Supp}_R(M) &= \bigcup_{i=1}^{n} \operatorname{Supp}_R\left(\frac{M_i}{M_{i-1}}\right) && 3.2.27 \\ &= \bigcup_{i=1}^{n} \operatorname{Supp}_R\left(\frac{R}{\mathfrak{p}_i}\right) && \frac{M_i}{M_{i-1}} \cong \frac{R}{\mathfrak{p}_i} \\ &= \bigcup_{i=1}^{n} V(\mathfrak{p}_i) && 3.2.11 \end{aligned}$$

□

Corollary 3.3.4. *If R is noetherian and M is a finitely generated R-module, then $|\operatorname{Ass}_R(M)| < \infty$.*

Remark 3.3.5. The finitely generated assumption for M in the above corollary is indeed necessary. There exist noetherian rings such as $R = \mathbb{K}[x]$ or $R = \mathbb{Z}$ for which $|\operatorname{Spec}(R)| = \infty$. In such cases we can define a module

$$M = \bigoplus_{i=1}^{\infty} \frac{R}{\mathfrak{p}_i}$$

such that $\mathfrak{p}_1, \mathfrak{p}_2, \dots \in \operatorname{Spec}(R)$ are all distinct. Since $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots\} \subseteq \operatorname{Ass}_R(M)$ by Proposition 3.2.25, we know $|\operatorname{Ass}_R(M)| = \infty$.

We next use prime filtrations to give another verification of Example 3.2.15.

Example 3.3.6. Assume R is a unique factorization domain and let $x \in R \setminus R^\times$ be non-zero. Then $x = p_1 \cdots p_n$ for some primes $p_1, \dots, p_n \in R$. Defining the module $M = R/xR$ we have the filtration

$$0 \hookrightarrow \frac{R}{p_1 R} \hookrightarrow \cdots \hookrightarrow \frac{R}{p_1 \cdots p_{n-2} R} \overset{\phi_2}{\hookrightarrow} \frac{R}{p_1 \cdots p_{n-1} R} \overset{\phi_1}{\hookrightarrow} \frac{R}{p_1 \cdots p_n R} = \frac{R}{xR} = M$$

where $\phi_1(\overline{r}) := \overline{p_n r}$ and hence $\operatorname{Im}(\phi_1) \leq R/xR$ is the ideal generated over R by $\overline{p_n}$. The maps ϕ_i for $i \geq 2$ are defined similarly. Using a clever re-write of the submodule $\operatorname{Im}(\phi_1)$ (cf. Note 3.3.7), we apply the Third Isomorphism Theorem to write

$$\frac{R/xR}{\operatorname{Im}(\phi_1)} = \frac{R/p_1 \cdots p_n R}{\overline{p_n} \cdot R/(p_1 \cdots p_n R)} = \frac{R/p_1 \cdots p_n R}{p_n R/p_1 \cdots p_n R} \cong \frac{R}{p_n R}$$

and similarly

$$\frac{R/p_1 \cdots p_i R}{\operatorname{Im}(\phi_{n-i+1})} \cong \frac{R}{p_i R}$$

for any $i \in [n-1]$. Therefore R/xR has a prime filtration with $\mathfrak{p}_i = p_i R$ and from Theorem 3.3.3 we know

$$\operatorname{Ass}_R(R/xR) \subseteq \{p_1 R, \dots, p_n R\}. \tag{3.3.6.1}$$

Moreover for any $i \in [n]$ we can define the injection

$$\begin{aligned} R/p_i R &\hookrightarrow R/xR \\ \overline{r} &\longmapsto \overline{r p_1 \cdots p_{i-1} p_{i+1} \cdots p_n} \end{aligned}$$

implying by Remark 3.2.18 that $p_i R \in \operatorname{Ass}_R(R/xR)$. Hence we have equality in (3.3.6.1).

Note 3.3.7. In the previous example we rely on the following general fact. Let $J \leq R$, M an R-module, and $N \subseteq M$ a submodule. Then we have

$$\begin{aligned} J \cdot \frac{M}{N} &= \langle j \cdot (m + N) \mid j \in J,\ m \in M \rangle \\ &= \langle jm + N \mid j \in J,\ m \in M \rangle \\ &= \langle (jm + n) + N \mid j \in J,\ m \in M,\ n \in N \rangle \\ &= \frac{JM + N}{N} \end{aligned} \tag{3.3.7.1}$$

In the special case when $N \subseteq JM$ we have

$$JM + N = \langle jm + n \mid j \in J,\ m \in M,\ n \in N \rangle = \langle jm \mid j \in J,\ m \in M \rangle = JM$$

so (3.3.7.1) above simplifies to give the equality

$$J \cdot \frac{M}{N} = \frac{JM}{N}.$$

We now introduce some notation and a lemma in order to simplify the proof of Example 3.3.10.

Definition 3.3.8. Let A be a commutative ring with identity and define the polynomial ring $R = A[X_1, \ldots, X_d]$. A *monomial* in R is an element of the form $X_1^{n_1} \cdots X_d^{n_d}$ where $n_1, \ldots, n_d \in \mathbb{N}_0$. The collection of all monomials in a subset $S \subseteq R$ is denoted $[\![S]\!]$ and an ideal $I \leq R$ is a *monomial ideal* if it there exists a set $T \subseteq [\![R]\!]$ such that $I = \langle T \rangle$. Let $\mathfrak{X} \leq R$ denote the ideal generated by the variables, i.e., $\mathfrak{X} = \langle X_1, \ldots, X_d \rangle$.

Lemma 3.3.9. *Let k be a field, let $R = k[X_1, \ldots, X_d]$ be a polynomial ring, and let I be a monomial ideal of R such that $X_i^{m_i} \in I$ for all $i \in [d]$. Then there exists a chain of ideals*

$$I = I_0 \subset I_1 \subset \cdots \subset I_A = R$$

where A is the dimension of R/I as a k-vector space, such that $I_j/I_{j-1} \cong R/\mathfrak{X}$ for all $j \in [A]$.

Proof. We induct on A. If $A = 1$, then noting that $A = |[\![R]\!] \setminus [\![I]\!]|$, this implies $[\![R]\!] \setminus [\![I]\!] = \{1\}$, so $I = \mathfrak{X}$ and the chain $I \subset R$ satisfies the claim.

Assume $A > 1$ and the claim holds for any monomial ideal $J \supset \{X_1^{n_1}, \ldots, X_d^{n_d}\}$ for some non-zero $n_1, \ldots, n_d \in \mathbb{N}$ for which $\dim_k(R/J) = A - 1$. Let $f \in [\![R]\!]$ such that $f \notin I$, but $X_i f \in I$ for every $i \in [d]$. (These are called the *corner elements* of I.) We have an R-module homomorphism

$$\begin{aligned} \tau : R &\longrightarrow \frac{I + fR}{I} \\ r &\longmapsto \overline{0 + rf} \end{aligned}$$

which is non-zero and surjective since $\tau(1) = \overline{f}$, $f \notin I$, and $\overline{f}$ generates the codomain. However, $X_i \in \operatorname{Ker}(\tau)$ for every $i \in [d]$ by definition of a corner element, so $X_1, \ldots, X_d \in \operatorname{Ker}(\tau)$. Hence $\mathfrak{X} \subset \operatorname{Ker}(\tau) \subsetneq R$ so $\operatorname{Ker}(\tau) = \mathfrak{X}$ by the maximality of $\mathfrak{X}$ and therefore $R/\mathfrak{X} \cong (I + fR)/I$. We write $I \subsetneq I + fR$ to get the beginning of our chain.

Now we claim $\dim_k(R/(I + fR)) = A - 1$. We have a short exact sequence

$$0 \longrightarrow \frac{I + fR}{I} \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{I + fR} \longrightarrow 0$$

with the natural injection and surjection. This is exact as a sequence of k-modules (i.e., vector spaces), so it splits to yield

$$\frac{R}{I} \cong \frac{I+fR}{I} \oplus \frac{R}{I+fR}$$

and therefore to justify our claim we need only point out that

$$\frac{I+fR}{I} \cong \frac{R}{\mathfrak{X}} \cong k.$$

Therefore by our induction hypothesis we have a sequence

$$I + fR = I_1 \subset I_2 \subset \cdots \subset I_A = R$$

such that $I_j/I_{j-1} \cong R/\mathfrak{X}$ for $j = 2, \ldots, A$. Splicing on $I \subset I_1$ we have the desired chain. ☐

Example 3.3.10. Let k be a field and consider the unique factorization domain $R = k[x, y]$.

(a) Define $I = (x^m, y^n)R$ where $m, n \in \mathbb{N}$ and consider R/I as a finitely generated R-module. From Example 3.2.17 we already know

$$\operatorname{Ass}_R\left(\frac{R}{(x^m, y^n)R}\right) = \{(x, y)R\}$$

so we want to build a prime filtration such that each subquotient is isomorphic to $R/(x, y)R$. By Lemma 3.3.9 we know there exists a chain of ideals

$$I = I_0 \subset I_1 \subset \cdots \subset I_A = R$$

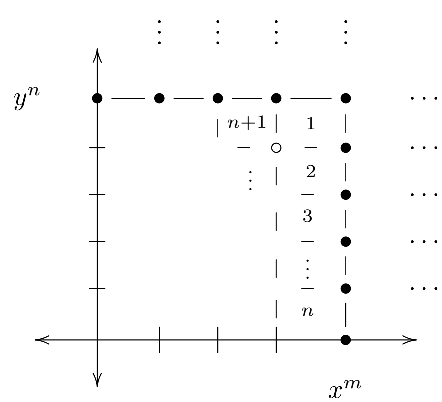
such that $I_j/I_{j-1} \cong R/(x, y)R$ for all $j \in [A]$ where $A = \dim_k(R/I)$. Considering that $I \subset I_j$ for all j we also have the chain

$$0 \subset I_1/I \subset I_2/I \subset \cdots \subset I_A/I = R/I$$

and by the Third Isomorphism Theorem

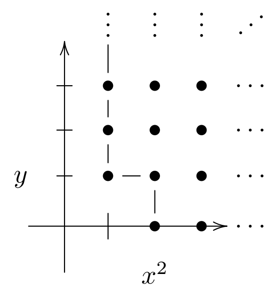
$$\frac{I_j/I}{I_{j-1}/I} \cong \frac{R}{(x, y)R}.$$

Since $(x, y)R \in \operatorname{Spec}(R)$, this chain is a prime filtration for R/I. In the proof of the lemma, we saw that the original chain from I to R is built by "throwing in" successive corner elements of I. In practice we can proceed in a methodical fashion as depicted in the lattice diagram below



where the element $x^{m-1}y^{n-1}$, marked with '∘', represents a generator of I_1/I. We have thus demonstrated the existence of the prime filtration of R/I guaranteed by Theorem 3.3.1.

(b) Next consider the ideal $L = (x^2, xy)R$ with the following lattice diagram.



For elements $x + L,\ y + L \in R/L$, we already know $(x, y)R = \mathrm{Ann}_R(x + L)$ and $(x)R = \mathrm{Ann}_R(y + L)$, so

$$(x, y)R,\ (y)R \in \mathrm{Ass}_R(R/L) \tag{3.3.10.1}$$

So we want to show R/L has a prime filtration

$$0 \subseteq \langle x + L\rangle \subseteq R/L$$

To check the subsequent quotients, we first point out that by the argument in part (a) of this example

$$\frac{\langle x + L\rangle}{0} \cong \langle x + L\rangle \cong \frac{R}{(x, y)R}$$

so the condition for a prime filtration is satisfied for the first containment. To check the condition on the second containment we define the surjection

$$\begin{aligned} R/L &\longrightarrow R/(x)R \\ r + L &\longmapsto \overline{r} \end{aligned}$$

with kernel $\langle x+L\rangle$, so the condition is verified by the First Isomorphism Theorem and because $(x)R \in \mathrm{Spec}(R)$. (Well-definedness of the above map holds since $L \subseteq (x)R$.) Thus we have a prime filtration with $\mathfrak{p}_1 = (x, y)R$ and $\mathfrak{p}_2 = (x)R$, so $\mathrm{Ass}_R(R/L) \subseteq \{(x, y)R,\ (x)R\}$ by Theorem 3.3.3 and in fact $\mathrm{Ass}_R(R/L) = \{(x, y)R,\ (x)R\}$ by (3.3.10.1).

Proposition 3.3.11. *Let R be a non-zero commutative ring with identity, let M be an R-module, and let $U \subseteq R$ be a multiplicatively closed subset.*

(a) $\mathrm{Supp}_{U^{-1}R}(U^{-1}M) = \left\{ U^{-1}\mathfrak{p} \,\middle|\, \mathfrak{p} \in \mathrm{Supp}_R(M) \text{ and } \mathfrak{p} \cap U = \emptyset \right\}$

(b) $\mathrm{Ass}_{U^{-1}R}(U^{-1}M) \supseteq \left\{ U^{-1}\mathfrak{p} \,\middle|\, \mathfrak{p} \in \mathrm{Ass}_R(M) \text{ and } \mathfrak{p} \cap U = \emptyset \right\}$

(c) If R is also noetherian, then we have equality in (b).

Proof. (a). The prime ideals of $U^{-1}R$ are described as follows.

$$\mathrm{Spec}(U^{-1}R) = \left\{ U^{-1}\mathfrak{p} \,\middle|\, \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \mathfrak{p} \cap U = \emptyset \right\}$$

For any ideal $U^{-1}\mathfrak{p}$, we know $M_{\mathfrak{p}} \cong (U^{-1}M)_{U^{-1}\mathfrak{p}}$ by the map

$$\begin{aligned} \Phi\colon M_{\mathfrak{p}} &\longrightarrow (U^{-1}M)_{U^{-1}\mathfrak{p}} \\ \frac{m}{v} &\longmapsto \frac{m/1}{v/1} \end{aligned}$$

and thus

$$(U^{-1}M)_{U^{-1}\mathfrak{p}} \neq 0 \iff M_{\mathfrak{p}} \neq 0$$

completing the proof of (a).

(b). Let $\mathfrak{p} \in \operatorname{Ass}_R(M)$ such that $U \cap \mathfrak{p} = \emptyset$. By Remark 3.2.18 there exists a monomorphism

$$R/\mathfrak{p} \xhookrightarrow{\varphi} M$$

and therefore we also have the following monomorphism.

$$U^{-1}(R/\mathfrak{p}) \xhookrightarrow{U^{-1}\varphi} U^{-1}M$$

Hence $U^{-1}\mathfrak{p} \in \operatorname{Ass}_{U^{-1}R}(U^{-1}M)$ by the isomorphism

$$\frac{U^{-1}R}{U^{-1}\mathfrak{p}} \cong U^{-1}(R/\mathfrak{p}).$$

(c). Assume R is noetherian and let $U^{-1}\mathfrak{p} \in \operatorname{Ass}_{U^{-1}R}(U^{-1}M)$, where we know the form of such elements by the first line of the proof of part (a). We need to show $\mathfrak{p} \in \operatorname{Ass}_R(M)$. Since R is noetherian let $x_1, \ldots, x_n \in R$ such that $\mathfrak{p} = (x_1, \ldots, x_n)R$ and by virtue of being an associated prime we also know $U^{-1}\mathfrak{p} = \operatorname{Ann}_{U^{-1}R}(m/u)$ for some $m/u \in U^{-1}M$. For any $i \in [n]$, $x_i/1 \in U^{-1}\mathfrak{p}$ so

$$\frac{x_i}{1} \cdot \frac{m}{v} = 0$$

and thus there exist $u_1, \ldots, u_n \in U$ such that $u_i x_i m = 0$ for each $i \in [n]$. We define $u' = u_1 \cdots u_n \in U$ for which we have

$$x_i u' m = x_i u_1 \cdots u_i \cdots u_n m = 0$$

for every $i \in [n]$, implying

$$\mathfrak{p} = (x_1, \ldots, x_n)R \subseteq \operatorname{Ann}_R(u'm).$$

Now recall to prove $\mathfrak{p} \in \operatorname{Ass}_R(M)$ it suffices to find a monomorphism mapping from $R/\mathfrak{p}$ to M. Define the R-module homomorphism

$$\begin{aligned} \phi\colon R &\longrightarrow M \\ r &\longmapsto ru'm \end{aligned}$$

and notice that $\mathfrak{p} \subseteq \operatorname{Ker}(\phi)$. Then by the Universal Mapping Property for quotients, the map

$$\begin{aligned} \alpha\colon R/\mathfrak{p} &\longrightarrow M \\ \overline{r} &\longmapsto ru'm \end{aligned}$$

is a well-defined R-module homomorphism as well. To show α is one-to-one, we consider a commutative diagram

$$\begin{array}{ccc} R/\mathfrak{p} & \xrightarrow{\alpha} & M \\ \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\gamma} \\ U^{-1}(R/\mathfrak{p}) & \xrightarrow{U^{-1}\alpha} & U^{-1}(M) \end{array}$$

and we will show both β and $U^{-1}\alpha$ are injective. First we verify the commutivity of the diagram. Indeed for any $r + \mathfrak{p} \in R/\mathfrak{p}$ we have

$$(U^{-1}\alpha \circ \beta)(r+\mathfrak{p}) = U^{-1}\alpha\left(\frac{r+\mathfrak{p}}{1}\right) = \frac{\alpha(r+\mathfrak{p})}{1}$$
$$(\gamma \circ \alpha)(r+\mathfrak{p}) = \gamma(\alpha(r+\mathfrak{p})) = \frac{\alpha(r+\mathfrak{p})}{1}.$$

To show β is one-to-one, let $r + \mathfrak{p} \in R/\mathfrak{p}$ such that $\beta(r+\mathfrak{p}) = 0$. This implies

$$\frac{r+\mathfrak{p}}{1} = \frac{0+\mathfrak{p}}{v}$$

for any $v \in U$. By our definition of equality in the ring of fractions, there is some $w \in U$ such that

$$0 + \mathfrak{p} = (w \cdot 1 \cdot 0) + \mathfrak{p} = (wrv) + \mathfrak{p}$$

so $wrv \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, from the conditions $w, v \in U$ and $U \cap \mathfrak{p} = \emptyset$, we conclude $r \in \mathfrak{p}$ and $r + \mathfrak{p} = 0 + \mathfrak{p}$. Hence β is one-to-one.

Turning our attention to $U^{-1}\alpha$, we first note that since u/m and $u'm/1$ differ only by the unit u'/u we have

$$U^{-1}\mathfrak{p} = \operatorname{Ann}_{U^{-1}R}\left(\frac{u'm}{1}\right).$$

Next consider an element in the kernel of $U^{-1}\alpha$:

$$\begin{aligned}(U^{-1}\alpha)\left(\frac{r+\mathfrak{p}}{v}\right) = 0 &\iff \frac{ru'm}{v} = \frac{0}{v}\\ &\iff \frac{r}{v} \in U^{-1}\mathfrak{p}\\ &\iff r \in \mathfrak{p}\\ &\iff \frac{r+\mathfrak{p}}{v} = 0 \in U^{-1}(R/\mathfrak{p}).\end{aligned}$$

Hence $U^{-1}\alpha$ is one-to-one and therefore so is the composition $(U^{-1}\alpha) \circ \beta$. By the commutivity of the diagram the composition $\gamma \circ \alpha$ must also be one-to-one, so we conclude α is injective. ☐

Corollary 3.3.12. *Let $Q \in \operatorname{Spec}(R)$.*

(a) $\operatorname{Supp}_R(M_Q) = \{\mathfrak{p}_Q \mid \mathfrak{p} \in \operatorname{Supp}_R(M),\ \mathfrak{p} \subseteq Q\}$

(b) $\operatorname{Ass}_R(M_Q) \supseteq \{\mathfrak{p}_Q \mid \mathfrak{p} \in \operatorname{Ass}_R(M),\ \mathfrak{p} \subseteq Q\}$

(c) If R is noetherian, then we have equality in (b).

Proof. Set $U = R \setminus Q$ in the context of Proposition 3.3.11. Then $\mathfrak{p} \cap U = \emptyset$ if and only if $\mathfrak{p} \subseteq Q$. ☐

Proposition 3.3.13. *Assume R is noetherian and $M \neq 0$ is a finitely generated R-module with prime filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for every $i \in [n]$.*

(a) $\operatorname{Min}(\operatorname{Ass}_R(M)) = \operatorname{Min}\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \operatorname{Min}(\operatorname{Supp}_R(M))$

(b) $|\operatorname{Min}(\operatorname{Supp}_R(M))| < \infty$ *and for all* $\mathfrak{p} \in \operatorname{Supp}_R(M)$*, there exists some* $\mathfrak{p}' \in \operatorname{Min}(\operatorname{Supp}_R(M))$ *such that* $\mathfrak{p}' \subseteq \mathfrak{p}$*.*

(c) $|\operatorname{Min}(\operatorname{Spec}(R))| < \infty$ *and for all* $\mathfrak{p} \in \operatorname{Spec}(R)$*, there exists some* $\mathfrak{p}' \in \operatorname{Min}(\operatorname{Spec}(R))$ *such that* $\mathfrak{p}' \subseteq \mathfrak{p}$*.*

Proof. (a). Hypothetically, let $\mathfrak{p} \in \operatorname{Min}(\operatorname{Supp}_R(M))$ and thus $M_\mathfrak{p} \neq 0$. Since R is noetherian we know $R_\mathfrak{p}$ is noetherian as well, so $\operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq \emptyset$ by Corollary 3.2.21. Therefore by Corollary 3.3.12 we let $\mathfrak{q} \in \operatorname{Ass}_R(M)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q}_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Since $\operatorname{Ass}_R(M) \subseteq \operatorname{Supp}_R(M)$ by Theorem 3.3.3, the minimality of $\mathfrak{p}$ in $\operatorname{Supp}_R(M)$ combined with the fact that $\mathfrak{q} \subseteq \mathfrak{p}$ implies $\mathfrak{p} = \mathfrak{q} \in \operatorname{Ass}_R(M)$. We claim $\mathfrak{p} \in \operatorname{Min}(\operatorname{Ass}_R(M))$. Since $\operatorname{Ass}_R(M)$ is finite also by Theorem 3.3.3, there exists an ideal $\mathfrak{p}' \in \operatorname{Min}(\operatorname{Ass}_R(M))$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$. Again using the minimality of $\mathfrak{p} \in \operatorname{Supp}_R(M)$ we conclude $\mathfrak{p} = \mathfrak{p}' \in \operatorname{Min}(\operatorname{Ass}_R(M))$ and we have thus shown

$$\operatorname{Min}(\operatorname{Supp}_R(M)) \subseteq \operatorname{Min}(\operatorname{Ass}_R(M)). \tag{3.3.13.1}$$

Next let

$$\mathfrak{p}_i \in \operatorname{Min}\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \overset{(1)}{\subseteq} \operatorname{Supp}_R(M)$$

the existence of which is guaranteed, because we are taking a minimal element of a finite set, and (1) is given by Theorem 3.3.3. We want to show $\mathfrak{p}_i$ is a minimal element of $\operatorname{Supp}_R(M)$. Suppose there is an ideal $\mathfrak{p}' \in \operatorname{Supp}_R(M)$ such that $\mathfrak{p}' \subseteq \mathfrak{p}_i$. By Theorem 3.3.3 there exists an ideal $\mathfrak{p}_j \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ such that

$$\mathfrak{p}_j \subseteq \mathfrak{p}' \subseteq \mathfrak{p}_i$$

and the minimality of $\mathfrak{p}_i$ yields

$$\mathfrak{p}_i \subseteq \mathfrak{p}_j \subseteq \mathfrak{p}' \subseteq \mathfrak{p}_i.$$

Hence $\mathfrak{p}' = \mathfrak{p}_i$, so $\mathfrak{p}_i$ is minimal and we have shown

$$\operatorname{Min}\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subseteq \operatorname{Min}(\operatorname{Supp}_R(M)). \tag{3.3.13.2}$$

Note this containment verifies the legitimacy of the hypothetical element we took from $\operatorname{Min}(\operatorname{Supp}_R(M))$ in the beginning of the proof. Finally let

$$\mathfrak{p} \in \operatorname{Min}(\operatorname{Ass}_R(M)) \subseteq \operatorname{Ass}_R(M) \overset{3.3.3}{\subseteq} \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}.$$

This implies $\mathfrak{p} = \mathfrak{p}_i$ for some $i \in [n]$ and since $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is finite, there exists some $\mathfrak{p}_j \in \operatorname{Min}\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ such that

$$\mathfrak{p}_j \subseteq \mathfrak{p}_i = \mathfrak{p}.$$

Using the results from (3.3.13.1) and (3.3.13.2) we have shown

$$\mathfrak{p}_j \in \operatorname{Min}\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subseteq \operatorname{Min}(\operatorname{Supp}_R(M)) \subseteq \operatorname{Min}(\operatorname{Ass}_R(M))$$

whereby we conclude $\mathfrak{p} = \mathfrak{p}_j$ by the minimality of $\mathfrak{p}$. This shows

$$\operatorname{Min}(\operatorname{Ass}_R(M)) \subseteq \operatorname{Min}\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}.$$

(b). Let $\mathfrak{p} \in \operatorname{Supp}_R(M)$ and we have $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some $i \in [n]$ by Theorem 3.3.3. Therefore there exists some

$$\mathfrak{p}_j \in \operatorname{Min}\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \operatorname{Min}(\operatorname{Supp}_R(M))$$

such that

$$\mathfrak{p} \supseteq \mathfrak{p}_i \supseteq \mathfrak{p}_j$$

so taking $\mathfrak{p}_j$ as our $\mathfrak{p}'$ this proves the second part. The first part is verified simply as follows.

$$|\operatorname{Min}(\operatorname{Supp}_R(M))| = |\operatorname{Min}\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}| \leq n < \infty$$

(c). This is more or less a corollary. Set $M = R$, note $\operatorname{Supp}_R(R) = \operatorname{Spec}(R)$, and apply (b). □

We provide here two examples demonstrating that minimal associated primes may or may not be unique.

Example 3.3.14. Consider the ring of polynomials with complex coefficients $R = \mathbb{C}[x]$ and define the R-module

$$M = \frac{\mathbb{C}[x]}{x(x-1)\cdots(x-9)}.$$

From our work in Example 3.3.6 we know

$$\mathrm{Ass}_R(M) = \{xR, (x-1)R, \ldots, (x-9)R\}.$$

Since $(x-i)R \not\subseteq (x-j)R$ for any $i \neq j$, we conclude

$$\mathrm{Min}(\mathrm{Ass}_R(M)) = \mathrm{Ass}_R(M).$$

Example 3.3.15. Consider the ring of polynomials in two variables with coefficients in an arbitrary field, $R = \mathbb{K}[x, y]$, and define the R-module

$$M = \frac{\mathbb{K}[x,y]}{(x^2, xy)R}.$$

We already know from Example 3.3.10 that

$$\mathrm{Ass}_R(M) = \{(x)R, (x,y)R\}$$

and since $(x)R \subset (x,y)R$ we have

$$\mathrm{Min}(\mathrm{Ass}_R(M)) = \{(x)R\}.$$

Definition 3.3.16. With R and M as in Proposition 3.3.13, define

$$\mathrm{Min}_R(M) = \mathrm{Min}(\mathrm{Ass}_R(M)).$$

Any ideal $\mathfrak{p} \in \mathrm{Min}_R(M)$ is a *minimal prime of M* or a *minimal associated prime of M*. If $\mathfrak{q} \in \mathrm{Ass}_R(M) \setminus \mathrm{Min}_R(M)$, then $\mathfrak{q}$ is an *embedded prime of M*.

Note 3.3.17. Let R be a noetherian ring. For any ideals $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$, we have $V(\mathfrak{p}) \subseteq V(\mathfrak{q})$. Applying Theorem 3.3.3 and Proposition 3.3.13, it follows that

$$\mathrm{Supp}_R(M) = \bigcup_{i=1}^{n} V(\mathfrak{p}_i) = \bigcup_{\mathfrak{p}_i \in \mathrm{Min}(\mathrm{Supp}_R(M))} V(\mathfrak{p}_i)$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the prime ideals from some prime filtration of M.

3.4 Prime Avoidance and Nakayama's Lemma

In this section we prove two major results from abstract algebra. Prime Avoidance gives us more insight into the associated primes of finitely generated R-modules, especially when R is noetherian, as we will see in Corollaries 3.4.2 and 3.4.3. Nakayama's Lemma has a number of corollaries, some of which we will produce here. Perhaps most importantly, Nakayama's Lemma gives us Lemma 3.4.19 which we will use directly in the proof of Theorem 3.5.16.

Lemma 3.4.1 (Prime Avoidance)**.** *Let R be a non-zero commutative ring with identity and let $I_1, \ldots, I_n, J \leq R$ be ideals. Assume one of the following:*

1. *R contains an infinite field as a subring, or*

2. *the ideals $I_1, \dots, I_{n-2}$ are prime.*

Then whenever $J \subseteq \bigcup_{i=1}^n I_i$, we have $J \subseteq I_i$ for some $i \in [n]$. Equivalently, if $J \not\subseteq I_i$ for all $i \in [n]$, then $J \not\subseteq \bigcup_{i=1}^n I_i$.

Proof. Assume $\mathbb{K} \subseteq R$ as a subring with $\mathbb{K}$ an infinite field. For an arbitrary vector space V over $\mathbb{K}$, we have this fact:

$$V_1, \dots, V_n \subsetneq V \text{ proper subspaces} \implies \bigcup_{i=1}^n V_i \subsetneq V. \tag{3.4.1.1}$$

Assume $J \not\subseteq I_j$ for any $j \in [n]$, which implies $J \cap I_j \subsetneq J$. Since ideals of R are $\mathbb{K}$-vector spaces, (3.4.1.1) gives

$$J \cap \left(\bigcup_{j=1}^n I_j \right) = \bigcup_{j=1}^n (J \cap I_j) \subsetneq J$$

and hence

$$J \not\subseteq \bigcup_{j=1}^n I_j.$$

Now assume $I_1, \dots, I_{n-2}$ are prime and we will argue by induction on n. For the base case $n = 1$, the hypothesis is vacuous and the conclusion holds trivially. For the base case $n = 2$, suppose $J \not\subseteq I_1, I_2$ and suppose for the sake of contradiction that $J \subseteq (I_1 \cup I_2)$. Then there exist elements $x_1, x_2 \in J$ such that $x_1 \notin I_1$ and $x_2 \notin I_2$. We observe

$$x_1 \in J \subseteq (I_1 \cup I_2) \implies x_1 \in I_2$$
$$x_2 \in J \subseteq (I_1 \cup I_2) \implies x_2 \in I_1.$$

We know also that $x_1 + x_2 \in J$. If $x_1 + x_2 \in I_1$, then $x_1 = (x_1 + x_2) - x_2 \in I_1$, a contradiction. An identical contradiction lets us conclude $x_1 + x_2 \notin I_2$, giving us

$$x_1 + x_2 \in J \setminus (I_1 \cup I_2)$$

which is a contradiction, proving the second base case.

For the induction step, assume $n \geq 3$ and assume the result holds for lists of length $n - 1$. If $J \subseteq \bigcup_{i \neq l} I_i$ for some $l \in [n]$, then by the induction hypothesis $J \subseteq I_i$ for some $i \neq l$ and we are done. Therefore assume without loss of generality

$$J \not\subseteq \bigcup_{i \neq l} I_i$$

for all $l \in [n]$. For each $l \in [n]$, fix an element

$$x_l \in J \setminus \bigcup_{i \neq l} I_i \tag{3.4.1.2}$$

and consider the element $x' = x_1 + (x_2 \cdots x_n) \in J$. Suppose $J \subseteq \bigcup_{i=1}^n I_i$, implying $x' \in I_i$ for some i. If we first suppose $x' \in I_1$, it follows that $x_2 \cdots x_n = x' - x_1 \in I_1$. Since I_1 is a prime ideal, this implies $x_j \in I_1$ for some $j \geq 2$, contradicting our choices in (3.4.1.2). Second, if we suppose $x' \in I_i$ for some $i \geq 2$, then we have $x_2 \cdots x_n \in I_i$ and therefore $x_1 = x' - x_2 \cdots x_n \in I_i$, again contradicting (3.4.1.2). □

Corollary 3.4.2. *Let R be a noetherian ring and let M be a non-zero, finitely generated R-module. If $J \leq R$ consists entirely of zero-divisors on M, then there exists some ideal $\mathfrak{p} \in \operatorname{Ass}_R(M)$ such that $J \subseteq \mathfrak{p}$.*

Proof. Since R is noetherian and M is finitely generated, Theorem 3.3.3 implies $\operatorname{Ass}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ for some $n \geq 1$. By Proposition 3.2.19 we have

$$J \subseteq \{\text{zero-divisors on } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p}.$$

Therefore $J \subseteq \mathfrak{p}_i$ for some $i \in [n]$ by Lemma 3.4.1. □

Corollary 3.4.3. *Assume R is noetherian, let M be a non-zero finitely generated R-module, and let $\mathfrak{m} \lneq R$ be a maximal ideal. Then*

$$\mathfrak{m} \text{ contains a non-zero-divisor on } M \iff \mathfrak{m} \notin \operatorname{Ass}_R(M)$$

or equivalently

$$\mathfrak{m} \text{ consists entirely of zero-divisors on } M \iff \mathfrak{m} \in \operatorname{Ass}_R(M).$$

Proof. If $\mathfrak{m}$ consists entirely of zero-divisors on M, then $\mathfrak{m} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R(M)$ by Corollary 3.4.2. Since we have a maximal ideal inside a proper ideal, $\mathfrak{m} = \mathfrak{p} \in \operatorname{Ass}_R(M)$, proving one direction. On the other hand, if we suppose $\mathfrak{m} = \operatorname{Ann}_R(m)$ for some $0 \neq m \in M$, then $\mathfrak{m}$ consists entirely of zero-divisors on M. □

Definition 3.4.4. Let $I \leq R$ and M an R-module. IM is the submodule of M defined

$$IM = \langle im \in M \mid i \in I,\ m \in M \rangle = \left\{ \sum_{j=1}^{n} i_j m_j \;\middle|\; i_j \in I,\ m_j \in M,\ \forall j \in [n];\ n \in \mathbb{N} \right\}.$$

Fact 3.4.5. *We can make M/IM into an R/I-module by the operation*

$$(r + I)(m + IM) = (rm) + IM.$$

Proof. Checking the module axioms is straight-forward. For instance, we verify two questions of well-definedness here. Let $r_1, r_2 \in R$ such that $r_1 + I = r_2 + I$ and let $m \in M$. Therefore $r_1 - r_2 \in I$ and it follows that $(r_1 m) - (r_2 m) = (r_1 - r_2)m \in IM$. Thus

$$(r_1 + I)(m + IM) = (r_1 m) + IM = (r_2 m) + IM = (r_2 + I)(m + IM)$$

verifying one question. Now let $r \in R$ and let $m_1, m_2 \in M$ such that $m_1 + IM = m_2 + IM$. Then there exist $i \in I$ and $n \in M$ such that $m_1 - m_2 = in$. Therefore $(rm_1) - (rm_2) = r(m_1 - m_2) = r(in) = (ri)n \in IM$ and it follows that

$$(r + I)(m_1 + IM) = (rm_1) + IM = (rm_2) + IM = (r + I)(m_2 + IM)$$

verifying the second question, so the operation is well-defined. □

Definition 3.4.6. R is a *local* ring if it has a unique maximal ideal. If the unique maximal ideal is $\mathfrak{m} \lneq R$, then we say $(R, \mathfrak{m})$ is a local ring. Some texts would say this is a *quasi-local* ring, because we are not assuming R is noetherian.

Fact 3.4.7. *Let R be a commutative ring with identity.*

(a) If $\mathfrak{m} \lneq R$ is maximal such that $1 + \mathfrak{m} \subseteq R^\times$, then $(R, \mathfrak{m})$ is local.

(b) The following are equivalent.

(i) $(R, \mathfrak{m})$ is local.

(ii) $R \setminus R^\times$ is a proper ideal of R.

(iii) There exists a proper ideal $\mathfrak{a} \lneq R$ such that $R \setminus \mathfrak{a} \subseteq R^\times$.

(c) When the conditions of (b) are satisfied, we have $\mathfrak{m} = \mathfrak{a} = R \setminus R^\times$.

Proof. (b). Part (c) will follow from this argument as well. First assume $(R, \mathfrak{m})$ is local and we claim $\mathfrak{m} = R \setminus R^\times$, for which it suffices to show that $R \setminus \mathfrak{m} = R^\times$. For any $u \in R^\times$, $R = \langle u \rangle$, so $u \notin \mathfrak{m}$ since $\mathfrak{m}$ is a proper ideal. Thus $R^\times \subseteq R \setminus \mathfrak{m}$. For any $x \in R \setminus R^\times$, we have $\langle x \rangle \lneq R$. Since every proper ideal is contained in a maximal ideal, $\langle x \rangle \subseteq \mathfrak{m}$ and therefore $x \in \mathfrak{m}$. Having shown the contrapositive, we conclude $R^\times \supseteq R \setminus \mathfrak{m}$. Therefore (i) implies (ii).

If we assume $R \setminus R^\times \lneq R$ and set $\mathfrak{a} = R \setminus R^\times$, then it is immediate that $R \setminus \mathfrak{a} = R^\times$. Hence (ii) implies (iii).

Finally, assume $\mathfrak{a} \lneq R$ is such that $R \setminus \mathfrak{a} \subseteq R^\times$. Taking the complement we have $\mathfrak{a} \supseteq R \setminus R^\times$. On the other hand, if we let $a \in \mathfrak{a}$, then $a \notin R^\times$, because $\mathfrak{a}$ is a proper ideal. Therefore $a \in R \setminus R^\times$ and thus $\mathfrak{a} \subseteq R \setminus R^\times$. Hence $\mathfrak{a} = R \setminus R^\times$. We claim for every maximal ideal $\eta \lneq R$ we have $\eta = \mathfrak{a}$. For any $y \in \eta$, since y does not generate the entire ring, $y \notin R^\times$. Therefore $y \in \mathfrak{a}$ and thus $\eta \subseteq \mathfrak{a} \subsetneq R$. Since η is maximal, $\eta = \mathfrak{a}$, completing the proof of (b) and (c).

(a). By part (b) it suffices to show $R \setminus \mathfrak{m} \subseteq R^\times$. Let $x \in R \setminus \mathfrak{m}$ and set $\langle x, \mathfrak{m} \rangle = \langle \{x\} \cup \mathfrak{m} \rangle$. It is straightforward to show

$$\langle x, \mathfrak{m} \rangle = \langle ax + m \mid a \in R, \, m \in \mathfrak{m} \rangle .$$

Then $\mathfrak{m} \subseteq \langle x, \mathfrak{m} \rangle$. Since $x \in \langle x, \mathfrak{m} \rangle$ and $x \notin \mathfrak{m}$ we have $\mathfrak{m} \subsetneq \langle x, \mathfrak{m} \rangle \subseteq R$. Therefore $\langle x, \mathfrak{m} \rangle = R$ by the maximality of $\mathfrak{m}$ and it follows that $1 \in \langle x, \mathfrak{m} \rangle$. Then we let $a \in R$ and $m \in \mathfrak{m}$ such that $1 = ax + m$. Therefore

$$ax = 1 - m \in 1 + \mathfrak{m} \subseteq R^\times$$

and we conclude $a, x \in R^\times$. □

Lemma 3.4.8 (Nakayama's Lemma). *Assume $(R, \mathfrak{m})$ is a local ring and M is a finitely generated R-module. The following conditions are equivalent.*

(i) $M = 0$

(ii) $M = \mathfrak{m}M$

(iii) $M/\mathfrak{m}M = 0$

Proof. It is clear that (i) implies (iii) and it is also clear that conditions (ii) and (iii) are equivalent, so we need only show that (ii) implies (i). Assume $M = \mathfrak{m}M$, let $m_1, \ldots, m_n \in M$ be a generating sequence for M, and assume no proper subsequence of $m_1, \ldots, m_n$ generates M. Suppose for the sake of contradiction that $n \geq 1$. Then $m_1 \in M = \mathfrak{m}M$ can be written $m_1 = \sum_{i=1}^{n} r_i m_i$ for some $r_1, \ldots, r_n \in \mathfrak{m}$. Therefore

$$(1 - r_1)m_1 = m_1 - r_1 m_1 = \sum_{i=2}^{n} r_i m_i.$$

Since $r_1 \in \mathfrak{m}$, Fact 3.4.7 above implies $1 - r_1 \in R^\times$ so $m_1 \in \langle m_2, \ldots, m_n \rangle$. In other words

$$M = \langle m_1, \ldots, m_n \rangle \subseteq \langle m_2, \ldots, m_n \rangle \subseteq M$$

giving equality at every step, which contradicts the minimality of our generating sequence. Therefore $n = 0$ and $M = \langle \emptyset \rangle = 0$. □

Example 3.4.9. Let k be a field and consider the ring $R = k \times k$. We can define the projection

$$P_1 : R \twoheadrightarrow k$$
$$(a, b) \mapsto a$$

for which the kernel $\text{Ker}\,(P_1) = 0 \times k$ is a maximal ideal and we denote it $\mathfrak{m} = \text{Ker}\,(P_1)$. Notice this maximal ideal is not unique. Consider the cyclic R-module $M = 0 \times k = \langle (0, 1) \rangle$. In this case we have

$$\mathfrak{m}M = (0 \times k)(0 \times k) = (0 \times k) = M$$

but $M \neq 0$. The point here is that in order to use a maximal ideal in Nakayama's Lemma, we really do need that ideal to be unique.

Example 3.4.10. Let $(R, \mathfrak{m})$ be a local integral domain, but not a field. Such rings could be

$$\mathbb{Z}_{\langle p \rangle} \qquad \text{or} \qquad \mathbb{K}[X]_{\langle X \rangle}$$

where $p \in \mathbb{N}$ a prime. Let $M = Q(R) \neq 0$ be the field of fractions of R. Since R is not a field, $\mathfrak{m} \neq 0$ and one can check that $\mathfrak{m} \cdot Q(R) = Q(R)$. So M must be finitely generated in Nakayama's Lemma.

Corollary 3.4.11. *If $(R, \mathfrak{m})$ is local, noetherian, and not a field, then $\mathfrak{m}^2 \subsetneq \mathfrak{m}$.*

Proof. Since R is noetherian, we know $\mathfrak{m}$ is finitely generated, so by Nakayama's Lemma, if $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}^2 = \mathfrak{m}$, then $\mathfrak{m} = 0$, which is a contradiction, since R is not a field. $\square$

Corollary 3.4.12. *Assume R is noetherian and M is a non-zero, finitely generated R-module.*

(a) If $(R, \mathfrak{m})$ is local and not a field with $\mathfrak{m} \notin \text{Ass}_R(M)$, then $\mathfrak{m} \setminus \mathfrak{m}^2$ contains a non-zero-divisor on M.

(b) If $\mathfrak{m} \lneq R$ a maximal ideal such that $\mathfrak{m}^2 \neq \mathfrak{m}$ and $\mathfrak{m} \notin \text{Ass}_R(M)$, then $\mathfrak{m} \setminus \mathfrak{m}^2$ contains a non-zero-divisor on M.

Proof. We will prove part (b) and part (a) will follow by Corollary 3.4.11. By Proposition 3.2.19, the set of associated primes is nonempty and by Theorem 3.3.3 we write $\text{Ass}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ for some $n \geq 1$. We will apply Prime Avoidance (Lemma 3.4.1) to the list $\mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{m}^2, \mathfrak{m}$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are all prime. Since we are assuming $\mathfrak{m} \neq \mathfrak{m}^2$, it follows that $\mathfrak{m} \not\subseteq \mathfrak{m}^2$ (maximality of $\mathfrak{m}$ would force equality in this case). We are also assuming $\mathfrak{m} \notin \text{Ass}_R(M)$, so by prime avoidance

$$\mathfrak{m} \not\subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n \cup \mathfrak{m}^2.$$

Therefore there exists an element $x \in \mathfrak{m}$ such that $x \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n = \text{ZD}_R(M)$ and $x \notin \mathfrak{m}^2$ (cf. Proposition 3.2.19). Hence we have found an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ that is a non-zero-divisor on M. $\square$

Corollary 3.4.13. *Let $(R, \mathfrak{m})$ be local, M an R-module, and $N \subseteq M$ a submodule such that M/N is finitely generated over R. If $M = N + \mathfrak{m}M$, then $M = N$. (Note the stronger assumption that M is finitely generated would be sufficient to conclude M/N finitely generated.)*

Proof. If $M = N + \mathfrak{m}M$, then we have

$$\mathfrak{m}\left(\frac{M}{N}\right) = \frac{\mathfrak{m}M + N}{N} = \frac{M}{N}.$$

Since M/N is finitely generated, we may apply Nakayama's Lemma to conclude $M = N$. $\square$

Definition 3.4.14. Let M be a finitely generated R-module. A *minimal generating sequence* for M is a generating sequence $m_1, \dots, m_n \in M$ such that no proper subsequence generates M.

Example 3.4.15. For the ring $R = \mathbb{K}[x, y]$, the elements $x, y \in R$ form a minimal generating sequence for $\langle x, y \rangle$.

Remark 3.4.16. Note in our definition above, we make no claim on our ability to find a shorter generating sequence, but rather we cannot shorten this *particular* sequence without disrupting its generating property for M.

Corollary 3.4.17. *Let $(R, \mathfrak{m}, \mathfrak{K})$ be local and M a finitely generated R-module. Let $m_1, \dots, m_n \in M$.*

(a) $M/\mathfrak{m}M$ is a finite-dimensional vector space over $\mathfrak{K}$ via the scalar multiplication

$$\overline{r} \cdot \overline{m} = \overline{rm}.$$

(b) The sequence $m_1, \dots, m_n$ generates M as an R-module if and only if $\overline{m}_1, \dots, \overline{m}_n \in M/\mathfrak{m}M$ spans $M/\mathfrak{m}M$ as a $\mathfrak{K}$-vector space.

(c) $m_1, \dots, m_n \in M$ is a minimal generating sequence for M if and only if $\overline{m}_1, \dots, \overline{m}_n \in M/\mathfrak{m}M$ is a basis for $M/\mathfrak{m}M$ over $\mathfrak{K}$. In particular, every minimal generating sequence for M has the same number of elements, namely

$$\dim_{\mathfrak{K}}(M/\mathfrak{m}M).$$

Proof. $M/\mathfrak{m}M$ is a $\mathfrak{K}$-module by Fact 3.4.5. The $\mathfrak{K}$-vector space axioms follow directly from the R-module axioms. For example we have

$$\overline{r}(\overline{s} \cdot \overline{m}) = \overline{r} \cdot (\overline{sm}) = \overline{r(sm)} = \overline{(rs)m} = \overline{rs} \cdot \overline{m} = (\overline{r}\,\overline{s}) \cdot \overline{m}.$$

Since M is finitely generated we let $m_1, \dots, m_n$ generate M over R. For any $m \in M$, there exist $r_1, \dots, r_n \in R$ such that $m = \sum_{i=1}^n r_i m_i$. Therefore for any $\overline{m} \in M/\mathfrak{m}M$ there exist $\overline{r_1}, \dots, \overline{r_n} \in \mathfrak{K}$ such that

$$\overline{m} = \overline{\sum_{i=1}^n r_i m_i} = \sum_{i=1}^n \overline{r}_i \overline{m}_i$$

so $\overline{m}_1, \dots, \overline{m}_n$ spans $M/\mathfrak{m}M$ over $\mathfrak{K}$, completing the proof of part (a) as well as the forward implication of part (b).

Now assume $\overline{m}_1, \dots, \overline{m}_n$ spans $M/\mathfrak{m}M$ over $\mathfrak{K}$. We claim $M = \langle m_1, \dots, m_n \rangle + \mathfrak{m}M$. Certainly the reverse containment holds, because $\langle m_1, \dots, m_n \rangle, \mathfrak{m}M \subseteq M$ as submodules. On the other hand, if we let $m \in M$ be arbitrary, then our spanning assumption implies there exist $\overline{r}_1, \dots, \overline{r}_n \in \mathfrak{K}$ such that

$$\overline{m} = \sum_{i=1}^n \overline{r}_i \overline{m}_i = \overline{\sum_{i=1}^n r_i m_i}.$$

This yields

$$\begin{aligned} & m - \sum_{i=1}^n r_i m_i \in \mathfrak{m}M \\ \Longrightarrow\ & m = \sum_{i=1}^n r_i m_i + y && \text{for some } y \in \mathfrak{m}M \\ \Longrightarrow\ & m \in \langle m_1, \dots, m_n \rangle + \mathfrak{m}M \end{aligned}$$

proving the claim. Therefore we apply Corollary 3.4.13 to conclude $M = \langle m_1, \ldots, m_n \rangle$, so M is finitely generated, proving part (b).

For the forward direction of part (c), assume $m_1, \ldots, m_n$ is a minimal generating sequence for M. Part (b) then implies $\overline{m}_1, \ldots, \overline{m}_n$ spans $M/\mathfrak{m}M$. If we suppose for the sake of contradiction that $\overline{m}_1, \ldots, \overline{m}_n$ is not a basis, then we can rearrange them if necessary to assume $\overline{m}_2, \ldots, \overline{m}_n$ spans $M/\mathfrak{m}M$. Another application of part (b) implies $m_2, \ldots, m_n$ is a generating sequence for M over R, contradicting our minimality assumption. Therefore $\overline{m}_1, \ldots, \overline{m}_n$ is a basis for $M/\mathfrak{m}M$ as a $\mathfrak{K}$-vector space.

On the other hand, let us assume that $\overline{m}_1, \ldots, \overline{m}_n$ is a basis for $M/\mathfrak{m}M$ over $\mathfrak{K}$. Equivalently, we have $\overline{m}_1, \ldots, \overline{m}_n$ is a minimally spanning set for $M/\mathfrak{m}M$ over $\mathfrak{K}$. Then part (b) implies $m_1, \ldots, m_n$ is a generating sequence for M as an R-module. Suppose $m_1, \ldots, m_n$ is not minimal. That is, assume $m_2, \ldots, m_n$ is a generating sequence for M, after some rearrangement of the m_i's if necessary. Again applying part (b) we know $\overline{m}_2, \ldots, \overline{m}_n$ spans $M/\mathfrak{m}M$ as a $\mathfrak{K}$-vector space, contradicting the fact that $\overline{m}_1, \ldots, \overline{m}_n$ is a minimal spanning set. □

Corollary 3.4.18. *Assume $(R, \mathfrak{m}, \mathfrak{K})$ is local and P is a finitely generated projective R-module. Then P is free with $P \cong R^n$ where $n = \dim_{\mathfrak{K}}(P/\mathfrak{m}P)$.*

Proof. By Corollary 3.4.13 there exist $p_1, \ldots, p_n \in P$ such that they form a minimal generating sequence for P, where $n = \dim_{\mathfrak{K}}(P/\mathfrak{m}P)$. Note this implies $P/\mathfrak{m}P$ is an n-dimensional $\mathfrak{K}$-vector space and therefore $P/\mathfrak{m}P \cong \mathfrak{K}^n$. We therefore have the following well-defined, surjective R-module homomorphism.

$$\begin{aligned} \tau : R^n &\twoheadrightarrow P \\ e_i &\longmapsto p_i \\ \textstyle\sum_{i=1}^n r_i e_i &\longmapsto \textstyle\sum_{i=1}^n r_i p_i \end{aligned}$$

We may therefore define a short exact sequence $0 \longrightarrow \operatorname{Ker}(\tau) \xrightarrow{\subseteq} R^n \xrightarrow{\tau} P \longrightarrow 0$ and from here it suffices to show that $\operatorname{Ker}(\tau) = \{0\}$, thereby proving P is isomorphic to a free module. Let $K = \operatorname{Ker}(\tau)$. Since P is projective, the short exact sequence splits and we write

$$\begin{aligned} R^n \cong K \oplus P &\xrightarrow{\pi} K \\ (x, p) &\longmapsto x \end{aligned}$$

where π is a well-defined, surjective R-module homomorphism, so K is also finitely generated. We will use this to apply Nakayama's Lemma. We have a string of isomorphisms.

$$\mathfrak{K}^n \cong \left(\frac{R}{\mathfrak{m}R}\right)^n \cong \frac{R^n}{\mathfrak{m}R^n} \cong \frac{K \oplus P}{\mathfrak{m}(K \oplus P)} \cong \frac{K}{\mathfrak{m}K} \oplus \frac{P}{\mathfrak{m}P} \cong \frac{K}{\mathfrak{m}K} \oplus \mathfrak{K}^n$$

Since isomorphic vector spaces have the same dimension we have $n = \dim_{\mathfrak{K}}(K/\mathfrak{m}K) + n$, implying $\dim_{\mathfrak{K}}(K/\mathfrak{m}K) = 0$ and therefore $K/\mathfrak{m}K = 0$. It follow from Nakayama's Lemma that $\operatorname{Ker}(\tau) = K = 0$. □

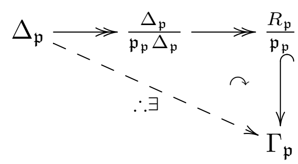
Lemma 3.4.19. *Assume R is noetherian, let Γ and Δ be non-zero, finitely generated R-modules, and let $I \leq R$ such that $\operatorname{Supp}_R(\Delta) = V(I)$. If $I \subseteq \operatorname{ZD}_R(\Gamma)$, then $\operatorname{Hom}_R(\Delta, \Gamma) \neq 0$.*

Proof. By Corollary 3.4.2 there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass}_R(\Gamma)$ such that $I \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is an associated prime, by Remark 3.2.18 there exists an injective R-module homomorphism $R/\mathfrak{p} \hookrightarrow \Gamma$ and the exactness of localization gives the existence of an injection $(R/\mathfrak{p})_{\mathfrak{p}} \hookrightarrow \Gamma_{\mathfrak{p}}$. Moreover, by Theorem 3.1.4 we have $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \cong (R/\mathfrak{p})_{\mathfrak{p}}$ and therefore we have an injection $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \hookrightarrow \Gamma_{\mathfrak{p}}$.

Since $I \subseteq \mathfrak{p}$, we know $\mathfrak{p}$ is an element of the variety $V(I) = \operatorname{Supp}_R(\Delta)$. Therefore $\Delta_\mathfrak{p} \neq 0$. This is finitely generated over $R_\mathfrak{p}$ (since Δ finitely generated over R), so by Nakayama's Lemma we have

$$\frac{\Delta_\mathfrak{p}}{\mathfrak{p}_\mathfrak{p}\Delta_\mathfrak{p}} \neq 0 \tag{3.4.19.1}$$

Similar to the context of Corollary 3.4.17, the module in (3.4.19.1) gives a vector space over $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$. Since one can always surject from a non-zero vector space onto the underlying field, we have the following commutative diagram.



Since $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is non-zero, we have exhibited a non-zero R-module homomorphism in $\operatorname{Hom}_{R_\mathfrak{p}}(\Delta_\mathfrak{p}, \Gamma_\mathfrak{p})$ which is therefore non-zero. Therefore $\operatorname{Hom}_R(\Delta, \Gamma) \neq 0$ since $\operatorname{Hom}_{R_\mathfrak{p}}(\Delta_\mathfrak{p}, \Gamma_\mathfrak{p}) \cong \operatorname{Hom}_R(\Delta, \Gamma)$. □

Corollary 3.4.20. *Let R be noetherian, let M be a non-zero, finitely generated R-module, and let $I \lneq R$ be a proper ideal. If $\operatorname{depth}_R(I; M) = 0$, then $\operatorname{Hom}_R(R/I, M) \neq 0$.*

Proof. Since $\operatorname{depth}_R(I; M) = 0$, it follows that I is composed entirely of zero-divisors on M. Since $I \neq R$, the module R/I is non-zero and is also finitely generated over R. Therefore by Remark 3.2.11 we have $\operatorname{Supp}_R(R/I) = V(I)$ and hence $\operatorname{Hom}_R(R/I, M)$ is non-zero by Lemma 3.4.19. □

Lemma 3.4.21. *Let R be noetherian and let M and N be R-modules such that M is finitely generated. Then*

$$\operatorname{Ass}_R(\operatorname{Hom}_R(M, N)) = \operatorname{Supp}_R(M) \cap \operatorname{Ass}_R(N).$$

Proof. If $M = 0$, then $\operatorname{Hom}_R(M, N) = 0$, so $\operatorname{Ass}_R(\operatorname{Hom}_R(M, N)) = \operatorname{Ass}_R(0) = \emptyset$, $\operatorname{Supp}_R(M) = \emptyset$, and therefore $\operatorname{Supp}_R(M) \cap \operatorname{Ass}_R(N) = \emptyset$. Hence the conclusion holds in this case. The conclusion holds by the same reasoning if $N = 0$, so assume without loss of generality that $M, N \neq 0$.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \notin \operatorname{Supp}_R(M)$ and we will argue that $\mathfrak{p} \notin \operatorname{Ass}_R(\operatorname{Hom}_R(M, N))$. Since $M_\mathfrak{p} = 0$ we have $\operatorname{Hom}_R(M, N)_\mathfrak{p} \cong \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$ by Proposition 3.1.8. Therefore

$$\mathfrak{p} \notin \operatorname{Supp}_R(\operatorname{Hom}_R(M, N)) \supseteq \operatorname{Ass}_R(\operatorname{Hom}_R(M, N)).$$

We've shown $\mathfrak{p}$ not in the support of M implies $\mathfrak{p}$ not in the associated primes of the homomorphism module. This is the contrapositive of

$$\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Hom}_R(M, N)) \implies \mathfrak{p} \in \operatorname{Supp}_R(M)$$

so $\operatorname{Ass}_R(\operatorname{Hom}_R(M, N)) \subseteq \operatorname{Supp}_R(M)$.

Now notice M finitely generated implies there exists some $t \geq 1$ such that we can map surjectively from R^t onto M. The left-exactness of $\operatorname{Hom}_R(-, N)$ along with Hom-cancellation and Example 3.1.2 gives

$$\operatorname{Hom}_R(M, N) \hookrightarrow \operatorname{Hom}_R(R^t, N) \cong N^t.$$

Therefore by Remark 3.2.26 and Lemma 3.2.28 we have

$$\operatorname{Ass}_R(\operatorname{Hom}_R(M, N)) \subseteq \operatorname{Ass}_R(N^t) = \operatorname{Ass}_R(N)$$

which proves $\operatorname{Ass}_R(\operatorname{Hom}_R(M, N)) \subseteq \operatorname{Supp}_R(M) \cap \operatorname{Ass}_R(N)$.

For the reverse containment, first consider that in the degenerate case when $\mathrm{Supp}_R(M) \cap \mathrm{Ass}_R(N)$ is empty, we have nothing to show, since every set contains the empty set. Now assume without loss of generality that $\mathfrak{p}$ lies in the intersection. We claim

$$\mathrm{Hom}_R(M, R/\mathfrak{p}) \neq 0. \tag{3.4.21.1}$$

To put things in the notation of Lemma 3.4.19, set $I = \mathrm{Ann}_R(M)$, $\Gamma = R/\mathfrak{p}$, and $\Delta = M$. By Remark 3.2.11 we have

$$\mathrm{Supp}_R(\Delta) = V(\mathrm{Ann}_R(M)) = \mathrm{Supp}_R(M)$$

which implies $I \subseteq \mathfrak{p}$, since $\mathfrak{p} \in \mathrm{Supp}_R(M)$. Therefore $I \cdot R/\mathfrak{p} = 0$, so $I \subseteq \mathrm{ZD}(R/\mathfrak{p})$, and thus (3.4.21.1) holds by Lemma 3.4.19.

Next we claim if $\alpha \in \mathrm{Hom}_R(M, R/\mathfrak{p})$ is non-zero, then $\mathrm{Ann}_R(\alpha) = \mathfrak{p}$. If $x \in \mathfrak{p}$, then $x \cdot R/\mathfrak{p} = 0$ and in particular $x \cdot \alpha(m) = 0$ for all $m \in M$. Equivalently, this means $(x\alpha)(m) = 0$ for all $m \in M$, so $x \in \mathrm{Ann}_R(\alpha)$ and we have shown $\mathfrak{p} \subseteq \mathrm{Ann}_R(\alpha)$. On the other hand, since $\alpha \neq 0$, let $m \in M$ such that $\alpha(m) \neq 0$. Notice also that since $\mathfrak{p}$ is prime and $0 \neq \alpha(m) \in R/\mathfrak{p}$, then $\mathrm{Ann}_R(\alpha(m)) = \mathfrak{p}$. Now for any $y \in \mathrm{Ann}_R(\alpha)$, we have $y \cdot \alpha(m) = (y\alpha)(m) = 0$, implying $y \in \mathrm{Ann}_R(\alpha(m)) = \mathfrak{p}$ and we conclude $\mathrm{Ann}_R(\alpha) = \mathfrak{p}$ by mutual containment.

It now suffices to show $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Hom}_R(M, N))$. By (3.4.21.1), there exists some element $\alpha \in \mathrm{Hom}_R(M, R/\mathfrak{p}) \setminus \{0\}$ and by our second claim $\mathrm{Ann}_R(\alpha) = \mathfrak{p}$. Define the R-module homomorphism

$$\begin{aligned} \phi : R &\longrightarrow \mathrm{Hom}_R(M, R/\mathfrak{p}) \\ r &\longmapsto r\alpha \end{aligned}$$

and note $\mathrm{Ker}(\phi) = \mathrm{Ann}_R(\alpha) = \mathfrak{p}$. Therefore by the First Isomorphism Theorem we have the injective R-module homomorphism

$$\begin{aligned} \overline{\phi} \colon R/\mathfrak{p} &\longrightarrow \mathrm{Hom}_R(M, R/\mathfrak{p}) \\ \overline{r} &\longmapsto r\alpha. \end{aligned}$$

Moreover since $\mathfrak{p} \in \mathrm{Ass}_R(N)$, there also exists an injective R-module homomorphism $R/\mathfrak{p} \hookrightarrow N$ (3.2.18). Since $\mathrm{Hom}_R(M, -)$ is right exact we have the horizontal injection in the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(M, R/\mathfrak{p}) & \hookrightarrow & \mathrm{Hom}_R(M, N) \\ \uparrow & \curvearrowright & \nearrow \therefore\exists \\ R/\mathfrak{p} & & \end{array}$$

and we conclude $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Hom}_R(M, N))$ by Remark 3.2.18. □

Example 3.4.22. Recall the running example we began in Note 3.2.16. We can see our new lemma in action.

	Supp_R	Ass_R
$R = \mathbb{K}[x, y]$	$\mathrm{Spec}(R)$	$\{0\}$
$A = R/((x, y)R)^n$	$\{(x, y)R\}$	$\{(x, y)R\}$
$B = R/(x^n, y^m)R$	$\{(x, y)R\}$	$\{(x, y)R\}$
$C = R/(x^2, xy)R$	$V(xR)$	$\{xR, (x, y)R\}$

$$\mathrm{Ass}_R(\mathrm{Hom}_R(A, R)) = \mathrm{Supp}_R(A) \cap \mathrm{Ass}_R(R) = \{(x, y)\} \cap \{0\} = \emptyset$$
$$\therefore \mathrm{Hom}_R(A, R) = 0 \quad (3.2.21)$$
$$\mathrm{Ass}_R(\mathrm{Hom}_R(B, C)) = \mathrm{Supp}_R(B) \cap \mathrm{Ass}_R(C) = \{(x, y)R\} \cap \{xR, (x, y)R\} = \{(x, y)R\} \neq \emptyset$$
$$\therefore \mathrm{Hom}_R(B, C) \neq 0 \quad (3.2.21)$$

3.5 Regular Sequences and Ext

We begin the section with a re-characterization of regular sequences and a characterization of the radical ideal in terms of the intersection of prime ideals (see Lemma 3.5.7). After stating four facts, we use them to prove Lemma 3.5.13 before finally achieving our goal of the chapter by proving Theorem 3.5.16.

Discussion 3.5.1. We have already defined *M-regular* elements and sequences (Definition 2.2.1). We can also give a different characterization of M-regular sequences.

Assume $M \neq 0$ is a finitely generated R-module. We claim a sequence $a_1, \ldots, a_n \in R$ is M-regular if and only if $a_1 \notin \mathrm{ZD}_R(M)$, $a_i \notin \mathrm{ZD}_R(M/(a_1, \ldots, a_{i-1})M)$ for all $i = 2, \ldots, n$, and $(a_1, \ldots, a_n)M \neq M$.

Proof. One implication is trivial from Definition 2.2.1, so assume $a_1 \notin \mathrm{ZD}_R(M)$, furthermore that $a_i \notin \mathrm{ZD}_R(M/(a_1, \ldots, a_{i-1})M)$ for all $i = 2, \ldots, n$, and $(a_1, \ldots, a_n)M \neq M$. Suppose for the sake of contradiction that $a_1 M = M$. This implies

$$(a_1, \ldots, a_n)M \subseteq M = a_1 M \subseteq (a_1, \ldots, a_n)M.$$

This contradicts the assumption that $(a_1, \ldots, a_n)M \neq M$. Thus $a_1 M \neq M$.

Now suppose $a_i(M/(a_1, \ldots, a_{i-1})M) = M/(a_1, \ldots, a_{i-1})M$ for some $i \geq 2$. Then we have

$$\frac{M}{(a_1, \ldots, a_{i-1})M} = a_i \cdot \frac{M}{(a_1, \ldots, a_{i-1})M} = \frac{(a_1, \ldots, a_i)M}{(a_1, \ldots, a_{i-1})M}$$

which implies

$$(a_1, \ldots, a_n)M \subseteq M = (a_1, \ldots, a_i)M \subseteq (a_1, \ldots, a_n)M.$$

However, this again contradicts the assumption $(a_1, \ldots, a_n)M \neq M$ and we conclude

$$a_i(M/(a_1, \ldots, a_{i-1})M) \neq M/(a_1, \ldots, a_{i-1})M$$

for all $i \in [n]$. Hence, $a_1, \ldots, a_n$ is M-regular. □

Discussion 3.5.2. If $(R, \mathfrak{m})$ is a local ring and $M \neq 0$ is a finitely generated R-module, then Nakayama's Lemma implies $\mathfrak{a}M \neq M$ for all $\mathfrak{a} \lneq R$. In particular, a sequence $a_1, \ldots, a_n \in \mathfrak{a} \lneq R$ is M-regular if and only if $a_1 \notin \mathrm{ZD}_R(M)$ and $a_i \notin \mathrm{ZD}_R(M/(a_1, \ldots, a_{n-1})M)$ for all $i = 2, \ldots, n$, by Discussion 3.5.1.

Example 3.5.3. Let $\mathbb{K}$ be a field.

(a) Let $R = \mathbb{K}[X_1, \ldots, X_d]$ for some $d \geq 1$. We claim for any $n \leq d$, the sequence $X_1, \ldots, X_n \in R$ is R-regular. In the case when $n = 1$, note that $X_1 \notin \mathrm{ZD}_R(R)$ and $R/X_1R \cong \mathbb{K}[X_2, \ldots, X_n] \neq 0$, so $X_1 R \neq R$ and thus X_1 is R-regular.

Now assume $d \geq n \geq 2$ and the sequence $X_1, \ldots, X_{n-1}$ is R-regular. To show $X_1, \ldots, X_n$ to be regular we need only point out that $X_n \notin \mathrm{ZD}_R(R/(X_1, \ldots, X_{n-1})R)$ and that

$$\frac{R}{(X_1, \ldots, X_n)R} \cong \mathbb{K}[X_{n+1}, \ldots, X_d] \neq 0$$

implying by Nakayama's Lemma that

$$(X_1, \ldots, X_n)R \neq R.$$

Therefore $X_1, \ldots, X_n$ is R-regular by Discussion 3.5.1.

(b) Consider the ring $\mathbb{Z}$. For any $n \in \mathbb{Z}$ with $n \geq 2$, n is $\mathbb{Z}$-regular, because n is a non-zero, non-unit element of an integral domain. However, we can show that $\mathbb{Z}$ has no regular sequences of length two.

Suppose $m, n \in \mathbb{Z}$ is $\mathbb{Z}$-regular. Then $m \neq 0$ is a non-unit and to build a regular sequence n must be $\mathbb{Z}/m\mathbb{Z}$-regular. Two things can go wrong:

(1) If $(m, n) = 1$, then
$$n \cdot \mathbb{Z}/m\mathbb{Z} = (m, n)\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/m\mathbb{Z}$$
so n is not $\mathbb{Z}/m\mathbb{Z}$-regular.

(2) If $(m, n) = d \geq 2$, then $d|m$, $d|n$, and we have cases to check. For instance if $m|n$, then $n \cdot \mathbb{Z}/m\mathbb{Z} = 0$ so m, n is not regular. If $m \nmid n$, then $\overline{d} \in \mathbb{Z}/m\mathbb{Z}$ is non-zero and $n \cdot \overline{d} = \overline{0} \in \mathbb{Z}/m\mathbb{Z}$. Therefore n is a zero-divisor and thus m, n fails to be regular.

(c) Any field $\mathbb{K}$ has no regular sequences, because $\mathbb{K} \setminus \{0\} = \mathbb{K}^\times$, so $k\mathbb{K} = \mathbb{K}$ for all $k \neq 0$.

(d) The quotient ring $R = \mathbb{K}[x]/(x^2)$ has no regular sequences, because the non-units of R are of the form $a\overline{x}$ for some $a \in \mathbb{K}$ and $a\overline{x} \cdot \overline{x} = 0$ implies $a\overline{x} \in \mathrm{ZD}_R(R)$.

Definition 3.5.4. An M-regular sequence $x_1, \ldots, x_n \in I \leq R$ is a *maximal M-regular sequence* in I if for any $x_{n+1} \in I$, the sequence $x_1, \ldots, x_{n+1}$ is not M-regular.

Remark 3.5.5. If R is noetherian and M an R-module, then for any $I \lneq R$, M has a maximal regular sequence in I. Moreover every M-regular sequence in $I \leq R$ extends to a maximal M-regular sequence in I, which we prove here.

Proof. Let $\underline{x} = x_1, \ldots, x_n \in I$ be an M-regular sequence and suppose for any $N > n$ and any $x_{n+1}, \ldots, x_N \in I$ such that the sequence $\underline{\underline{x}} = x_1, \ldots, x_N$ is regular, $\underline{\underline{x}}$ is not maximal. Then for any $N > n$ there exists some $x_{N+1} \in I$ such that $\underline{\underline{x}}, x_{N+1}$ is regular. Define $I_k = \langle x_1, \ldots, x_k \rangle$ and consider the chain
$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
which we claim is made of proper containments. We need to show $x_{k+1} \notin I_k$ for any k. If we suppose otherwise, then $x_{k+1} \cdot M/I_kM = 0$, which violates the regularity of the sequence in a big way. Therefore we have exhibited a chain
$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots \subseteq R$$
that does not stabilize, contradicting the fact that R is noetherian. Therefore $\underline{x}$ must extend to a maximal regular sequence. □

Remark 3.5.6. If $(R, \mathfrak{m})$ is a local, noetherian, ring and $M \neq 0$ is a finitely generated R-module, then we have an algorithm for finding maximal M-regular sequences.

Step 1: If $\mathfrak{m} \in \mathrm{Ass}_R(M)$, then $\mathfrak{m} \subseteq \mathrm{ZD}_R(M)$ by Corollary 3.4.3, so the empty set is a maximal M-regular sequence and we can therefore stop.

Step 2: Assume $\mathfrak{m} \notin \mathrm{Ass}_R(M)$, i.e., $\mathfrak{m} \not\subseteq \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Ass}_R(M)$. Then by prime avoidance
$$\mathfrak{m} \not\subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p} \quad \Longrightarrow \quad \mathrm{ZD}_R(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p} \subsetneq \mathfrak{m}.$$
Hence there exits an element $x_1 \in \mathfrak{m} \setminus \mathrm{ZD}_R(M)$ (i.e., $x_1 \in \mathfrak{m} \setminus \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R(M)$).

Step 3: Repeat Steps 1 and 2 with the module M/x_1M in place of M. If $\mathfrak{m} \in \mathrm{Ass}_R(M/x_1M)$, then x_1 is a maximal M-regular sequence, so we stop. Otherwise there exists some $x_2 \in \mathfrak{m}$ such that $x_2 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R(M/x_1M)$.

Step 4: Repeat Steps 1 and 2 with $M/(x_1, x_2)M$. And so on.

By the proof of Remark 3.5.5, this process must terminate after finitely many steps.

Associated primes are indispensable for the proof of Theorem 3.5.16, though their use is a bit hidden. The point of the following lemma is to give in part (b) a context in which they are a bit easier to write down.

Lemma 3.5.7. *Let R be a non-zero, noetherian, commutative ring with identity and let $I \lneq R$ be a proper ideal.*

(a) $\operatorname{rad}(I) = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(R/I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \operatorname{Min}_R(R/I)} \mathfrak{p}$

(b) If I is the intersection of a finite number of prime ideals, then

$$\operatorname{Ass}_R(R/I) = \operatorname{Min}_R(R/I) = \{\textit{minimum elements in the intersection defining } I\}.$$

(c) If I is an intersection of prime ideals, then it is the intersection of a finite number of prime ideals.

Proof. (a). This is justified by the following string of containments.

$$\begin{aligned}
\operatorname{rad}(I) &= \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} && 3.2.6 \\
&= \bigcap_{\mathfrak{p} \in \operatorname{Supp}_R(R/I)} \mathfrak{p} && 3.2.11 \\
&\subseteq \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(R/I)} \mathfrak{p} && 3.2.19 \\
&\subseteq \bigcap_{\mathfrak{p} \in \operatorname{Min}_R(R/I)} \mathfrak{p} && 3.3.16 \\
&\subseteq \bigcap_{\mathfrak{p} \in \operatorname{Supp}_R(R/I)} \mathfrak{p} && 3.3.13
\end{aligned}$$

Hence we have equality at every step.

(b). Assume $I = \bigcap_{i=1}^{n} \mathfrak{p}_i$ and re-order if necessary to assume $\mathfrak{p}_1, \ldots, \mathfrak{p}_j$ are the minimal elements in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ with respect to containment. Therefore $I = \bigcap_{i=1}^{j} \mathfrak{p}_i$ and we first claim $\mathfrak{p}_1, \ldots, \mathfrak{p}_j \in \operatorname{Min}_R(R/I)$. By 4 in Remark 3.2.11 and Proposition 3.3.13 we have

$$\operatorname{Min}_R(R/I) = \operatorname{Min}(\operatorname{Supp}_R(R/I)) = \operatorname{Min}(V(I))$$

(see also Definition 3.3.16) so it suffices to show $\mathfrak{p}_i$ is minimal in $V(I)$ for each $i = [j]$. We know $\mathfrak{p}_k \in V(I)$ for any $k \in [j]$ since each contains the intersection $\cap_{i=1}^{j} \mathfrak{p}_i$, which is precisely I. To show minimality suppose $\mathfrak{p} \in V(I)$ such that $\mathfrak{p} \subseteq \mathfrak{p}_i$. Observe the following with the product of prime ideals.

$$\mathfrak{p}_1 \cdots \mathfrak{p}_j \subseteq \bigcap_{k=1}^{j} \mathfrak{p}_k = I \subseteq \mathfrak{p} \subseteq \mathfrak{p}_i$$

The fact that $\mathfrak{p}$ is prime implies there exists some index $l \in [j]$ such that $\mathfrak{p}_l \subseteq \mathfrak{p} \subseteq \mathfrak{p}_i$, but since $\mathfrak{p}_i$ is minimal among the $\mathfrak{p}_k$'s, we know $\mathfrak{p}_i \subseteq \mathfrak{p}_l$. Therefore we have

$$\mathfrak{p}_l \subseteq \mathfrak{p} \subseteq \mathfrak{p}_i \subseteq \mathfrak{p}_l$$

forcing equality at every step and hence $\mathfrak{p} = \mathfrak{p}_i$. By this argument and by Definition 3.3.16, we have shown

$$\begin{aligned}\{\text{minimal elements in the intersection defining } I\} &= \{\mathfrak{p}_1, \ldots, \mathfrak{p}_j\} \\ &\subseteq \operatorname{Min}(V(I)) \\ &= \operatorname{Min}_R(R/I) \\ &\subseteq \operatorname{Ass}_R(R/I).\end{aligned}$$

So to complete the proof of this part it suffices to show $\operatorname{Ass}_R(R/I) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_j\}$. Let $\mathfrak{p} \in \operatorname{Ass}_R(R/I)$ and by definition of being an associated prime there exists some $\overline{0} \neq \overline{x} \in R/I$ such that $\mathfrak{p} = \operatorname{Ann}_R(\overline{x})$. By definition of what it means to be zero in the quotient module R/I we have

$$\mathfrak{p}x \subseteq I = \bigcap_{i=1}^{j} \mathfrak{p}_j.$$

Since $\overline{x}$ is non-zero, $x \notin I$, so there exists some $k \in [j]$ such that $x \notin \mathfrak{p}_k$. Rewriting our last line we have

$$\mathfrak{p}x \subseteq I = \bigcap_{i=1}^{j} \mathfrak{p}_j \subseteq \mathfrak{p}_k$$

where the fact that $\mathfrak{p}_k$ is prime implies $\mathfrak{p} \subseteq \mathfrak{p}_k$ (if x isn't in $\mathfrak{p}_k$, then $\mathfrak{p}$ must be). Since we have already shown the minimal elements of the intersection defining I ($\mathfrak{p}_k$ in particular) are also in $\operatorname{Min}_R(R/I)$, this implies $\mathfrak{p} = \mathfrak{p}_k$ and completes the proof of part (b).

(c). If I is the intersection of prime ideals, say $\mathfrak{p}_\lambda$ for $\lambda \in \Lambda$, then we have

$$\bigcap_{\lambda \in \Lambda} \mathfrak{p}_\lambda = I \subseteq \operatorname{rad}(I) = \bigcap_{\mathfrak{p} \in \operatorname{Min}_R(R/I)} \mathfrak{p} \subseteq \bigcap_{\lambda \in \Lambda} \mathfrak{p}_\lambda$$

and equality at every step follows. Since $\operatorname{Min}_R(R/I)$ is finite by Corollary 3.3.4, part (c) holds. □

Example 3.5.8. Let $\mathbb{K}$ be a field, let $R = \mathbb{K}[x,y]_{(x,y)}$, which is a local ring with unique maximal ideal $\mathfrak{m} = (x,y)R$, and define the R-module $M = R/(xy)R$. We will use the steps in Remark 3.5.6 to find a maximal M-regular sequence. We know $(x)R = \operatorname{Ann}_R(\overline{y})$ and $(y)R = \operatorname{Ann}_R(\overline{x})$, for $\overline{x}, \overline{y} \in M$, so $(x)R$, $(y)R \in \operatorname{Ass}_R(M)$. It is straight forward to show $(xy)R = (x)R \cap (y)R$, so we actually have $\operatorname{Ass}_R(M) = \{(x)R,\ (y)R\}$ by Lemma 3.5.7(b). Since $\mathfrak{m} \notin \operatorname{Ass}_R(M)$, we need to find an element

$$a_1 \in (x,y)R \setminus [(x)R \cup (y)R].$$

That is, we want to find $a_1 = fx + gy$ such that $x \nmid a_1$ and $y \nmid a_1$, for some $f, g \in R$. In particular we can take $a_1 = x - y$.

For Step 3, we first repeat Step 1 with the module M/a_1M, so we need to determine if $\mathfrak{m} \in \operatorname{Ass}_R(M/a_1M)$. Observe

$$\frac{M}{a_1M} = \frac{R/(xy)R}{(x-y)\cdot R/(xy)R} \cong \frac{R}{(x-y, xy)R} \cong \frac{R/(x-y)R}{(xy)\cdot R/(x-y)R}$$

where

$$\frac{R}{(x-y)R} = \frac{\mathbb{K}[x,y]_{(x,y)}}{(x-y)\mathbb{K}[x,y]_{(x,y)}} \cong \mathbb{K}[x]_{(x)}.$$

Colloquially, the last isomorphism above holds because setting $x - y = 0$ is the same as setting $x = y$. This gives

$$\frac{M}{a_1M} \cong \frac{\mathbb{K}[x]_{(x)}}{x^2 \cdot \mathbb{K}[x]_{(x)}}. \tag{3.5.8.1}$$

We will now argue

$$\operatorname{Ann}_R\left(\overline{x} \in \frac{M}{a_1 M}\right) = (x,y)R$$

and will thereby have showed $\mathfrak{m} \in \operatorname{Ass}_R(M/a_1M)$. Since $x = y$ we have

$$y \cdot \overline{x} = x \cdot \overline{x} = \overline{x^2} = 0 \in M/a_1M$$

by (3.5.8.1) and therefore $(x,y)R \subseteq \operatorname{Ann}_R(\overline{x})$. Moreover we know $\overline{x} \neq \overline{0} \in M/a_1M$ by (3.5.8.1), implying $\operatorname{Ann}_R(\overline{x}) \neq R$ and in fact the maximality of $(x,y)R$ implies $\mathfrak{m} = (x,y)R = \operatorname{Ann}_R(\overline{x})$. Hence $\mathfrak{m} \in \operatorname{Ass}_R(M/a_1M)$ and we can stop. That is, $x - y$ is a maximal M-regular sequence in M of length 1.

We shall assume the following fact without proof (for the moment) and use it to prove the subsequent fact. (See Discussion 5.2.3 for existence of the induced maps.)

Fact 3.5.9. *Let $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ R-module homomorphisms. For any $i \geq 0$, there exist R-module homomorphisms*

$$\operatorname{Ext}^i_R(A,g) : \operatorname{Ext}^i_R(A,B) \longrightarrow \operatorname{Ext}^i_R(A,B')$$
$$\operatorname{Ext}^i_R(f,B) : \operatorname{Ext}^i_R(A',B) \longrightarrow \operatorname{Ext}^i_R(A,B).$$

If $f' : A' \longrightarrow A''$ and $g' : B' \longrightarrow B''$ are also two R-module homomorphisms, then the following diagrams commute.

$$\begin{array}{ccc} \operatorname{Ext}^i_R(A,B) & \xrightarrow{\operatorname{Ext}^i_R(A,g)} & \operatorname{Ext}^i_R(A,B') \\ & \searrow^{\operatorname{Ext}^i_R(A,g'\circ g)} \quad \curvearrowright & \downarrow{\scriptstyle \operatorname{Ext}^i_R(A,g')} \\ & & \operatorname{Ext}^i_R(A,B'') \end{array} \qquad \begin{array}{ccc} \operatorname{Ext}^i_R(A'',B) & \xrightarrow{\operatorname{Ext}^i_R(f',B)} & \operatorname{Ext}^i_R(A',B) \\ & \searrow^{\operatorname{Ext}^i_R(f'\circ f,B)} \quad \curvearrowright & \downarrow{\scriptstyle \operatorname{Ext}^i_R(f,B)} \\ & & \operatorname{Ext}^i_R(A,B) \end{array}$$

Colloquially, we are saying that $\operatorname{Ext}^i_R(A,-)$ and $\operatorname{Ext}^i_R(-,B)$ each respect compositions.

Next, we use this fact to establish the following.

Fact 3.5.10. *If A, A', B, and B' are all R-modules, then*

$$\operatorname{Ext}^i_R(A, 0^B_{B'}) = 0 = \operatorname{Ext}^i_R(0^A_{A'}, B)$$

where $0^B_{B'}$ denotes the zero map from B into B' and $0^A_{A'}$ denotes the zero map from A into A'.

Proof. As silly as it looks to write down, we begin with the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{0^A_0} & 0 \\ & \searrow^{0^A_{A'}} \; \curvearrowright & \downarrow{\scriptstyle 0^0_{A'}} \\ & & A' \end{array}$$

From Fact 3.5.9, the following diagram also commutes.

$$\begin{array}{ccc} \mathrm{Ext}^i_R(A',B) & \xrightarrow{\mathrm{Ext}^i_R\left(0^0_{A'},B\right)} & \mathrm{Ext}^i_R(0,B) \\ & {\scriptstyle \mathrm{Ext}^i_R\left(0^A_{A'},B\right)} \searrow \quad \curvearrowright & \Big\downarrow {\scriptstyle \mathrm{Ext}^i_R\left(0^A_0,B\right)} \\ & & \mathrm{Ext}^i_R(A,B) \end{array}$$

Since $\mathrm{Ext}^i_R(0,B) = 0$ by Proposition 2.1.12, the commutivity of the diagram forces $\mathrm{Ext}^i_R(0^A_{A'},B) = 0$ as well.

In a similar fashion, we have two more commutative diagrams.

$$\begin{array}{ccc} B & \xrightarrow{0^B_0} & 0 \\ & {\scriptstyle 0^B_{B'}} \searrow \ \curvearrowright & \Big\downarrow {\scriptstyle 0^0_{B'}} \\ & & B' \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \mathrm{Ext}^i_R(A,B) & \xrightarrow{\mathrm{Ext}^i_R(A,0)} & \mathrm{Ext}^i_R(A,0) \\ & {\scriptstyle \mathrm{Ext}^i_R\left(A,0^B_{B'}\right)} \searrow \quad \curvearrowright & \Big\downarrow {\scriptstyle \mathrm{Ext}^i_R(A,0)} \\ & & \mathrm{Ext}^i_R(A,B') \end{array}$$

From the second diagram we conclude $\mathrm{Ext}^i_R(A, 0^B_{B'}) = 0$ as desired, using similar reasoning as before. $\square$

Similarly, we assume the next fact without proof and use it to prove the subsequent one. See Discussion 5.2.4 for some justification of Fact 3.5.11.

Fact 3.5.11. *Let $r \in R$ and let A, B be R-modules. The multiplication map*

$$\mu^B_r : B \longrightarrow B$$
$$b \longmapsto rb$$

is a well-defined R-module homomorphism (cf. Notation 2.1.8). The induced maps

$$\mathrm{Ext}^i_R(A, \mu^B_r) : \mathrm{Ext}^i_R(A,B) \xrightarrow{r\cdot} \mathrm{Ext}^i_R(A,B)$$
$$\mathrm{Ext}^i_R(\mu^A_r, B) : \mathrm{Ext}^i_R(A,B) \xrightarrow{r\cdot} \mathrm{Ext}^i_R(A,B)$$

from Fact 3.5.9 are the multiplication maps $\mu_r^{\mathrm{Ext}^i_R(A,B)}$ and $\mu_r^{\mathrm{Ext}^i_R(A,B)}$. That is, the map on Ext induced by a multiplication map is itself a multiplication map.

Fact 3.5.12. *Given the two identity maps $\mathrm{id}_A : A \longrightarrow A$ and $\mathrm{id}_B : B \longrightarrow B$, we have*

$$\mathrm{Ext}^i_R(\mathrm{id}_A, B) = \mathrm{id}_{\mathrm{Ext}^i_R(A,B)} = \mathrm{Ext}^i_R(A, \mathrm{id}_B).$$

Proof. This is essentially a corollary of Fact 3.5.11.

$$\mathrm{Ext}^i_R(\mathrm{id}_A, B) = \mathrm{Ext}^i_R(\mu^A_1, B) = \mu_1^{\mathrm{Ext}^i_R(A,B)} = \mathrm{id}_{\mathrm{Ext}^i_R(A,B)}$$
$$\mathrm{Ext}^i_R(A, \mathrm{id}_B) = \mathrm{Ext}^i_R(A, \mu^B_1) = \mu_1^{\mathrm{Ext}^i_R(A,B)} = \mathrm{id}_{\mathrm{Ext}^i_R(A,B)}$$

$\square$

We will see all four of the above facts again in Section 5.3, where we will justify Facts 3.5.9 and 3.5.11, and where we will give alternative proofs of Facts 3.5.10 and 3.5.12. For now, we use them to prove a lemma.

Lemma 3.5.13. *If M and N are R-modules, then for all $i \geq 0$ we have*

$$(\mathrm{Ann}_R(M) \cup \mathrm{Ann}_R(N)) \subseteq \mathrm{Ann}_R\left(\mathrm{Ext}^i_R(M,N)\right).$$

That is, if $x \in R$ such that $xM = 0$ or $xN = 0$, then $x \cdot \mathrm{Ext}^i_R(M,N) = 0$.

Proof. Let $x \in R$ and assume $xM = 0$. Therefore $\mu^M_x = 0^M_M$ and applying Facts 3.5.10 and 3.5.11 we have

$$\mu_x^{\mathrm{Ext}^i_R(M,N)} = \mathrm{Ext}^i_R(\mu^M_x, N) = \mathrm{Ext}^i_R(0^M_M, N) = 0.$$

The proof is done similarly if $y \in R$ such that $yN = 0$. □

Example 3.5.14. If $m, n \in \mathbb{Z}$ (not both 0) and $g = \gcd(m,n)$, then we first claim that

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}.$$

We take a projective resolution of $\mathbb{Z}/m\mathbb{Z}$, truncate it, and apply the functor $\mathrm{Hom}_R(-, \mathbb{Z}/n\mathbb{Z})$.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z} \xrightarrow{\tau} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{(m\cdot)^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow 0$$

By Hom-cancellation the final sequence is isomorphic to

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \longrightarrow \cdots.$$

Thus we compute

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) &= \frac{\mathrm{Ker}\left(\ \mathbb{Z}/n\mathbb{Z} \longrightarrow 0\ \right)}{\mathrm{Im}\left(\ \mathbb{Z}/n\mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z}/m\mathbb{Z}\ \right)} \\ &= \frac{\mathbb{Z}/n\mathbb{Z}}{m \cdot (\mathbb{Z}/n\mathbb{Z})} = \frac{\mathbb{Z}/n\mathbb{Z}}{(m,n)\mathbb{Z}/n\mathbb{Z}} = \frac{\mathbb{Z}/n\mathbb{Z}}{g\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/g\mathbb{Z}.\end{aligned}$$

Note since $\mathbb{Z}/g\mathbb{Z}$ is annihilated by both m and n, it is also annihilated by $m\mathbb{Z} \cup n\mathbb{Z}$, so as separate verification of the conclusion of Lemma 3.5.13 in this special case, we note

$$\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}) \cup \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = m\mathbb{Z} \cup n\mathbb{Z} \subseteq \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/g\mathbb{Z}) \subseteq \mathrm{Ann}_{\mathbb{Z}}(\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})).$$

Remark 3.5.15. In general, if M and N are R-modules, then we at least have

$$\mathrm{Ann}_R(M) \cup \mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R(\mathrm{Ext}^i_R(M,N))$$

but we cannot assume equality here. However, since this is true for all $i \in \mathbb{Z}$, we can strengthen the conclusion of the lemma to write

$$\mathrm{Ann}_R(M) \cup \mathrm{Ann}_R(N) \subseteq \bigcap_{i=0}^{\infty} \mathrm{Ann}_R(\mathrm{Ext}^i_R(M,N)).$$

For instance, in Example 3.5.14 we have

$$\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$$

and therefore

$$\mathrm{Ann}_{\mathbb{Z}}\left(\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})\right) = \mathbb{Z}.$$

Yet notice

$$\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}) + \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = m\mathbb{Z} + n\mathbb{Z} = g\mathbb{Z}.$$

So if m and n are relatively prime, then $g = 1$ and we achieve the equality

$$\mathrm{Ann}_R(\mathbb{Z}/m\mathbb{Z}) \cup \mathrm{Ann}_R(\mathbb{Z}/n\mathbb{Z}) = g\mathbb{Z} = \mathbb{Z} = \mathrm{Ann}_R(\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})).$$

In general, however, $g\mathbb{Z} \neq \mathbb{Z}$. Thus equality in Lemma 3.5.13 is achievable, but does not hold in general.

Here we finally achieve the goal of the chapter by characterizing depth in terms of vanishing Ext modules.

Theorem 3.5.16. *Assume R is noetherian, let $I \leq R$ be an ideal, and assume M is a finitely generated R-module such that $IM \neq M$. Let $n \geq 0$. The following are equivalent.*

(i) $\mathrm{Ext}^i_R(N, M) = 0$ *for all* $i < n$ *and all finitely generated R-modules N satisfying* $\mathrm{Supp}_R(N) \subseteq V(I)$.

(ii) $\mathrm{Ext}^i_R(R/I, M) = 0$ *for all* $i < n$.

(iii) $\mathrm{Ext}^i_R(N, M) = 0$ *for all* $i < n$ *and for <u>some</u> finitely generated R-module N satisfying* $\mathrm{Supp}_R(N) = V(I)$.

(iv) Every M-regular sequence in I of length no greater than n can be extended to an M-regular sequence in I of length equal to n.

(v) M has a regular sequence in I of length n.

Proof. Since $\mathrm{Supp}_R(R/I) = V(I)$ by Remark 3.2.11, (*i*) implies (*ii*) and (*ii*) implies (*iii*) are already done (consider $N = R/I$). To show (*iii*) implies (*iv*), assume N is a finitely generated R-module with $\mathrm{Supp}_R(N) = V(I)$ such that $\mathrm{Ext}^i_R(N, M) = 0$ for all $i < n$. Since the case $n = 0$ is trivial we assume $n \geq 1$. Therefore by assumption $\mathrm{Hom}_R(N, M) \cong \mathrm{Ext}^0_R(N, M) = 0$ and it follows from Lemma 3.4.19 that $I \not\subseteq \mathrm{ZD}_R(M)$, i.e., there exists $a_1 \notin \mathrm{ZD}_R(M) \cap I$. Now we induct on n. If $n = 1$, then we're done since if we start with a sequence of length 0, we can extend to a_1 and if we start with a sequence of length 1, we needn't extend at all.

Assume $n \geq 2$ and the result holds for all finitely generated R-modules M' satisfying $\mathrm{Ext}^i_R(N, M') = 0$ for all $i < n - 1$. Let $a_1, \ldots, a_k \in I$ be an M-regular sequence with $k \leq n$. If $k = n$, then we're done. If $k = 0$, then we already know there exists an M-regular element $a_1 \in I$ from which we would start our sequence, so assume $1 \leq k \leq n - 1$. The sequence

$$0 \longrightarrow M \xrightarrow{a_1 \cdot} M \longrightarrow M/a_1M \longrightarrow 0$$

is exact and yields the long exact sequence (3.5.16.1) from which it follows

$$\mathrm{Ext}^i_R(N, M/a_1M) = 0$$

for all $i < n - 1$ (i.e., whenever $i + 1 < n$). We assumed $a_1, \ldots, a_k$ is M-regular, so $a_2, \ldots, a_k$ is a M/a_1M-regular sequence of length $k - 1 < n - 1$. Therefore under our induction hypothesis we may extend to a M/a_1M-regular sequence $a_2, \ldots, a_k, \ldots, a_n$ of length $n-1$. Hence to conclude $a_1, \ldots, a_n \in I$ is M-regular of length n and thus complete the proof of this implication, it suffices to show $I \cdot M/a_1M \neq M/a_1M$. Indeed since $IM \neq M$ we have

$$I \cdot \frac{M}{a_1M} = \frac{IM}{a_1M} \neq \frac{M}{a_1M}.$$

$$\begin{aligned}
0 \longrightarrow \operatorname{Hom}_R(N,M) \xrightarrow{a_1\cdot} \operatorname{Hom}_R(N,M) \longrightarrow \operatorname{Hom}_R(N,M/a_1M) \\
\longrightarrow \operatorname{Ext}^1_R(N,M) \xrightarrow{a_1\cdot} \cdots
\end{aligned} \tag{3.5.16.1}$$

$$\begin{aligned}
\cdots \xrightarrow{a_1\cdot} \underset{=0}{\operatorname{Ext}^i_R(N,M)} \longrightarrow \underset{\therefore=0}{\operatorname{Ext}^i_R(N,M/a_1M)} \\
\longrightarrow \underset{=0}{\operatorname{Ext}^{i+1}_R(N,M)} \xrightarrow{a_1\cdot} \cdots
\end{aligned}$$

$$\begin{aligned}
\cdots \xrightarrow{a_1\cdot} \operatorname{Ext}^{n-1}_R(N,M) \longrightarrow \operatorname{Ext}^{n-1}_R(N,M/a_1M) \\
\longrightarrow \operatorname{Ext}^n_R(N,M) \longrightarrow \cdots
\end{aligned}$$

To show (iv) implies (v), we simply point out that if we assume (iv), then the empty sequence can be extended to an M-regular sequence of length n.

Proving (v) implies (i) is again by induction. Assume M has a regular sequence $a_1, \ldots, a_n \in I$ and let N be a finitely generated R-module such that $\operatorname{Supp}_R(N) \subseteq V(I)$. We want to show $\operatorname{Ext}^i_R(N,M) = 0$ for all $i < n$. In the case when $n = 1$, our sequence is merely the element $a_1 \in I$. By Remark 3.2.11 and our assumption we have

$$V(\operatorname{Ann}_R(N)) = \operatorname{Supp}_R(N) \subseteq V(I)$$

so by Lemma 3.2.7 we have $a_1^t \in I^t \subseteq \operatorname{Ann}_R(N)$ (i.e., $a_1^tN = 0$) for all t sufficiently large. By construction the sequence

$$0 \longrightarrow M \xrightarrow{a_1\cdot} M \longrightarrow M/a_1M \longrightarrow 0$$

is exact. By the left exactness of $\operatorname{Hom}_R(N,-)$, this implies the sequence

$$0 \longrightarrow \operatorname{Hom}_R(N,M) \xhookrightarrow{a_1\cdot} \operatorname{Hom}_R(N,M)$$

is also exact, as is the sequence

$$0 \longrightarrow \operatorname{Hom}_R(N,M) \xhookrightarrow{a_1^t\cdot} \operatorname{Hom}_R(N,M)$$

for all $t \geq 1$, since the composition of injective functions is still injective. Since $a_1^t\cdot$ is the (injective) zero map for all sufficiently large t by Lemma 3.5.13, we conclude $\operatorname{Hom}_R(N,M) = 0$ as desired and we let this serve as the base case.

Assume $n \geq 2$ and the result holds for all $i < n-1$. We want to show $\mathrm{Ext}_R^{n-1}(N, M) = 0$. Let $a_1, \ldots, a_{n-1} \in I$ be an M-regular sequence guaranteed by (v). By our induction hypothesis $\mathrm{Ext}_R^i(N, M) = 0$ for all $i < n-1$ and we also know I contains an M/a_1M-regular sequence of length $n-1$, namely $a_2, \ldots, a_n$. Our induction hypothesis again implies

$$\mathrm{Ext}_R^i(N, M/a_1M) = 0$$

for all $i < n-1$. In particular, $\mathrm{Ext}_R^{n-2}(N, M/a_1M) = 0$ and therefore we have

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Ext}_R^{n-1}(N, M) \xhookrightarrow{a_1 \cdot} \mathrm{Ext}_R^{n-1}(N, M) \longrightarrow \cdots$$

by the exactness of the sequence (see (3.5.16.1)). Since the composition of injective functions yields an injective function, the sequence

$$0 \longrightarrow \mathrm{Ext}_R^{n-1}(N, M) \xhookrightarrow{a_1^t \cdot} \mathrm{Ext}_R^{n-1}(N, M)$$

is also exact for any $t > 0$. As in the base case we can take $t \gg 0$ such that $a_1^t \cdot N = 0$ and by Lemma 3.5.13, $a_1^t \cdot \mathrm{Ext}_R^{n-1}(N, M) = 0$ as well. Thus we have an injective zero map, implying the domain must be zero, i.e., $\mathrm{Ext}_R^{n-1}(N, M) = 0$ as desired. $\square$

Chapter 4

Homology

The ultimate goal of the next three chapters is to prove the well-definedness of Ext and the existence of long exact sequences as described in Theorem 2.1.1. We begin by introducing some of the basics of homology.

4.1 Chain Complexes and Homology

In this section we define chain complexes and homology modules. We present some of their basic characteristics and show in Theorem 4.1.9 that they play well with Hom modules. We also see in Example 4.1.7 is that Ext modules are specific homology modules.

Definition 4.1.1. A *chain complex of R-modules and R-module homomorphisms*, also known as an *R-complex*, is a sequence

$$M_\bullet = \qquad \cdots \xrightarrow{\partial_{i+2}^M} M_{i+1} \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} M_{i-1} \xrightarrow{\partial_{i-1}^M} \cdots$$

of R-modules and R-module homomorphisms such that $\partial_n^M \circ \partial_{n+1}^M = 0$ for all $n \in \mathbb{Z}$. The map ∂_i^M is the i^{th} *differential* of the complex and

$$H_i(M_\bullet) = \frac{\operatorname{Ker}\left(\partial_i^M\right)}{\operatorname{Im}\left(\partial_{i+1}^M\right)}$$

is the i^{th} *homology module* of $M_\bullet$. Note this quotient makes sense to write down since $\operatorname{Im}\left(\partial_{i+1}^M\right) \subseteq \operatorname{Ker}\left(\partial_i^M\right)$ if and only if $\partial_i^M \circ \partial_{i+1}^M = 0$.

Remark 4.1.2. An R-complex $M_\bullet$ is exact if and only if $H_i(M_\bullet) = 0$ for all $i \in \mathbb{Z}$. Colloquially, $H_i(M_\bullet)$ measures how far $M_\bullet$ is from being exact at the i^{th} position, M_i.

Example 4.1.3. Let M be an R-module and let $P_\bullet^+$ be an augmented projective resolution of M.

$$\begin{aligned} P_\bullet^+ = &\qquad \cdots \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0 \\ P_\bullet = &\qquad \cdots \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \longrightarrow 0 \end{aligned}$$

Since $P_\bullet^+$ is exact, it is also an R-complex with $H_i(P_\bullet^+) = 0$ for all $i \in \mathbb{Z}$. On the other hand, while $P_\bullet$ is not exact, it is still an R-complex since $\partial_i^P \circ \partial_{i+1}^P = 0$ for all positive i and for all negative i

we have $\partial_i^P \circ \partial_{i+1}^P = 0 \circ \partial_{i+1}^P = 0$. Therefore $H_i(P_\bullet) = 0$ for all $i \neq 0$ and leveraging the exactness of $P_\bullet^+$ we have

$$H_0(P_\bullet) = \frac{\operatorname{Ker}(P_0 \longrightarrow 0)}{\operatorname{Im}\left(\partial_1^P\right)} = \frac{P_0}{\operatorname{Ker}(\tau)} \cong \operatorname{Im}(\tau) = M.$$

Conversely, if the sequence

$$Q_\bullet = \qquad \cdots \xrightarrow{\partial_3^Q} Q_2 \xrightarrow{\partial_2^Q} Q_1 \xrightarrow{\partial_1^Q} Q_0 \longrightarrow 0$$

is an R-complex such that each Q_i is projective and $H_i(Q_\bullet) = 0$ for all $i \neq 0$, then $Q_\bullet$ is a projective resolution of the homology module $H_0(Q_\bullet)$.

Proof. Note that

$$H_0(Q_\bullet) = \frac{\operatorname{Ker}(Q_0 \longrightarrow 0)}{\operatorname{Im}\left(\partial_1^Q\right)} = \frac{Q_0}{\operatorname{Im}\left(\partial_1^Q\right)}.$$

Denote the natural epimorphism

$$\pi : Q_0 \longrightarrow \frac{Q_0}{\operatorname{Im}\left(\partial_1^Q\right)}$$

which once adjoined to $Q_\bullet$ gives the exact sequence

$$Q_\bullet^+ = \qquad \cdots \xrightarrow{\partial_3^Q} Q_2 \xrightarrow{\partial_2^Q} Q_1 \xrightarrow{\partial_1^Q} Q_0 \xrightarrow{\pi} \frac{Q_0}{\operatorname{Im}\left(\partial_1^Q\right)} \longrightarrow 0$$

completing the proof. □

Definition 4.1.4. Let $M_\bullet$ be an R-complex and let N be an R-module. We define *lower star* and *upper star* on R-complexes as we did with exact sequences. Define

$$M_{\bullet *} = \operatorname{Hom}_R(N, M_\bullet) = \qquad \cdots \xrightarrow{\left(\partial_{i+2}^M\right)_*} \underbrace{\operatorname{Hom}_R(N, M_{i+1})}_{=(M_{i+1})_*} \xrightarrow[\partial_{i+1}^M \circ (-)]{\left(\partial_{i+1}^M\right)_*} \underbrace{\operatorname{Hom}_R(N, M_i)}_{=(M_i)_*} \xrightarrow{\left(\partial_i^M\right)_*} \cdots$$

$$M_\bullet^* = \operatorname{Hom}_R(M_\bullet, N) = \qquad \cdots \xrightarrow{\left(\partial_{i-1}^M\right)^*} \underbrace{\operatorname{Hom}_R(M_{i-1}, N)}_{=(M_{i-1})^*} \xrightarrow[(-) \circ \partial_i^M]{\left(\partial_i^M\right)^*} \underbrace{\operatorname{Hom}_R(M_i, N)}_{=(M_i)^*} \xrightarrow{\left(\partial_{i+1}^M\right)^*} \cdots$$

where

$$\left(\partial_i^M\right)_* = \operatorname{Hom}_R(N, \partial_i^M) \qquad\qquad \left(\partial_i^M\right)^* = \operatorname{Hom}_R(\partial_i^M, N).$$

Proposition 4.1.5. *Both $M_{\bullet *}$ and $M_\bullet^*$ are R-complexes.*

Proof. The argument is written succinctly as follows.

$$(\partial_i^M)^* \circ (\partial_{i-1}^M)^* = (\partial_{i-1}^M \circ \partial_i^M)^* = 0^* = 0$$
$$(\partial_i^M)_* \circ (\partial_{i+1}^M)_* = (\partial_{i+1}^M \circ \partial_i^M)_* = 0_* = 0$$

□

Notation 4.1.6. We add some more short-hand.

$$(M_*)_i = M_{i*} \qquad\qquad \partial_i^{M_*} = (\partial_i^M)_*$$
$$(M^*)_j = (M_{-j})^* \qquad\qquad \partial_j^{M^*} = (\partial_{-j+1}^M)^*$$

Example 4.1.7. Let M and N be R-modules and let $P_\bullet$ be a projective resolution of M. Observe that the indices for the projective resolution are decreasing, whereas after applying $\operatorname{Hom}_R(P_\bullet, N)$ to get $P_\bullet^*$ the indices are increasing.

$$\begin{array}{ll} P_\bullet = & \cdots \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \longrightarrow 0 \\ P_\bullet^* = & 0 \longrightarrow P_0^* \xrightarrow{(\partial_1^P)^*} P_1^* \xrightarrow{(\partial_2^P)^*} P_2^* \xrightarrow{(\partial_3^P)^*} \cdots \end{array}$$

From Definition 4.1.1, when calculating homology modules we require decreasing indices, so in the case when we want $H_j(P_\bullet)$, the indices line up nicely.

$$H_j(P_\bullet) = \frac{\operatorname{Ker}\left(\partial_j^P\right)}{\operatorname{Im}\left(\partial_{j+1}^P\right)}$$

To put $P_\bullet^*$ in the form of having decreasing indices, we can define $k = -i$ and set $M_k = P_{-k}^*$ in order to write the following.

$$\begin{array}{lccccccccc} P_\bullet^* = & 0 & \longrightarrow & P_0^* & \xrightarrow{(\partial_1^P)^*} & P_1^* & \xrightarrow{(\partial_2^P)^*} & P_2^* & \xrightarrow{(\partial_3^P)^*} & \cdots \\ & \| & & \| & & \| & & \| & & \\ & M_1 & & M_0 & & M_{-1} & & M_{-2} & & \end{array}$$

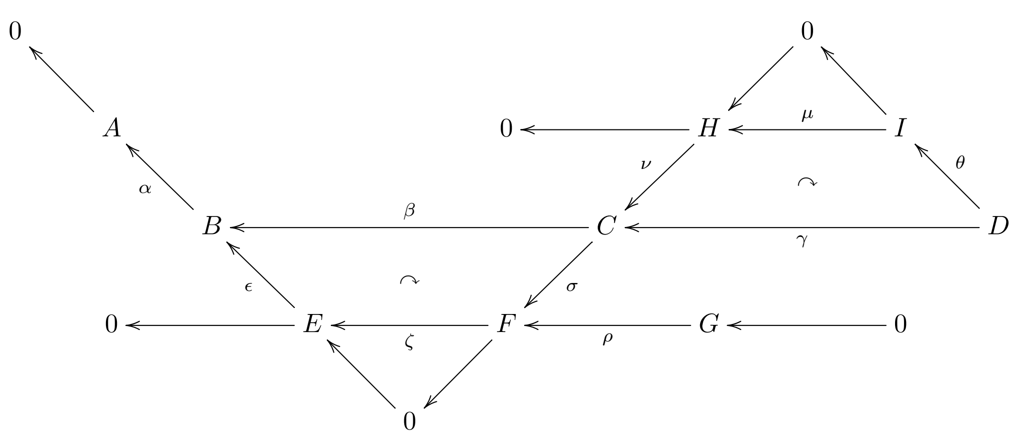
Hence building the i^{th} homology module from these M_i amounts to finding $\operatorname{Ext}_R^{-i}(M, N)$, i.e.,

$$H_i(P_\bullet^*) = H_i(\operatorname{Hom}_R(P_\bullet, N)) = \operatorname{Ext}_R^{-i}(M, N).$$

We can align the homology modules with their respective projective modules in $P_\bullet^*$ to make this even clearer.

$$\begin{array}{lccccccccccc} P_\bullet^* = & 0 & \longrightarrow & P_0^* & \longrightarrow & P_1^* & \longrightarrow & P_2^* & \longrightarrow \cdots \longrightarrow & P_i^* & \longrightarrow & \cdots \\ & & & H_0(P_\bullet^*) & & H_{-1}(P_\bullet^*) & & H_{-2}(P_\bullet^*) & & H_{-i}(P_\bullet^*) & & \end{array}$$

Lemma 4.1.8. *Assume we have the following commutative diagram of R-modules and R-module homomorphisms where the upper and lower horizontal sequences are exact, as are the diagonal sequences.*



If $\beta \circ \gamma = 0$, then $G \cong \dfrac{\operatorname{Ker}(\beta)}{\operatorname{Im}(\gamma)}$.

Proof. First, we claim that σ with domain restricted to $\operatorname{Ker}(\beta)$ surjects onto $\operatorname{Ker}(\zeta)$. We denote σ with such a restriction as $\hat{\sigma}\colon \operatorname{Ker}(\beta) \longrightarrow F$ and we want to show by mutual containment that

$$\operatorname{Im}(\hat{\sigma}) = \operatorname{Ker}(\zeta). \tag{4.1.8.1}$$

Let $c \in \operatorname{Ker}(\beta)$ and since $\beta(c) = 0$, by the commutivity of the diagram we also have $\epsilon(\zeta(\hat{\sigma}(c))) = 0$. Since ϵ is injective we have $\zeta(\hat{\sigma}(c)) = 0$ and therefore $\hat{\sigma}(c) \in \operatorname{Ker}(\zeta)$. This takes care of the forward containment of (4.1.8.1).

Let $f \in \operatorname{Ker}(\zeta)$ and it follows $\epsilon(\zeta(f)) = \epsilon(0) = 0 \in B$. Since σ is surjective, there exists some $c \in C$ such that $\sigma(c) = f$. By the commutivity of the diagram $\beta(c) = 0$, so $\hat{\sigma}$ is defined at c and moreover $f = \widehat{\sigma(c)} \in \operatorname{Im}(\hat{\sigma})$. This handles the reverse containment and thus (4.1.8.1) is proven.

For our second claim, we will show

$$\operatorname{Ker}(\hat{\sigma}) = \operatorname{Im}(\gamma) \tag{4.1.8.2}$$

again by mutual containment. Let $c \in \operatorname{Ker}(\hat{\sigma}) \subseteq \operatorname{Ker}(\sigma) = \operatorname{Im}(\nu)$ and let $h \in H$ such that $\nu(h) = c$. Since μ and θ are each surjective, choose $i \in I$ such that $\mu(i) = h$ and choose $d \in D$ such that $\theta(d) = i$. By the commutivity of the diagram we have $\gamma(d) = \nu(\mu(\theta(d))) = c$ and therefore $c \in \operatorname{Im}(\gamma)$, justifying the forward containment of (4.1.8.2).

Now let $d \in D$ and set $c = \gamma(d)$. By the commutivity of the diagram $\nu(\mu(\theta(d))) = c$, so $c \in \operatorname{Im}(\nu) = \operatorname{Ker}(\sigma)$ and $\sigma(c) = 0$. Moreover, $\beta(c) = \epsilon(\zeta(\sigma(c))) = 0$ by the commutivity of the diagram as well. Thus $c \in \operatorname{Ker}(\beta)$ so $\hat{\sigma}$ is defined at c with $\hat{\sigma}(c) = 0$ and therefore $c \in \operatorname{Ker}(\hat{\sigma})$. This completes our justification of (4.1.8.2).

By the First Isomorphism Theorem, (4.1.8.1), and (4.1.8.2), we have

$$\frac{\operatorname{Ker}(\beta)}{\operatorname{Im}(\gamma)} = \frac{\operatorname{Ker}(\beta)}{\operatorname{Ker}(\hat{\sigma})} \cong \operatorname{Im}(\hat{\sigma}) = \operatorname{Ker}(\zeta).$$

By the exactness of the bottom row we also have

$$\frac{\operatorname{Ker}(\beta)}{\operatorname{Im}(\gamma)} \cong \operatorname{Im}(\rho)$$

and since ρ is injective the apply the First Isomorphism Theorem again to conclude

$$\frac{\operatorname{Ker}(\beta)}{\operatorname{Im}(\gamma)} \cong G.$$

□

Theorem 4.1.9. *Let $M_\bullet$ be an R-complex and let N be an R-module.*

(a) If N is projective, then $H_i(\operatorname{Hom}_R(N, M_\bullet)) \cong \operatorname{Hom}_R(N, H_i(M_\bullet))$.

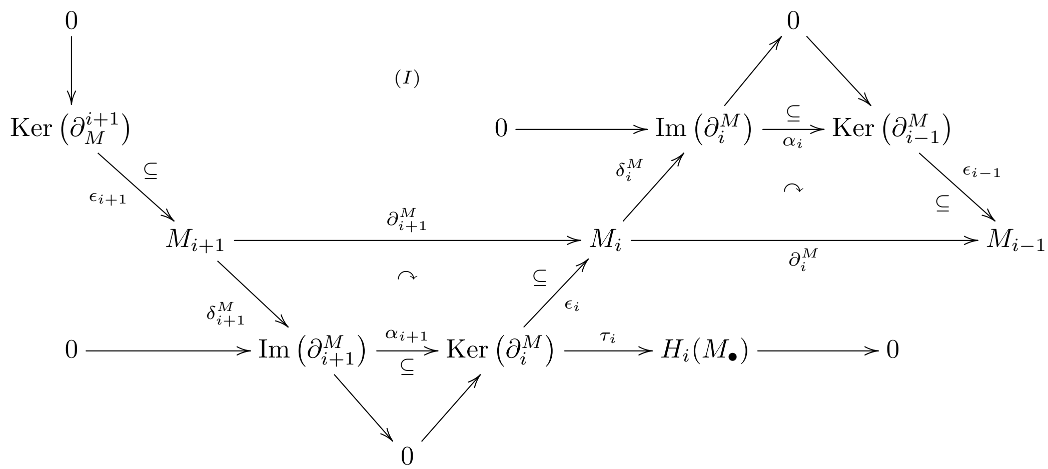
(b) If N is injective, then $H_i(\operatorname{Hom}_R(M_\bullet, N)) \cong \operatorname{Hom}_R(H_{-i}(M_\bullet), N)$.

We also write each of these colloquially.

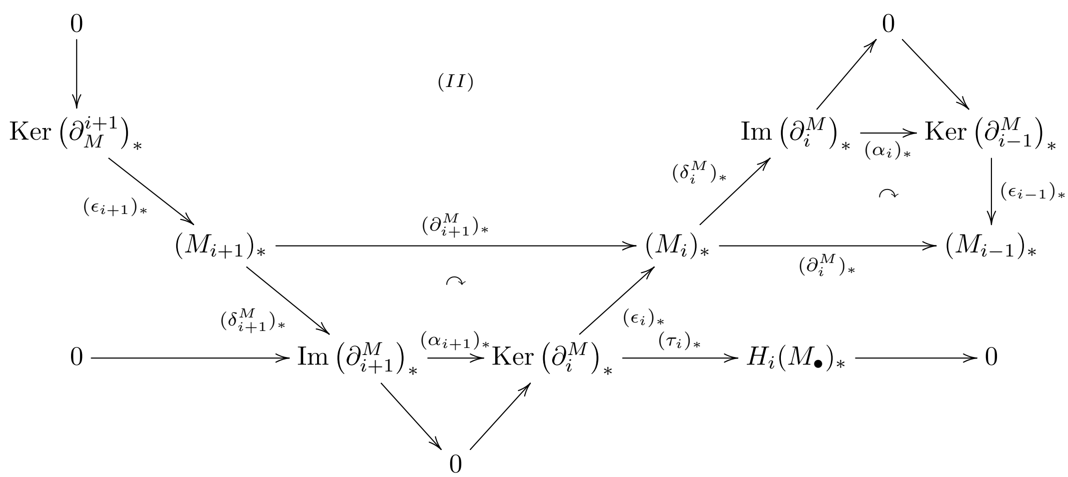
(a) 'Homming' with a projective module in the first slot commutes with taking homology.

(b) 'Homming' with an injective module in the second slot commutes with taking homology, as long as we are careful about indices.

Proof. (a). Consider the following diagram.



where δ_i^M is the map induced by ∂_i^M and τ_i is the natural surjection. The diagonals are all exact, as is the lower horizontal sequence. Moreover, $\mathrm{Hom}_R(N, -) = (-)_*$ is exact, because N is projective. Therefore we have the commutative diagram given below.



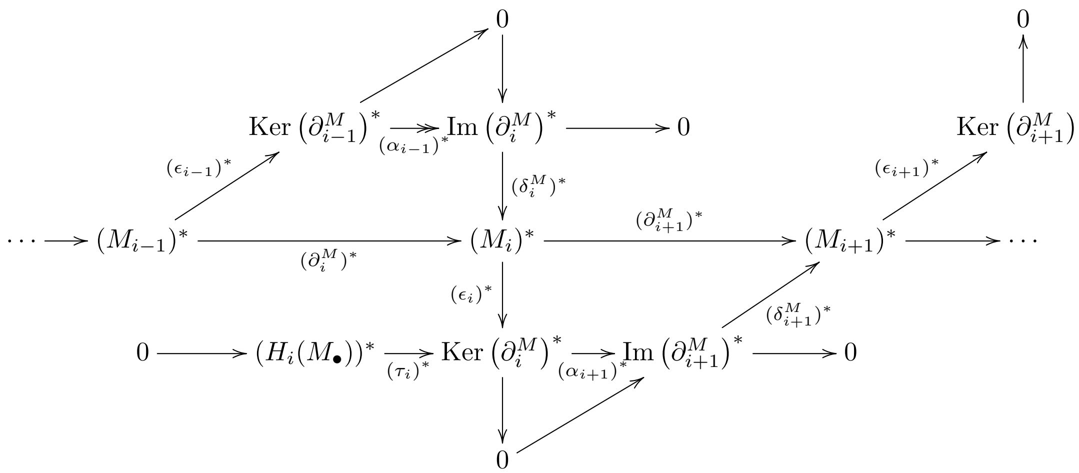
Since the exactness of the lower horizontal sequence is preserved, we claim the following is an isomorphism of short exact sequences, where ϵ and π are the natural injection and surjection, respectively.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}\left(\partial_{i+1}^M\right)_* & \xrightarrow{(\alpha_{i+1})_*} & \mathrm{Ker}\left(\partial_i^M\right)_* & \xrightarrow{(\tau)_*} & H_i(M_\bullet)_* & \longrightarrow & 0 \\ & & \cong \downarrow \beta & \circlearrowright & \cong \downarrow \gamma & \text{s.t.} \circlearrowright & \cong \downarrow \therefore \exists \theta & & \\ 0 & \longrightarrow & \mathrm{Im}\left((\partial_{i+1}^M)_*\right) & \xrightarrow{\epsilon} & \mathrm{Ker}\left((\partial_i^M)_*\right) & \xrightarrow{\pi} & H_i(M_{\bullet *}) & \longrightarrow & 0 \end{array} \qquad (4.1.9.1)$$

The map γ is induced by $(\epsilon_i)_*$ and is well-defined, because of diagram (II) above. It is also a monomorphism, because $(\epsilon_i)_*$ is a monomorphism. Moreover it is onto, which one can see from a standard diagram chase. The map β is induced by $(\epsilon_i)_* \circ (\alpha_{i+1})_*$ and is an isomorphism for similar reasons as γ. Also, it is straightforward to show that the left-hand square of (4.1.9.1) commutes. It follows that there exists some θ making the right-hand square commute and θ is an isomorphism by the Short-Five Lemma. Since $(M_\bullet)_* = \mathrm{Hom}_R(N, M_\bullet)$, we have

$$H_i(\mathrm{Hom}_R(N, M_\bullet)) = H_i(M_{\bullet *}) \cong H_i(M_\bullet)_* = \mathrm{Hom}_R(N, H_i(M_\bullet)).$$

(b). Let $i \in \mathbb{Z}$ be given and we apply $\mathrm{Hom}_R(-, N)$ to commutative diagram (I), which preserves the exactness and flips everything.



Therefore we have

$$\begin{aligned}
(H_i(M_\bullet))^* = \mathrm{Hom}_R(H_i(M_\bullet), N) &\cong \frac{\mathrm{Ker}\left((\partial_{i+1}^M)^*\right)}{\mathrm{Im}\left((\partial_i^M)^*\right)} && \text{Lemma 4.1.8} \\
&= \frac{\mathrm{Ker}\left(\partial_{-i}^{M^*}\right)}{\mathrm{Im}\left(\partial_{-i+1}^{M^*}\right)} && \text{Notation 4.1.6} \\
&= H_{-i}(M_\bullet^*) \\
&= H_{-i}(\mathrm{Hom}_R(M_\bullet, N)) && \text{Definition 4.1.4}
\end{aligned}$$

completing the proof of part (b). ☐

4.2 Ext Modules

There are two main propositions in this section. We state formally in Proposition 4.2.3 why one says Ext detects whether a given module is projective. In Proposition 4.2.8 we give conditions under which we know Ext modules are finitely generated.

Discussion 4.2.1. We have already put a fair amount of time into describing Ext_R^i, so in this section we add only a few more things. Let M be an R-module and $P_\bullet$ a projective resolution of M. That is

$$P_\bullet = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \longrightarrow 0$$

$$P_\bullet^+ = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

where all P_i are projective and $P_\bullet^+$ is exact. We saw in Example 4.1.7 that

$$\mathrm{Ext}_R^i(M, N) = H_{-i}(\mathrm{Hom}_R(P_\bullet, N))$$

for all $i \in \mathbb{Z}$, where

$$\operatorname{Hom}_R(P_\bullet, N) = P_\bullet^* = \cdots \longrightarrow 0 \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow P_2^* \longrightarrow \cdots$$

$$\text{position:} \qquad -1 \qquad 0 \qquad 1 \qquad 2$$

and the $-i^{th}$ module is built from the i^{th} position of $P_\bullet^*$, i.e.,

$$(P_\bullet^*)_i = P_{-i}^*.$$

The following theorem was previously stated as Fact 2.1.13 and will be proven in Theorem 6.5.2.

Theorem 4.2.2. *If $P_\bullet$ and $Q_\bullet$ are two projective resolutions of M, then*

$$H_{-i}(\operatorname{Hom}_R(P_\bullet, N)) \cong H_{-i}(\operatorname{Hom}_R(Q_\bullet, N))$$

for all $i \in \mathbb{Z}$. The slogan here is '$\operatorname{Ext}_R^i(M, N)$ is independent of our choice of projective resolution.'

Proposition 4.2.3. *Let M and N be R-modules.*

(a) If M is projective, then $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > 0$.

(b) If N is injective, then $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > 0$.

Proof. (a). Since M is projective, the augmented projective resolution and projective resolution are as follows.

$$P_\bullet^+ = \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \xrightarrow{\text{id}} M \longrightarrow 0$$

$$P_\bullet = \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \longrightarrow 0$$

In practice, we stop writing terms for the projective resolution, but in reality we may write more completely

$$P_\bullet = \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots.$$

$$\text{position:} \qquad 2 \qquad 1 \qquad 0 \qquad -1$$

Since $\operatorname{Hom}_R(-, N)$ is arrow-reversing, this gives

$$P_\bullet^* = \qquad \cdots \longrightarrow 0 \longrightarrow M^* \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots.$$

$$\text{position:} \qquad -1 \qquad 0 \qquad 1 \qquad 2$$

Therefore for all $i > 0$ we have as desired, namely

$$\operatorname{Ext}_R^i(M, N) = H_{-i}(0) = 0.$$

(b). Let $P_\bullet$ be a projective resolution of M and we have the following.

$$\begin{aligned}\operatorname{Ext}_R^i(M, N) = H_{-i}(\operatorname{Hom}_R(P_\bullet, N)) &\overset{4.1.9}{\cong} \operatorname{Hom}_R(H_i(P_\bullet), N) \\ &\overset{4.1.3}{\cong} \operatorname{Hom}_R(0, N), \text{ for all } i > 0 \\ &\overset{2.1.12}{=} 0\end{aligned}$$

□

The next several results set up the proof of Proposition 4.2.8. We begin with a definition.

Definition 4.2.4. Let R be a non-zero commutative ring with identity and let M be an R module. M is a *noetherian* module if it satisfies the following equivalent conditions.

(i) Every submodule of M is finitely generated.

(ii) M satisfies the ascending chain condition for submodules.

(iii) Every nonempty set S of R-submodules of M has a maximal element. That is, there exists an element $N \in S$ such that for all $N' \in S$, if $N \subseteq N'$, then $N = N'$.

Note 4.2.5. R is a noetherian ring if and only if R is noetherian as an R-module.

Proposition 4.2.6. *Let R be a non-zero commutative ring with identity and consider an exact sequence*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

of R-modules and R-module homomorphisms. In this setting M is a noetherian module over R if and only if M' and M'' are noetherian over R.

Proof. First assume that M is a noetherian R-module and let $N' \subset M'$ be a submodule. $f(N') \subset M$ is a finitely generated submodule, since M is noetherian. Therefore since f is injective, N' is finitely generated by the First Isomorphism Theorem. Since N' was arbitrarily taken, M' is noetherian.

Now consider an chain $N_1'' \subset N_2'' \subset \dots$ of submodules of M''. Then there is a chain $N_1 \subset N_2 \subset \dots$ of submodules of M with $N_i = g^{-1}(N_i'')$. Since M is noetherian, there is some $k \in \mathbb{N}$ such that $N_j = N_k$ for all $j \geq k$. Since g is surjective, $g(N_j) = N_j''$ for all j and we have

$$N_j'' = g(N_j) = g(N_k) = N_k''$$

for all $j \geq k$. Hence the chain $N_1'' \subset N_2'' \subset \dots$ stabilizes and we conclude M'' is noetherian.

Second, we instead assume both M' and M'' are noetherian. We want to show an arbitrary submodule $N \subseteq M$ is finitely generated. Since $g(N) \subseteq M''$ as a submodule and M'' is noetherian, it is finitely generated. We let $n_1, \dots, n_p \in N$ such that $g(N) = \langle g(n_1), \dots, g(n_p) \rangle$. Similarly, $f^{-1}(N) \subseteq M'$ as a submodule and we let $m'_{p+1}, \dots, m'_q \in M'$ such that $f^{-1}(N) = \left\langle m'_{p+1}, \dots, m'_q \right\rangle$. We claim $N = \langle n_1, \dots, n_q \rangle$ where $n_i = f(m'_i)$ for every $i = p+1, \dots, q$. Since one containment is by choice of n_i, it suffices to show $N \subseteq \langle n_1, \dots, n_q \rangle$. Let $n \in N$ be given. Then there exist $r_1, \dots, r_p \in R$ such that

$$g(n) = \sum_{i=1}^{p} r_i g(n_i) = g\left(\sum_{i=1}^{p} r_i n_i\right).$$

Since g is an R-module homomorphism it follows that

$$n - \sum_{i=1}^{p} r_i n_i \in \operatorname{Ker}(g) = \operatorname{Im}(f).$$

Moreover, since $n - \sum_{i=1}^{p} r_i n_i \in N$ as well, we have an element $x \in f^{-1}(N) = \left\langle m'_{p+1}, \dots, m'_q \right\rangle$ such that $f(x) = n - \sum_{i=1}^{p} r_i n_i$. So there are $r_{p+1}, \dots, r_q \in R$ such that $x = \sum_{i=p+1}^{q} r_i m'_i$. It follows that

$$n - \sum_{i=1}^{p} r_i n_i = f\left(\sum_{i=p+1}^{q} r_i m'_i\right) = \sum_{i=p+1}^{q} r_i f(m'_i) = \sum_{i=p+1}^{q} r_i n_i$$

and therefore $n = \sum_{i=1}^{q} r_i n_i$. $\square$

Proposition 4.2.7. *Let R be a non-zero commutative ring with identity and let M be an R-module.*

1. *The following are equivalent.*
   (a) *M is noetherian over R.*
   (b) *M^n is noetherian over R for all $n \in \mathbb{N}$.*
   (c) *M^n is noetherian over R for some $n \in \mathbb{N}$.*
2. *The following are equivalent.*
   (a) *R is a noetherian ring.*
   (b) *R^n is noetherian over R for all $n \in \mathbb{N}$.*
   (c) *R^n is noetherian over R for some $n \in \mathbb{N}$.*
3. *In the case when R is a noetherian ring, the following are equivalent.*
   (a) *M is finitely generated over R.*
   (b) *M is noetherian over R.*
   (c) *M has a* degree-wise finite free resolution, *that is, there is an exact sequence*

$$\cdots \longrightarrow R^{\beta_2} \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0$$

with each $\beta_i \in \mathbb{N}_0$.

Proof. 1. Consider the following short exact sequence for any $n > 1$.

$$0 \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M \longrightarrow 0$$

$$\begin{pmatrix} m_1 \\ \vdots \\ m_{n-1} \end{pmatrix} \longmapsto \begin{pmatrix} m_1 \\ \vdots \\ m_{n-1} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \longmapsto m_n$$

If M is noetherian, then we apply Proposition 4.2.6 to the short exact sequence above to conclude by induction on n that M^n is noetherian for any $n \geq 1$. Therefore (a) implies (b). The implication (b) implies (c) is trivial. If we assume M^n is noetherian for some $n \in \mathbb{N}$, then applying the same exercise to the same short exact sequence we conclude M is noetherian, so (c) implies (a).

2. By Note 4.2.5, this is a corollary of part 1.

3. M is noetherian over R if and only if every submodule of M is finitely generated over R. In particular, M is finitely generated since it is a submodule of itself, so (b) implies (a). From the exact sequence

$$\cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0$$

we can build a short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\tau) \xrightarrow{\subset} R^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0\ .$$

Since R^{β_0} is noetherian by part 2, M is noetherian as well by Proposition 4.2.6. Thus (c) implies (b).

Now we assume M is finitely generated over R and we want to build a degree-wise finite free resolution of M. Let $m_1, \ldots, m_{\beta_0} \in M$ be a set of generators for M and define the surjection

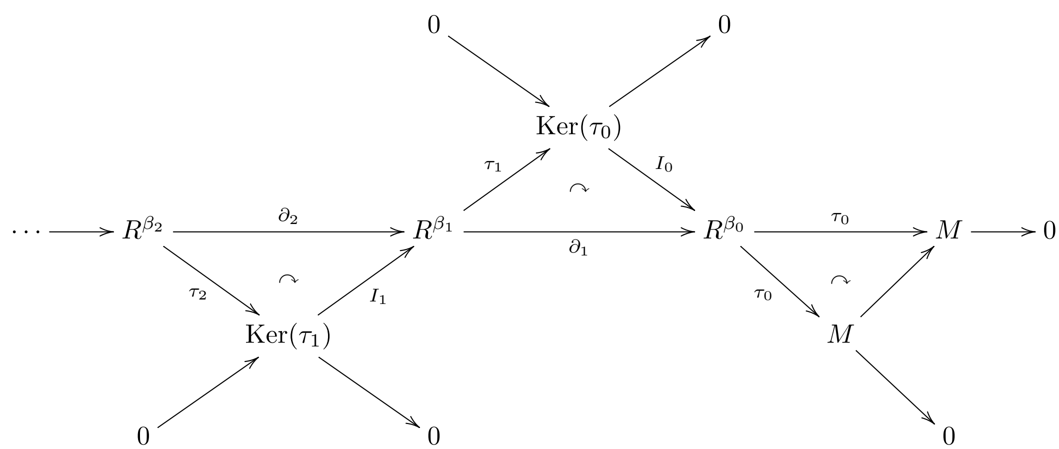
$$\begin{aligned} \tau_0 : R^{\beta_0} &\longrightarrow M \\ \textstyle\sum_{i=1}^{\beta_0} r_i e_i &\longmapsto \textstyle\sum_{i=1}^{\beta_0} r_i m_i \end{aligned}$$

where $e_1, \ldots, e_{\beta_0}$ is the standard basis of the free module R^{β_0}, which is noetherian by part 2. Therefore the submodule $\mathrm{Ker}(\tau_0) \subset R^{\beta_0}$ is finitely generated and we write $\mathrm{Ker}(\tau_0) = (f_1, \ldots, f_{\beta_1})R^{\beta_0}$ for some $f_1, \ldots, f_{\beta_1} \in R^{\beta_0}$. We may then approximate $\mathrm{Ker}(\tau_0)$ by the free module R^{β_1} using the surjection

$$\begin{aligned} \tau_1 : R^{\beta_1} &\longrightarrow \mathrm{Ker}(\tau_0) \\ \textstyle\sum_{i=1}^{\beta_1} r_i e'_i &\longmapsto \textstyle\sum_{i=1}^{\beta_1} r_i f_i \end{aligned}$$

where $e'_1, \ldots, e'_{\beta_1}$ is the standard basis. Since R^{β_1} is again noetherian, $\mathrm{Ker}(\tau_1)$ is again finitely generated and this process may continue.

For any $j \geq 1$ define $\partial_j = \tau_j \circ I_{j-1}$ where for any $k \geq 0$ we define I_k to be the containment map from $\mathrm{Ker}(\tau_k)$ into R^{β_k}. Then we can build the following commutative diagram where the row is exact, because the diagonals are exact by construction.



□

Proposition 4.2.8. *Let R be noetherian. If M and N are finitely generated R-modules, then $\mathrm{Ext}^i_R(M, N)$ is finitely generated for all $i \in \mathbb{Z}$.*

Proof. Since R is noetherian and M is finitely generated, by Proposition 4.2.7 M has a projective resolution of the form

$$P_\bullet = \qquad \cdots \longrightarrow R^{\beta_2} \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow 0$$

where $\beta_i \in \mathbb{N}_0$ for all $i \in \mathbb{N}$. Therefore from Fact 3.1.1 and from Hom-cancellation we have

$$\mathrm{Hom}_R(R^{\beta_i}, N) \cong \mathrm{Hom}_R(R, N)^{\beta_i} \cong N^{\beta_i}.$$

Since N is finitely generated and R is noetherian, N^{β_i} is also finitely generated and noetherian. Therefore the submodule $\mathrm{Ker}\left(\partial_{-i}^{P_\bullet^*}\right)$ is finitely generated and hence so is the following.

$$\frac{\mathrm{Ker}\left(\partial_{-i}^{P_\bullet^*}\right)}{\mathrm{Im}\left(\partial_{-i+1}^{P_\bullet^*}\right)} = H_{-i}(P_\bullet^*) = \mathrm{Ext}_R^i(M, N)$$

□

Chapter 5

Chain Maps and Induced Maps on Ext

In this chapter we continue to build the technology needed to prove that Ext is well-defined and to establish long exact sequences.

5.1 Chain Maps

In this section we introduce chain maps and show in Proposition 5.1.3 that these induce maps on homology modules. We will use this fact heavily when we prove the existence of the mother of all long exact sequences (Theorem 6.1.2).

Definition 5.1.1. Let $M_\bullet$ and $N_\bullet$ be R-complexes. A *chain map* from $M_\bullet$ into $N_\bullet$ is a sequence of R-module homomorphisms

$$F_\bullet = \{ F_i : M_i \longrightarrow N_i \mid i \in \mathbb{Z}\}$$

such that the following diagram commutes.

$$\begin{array}{cccccccc}
M_\bullet = & \cdots \longrightarrow & M_i & \xrightarrow{\partial_i^M} & M_{i-1} & \longrightarrow \cdots \\
\downarrow{\scriptstyle F_\bullet} & & \downarrow{\scriptstyle F_i} & \curvearrowright & \downarrow{\scriptstyle F_{i-1}} & \\
N_\bullet = & \cdots \longrightarrow & N_i & \xrightarrow[\partial_i^N]{} & N_{i-1} & \longrightarrow \cdots
\end{array}$$

We denote such a sequence as

$$F_\bullet : M_\bullet \longrightarrow N_\bullet.$$

Chain maps are also known as *commutative ladder diagrams.* An *isomorphism* from $M_\bullet$ to $N_\bullet$ is a chain map such that each F_i is an isomorphism.

Example 5.1.2. Consider the ring $R = \mathbb{Z}_{12} = \mathbb{Z}/12\mathbb{Z}$ and let $M_\bullet$ and $N_\bullet$ each be the constant sequence of copies of R with the R-module homomorphisms defined below. Defining various multiplication maps from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$ (vertically) we have a chain map from $M_\bullet$ to $N_\bullet$.

$$\begin{array}{cccccccccc}
M_\bullet = & \cdots \xrightarrow{6\cdot} & \mathbb{Z}_{12} & \xrightarrow{4\cdot} & \mathbb{Z}_{12} & \xrightarrow{6\cdot} & \mathbb{Z}_{12} & \xrightarrow{4\cdot} \cdots \\
 & \curvearrowright & \downarrow{\scriptstyle 2\cdot} & \curvearrowright & \downarrow{\scriptstyle 3\cdot} & \curvearrowright & \downarrow{\scriptstyle 2\cdot} & \curvearrowright \\
N_\bullet = & \cdots \xrightarrow[4\cdot]{} & \mathbb{Z}_{12} & \xrightarrow[6\cdot]{} & \mathbb{Z}_{12} & \xrightarrow[4\cdot]{} & \mathbb{Z}_{12} & \xrightarrow[6\cdot]{} \cdots
\end{array}$$

Proposition 5.1.3. *Let* $F_\bullet : M_\bullet \longrightarrow N_\bullet$ *be a chain map.*

(a) $F_i\left(\operatorname{Ker}\left(\partial_i^M\right)\right) \subseteq \operatorname{Ker}\left(\partial_i^N\right)$

(b) $F_i\left(\operatorname{Im}\left(\partial_{i+1}^M\right)\right) \subseteq \operatorname{Im}\left(\partial_{i+1}^N\right)$

(c) F_i *induces a well-defined* R*-module homomorphism from* $H_i(M_\bullet)$ *to* $H_i(N_\bullet)$ *given by*

$$\begin{array}{rrcl} H_i(F_\bullet): & H_i(M_\bullet) & \longrightarrow & H_i(N_\bullet) \\ & \overline{m} & \longmapsto & \overline{F_i(m)} \\ & m + \operatorname{Im}\left(\partial_{i+1}^M\right) & \longmapsto & F_i(m) + \operatorname{Im}\left(\partial_{i+1}^N\right). \end{array}$$

To put it yet another way

$$H_i(F_\bullet)(\overline{m}) = \overline{F_i(m)}.$$

Proof. (a). For any $\alpha \in \operatorname{Ker}(\partial_i^M)$ we have

$$\partial_i^N(F_i(\alpha)) = F_{i-1}(\partial_i^M(\alpha)) = F_{i-1}(0) = 0$$

because $F_\bullet$ is a chain map, completing this part.

(b). For any $\beta \in \operatorname{Im}\left(\partial_{i+1}^M\right)$ we can lift to some $\gamma \in M_{i+1}$ such that $\partial_{i+1}^M(\gamma) = \beta$. Then since $F_\bullet$ is a chain map we have

$$\partial_{i+1}^N(F_i(\gamma)) = F_i(\partial_{i+1}^M(\gamma)) = F_i(\beta).$$

(c). This is a corollary. Part (a) ensures that $H_i(F_\bullet)$ lands well, part (b) ensures that $H_i(F_\bullet)$ preserves equality, and the R-linearity of F_i gives the R-linearity of $H_i(F_\bullet)$. □

Remark 5.1.4. The construction of $H_i(F_\bullet)$ is summarized in the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im}\left(\partial_{i+1}^M\right) & \xrightarrow{\subseteq} & \operatorname{Ker}\left(\partial_i^M\right) & \longrightarrow & H_i(M_\bullet) & \longrightarrow & 0 \\ & & \downarrow \alpha_i & \curvearrowright & \downarrow \beta_i & \curvearrowright & \downarrow \therefore \exists! H_i(F_\bullet) & & \\ 0 & \longrightarrow & \operatorname{Im}\left(\partial_{i+1}^N\right) & \xrightarrow[\subseteq]{} & \operatorname{Ker}\left(\partial_i^N\right) & \longrightarrow & H_i(N_\bullet) & \longrightarrow & 0 \end{array}$$

Here α_i and β_i are each induced by F_i (by parts (b) and (a) of Proposition 5.1.3, respectively).

Example 5.1.5. Recall $F_\bullet$, $M_\bullet$, and $N_\bullet$ from Example 5.1.2. We have the homology modules

$$H_0(M_\bullet) = \frac{\operatorname{Ker}(6\cdot)}{\operatorname{Im}(4\cdot)} = \frac{2\cdot\mathbb{Z}_{12}}{4\cdot\mathbb{Z}_{12}} \cong \frac{2\mathbb{Z}}{4\mathbb{Z}} \cong \mathbb{Z}_2$$
$$H_0(N_\bullet) = \frac{\operatorname{Ker}(4\cdot)}{\operatorname{Im}(6\cdot)} = \frac{3\cdot\mathbb{Z}_{12}}{6\cdot\mathbb{Z}_{12}} \cong \mathbb{Z}_2$$

and the following map induced by $F_0 = 3\cdot$.

$$\begin{array}{rrcl} H_0(F_\bullet): & H_0(M_\bullet) & \longrightarrow & H_0(N_\bullet) \\ & \frac{2\mathbb{Z}_{12}}{4\mathbb{Z}_{12}} & \xrightarrow{3\cdot} & \frac{3\mathbb{Z}_{12}}{6\mathbb{Z}_{12}} \\ & \overline{2n} & \longmapsto & \overline{3\cdot 2n} = \overline{0} \end{array}$$

Note this implies $H_0(F_\bullet)$ is actually the zero map. The point is one might suspect this induced map to be multiplication by 3 from $\mathbb{Z}_2$ into $\mathbb{Z}_2$, but it can't be, because that would be an isomorphism and what we have found clearly is not.

In a similar fashion, we can study the induced map $H_1(F_\bullet)$.

$$\begin{array}{rccc} H_1(F_\bullet): & H_1(M_\bullet) & \longrightarrow & H_1(N_\bullet) \\ & \frac{3\mathbb{Z}_{12}}{6\mathbb{Z}_{12}} & \xrightarrow{2\cdot} & \frac{2\mathbb{Z}_{12}}{4\mathbb{Z}_{12}} \\ & \overline{3k} & \longmapsto & \overline{2\cdot 3k} = \overline{6k} \end{array}$$

Note this is an isomorphism since it sends 0 to 0 and sends $\overline{3}$ to $\overline{6} = \overline{2}$. That is, it sends the generator of an order-2 cyclic group to the generator of another order-2 cyclic group.

5.2 Liftings and Resolutions

In this section we show that an R-module homomorphism can be extended to produce a chain map on projective resolutions. Then we give some justification for Facts 3.5.9 and 3.5.11, as promised.

Lemma 5.2.1. *Consider the following diagram of R-modules and R-module homomorphisms with exact rows.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\alpha} & P & \xrightarrow{\gamma} & M & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & N' & \xrightarrow[\delta]{} & Q & \xrightarrow[\sigma]{} & N & \longrightarrow & 0 \end{array}$$

If P is projective, then there exist R-module homomorphisms f' and F making the following diagram commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\alpha} & P & \xrightarrow{\gamma} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & \curvearrowright & \downarrow{\scriptstyle F} & \curvearrowright & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & N' & \xrightarrow[\delta]{} & Q & \xrightarrow[\sigma]{} & N & \longrightarrow & 0 \end{array}$$

Before proving this lemma, we give the following application.

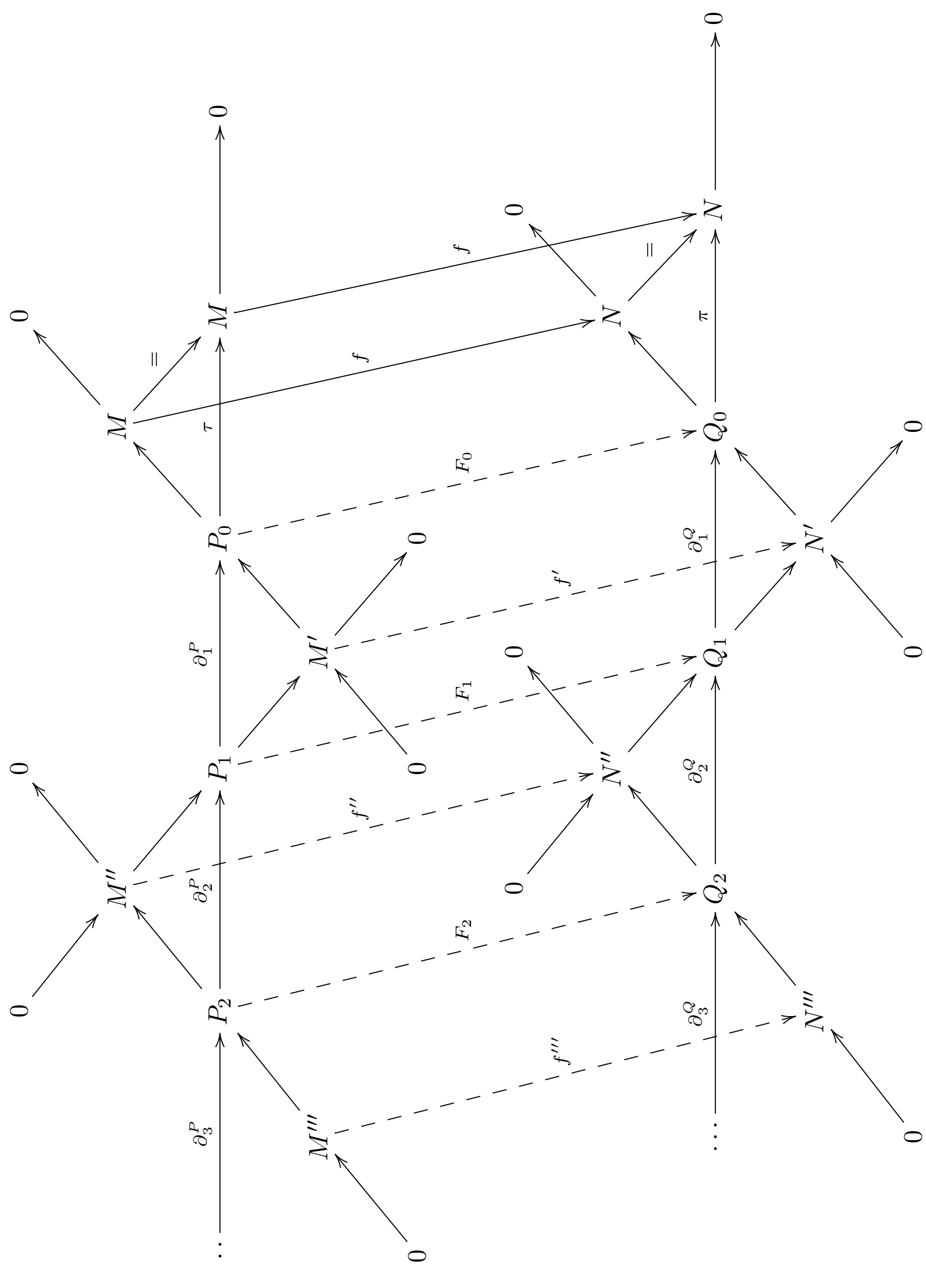
Proposition 5.2.2. *Let $P_\bullet^+$ be an augmented projective resolution of M and let $Q_\bullet^+$ be a "left resolution of N", i.e., an exact sequence*

$$Q_\bullet^+ = \qquad \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

where the modules $Q_0, Q_1, \ldots$ may not be projective. In this case, for every R-module homomorphism $f\colon M \longrightarrow N$, there exists a commutative diagram

$$\begin{array}{ccccccccccc} P_\bullet^+ = & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ {\scriptstyle F_\bullet^+}\downarrow & \cdots & \curvearrowright & \downarrow{\scriptstyle F_1} & \curvearrowright & \downarrow{\scriptstyle F_0} & \curvearrowright & \downarrow{\scriptstyle f} & & \\ Q_\bullet^+ = & \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

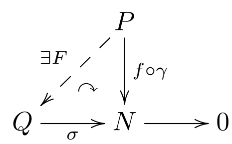
Proof. We give a convincing diagram (5.2.2.1) and the general idea. (One may also want to revisit the construction in Discussion 2.1.2.) The maps F_0 and f' come from Lemma 5.2.1. Then the maps F_1 and f'' come from the same lemma, and so on, inductively. A diagram chase shows the larger rectangles commute as well. $\square$



(5.2.2.1)

We now prove Lemma 5.2.1.

Proof of Lemma 5.2.1. Since P is projective, Definition 1.1.14(b) gives the existence of a function F such that the following diagram commutes.



This is precisely one of the functions we seek. We can also prove this using another characterization of projective modules. Specifically $\text{Hom}_R(P, -)$ is exact by Definition 1.1.14(a) so applying it to the bottom row of the diagram we get the following short exact sequence.

$$0 \longrightarrow \text{Hom}_R(P, N') \longrightarrow \text{Hom}_R(P, Q) \overset{\sigma_*}{\twoheadrightarrow} \text{Hom}_R(P, N) \longrightarrow 0$$

Noting the surjectivity of σ_*, there exists an R-module homomorphism $F \in \text{Hom}_R(P, Q)$ such that

$$\sigma_*(F) = f \circ \gamma \in \text{Hom}_R(P, N).$$

Since $\sigma_*(F) = \sigma \circ F$, this also yields the desired map.

Proving the existence of f' takes a bit more work. For any $m' \in M'$, the commutivity afforded by F and the exactness of the rows give

$$\sigma(F(\alpha(m'))) = f(\gamma(\alpha(m'))) = f(0) = 0.$$

Therefore $F(\alpha(m')) \in \text{Ker}\,(\sigma) = \text{Im}\,(\delta)$ and there exists some $n' \in N'$ such that $F(\alpha(m')) = \delta(n')$. In fact, since δ is injective, this n' is unique. Therefore we have the well-defined map

$$f' : M' \longrightarrow N'$$
$$m' \longmapsto n'$$

which we claim is an R-module homomorphism. To check R-linearity, first let $m' \in M'$ and let $r \in R$. Then there exists some $n' \in N'$ such that $F(\alpha(m')) = \delta(n')$ and we consider $rn' \in N'$ to find

$$\delta(rn') = r \cdot \delta(n') = r \cdot F(\alpha(m')) = F(\alpha(rm')).$$

Therefore

$$f'(rm') = rn' = r \cdot f'(m').$$

We prove the additivity of f' in a similar fashion. Let $m_1', m_2' \in M'$ and there exist n_1', $n_2' \in N'$ such that

$$F(\alpha(m_1')) = \delta(n_1') \qquad F(\alpha(m_2')) = \delta(n_2')$$

Therefore considering the element $n_1' + n_2' \in N'$ we have

$$\delta(n_1' + n_2') = \delta(n_1') + \delta(n_2') = F(\alpha(m_1')) + F(\alpha(m_2')) = F(\alpha(m_1' + m_2'))$$

and hence

$$f'(m_1' + m_2') = n_1' + n_2' = f'(m_1') + f'(m_2').$$

□

Here we construct the induced maps on Ext from Fact 3.5.9, but we will still put off some questions of well-definedness.

Discussion 5.2.3. Consider R-module homomorphisms

$$f : M \longrightarrow M' \qquad\qquad g : N \longrightarrow N'$$

and we want to derive, though by no means completely at this point, the following induced maps.

$$\operatorname{Ext}^i_R(M,N) \xrightarrow{\operatorname{Ext}^i_R(M,g)} \operatorname{Ext}^i_R(M,N')$$
$$\operatorname{Ext}^i_R(M',N) \xrightarrow{\operatorname{Ext}^i_R(f,N)} \operatorname{Ext}^i_R(M,N)$$

The two maps we seek between Ext's come from chain maps

$$\operatorname{Hom}_R(P_\bullet,N) \xrightarrow{\operatorname{Hom}_R(P_\bullet,g)} \operatorname{Hom}_R(P_\bullet,N')$$
$$\operatorname{Hom}_R(P'_\bullet,N) \xrightarrow{\operatorname{Hom}_R(F_\bullet,N)} \operatorname{Hom}_R(P_\bullet,N)$$

where $P_\bullet$ is a projective resolution of M, $P'_\bullet$ is a projective resolution of M', and $F_\bullet : P_\bullet \longrightarrow P'_\bullet$ is a "lift" of f. That is, given the two augmented projective resolutions, because of the map f, there exist maps F_0, F_1, and so on that make each of the successive diagrams commute.

$$\begin{array}{lccccccccc}
P^+_\bullet = & \cdots & \xrightarrow{\partial^P_2} & P_1 & \xrightarrow{\partial^P_1} & P_0 & \xrightarrow{\tau} & M & \longrightarrow & 0 \\
& \cdots & & \downarrow{\scriptstyle \exists F_1} & \curvearrowright & \downarrow{\scriptstyle \exists F_0} & \curvearrowright & \downarrow{\scriptstyle f} & & \\
(P'_\bullet)^+ = & \cdots & \xrightarrow[\partial^{P'}_2]{} & P'_1 & \xrightarrow[\partial^{P'}_1]{} & P'_0 & \xrightarrow[\tau']{} & M' & \longrightarrow & 0
\end{array}$$

Therefore restricting down to the projective resolutions we have

$$\begin{array}{lccccccc}
P_\bullet = & \cdots & \xrightarrow{\partial^P_2} & P_1 & \xrightarrow{\partial^P_1} & P_0 & \longrightarrow & 0 \\
& \cdots & \curvearrowright & \downarrow{\scriptstyle F_1} & \curvearrowright & \downarrow{\scriptstyle F_0} & & \\
P'_\bullet = & \cdots & \xrightarrow[\partial^{P'}_2]{} & P'_1 & \xrightarrow[\partial^{P'}_1]{} & P'_0 & \longrightarrow & 0.
\end{array}$$

We say $F_\bullet = \{F_0, F_1, F_2, \ldots\}$ *is a chain map compatible with* f. Or to put it another way, $F_\bullet$ is a chain map such that

$$\begin{array}{ccc}
H_0(P_\bullet) & \xrightarrow{H_0(F_\bullet)} & H_0(P'_\bullet) \\
{\scriptstyle \cong}\downarrow & \curvearrowright & \downarrow{\scriptstyle \cong} \\
M & \xrightarrow[f]{} & M'
\end{array}$$

using the induced map from Proposition 5.1.3. We want to show that the ladder diagrams $\operatorname{Hom}_R(P_\bullet, g)$

and $\mathrm{Hom}_R(F_\bullet, N)$ commute. First we consider $\mathrm{Hom}_R(P_\bullet, g)$.

$$\begin{array}{cccccc}
\mathrm{Hom}_R(P_\bullet, N) = & \cdots \longrightarrow & \mathrm{Hom}_R(P_i, N) & \xrightarrow{\mathrm{Hom}_R(\partial^P_{i+1}, N)} & \mathrm{Hom}_R(P_{i+1}, N) & \longrightarrow \cdots \\
{\scriptstyle \mathrm{Hom}_R(P_\bullet, g)}\downarrow & & {\scriptstyle \mathrm{Hom}_R(P_i, g)}\downarrow & \curvearrowright ? & \downarrow{\scriptstyle \mathrm{Hom}_R(P_{i+1}, g)} & \\
\mathrm{Hom}_R(P_\bullet, N') = & \cdots \longrightarrow & \mathrm{Hom}_R(P_i, N') & \xrightarrow[\mathrm{Hom}_R(\partial^P_{i+1}, N')]{} & \mathrm{Hom}_R(P_{i+1}, N') & \longrightarrow \cdots
\end{array}$$

To check commutivity we track an arbitrary $\phi \in \mathrm{Hom}_R(P_i, N)$.

$$\begin{array}{ccccc}
\phi & \xmapsto{\mathrm{Hom}_R(\partial^P_{i+1}, N)} & \phi \circ \partial^P_{i+1} & & \\
{\scriptstyle \mathrm{Hom}_R(P_i, g)}\downarrow & & & \searrow^{\mathrm{Hom}_R(P_{i+1}, g)} & \\
g \circ \phi & \xmapsto[\mathrm{Hom}_R(\partial^P_{i+1}, N')]{} & (g \circ \phi) \circ \partial^P_{i+1} & = & g \circ (\phi \circ \partial^P_{i+1})
\end{array}$$

Therefore the diagram commutes by the associativity of function composition and we define the first of our two maps as

$$\mathrm{Ext}^i_R(M, g) = H_{-i}(\mathrm{Hom}_R(P_\bullet, g))$$

where

$$\mathrm{Ext}^i_R(M, g)\left(\overline{\phi}\right) = \overline{\mathrm{Hom}_R(P_\bullet, g)_{-i}(\phi)} = \overline{\mathrm{Hom}_R(P_i, g)(\phi)} = \overline{g \circ \phi}.$$

The chain map $\mathrm{Hom}_R(F_\bullet, N)$ also arises from maps between the chain complexes used to define the Ext's of the domain and codomain.

$$\mathrm{Hom}_R(F_\bullet, N) : \mathrm{Hom}_R(P'_\bullet, N) \longrightarrow \mathrm{Hom}_R(P_\bullet, N)$$

As with the first map, there is a question of commutivity in a particular diagram we need answered in order to verify we have a chain map.

$$\begin{array}{cccccc}
\mathrm{Hom}_R(P'_\bullet, N) = & \cdots \longrightarrow & \mathrm{Hom}_R(P'_i, N) & \xrightarrow{\mathrm{Hom}_R(\partial^{P'}_{i+1}, N)} & \mathrm{Hom}_R(P'_{i+1}, N) & \longrightarrow \cdots \\
{\scriptstyle \mathrm{Hom}_R(F_\bullet, N)}\downarrow & & {\scriptstyle \mathrm{Hom}_R(F_i, N)}\downarrow & \curvearrowright ? & \downarrow{\scriptstyle \mathrm{Hom}_R(F_{i+1}, N)} & \\
\mathrm{Hom}_R(P_\bullet, N) = & \cdots \longrightarrow & \mathrm{Hom}_R(P_i, N) & \xrightarrow[\mathrm{Hom}_R(\partial^P_{i+1}, N)]{} & \mathrm{Hom}_R(P_{i+1}, N) & \longrightarrow \cdots
\end{array}$$

We again ignore well-definedness and check commutivity.

$$\begin{array}{ccccc}
\psi & \xmapsto{\mathrm{Hom}_R(\partial^{P'}_{i+1}, N)} & \psi \circ \partial^{P'}_{i+1} & & \\
{\scriptstyle \mathrm{Hom}_R(F_i, N)}\downarrow & & & \searrow^{\mathrm{Hom}_R(F_{i+1}, N)} & \\
\psi \circ F_i & \xmapsto[\mathrm{Hom}_R(\partial^P_{i+1}, N)]{} & (\psi \circ F_i) \circ \partial^P_{i+1} & = & (\psi \circ \partial^{P'}_{i+1}) \circ F_{i+1}
\end{array}$$

Where the equality holds since $F_\bullet$ is a chain map. Therefore we define the second map below.

$$\mathrm{Ext}^i_R(f, N)(\overline{\psi}) = \overline{\mathrm{Hom}_R(F_\bullet, N)_{-i}(\psi)} = \overline{\psi \circ F_i}$$

In all reality, one also needs to show this construction is independent of choice of $P_\bullet$, $P'_\bullet$, and $F_\bullet$, but we will end our discussion for now.

Here we give some justification for Fact 3.5.11.

Discussion 5.2.4. Let $r \in R$, let $L_\bullet$ be an R-complex, and define the map

$$\begin{aligned} \mu_r^M : M &\longrightarrow M \\ m &\longmapsto rm \end{aligned}$$

where M is any R-module. Notice that we can build a chain map from $L_\bullet$ to itself out of such R-module homomorphisms.

$$\begin{array}{lccccccc} L_\bullet = & \cdots \longrightarrow & L_i & \xrightarrow{\partial_i^L} & L_{i-1} & \longrightarrow \cdots \\ & & \downarrow \mu_r^{L_i} & \curvearrowright & \downarrow \mu_r^{L_{i-1}} & \\ L_\bullet = & \cdots \longrightarrow & L_i & \xrightarrow[\partial_i^L]{} & L_{i-1} & \longrightarrow \cdots \end{array}$$

We confirm the commutivity of the diagram by tracking an arbitrary element $\ell \in L_i$.

$$\begin{array}{ccccc} \ell & \longmapsto & \partial_i^L(\ell) & & \\ \downarrow & & & \searrow & \\ r\ell & \longmapsto & \partial_i^L(r\ell) & = & r \cdot \partial_i^L(\ell) \end{array}$$

Hence we say

$$(\mu_r^{L_\bullet})_\bullet : L_\bullet \longrightarrow L_\bullet.$$

Furthermore, the map induced on homologies is also a multiplication map. That is

$$H_i((\mu_r^{L_\bullet})_\bullet) = \mu_r^{H_i(L_\bullet)}$$

because of the following.

$$H_i((\mu_r^{L_\bullet})_\bullet)(\ell) = \overline{(\mu_r^{L_\bullet})_i(\ell)} = \overline{r\ell} = r \cdot \overline{\ell} = \mu_r^{H_i(L_\bullet)}\left(\overline{\ell}\right)$$

We now claim

$$\operatorname{Ext}_R^i(\mu_r^M, N) = \mu_r^{\operatorname{Ext}_R^i(M,N)} = \operatorname{Ext}_R^i(M, \mu_r^N).$$

Indeed the second equality in our claim follows from

$$\operatorname{Ext}_R^i(M, \mu_r^N)(\overline{\phi}) = \overline{\mu_r^N \circ \phi} \underset{\ddagger}{=} \overline{r\phi} = r \cdot \overline{\phi} = \mu_r^{\operatorname{Ext}_R^i(M,n)}\left(\overline{\phi}\right) \tag{5.2.4.1}$$

where ‡ holds since $(\mu_r^N \circ \phi)(x) = r \cdot \phi(x) = (r\phi)(x)$. For the first equality in our claim, we need $F_\bullet$.

$$\begin{array}{lcccccccc} P_\bullet^+ = & \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ \left(\mu_r^{P_\bullet^+}\right)_\bullet \Big\downarrow & \cdots \quad \curvearrowright & \Big\downarrow \mu_r^{P_1} & \curvearrowright & \Big\downarrow \mu_r^{P_0} & \curvearrowright & \Big\downarrow \mu_r^M & & \\ P_\bullet^+ = & \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

It is straightforward to show that this diagram commutes, i.e., it satisfies the conclusion of Proposition 5.2.2. Thus, we have the following.

$$\operatorname{Ext}_R^i(\mu_r^M, N)(\overline{\psi}) = \overline{\psi \circ \mu_r^{P_i}} = \overline{r \cdot \psi} = r \cdot \overline{\psi} = \mu_r^{\operatorname{Ext}_R^i(M,N)}$$

Example 5.2.5. Let $R = \mathbb{Z}_{12}$ and define R-modules $M = \mathbb{Z}_6$ and $N = \mathbb{Z}_3$. We then have the following chain map $F_\bullet$, where τ, π, and ρ are all natural surjections.

$$\begin{array}{lccccccccccccc}
P_\bullet^+ = & \cdots & \xrightarrow{2\cdot} & \mathbb{Z}_{12} & \xrightarrow{6\cdot} & \mathbb{Z}_{12} & \xrightarrow{2\cdot} & \mathbb{Z}_{12} & \xrightarrow{6\cdot} & \mathbb{Z}_{12} & \xrightarrow{\tau} & \mathbb{Z}_6 & \longrightarrow & 0 \\
 & \cdots & \curvearrowright & F_3 \downarrow 2\cdot & \curvearrowright & F_2 \downarrow 1\cdot & \curvearrowright & F_1 \downarrow 2\cdot & \curvearrowright & F_0 \downarrow 1\cdot & \curvearrowright & \downarrow \rho & & \\
Q_\bullet^+ = & \cdots & \xrightarrow{4\cdot} & \mathbb{Z}_{12} & \xrightarrow{3\cdot} & \mathbb{Z}_{12} & \xrightarrow{4\cdot} & \mathbb{Z}_{12} & \xrightarrow{3\cdot} & \mathbb{Z}_{12} & \xrightarrow{\pi} & \mathbb{Z}_3 & \longrightarrow & 0
\end{array}$$

Reducing from the augmented resolutions, we lose our exactness on the right side, but we still have a chain map.

$$\begin{array}{lccccccccccc}
P_\bullet = & \cdots & \xrightarrow{2\cdot} & \mathbb{Z}_{12} & \xrightarrow{6\cdot} & \mathbb{Z}_{12} & \xrightarrow{2\cdot} & \mathbb{Z}_{12} & \xrightarrow{6\cdot} & \mathbb{Z}_{12} & \longrightarrow & 0 \\
F_\bullet \downarrow & \cdots & \curvearrowright & F_3 \downarrow 2\cdot & \curvearrowright & F_2 \downarrow 1\cdot & \curvearrowright & F_1 \downarrow 2\cdot & \curvearrowright & F_0 \downarrow 1\cdot & & \\
Q_\bullet = & \cdots & \xrightarrow{4\cdot} & \mathbb{Z}_{12} & \xrightarrow{3\cdot} & \mathbb{Z}_{12} & \xrightarrow{4\cdot} & \mathbb{Z}_{12} & \xrightarrow{3\cdot} & \mathbb{Z}_{12} & \longrightarrow & 0
\end{array}$$

We want to compute maps on Ext induced by ρ. Specifically, we want to compute the maps

$$\begin{array}{ccccc}
\operatorname{Ext}^i_{\mathbb{Z}_{12}}(\rho, \mathbb{Z}_{12}) & : & \operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_{12}) & \longrightarrow & \operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_6, \mathbb{Z}_{12}) \\
\| & & \| & & \| \\
H_{-i}(\operatorname{Hom}_{\mathbb{Z}_{12}}(F_\bullet, \mathbb{Z}_{12})) & & H_{-i}(\operatorname{Hom}_{\mathbb{Z}_{12}}(Q_\bullet, \mathbb{Z}_{12})) & & H_{-i}(\operatorname{Hom}_{\mathbb{Z}_{12}}(P_\bullet, \mathbb{Z}_{12})).
\end{array}$$

From Discussion 5.2.3, we know exactly how this map behaves for any given index i.

$$H_{-i}(\operatorname{Hom}_{\mathbb{Z}_{12}}(F_\bullet, \mathbb{Z}_{12}))(\overline{\phi}) = \overline{\phi \circ F_i}$$

In order to understand this better, we apply the functor $\operatorname{Hom}_{\mathbb{Z}_{12}}(-, \mathbb{Z}_{12})$ to the chain map above.

$$\begin{array}{lccccccccc}
P_\bullet^* = & 0 & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) & \xrightarrow{6\cdot} & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) & \xrightarrow{2\cdot} & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) & \xrightarrow{6\cdot} & \cdots \\
 & & & \uparrow 1\cdot & & \uparrow 2\cdot & & \uparrow 1\cdot & & \\
Q_\bullet^* = & 0 & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) & \xrightarrow[3\cdot]{} & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) & \xrightarrow[4\cdot]{} & \operatorname{Hom}_{\mathbb{Z}_{12}}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) & \xrightarrow[3\cdot]{} & \cdots
\end{array}$$

Note we still have multiplication maps (see our justification for ‡ in Equation 5.2.4.1). By Hom-cancellation we have the following.

$$\begin{array}{lccccccccc}
P_\bullet^* \cong & 0 & \longrightarrow & \mathbb{Z}_{12} & \xrightarrow{6\cdot} & \mathbb{Z}_{12} & \xrightarrow{2\cdot} & \mathbb{Z}_{12} & \xrightarrow{6\cdot} & \cdots \\
 & & & \uparrow 1\cdot & & \uparrow 2\cdot & & \uparrow 1\cdot & & \\
Q_\bullet^* \cong & 0 & \longrightarrow & \mathbb{Z}_{12} & \xrightarrow[3\cdot]{} & \mathbb{Z}_{12} & \xrightarrow[4\cdot]{} & \mathbb{Z}_{12} & \xrightarrow[3\cdot]{} & \cdots
\end{array}$$

Noticing this ladder diagram is merely the second diagram in this example with the arrows reversed, we know there is only one place where the rows are not exact, namely at the 0^{th} index. Therefore the Ext^i's vanish for all $i > 0$. So we write

$$\operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_6, \mathbb{Z}_{12}) = 0 = \operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_{12})$$

for all $i > 0$ and hence

$$\operatorname{Ext}^i_{Z_{12}}(\rho, \mathbb{Z}_{12}) : 0 \longrightarrow 0$$

is the zero map. At the $i = 0$ position we have

$$\operatorname{Ext}^0_R(\mathbb{Z}_6, \mathbb{Z}_{12}) \cong \operatorname{Ker}\left(\mathbb{Z}_{12} \xrightarrow{6\cdot} \mathbb{Z}_{12} \right) \cong \langle \overline{2} \rangle$$

$$\operatorname{Ext}^0_R(\mathbb{Z}_3, \mathbb{Z}_{12}) \cong \operatorname{Ker}\left(\mathbb{Z}_{12} \xrightarrow{3\cdot} \mathbb{Z}_{12} \right) \cong \langle \overline{4} \rangle$$

Therefore the map induced by $\rho = \left(\mathbb{Z}_6 \xrightarrow{1\cdot} \mathbb{Z}_6 \right)$,

$$\operatorname{Ext}^0_R(\rho, \mathbb{Z}_{12})\colon\ \operatorname{Ext}^0_R(\mathbb{Z}_3, \mathbb{Z}_{12}) \longrightarrow \operatorname{Ext}^0_R(\mathbb{Z}_6, \mathbb{Z}_{12})$$

is just the inclusion map $\langle \overline{4} \rangle \xhookrightarrow{\subseteq} \langle \overline{2} \rangle$.

Example 5.2.6. Next, we generalize the previous example by computing $\operatorname{Ext}^i_{Z_{12}}(\rho, \mathbb{Z}_n)$ for several n satisfying $n|12$.

First we handle the $n = 2$ and $n = 4$ cases. Since $2{\cdot}\mathbb{Z}_2 = 0$ and $3{\cdot}\mathbb{Z}_3 = 0$, by Discussion 5.2.4 we know

$$2 \cdot \operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_2) = 0 = 3 \cdot \operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_2)$$

and therefore

$$1 \cdot \operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_2) = (3-2) \cdot \operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_2) = 0.$$

Thus $\operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_2) = 0$ for all $i \in \mathbb{Z}$ and for almost identical reasons $\operatorname{Ext}^i_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_4) = 0$ for all $i \in \mathbb{Z}$ as well, so we need not endeavor any further to study the induced maps on homologies in these cases (maps between zeros are boring).

For the case when $n = 6$, much of the derivation is a replication of Example 5.2.5, so we will not reproduce it here, but the resulting ladder diagram is below.

$$\begin{array}{ccccccccc}
P^*_\bullet \cong & 0 \longrightarrow & \mathbb{Z}_6 & \xrightarrow{6\cdot} & \mathbb{Z}_6 & \xrightarrow{2\cdot} & \mathbb{Z}_6 & \xrightarrow{6\cdot} & \cdots \\
& & \uparrow{\scriptstyle 1\cdot} & & \uparrow{\scriptstyle 2\cdot} & & \uparrow{\scriptstyle 1\cdot} & & \\
Q^*_\bullet \cong & 0 \longrightarrow & \mathbb{Z}_6 & \xrightarrow[3\cdot]{} & \mathbb{Z}_6 & \xrightarrow[4\cdot]{} & \mathbb{Z}_6 & \xrightarrow[3\cdot]{} & \cdots
\end{array}$$

At the $i = 0$ position we have the following homology modules.

$$H_0(P^*_\bullet) = \frac{\operatorname{Ker}\left(\mathbb{Z}_6 \xrightarrow{0\cdot} \mathbb{Z}_6 \right)}{\operatorname{Im}\left(0 \longrightarrow \mathbb{Z}_6 \right)} = \frac{\mathbb{Z}_6}{0} \cong \mathbb{Z}_6$$

$$H_0(Q^*_\bullet) = \frac{\operatorname{Ker}\left(\mathbb{Z}_6 \xrightarrow{3\cdot} \mathbb{Z}_6 \right)}{\operatorname{Im}\left(0 \longrightarrow \mathbb{Z}_6 \right)} \cong \frac{2 \cdot \mathbb{Z}_6}{0} \cong 2 \cdot \mathbb{Z}_6$$

Therefore the multiplication map $1\cdot$ is essentially a containment map.

$$H_0(F_\bullet) \ :\ H_0(Q^*_\bullet) \xrightarrow{1\cdot} H_0(P^*_\bullet)$$

$$2 \cdot \mathbb{Z}_6 \xrightarrow[\subseteq]{1\cdot} \mathbb{Z}_6$$

That is, it is injective, and is neither onto nor the zero map. On the other hand, at the $i = -1$ position we have

$$H_{-1}(P_\bullet^*) = \operatorname{Ext}^1_{\mathbb{Z}_{12}}(\mathbb{Z}_6, \mathbb{Z}_6) = \frac{\operatorname{Ker}\left(\mathbb{Z}_6 \xrightarrow{2\cdot} \mathbb{Z}_6 \right)}{\operatorname{Im}\left(\mathbb{Z}_6 \xrightarrow{0\cdot} \mathbb{Z}_6 \right)} = \frac{3 \cdot \mathbb{Z}_6}{0} \cong 3\mathbb{Z}_6$$

and

$$H_{-1}(Q_\bullet^*) = \operatorname{Ext}^1_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_6) = \frac{\operatorname{Ker}\left(\mathbb{Z}_6 \xrightarrow[=-2\cdot]{4\cdot} \mathbb{Z}_6 \right)}{\operatorname{Im}\left(\mathbb{Z}_6 \xrightarrow{3\cdot} \mathbb{Z}_6 \right)} \cong \frac{3 \cdot \mathbb{Z}_6}{3 \cdot \mathbb{Z}_6} = 0.$$

Therefore the induced map is the zero map and by the periodicity of our diagram, the same will hold for all odd i. Similarly

$$H_{-2}(Q_\bullet^*) = \operatorname{Ext}^2_{\mathbb{Z}_{12}}(\mathbb{Z}_3, \mathbb{Z}_6) = \frac{\operatorname{Ker}\left(\mathbb{Z}_6 \xrightarrow{3\cdot} \mathbb{Z}_6 \right)}{\operatorname{Im}\left(\mathbb{Z}_6 \xrightarrow[=-2\cdot]{4\cdot} \mathbb{Z}_6 \right)} = \frac{2 \cdot \mathbb{Z}_6}{2 \cdot \mathbb{Z}_6} = 0$$

so the periodicity of our ladder diagram lets us conclude $\operatorname{Ext}^i_{\mathbb{Z}_{12}}(\rho, \mathbb{Z}_6) = 0$ for all $i > 0$.

Chapter 6

Long Exact Sequences

In this chapter we achieve the goal set in Section 2.1 by proving the existence of long exact sequences for Ext and the well-definedness of Ext (see Theorems 6.2.1, 6.3.3, and 6.5.2).

6.1 The Mother of All Long Exact Sequences

In this section we prove the existence of long exact sequences in general and we prove the Snake Lemma as a corollary, which we will need for future results, such as Lemmas 6.3.1 and 6.3.2.

Definition 6.1.1. Let $M_\bullet$, $M'_\bullet$, and $M''_\bullet$ be R-complexes. A diagram of chain maps

$$0 \longrightarrow M'_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} M''_\bullet \longrightarrow 0$$

is a *short exact sequence* R-complexes if each row in the ladder is exact.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial^{M'}_{i+1}} & & \downarrow{\scriptstyle \partial^{M}_{i+1}} & & \downarrow{\scriptstyle \partial^{M''}_{i+1}} & & \\
0 & \longrightarrow & M'_i & \xrightarrow{f_i} & M_i & \xrightarrow{g_i} & M''_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^{M'}_{i}} & \curvearrowright & \downarrow{\scriptstyle \partial^{M}_{i}} & \curvearrowright & \downarrow{\scriptstyle \partial^{M''}_{i}} & & \\
0 & \longrightarrow & M'_{i-1} & \xrightarrow[f_{i-1}]{} & M_{i-1} & \xrightarrow[g_{i-1}]{} & M''_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^{M'}_{i-1}} & & \downarrow{\scriptstyle \partial^{M}_{i-1}} & & \downarrow{\scriptstyle \partial^{M''}_{i-1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Theorem 6.1.2. *Consider the following short exact sequence of R-complexes.*

$$0 \longrightarrow M'_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} M''_\bullet \longrightarrow 0$$

Then for every $i \in \mathbb{Z}$ there exists an R-module homomorphism

$$\eth_i : H_i(M''_\bullet) \longrightarrow H_{i-1}(M'_\bullet)$$

$$\overline{m''_i} \longmapsto \overline{m'_{i-1}}$$

such that the following sequence is exact.

$$\cdots \xrightarrow{\eth_{i+1}} H_i(M'_\bullet) \xrightarrow{H_i(f_\bullet)} H_i(M_\bullet) \xrightarrow{H_i(g_\bullet)} H_i(M''_\bullet) \xrightarrow{\eth_i} H_{i-1}(M'_\bullet) \xrightarrow{H_{i-1}(f_\bullet)} \cdots$$

We call $\eth_i$ *a* connecting homomorphism.

Proof. We will prove this in nine steps.

Step 1: Let us construct $\eth_i$. Let $\xi \in H_i(M''_\bullet) = \operatorname{Ker}\left(\partial_i^{M''}\right) / \operatorname{Im}\left(\partial_{i+1}^{M''}\right)$ and let $\alpha \in \operatorname{Ker}\left(\partial_i^{M''}\right)$ such that $\xi = \overline{\alpha} \in H_i(M''_\bullet)$. Since g_i is surjective, let $\beta \in M_i$ such that $g_i(\beta) = \alpha$. Since $g_\bullet$ is a chain map (i.e., since the partials and g_i's commute) and by definition of β we have

$$g_{i-1}\left(\partial_i^M(\beta)\right) = \partial_i^{M''}\left(g_i(\beta)\right) = \partial_i^{M''}(\alpha) = 0.$$

Therefore $\partial_i^M(\beta) \in \operatorname{Ker}(g_{i-1}) = \operatorname{Im}(f_{i-1})$, so we let $\gamma \in M'_{i-1}$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta)$. We define $\eth_i$ in terms of this element γ.

$$\eth_i(\xi) = \overline{\gamma} \in \frac{\operatorname{Ker}\left(\partial_{i-1}^{M'}\right)}{\operatorname{Im}\left(\partial_i^{M'}\right)} = H_{i-1}(M'_\bullet)$$

We need to show $\gamma \in \operatorname{Ker}\left(\partial_{i-1}^{M'}\right)$, which we will do first in the next step.

Step 2: We show $\eth_i$ is well-defined. First we have

$$\begin{aligned} f_{i-2}\left(\partial_{i-1}^{M'}(\gamma)\right) &= \partial_{i-1}^M\left(f_{i-1}(\gamma)\right) && F_\bullet \text{ a chain map} \\ &= \partial_{i-1}^M\left(\partial_i^M(\beta)\right) && \text{definition of } \gamma \\ &= 0. && M_\bullet \text{ an } R\text{-complex} \end{aligned}$$

Since f_{i-2} is injective, this implies $\partial_{i-1}^{M'}(\gamma) = 0$, i.e., $\gamma \in \operatorname{Ker}\left(\partial_{i-1}^{M'}\right)$, as desired.

Second we will show $\overline{\gamma} \in H_{i-1}(M'_\bullet)$ is independent of any choices made in Step 1. Let $\alpha, \alpha' \in \operatorname{Ker}\left(\partial_i^{M''}\right)$ such that $\overline{\alpha} = \xi = \overline{\alpha'}$, let $\beta, \beta' \in M_i$ such that $g_i(\beta) = \alpha$ and $g_i(\beta') = \alpha'$, and let $\gamma, \gamma' \in M'_{i-1}$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta)$ and $f_{i-1}(\gamma') = \partial_i^M(\beta')$. We need to show $\overline{\gamma} = \overline{\gamma'}$ in $H_{i-1}(M'_\bullet) = \operatorname{Ker}\left(\partial_{i-1}^{M'}\right) / \operatorname{Im}\left(\partial_i^{M'}\right)$, or in other words, we need to show $\gamma - \gamma' \in \operatorname{Im}\left(\partial_i^{M'}\right)$.

By assumption $\overline{\alpha} = \overline{\alpha'} \in H_i(M'_\bullet) = \operatorname{Ker}\left(\partial_i^{M'}\right) / \operatorname{Im}\left(\partial_{i+1}^{M'}\right)$, so $\alpha - \alpha' \in \operatorname{Im}\left(\partial_{i+1}^{M'}\right)$ and we let $\eta \in M''_{i+1}$ such that $\partial_{i+1}^{M''}(\eta) = \alpha - \alpha'$. Since g_{i+1} is surjective, we may let $\nu \in M_{i+1}$ such that $g_{i+1}(\nu) = \eta$ and we compute the following.

$$g_i(\beta - \beta' - \partial_{i+1}^M(\nu)) = g_i(\beta) - g_i(\beta') - (g_i \circ \partial_{i+1}^M)(\nu) = \alpha - \alpha' - (\alpha - \alpha') = 0$$

In the above calculation we rely only on the definitions of our elements and the linearity of g_i. By this calculation we know $\beta - \beta' - \partial_{i+1}^M(\nu) \in \operatorname{Ker}(g_i) = \operatorname{Im}(f_i)$ so let $\omega \in M'_i$ such that $f_i(\omega) =$

$\beta - \beta' - \partial_{i+1}^M(\nu)$. Since $\gamma, \gamma', \partial_i^{M'}(\omega) \in M'_{i-1}$, we compute as follows.

$$\begin{aligned}
f_{i-1}\left(\partial_i^{M'}(\omega) - (\gamma - \gamma')\right) &= \left(f_{i-1} \circ \partial_i^{M'}\right)(\omega) - f_{i-1}(\gamma) + f_{i-1}(\gamma') && \text{linearity} \\
&= \left(\partial_i^M \circ f_i\right)(\omega) - \partial_i^M(\beta) + \partial_i^M(\beta') && f_\bullet \text{ a chain map} \\
&= \partial_i^M\left(\beta - \beta' - \partial_{i+1}^M(\nu)\right) - \partial_i^M(\beta) + \partial_i^M(\beta') && \text{definition of } \omega \\
&= \partial_i^M\left(\beta - \beta' - \partial_{i+1}^M(\nu) - \beta + \beta'\right) && \text{linearity} \\
&= -\left(\partial_i^M \circ \partial_{i+1}^M\right)(\nu) \\
&= 0 && M_\bullet \text{ an } R\text{-complex}
\end{aligned}$$

Since f_{i-1} is injective, this implies $\partial_i^{M'}(\omega) - (\gamma - \gamma') = 0$ or equivalently

$$\gamma - \gamma' = \partial_i^{M'}(\omega) \in \operatorname{Im}\left(\partial_i^{M'}\right)$$

completing this step.

Step 3: Here we prove $\eth_i$ is an R-module homomorphism. Let $\xi, \xi' \in H_i(M''_\bullet)$ and $r \in R$. Also let $\alpha, \alpha' \in \operatorname{Ker}\left(\partial_i^{M''}\right)$ such that $\overline{\alpha} = \xi$ and $\overline{\alpha'} = \xi'$, let $\beta, \beta' \in M_i$ such that $g_i(\beta) = \alpha$ and $g_i(\beta') = \alpha'$, and let $\gamma, \gamma' \in M'_{i-1}$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta)$ and $f_{i-1}(\gamma') = \partial_i^M(\beta')$.

Notice that $r\alpha + \alpha' \in \operatorname{Ker}\left(\partial_i^{M''}\right)$ and hence it makes sense to write $\overline{r\alpha + \alpha'} = r\xi + \xi'$. Notice also that $r\beta + \beta' \in M_i$ so we have

$$g_i(r\beta + \beta') = g_i(r\beta) + g_i(\beta') = r \cdot g_i(\beta) + g_i(\beta') = r\alpha + \alpha'.$$

Finally note that $r\gamma + \gamma' \in M'_{i-1}$ for which we have

$$\begin{aligned}
f_{i-1}(r\gamma + \gamma') &= f_{i-1}(r\gamma) + f_{i-1}(\gamma') = r \cdot f_{i-1}(\gamma) + f_{i-1}(\gamma') \\
&= r \cdot \partial_i^M(\beta) + \partial_i^M(\beta') = \partial_i^M(r\beta) + \partial_i^M(\beta') = \partial_i^M(r\beta + \beta').
\end{aligned}$$

Therefore we have an element satisfying the definition of $\eth_i$ described in Step 1 so we conclude this step in the following display.

$$\eth_i(r\xi + \xi') = \overline{r\gamma + \gamma'} = r \cdot \overline{\gamma} + \overline{\gamma} = r \cdot \eth_i(\xi) + \eth_i(\xi)$$

Step 4: We tackle the first of several questions of exactness. Here we show $\operatorname{Im}(H_i(f_\bullet)) \subseteq \operatorname{Ker}(H_i(g_\bullet))$. Let $\delta \in H_i(M'_\bullet)$ and let $\rho \in \operatorname{Ker}\left(\partial_i^{M'}\right)$ such that $\overline{\rho} = \delta$. Therefore we have

$$H_i(g_\bullet)\left(H_i(f_\bullet)(\delta)\right) = H_i(g_\bullet)\left(\overline{f_i(\rho)}\right) = \overline{(g_i \circ f_i)(\rho)} = \overline{0} = 0$$

where the third equality comes from the exactness of the original sequence of chain maps.

Step 5: We now show $\operatorname{Im}(H_i(f_\bullet)) \supseteq \operatorname{Ker}(H_i(g_\bullet))$. Let $\delta \in \operatorname{Ker}(H_i(g_\bullet))$ and let $\rho \in \operatorname{Ker}\left(\partial_i^M\right)$ such that $\overline{\rho} = \delta$. This gives

$$0 = H_i(g_\bullet)(\overline{\rho}) = \overline{g_i(\rho)} \in H_i(M''_\bullet) = \frac{\operatorname{Ker}\left(\partial_i^{M''}\right)}{\operatorname{Im}\left(\partial_{i+1}^{M''}\right)}.$$

Therefore $g_i(\rho) \in \operatorname{Im}\left(\partial_{i+1}^{M''}\right)$ so we lift to some $\mu \in M''_{i+1}$ such that $\partial_{i+1}^{M''}(\mu) = g_i(\rho)$ and lift again to some $\sigma \in M_{i+1}$ such that $g_{i+1}(\sigma) = \mu$ (since g_{i+1} is surjective). Since $\rho, \partial_{i+1}^M(\sigma) \in M_i$, we consider the element $\rho - \partial_{i+1}^M(\sigma) \in M_i$. Using linearity and the fact that $g_\bullet$ is a chain map we compute

$$g_i(\rho - \partial_{i+1}^M(\sigma)) = g_i(\rho) - (g_i \circ \partial_{i+1}^M)(\sigma) = g_i(\rho) - (\partial_{i+1}^{M''} \circ g_{i+1})(\sigma) = g_i(\rho) - \partial_{i+1}^{M''}(\mu) = 0.$$

Hence $\rho - \partial_{i+1}^{M}(\sigma) \in \text{Ker}(g_i) = \text{Im}(f_i)$ and we let $\tau \in M_i'$ such that $f_i(\tau) = \rho - \partial_{i+1}^{M}(\sigma)$. We claim $\tau \in \text{Ker}\left(\partial_i^{M'}\right)$ and point out it suffices to show $\left(f_{i-1} \circ \partial_i^{M'}\right)(\tau) = 0$ since f_{i-1} is injective. We compute

$$\left(f_{i-1} \circ \partial_i^{M'}\right)(\tau) = \partial_i^M(f_i(\tau)) = \partial_i^M(\rho - \partial_{i+1}^M(\sigma)) = \partial_i^M(\rho) - \left(\partial_i^M \circ \partial_{i+1}^M\right)(\sigma) = 0$$

where the last equality holds by definition of ρ and because $M_\bullet$ is a chain complex.

We consider $\rho, \partial_{i+1}^M(\sigma) \in \text{Ker}\left(\partial_i^M\right)$ and $\tau \in \text{Ker}\left(\partial_i^{M'}\right)$, which represent the cosets $\overline{\rho}, \overline{\partial_{i+1}^M(\sigma)} \in H_i(M_\bullet)$ and $\overline{\tau} \in H_i(M_\bullet')$. Therefore it makes sense to compute

$$H_i(f_\bullet)(\overline{\tau}) = \overline{f_i(\tau)} = \overline{\rho - \partial_{i+1}^M(\sigma)} = \overline{\rho} - \overline{\partial_{i+1}^M(\sigma)} = \overline{\rho} - \overline{0} = \overline{\rho} = \delta.$$

Hence $\delta \in \text{Im}(H_i(f_\bullet))$, completing this step.

Step 6: Continuing our proof of exactness, we show here that $\text{Im}(H_i(g_\bullet)) \subseteq \text{Ker}(\eth_i)$. Let $\zeta \in H_i(M_\bullet)$ and let $\beta \in \text{Ker}\left(\partial_i^M\right)$ such that $\overline{\beta} = \zeta$. We want to show that $(\eth_i \circ H_i(g_\bullet))(\overline{\beta}) = 0$. Define $\alpha = g_i(\beta)$ and we have

$$H_i(g_\bullet)(\overline{\beta}) = \overline{g_i(\beta)} = \overline{\alpha}.$$

Computing $\eth_i(H_i(g_\bullet)(\overline{\beta})) = \eth_i(\overline{\alpha})$ requires some $\gamma \in \text{Ker}\left(\partial_{i-1}^{M'}\right)$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta)$. Since $\beta \in \text{Ker}\left(\partial_i^M\right)$ by assumption, $\partial_i^M(\beta) = 0 = f_{i-1}(0)$, so setting $\gamma = 0$ we get

$$\eth_i(\overline{\alpha}) = \overline{\gamma} = \overline{0} = 0.$$

Step 7: We now show $\text{Im}(H_i(g_\bullet)) \supseteq \text{Ker}(\eth_i)$. Let $\xi \in \text{Ker}(\eth_i) \subseteq H_i(M_\bullet'')$ and let $\alpha \in \text{Ker}\left(\partial_i^{M''}\right)$ such that $\xi = \overline{\alpha}$. Fix some $\beta \in M_i$ such that $g_i(\beta) = \alpha$ and some $\gamma \in M_{i-1}'$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta) \in \text{Ker}(g_{i-1}) = \text{Im}(f_{i-1})$. Our construction in Step 1 implies $\eth_i(\xi) = \overline{\gamma}$ so we have

$$0 = \eth_i(\xi) = \overline{\gamma} \in H_{i-1}(M_\bullet') = \frac{\text{Ker}\left(\partial_{i-1}^{M'}\right)}{\text{Im}\left(\partial_i^{M'}\right)}.$$

Hence $\gamma \in \text{Im}\left(\partial_i^{M'}\right)$ and we let $\omega \in M_i'$ such that $\partial_i^{M'}(\omega) = \gamma$. Moreover, $f_i(\omega), \beta \in M_i$ so we compute the following.

$$\begin{aligned}
\partial_i^M(\beta - f_i(\omega)) &= \partial_i^M(\beta) - \left(\partial_i^M \circ f_i\right)(\omega) && \text{linearity} \\
&= \partial_i^M(\beta) - \left(f_{i-1} \circ \partial_i^{M'}\right)(\omega) && f_\bullet \text{ a chain complex} \\
&= \partial_i^M(\beta) - f_{i-1}(\gamma) && \text{definition of } \omega \\
&= \partial_i^M(\beta) - \partial_i^M(\beta) && \text{definition of } \gamma \\
&= 0
\end{aligned}$$

Therefore $\beta - f_i(\omega) \in \text{Ker}\left(\partial_i^M\right)$ and hence $\overline{\beta - f_i(\omega)} \in H_i(M_\bullet)$. We may also compute

$$H_i(g_\bullet)(\overline{\beta - f_i(\omega)}) = \overline{g_i(\beta - f_i(\omega))} = \overline{g_i(\beta) - (g_i \circ f_i)(\omega)} = \overline{g_i(\beta)} = \overline{\alpha} = \xi$$

where the third equality holds by the exactness of the i^{th} row of the given diagram. Hence $\xi \in \text{Im}(H_i(g_\bullet))$, which completes this step.

Step 8: Here we show $\text{Im}(\eth_i) \subseteq \text{Ker}(H_{i-1}(f_\bullet))$. Let $\xi \in H_i(M_\bullet'')$ and let $\alpha \in \text{Ker}\left(\partial_i^{M''}\right)$ such that $\xi = \overline{\alpha}$. We want to show that $H_{i-1}(f_\bullet)(\eth_i(\overline{\alpha})) = 0$. Since g_i is surjective, let $\beta \in M_i$ such that

$g_i(\beta) = \alpha$ and since $\partial_i^M(\beta) \in \operatorname{Ker}(g_{i-1}) = \operatorname{Im}(f_{i-1})$, let $\gamma \in M'_{i-1}$ such that $f_{i-1}(\gamma) = \partial_i^M(\beta)$. We therefore have

$$H_{i-1}(f_\bullet)(\eth_i(\overline{\alpha})) = H_{i-1}(f_\bullet)(\overline{\gamma}) = \overline{f_{i-1}(\gamma)} = \overline{\partial_i^M(\beta)} = 0$$

which completes this step.

Step 9: We finally show that $\operatorname{Im}(\eth_i) \supseteq \operatorname{Ker}(H_{i-1}(f_\bullet))$. Let $\lambda \in \operatorname{Ker}(H_{i-1}(f_\bullet))$ and fix some element $\gamma \in \operatorname{Ker}\left(\partial_{i-1}^{M'}\right)$ such that $\lambda = \overline{\gamma} \in H_{i-1}(M'_\bullet)$. By assumption we have

$$0 = H_{i-1}(f_\bullet)(\lambda) = H_{i-1}(f_\bullet)(\overline{\gamma}) = \overline{f_{i-1}(\gamma)} \in H_{i-1}(M_\bullet) = \frac{\operatorname{Ker}\left(\partial_{i-1}^M\right)}{\operatorname{Im}\left(\partial_i^M\right)}.$$

It follows that $f_{i-1}(\gamma) \in \operatorname{Im}\left(\partial_i^M\right)$, so we may let $\beta \in M_i$ such that $\partial_i^M(\beta) = f_{i-1}(\gamma)$. Denote $g_i(\beta) = \alpha$ and notice by our construction in Step 1, this element is a good candidate on which to apply $\eth_i$. Observe that

$$\partial_i^{M''}(\alpha) = \partial_i^{M''}(g_i(\beta)) = g_{i-1}(\partial_i^M(\beta)) = (g_{i-1} \circ f_{i-1})(\gamma) = 0$$

so $\alpha \in \operatorname{Ker}\left(\partial_i^{M''}\right)$. Therefore $\overline{\alpha} \in H_i(M''_\bullet)$ and

$$\eth_i(\overline{\alpha}) = \overline{\gamma} = \lambda.$$

This completes the proof of the theorem. □

Corollary 6.1.3 (Snake Lemma)**.** *Consider a commutative diagram of R-modules and R-module homomorphisms with exact rows.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & M''_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial'_1} & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial''_1} & & \\
0 & \longrightarrow & M'_0 & \xrightarrow{f_1} & M_0 & \xrightarrow{g_1} & M''_0 & \longrightarrow & 0
\end{array}$$

There exists an exact sequence

$$\begin{array}{l}
0 \longrightarrow \operatorname{Ker}(\partial'_1) \longrightarrow \operatorname{Ker}(\partial_1) \longrightarrow \operatorname{Ker}(\partial''_1) \longrightarrow \\
\longrightarrow \operatorname{Coker}(\partial'_1) \longrightarrow \operatorname{Coker}(\partial_1) \longrightarrow \operatorname{Coker}(\partial''_1) \longrightarrow 0.
\end{array}$$

Proof. From the given diagram, we extend to form the following short exact sequence of R-complexes.

$$0 \longrightarrow M'_\bullet \longrightarrow M_\bullet \longrightarrow M''_\bullet \longrightarrow 0$$

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow{\scriptstyle \epsilon'_1} & & \downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \epsilon''_1} & & \\
0 & \longrightarrow & M'_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & M''_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial'_1} & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial''_1} & & \\
0 & \longrightarrow & M'_0 & \xrightarrow{f_0} & M_0 & \xrightarrow{g_0} & M''_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau'_1} & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau''_1} & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Note that the columns in this diagram are R-complexes, because

$$\mathrm{Im}\,(\epsilon'_1) = \{0\} \subseteq \mathrm{Ker}\,(\partial'_1)$$
$$\mathrm{Im}\,(\partial'_1) \subseteq M'_0 = \mathrm{Ker}\,(\tau'_1)$$

and similarly for the other two columns. By Theorem 6.1.2, we have the following long exact sequence.

$$\begin{array}{l}
\cdots \longrightarrow H_2(M''_\bullet) \longrightarrow \\
H_1(M'_\bullet) \longrightarrow H_1(M_\bullet) \longrightarrow H_1(M''_\bullet) \longrightarrow \\
H_0(M'_\bullet) \longrightarrow H_0(M_\bullet) \longrightarrow H_0(M''_\bullet) \longrightarrow \\
H_{-1}(M'_\bullet) \longrightarrow \cdots
\end{array}$$

By construction

$$H_i(M'_\bullet) = H_i(M_\bullet) = H_i(M''_\bullet) = 0$$

for all $i > 1$ and all $i < 0$. Checking definitions of the remaining six homology modules verifies the claim. $\square$

Remark 6.1.4. In the context of Corollary 6.1.3, we know for each $i = 0, 1, 2$, $\partial_1^{(i)}$ is injective if and only if $\mathrm{Ker}\left(\partial_1^{(i)}\right) = 0$. For a consequence of this, suppose ∂''_1 is injective. Then in our long exact sequence we have

$$0 \longrightarrow \mathrm{Ker}\,(\partial'_1) \longrightarrow \mathrm{Ker}\,(\partial_1) \longrightarrow \mathrm{Ker}\,(\partial''_1) = 0$$

and from Fact 3.2.24, it follows that $\operatorname{Ker}(\partial_1') = 0$ if and only if $\operatorname{Ker}(\partial_1) = 0$, i.e., ∂_1' is injective if and only if ∂_1 is injective. In a similar fashion, if we suppose that ∂_1' is surjective (i.e., $\operatorname{Coker}(\partial_1') = 0$), then ∂_1 is surjective if and only if ∂_1'' is surjective. The proof of this is analogous using the latter half of the long exact sequence in Corollary 6.1.3.

6.2 The First Long Exact Sequence in Ext

In this section we use Theorem 6.1.2 to establish the first of two long exact sequences of Ext modules associated to a given short exact sequence of R-modules. We also motivate another long exact sequence in Discussion 6.2.2.

Theorem 6.2.1. *Let L be an R-module and let*

$$0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$$

be a short exact sequence of R-modules. There exists the following long exact sequence associated to $\operatorname{Ext}^i_R(L, -)$.

$$\begin{array}{l}
0 \longrightarrow \operatorname{Hom}_R(L, N') \longrightarrow \operatorname{Hom}_R(L, N) \longrightarrow \operatorname{Hom}_R(L, N'') \\
\longrightarrow \operatorname{Ext}^1_R(L, N') \longrightarrow \operatorname{Ext}^1_R(L, N) \longrightarrow \operatorname{Ext}^1_R(L, N'') \\
\longrightarrow \cdots \qquad\qquad \cdots \longrightarrow \operatorname{Ext}^{i-1}_R(L, N'') \\
\longrightarrow \operatorname{Ext}^i_R(L, N') \longrightarrow \operatorname{Ext}^i_R(L, N) \longrightarrow \operatorname{Ext}^i_R(L, N'') \\
\longrightarrow \operatorname{Ext}^{i+1}_R(L, N') \longrightarrow \cdots
\end{array}$$

Proof. Let $P_\bullet$ be a projective resolution for L. We claim that the R-complexes $\operatorname{Hom}_R(P_\bullet, N')$, $\operatorname{Hom}_R(P_\bullet, N)$, and $\operatorname{Hom}_R(P_\bullet, N'')$ form a short exact sequence of complexes, to which we may apply Theorem 6.1.2 to achieve the desired result. See the diagram on the following page.

Since P_i is projective for all i, $\operatorname{Hom}_R(P_i, -)$ is exact for all i and therefore the rows are all exact. Furthermore the diagrams commute by the associativity of function composition. Hence we have a short exact sequence of R-complexes and the associated long exact sequence has the desired shape, since

$$H_{-i}(\operatorname{Hom}_R(P_\bullet, N^{(j)})) = \operatorname{Ext}^i_R(L, N^{(j)})$$

for all i and $j = 0, 1, 2$.

$$0 \longrightarrow \operatorname{Hom}_R(P_\bullet, N') \xrightarrow{\operatorname{Hom}_R(P_\bullet,\alpha)} \operatorname{Hom}_R(P_\bullet, N) \xrightarrow{\operatorname{Hom}_R(P_\bullet,\beta)} \operatorname{Hom}_R(P_\bullet, N'') \longrightarrow 0$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_R(P_0, N') & \underset{\alpha_*}{\xrightarrow{\operatorname{Hom}_R(P_0,\alpha)}} & \operatorname{Hom}_R(P_0, N) & \underset{\beta_*}{\xrightarrow{\operatorname{Hom}_R(P_0,\beta)}} & \operatorname{Hom}_R(P_0, N'') & \longrightarrow & 0 \\
 & & {\scriptstyle \operatorname{Hom}_R(\partial_1^P,N')}\downarrow{\scriptstyle (\partial_1^P)^*} & & {\scriptstyle \operatorname{Hom}_R(\partial_1^P,N)}\downarrow{\scriptstyle (\partial_1^P)^*} & & {\scriptstyle \operatorname{Hom}_R(\partial_1^P,N'')}\downarrow{\scriptstyle (\partial_1^P)^*} & & \\
0 & \longrightarrow & \operatorname{Hom}_R(P_1, N') & \xrightarrow{\operatorname{Hom}_R(P_1,\alpha)} & \operatorname{Hom}_R(P_1, N) & \xrightarrow{\operatorname{Hom}_R(P_1,\beta)} & \operatorname{Hom}_R(P_1, N'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_R(P_i, N') & \longrightarrow & \operatorname{Hom}_R(P_i, N) & \longrightarrow & \operatorname{Hom}_R(P_i, N'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

□

Discussion 6.2.2. Here we describe how one might obtain the other long exact sequence from Theorem 2.1.1, namely

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{Hom}_R(N'', L) & \longrightarrow & \operatorname{Hom}_R(N, L) & \longrightarrow & \operatorname{Hom}_R(N', L) & \\
\longrightarrow & \operatorname{Ext}^i_R(N'', L) & \longrightarrow & \operatorname{Ext}^i_R(N, L) & \longrightarrow & \operatorname{Ext}^i_R(N', L) & \longrightarrow \cdots \\
 & \| & & \| & & \| & \\
 & H_{-i}(\operatorname{Hom}_R(Q''_\bullet, L)) & & H_{-i}(\operatorname{Hom}_R(Q_\bullet, L)) & & H_{-i}(\operatorname{Hom}_R(Q'_\bullet, L)) &
\end{array}$$

where $Q''_\bullet$, $Q_\bullet$, and $Q'_\bullet$ are projective resolutions of N'', N, and N', respectively. For this we would need a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(Q''_\bullet, L) \longrightarrow \operatorname{Hom}_R(Q_\bullet, L) \longrightarrow \operatorname{Hom}_R(Q'_\bullet, L) \longrightarrow 0$$

which requires a short exact sequence

$$0 \longrightarrow Q'_\bullet \longrightarrow Q_\bullet \longrightarrow Q''_\bullet \longrightarrow 0 \tag{†}$$

such that $\operatorname{Hom}_R(\dagger, L)$ is exact. Note that if there exists a short exact sequence (†), then it actually follows that $\operatorname{Hom}_R(\dagger, L)$ is exact by the following. Consider an arbitrary row of (†).

$$0 \longrightarrow Q'_i \longrightarrow Q_i \longrightarrow Q''_i \longrightarrow 0 \tag{‡}$$

Since Q_i'' is projective, the sequence (‡) splits, so $\mathrm{Hom}_R(\ddagger, L)$ is split exact (and therefore exact). So given a short exact sequence $0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$, we want to construct a short exact sequence of projective resolutions as in (†). The good news is we already have a means of lifting α and β to chain maps $Q'_\bullet \xrightarrow{A} Q_\bullet$ and $Q_\bullet \xrightarrow{B} Q''_\bullet$, respectively. However, the resulting short sequence is not exact in general. We let this serve as motivation for the horseshoe lemma in the next section.

6.3 The Horseshoe Lemma and Second Long Exact Sequence in Ext

In this section we prove the Horseshoe Lemma (Lemma 6.3.2) and use it to prove the existence of the long exact sequence described in Discussion 6.2.2.

Lemma 6.3.1. *Consider a short exact sequence of R-modules and R-module homomorphisms.*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

Let $\tau' : P' \to M'$ and $\tau'' : P'' \to M''$ be surjections where P' and P'' are projective. There is a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \xrightarrow{\epsilon} & P' \oplus P'' & \xrightarrow{\pi} & P'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau''} & & \\ 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{6.3.1.1}$$

where ϵ and π are the natural injection and surjection, respectively.

Proof. Use the fact that P'' is projective (see Definition 1.1.14) to find an R-module homomorphism $h : P'' \to M$ making the following diagram commute.

$$\begin{array}{ccccc} & & P'' & & \\ & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle \tau''} & & \\ M & \xrightarrow{g} & M'' & \longrightarrow & 0 \end{array}$$

Define $\tau : P' \oplus P'' \to M$ by the formula

$$\tau(p', p'') = f(\tau'(p')) + h(p'').$$

The map τ is well defined by construction. Let $\alpha, \beta \in P'$, $\xi, \zeta \in P''$, and $r \in R$. We check that τ is an R-module homomorphism below.

$$\begin{aligned} \tau(r(\alpha,\xi) + (\beta,\zeta)) &= \tau(r\alpha + \beta\ ,\ r\xi + \zeta) \\ &= f(\tau'(r\alpha + \beta)) \ + \ h(r\xi + \zeta) \\ &= f(r \cdot \tau'(\alpha) + \tau'(\beta)) \ + \ r \cdot h(\xi) + h(\zeta) \\ &= r \cdot f(\tau'(\alpha)) + f(\tau'(\beta)) \ + \ r \cdot h(\xi) + h(\zeta) \\ &= r \cdot [f(\tau'(\alpha)) + h(\xi)] \ + \ [f(\tau'(\beta)) + h(\zeta)] \\ &= r \cdot \tau(\alpha,\xi) \ + \ \tau(\beta,\zeta) \end{aligned}$$

We also show τ makes (6.3.1.1) commute. For any $p' \in P'$ we have

$$\tau(\epsilon(p')) = \tau(p', 0) = f(\tau'(p'))$$

so the square on the left side commutes. For any $(p', p'') \in P' \oplus P''$ we have

$$\begin{aligned}
&\tau''(\pi(p', p'')) = \tau''(p'') \\
&g[\tau(p', p'')] = g[f(\tau'(p')) + h(p'')] = (g \circ f)(\tau'(p')) + g[h(p'')] = 0 + g(h(p''))
\end{aligned}$$

where the zero in the last step comes from the exactness of the given short exact sequence. The two results are equal by definition of the map h. Therefore the square on the right in (6.3.1.1) commutes.

Since τ' and τ'' are each surjective the left and right columns of (6.3.1.1) are exact. Moreover, the Snake Lemma (see Remark 6.1.4) shows that τ must be surjective as well and the center column is exact, completing the proof. □

Lemma 6.3.2 (Horseshoe Lemma)**.** *Consider the short exact sequence of R-modules and R-module homomorphisms.*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

Let $P'_\bullet$ and $P''_\bullet$ be projective resolutions of M' and M'', respectively. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial_2^{P'}} & & \downarrow{\scriptstyle \partial_2^{P}} & & \downarrow{\scriptstyle \partial_2^{P''}} & & \\
0 & \longrightarrow & P'_1 & \xrightarrow{F_1} & P_1 & \xrightarrow{G_1} & P''_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_1^{P'}} & & \downarrow{\scriptstyle \partial_1^{P}} & & \downarrow{\scriptstyle \partial_1^{P''}} & & \\
0 & \longrightarrow & P'_0 & \xrightarrow{F_0} & P_0 & \xrightarrow{G_0} & P''_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau''} & & \\
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

such that the middle column is an augmented projective resolution of M.

Proof. Note that each row of the diagram, aside from the bottom row, will be split since each P''_i is projective for all $i \in \mathbb{N}$. Using Lemma 6.3.1 we construct a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P'_0 & \xrightarrow{F_0} & P'_0 \oplus P''_0 & \xrightarrow{G_0} & P''_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau''} & & \\
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where F_0 and G_0 are the natural injection and surjection (ϵ and π from the lemma), respectively. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(\tau') & \xrightarrow{f_1} & \mathrm{Ker}(\tau) & \xrightarrow{g_1} & \mathrm{Ker}(\tau'') & \longrightarrow & 0 \\
 & & {\scriptstyle\subseteq}\downarrow & & {\scriptstyle\subseteq}\downarrow & & {\scriptstyle\subseteq}\downarrow & & \\
0 & \longrightarrow & P_0' & \xrightarrow{F_0} & P_0 & \xrightarrow{G_0} & P_0'' & \longrightarrow & 0 \\
 & & {\scriptstyle\tau'}\downarrow & & {\scriptstyle\tau}\downarrow & & {\scriptstyle\tau''}\downarrow & & \\
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{6.3.2.1}$$

where f_1 and g_1 are induced by f and g, respectively, and $P_0 = P_0' \oplus P_0''$. The columns are exact by construction and the top row is exact by the Snake Lemma (6.1.3), because the cokernel of a surjection is zero. Hence we have exactness everywhere.

For ease of notation, let $M_1' = \mathrm{Ker}(\tau')$, $M_1 = \mathrm{Ker}(\tau)$, and $M_1'' = \mathrm{Ker}(\tau'')$. We may apply Lemma 6.3.1 again to build another commutative diagram with exact rows and columns, defining M_2', M_2, and M_2'' similarly.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_2' & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & M_2'' & \longrightarrow & 0 \\
 & & {\scriptstyle\subseteq}\downarrow & & {\scriptstyle\subseteq}\downarrow & & {\scriptstyle\subseteq}\downarrow & & \\
0 & \longrightarrow & P_1' & \xrightarrow{F_1} & P_1 & \xrightarrow{G_1} & P_1'' & \longrightarrow & 0 \\
 & & {\scriptstyle\tau_1'}\downarrow & & {\scriptstyle\tau_1}\downarrow & & {\scriptstyle\tau_1''}\downarrow & & \\
0 & \longrightarrow & M_1' & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & M_1'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{6.3.2.2}$$

Splicing (6.3.2.1) and (6.3.2.2) together, we obtain a slightly larger diagram with rows and columns still exact.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_2' & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & M_2'' & \longrightarrow & 0 \\
 & & \subseteq\downarrow & & \subseteq\downarrow & & \subseteq\downarrow & & \\
0 & \longrightarrow & P_1' & \xrightarrow{F_1} & P_1 & \xrightarrow{G_1} & P_1'' & \longrightarrow & 0 \\
 & & \tau_1'\downarrow & & \tau_1\downarrow & & \tau_1''\downarrow & & \\
0 & \longrightarrow & P_0' & \xrightarrow{F_0} & P_0 & \xrightarrow{G_0} & P_0'' & \longrightarrow & 0 \\
 & & \tau'\downarrow & & \tau\downarrow & & \tau''\downarrow & & \\
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We have cheated a bit by using the names τ_1', τ_1, and τ_1'', but note that there are copies of M_1', M_1, and M_1'' sitting inside of P_0', P_0, and P_0'', respectively. We may repeat this construction inductively to achieve the desired diagram. □

With the Horseshoe Lemma established, we are able to give the long exact sequence we described in Discussion 6.2.2.

Theorem 6.3.3. *Let L be an R-module and let*

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

be a sequence of R-modules. There exists the following long exact sequence associated to $\operatorname{Ext}_R^i(-, L)$.

$$\begin{array}{l}
0 \longrightarrow \operatorname{Hom}_R(N'', L) \longrightarrow \operatorname{Hom}_R(N, L) \longrightarrow \operatorname{Hom}_R(N', L) \\
\longrightarrow \operatorname{Ext}_R^1(N'', L) \longrightarrow \operatorname{Ext}_R^1(N, L) \longrightarrow \operatorname{Ext}_R^1(N', L) \\
\longrightarrow \cdots \qquad \cdots \longrightarrow \operatorname{Ext}_R^{i-1}(N', L) \\
\longrightarrow \operatorname{Ext}_R^i(N'', L) \longrightarrow \operatorname{Ext}_R^i(N, L) \longrightarrow \operatorname{Ext}_R^i(N', L) \\
\longrightarrow \operatorname{Ext}_R^{i+1}(N'', L) \longrightarrow \cdots
\end{array}$$

Proof. By Discussion 6.2.2 we need only justify the existence of a short exact sequence

$$0 \longrightarrow Q'_\bullet \longrightarrow Q_\bullet \longrightarrow Q''_\bullet \longrightarrow 0$$

of R-complexes where $Q'_\bullet$, $Q_\bullet$, and $Q''_\bullet$ are projective resolutions of N', N, and N'', respectively. This has just been shown in the Horseshoe Lemma above, so the proof is done. □

6.4 Mapping Cones

In this section we explore mapping cones and quasiisomorphisms. Both are needed for Lemmas 6.5.1 and 6.5.3, which are each used directly to prove Ext is well-defined (Theorem 6.5.2). Proposition 6.4.9 and Lemma 6.4.13 from Schanuel are also used directly in the proof of the well-definedness of Ext.

Definition 6.4.1. Let $X_\bullet$ be an R-complex. The *shift of* $X_\bullet$, or the *suspension of* $X_\bullet$, is denoted $\Sigma X_\bullet$ where

$$(\Sigma X)_i = X_{i-1} \qquad \text{and} \qquad \partial_i^{\Sigma X} = -\partial_{i-1}^X.$$

Remark 6.4.2. We line up the R-complex $X_\bullet$ with its shift.

$$X_\bullet = \qquad \cdots \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \xrightarrow{\partial_{i-1}^X} \cdots$$

$$\Sigma X_\bullet = \qquad \cdots \xrightarrow{-\partial_i^X} X_{i-1} \xrightarrow{-\partial_{i-1}^X} X_{i-2} \xrightarrow{-\partial_{i-2}^X} \cdots$$

We now verify that the shift of $X_\bullet$ is itself an R-complex and that

$$H_i(\Sigma X_\bullet) = H_{i-1}(X_\bullet).$$

Colloquially, we want to verify that the homology of a shift is just a shift in the homology. Certainly $\Sigma X_\bullet$ is a sequence of R-module homomorphisms and since $X_\bullet$ is an R-complex we also have

$$-\partial_{i-1}^X \circ -\partial_i^X = \partial_{i-1}^X \circ \partial_i^X = 0.$$

Hence $\Sigma X_\bullet$ is an R-complex. By definition of homology we have

$$H_{i-1}(X_\bullet) = \frac{\operatorname{Ker}\left(\partial_{i-1}^X\right)}{\operatorname{Im}\left(\partial_i^X\right)} \qquad\qquad H_i(\Sigma X_\bullet) = \frac{\operatorname{Ker}\left(-\partial_{i-1}^X\right)}{\operatorname{Im}\left(-\partial_i^X\right)}$$

and these two are equal since $\operatorname{Ker}\left(-\partial_{i-1}^X\right) = \operatorname{Ker}\left(\partial_{i-1}^X\right)$ and $\operatorname{Im}\left(-\partial_i^X\right) = \operatorname{Im}\left(\partial_i^X\right)$.

Definition 6.4.3. Let $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ be a chain map. We define the *mapping cone* of $f_\bullet$ as

$$\operatorname{Cone}(f_\bullet) = \qquad \cdots \longrightarrow \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix}} \begin{matrix} Y_{i-1} \\ \oplus \\ X_{i-2} \end{matrix} \longrightarrow \cdots$$

where for every $i \in \mathbb{Z}$

$$\operatorname{Cone}(f_\bullet)_i = \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix}$$

and

$$\partial_i^{\operatorname{Cone}(f_\bullet)} \begin{pmatrix} y_i \\ x_{i-1} \end{pmatrix} = \begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix} \begin{pmatrix} y_i \\ x_{i-1} \end{pmatrix} = \begin{pmatrix} \partial_i^Y(y_i) + f_{i-1}(x_{i-1}) \\ -\partial_{i-1}^X(x_{i-1}) \end{pmatrix}.$$

Proposition 6.4.4. *If $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ is a chain map, then* $\operatorname{Cone}(f_\bullet)$ *is an R-complex.*

Proof. First we verify that the cone is a sequence of R-module homomorphisms. Each element $\operatorname{Cone}(f_\bullet)_i$ is a direct sum of two R-modules so is itself an R-module. Taking an arbitrary $r \in R$ and two elements from $\operatorname{Cone}(f_\bullet)_i$ we observe

$$\begin{aligned}
\partial_i^{\operatorname{Cone}(f_\bullet)}\left(r\begin{pmatrix} y_i \\ x_{i-1}\end{pmatrix} + \begin{pmatrix} y_i' \\ x_{i-1}'\end{pmatrix}\right) &= \begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X\end{pmatrix}\begin{pmatrix} ry_i + y_i' \\ rx_{i-1} + x_{i-1}'\end{pmatrix} \\
&= \begin{pmatrix} \partial_i^Y(ry_i + y_i') + f_{i-1}(rx_{i-1} + x_{i-1}') \\ -\partial_{i-1}^X(rx_{i-1} + x_{i-1}')\end{pmatrix} \\
&= \begin{pmatrix} r\partial_i^Y(y_i) + \partial_i^Y(y_i') + rf_{i-1}(x_{i-1}) + f_{i-1}(x_{i-1}') \\ r \cdot -\partial_{i-1}^X(x_{i-1}) - \partial_{i-1}^X(x_{i-1}')\end{pmatrix} \\
&= \begin{pmatrix} r\partial_i^Y(y_i) + rf_{i-1}(x_{i-1}) \\ r \cdot -\partial_{i-1}^X(x_{i-1})\end{pmatrix} + \begin{pmatrix} \partial_i^Y(y_i') + f_{i-1}(x_{i-1}') \\ -\partial_{i-1}^X(x_{i-1}')\end{pmatrix} \\
&= r \cdot \begin{pmatrix} \partial_i^Y(y_i) + f_{i-1}(x_{i-1}) \\ -\partial_{i-1}^X(x_{i-1})\end{pmatrix} + \begin{pmatrix} \partial_i^Y(y_i') + f_{i-1}(x_{i-1}') \\ -\partial_{i-1}^X(x_{i-1}')\end{pmatrix} \\
&= r \cdot \partial_i^{\operatorname{Cone}(f_\bullet)}\begin{pmatrix} y_i \\ x_{i-1}\end{pmatrix} + \partial_i^{\operatorname{Cone}(f_\bullet)}\begin{pmatrix} y_i' \\ x_{i-1}'\end{pmatrix}.
\end{aligned}$$

Since the well-definedness of $\partial_i^{\operatorname{Cone}(f_\bullet)}$ is a direct consequence of the well-definedness of the maps ∂_i^Y, f_{i-1}, and ∂_{i-1}^X for each $i \in \mathbb{Z}$, we conclude each $\partial_i^{\operatorname{Cone}(f_\bullet)}$ is an R-module homomorphism. Moreover

$$\begin{aligned}
\partial_i^{\operatorname{Cone}(f_\bullet)} \circ \partial_{i+1}^{\operatorname{Cone}(f_\bullet)} &= \begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X\end{pmatrix}\begin{pmatrix} \partial_{i+1}^Y & f_i \\ 0 & -\partial_i^X\end{pmatrix} \\
&= \begin{pmatrix} \partial_i^Y \circ \partial_{i+1}^Y & \partial_i^Y \circ f_i - f_{i-1} \circ \partial_i^X \\ 0 & \partial_{i-1}^X \circ \partial_i^X\end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}.
\end{aligned}$$

The $(1,1)$-entry of the composition is zero, because $Y_\bullet$ is an R-complex and similarly for the $(2,2)$-entry. The $(1,2)$-entry is zero, because $f_\bullet$ is a chain map. This concludes the proof. □

Example 6.4.5. Here we introduce some special cases of the *Koszul complex*. (See Section 7.3 for more on this topic.) Fix an element $x \in R$ and define the R-complex

$$X_\bullet \qquad 0 \longrightarrow R \xrightarrow{x\cdot} R \longrightarrow 0\,.$$

Fix another element $y \in R$ and define the following chain map.

$$\begin{array}{ccccccccccc}
X_\bullet & & & 0 & \longrightarrow & R & \xrightarrow{x\cdot} & R & \longrightarrow & 0 & \dashrightarrow 0 \\
\downarrow{\scriptstyle y_\bullet} & & & & & \downarrow{\scriptstyle y\cdot} & \curvearrowright & \downarrow{\scriptstyle y\cdot} & & & \\
X_\bullet & & 0 \dashrightarrow & 0 & \longrightarrow & R & \xrightarrow{x\cdot} & R & \longrightarrow & 0 &
\end{array}$$

We can compute the mapping cone of $f_\bullet$.

$$\begin{aligned}
\operatorname{Cone}(y_\bullet) = \quad & \begin{matrix}0\\ \oplus \\ 0\end{matrix} \longrightarrow \begin{matrix}0\\ \oplus \\ R\end{matrix} \xrightarrow{\begin{pmatrix} 0 & y \\ 0 & -x\end{pmatrix}} \begin{matrix}R\\ \oplus \\ R\end{matrix} \xrightarrow{\begin{pmatrix} x & y \\ 0 & 0\end{pmatrix}} \begin{matrix}R\\ \oplus \\ 0\end{matrix} \longrightarrow \begin{matrix}0\\ \oplus \\ 0\end{matrix} \\
\cong \quad & 0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ -x\end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y\end{pmatrix}} R \longrightarrow 0
\end{aligned}$$

It is sensible that we should call this a complex, since

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} y \\ -x \end{pmatrix} = xy - xy = 0.$$

Proposition 6.4.6. *Let $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ be a chain map.*

(a) There is a chain map $\epsilon_\bullet : Y_\bullet \longrightarrow \mathrm{Cone}(f_\bullet)$ defined as the sequence of natural injections

$$\epsilon_i : Y_i \hookrightarrow \begin{array}{c} Y_i \\ \oplus \\ X_{i-1} \end{array} = \mathrm{Cone}(f_\bullet)_i \ .$$

(b) There is a chain map $\tau_\bullet : \mathrm{Cone}(f_\bullet) \longrightarrow \Sigma X_\bullet$ defined as the sequence of natural surjections

$$\tau_i : \mathrm{Cone}(f_\bullet) = \begin{array}{c} Y_i \\ \oplus \\ X_{i-1} \end{array} \longrightarrow X_{i-1} = (\Sigma X_\bullet)_i \ .$$

(c) The following sequence is exact.

$$0 \longrightarrow Y_\bullet \xrightarrow{\epsilon_\bullet} \mathrm{Cone}(f_\bullet) \xrightarrow{\tau_\bullet} \Sigma X_\bullet \longrightarrow 0$$

(d) In the associated long exact sequence, the connecting map

$$\eth_i : H_i\left(\Sigma X_\bullet\right) \longrightarrow H_{i-1}(Y_\bullet)$$

is equal to $H_{i-1}(f_\bullet)$.

Proof. To prove the first three parts, it suffices to fix an arbitrary $i \in \mathbb{Z}$ and show the following diagram is commutative with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y_i & \xrightarrow[\epsilon_i]{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \begin{array}{c} Y_i \\ \oplus \\ X_{i-1} \end{array} & \xrightarrow[\tau_i]{\begin{pmatrix} 0 & 1 \end{pmatrix}} & X_{i-1} & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \partial_i^Y} & & \Big\downarrow {\scriptstyle \begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix}} & & \Big\downarrow {\scriptstyle -\partial_{i-1}^X} & & \\
0 & \longrightarrow & Y_{i-1} & \xrightarrow[\epsilon_{i-1}]{} & \begin{array}{c} Y_{i-1} \\ \oplus \\ X_{i-2} \end{array} & \xrightarrow[\tau_{i-2}]{} & X_{i-2} & \longrightarrow & 0
\end{array}$$

The rows are exact by Example 1.1.4. We check commutivity of the two squares by tracking arbitrary elements around each and thereby complete the proof of parts (a), (b), and (c).

$$\begin{array}{ccc}
y & \longmapsto & \begin{pmatrix} y \\ 0 \end{pmatrix} \\
\Big\downarrow & & \Big\downarrow \\
\partial_i^Y(y) & \longrightarrow & \begin{pmatrix} \partial_i^Y(y) \\ 0 \end{pmatrix}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\begin{pmatrix} y \\ x \end{pmatrix} & \longmapsto & x \\
\Big\downarrow & & \Big\downarrow \\
\begin{pmatrix} \partial_i^Y(y) + f_{i-1}(x) \\ -\partial_{i-1}^X(x) \end{pmatrix} & \longmapsto & -\partial_{i-1}^X(x)
\end{array}$$

(d). Note that $H_i(\Sigma X_\bullet) = H_{i-1}(X_\bullet)$ and let $\overline{x} \in H_{i-1}(X_\bullet)$ be arbitrary. Thus $x \in \operatorname{Ker}\left(-\partial_{i-1}^X\right) \subseteq X_{i-1}$ and we begin what has become standard for calculating $\eth$. We lift back to the element

$$\begin{pmatrix} 0 \\ x \end{pmatrix} \in \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix}$$

which is a preimage of x under τ_i. It also holds

$$\begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} f_{i-1}(x) \\ -\partial_i^X(x) \end{pmatrix} = \begin{pmatrix} f_{i-1}(x) \\ 0 \end{pmatrix}$$

so we may lift to the element $f_{i-1}(x) \in Y_{i-1}$ for which we have

$$\epsilon_{i-1}(f_{i-1}(x)) = \begin{pmatrix} f_{i-1}(x) \\ 0 \end{pmatrix}.$$

Hence $\eth_i(\overline{x}) = \overline{f_{i-1}(x)} = H_{i-1}(f_\bullet)(\overline{x})$ as desired. □

Definition 6.4.7. A chain map $f_\bullet \colon X_\bullet \longrightarrow Y_\bullet$ is a *quasiisomorphism* if the induced map on homology

$$H_i(f_\bullet) \colon H_i(X_\bullet) \longrightarrow H_i(Y_\bullet)$$

is an isomorphism, for all $i \in \mathbb{Z}$.

Example 6.4.8. If $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ is an isomorphism, then it is also a quasiisomorphism. To see the reason for this, consider that if $g_\bullet : Y_\bullet \longrightarrow X_\bullet$ is a two-sided inverse for $f_\bullet$, then the induced map on homology $H_i(g_\bullet) : H_i(Y_\bullet) \longrightarrow H_i(X_\bullet)$ is a two-sided inverse for $H_i(f_\bullet)$.

The converse of this, however, fails in general. By way of demonstration, let M be an R-module and let $P_\bullet$ be a projective resolution of M.

$$P_\bullet^+ = \qquad \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

We may define also the following chain map, call it $\tau_\bullet$.

$$\begin{array}{ccccccccc} P_\bullet = & & \cdots & \xrightarrow{\partial_2^P} & P_1 & \xrightarrow{\partial_1^P} & P_0 & \longrightarrow & 0 \\ \tau_\bullet \big\downarrow & & & & \big\downarrow & & \big\downarrow \tau & & \\ M_\bullet = & & \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

While $\tau_\bullet$ is not an isomorphism (since $P_1 \neq 0$ and M not projective, in general), we claim $\tau_\bullet$ is a quasiisomorphism. Since $P_\bullet$ is exact at P_i for all $i \neq 0$, for these i we have the silly isomorphism below.

$$H_i(\tau_\bullet) : 0 \longrightarrow 0$$

It suffices then to study the $i = 0$ position.

$$\begin{array}{ccc} H_0(\tau_\bullet) : H_0(P_\bullet) & \longrightarrow & H_0(M_\bullet) \\ \| & & \| \\ \dfrac{P_0}{\operatorname{Im}\left(\partial_1^P\right)} & \xrightarrow{\hat{\tau}} & \dfrac{M}{0} \end{array}$$

Here $\hat{\tau}$ denotes the map induced by τ (see Proposition 5.1.3). Since τ is surjective, $\hat{\tau}$ must also be surjective. Since $P_\bullet^+$ is exact, $\operatorname{Ker}(\tau) = \operatorname{Im}\left(\partial_1^P\right)$ and therefore $\hat{\tau}$ is also injective. Hence $H_0(\tau_\bullet)$ is an isomorphism and $\tau_\bullet$ is a quasiisomorphism as claimed.

Proposition 6.4.9. *A chain map* $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ *is a quasiisomorphism if and only if* $\mathrm{Cone}(f_\bullet)$ *is exact.*

Proof. Consider the long exact sequence

$$\cdots \longrightarrow H_i(\mathrm{Cone}(f_\bullet)) \longrightarrow H_{i-1}(X_\bullet) \longrightarrow H_{i-1}(Y_\bullet) \longrightarrow H_{i-1}(\mathrm{Cone}(f_\bullet)) \longrightarrow H_{i-2}(X_\bullet) \longrightarrow H_{i-2}(Y_\bullet) \longrightarrow \cdots$$

from the mapping cone (see Proposition 6.4.6 and Theorem 6.1.2), where the connecting homomorphisms are $\eth_i = H_{i-1}(f_\bullet)$. If we suppose $f_\bullet$ is a quasiisomorphism, then by definition $\eth_i$ is an isomorphism for all i and it follows from Lemma 6.4.10 that

$$H_{i-1}(\mathrm{Cone}(f_\bullet)) = 0.$$

On the other hand, if we suppose $\mathrm{Cone}(f_\bullet)$ is exact, then each section of our long exact sequence is of the form

$$0 \longrightarrow H_{i-1}(X_\bullet) \xrightarrow{H_i(f_\bullet)} H_{i-1}(Y_\bullet) \longrightarrow 0$$

where exactness at $H_{i-1}(X_\bullet)$ and $H_{i-1}(Y_\bullet)$ forces $\mathrm{Ker}\,(H_i(f_\bullet)) = 0$ and $\mathrm{Im}\,(H_i(f_\bullet)) = G$, respectively. Hence $H_i(f_\bullet)$ is an isomorphism and $f_\bullet$ is a quasiisomorphism by definition. □

Lemma 6.4.10. *Given A, B, C, D, and E are R-modules and given the exact sequence*

$$A \xrightarrow{\cong} B \xrightarrow{f} C \xrightarrow{g} D \xrightarrow{\cong} E$$

it follows that $C = 0$.

Proof. The isomorphism on the left forces $\mathrm{Ker}(f) = B$, implying $\mathrm{Ker}(g) = \mathrm{Im}\,(f) = 0$. The other isomorphism forces $\mathrm{Im}\,(g) = 0$ and it follows that $C = 0$. □

Lemma 6.4.11. *Consider the following exact sequence with $n \geq 1$.*

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

If $P_0, \ldots, P_{n-1}$ are all projective, then K_n is projective as well.

Proof. We tackle a few base cases first. If $n = 1$, then the exactness of the sequence implies $K_1 \cong P_0$ and K_1 is therefore projective. If $n = 2$, then since P_0 is projective, the sequence below splits.

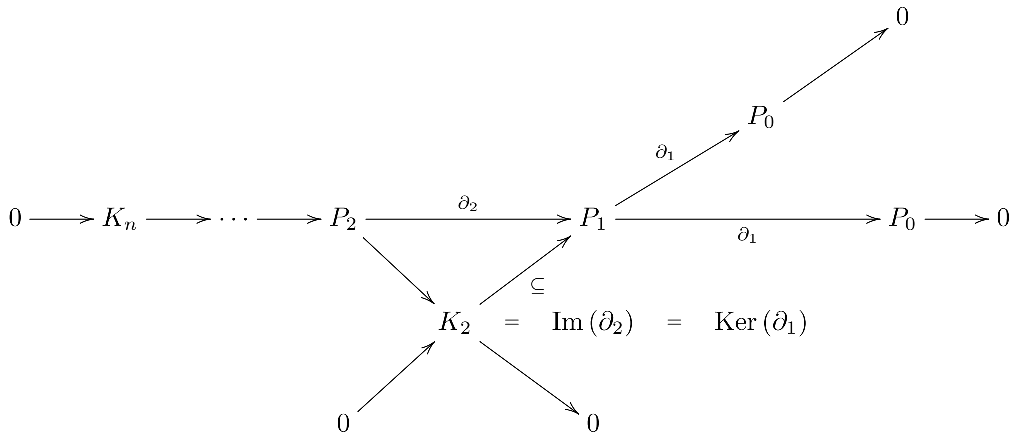
$$0 \longrightarrow K_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

That is, $P_1 \cong K_2 \oplus P_0$. Since P_1 is projective, it follows that K_2 must also be projective (see Lemma 6.4.12).

Assume now that $n \geq 3$ and the result holds for all sequences of length $n - 1$. Our exact sequence is therefore of the form

$$0 \longrightarrow K_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow 0$$

which we may 'slice' using the kernel of the 1^{st} differential.



The diagonal we have constructed guarantees K_2 is projective by the $n = 2$ base case. Since the exact sequence

$$0 \longrightarrow K_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow K_2 \longrightarrow 0$$

is therefore covered under the induction hypothesis, we conclude K_n is projective as well. $\square$

Lemma 6.4.12. *Two R-modules A and B are projective if and only if $A \oplus B$ is projective.*

Proof. Let $\mathcal{S}$ be an arbitrary exact sequence of R-modules.

$$\mathcal{S} = \qquad \cdots \xrightarrow{\partial^{\mathcal{S}}_{i+2}} S_{i+1} \xrightarrow{\partial^{\mathcal{S}}_{i+1}} S_i \xrightarrow{\partial^{\mathcal{S}}_{i}} S_{i-1} \xrightarrow{\partial^{\mathcal{S}}_{i-1}} \cdots$$

We will show

$$\mathrm{Hom}_R(A \oplus B, \mathcal{S}) \cong \mathrm{Hom}_R(A, \mathcal{S}) \oplus \mathrm{Hom}_R(B, \mathcal{S})$$

as R-complexes. For each $i \in \mathbb{Z}$ define the map

$$F_i : \mathrm{Hom}_R(A \oplus B, S_i) \longrightarrow \mathrm{Hom}_R(A, S_i) \oplus \mathrm{Hom}_R(B, S_i)$$

$$\rho \longmapsto (\rho_A, \rho_B)$$

where

$$\rho_A : A \longrightarrow S_i \qquad\qquad \rho_B : B \longrightarrow S_i$$

$$a \longmapsto \rho(a, 0) \qquad\qquad b \longmapsto \rho(0, b).$$

Since ρ_A and ρ_B are compositions of ρ with natural inclusions, each is a well-defined R-module homomorphism and therefore F_i is also a well-defined function. It is straightforward to show that F_i is also R-linear. Each F_i is also surjective since for any $(\alpha, \beta) \in \mathrm{Hom}_R(A, S_i) \oplus \mathrm{Hom}_R(B, S_i)$ we may define

$$\gamma : A \oplus B \longrightarrow S_i$$

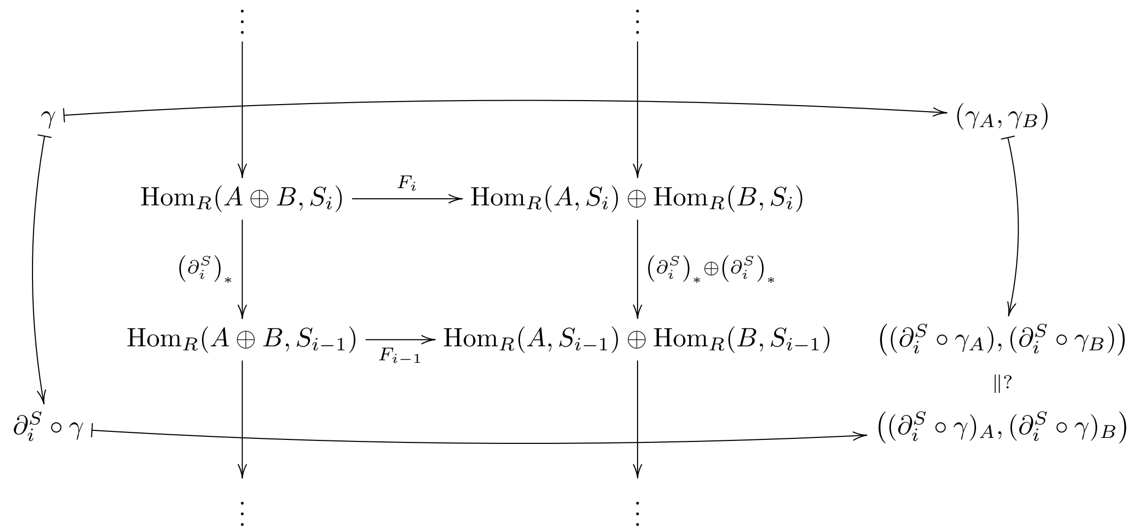
$$(a, b) \longmapsto \alpha(a) + \beta(b)$$

for which

$$F_i(\gamma) = (\gamma_A, \gamma_B) = (\alpha, \beta).$$

Consider also that if $F_i(\rho) = 0$, then $\rho_A = 0^A_{S_i}$ and $\rho_B = 0^B_{S_i}$ and hence $\rho = 0^{A \oplus B}_{S_i}$, so F_i is injective (refer to Fact 3.5.10 for 0^-_- notation).

Therefore the isomorphism of R-complexes will follow once we have verified the commutivity of the following diagram.



To this end, consider that for any $a \in A$ we have

$$(\partial_i^S \circ \gamma_A)(a) = \partial_i^S(\gamma_A(a)) = \partial_i^S(\gamma(a,0)) = (\partial_i^S \circ \gamma)(a,0) = (\partial_i^S \circ \gamma)_A(a).$$

Similarly, for any $b \in B$ we have

$$(\partial_i^S \circ \gamma_B)(b) = \partial_i^S(\gamma_B(b)) = \partial_i^S(\gamma(b,0)) = (\partial_i^S \circ \gamma)(b,0) = (\partial_i^S \circ \gamma)_B(b).$$

Hence $F_\bullet$ is an isomorphism of R-complexes. Therefore we have

$$\begin{aligned} \mathrm{Hom}_R(A \oplus B, \mathcal{S}) \text{ exact } &\iff \mathrm{Hom}_R(A, \mathcal{S}) \oplus \mathrm{Hom}_R(B, \mathcal{S}) \text{ exact} \\ &\iff \mathrm{Hom}_R(A, \mathcal{S}),\ \mathrm{Hom}_R(B, \mathcal{S}) \text{ both exact} \end{aligned}$$

and therefore $A \oplus B$ is projective if and only if A and B are both projective. □

Lemma 6.4.13 (Schanuel)**.** *Consider exact sequences*

$$0 \longrightarrow K_n \xrightarrow{\partial_n^P} P_{n-1} \xrightarrow{\partial_{n-1}^P} \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

$$0 \longrightarrow L_n \xrightarrow{\partial_n^Q} Q_{n-1} \xrightarrow{\partial_{n-1}^Q} \cdots \xrightarrow{\partial_2^Q} Q_1 \xrightarrow{\partial_1^Q} Q_0 \xrightarrow{\pi} M \longrightarrow 0$$

such that $P_0, \ldots, P_{n-1}, Q_0, \ldots, Q_{n-1}$ are all projective. Then

$$K_n \text{ projective} \iff L_n \text{ projective.}$$

Proof. By the proof of Proposition 5.2.2, we can lift the identity map id_M to build a chain map between the two sequences. That is, there exist R-module homomorphisms $f_0, \ldots, f_n$ that make the following diagram commute.

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & K_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ 0 & \longrightarrow & L_n & \longrightarrow & Q_{n-1} & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Let f_i be the zero map for all $i \notin \{0, \dots, n\}$ and truncate the two resolutions. We have a chain map $f_\bullet \colon P_\bullet \longrightarrow Q_\bullet$ in the display below.

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & K_n & \longrightarrow & P_{n-1} & \longrightarrow \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & L_n & \longrightarrow & Q_{n-1} & \longrightarrow \cdots \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

As in Example 6.4.8, one can check that $f_\bullet$ is a quasiisomorphism and thus by Proposition 6.4.9 we know $\mathrm{Cone}(f_\bullet)$ is exact, which we write below.

$$0 \longrightarrow K_n \longrightarrow \begin{matrix} L_n \\ \oplus \\ P_{n-1} \end{matrix} \longrightarrow \begin{matrix} Q_{n-1} \\ \oplus \\ P_{n-2} \end{matrix} \longrightarrow \cdots \longrightarrow \begin{matrix} Q_1 \\ \oplus \\ P_0 \end{matrix} \longrightarrow Q_0 \longrightarrow 0$$

If we assume L_n is projective, then $L_n \oplus P_{n-1}$ is projective and K_n must also be projective under Lemma 6.4.11.

Running this entire argument again having placed $Q_\bullet^+$ in the top of our ladder diagram would yield an identical result, so the forward implication is proven by symmetry. □

6.5 Well-Definedness of Ext

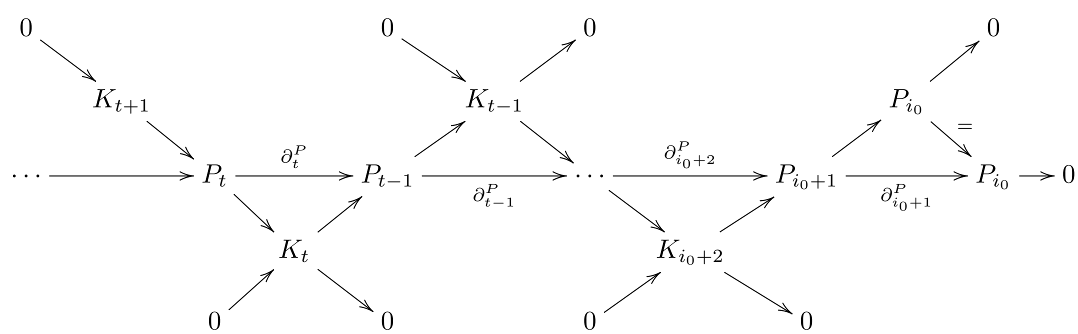
With all the necessary tools now in place, we finally prove that Ext is well-defined.

Lemma 6.5.1. *If $P_\bullet$ is an exact R-complex such that each P_i is projective and $P_i = 0$ for all $i < i_0$ for some fixed $i_0 \in \mathbb{Z}$, then for any R-module N, $\mathrm{Hom}_R(P_\bullet, N)$ is exact.*

Proof. The given complex has the following form around the i_0 position.

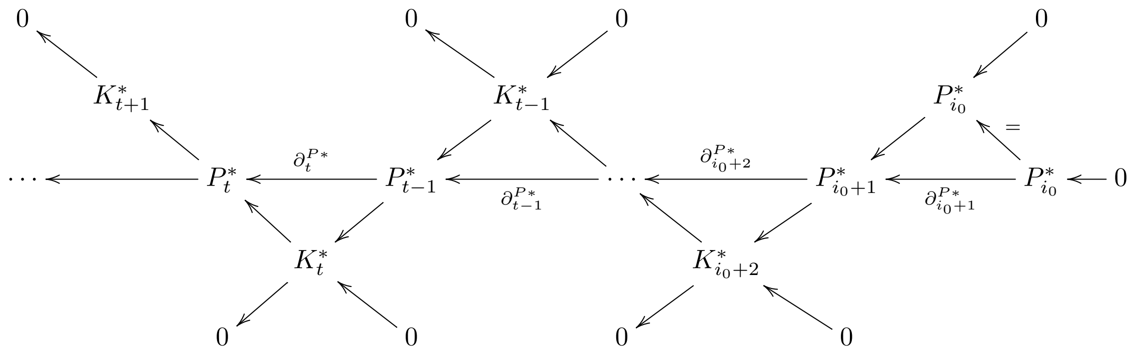
$$P_\bullet = \qquad \cdots \xrightarrow{\partial^P_{i_0+2}} P_{i_0+1} \xrightarrow{\partial^P_{i_0+1}} P_{i_0} \longrightarrow 0 \longrightarrow \cdots$$

Let K_t denote $\mathrm{Ker}\left(\partial^P_{t-1}\right)$ and 'slice' the above exact sequence.



(6.5.1.1)

The diagonals are all exact and both P_{i_0}, P_{i_0+1} are projective, so by Lemma 6.4.11, the module K_{i_0+2} is projective. If we let $t > i_0 + 2$ and assume K_{t-1} is projective, then since P_{t-1} is projective, the same lemma guarantees K_t is projective as well. Hence by induction K_t is projective for all $t \geq i_0 + 2$, implying $\mathrm{Hom}_R(D, N)$ is split exact for any R-module N, where D is any diagonal sequence in (6.5.1.1). Let $(-)^* = \mathrm{Hom}_R(-, N)$ and we have the following commutative diagram.



Since the diagonals are exact and the diagrams all commute, a diagram chase shows that the row must also be exact. That is, $P_\bullet^* = \operatorname{Hom}_R(P_\bullet, N)$ is exact, as desired. $\square$

Ladies and gentlemen, we have arrived:

Theorem 6.5.2. Ext *is independent of choice of projective resolution.*

Proof. Let $P_\bullet$ and $Q_\bullet$ be two projective resolutions of an R-module M and let $f_\bullet \colon P_\bullet \longrightarrow Q_\bullet$ be a lift of the identity on M (see Proposition 5.2.2). From the work done in the proof of Lemma 6.4.13, this implies $\operatorname{Cone}(f_\bullet)$ is exact.

$$\operatorname{Cone}(f_\bullet) = \qquad \cdots \longrightarrow \begin{matrix} Q_2 \\ \oplus \\ P_1 \end{matrix} \longrightarrow \begin{matrix} Q_1 \\ \oplus \\ P_0 \end{matrix} \longrightarrow Q_0 \longrightarrow 0$$

Since every module $Q_i \oplus P_{i-1}$ is projective, by Lemma 6.5.1 we have $\operatorname{Hom}_R(\operatorname{Cone}(f_\bullet), N)$ exact for any R-module N. Moreover the shift $\Sigma \operatorname{Hom}_R(\operatorname{Cone}(f_\bullet), N)$ is also exact since

$$H_i\left(\Sigma\star\right) = H_{i-1}(\star).$$

By Lemma 6.5.3

$$\Sigma \operatorname{Hom}_R(\operatorname{Cone}(f_\bullet), N) \cong \operatorname{Cone}\left(\operatorname{Hom}_R(f_\bullet, N)\right).$$

Hence by Proposition 6.4.9 it follows that $\operatorname{Hom}_R(f_\bullet, N)$ is a quasiisomorphism and therefore the following is an isomorphism for any $i \in \mathbb{Z}$.

$$H_{-i}(\operatorname{Hom}_R(f_\bullet, N)) \ : \ H_{-i}(\operatorname{Hom}_R(Q_\bullet, N)) \xrightarrow{\cong} H_{-i}(\operatorname{Hom}_R(P_\bullet, N))\ .$$

This completes the proof. $\square$

Lemma 6.5.3. *Let R be a commutative ring with identity, let M be an R-module, and consider a chain map $F_\bullet : X_\bullet \to Y_\bullet$. Then*

$$\operatorname{Cone}(\operatorname{Hom}_R(F_\bullet, M)) \cong \Sigma \operatorname{Hom}_R(\operatorname{Cone}(F_\bullet), M).$$

Proof. From our chain map

$$\begin{array}{ccccccccc}
X_\bullet & \qquad & \cdots & \xrightarrow{\partial_{i+1}^X} & X_i & \xrightarrow{\partial_i^X} & X_{i-1} & \xrightarrow{\partial_{i-1}^X} & \cdots \\
\downarrow{\scriptstyle F_\bullet} & & & & \downarrow{\scriptstyle F_i} & & \downarrow{\scriptstyle F_{i-1}} & & \\
Y_\bullet & & \cdots & \xrightarrow[\partial_{i+1}^Y]{} & Y_i & \xrightarrow[\partial_i^Y]{} & Y_{i_1} & \xrightarrow[\partial_{i-1}^Y]{} & \cdots
\end{array}$$

we are able to write

$$\mathrm{Cone}(F_\bullet) = \quad \cdots \longrightarrow \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_i^Y & F_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix}} \begin{matrix} Y_{i-1} \\ \oplus \\ X_{i-2} \end{matrix} \longrightarrow \cdots .$$

Applying the contravariant functor $\mathrm{Hom}_R(-, M)$ we write $\mathrm{Hom}_R(\mathrm{Cone}(F_\bullet), M)$ below.

$$\cdots \longrightarrow \mathrm{Hom}_R\left(\begin{matrix} Y_{-i} \\ \oplus \\ X_{-i-1} \end{matrix}, M\right) \xrightarrow{\begin{pmatrix} \partial_{-i+1}^Y & F_{-i} \\ 0 & -\partial_{-i}^X \end{pmatrix}^*} \mathrm{Hom}_R\left(\begin{matrix} Y_{-i+1} \\ \oplus \\ X_{-i} \end{matrix}, M\right) \longrightarrow \cdots$$

The shift $\Sigma \, \mathrm{Hom}_R(\mathrm{Cone}(F_\bullet), M)$ follows readily, which we write below.

$$\cdots \longrightarrow \mathrm{Hom}_R\left(\begin{matrix} Y_{-i+1} \\ \oplus \\ X_{-i} \end{matrix}, M\right) \xrightarrow{-\begin{pmatrix} \partial_{-i+2}^Y & F_{-i+1} \\ 0 & -\partial_{-i+1}^X \end{pmatrix}^*} \mathrm{Hom}_R\left(\begin{matrix} Y_{-i+2} \\ \oplus \\ X_{-i+1} \end{matrix}, M\right) \longrightarrow \cdots$$

Now we write down $\mathrm{Cone}(\mathrm{Hom}_R(F_\bullet, M))$. We begin with the induced chain map

$$\begin{array}{ccc} \mathrm{Hom}_R(Y_\bullet, M) & & \cdots \xrightarrow{(\partial_{-i}^Y)^*} \mathrm{Hom}_R(Y_{-i}, M) \xrightarrow{(\partial_{-i+1}^Y)^*} \mathrm{Hom}_R(Y_{-i+1}, M) \xrightarrow{(\partial_{-i+2}^Y)^*} \cdots \\ \downarrow {\scriptstyle \mathrm{Hom}_R(F_\bullet, M)} & & \qquad \downarrow {\scriptstyle F_{-i}^*} \qquad\qquad\qquad \downarrow {\scriptstyle F_{-i+1}^*} \\ \mathrm{Hom}_R(X_\bullet, M) & & \cdots \xrightarrow[(\partial_{-i}^X)^*]{} \mathrm{Hom}_R(X_{-i}, M) \xrightarrow[(\partial_{-i+1}^X)^*]{} \mathrm{Hom}_R(X_{-i+1}, M) \xrightarrow[(\partial_{-i+2}^X)^*]{} \cdots \end{array}$$

and take the cone

$$\mathrm{Cone}(\mathrm{Hom}_R(F_\bullet, M)) = \cdots \to \begin{matrix} \mathrm{Hom}_R(X_{-i}, M) \\ \oplus \\ \mathrm{Hom}_R(Y_{-i+1}, M) \end{matrix} \xrightarrow{\begin{pmatrix} \left(\partial_{-i+1}^X\right)^* & F_{-i+1}^* \\ 0 & -\left(\partial_{-i+2}^Y\right)^* \end{pmatrix}} \begin{matrix} \mathrm{Hom}_R(X_{-i+1}, M) \\ \oplus \\ \mathrm{Hom}_R(Y_{-i+2}, M) \end{matrix} \to \cdots$$

On the next page we write down explicitly the isomorphism between these two complexes, because what this document needs is another large commutative diagram. The vertical maps send $\begin{pmatrix} a & b \end{pmatrix}$ to $\begin{pmatrix} b \\ -a \end{pmatrix}$ and each is an isomorphism. Since the commutivity of the diagram is depicted as well, the diagram completes the proof. $\square$

$$
\begin{array}{ccccccc}
 & & (\phi \quad \psi) & \longmapsto & \left(-\phi\circ\partial^Y_{-i+2} \quad -\phi\circ F_{-i+1}+\psi\circ\partial^X_{-i+1}\right) & & \\
 & & \rotatebox{90}{$\in$} & & & & \\
\Sigma\operatorname{Hom}_R(\operatorname{Cone}(F_\bullet),M) = & \cdots \longrightarrow & \operatorname{Hom}_R\left(\begin{matrix} Y_{-i+1} \\ \oplus \\ X_{-i}\end{matrix}, M\right) & \xrightarrow{\begin{pmatrix} -\partial^Y_{-i+2} & -F_{-i+1} \\ 0 & \partial^X_{-i+1}\end{pmatrix}^*} & \operatorname{Hom}_R\left(\begin{matrix} Y_{-i+2} \\ \oplus \\ X_{-i+1}\end{matrix}, M\right) & \longrightarrow \cdots & \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
\operatorname{Cone}(\operatorname{Hom}_R(F_\bullet,M)) = & \cdots \longrightarrow & \begin{matrix}\operatorname{Hom}_R(X_{-i},M) \\ \oplus \\ \operatorname{Hom}_R(Y_{-i+1},M)\end{matrix} & \xrightarrow[\begin{pmatrix} \left(\partial^X_{-i+1}\right)^* & F^*_{-i+1} \\ 0 & -\left(\partial^Y_{-i+2}\right)^*\end{pmatrix}]{} & \begin{matrix}\operatorname{Hom}_R(X_{-i+1},M) \\ \oplus \\ \operatorname{Hom}_R(Y_{-i+2},M)\end{matrix} & \longrightarrow \cdots & \\
 & & \begin{pmatrix}\psi \\ -\phi\end{pmatrix} & \longmapsto & \begin{pmatrix}\psi\circ\partial^X_{-i+1}-\phi\circ F_{-i+1} \\ \phi\circ\partial^Y_{-i+2}\end{pmatrix} = \begin{pmatrix}-\phi\circ F_{-i+1}+\psi\circ\partial^X_{-i+1} \\ \phi\circ\partial^Y_{-i+2}\end{pmatrix} & &
\end{array}
$$

Chapter 7

Additional Topics

In this chapter, we give a colloquial treatment of some further properties of Ext. We also briefly discuss the Koszul complex and some further homological constructions.

7.1 Other Derived Functors

To obtain $\mathrm{Ext}^i_R(M,N)$, we know to take a projective resolution of M, apply $\mathrm{Hom}_R(-,N)$ to the resolution, and take homology. More generally, given a functor $\mathfrak{F}$, one can take an appropriate resolution, apply $\mathfrak{F}$ to the resolution, and take homology. Here the type of resolution depends entirely on the type of exactness and the variance of the functor to be applied. In this section we explore some such functors.

Example 7.1.1. The functor we already know is Ext.

$$\mathrm{Ext}^i_R(M,N) = H_{-i}(\mathrm{Hom}_R(P_\bullet, N))$$

We say Ext is the *right-derived functor* of $\mathrm{Hom}_R(-,N)$ and we use i as a superscript, because $\mathrm{Hom}_R(-,N)$ is contravariant (i.e., arrow-reversing).

Example 7.1.2. Closely related to Ext is Tor, the *left-derived functor* of the tensor product $-\otimes_R N$.

$$\mathrm{Tor}^R_i(M,N) = H_{-i}(P_\bullet \otimes_R N)$$

Here we use i as a subscript, because $-\otimes_R N$ is covariant (i.e., arrow-preserving).

Other constructions require different resolutions, which we define next.

Definition 7.1.3. An *augmented injective resolution* of N is an exact sequence

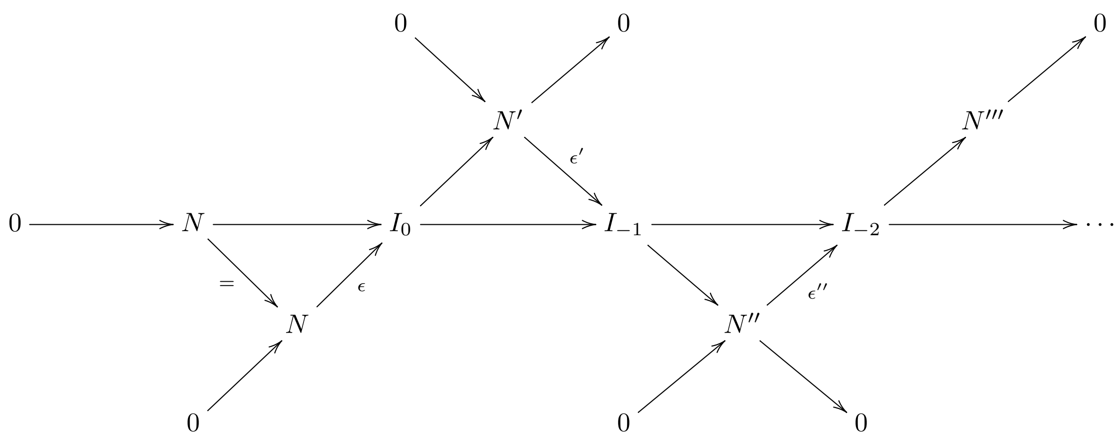
$$^+I_\bullet = \qquad 0 \longrightarrow N \xrightarrow{\epsilon} I_0 \xrightarrow{\partial^I_0} I_{-1} \xrightarrow{\partial^I_{-1}} I_{-2} \xrightarrow{\partial^I_{-2}} \cdots$$

where I_i is injective for all $i \in \mathbb{Z}$. The corresponding *truncated injective resolution* is

$$I_\bullet = \qquad 0 \longrightarrow I_0 \xrightarrow{\partial^I_0} I_{-1} \xrightarrow{\partial^I_{-1}} I_{-2} \xrightarrow{\partial^I_{-2}} \cdots$$

which is not exact in general.

Fact 7.1.4. *For all R-modules N, the exists an injective module I_0 and an injective R-module homomorphism $\epsilon : N \longrightarrow I_0$. Colloquially, we say every R-module N is a 'submodule' of an injective R-module. A consequence of this is the existence of an injective resolution for any R-module N, built inductively as the following diagram suggests.*



where $N^{(i)} = \text{Coker}\left(\epsilon^{(i-1)}\right)$.

Example 7.1.5. The i^{th} right-derived functor of $\text{Hom}_R(M, -)$ is $H_{-i}(\text{Hom}_R(M, I_\bullet))$.

The following result says we can compute Ext modules from injective resolutions as well as projective resolutions.

Theorem 7.1.6 (Balance for Ext)**.** *Let M and N by two R-modules, let $P_\bullet$ be a projective resolution for M, and let $I_\bullet$ be an injective resolution for N. Then $H_{-i}(\text{Hom}_R(M, I_\bullet)) \cong \text{Ext}^i_R(M, N)$.*

Proof. We give only a sketch of this proof. There exists a notion of $\text{Hom}_R(P_\bullet, I_\bullet)$ and one uses mapping cones as in Theorem 6.5.2 to show that the induced chain maps

$$\text{Hom}_R(P_\bullet, N) \xrightarrow{\simeq} \text{Hom}_R(P_\bullet, I_\bullet) \xleftarrow{\simeq} \text{Hom}_R(M, I_\bullet)$$

are quasiisomorphisms. From this one concludes directly that

$$\text{Ext}^i_R(M, N) \cong \text{Hom}_R(P_\bullet, N) \cong \text{Hom}_R(P_\bullet, I_\bullet) \cong \text{Hom}_R(M, I_\bullet).$$

□

Similarly, we have the following.

Theorem 7.1.7 (Balance for Tor)**.** *For any R-modules M and N with respective projective resolutions $P_\bullet$ and $Q_\bullet$, we have the following isomorphisms.*

$$H_i(P_\bullet \otimes_R N) \cong H_i(P_\bullet \otimes_R Q_\bullet) \cong H_i(M \otimes_R Q_\bullet)$$

Next, we consider Grothendieck's local cohomology.

Definition 7.1.8. Let $\mathfrak{a} \leq R$ be an ideal and let M be an R-module. The $\mathfrak{a}$-*torsion functor*, denoted $\Gamma_\mathfrak{a}$, is defined on modules as

$$\Gamma_\mathfrak{a}(M) = \{ m \in M \mid \mathfrak{a}^n m = 0, \ \forall n \gg 0 \}.$$

See Facts 7.1.10 and 7.1.11 for functorial properties.

Example 7.1.9. Given the ring $\mathbb{Z}$ and an ideal $p\mathbb{Z}$ the p-torsion functor can be written

$$\Gamma_{p\mathbb{Z}}(M) = \{m \in M \mid p^n m = 0, \ \forall n \gg 0\}.$$

In particular, let $p = 2$ and let $M = \mathbb{Z}/144\mathbb{Z}$. We compute the 2-torsion functor as follows.

$$\begin{aligned}
\Gamma_{p\mathbb{Z}}\left(\frac{\mathbb{Z}}{144\mathbb{Z}}\right) &\cong \Gamma_{p\mathbb{Z}}\left(\frac{\mathbb{Z}}{2^4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3^2\mathbb{Z}}\right) \\
&\cong \Gamma_{2\mathbb{Z}}\left(\frac{\mathbb{Z}}{2^4\mathbb{Z}}\right) \oplus \Gamma_{2\mathbb{Z}}\left(\frac{\mathbb{Z}}{3^2\mathbb{Z}}\right) \\
&\cong \frac{\mathbb{Z}}{2^4\mathbb{Z}} \oplus 0 \\
&\cong \frac{\mathbb{Z}}{2^4\mathbb{Z}}
\end{aligned}$$

Note that $\Gamma_{2\mathbb{Z}}(\mathbb{Z}/3^2\mathbb{Z}) \cong 0$ since 2^n acts as a unit on $\mathbb{Z}/3^2\mathbb{Z}$ for all $n \in \mathbb{N}$.

Fact 7.1.10. *For all R-module homomorphisms $\phi : M \longrightarrow M'$, we have*

$$\phi\left(\Gamma_{\mathfrak{a}}(M)\right) \subseteq \Gamma_{\mathfrak{a}}(M').$$

A result of this fact is the following commutative diagram, where $\Gamma_{\mathfrak{a}}(\phi)$ is induced from ϕ by restricting the domain and codomain.

$$\begin{array}{ccc}
M & \xrightarrow{\phi} & M' \\
\uparrow \subseteq & \curvearrowright & \uparrow \subseteq \\
\Gamma_{\mathfrak{a}}(M) & \dashrightarrow_{\Gamma_{\mathfrak{a}}(\phi)} & \Gamma_{\mathfrak{a}}(M')
\end{array}$$

Proof. Let $n \in \mathbb{N}$. If $\mathfrak{a}^n m = 0$, then we also have $0 = \phi(\mathfrak{a}^n m) = \mathfrak{a}^n \cdot \phi(m)$. □

Fact 7.1.11. *$\Gamma_{\mathfrak{a}}$ is a covariant functor and is left-exact.*

Example 7.1.12. The functor $\Gamma_{\mathfrak{a}}$ is not right-exact in general. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

to which we apply $\Gamma_{2\mathbb{Z}}$ to obtain

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_{2\mathbb{Z}}(\mathbb{Z}) & \xrightarrow{2\cdot} & \Gamma_{2\mathbb{Z}}(\mathbb{Z}) & \longrightarrow & \Gamma_{2\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
 & & 0 & & 0 & & \mathbb{Z}/2\mathbb{Z} \neq 0 & &
\end{array}$$

which is not exact.

Definition 7.1.13. Let $I_\bullet$ be an injective resolution of an R-module N. The i^{th} *local cohomology module associated to N with support in $\mathfrak{a} \leq R$* is the i^{th} right-derived functor of $\Gamma_{\mathfrak{a}}$.

$$H^i_{\mathfrak{a}}(N) = H_{-i}(\Gamma_{\mathfrak{a}}(I_\bullet))$$

Example 7.1.14. Let $\mathbb{Z}$ be both the ring and module in this example and let $\mathfrak{a} = 2\mathbb{Z}$. The following is an augmented injective resolution for $\mathbb{Z}$.

$$^{+}I_\bullet = \qquad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Applying $\Gamma_\mathfrak{a}$ to the truncated resolution we get the following.

$$\Gamma_\mathfrak{a}(I_\bullet) = \qquad 0 \longrightarrow \Gamma_{2\mathbb{Z}}(\mathbb{Q}) \longrightarrow \Gamma_{2\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

Since $2^n \in \mathbb{Q}$ is a unit for $n = 1, 2, 3, \ldots$ we write equivalently

$$\Gamma_\mathfrak{a}(I_\bullet) = \qquad 0 \longrightarrow 0 \longrightarrow \Gamma_{2\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

where $\Gamma_{2\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) = \left\{ \overline{(a/2^n)} \;\middle|\; a \in \mathbb{Z},\ n \in \mathbb{N} \right\} \neq 0$. We now compute the cohomology as follows.

$$H^i_{2\mathbb{Z}} = \begin{cases} 0 & i \neq 1 \\ \Gamma_{2\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) & i = 1 \end{cases}$$

7.2 Ext and Extensions

The point of this section is that one can define an equivalence relation on sets of short exact sequences in such a way that the set of equivalence classes is naturally in bijection with an Ext^1_R-module.

Definition 7.2.1. An *extension* of M by N is a short exact sequence

$$\zeta = \qquad 0 \longrightarrow N \xrightarrow{f} A \xrightarrow{g} M \longrightarrow 0.$$

We also define an equivalence relation on the set of extensions of M by N. If ζ' is another extension of M, then $\zeta \sim \zeta'$ if there exists a commutative diagram of the following form.

$$\begin{array}{cccccccccc} \zeta = & 0 & \longrightarrow & N & \xrightarrow{f} & A & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle 1} & \curvearrowright & \downarrow{\scriptstyle \phi} & \curvearrowright & \downarrow{\scriptstyle 1} & & \\ \zeta' = & 0 & \longrightarrow & N & \xrightarrow[f']{} & A' & \xrightarrow[g']{} & M & \longrightarrow & 0 \end{array}$$

The collection of all equivalence classes of such extensions is a set which we denote $\mathrm{E}^1_R(M, N)$.

Theorem 7.2.2 (Yoneda)**.** *For any R-modules M and N, there exists a bijective function*

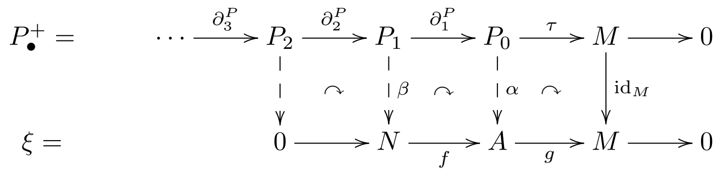
$$\Phi : \mathrm{E}^1_R(M, N) \longrightarrow \mathrm{Ext}^1_R(M, N)$$

which we construct next.

Construction 7.2.3. Let an extension ξ be given:

$$\xi = \qquad 0 \longrightarrow N \xrightarrow{f} A \xrightarrow{g} M \longrightarrow 0\ .$$

If $P_\bullet^+$ is a projective resolution of M, by Proposition 5.2.2 we can lift the identity map on M to build the following ladder diagram.



The commutivity of the diagram implies $0 = \beta \circ \partial_2^P = \left(\partial_2^P\right)^* (\beta)$ and therefore

$$\overline{\beta} \in \frac{\operatorname{Ker}\left(\left(\partial_2^P\right)^*\right)}{\operatorname{Im}\left(\left(\partial_1^P\right)^*\right)} = \operatorname{Ext}_R^1(M, N).$$

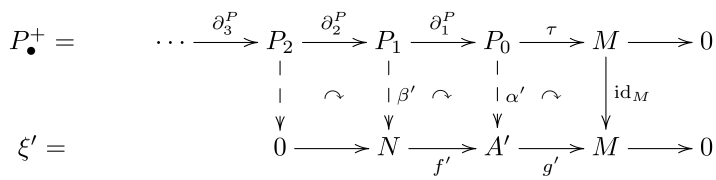
Hence we define the bijection proposed in Theorem 7.2.2 as follows.

$$\Phi([\xi]) = \overline{\beta}$$

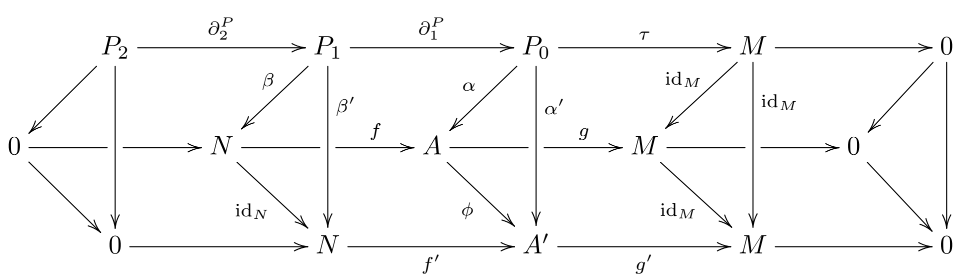
We give a sketch of the proof that this is well-defined. We suppose $\xi \sim \xi'$ and we want to show $\overline{\beta} = \overline{\beta'}$, where ξ' is

$$\xi' = \qquad 0 \longrightarrow N \xrightarrow{f'} A' \xrightarrow{g'} M \longrightarrow 0\ .$$

and β' is in the following ladder diagram.



For this it suffices to show $\beta - \beta' \in \operatorname{Im}\left((\partial_1^P)^*\right)$. Consider the following diagram, where all the rectangular diagrams commute, but the triangular ones need not commute.



Here the map ϕ comes from the equivalence $\xi \sim \xi'$. One can apply $\operatorname{Hom}_R(P_0, -)$ to ξ', which preserves exactness, and select a map $\gamma \in \operatorname{Hom}_R(P_0, N)$ such that $f' \circ \gamma = \phi \circ \alpha - \alpha'$. One then shows that $\beta - \beta' = (\partial_1^P)^*(\gamma)$.

Proving the injectivity and surjectivity of this map is beyond the scope of this document. The crux of the latter is that given any extension ζ we can lift the identity map on M to find an appropriate β.

One can obtain the next result as a corollary of Theorem 7.2.2. We present a partial alternate proof that uses technology we have developed completely.

Theorem 7.2.4. *For all R-modules M and N, the following are equivalent.*

(i) $\mathrm{Ext}^1_R(M,N) = 0$

(ii) Every short exact sequence $0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$ *splits.*

Proof. (i) $\Longrightarrow$ (ii). Consider the short exact sequence

$$0 \longrightarrow N \xrightarrow{\rho} X \xrightarrow{\phi} M \longrightarrow 0. \tag{7.2.4.1}$$

An associated long exact sequence is

$$0 \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,X) \xrightarrow{\phi_*} \mathrm{Hom}_R(M,M) \longrightarrow \mathrm{Ext}^1_R(M,N) \longrightarrow \cdots.$$

If we assume $\mathrm{Ext}^1_R(M,N) = 0$, then ϕ_* is surjective and for $\mathrm{id}_M \in \mathrm{Hom}_R(M,M)$, there exists some $\alpha \in \mathrm{Hom}_R(M,X)$ such that $\mathrm{id}_M = \phi_*(\alpha) = \phi \circ \alpha$. Therefore by Fact 1.1.10, the sequence (7.2.4.1) splits and (ii) holds.

□

7.3 The Koszul Complex

Here we introduce the Koszul complex in full generality (Defintion 7.3.5) and study its homology. In Theorem 7.3.17 we give a means of detecting regular sequences and in Theorem 7.3.21 we give three significant characteristics of R modulo a regular sequence.

Recall 7.3.1. In Proposition 6.4.6 we saw for any chain map $f_\bullet : M_\bullet \longrightarrow N_\bullet$ we have the following short exact sequence and associated long exact sequence.

$$0 \longrightarrow N_\bullet \xrightarrow{\epsilon_\bullet} \mathrm{Cone}(f_\bullet) \xrightarrow{\pi_\bullet} \Sigma M_\bullet \longrightarrow 0$$

$$\cdots \xrightarrow{H_i(\epsilon_\bullet)} H_i(\mathrm{Cone}(f_\bullet)) \xrightarrow{H_i(\pi_\bullet)} H_{i-1}(M_\bullet) \xrightarrow[H_{i-1}(f_\bullet)]{\eth_i =} H_{i-1}(N) \longrightarrow \cdots$$

$$H_{i-1}(M_\bullet) = H_i(\Sigma M_\bullet)$$

Here $\epsilon_\bullet$ and $\pi_\bullet$ are the natural injection and surjection, respectively. (See also Definition 6.4.3.)

Example 7.3.2. If M is an R-module, then we say

$$M_\bullet = \qquad 0 \longrightarrow M \longrightarrow 0$$

is a chain complex concentrated in degree zero. For any $r \in R$ we may also define a chain map

$$\begin{array}{ccc} M_\bullet = & & 0 \longrightarrow M \longrightarrow 0 \\ \mu^r_\bullet \big\downarrow & & \big\downarrow r\cdot \\ M_\bullet = & & 0 \longrightarrow M \longrightarrow 0 \end{array}$$

which yields the cone

$$\mathrm{Cone}(\mu^r_\bullet) = \qquad 0 \longrightarrow M \xrightarrow{r} M \longrightarrow 0.$$

Definition 7.3.3. For any R-module M and any $r \in R$, define the following submodule.

$$(0 \underset{M}{:} r) = \{ m \in M \mid rm = 0 \}$$

This is the largest submodule of M annihilated by r, called the *annihilator* of r in M. More generally, for any $S \subseteq R$ we have

$$(0 \underset{M}{:} S) = \{ m \in M \mid sm = 0, \, \forall s \in S \} .$$

Proposition 7.3.4. *Let $X_\bullet$ be an R-complex and let $r \in R$ be fixed. Consider the* homothety *map $\mu^r_\bullet : X_\bullet \longrightarrow X_\bullet$ defined as in Discussion 5.2.4 and the short exact sequence*

$$0 \longrightarrow X_\bullet \longrightarrow \mathrm{Cone}(\mu^r_\bullet) \longrightarrow \Sigma X_\bullet \longrightarrow 0 .$$

(a) In the associated long exact sequence, the connecting map is also a multiplication map, i.e.,

$$\eth_i(\overline{x_{i-1}}) = r \cdot \overline{x_{i-1}}.$$

(b) For any $i \in \mathbb{Z}$, there exists a short exact sequence

$$0 \longrightarrow \frac{H_i(X_\bullet)}{r \cdot H_i(X_\bullet)} \longrightarrow H_i\left(\mathrm{Cone}(\mu^r_\bullet)\right) \longrightarrow \left(0 \underset{H_{i-1}(X_\bullet)}{:} r \right) \longrightarrow 0 .$$

Proof. (a). This follows directly from the definition of the connecting map, properties of cosets, and Recall 7.3.1.

$$\eth_i(\overline{x_{i-1}}) = H_{i-1}(\mu^r_\bullet)(\overline{x_{i-1}}) = \overline{\mu^r_{i-1}(x_{i-1})} = \overline{rx_{i-1}} = r \cdot \overline{x_{i-1}}$$

(b). By part (a) and from our comments in 7.3.1, the associated long exact sequence is as follows.

$$\cdots \longrightarrow H_i(X_\bullet) \xrightarrow{r\cdot} H_i(X_\bullet) \xrightarrow{H_i(\epsilon_\bullet)} H_i(\mathrm{Cone}(\mu^r_\bullet)) \xrightarrow{H_i(\pi_\bullet)} H_{i-1}(X_\bullet) \xrightarrow{r\cdot} H_{i-1}(X_\bullet) \longrightarrow \cdots$$

From the First Isomorphism Theorem for modules we have

$$\mathrm{Im}\,(H_i(\epsilon_\bullet)) \cong \frac{H_i(X_\bullet)}{\mathrm{Ker}\,(H_i(\epsilon_\bullet))} = \frac{H_i(X_\bullet)}{r \cdot H_i(X_\bullet)}$$

since the kernel of $H_i(\epsilon_\bullet)$ is the image of $r\cdot$ by the exactness of the sequence. Therefore when we 'slice' the long exact sequence around $H_i(\mathrm{Cone}(\mu^r_\bullet))$ we get the following.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}\,(H_i(\epsilon_\bullet)) & \longrightarrow & H_i(\mathrm{Cone}(\mu^r_\bullet)) & \longrightarrow & \mathrm{Im}\,(H_i(\pi_\bullet)) & \longrightarrow & 0 \\ & & \wr\Vert & & & & \Vert & & \\ & & \dfrac{H_i(X_\bullet)}{r \cdot H_i(X_\bullet)} & & & & \mathrm{Ker}\,(r\cdot) & & \\ & & & & & & \Vert & & \\ & & & & & & \left(0 \underset{H_{i-1}(X_\bullet)}{:} r \right) & & \end{array}$$

□

Definition 7.3.5. Here we define a particular R-complex, called the *Koszul complex*. Given an R-module M and $x_1, \ldots, x_n \in R$, we define $\mathrm{K}_\bullet(\underline{x}; M)$ inductively on the length of the sequence. Let $\underline{x} = x_1, \ldots, x_n$ and $\underline{x}' = x_1, \ldots, x_{n-1}$.

$$
\begin{array}{lll}
n = 0 & \mathrm{K}_\bullet(\emptyset; M) = 0 \longrightarrow M \longrightarrow 0 & = M_\bullet \\
n = 1 & \mathrm{K}_\bullet(x_1; M) = 0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow 0 & = \mathrm{Cone}\left(M_\bullet \xrightarrow{x_1 \cdot} M_\bullet \right) \\
n \geq 2 & \mathrm{K}_\bullet(\underline{x}; M) = \mathrm{Cone}\left(\mathrm{K}_\bullet(\underline{x}'; M) \xrightarrow{x_n \cdot} \mathrm{K}_\bullet(\underline{x}'; M) \right) &
\end{array}
$$

We define also the following shorthand notations.

$$
H_i(\underline{x}; M) = H_i(\mathrm{K}_\bullet(\underline{x}; M)) \qquad \mathrm{K}_\bullet(\underline{x}) = \mathrm{K}_\bullet(\underline{x}; R) \qquad H_i(\underline{x}) = H_i(\underline{x}; R)
$$

We will use the above notation for $\underline{x}$ and $\underline{x}'$ throughout the rest of this section.

Example 7.3.6. By the previous definition $K_\bullet(x, y; M)$ is the cone of the following chain map.

$$
\begin{array}{ccccccccccc}
\mathrm{K}_\bullet(x; M) & & & & 0 & \longrightarrow & M & \xrightarrow{x\cdot} & M & \longrightarrow & 0 \longrightarrow 0 \\
\Big\downarrow{\scriptstyle y\cdot} & & & & & & \Big\downarrow{\scriptstyle y\cdot} & & \Big\downarrow{\scriptstyle y\cdot} & & \\
\mathrm{K}_\bullet(x; M) & & 0 & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow[x\cdot]{} & M & \longrightarrow & 0
\end{array}
$$

This yields

$$
\mathrm{K}_\bullet(x, y; M) = \quad
\begin{matrix} 0 \\ \oplus \\ 0 \end{matrix}
\xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}
\begin{matrix} 0 \\ \oplus \\ M \end{matrix}
\xrightarrow{\begin{pmatrix} 0 & y \\ 0 & -x \end{pmatrix}}
\begin{matrix} M \\ \oplus \\ M \end{matrix}
\xrightarrow{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}}
\begin{matrix} M \\ \oplus \\ 0 \end{matrix}
\xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}
\begin{matrix} 0 \\ \oplus \\ 0 \end{matrix}
$$

or more simply

$$
\mathrm{K}_\bullet(x, y; M) = \quad 0 \longrightarrow M \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} M^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} M \longrightarrow 0\,.
$$

To find $\mathrm{K}_\bullet(x, y, z; M)$ we take the cone of the following chain map.

$$
\begin{array}{ccccccccccccc}
\mathrm{K}_\bullet(x, y; M) = & & & & 0 & \longrightarrow & M & \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} & M^2 & \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} & M & \longrightarrow & 0 \longrightarrow 0 \\
\Big\downarrow{\scriptstyle z\cdot} & & & & \Big\downarrow & & \Big\downarrow{\scriptstyle z\cdot} & & \Big\downarrow{\scriptstyle z\cdot I_2} & & \Big\downarrow{\scriptstyle z\cdot} & & \Big\downarrow \\
\mathrm{K}_\bullet(x, y; M) = & & 0 & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow[\begin{pmatrix} y \\ -x \end{pmatrix}]{} & M^2 & \xrightarrow[\begin{pmatrix} x & y \end{pmatrix}]{} & M & \longrightarrow & 0
\end{array}
$$

So we have

$$
\mathrm{K}_\bullet(x, y, z; M) = \quad
\begin{matrix} 0 \\ \oplus \\ 0 \end{matrix}
\longrightarrow
\begin{matrix} 0 \\ \oplus \\ M \end{matrix}
\xrightarrow{\left(\begin{array}{c|c} 0 & z \\ \hline 0 & -y \\ 0 & x \end{array}\right)}
\begin{matrix} M \\ \oplus \\ M^2 \end{matrix}
\xrightarrow{\left(\begin{array}{c|cc} y & z & 0 \\ -x & 0 & z \\ \hline 0 & -x & -y \end{array}\right)}
\begin{matrix} M^2 \\ \oplus \\ M \end{matrix}
\xrightarrow{\left(\begin{array}{cc|c} x & y & z \\ \hline 0 & 0 & 0 \end{array}\right)}
\begin{matrix} M \\ \oplus \\ 0 \end{matrix}
\longrightarrow
\begin{matrix} 0 \\ \oplus \\ 0 \end{matrix}
$$

which we can simplify to write

$$\mathrm{K}_\bullet(x,y,z;M) = \qquad 0 \longrightarrow M \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} M^3 \xrightarrow[\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}]{} M^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} M \longrightarrow 0.$$

Example 7.3.7. Consider the polynomial ring $R = A[x]$ where A is a commutative ring with identity and x is an indeterminate. The Koszul complex for this singleton sequence is

$$\mathrm{K}_\bullet(x) = \qquad 0 \longrightarrow \underset{1}{R} \xrightarrow{x\cdot} \underset{0}{R} \longrightarrow 0$$

and we may calculate the homology modules of this complex. Since x is a non-zero-divisor

$$H_1(x) \cong \mathrm{Ker}(x\cdot) = 0.$$

At the only other position of any potential interest we have

$$H_0(x) = \frac{R}{\mathrm{Im}\,(x\cdot)} = \frac{R}{xR} \cong A.$$

Example 7.3.8. Now consider the polynomial ring in two variables $R = A[x,y]$, and we again calculate the homology modules of this complex.

$$\mathrm{K}_\bullet(x,y) = \qquad 0 \longrightarrow \underset{2}{R} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \underset{1}{R^2} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \underset{0}{R} \longrightarrow 0$$

The zero position and second position are each straightforward.

$$H_0(x,y) = \frac{R}{\mathrm{Im}\,\left(\begin{pmatrix} x & y \end{pmatrix}\right)} = \frac{R}{(x,y)} \cong A$$

$$H_2(x,y) = \frac{\mathrm{Ker}\,\left(\begin{pmatrix} y & -x \end{pmatrix}^T\right)}{0} \cong (0 \underset{R}{:} y) \cap (0 \underset{R}{:} x) = 0$$

We claim the homology is zero at the first position as well, for which it suffices to show $\mathrm{Ker}\,\left(\begin{pmatrix} x & y \end{pmatrix}\right) = \mathrm{Im}\left(\begin{pmatrix} y & -x \end{pmatrix}^T\right)$. The reverse containment holds because $K_\bullet(x,y)$ is an R-complex.

For any $\begin{pmatrix} f & g \end{pmatrix}^T \in \mathrm{Ker}\,\left(\begin{pmatrix} x & y \end{pmatrix}\right)$ we have $gy = -fx$, so $x|g$ and $y|f$. Therefore let $g_1, f_1 \in R$ such that $g = xg_1$ and $f = yf_1$. It follows that

$$xy(f_1 + g_1) = xyf_1 + xyg_1 = xf + yg = 0$$

and hence $f_1 + g_1 = 0$, so $g_1 = -f_1$ and $g = -xf_1$. Finally this gives

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} yf_1 \\ -xf_1 \end{pmatrix} = f_1 \begin{pmatrix} y \\ -x \end{pmatrix} \in \left\langle \begin{pmatrix} y \\ -x \end{pmatrix} \right\rangle = \mathrm{Im}\left(\begin{pmatrix} y \\ -x \end{pmatrix}\right)$$

so the forward containment holds.

Example 7.3.9. Let A be a field and define the ring

$$R = \frac{A[X,Y]}{(XY)}$$

where X and Y are indeterminates. Let $x, y \in R$ denote $\overline{X}$,$\overline{Y}$, respectively. The Koszul complex is then written the same as in the previous example.

$$\mathrm{K}_\bullet(x,y) = \qquad 0 \longrightarrow \underset{2}{R} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \underset{1}{R^2} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \underset{0}{R} \longrightarrow 0$$

Also as in the previous example, the homology modules in the zeroth and second positions are straightforward to calculate.

$$H_0(x,y) \cong \frac{R}{(x,y)R} \cong A$$

$$H_2(x,y) = \{r \in R \mid xr = 0 = yr\} = yR \cap xR = xyR = 0$$

We claim $H_1(x,y) \cong A$. As in Example 7.3.8, let $\begin{pmatrix} f & g \end{pmatrix}^T \in \mathrm{Ker}\left(\begin{pmatrix} x & y \end{pmatrix}\right)$. Using the canonical basis $\{1, x, y, x^2, y^2, \dots\}$ for R over A we may write f and g as the following finite sums.

$$f = a + x\sum_i b_i x^i + y \sum_j c_j y^j$$

$$g = d + x\sum_i e_i x^i + y \sum_j v_j y^j$$

By virtue of being in the kernel we have

$$\begin{aligned} 0 &= fx + gy \\ &= ax + x^2\sum_i b_i x^i + xy\sum_j c_j y^j + dy + yx\sum_i e_i x^i + y^2 \sum_j v_j y^j \\ &= ax + x^2 \sum_i b_i x^i + dy + y^2 \sum_j v_j y^j \end{aligned}$$

since $xy = 0 \in R$. Therefore by the linear independence of our basis we have $a, d, b_i, v_j = 0 \in A$ for all i and j, so we write $f = y\sum_j c_j y^j$ and $g = x\sum_i e_i x^i$. From this we have

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix} = \sum_j c_j y^j \begin{pmatrix} y \\ 0 \end{pmatrix} + \sum_i e_i x^i \begin{pmatrix} 0 \\ x \end{pmatrix} \in \left\langle \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix} \right\rangle$$

so $\mathrm{Ker}\left(\begin{pmatrix} x & y \end{pmatrix}\right) \subseteq \left\langle \begin{pmatrix} y & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & x \end{pmatrix}^T \right\rangle$. Since the generators of the right-hand side are in the kernel (because $xy = 0$), we actually have equality. Thus we compute

$$H_1(x,y) = \frac{\left\langle \begin{pmatrix} y & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & x \end{pmatrix}^T \right\rangle}{\left\langle \begin{pmatrix} y & -x \end{pmatrix}^T \right\rangle} = \frac{\left\langle \begin{pmatrix} y & -x \end{pmatrix}^T, \begin{pmatrix} 0 & x \end{pmatrix}^T \right\rangle}{\left\langle \begin{pmatrix} y & -x \end{pmatrix}^T \right\rangle}.$$

Hence $H_1(x,y)$ is cyclic generated by $\overline{\begin{pmatrix} 0 & x \end{pmatrix}^T}$, so we can surject onto $H_1(x,y)$ by the following R-module homomorphism.

$$\begin{aligned} R &\xrightarrow{\ \phi\ } H_1(x,y) \\ r &\longmapsto r\overline{\begin{pmatrix} 0 \\ x \end{pmatrix}} \end{aligned}$$

Since $\begin{pmatrix}0 & x\end{pmatrix}^T \notin \left\langle \begin{pmatrix}y & -x\end{pmatrix}^T \right\rangle$, we have $H_1(x,y) \neq 0$ and therefore $\mathrm{Ker}(\phi) \neq R$. On the other hand, $x, y \in \mathrm{Ker}(\phi)$ by the following.

$$\phi(x) = x\overline{\begin{pmatrix}0\\x\end{pmatrix}} = \overline{\begin{pmatrix}0\\x^2\end{pmatrix}} = \overline{\begin{pmatrix}-xy\\x^2\end{pmatrix}} = -x\overline{\begin{pmatrix}y\\-x\end{pmatrix}} = 0$$
$$\phi(y) = y\overline{\begin{pmatrix}0\\x\end{pmatrix}} = \overline{\begin{pmatrix}0\\xy\end{pmatrix}} = \overline{\begin{pmatrix}0\\0\end{pmatrix}} = 0$$

Therefore the ideal (x,y) is contained in the kernel of ϕ, which is strictly contained in the ring R. Since A is a field, (x,y) is maximal and is therefore equal to $\mathrm{Ker}(\phi)$. Hence

$$H_1(x,y) \cong \frac{R}{\mathrm{Ker}(\phi)} = \frac{R}{(x,y)} \cong A.$$

Proposition 7.3.10. *For any R-module M, $\mathrm{K}_i(\underline{x};M) \cong M^{\binom{n}{i}}$.*

Proof. This is proven by induction on n. The base cases $n = 0,1,2,3$ have already been seen in Definition 7.3.5 and Example 7.3.6. Assume $n \geq 4$ and the claim holds for $1,\dots,n-1$. Then we have

$$\begin{aligned}
\mathrm{K}_i(\underline{x};M) &\cong \mathrm{K}_i(\underline{x}';M) \oplus \mathrm{K}_{i-1}(\underline{x}';M) && \text{Definition 6.4.3}\\
&\cong M^{\binom{n-1}{i}} \oplus M^{\binom{n-1}{i-1}} && \text{induction hypothesis}\\
&\cong M^{\binom{n-1}{i}+\binom{n-1}{i-1}}\\
&= M^{\binom{n}{i}}.
\end{aligned}$$

□

Proposition 7.3.11. *Let M be an R-module.*

(a) The differential $\partial_1^{\mathrm{K}_\bullet(\underline{x};M)}$ is the following map.

$$M^n \xrightarrow{\begin{pmatrix}x_1 & x_2 & \cdots & x_n\end{pmatrix}} M$$

(b) The differential $\partial_n^{\mathrm{K}_\bullet(\underline{x};M)}$ is the following map.

$$M \xrightarrow{\begin{pmatrix}x_n & -x_{n-1} & \cdots & (-1)^{n-1}x_1\end{pmatrix}^T} M^n$$

(c) The homologies of the two 'ends' of the complex will be as follows.

$$H_0(\underline{x};M) \cong \frac{M}{(\underline{x})M}$$
$$H_n(\underline{x};M) \cong \bigcap_{i=1}^{n}(0 \underset{M}{:} x_i) = \{m \in M \mid x_i m = 0,\ \forall i = 1,\dots,n\} = (0 \underset{M}{:} \langle \underline{x} \rangle)$$

Proof. (a). We prove this by induction on n. The base cases $n = 1,2$ have already been seen in Definition 7.3.5 and Example 7.3.6, so assume $n \geq 3$ and that the claim holds for $\underline{x}' = x_1,\dots,x_{n-1}$.

By definition, the Koszul complex, $\mathrm{K}_\bullet(\underline{x};M)$, is the cone of

$$\begin{array}{ccccccccc}
\mathrm{K}_\bullet(\underline{x}';M) & & \cdots & \longrightarrow & M^{n-1} & \xrightarrow{(x_1\ \cdots\ x_{n-1})} & M & \longrightarrow & 0 \\
\downarrow{\scriptstyle x_n\cdot} & & & & \downarrow{\scriptstyle x_n\cdot I_{n-1}} & & \downarrow{\scriptstyle x_n\cdot} & & \\
\mathrm{K}_\bullet(\underline{x}';M) & & \cdots & \longrightarrow & M^{n-1} & \xrightarrow{(x_1\ \cdots\ x_{n-1})} & M & \longrightarrow & 0
\end{array}$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix. Taking the cone yields the top row of the following diagram.

$$\begin{array}{ccccccccc}
\mathrm{K}_\bullet(\underline{x};M) = & & \cdots & \longrightarrow & \begin{matrix} M^{n-1} \\ \oplus \\ M \end{matrix} & \xrightarrow{\begin{pmatrix} x_1 & \cdots & x_{n-1} & x_n \\ 0 & \cdots & 0 & 0 \end{pmatrix}} & \begin{matrix} M \\ \oplus \\ 0 \end{matrix} & \longrightarrow & 0 \\
 & & & & \downarrow\cong & \curvearrowright & \downarrow\cong & & \\
 & & & & M^n & \xrightarrow[(x_1\ \cdots\ x_n)]{} & M & \longrightarrow & 0
\end{array}$$

This proves part (a). Moreover, taking homology at the zeroth position of the bottom row (the M beneath $M\oplus 0$), the commutivity of the diagram and the isomorphisms depicted allow us to conclude

$$H_0(\underline{x};M) \cong \frac{M}{(x_1,\ldots,x_n)M}$$

which is among the claims of part (c).

(b). This is likewise proven by induction on n and the base cases $n=1,2$ have likewise already been shown. Therefore we assume $n\geq 3$ and that the claim holds for $\underline{x}'$. Again $\mathrm{K}_\bullet(\underline{x};M)$ is the cone of the chain map

$$\begin{array}{ccccccccc}
\mathrm{K}_\bullet(\underline{x}';M) = & & 0 & \longrightarrow & M & \xrightarrow{\xi} & M^{n-1} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle x_n\cdot} & & & & \downarrow{\scriptstyle x_n\cdot} & & \downarrow{\scriptstyle x_n I_{n-1}} & & \\
\mathrm{K}_\bullet(\underline{x}';M) = & & 0 & \longrightarrow & M & \xrightarrow[\xi]{} & M^{n-1} & \longrightarrow & \cdots
\end{array}$$

where $\xi = \begin{pmatrix} x_{n-1} & -x_{n-2} & \cdots & (-1)^{n-2}x_1 \end{pmatrix}^T$. Taking the cone yields

$$\begin{array}{ccccccc}
0 & \longrightarrow & \begin{matrix} 0 \\ \oplus \\ M \end{matrix} & \xrightarrow{\begin{pmatrix} 0 & x_n \\ 0 & -\xi \end{pmatrix}} & \begin{matrix} M \\ \oplus \\ M^{n-1} \end{matrix} & \longrightarrow & \cdots \\
 & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & M & \xrightarrow[\overline{\xi}=\begin{pmatrix} x_n \\ -\xi \end{pmatrix}]{} & M^n & \longrightarrow & \cdots
\end{array}$$

which proves the desired result. Taking homology at the n^{th} position allows us to complete the proof of part (c) as well:

$$H_n(\underline{x};M) \cong \operatorname{Ker}\left(\overline{\xi}\right) = \bigcap_{i=1}^{n} \{\, m\in M \mid x_i m = 0\} = (0 :_M \langle \underline{x}\rangle).$$

□

Remark 7.3.12. In the context of Proposition 7.3.11, a similar analysis shows that each differential $\partial_j^{K_\bullet(\underline{x};M)}$ can be expressed as a matrix consisting entirely of zeros and $\pm x_1, \ldots, \pm x_n$.

Proposition 7.3.13. *For every $i \in \mathbb{Z}$, there exists a short exact sequence*

$$0 \longrightarrow \frac{H_i(\underline{x}'; M)}{x_n \cdot H_i(\underline{x}'; M)} \longrightarrow H_i(\underline{x}; M) \longrightarrow (0 \underset{H_{i-1}(\underline{x}';M)}{:} x_n) \longrightarrow 0.$$

Proof. By part (b) of Proposition 7.3.4 and by the definitions of the mapping cone and the Koszul complex (6.4.3 and 7.3.5, respectively) it suffices to show there exists a short exact sequence of R-complexes

$$0 \longrightarrow K_\bullet(\underline{x}'; M) \longrightarrow K_\bullet(\underline{x}; M) \longrightarrow \Sigma K_\bullet(\underline{x}'; M) \longrightarrow 0 .$$

This is given by Proposition 6.4.6, so the proof is complete. □

The following fact is used with the preceding proposition to explain some annihilation properties of Koszul homology modules in the subsequent proposition.

Fact 7.3.14. *Consider the following exact sequence of R-modules.*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

If $r, s \in R$ annihilate A and C, respectively, then $rs \in R$ annihilates B.

Proof. Let $b \in B$ be given. Since s annihilates C we have

$$\beta(sb) = s\beta(b) = 0$$

so $sb \in \operatorname{Ker}(\beta) = \operatorname{Im}(\alpha)$. Let $a \in A$ such that $\alpha(a) = sb$ and we have

$$rsb = r\alpha(a) = \alpha(ra) = \alpha(0) = 0.$$

□

Proposition 7.3.15. *In the context of Proposition 7.3.11, for any $i \in [n]$ and any $j \in \mathbb{Z}$ one has*

$$x_i^{2^{n-1}} \cdot H_j(\underline{x}; M) = 0.$$

Proof. This is yet another proof by induction on n. When $n = 1$ we have $H_1(x; M) = (0 \underset{M}{:} x)$ and $H_0(x; M) = M/xM$ by Proposition 7.3.11(c) and $H_j(x; M) = 0$ for all $j \neq 0, 1$. Note these are indeed annihilated by x, so the result holds for the base case.

Assume $n \geq 2$ and that

$$x_i^{2^{n-2}} \cdot H_j(\underline{x}'; M) = 0$$

for any $i \leq n-1$ and any j. Let $i \in [n]$ and $j \in \mathbb{Z}$ be given and consider the short exact sequence given by Proposition 7.3.13.

$$\begin{array}{ccccccccc} 0 \longrightarrow & \dfrac{H_j(\underline{x}'; M)}{x_n \cdot H_j(\underline{x}'; M)} & \longrightarrow & H_j(\underline{x}; M) & \longrightarrow & (0 \underset{H_{j-1}(\underline{x}';M)}{:} x_n) & \longrightarrow 0 \\ & A & & B & & C & \end{array}$$

By the induction hypothesis, the two modules at the A and C positions are each annihilated by both x_n and $x_i^{2^{n-2}}$ for all $i \leq n-1$. Thus $H_j(\underline{x}; M)$ is annihilated by both x_n^2 and $x_i^{2^{n-1}}$ for all $i \leq n-1$, by Fact 7.3.14. □

Remark 7.3.16. The conclusion of Proposition 7.3.15 can be strengthened to say $x_i H_j(\underline{x}; M) = 0$. However, the proof of this stronger result requires technology beyond the scope of this document.

The next result leads to one of the most important properties of the Koszul complex. See Theorem 7.3.21.

Theorem 7.3.17. *If $\underline{x}$ is M-regular, then $H_i(\underline{x}; M) = 0$ for all non-zero i.*

Proof. Another proof by induction. The base case $n = 1$ follows from Proposition 7.3.11(c). Assume $n \geq 2$ and the claim holds for regular sequences of length $n-1$. If $\underline{x}$ is M-regular, then by definition of the shorter sequence $\underline{x}'$ is M-regular as well. Therefore by the induction hypothesis $H_i(\underline{x}'; M) = 0$ for all $i \neq 0$. Let $i \geq 1$ be given and consider the short exact sequence given in Proposition 7.3.13.

$$0 \longrightarrow \frac{H_i(\underline{x}'; M)}{x_n \cdot H_i(\underline{x}'; M)} \longrightarrow H_i(\underline{x}; M) \longrightarrow (0 \underset{H_{i-1}(\underline{x}'; M)}{:} x_n) \longrightarrow 0$$

By the induction hypothesis this can be rewritten

$$0 \longrightarrow 0 \longrightarrow H_i(\underline{x}; M) \longrightarrow (0 \underset{H_{i-1}(\underline{x}'; M)}{:} x_n) \longrightarrow 0.$$

Note also that as long as $i \geq 2$, by our induction hypothesis we have $(0 \underset{H_{i-1}(\underline{x}'; M)}{:} x_n) \subseteq H_{i-1}(\underline{x}'; M) = 0$, so by Fact 1.1.5 it suffices to show $(0 \underset{H_{i-1}(\underline{x}'; M)}{:} x_n) = 0$ when $i = 1$. In the case when $i = 1$ we have

$$H_{i-1}(\underline{x}'; M) = H_0(\underline{x}'; M) \cong M/(\underline{x}')M$$

by Proposition 7.3.11. Since $\underline{x}$ is M-regular, x_n is regular on $M/(\underline{x}')M$ and is therefore not a zero-divisor on $H_0(\underline{x}'; M)$. Hence

$$(0 \underset{H_0(\underline{x}'; M)}{:} x_n) = 0.$$

$\square$

Definition 7.3.18. An R-module M has *finite projective dimension* (written $\mathrm{pd}_R(M) < \infty$) if there exists an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that $P_0, \ldots, P_n$ are each projective. Given such a sequence we also write $\mathrm{pd}_R(M) \leq n$; we have equality in the case when the above is the shortest such sequence.

Example 7.3.19. By the above definition, an R-module M is projective if and only if its projective dimension is zero.

Example 7.3.20. We claim if M is a finitely generated abelian group (i.e., a finitely generated $\mathbb{Z}$-module), then $\mathrm{pd}_{\mathbb{Z}}(M) \leq 1$. If M has generators $m_1, \ldots, m_r$, then by the Fundamental Theorem of Finitely Generated Abelian Groups we write

$$M \cong \mathbb{Z}^{r-n} \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z} \tag{7.3.20.1}$$

for some integers $d_1, \ldots, d_n$. Hence one can surject onto M from the free module $\mathbb{Z}^r$:

$$\mathbb{Z}^r \xrightarrow{\tau} M \longrightarrow 0$$

where $\tau(e_i) = m_i$ for each standard basis vector e_i. Using the isomorphism (7.3.20.1) we complete the projective resolution as a short exact sequence.

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{D} \mathbb{Z}^r \xrightarrow{\tau} M \longrightarrow 0$$

Here D can be represented as a matrix mapping generators of $\mathbb{Z}^n$ to generators of $\operatorname{Ker}(\tau)$.

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

A takeaway from this example is that the Fundamental Theorem gives us a way to build free resolutions.

Theorem 7.3.21. *Assume $\underline{x}$ is R-regular.*

(a) $K_\bullet(\underline{x})$ *is a free resolution of* $R/(\underline{x})R$ *over* R.

(b) $\operatorname{Ext}^i_R(R/(\underline{x})R, R/(\underline{x})R) \cong (R/(\underline{x})R)^{\binom{n}{i}}$

(c) $\operatorname{pd}_R(R/(\underline{x})R) = n$

Proof. (a). Theorem 7.3.17 tells us we have vanishing homologies for all $i \neq 0$ and Proposition 7.3.11(c) tells us $H_0(\underline{x}) \cong R/(\underline{x})R$. It follows readily that the following augmented Koszul complex is exact.

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & R & \longrightarrow & R^n & \longrightarrow \cdots \longrightarrow & R^n & \longrightarrow & R & \xrightarrow{\tau} & \dfrac{R}{(\underline{x})R} & \longrightarrow 0 \\ & n & & n-1 & & 1 & & 0 & & -1 & \end{array}$$

Note we have incidentally shown $\operatorname{pd}_R(R/(\underline{x})) \leq n$.

(b). The free resolution of $R/(\underline{x})R$ from part (a) is a projective resolution so we consider

$$\begin{array}{ccccccccccc} \operatorname{Hom}_R(K_\bullet(\underline{x}), R/(\underline{x})) = & 0 \longrightarrow & R^* & \longrightarrow & (R^n)^* & \longrightarrow \cdots \longrightarrow & (R^n)^* & \longrightarrow & R^* & \longrightarrow 0 \\ & & 0 & & -1 & & -(n-1) & & -n & \end{array}$$

which is isomorphic to

$$\begin{array}{ccccccccccc} 0 \longrightarrow & R/(\underline{x}) & \longrightarrow & (R/(\underline{x}))^n & \longrightarrow \cdots \longrightarrow & (R/(\underline{x}))^{\binom{n}{i}} & \longrightarrow \cdots \longrightarrow & (R/(\underline{x}))^n & \longrightarrow & R/(\underline{x}) & \longrightarrow 0 \\ & 0 & & -1 & & i & & -(n-1) & & -n & \end{array}$$

by Hom-cancellation. Let the above complex be denoted $\diamond$. The differentials of $\diamond$ are the transposes of the matrices representing the differentials in the original free resolution, which are composed entirely of zeroes and $\pm x_1, \ldots, \pm x_n$. Hence every differential in $\diamond$ is a zero map and therefore

$$\operatorname{Ext}^i_R(R/(\underline{x}), R/(\underline{x})) \cong H_{-i}(\diamond) \cong (R/(\underline{x}))^{\binom{n}{i}}$$

for all i.

(c). Suppose the projective dimension of $R/(\underline{x})$ is less than n. Then there exists a projective resolution

$$0 \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0\ .$$

Since Ext is independent of choice of resolution by Theorem 6.5.2, this implies

$$0 \cong \operatorname{Ext}^n_R(R/(\underline{x}), R/(\underline{x})) \cong (R/(\underline{x}))^{\binom{n}{n}} \cong R/(\underline{x}) \neq 0$$

where the non-vanishing holds since $\underline{x}$ is R-regular. Hence part (c) is proven by contradiction. □

Example 7.3.22. Let $\mathbb{K}$ be a field and let R be one of the following rings.

$$\mathbb{K}[X_1, \ldots, X_n] \qquad \mathbb{K}[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)} \qquad \mathbb{K}[\![X_1, \ldots, X_n]\!]$$

In any case, the sequence $\underline{X} = X_1, \ldots, X_n$ is R-regular and as we saw in Theorem 7.3.21, the augmented Koszul complex is therefore exact

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & R & \longrightarrow & R^n & \longrightarrow & \cdots & \longrightarrow & R^n & \longrightarrow & R & \longrightarrow & R/(\underline{X}) & \longrightarrow & 0 \\ & & n & & n-1 & & & & 1 & & 0 & & & & \end{array}$$

and $\operatorname{pd}_R(R/(\underline{X})) = n$. This is a noteworthy example, because in general writing out projective resolutions is very hard. In fact, even detecting finite projective dimension is difficult.

7.4 Additional Discussions on Ext

In the first theorem of the section, we strengthen part of Proposition 4.2.3, which we will subsequently generalize in Theorem 7.4.3. This is related to the very important Hilbert Syzygy Theorem (7.4.4) and results of Auslander, Buchsbaum, and Serre (7.4.11), and Auslander and Bridger (7.4.18).

Theorem 7.4.1. *Let R be a commutative ring with identity and let M be an R-module. The following are equivalent.*

(i) M is a projective module over R.

(ii) $\operatorname{Ext}^i_R(M, N) = 0$ for all $i \geq 1$ and for all R-modules N.

(iii) $\operatorname{Ext}^1_R(M, N) = 0$ for all R-modules N.

Proof. It is obvious that (ii) implies (iii). The implication (i) implies (ii) is Proposition 4.2.3(a). The implication (iii) implies (i) follows from Theorem 7.2.4 and Definition 1.1.14(d). □

Lemma 7.4.2 (Dimension Shifting)**.** *Assume*

$$0 \longrightarrow A \xrightarrow{\epsilon} L_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} L_0 \xrightarrow{\tau} B \longrightarrow 0 \qquad (7.4.2.1)$$

is an exact sequence of R-modules and that L_i is projective for each $i = 0, \ldots, n-1$. Then for all $i \geq 1$ and for any R-module X we have

$$\operatorname{Ext}^{n+i}_R(B, X) \cong \operatorname{Ext}^i_R(A, X).$$

Proof. Let the following be a projective resolution of A, indexed rather suggestively.

$$\cdots \xrightarrow{d_{n+2}} L_{n+1} \xrightarrow{d_{n+1}} L_n \xrightarrow{\pi} A \longrightarrow 0$$

We can splice this with (7.4.2.1) to get

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow L_{n+1} & \xrightarrow{d_{n+1}} & L_n & & \xrightarrow{\quad d_n \quad} & & L_{n-1} & \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} & L_0 & \xrightarrow{\tau} & B & \longrightarrow & 0 \\
& & & {\scriptstyle\pi}\searrow & & \nearrow{\scriptstyle\epsilon} & & & & & & & \\
& & & & A & & & & & & & & \\
& & & \nearrow & & \searrow & & & & & & & \\
& & 0 & & & & 0 & & & & & &
\end{array}$$

where $d_n = \epsilon \circ \pi$. A diagram chase shows that the top row of this diagram is an augmented projective resolution of B. Calculating Ext using this we have

$$\operatorname{Ext}_R^{n+i}(B, X) = \frac{\operatorname{Ker}\left(L_{n+i}^* \xrightarrow{d_{n+i}^*} L_{n+i+1}^* \right)}{\operatorname{Im}\left(L_{n+i-1}^* \xrightarrow{d_{n+i-1}^*} L_{n+i}^* \right)} = \operatorname{Ext}_R^i(A, X)$$

for any $i \geq 1$. (Note that there is an alternative proof using long exact sequences associated with (7.4.2.1).) □

The following theorem generalizes Theorem 7.4.1.

Theorem 7.4.3. *Let $n \in \mathbb{N}$ and let M be an R-module. The following are equivalent.*

(i) *There exists an exact sequence*

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that each P_i is projective, i.e., $\operatorname{pd}_R(M) \leq n$.

(ii) $\operatorname{Ext}_R^i(M, -) = 0$ *for all* $i \geq n+1$.

(iii) $\operatorname{Ext}_R^{n+1}(M, -) = 0$.

(iv) *For every augmented projective resolution of M*

$$Q_\bullet^+ = \qquad \cdots \longrightarrow Q_{n+1} \xrightarrow{\partial_{n+1}^Q} Q_n \xrightarrow{\partial_n^Q} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

the module $\operatorname{Im}\left(\partial_n^Q\right)$ *is projective. That is, the augmented resolution above can be "softly truncated" to form a new projective resolution, written below.*

$$0 \longrightarrow \operatorname{Im}\left(\partial_n^Q\right) \xrightarrow{\subseteq} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

Proof. Showing (*iv*) implies (*i*) and showing (*ii*) implies (*iii*) are each trivial, and (*i*) implies (*ii*) follows from Note 2.1.7. So we will endeavor only to show that (*iii*) implies (*iv*). Assume (*iii*) holds and let $Q_\bullet^+$ be an augmented projective resolution of M. By Lemma 7.4.2 and our assumption we have

$$0 = \operatorname{Ext}_R^{n+1}(M, N) \cong \operatorname{Ext}_R^1\left(\operatorname{Im}\left(\partial_n^Q\right), N\right)$$

for all R-modules N. Therefore $\operatorname{Im}\left(\partial_n^Q\right)$ is projective by Theorem 7.4.1. □

The next result gives some rings over which all modules have finite projective dimension. Its proof is outside the scope of this document. See the subsequent example for rings that have modules of infinite projective dimension.

Theorem 7.4.4 (Hilbert Syzygy Theorem)**.** *Let $R = \mathbb{K}[x_1, \ldots, x_d]$ where $\mathbb{K}$ is a field, let M be an R-module, and let $P_\bullet^+$ be an augmented projective resolution of M. Under these assumptions $\operatorname{Im}\left(\partial_d^P\right)$ is projective. This is called a d^{th}* syzygy *of M. If $R = \mathbb{Z}[x_1, \ldots, x_d]$, then $\operatorname{Im}\left(\partial_{d+1}^P\right)$ is projective. If we localize either of these two rings, then the respective conclusions still hold.*

Example 7.4.5. Define the following two rings.

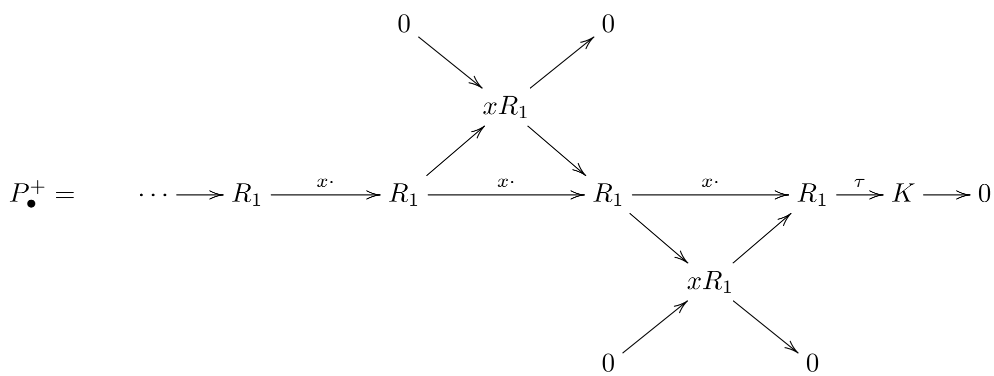
$$R_1 = \frac{\mathbb{K}[x]}{(x^2)} \qquad\qquad R_2 = \frac{\mathbb{K}[x,y]}{(xy)}$$

The rings R_1 and R_2 are not integral domains, so they are not (localizations of) polynomial rings over fields (and the hypotheses of Theorem 7.4.4 are therefore not satisfied). It is a fact (beyond the scope of this document) that if M_1 is an R_1-module and not free, then given a projective resolution $P_\bullet$ of M_1, the module $\operatorname{Im}\left(\partial_n^P\right)$ is never projective.

For example, consider the module

$$K = \frac{R_1}{xR_1}$$

for which we construct an augmented projective resolution.



The map τ is the natural surjection and we can observe immediately that at no point does this resolution terminate, which is a result of the fact that $\operatorname{Im}(x\cdot) = xR_1$ is not projective. Indeed if xR_1 were free, then $\operatorname{Ann}_R(xR_1) = \{0\}$, but $0 \neq x \in \operatorname{Ann}_R(xR_1)$ since $x^2 = 0$ implies $x \cdot xR_1 = 0$. Moreover, xR_1 is not projective by Corollary 3.4.18, because R_1 is local. Hence $xR_1 = \operatorname{Im}\left(\partial_1^P\right) = \operatorname{Im}\left(\partial_n^P\right)$ is not projective, for all $n \geq 1$.

Let us justify our claim that R_1 is not local. Recall the prime correspondence under quotients.

$$\begin{aligned}\{\mathfrak{p} \in \operatorname{Spec}(R_1)\} \rightleftarrows & \left\{\mathfrak{p} \in \operatorname{Spec}(\mathbb{K}[x]) \mid x^2 \in \mathfrak{p}\right\} \\ =& \{\mathfrak{p} \in \operatorname{Spec}(\mathbb{K}[x]) \mid x \in \mathfrak{p}\} \\ =& \{\mathfrak{p} \in \operatorname{Spec}(\mathbb{K}[x]) \mid (x) \subseteq \mathfrak{p}\}\end{aligned}$$

Since (x) is maximal in $\mathbb{K}[x]$, there is only one ideal on the right and therefore only one ideal on the left, so R_1 is local.

It follows that for all $n \in \mathbb{Z}$, there exists some R-module N_n such that $\operatorname{Ext}^n_{R_1}(K, N_n)$ is non-zero. In fact, for any n we may set $N_n = K$. Consider $\operatorname{Hom}_{R_1}(P_\bullet, K)$ below.

$$0 \longrightarrow \operatorname{Hom}_{R_1}(R_1, K) \xrightarrow{x\cdot^*} \operatorname{Hom}_{R_1}(R_1, K) \xrightarrow{x\cdot^*} \operatorname{Hom}_{R_1}(R_1, K) \xrightarrow{x\cdot^*} \cdots$$

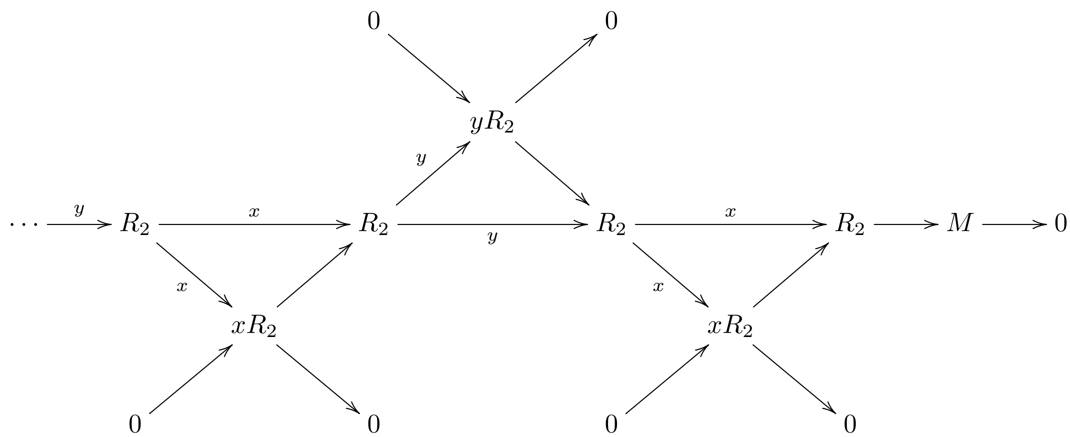
By Hom-cancellation this is isomorphic to the following.

$$0 \longrightarrow K \xrightarrow[=0]{x\cdot} K \xrightarrow[=0]{x\cdot} K \xrightarrow[=0]{x\cdot} \cdots$$

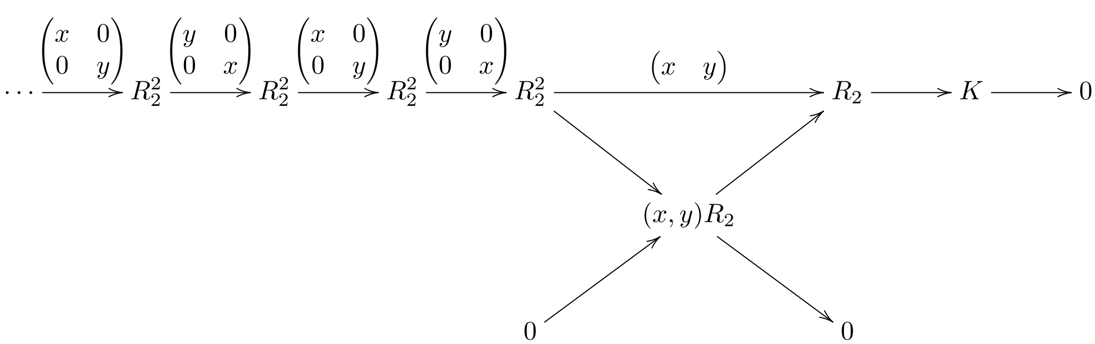
Therefore we may compute Ext for any $n \geq 1$.

$$\operatorname{Ext}^n_{R_1}(K, K) = \frac{\operatorname{Ker}\left(K \xrightarrow{0} K \right)}{\operatorname{Im}\left(K \xrightarrow{0} K \right)} = \frac{K}{0} \cong K \neq 0$$

Now let us play with R_2, defining the modules $K = R_2/(x, y)R_2$ and $M = R_2/xR_2 \cong K[y]$. Constructing an augmented projective resolution $P_\bullet^+$ of M, we observe a periodic behavior.



We may also construct an augmented projective resolution $Q_\bullet^+$ of K that exhibits a similar periodic behavior, but not immediately.



As above, we know

$$\operatorname{Ext}^n_{R_2}(M, K) \neq 0 \neq \operatorname{Ext}^n_{R_2}(K, K).$$

Applying the $\operatorname{Hom}_{R_2}(-, M)$ functor to $P_\bullet$, we compute $\operatorname{Ext}^n_{R_2}(M, M)$ precisely. Skipping over the Hom-cancellation step we have

$$\operatorname{Hom}_{R_2}(P_\bullet, M) = \qquad 0 \longrightarrow M \xrightarrow[=0]{x\cdot} M \xrightarrow[\neq 0]{y\cdot} M \xrightarrow[=0]{x\cdot} M \xrightarrow[\neq 0]{y\cdot} \cdots$$

so we compute as follows.

$$\operatorname{Ext}^0_{R_2}(M,M) = \frac{\operatorname{Ker}\left(M \xrightarrow{x\cdot} M \right)}{\operatorname{Im}\left(0 \longrightarrow M \right)} = \frac{M}{0} \cong M$$

$$\forall n \geq 1 \qquad \operatorname{Ext}^{2n-1}_{R_2}(M,M) = \operatorname{Ext}^1_{R_2}(M,M) = \frac{\operatorname{Ker}\left(M \xrightarrow{y\cdot} M \right)}{\operatorname{Im}\left(M \xrightarrow{x\cdot} M \right)} = \frac{0}{0} = 0$$

$$\forall n \geq 1 \qquad \operatorname{Ext}^{2n}_{R_2}(M,M) = \operatorname{Ext}^2_{R_2}(M,M) = \frac{\operatorname{Ker}\left(M \xrightarrow{x\cdot} M \right)}{\operatorname{Im}\left(M \xrightarrow{y\cdot} M \right)} = \frac{M}{yM} \cong \frac{K[y]}{yK[y]} \cong K$$

It is natural to ask whether one can say anything nice (as in Theorem 7.4.3(d)) about the image modules occurring in the resolutions from Example 7.4.5. In fact, we can, using the following notion; see Theorem 7.4.8.

Definition 7.4.6. An R-module G is *totally reflexive* if

(i) G is finitely generated and the map

$$\delta^G_R : G \xrightarrow{\cong} \operatorname{Hom}_R(\operatorname{Hom}_R(G,R),R)$$
$$g \longmapsto \Psi_g$$

is an R-module isomorphism, where

$$\Psi_g : \operatorname{Hom}_R(G,R) \longrightarrow R$$
$$\psi \longmapsto \psi(g).$$

(ii) For all $i \geq 1$ we have

$$\operatorname{Ext}^i_R(G,R) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(G,R),R).$$

Using the notation $(-)^* := \operatorname{Hom}_R(-,R)$ we may write more succinctly

(i) G is finitely generated and $\delta^G_R \colon G \xrightarrow{\cong} G^{**}$.

(ii) $\operatorname{Ext}^i_R(G,R) = 0 = \operatorname{Ext}^i_R(G^*,R)$ for all $i \geq 1$.

Example 7.4.7. Let $n \in \mathbb{N}$. The finitely generated free module R^n is totally reflexive as is any finitely generated projective R-module.

Theorem 7.4.8 (Auslander-Bridger)**.** *Let $\mathbb{K}$ be a field. If R is either of the two rings*

$$\mathbb{K}[x_0,\dots,x_d]/(f) \qquad\qquad \mathbb{Z}[x_1,\dots,x_d]/(f)$$

(where f is a non-zero, non-unit polynomial) or a localization of either of these, then there exists an exact sequence

$$0 \longrightarrow G_d \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

such that $G_0, \dots, G_d$ are each totally reflexive. Moreover for every projective resolution $P_\bullet$ of M such that each P_i is finitely generated, the module $\operatorname{Im}\left(\partial_d^P\right)$ is totally reflexive. Therefore the sequence

$$0 \longrightarrow \operatorname{Im}\left(\partial_d^P\right) \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is exact and $P_0, \dots, P_{d-1}, \operatorname{Im}\left(\partial_d^P\right)$ are all totally reflexive.

Now we return to projective dimension.

Fact 7.4.9. *For any local noetherian ring $(R, \mathfrak{m})$, the number of generators of $\mathfrak{m}$ is no smaller than the Krull dimension of R,*

$$\dim(R) = \sup\left\{ n \in \mathbb{N} \mid \exists \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \operatorname{Spec}(R) \right\}.$$

Next we restate part of Definition 2.3.4 and give a more complete version of Theorem 2.3.8.

Definition 7.4.10. A local noetherian ring $(R, \mathfrak{m})$ is *regular* if the number of generators of $\mathfrak{m}$ is equal to the Krull dimension of R.

Theorem 7.4.11 (Auslander, Buchsbaum, Serre)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. The following are equivalent.*

(i) *R is regular.*

(ii) $\operatorname{pd}_R(M) < \infty$ *for all R-modules M.*

(iii) $\operatorname{pd}_R(k) < \infty$.

(iv) $\operatorname{pd}_R(k) = \dim(R)$, *i.e., the finite projective dimension is equivalent to the Krull dimension.*

(v) $\operatorname{pd}_R(M) \leq \dim(R)$ *for all R-modules M.*

(vi) $\operatorname{Ext}_R^{d+1}(M, -) = 0$ *for all R-modules M, where $d = \dim(R)$.*

(vii) *For every R-module M and for every projective resolution $Q_\bullet$ of M, $\operatorname{Im}\left(\partial_d^Q\right)$ is projective, where $d = \dim(R)$.*

While R is always projective as an R-module (in fact, R^n is projective for all $n \geq 1$), R is rarely injective as an R-module (defined in 1.1.15), as we see next.

Theorem 7.4.12. *Assume R is noetherian and that R is either local or an integral domain. If R has a non-zero, finitely generated injective module, then R is artinian. That is, R satisfies the following equivalent conditions.*

(i) *R satisfies the descending chain condition on ideals.*

(ii) *R is a noetherian ring with Krull dimension zero.*

(iii) *R is a noetherian ring and every prime ideal is maximal.*

Example 7.4.13. If R is any field, then the only two ideals are R and 0, implying each of the following also hold.

(a) R satisfies both the ascending and descending chain conditions on ideals (the only non-trivial chain of ideals is $0 \subsetneq R$).

(b) The sole prime ideal of R is the zero ideal, so the Krull dimension of R is 0.

(c) Every R-module is a free module (i.e., of the form $R^{(\Lambda)}$) and therefore all R-modules are both injective and projective.

Example 7.4.14. Consider the ring of integers $R = \mathbb{Z}$, for which the ascending chain condition holds, but for which the descending chain condition fails. Ti see why the ascending chain condition holds, consider an arbitrary ascending chain of ideals.

$$n_1\mathbb{Z} \subseteq n_2\mathbb{Z} \subseteq n_3\mathbb{Z} \subseteq \dots$$

The integer n_1 has a finite list of prime factors. In order for the chain above to be one of proper containments, one must remove at least one prime factor from the list at each step. Since the list is finite, the chain has to stabilize.

We can confirm the descending chain condition fails by giving the following example.

$$\mathbb{Z} \supsetneq 10\mathbb{Z} \supsetneq 20\mathbb{Z} \supsetneq 40\mathbb{Z} \supsetneq \cdots \supsetneq 10 \cdot 2^k\mathbb{Z} \supsetneq \cdots$$

We can also confirm that $\mathbb{Z}$ is not injective as a $\mathbb{Z}$-module, which we do by showing $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ is not exact. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{13\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/13\mathbb{Z} \longrightarrow 0 \tag{7.4.14.1}$$

and apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$.

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/13\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{13\cdot} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0 \tag{7.4.14.2}$$

By Hom-cancellation the labeled map above can be written

$$\mathbb{Z} \overset{13\cdot}{\hookrightarrow} \mathbb{Z}\ .$$

Since the multiplication map is not onto, (7.4.14.2) is not exact.

Alternatively, our short exact sequence (7.4.14.1) is also an augmented projective resolution for the $\mathbb{Z}$-module $\mathbb{Z}/13\mathbb{Z}$, yielding the sequences

$$P_\bullet = \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{13\cdot} \mathbb{Z} \longrightarrow 0$$

$$\mathrm{Hom}_{\mathbb{Z}}(P_\bullet, \mathbb{Z}) \cong \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{13\cdot} \mathbb{Z} \longrightarrow 0$$

by Hom-cancellation. We calculate $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/13\mathbb{Z}, \mathbb{Z})$ below.

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/13\mathbb{Z}, \mathbb{Z}) = \frac{\mathrm{Ker}\left(\mathbb{Z} \longrightarrow 0 \right)}{\mathrm{Im}\left(\mathbb{Z} \xrightarrow{13\cdot} \mathbb{Z} \right)} = \frac{\mathbb{Z}}{13\mathbb{Z}} \neq 0$$

If $\mathbb{Z}$ were injective, then for an arbitrary $\mathbb{Z}$-module M, $\mathrm{Ext}^i_{\mathbb{Z}}(M, \mathbb{Z}) = 0$ for all $i \geq 1$ (Definition 1.1.15 part (d)). Since this has just been shown not to be the case, $\mathbb{Z}$ is not injective as a $\mathbb{Z}$-module.

Remark 7.4.15. A similar result as in the previous example can be obtained for any integral domain that is not a field, as the construction requires only that we have a ring R with a non-zero-divisor. In general R is not injective as an R-module. Moreover, in general there does not exist an exact sequence

$$0 \longrightarrow R \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0$$

such that $I_0, \dots, I_n$ are injective. This prompts the following definition.

Definition 7.4.16. A noetherian ring R is *Gorenstein* if there exists an exact sequence

$$0 \longrightarrow R \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0$$

such that $I_0, \ldots, I_n$ are injective, i.e., R has finite injective dimension and we write $\mathrm{id}_R(R) < \infty$.

Example 7.4.17. Each of the following rings are Gorenstein.

$$\frac{\mathbb{K}[x_0, \ldots, x_n]}{(f)} \qquad\qquad \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(f)}$$

In Theorem 7.4.11 we see that given a regular local ring R, one can take any R-module M along with any projective resolution $P_\bullet$ of M, and it follows that the kernel of every differential past the d^{th} spot will be projective, where $d = \dim(M)$. What if the ring is only Gorenstein? The answer comes in the next result by Auslander and Bridger. Compare it to Theorem 7.4.8 with Example 7.4.17 in mind.

Theorem 7.4.18 (Auslander-Bridger)**.** *If $(R, \mathfrak{m}, k)$ is a local noetherian ring, then the following are equivalent.*

(i) R is a Gorenstein ring.

(ii) For every finitely generated R-module M, there exists an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

such that $G_0, \ldots, G_n$ are all totally reflexive.

(iii) For every finitely generated R-module M and for every projective resolution $P_\bullet$ where each P_i is finitely generated, the module $\mathrm{Im}\left(\partial_d^P\right)$ is totally reflexive.

(iv) There exists an exact sequence

$$0 \longrightarrow G_d \longrightarrow G_{d-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow k \longrightarrow 0$$

such that $G_0, \ldots, G_d$ are totally reflexive and where $d = \dim(R)$ is the Krull dimension of R.

Chapter 8

Connections With Graphs

In this chapter, we show how to construct rings from simple graphs in such a way that algebraic properties of the ring correspond with combinatorial properties of the graph from which it was built. One important example of this is the fact that rings so constructed from K_1-coronas of finite simple graphs are Cohen-Macaulay (Section 8.3). We present this chapter rather colloquially.

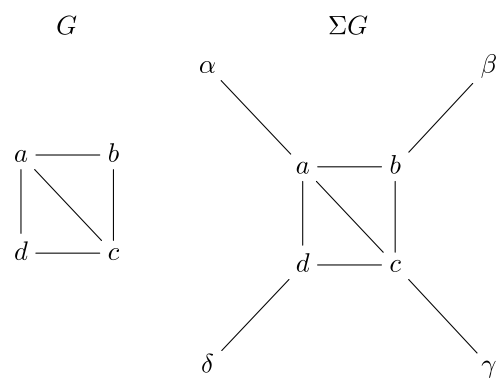
8.1 Introduction

Notation 8.1.1. In this chapter, G will denote a *finite simple graph*, which is a graph with a finite number of vertices, no duplicate edges, no directional edges, and no loops (edges that begin and terminate at the same vertex). Let V denote the set of vertices of G, either $\{v_1, \ldots, v_d\}$ or $\{a, b, c, \ldots\}$. Let E denote the set of edges of G, in which $v_i v_j$ denotes the edge connecting vertices v_i and v_j. That is, $v_i v_j \in E$ if and only if v_i and v_j are adjacent in G.

We may also denote a graph along with its vertex and edge sets as $G = \{V, E\}$. The $\mathcal{K}_1$-*corona* of G is the graph $\Sigma G = \{U, F\}$ where $|V| = d$, $U = V \cup \{u_1, \ldots, u_d\}$, and $F = E \cup \{\, u_i v_i \mid i = 1, \ldots, d\}$. A graph $\mathcal{K}_d = \{V, E_d\}$ is a *complete graph* with d vertices if $vu \in E_d$ for every $v, u \in V$. Finally we define $G^C = \{V, E'\}$ to be the *complement* of G, where $E' = E_d \setminus E$. One may wish to consult [3] for background on graph theory, though our treatment here is mostly self-contained.

If $n \in \mathbb{N}$, then let $[n]$ denote the set $\{1, \ldots, n\}$.

Example 8.1.2. The following simple graphs depict a graph G and its $\mathcal{K}_1$-corona ΣG.



We shall see that ΣG is particularly nice from an algebraic standpoint.

Definition 8.1.3. Let $G = \{V, E\}$ be simple graph. For any subset $U \subseteq V$, the *subgraph of G induced by U*, denoted $[U]$, is the subgraph $H = \{U, F\}$ where $vv' \in F \subseteq E$ if and only if $v, v' \in U$. We say $[U]$ is a *proper* subgraph if $U \neq V$.

Definition 8.1.4. Let $G = \{V, E\}$ be a simple graph and let $U \subset V$ be a subset. The subgraph $[U]$ is a *clique* of G if it is a complete graph. A subgraph $[U]$ is a *maximal clique* if it is a clique that is not a proper subgraph of another clique of G.

Example 8.1.5. Consider the following simple graph G.

Definition 8.1.6. Let K be a non-zero commutative ring with identity, let G be a simple graph, and V its vertex set. In this setting consider the polynomial ring $K[V] = K[v_1, \ldots, v_d]$. Then the ideal

$$I(G) = \langle v_i v_j \mid v_i, v_j \text{ adjacent in } G \rangle$$

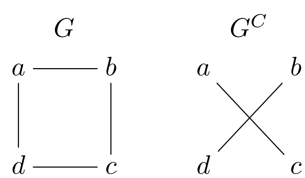
is called the *edge ideal of G*. Often we will take K to be a field, in which case we will write $\mathbb{K}$, but a ring is all we need for this definition. Define also the quotient ring $K[G] = K[V]/I(G)$.

8.2 Basic Examples and Facts

The idea here is some of the combinatorial properties of G are in correspondence with the algebraic properties of $I(G)$ and of $K[G]$. One thing we would like to do is find interesting examples of rings by identifying corresponding properties of the graph and then building a graph with those properties. This is the whole point of the rest of this chapter.

Fact 8.2.1. *The edge set of G^C, denoted $E(G^C)$, forms a linearly independent subset of $K[G]$, in fact, it represents the set of non-zero, square-free quadratic monomials in $K[G]$.*

Example 8.2.2. Consider the graph G and its complement.

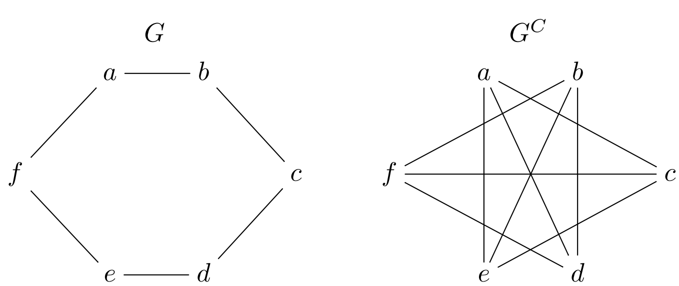


In this case

$$K[G] = \frac{K[a, b, c, d]}{(ab, bc, cd, ad)}$$

and $E(G^C) = \{ac, bd\} \subset K[G]$ is a linearly independent subset.

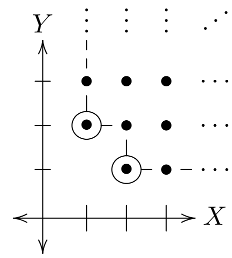
Example 8.2.3. Consider another graph and its complement.



Note the elements corresponding to each $\mathcal{K}_3 \subset G^C$ such as ace are non-zero in $K[G]$.

Fact 8.2.4. *Let $g, f_1, \ldots, f_m$ be (monic) monomials in the ring $K[\underline{X}]$ where $\underline{X} = X_1, \ldots, X_n$ and let $I = \langle f_1, \ldots, f_m \rangle \leq K[\underline{X}]$. Then $g \in I$ if and only if $\overline{g} = 0$ in $K[\underline{X}]/I$ if and only if $g = f_i h$ for some $i \in [m]$ and some monomial $h \in K[\underline{X}]$. This is Theorem 1.1.9 in [4].*

Example 8.2.5. The ideal $I = \left\langle XY^2, X^2Y \right\rangle \subset K[\underline{X}]$ has the following visual representation



where '$\bullet$' denotes a monomial in I and the two generators have been circled. Note that $I = \left\langle XY^2, X^2Y, X^3Y^4 \right\rangle$, since X^3Y^4 is redundant as a generator. One can see this visually as X^3Y^4 corresponds to a non-circled point $\bullet$ in the lattice representing I.

8.3 Why Are $\mathcal{K}_1$-Coronas Cohen-Macaulay?

We want to motivate our investigation into rings of the form $\mathbb{K}[\Gamma]$ where $\mathbb{K}$ is a field and Γ is the $\mathcal{K}_1$-corona of a simple graph. In any polynomial ring, let $\mathfrak{X}$ denote the ideal generated by all the variables.

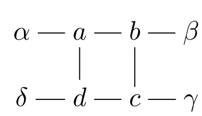
Fact 8.3.1. *Let $R = \mathbb{K}[X_1, \ldots, X_d]$ and let $I \lneq R$ be an ideal in the polynomial ring. Assume for each $i \in [d]$ there exists some $e_i \geq 1$ such that $X_i^{e_i} \in I$. Then* $\operatorname{Spec}(R/I) = \{\mathfrak{X}/I\}$*, so* $\dim(R/I) = 0$ *(Krull dimension).*

Definition 8.3.2. Let $R = \mathbb{K}[X_1, \ldots, X_d]$ and let $J \lneq R$ be an ideal of the polynomial ring generated by homogeneous polynomials. If R/J has a homogeneous regular sequence $f_1, \ldots, f_n$ such that

$$\dim\left(\frac{R/J}{\langle f_1, \ldots, f_n \rangle}\right) = 0$$

then R/J is *Cohen-Macaulay*. In this case $f_1, \ldots, f_n$ is a maximal R/J-regular sequence in $\mathfrak{X}$ and $n = \dim(R/J) = \operatorname{depth}(R/J)$.

Example 8.3.3. Let $R = \mathbb{K}[a, b, c, d, \alpha, \beta, \gamma, \delta]$ and let ΣG be the following simple graph.



Let $J = (ab, bc, cd, ad, a\alpha, b\beta, c\gamma, d\delta)R$ be the edge ideal of ΣG and let $I = I(G) + (a^2, b^2, c^2, d^2)R'$, where $R' = \mathbb{K}[a, b, c, d]$. Then R/J has a regular sequence $a - \alpha, b - \beta, c - \gamma, d - \delta$ such that

$$\frac{R/J}{\langle a - \alpha, b - \beta, c - \gamma, d - \delta \rangle} \cong \frac{R'}{I} \tag{8.3.3.1}$$

since on the left we essentially set $a = \alpha$, $b = \beta$, $c = \gamma$, and $d = \delta$. More precisely, if we consider the map sending $a, b, c, d \in R/J$ to themselves in R'/I and $\alpha, \beta, \gamma, \delta$ to a, b, c, d respectively, then (8.3.3.1) follows from the First Isomorphism Theorem. By Fact 8.3.1, we have $\dim(R'/I) = 0$ and by Definition 8.3.2 we conclude R/J is Cohen-Macaulay.

Remark 8.3.4. Note that in the above example we do not rely on any properties of G. In general if $\Sigma G = \{V, E\}$ is a $\mathcal{K}_1$-corona and J is its edge ideal in $\mathbb{K}[V]$, we always have a regular sequence that equates each vertex in G to its leaf in the $\mathcal{K}_1$-corona, so we will have an isomorphism like (8.3.3.1) with I satisfying the conditions of Fact 8.3.1 and therefore R/J will be Cohen-Macaulay. In short, rings of the form $\mathbb{K}[\Sigma G]$ are Cohen-Macaulay. This is a theme in the study of edge ideals: nice graphs have edge ideals with nice properties.

Here is another way to obtain the conclusion of Remark 8.3.4.

Example 8.3.5. Let $I = \langle X^3, XY, Y^2 \rangle$ be an ideal in $R = \mathbb{K}[X, Y]$. We call the square-free ideal $J = \langle X_1X_2X_3, X_1Y_1, Y_1Y_2 \rangle$ in $R' = \mathbb{K}[X_1, X_2, X_3, Y_1, Y_2]$ the *polarization* of I, where we replace each non-square-free generator by simply 'splitting' each of the offending variables into multiple variables. Similar to Example 8.3.3 we can mod out by a regular sequence to get

$$\frac{\mathbb{K}[X_1, X_2, X_3, Y_1, Y_2]/J}{\langle X_1 - X_2, X_1 - X_3, Y_1 - Y_2 \rangle} \cong \frac{\mathbb{K}[X, Y]}{I}.$$

Note once more that by Fact 8.3.1 and Definition 8.3.2, the ring R'/J is Cohen-Macaulay.

Remark 8.3.6. For any simple graph G and its $\mathcal{K}_1$-corona ΣG, the edge ideal $I(\Sigma G)$ will be the polarization of the ideal $I(G) + \langle a_1^2, \ldots, a_d^2 \rangle$, so $R/I(\Sigma G)$ is Cohen-Macaulay.

One thing that motivates the study of edge ideals is that one can see interesting algebraic information about $\mathbb{K}[G]$ from G. We see another instance of this in the next proposition. (See Section 8.4 for more about this.)

Definition 8.3.7. Set $R = \mathbb{K}[X_1, \ldots, X_d]$, let $[\![R]\!]$ denote the set of (monic) monomials in R, and let J be a monomial ideal in R (cf. Definition 3.3.8). A monomial $z \in [\![R]\!]$ is a *J-corner element* if $z \notin J$ and $X_1z, \ldots, X_dz \in J$. The set of J-corner elements is denoted $C_R(J)$.

Definition 8.3.8. Let M be a module over a local ring $(R, \mathfrak{m}, k)$. *The socle of M* is

$$\operatorname{Soc}(M) = (0 \underset{M}{:} \mathfrak{m}) = \{ m \in M \mid \mathfrak{m}m = 0 \}.$$

This is Definition 1.2.18 in [2].

Definition 8.3.9. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let M be a finitely generated R-module with $\operatorname{depth}(M) = t$. The *type* of M is $\dim_k \operatorname{Ext}_R^t(k, M)$.

Definition 8.3.10. Let $R = K[X_1, \ldots, X_d]$. A monomial ideal $J \lneq R$ is *m-reducible* if there are monomial ideals $J_1, J_2 \neq J$ such that $J = J_1 \cap J_2$. A monomial ideal $J \lneq R$ is *m-irreducible* if it is not m-reducible. (This is Definition 3.1.1 in [4].)

Example 8.3.11. In the polynomial ring $R = \mathbb{K}[X, Y]$, the ideal $J = (X^3, XY, Y^2)R$ is m-reducible. Let $J_1 = (X, Y^2)R$ and $J_2 = (X^3, Y)R$, which are m-irreducible. Then $X \notin J \subsetneq J_1$, $Y \notin J \supsetneq J_2$, and $J = J_1 \cap J_2$.

Definition 8.3.12. Let $R = K[X_1, \ldots, X_d]$ and let J be a monomial ideal. An *m-irreducible decomposition* is $J = \bigcap_{i=1}^n J_i$, where each J_i is an m-irreducible monomial ideal of R. An m-irreducible decomposition $J = \bigcap_{i=1}^n J_i$ is *redundant* if there exists some $j \in [n]$ such that $J = \bigcap_{i \neq j} J_i$. An m-irreducible decomposition is *irredundant* if it is not redundant. (These are Definitions 3.3.1 and 3.3.4 in [4].)

Proposition 8.3.13. *Let G be a simple graph with vertices $a_1, \ldots, a_d$ and define the polynomial ring $R' = \mathbb{K}[a_1, \ldots, a_d]$. Let ΣG be the $\mathcal{K}_1$-corona of G with vertices $a_1, \ldots, a_d, \alpha_1, \ldots, \alpha_d$ and define* *another polynomial ring* $R = \mathbb{K}[a_1, \ldots, a_d, \alpha_1, \ldots, \alpha_d]$. *Let* J *denote the edge ideal of* ΣG *in* R, *let* $I = I(G) + \langle a_1^2, \ldots, a_d^2 \rangle \leq R'$, *and let* $k = R'/\mathfrak{X}$.

$$\begin{aligned}
\operatorname{type}(R/J) &\overset{(i)}{=} \dim_k \left(\operatorname{Ext}^d_R(k, R/J) \right) \\
&\overset{(ii)}{=} \dim_k \left(\frac{(I : \mathfrak{X})}{I} \right) \\
&\overset{(iii)}{=} \textit{number of corner elements of } I \textit{ in } R' \\
&\overset{(iv)}{=} \textit{number of ideals in an irredundant, m-irreducible decomposition of } I \\
&\overset{(v)}{=} \textit{number of maximal cliques in } G^C
\end{aligned}$$

Proof. Set $r = \operatorname{type}(R/J)$.

(i). By Definition 1.2.15 in [2], it suffices to show the maximal R/J-regular sequences in $\mathfrak{X}$ have length d. As in Example 8.3.3, we have the R/J-regular sequence $a_1 - \alpha_1, \ldots, a_d - \alpha_d \in \mathfrak{X}$. By Remark 8.3.6, the ideal J is the polarization of I via the above regular sequence and we have the following isomorphism as in Example 8.3.5.

$$\frac{R/J}{\langle a_1 - \alpha_1, \ldots, a_d - \alpha_d \rangle} \cong \frac{R'}{I}$$

So $d = \dim(R/J)$ as we observed in Definition 8.3.2.

(ii). By Lemma 1.2.19 in [2] we have

$$\operatorname{type}(R/J) = \dim_k \left[\operatorname{Soc} \left(\frac{R/J}{\langle a_1 - \alpha_1, \ldots, a_d - \alpha_d \rangle} \right) \right].$$

There is a surjection $R/J \twoheadrightarrow R'/I$ which sends cosets $\overline{f}$ in $2d$ variables to cosets in d variables by replacing each α_i with a_i. The kernel of this map is precisely $\langle a_1 - \alpha_1, \ldots, a_d - \alpha_d \rangle$, so by the First Isomorphism Theorem we have

$$\operatorname{type}(R/J) = \dim_k \left[\operatorname{Soc}(R'/I) \right].$$

Note in the definition of the socle, we require that the module be over a local ring. By Fact 8.3.1, the unique maximal ideal of R'/I is $\mathfrak{m} = \mathfrak{X}/I$. That is, R'/I is local, and by the isomorphism just discussed, so is $(R/J)/\langle a_1 - \alpha_1, \ldots, a_d - \alpha_d \rangle$.

Now using only definitions and properties of cosets we have the following.

$$\begin{aligned}
\operatorname{Soc}(R'/I) = (0 \underset{R'/I}{:} \mathfrak{m}) = (0 \underset{R'/I}{:} \{\overline{a_1}, \ldots, \overline{a_d}\}) &= \left\{ \overline{f} \in R'/I \;\middle|\; \overline{a_i} \cdot \overline{f} = 0 \in R'/I, \ \forall i \in [d] \right\} \\
&= \left\{ \overline{f} \in R'/I \;\middle|\; a_i f \in I, \ \forall i \in [d] \right\} \\
&= \left\{ \overline{f} \in R'/I \;\middle|\; f \in (I \underset{R'}{:} \{a_1, \ldots, a_d\}) \right\} \\
&= \frac{(I \underset{R'}{:} \{a_1, \ldots, a_d\})}{I} \\
&= \frac{(I \underset{R'}{:} \mathfrak{X})}{I}
\end{aligned}$$

(iii). This follows from Proposition 6.2.3 in [4]. Given a monomial ideal I, the proposition states $C_{R'}(I) = [\![(I : \mathfrak{X})]\!] \setminus [\![I]\!]$. It follows that we have an isomorphism of k-vector spaces that justifies the third equality:

$$\frac{(I : \mathfrak{X})}{I} \cong (C_{R'}(I))k.$$

Note that each corner element is monic, so $C_{R'}(I)$ is linearly independent over k.

(iv). Since m-rad$(I) = \mathfrak{X}$, Theorem 6.1.5 in [4] implies I has an irredundant parametric decomposition (see Definitions 6.1.1 and 6.1.4 in the same text). Moreover, in the proof of that theorem, we see an irredundant, m-irreducible decomposition of I is that parametric decomposition. From Corollary 6.2.5 also in [4], the ideals in the decomposition are in 1-1 correspondence with the corner elements of I.

(v). Recall that $V' \subset V$ is *independent* if for every pair $u, v \in V'$, uv is not an edge of G, so the maximal cliques of G^C are in bijection with the maximal independent subsets in G. Therefore it suffices to show the corner elements of I in R' are in bijection with the maximal independent subsets in G. That is, we aim to show

$$C_{R'}(I) = \{\mathbf{a} := a_{i_1} \cdots a_{i_t} \mid \{a_{i_1}, \dots, a_{i_t}\} \subset V \text{ is a maximal independent subset}\}.$$

First, let us assume $\{a_{i_1}, \dots, a_{i_t}\}$ is a maximal independent subset of $V = \{a_1, \dots, a_d\}$. We assume the list $a_{i_1}, \dots, a_{i_t}$ contains no repetitions, so immediately $\mathbf{a}$ is squarefree and cannot be in $\langle a_1^2, \dots, a_d^2 \rangle$. Since $\{a_{i_1}, \dots, a_{i_t}\}$ is independent, the product $\mathbf{a}$ does not lie in the edge ideal $I(G)$ either. Since sums of finitely generated ideals are generated by the union of their generating sets, we have established that $\mathbf{a}$ is not in I, by Fact 8.2.4.

The maximal independence guarantees that for any $l \in [d] \setminus \{i_1, \dots, i_t\}$, we must have $\mathbf{a} \cdot a_l \in I(G)$. Note also that $\mathbf{a} \cdot a_l \in \langle a_1^2, \dots, a_d^2 \rangle$ for any $l \in \{i_1, \dots, i_t\}$. Hence for any $l \in [d]$ we must have $\mathbf{a} \cdot a_l \in I$, so $\mathbf{a} \in C_{R'}(I)$.

Conversely, let us assume $\mathbf{a} \in C_{R'}(I)$. It follows that $\mathbf{a} \notin \langle a_1^2, \dots, a_d^2 \rangle$, so $a_{i_j} \neq a_{i_k}$ for each $j \neq k$ and hence $a_{i_1}, \dots, a_{i_t}$ contains no repetitions. Moreover, $\mathbf{a} \notin I(G)$ implies $a_{i_j} a_{i_k} \notin I(G)$ for any $j, k \in [t]$, so $a_{i_1}, \dots, a_{i_t}$ is independent in G. Now, since $\mathbf{a}$ is a corner element, for any $l \in [d]$, we know $a_l \cdot \mathbf{a} \in I$, so either $a_l \cdot \mathbf{a} \in I(G)$ or $a_l \cdot \mathbf{a} \in \langle a_1^2, \dots, a_d^2 \rangle$. The former case happens precisely when $l \in [d] \setminus \{i_1, \dots, i_t\}$, because we do not permit loops in our simple graphs. In other words, for any $l \notin \{i_1, \dots, i_t\}$, there must exist $a_{i_j} \in \{a_{i_1}, \dots, a_{i_t}\}$ such that $a_{i_j} a_l$ is an edge of G. This is precisely what it means for $\{a_{i_1}, \dots, a_{i_t}\}$ to be maximally independent. $\square$

Example 8.3.14. Consider a simple graph G and its complement.

Let $R = \mathbb{K}[a, b, c, d]$ and let $I = I(G) + (a^2, b^2, c^2, d^2)R = (ab, bc, cd, ad, a^2, b^2, c^2, d^2)R$. Then $ac, bd \notin I$ give non-zero monomials in $(I : \mathfrak{X})/I$, since by our proposition corner elements of R/I are given by maximal cliques in G^C. Moreover, this implies the type of the $\mathcal{K}_1$-corona given in Example 8.3.3 is 2.

8.4 Application: Localization and Semidualizing Modules

In this section we introduce semidualizing modules and pose several questions about the set of such modules for a given ring. The main question of the section is 8.4.17, the answer to which will be the subject of Section 8.5.

Definition 8.4.1. Assume R is a noetherian ring. A finitely generated R-module C is *semidualizing* if $\operatorname{Ext}_R^i(C,C) = 0$ for all $i \geq 1$ and the homothety map

$$\chi_C^R : R \longrightarrow \operatorname{Hom}_R(C,C)$$

$$r \longmapsto \mu_r$$

is an isomorphism, where $\mu_r(c) = rc$ for any $c \in C$. Define also the set

$$\mathcal{S}_0(R) = \{\text{isomorphism classes of semidualizing } R\text{-modules}\}.$$

Example 8.4.2. The ring itself is always in $\mathcal{S}_0(R)$, as $\operatorname{Hom}_R(R,R) \cong R$ and $\operatorname{Ext}_R^i(R,R) = 0$ for all $i \geq 1$.

Open Problem 8.4.3. Given G, what are all the semidualizing modules over $K[G]$? That is, what are the elements of $\mathcal{S}_0(K[G])$? Even for specific classes of graphs like $\mathcal{K}_n$, $\mathcal{C}_n$ (cycles), or $\mathcal{P}_n$ (paths), this is an open problem.

Fact 8.4.4. *If R is local, then $|\mathcal{S}_0(R)| < \infty$. Unfortunately, the ring $\mathbb{K}[G]$ in which we are interested is not local in general. We can consider, however, the local ring $\mathbb{K}[G]_{\mathfrak{m}}$ where $\mathfrak{m}$ is the maximal ideal generated by the variables, or the ring $\mathbb{K}[\![G]\!]$, the formal power series version. (If A is a local noetherian commutative ring with identity, then*

$$A[\![X_1,\ldots,X_d]\!] = \left\{ \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_d=0}^{\infty} a_{i_1,i_2,\ldots,i_d} X_1^{i_1}X_2^{i_2}\cdots X_d^{i_d} \;\middle|\; a_{i_1,i_2,\ldots,i_d} \in A \right\}$$

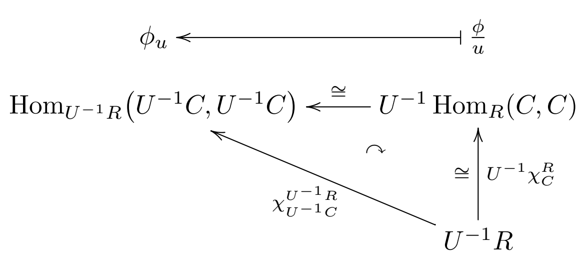
is a local noetherian commutative ring with identity as well.)

Open Problem 8.4.5. Assuming R is local, what is $|\mathcal{S}_0(R)|$? If we can't write down all the semidualizing modules, then maybe we can at least say how many there are. This seems vastly simpler to do than the problem stated in 8.4.3, but it can still be difficult to do. All current evidence suggests that $|\mathcal{S}_0(R)| = 2^n$ for some $n \in \mathbb{N}_0$.

Open Problem 8.4.6. What is the 'easiest' way to find a ring such that $|\mathcal{S}_0(R)| = 2^n$ for a given $n \in \mathbb{N}_0$?

The following fact says that semidualizing modules localize.

Fact 8.4.7. *Let $C \in \mathcal{S}_0(R)$ and let $U \subset R$ be a multiplicatively closed subset. Then $U^{-1}C \in \mathcal{S}_0(U^{-1}R)$. The following commutative diagram ensures the homothety map is an isomorphism.*



The horizontal map is an isomorphism by Proposition 3.1.8. The fact that R is noetherian and C is finitely generated ensures

$$\operatorname{Ext}_{U^{-1}R}^i(U^{-1}C, U^{-1}C) \cong U^{-1}\operatorname{Ext}_R^i(C,C) = 0$$

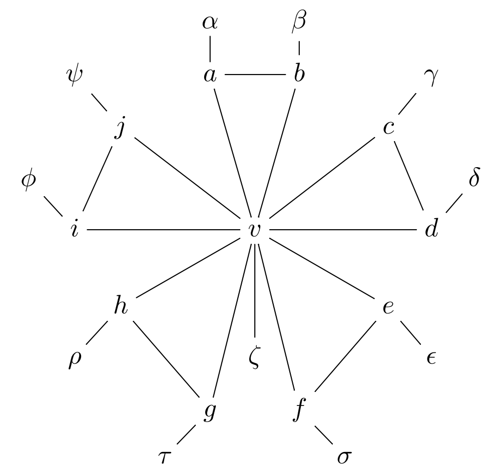
for all $i \geq 1$, which is argued similarly as in the proof of Proposition 3.1.8.

Remark 8.4.8. By the above fact the following map is well-defined.

$$\begin{aligned} \mathcal{S}_0(R) &\longrightarrow \mathcal{S}_0(U^{-1}R) \\ C &\longmapsto U^{-1}C \end{aligned} \tag{8.4.8.1}$$

It was previously guessed that if R is local, $\mathfrak{p} \in \operatorname{Spec}(R)$, and $U = R \setminus \mathfrak{p}$, then this map is onto, but that turned out to be false.

Let $n \in \mathbb{N}_{\geq 2}$. There is a ring we can build such that $|\mathcal{S}_0(R)| = 2$ and $|\mathcal{S}_0(R_\mathfrak{p})| = 2^n$, which we do by drawing a graph. For instance if $n = 5$, we may consider the $\mathcal{K}_1$-corona Γ (in 8.4.8.2). For the localized ring $R = \mathbb{K}[\Gamma]_\mathfrak{q}$, we have $|\mathcal{S}_0(R)| = 2$ where $\mathfrak{q}$ is generated by all the variables, while $|\mathcal{S}_0(R_\mathfrak{p})| = 2^5$ where $\mathfrak{p} = (a, \ldots, j, \alpha, \ldots, \psi, v)R$.



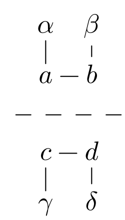
(8.4.8.2)

We give some indication in Example 8.4.11 and Fact 8.4.12 how the construction in the previous remark works, but first we give two more open problems related to the map (8.4.8.1).

Open Problem 8.4.9. If $R = K[G]$ and $U = R \setminus \mathfrak{X}$, then is the map (8.4.8.1) bijective?

Open Problem 8.4.10. If R is a Cohen-Macaulay, local, integral domain with $\mathfrak{p} \in \operatorname{Spec}(R)$ and $U = R \setminus \mathfrak{p}$, then is the map (8.4.8.1) bijective?

Example 8.4.11. Let $\mathbb{K}$ be a field and consider the following graph G.



In this case we have

$$\mathbb{K}[G] = \frac{\mathbb{K}[a,b,c,d,\alpha,\beta,\gamma,\delta]}{(ab, cd, a\alpha, b\beta, c\gamma, d\delta)} \cong \underbrace{\frac{\mathbb{K}[a,b,\alpha,\beta]}{(ab, a\alpha, b\beta)}}_{|\mathcal{S}_0|\geq 2} \otimes_{\mathbb{K}} \underbrace{\frac{\mathbb{K}[c,d,\gamma,\delta]}{(cd, c\gamma, d\delta)}}_{|\mathcal{S}_0|\geq 2} \tag{8.4.11.1}$$

where we have $|\mathcal{S}_0| \geq 2$ for the two rings in the tensor product by the following. Let R_1 and R_2 denote the two rings in the tensor product in (8.4.11.1). Since R_1 and R_2 are each a field adjoined to a $\mathcal{K}_1$-corona, they are both Cohen-Macaulay by Remark 8.3.6 and they each admit a dualizing module by Theorem 3.3.6 in [2]. Moreover by Proposition 8.3.13 the type of each of these rings is 2, so by Theorem 3.2.10 in [2] neither R_1 nor R_2 is Gorenstein (cf. Definition 3.1.18 in [2]). Therefore $D_1 \ncong R_1$ and $D_2 \ncong R_2$, where D_i is the dualizing module of R_i (Theorem 3.3.7 in [2]). Hence $|\mathcal{S}_0(R_i)| \geq 2$ for $i = 1, 2$.

We also have

$$\begin{aligned} \mathcal{S}_0(R) \times \mathcal{S}_0(S) &\hookrightarrow \mathcal{S}_0(R \underset{\mathbb{K}}{\otimes} S) \\ (C, D) &\longmapsto C \underset{\mathbb{K}}{\otimes} D. \end{aligned}$$

Therefore $\left|\mathcal{S}_0\left(R \underset{\mathbb{K}}{\otimes} S\right)\right| \geq |\mathcal{S}_0(R)| \cdot |\mathcal{S}_0(S)| \geq 4$. The point here is that the two disjoint pieces in G imply the ring it determines is a tensor product of two rings for which the order of $\mathcal{S}_0$ is known. (We are omitting a rather substantial argument that establishes the well-definedness and injectivity of the proposed map; this follows from [1].)

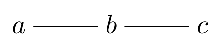
Fact 8.4.12. *If G has a star vertex (i.e., there exists some v adjacent to u for all $u \neq v$), then*

$$|\mathcal{S}_0\left(\mathbb{K}\left[\Sigma G\right]_{\mathfrak{X}}\right)| \leq 2.$$

See Corollary 4.6 in [5].

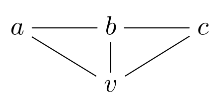
Example 8.4.13. Let the following serve as an outline of a localization process.

H :



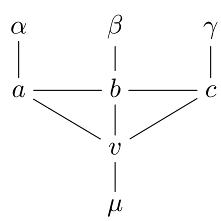
begin with a simple graph

G :



add a star vertex

ΣG :



add whiskers to form a $\mathcal{K}_1$-corona

Let $\Gamma = \Sigma G$, let $\mathbb{K}$ be a field, and consider the following ring and ideal.

$$\mathbb{K}[\Gamma] = \frac{\mathbb{K}[a, \alpha, b, \beta, c, \gamma, v, \mu]}{(ab, bc, av, bv, cv, a\alpha, b\beta, c\gamma, v\mu)} \qquad\qquad P = (a, \alpha, b, \beta, c, \gamma, v)$$

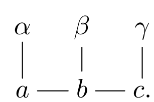
Since $\mu \notin P$, it is a unit in $\mathbb{K}[\Gamma]_P$, in which we allow everything in $\mathbb{K}[\Gamma] \setminus P$ to be a denominator.

First we will invert μ, and then everything else not in P.

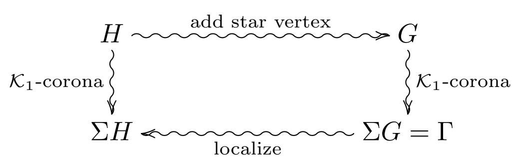
$$\begin{aligned}
\mathbb{K}[\Gamma]_\mu &= \frac{\mathbb{K}[a,\alpha,b,\beta,c,\gamma,v,\mu]_\mu}{(ab,bc,av,bv,cv,a\alpha,b\beta,c\gamma,v\mu)} \\
&= \frac{\mathbb{K}[a,\alpha,b,\beta,c,\gamma,v,\mu]_\mu}{(ab,bc,av,bv,cv,a\alpha,b\beta,c\gamma,v)} && \because \mu \text{ a unit} \\
&= \frac{\mathbb{K}[a,\alpha,b,\beta,c,\gamma,v,\mu]_\mu}{(ab,bc,a\alpha,b\beta,c\gamma,v)} && \text{drop redundant generators} \\
&\cong \frac{\mathbb{K}[a,\alpha,b,\beta,c,\gamma,\mu]_\mu}{(ab,bc,a\alpha,b\beta,c\gamma)} && \text{drop the dead variable} \qquad (8.4.13.1)
\end{aligned}$$

$$\begin{aligned}
\mathbb{K}[\Gamma]_P &\cong \frac{\mathbb{K}(\mu)[a,\alpha,b,\beta,c,\gamma]_{(a,\alpha,b,\beta,c,\gamma)}}{(ab,bc,a\alpha,b\beta,c\gamma)} \\
&\cong \mathbb{K}(\mu)[\Sigma H]_{(a,\alpha,b,\beta,c,\gamma)}
\end{aligned}$$

Line (8.4.13.1) corresponds very closely to the $\mathcal{K}_1$-corona of H:



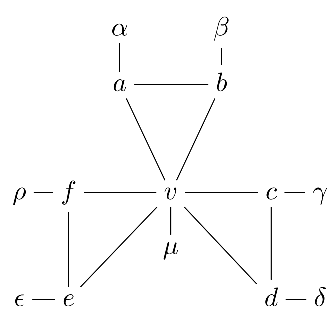
Hence the localization process can be depicted by the following diagram.



The following example and subsequent remark are leading us to the primary question in the section, Question 8.4.17. For more on the interest in this question, see Theorem B and Remark 4.2 in [6].

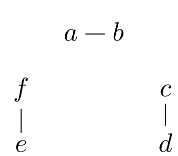
Example 8.4.14. Consider the graph Γ below obtained by suspending a graph G with a star vertex v.

Γ :



We can apply the process presented in Example 8.4.13 to compute $|\mathcal{S}_0(\mathbb{K}[\Gamma]_P)|$ via ΣH, where P is generated by the variables in $\mathbb{K}[\Gamma]$ other than μ. We acquire H by removing the star vertex v and all the whiskers.

H :

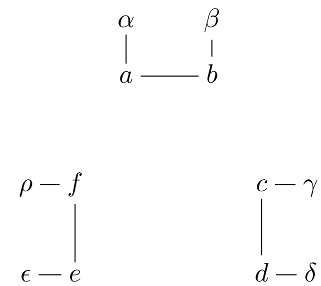


Hence the graph corresponding to the localization of $\mathbb{K}[\Gamma]$ at the prime ideal

$$P = (a, \alpha, b, \beta, c, \gamma, d, \delta, e, \epsilon, f, \rho, v)$$

is the following.

ΣH :

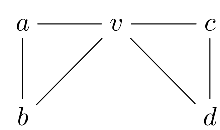


For the above graph, because there are three pairwise disjoint, non-trivial subgraphs as in Example 8.4.11, we have

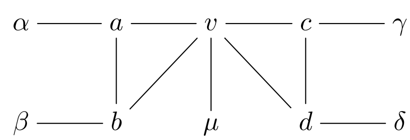
$$|\mathcal{S}_0(\mathbb{K}[\Gamma]_P)| = \left|\mathcal{S}_0(\tilde{\mathbb{K}}[\Sigma H]_{\tilde{P}})\right| = 2^3$$

where $\tilde{\mathbb{K}} = \mathbb{K}(\mu)$ and $\tilde{P} = (a, \alpha, \ldots, f, \rho)\tilde{\mathbb{K}}[\Sigma H]$.

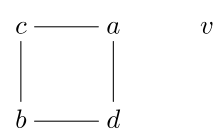
Example 8.4.15. Consider the following graph G.



We then make a $\mathcal{K}_1$-corona ΣG, displayed below.



Note that $|\mathcal{S}_0(\mathbb{K}[\Sigma G]_{\mathfrak{X}})| = 2$ by Corollary 4.6 in [5], where $\mathfrak{X}$ is generated by all the variables. Now consider G^C below.

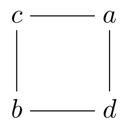


We summarize the maximal cliques in G^C in the following table, where the type is given in the third row as the total number of the maximal cliques (Proposition 8.3.13).

$\mathcal{K}_1$	1
$\mathcal{K}_2$	4
r	5

Now we localize $\mathbb{K}[\Sigma G]_{\mathfrak{X}}$ at the prime ideal $P = (a, \alpha, b, \beta, c, \gamma, d, \delta, v)_{\mathfrak{X}}$ to get $(\mathbb{K}[\Sigma G]_{\mathfrak{X}})_P \cong \tilde{\mathbb{K}}[\Sigma H]_{\tilde{P}}$ where $\tilde{\mathbb{K}} = \mathbb{K}(\mu)$, $\tilde{P} = (a, \alpha, b, \beta, c, \gamma, d, \delta)$, and H is below.

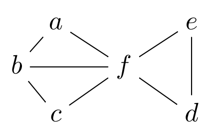
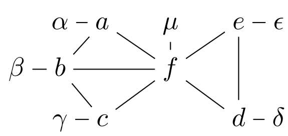
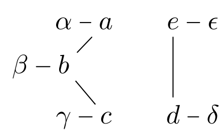
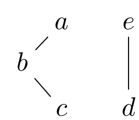
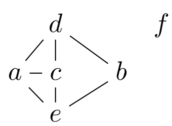
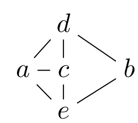
Next we compute H^C



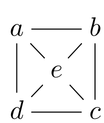
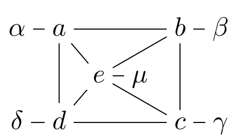
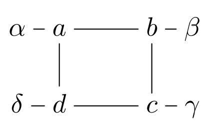
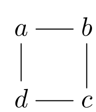
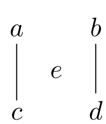
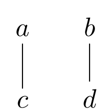
and the numbers of maximal cliques of each size in H^C

1	0
2	4
r	4

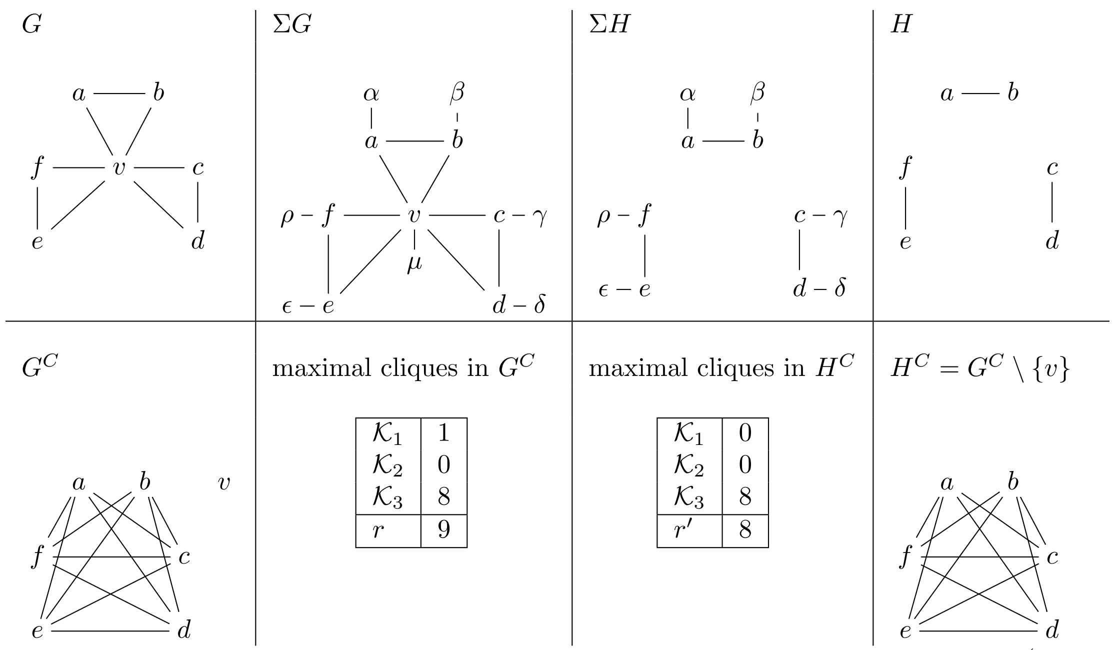
to get that the type of $(\mathbb{K}[\Sigma G]_{\mathfrak{x}})_P$ is 4. As in Example 8.4.14, we have $|\mathcal{S}_0(\mathbb{K}[\Sigma G]_P)| = 4$. We can run the same process on any simple graph G, as we demonstrate in tables (8.4.15.1), (8.4.15.2), and (8.4.15.3).

<table>
<tr><td>G
</td><td>ΣG
</td><td>ΣH
</td><td>H
</td></tr>
<tr><td>G^C
</td><td>maximal cliques in G^C
<table><tr><td>$\mathcal{K}_1$</td><td>1</td></tr><tr><td>$\mathcal{K}_2$</td><td>2</td></tr><tr><td>$\mathcal{K}_3$</td><td>2</td></tr><tr><td>r</td><td>5</td></tr></table></td><td>maximal cliques in H^C
<table><tr><td>$\mathcal{K}_1$</td><td>0</td></tr><tr><td>$\mathcal{K}_2$</td><td>2</td></tr><tr><td>$\mathcal{K}_3$</td><td>2</td></tr><tr><td>r'</td><td>4</td></tr></table></td><td>$H^C = G^C \setminus \{f\}$
</td></tr>
</table>

(8.4.15.1)

<table>
<tr><td>G
</td><td>ΣG
</td><td>ΣH
</td><td>H
</td></tr>
<tr><td>G^C
</td><td>maximal cliques in G^C
<table><tr><td>$\mathcal{K}_1$</td><td>1</td></tr><tr><td>$\mathcal{K}_2$</td><td>2</td></tr><tr><td>r</td><td>3</td></tr></table></td><td>maximal cliques in H^C
<table><tr><td>$\mathcal{K}_1$</td><td>0</td></tr><tr><td>$\mathcal{K}_2$</td><td>2</td></tr><tr><td>r'</td><td>2</td></tr></table></td><td>$H^C = G^C \setminus \{e\}$
</td></tr>
</table>

(8.4.15.2)



$\mathcal{K}_1$	1
$\mathcal{K}_2$	0
$\mathcal{K}_3$	8
r	9

$\mathcal{K}_1$	0
$\mathcal{K}_2$	0
$\mathcal{K}_3$	8
r'	8

(8.4.15.3)

In each of the above examples, we have $|\mathcal{S}_0(\mathbb{K}[\Sigma G]_{\mathfrak{X}})| = 2$ by Corollary 4.6 in [5]. It can be shown that in (8.4.15.1) localizing at $P = (a, \alpha, b, \beta, c, \gamma, d, \delta, e, \epsilon, f)_{\mathfrak{X}}$ gives $|\mathcal{S}_0((\mathbb{K}[\Sigma G]_{\mathfrak{X}})_P)| \geq 4$ as in Example 8.4.11. In (8.4.15.2), because the localized ring has type 2 (which is prime), by Theorem C(b) in [7] we have $|\mathcal{S}_0((\mathbb{K}[\Sigma G]_{\mathfrak{X}})_P)| = 2$ where $P = (a, \alpha, b, \beta, c, \gamma, d, \delta, e)_{\mathfrak{X}}$. By Example 8.4.14, we have $|\mathcal{S}_0((\mathbb{K}[\Sigma G]_{\mathfrak{X}})_P)| = 8$ in (8.4.15.3), where $P = (a, \alpha, b, \beta, c, \gamma, d, \delta, e, \epsilon, f, \rho, v)_{\mathfrak{X}}$.

Remark 8.4.16. In principle, one might be able to do better than the first graph in the preceding example. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. It is a fact that if $|\mathcal{S}(R_{\mathfrak{p}})| > 2$, then $r(R_{\mathfrak{p}}) \geq 4$ by Theorem C(b) in [7]. It follows from Corollary 3.3.12 in [2] that $r(R) \geq r(R_{\mathfrak{p}}) \geq 4$. Thus, the question of interest is the following.

Question 8.4.17. Does there exist a finite simple graph G and a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ where $R = \mathbb{K}[\Sigma G]$ such that $|\mathcal{S}_0(R)| = 2$, $|\mathcal{S}_0(R_{\mathfrak{p}})| > 2$, and $r(R) = r(R_{\mathfrak{p}}) = 4$?

8.5 Type and Localization of Edge Ideals

The point of the rest of the document is to show that the answer to Question 8.4.17 is "no" when G is obtained from H by adding a star vertex v and P is a prime ideal generated by variables other than μ, where μ is the endpoint of the whisker adjacent to v. The reason it works is essentially because of the interpretation of localization given in Example 8.4.15.

Lemma 8.5.1. *Let $H = \{V, E\}$ be a simple graph and let $H' = \{V', E'\}$ be such that $V' = V \cup \{v^*\}$ for some $v^* \notin V$ and $E' \supset E$ where every element of $E' \setminus E$ is of the form v^*v for some $v \in V$. If r and r' denote the number of maximal cliques in H and H', respectively, then $r' \geq r$.*

Proof. Define the map $\tau : \{\text{maximal cliques of } H\} \longrightarrow \{\text{maximal cliques of } H'\}$ as follows.

$$\tau([U]) = \begin{cases} [U, v^*] & uv^* \in E' \text{ for all } u \in U \\ [U] & \text{else} \end{cases}$$

It suffices then to show τ is a well-defined injection. If $[U]$ is a maximal clique of H, then it is a complete subgraph such that $[U, v]$ is not complete for any $v \in V \setminus U$. We claim $\tau([U])$ is a maximal complete subgraph of H'.

If $uv^* \in E'$ for all $u \in U$, then $\tau([U]) = [U, v^*]$ is a complete subgraph of H'. Moreover, it is maximal since to suppose otherwise contradicts the maximality of $[U]$. That is, if we suppose there exists some $v' \in V' \setminus (U \cup \{v^*\})$ such that $[U, v^*, v']$ is a complete subgraph of H', then $[U, v']$ is a complete subgraph of H, a contradiction.

If there exists some $u \in U$ such that $uv^* \notin E'$, then $\tau([U]) = [U]$ is clearly still a complete subgraph. A contradiction identical to that found in the previous case proves $\tau([U])$ is also maximal. Therefore τ lands well.

If $[U], [W]$ are both maximal cliques of H (not necessarily distinct), then we have

$$\begin{aligned}\tau([U]) = \tau([W]) &\iff [U] = [W] \text{ or } [U, v^*] = [W, v^*]\\ &\iff [U] = [W].\end{aligned}$$

We are justified in our neglect of the possibility that $[U, v^*] = \tau([U]) = \tau([W]) = [W]$ or $[W, v^*] = \tau([W]) = \tau([U]) = [U]$ since either one implies $v^* \in V$, a contradiction. □

Corollary 8.5.2. *Let $G = \{V, E\}$ be a simple graph and let $v_0 \in V$. Define $V' = V \setminus \{v_0\}$ and $G' = [V']$. If r and r' denote the number of maximal cliques in G^C and G'^C, respectively, then $r' \leq r$.*

Proof. This is by Lemma 8.5.1 with $H' = G^C$ and $H = (G')^C$. □

Theorem 8.5.3. *Let H be a finite simple graph and let G be obtained from H by adding a star vertex v. Consider the $\mathcal{K}_1$-corona ΣG and let μ be the endpoint of the whisker adjacent to v in ΣG. Let $P \in \operatorname{Spec}(\mathbb{K}[\Sigma G])$ be generated by all the variables of $\mathbb{K}[\Sigma G]$ except for μ and let $\mathfrak{p} \in \operatorname{Spec}(\mathbb{K}[\Sigma G])$ be such that $\mathfrak{p} \subseteq P$. Then* $\operatorname{type}(\mathbb{K}[\Sigma G]_{\mathfrak{p}}) < \operatorname{type}(\mathbb{K}[\Sigma G])$ *and* $\operatorname{type}(\mathbb{K}[\Sigma G]_P) = \operatorname{type}(\mathbb{K}[\Sigma G]) - 1$.

Proof. We first show that $\operatorname{type}(K[\Sigma G]_P) < \operatorname{type}(K[\Sigma G])$. The effect of localization on G described in the preceding examples show that it suffices to show that the number of maximal cliques r in G^C is more than the number of maximal cliques r' in $(G')^C$ where $G' = [V \setminus \{v\}]$. Corollary 8.5.2 implies that $r' \leq r$. (This inequality has nothing to do with the specific choice of v.) By construction, the vertex v is isolated in G^C, so it forms a maximal clique in G^C. It is straightforward to show that this clique is not in the image of the injective map

$$\tau\colon \left\{\text{maximal cliques in } (G')^C\right\} \longrightarrow \left\{\text{maximal cliques in } G^C\right\}.$$

Thus, τ is not surjective, so $r' < r$.

To complete the proof of this part, the last step in the following display is from the preceding paragraph, and the middle step is from Corollary 3.3.12 in [2]:

$$\operatorname{type}(\mathbb{K}[\Sigma G]_{\mathfrak{p}}) = \operatorname{type}(((\mathbb{K}[\Sigma G])_P)_{\mathfrak{p}_P}) \leq \operatorname{type}(\mathbb{K}[\Sigma G]_P) < \operatorname{type}(\mathbb{K}[\Sigma G]).$$

In the special case when $\mathfrak{p} = P$, the second part follows from the preceding section. In particular, under the localization process described, the maximal clique formed by the star vertex is deleted from the graph, so the type decreases by exactly one. □

8.6 Further Study

Discussion 8.6.1. Here we present another open question that we would like to study in the future. If $G = \{V, E\}$ is a simple graph with Roman vertices and $\Gamma = \Sigma G$ is a $\mathcal{K}_1$-corona of G by Greek vertices, then for $a \in V$ we have the short exact sequence

$$0 \longrightarrow \langle a \rangle \longrightarrow \mathbb{K}[\Gamma] \longrightarrow \mathbb{K}[\Gamma]/\langle a \rangle \longrightarrow 0$$

where $\mathbb{K}$ is a field. Denote the ideal $J = \underset{\mathbb{K}[\Gamma]}{(0\colon a)}$ and the ring $S = \mathbb{K}[a, \ldots, z, \alpha, \ldots, \omega]$. Since $\mathbb{K}[\Gamma]/J \cong \langle a \rangle$ by the First Isomorphism Theorem, we have the equivalent short exact sequence

$$0 \longrightarrow \mathbb{K}[\Gamma]/J \longrightarrow \mathbb{K}[\Gamma] \longrightarrow \mathbb{K}[\Gamma]/\langle a \rangle \longrightarrow 0.$$

Moreover since $J \leq \mathbb{K}[\Gamma] = S/I(\Gamma)$ we know J is of the form $\mathfrak{a}/I(\Gamma)$ for some $\mathfrak{a} \leq S$. Note also that

$$J = \frac{\left(I(\Gamma) \underset{S}{:} a\right)}{I(\Gamma)}$$

and $(I(\Gamma) \underset{S}{:} a)$ is a monomial ideal of S containing $I(\Gamma)$. We raise a question: Is there a simple graph $\tilde{G}$ with $\mathcal{K}_1$-corona $\tilde{\Gamma}$ for which $\left(I(\Gamma) \underset{S}{:} a\right) = I\left(\tilde{\Gamma}\right)$? That is, is there a simple graph $\tilde{G}$ with $\mathcal{K}_1$-corona $\tilde{\Gamma}$ for which $\mathbb{K}[\tilde{\Gamma}] = \mathbb{K}[\Gamma]/J$?

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