5-2018

Robust Solutions to Uncertain Multiobjective Programs

Garrett M. Dranichak
Clemson University, gdranic@g.clemson.edu

Robust Solutions to Uncertain Multiobjective Programs

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical Sciences

by
Garrett M. Dranichak
May 2018

Accepted by:
Dr. Margaret Wiecek, Committee Chair
Dr. Warren Adams
Dr. Herve Kerivin
Dr. Matthew Saltzman
Dr. Cole Smith

Abstract

Decision making in the presence of uncertainty and multiple conflicting objectives is a real-life issue, especially in the fields of engineering, public policy making, business management, and many others. The conflicting goals may originate from the variety of ways to assess a system's performance such as cost, safety, and affordability, while uncertainty may result from inaccurate or unknown data, limited knowledge, or future changes in the environment. To address optimization problems that incorporate these two aspects, we focus on the integration of robust and multiobjective optimization.

Although the uncertainty may present itself in many different ways due to a diversity of sources, we address the situation of objective-wise uncertainty only in the coefficients of the objective functions, which is drawn from a finite set of scenarios. Among the numerous concepts of robust solutions that have been proposed and developed, we concentrate on a strict concept referred to as highly robust efficiency in which a feasible solution is highly robust efficient provided that it is efficient with respect to every realization of the uncertain data. The main focus of our study is uncertain multiobjective linear programs (UMOLPs), however, nonlinear problems are discussed as well.

In the course of our study, we develop properties of the highly robust efficient set, provide its characterization using the cone of improving directions associated with the UMOLP, derive several bound sets on the highly robust efficient set, and present a robust counterpart for a class of UMOLPs. As various results rely on the polar and strict polar of the cone of improving directions, as well as the acuteness of this cone, we derive properties and closed-form representations of the (strict) polar and also propose methods to verify the property of acuteness. Moreover, we undertake the computation of highly robust efficient solutions. We provide methods for checking whether or not the highly robust efficient set is empty, computing highly robust efficient points, and determining whether a given solution of interest is highly robust efficient. An application in the area of bank management is included.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

[The contents of this dissertation include material from the 2016 paper published by *The Sheridan Press* titled “Robust multiobjective optimization for decision making under uncertainty and conflict” by M.M. Wiecek and G.M. Dranichak, the 2017 technical report titled “On highly robust efficient solutions to uncertain multiobjective linear programs” by G.M. Dranichak and M.M. Wiecek, and the 2018 technical report titled “On computing highly robust efficient solutions” by G.M. Dranichak and M.M. Wiecek. Both technical reports have been submitted for publication. This dissertation includes additional material not included in the above papers. As the words in the above papers are our own, we do not provide direct quotations.]

Decisions are a part of everyday life. Some decisions are ordinary like, “What am I going to wear to work today?” Others are more significant like, “What job should I pursue?” In all decision making, difficulty arises from a multitude of options and their relative importance, the objectives of the decision, and the constraints surrounding the decision. Balancing all of these aspects, mathematical programming or optimization acts as an aid in the decision-making process. Depending on the problem, different techniques or approaches may be preferred, e.g., linear programming in which the model only contains linear functions, multiobjective programming that exploits multiple goals, and robust optimization that handles uncertainty.

Within optimization, consideration of a single objective function for certain problems is not always adequate. Practical problems in the fields of engineering, business, and management, as well as many others, often involve conflicting goals present during the decision-making process such as cost, performance, reliability, safety, productivity, and affordability (e.g., Rangaiah [120], Barba [2], Mostashari [112], Xidonas et al. [143], and Figueira et al. [61]). For example, consider a water dam construction problem where we want to simultaneously minimize the cost and maximize the storage capacity of the resulting reservoir. One approach to this problem is using multiple objective functions (or goals), namely two in this case. Similarly to many practical problems, the construction cost and other goals are conflicting in the sense that a gain in one is at the expense of another.

Independently of conflict, real-world problems, such as those in the fields of portfolio management (e.g., Lobo and Boyd [103] and Tütüncü and Koenig [135]), supply chain management (e.g., Bertsimas and Thiele [14]), structural design (e.g., Ben-Tal and Nemirovski [6]), circuit design (e.g., Boyd et al. [21]), and power control (e.g., Hsiung et al. [72]), may involve some uncertainty and require robust solutions, that is, solutions that are "best" for all realizations of the uncertain data. As an example, again consider the water dam construction problem. One possible uncertainty that arises in the problem is the variable conditions at the different possible locations of the dam. Different locations of the dam on the river may have, e.g., dramatically different weather conditions and therefore cause the coefficients in the objective functions to be uncertain. If uncertainty exists, we intend to find solutions that are "best" for all possible uncertainties. That is, in the context of our example, we want solutions that remain "best" regardless of the location of the dam.

As another example of a problem that includes both multiple criteria and uncertainty, consider the production problem of manufacturing golf balls. Possible goals include minimizing production cost and minimizing production time. It is clear that with the exception of technological advances, the production cost and the production time are conflicting since decreases in production time require increases in production cost, so a multiobjective approach is necessary. Moreover, uncertainty can arise in this production problem in many ways. One such possibility is the cost of the production materials. Today, most golf balls are produced with a variety of polyurethanes. In order to have the most player-friendly golf balls, companies must constantly alter the materials used to produce the golf balls, and as a result, there is great uncertainty in the cost of the materials used. Hence, a robust approach is also necessary.

Other problems involving uncertainty and multiple criteria are numerous. For example, authors have explored problems ranging from portfolio management to transportation planning to crop irrigation strategy. Fliege and Werner [52], for instance, consider the well-known Markowitz portfolio (i.e., financial asset portfolio) optimization problem in which conflicting objectives such as revenue and risk are optimized under uncertain future expected returns. Similarly, Kuhn et al. [95] examine two types of transportation problems: aircraft route guidance and hazardous materials transportation. In the first, the goals to be optimized are travel distance and risk posed by weather under the unpredictability of the weather for a given route; in the second, travel time, distance, and fuel cost must be minimized under unknown travel times for a given plan. Moreover, Crespo et al. [28] inspect crop irrigation strategy in which objectives such as revenue, resource usage, and sustainability are optimized subject to the variability of the weather.

More specifically, practical applications often suffer from uncertainty in the coefficients of the objective functions for any of several reasons including that they are subjectively specified by a decision maker(s) or that they are estimated (possibly by linear regression). One well-known example where multiple criteria are needed and imprecision of the objective function coefficients is known to exist is public-sector decision problems due to the complexity of the issues under consideration and the difficulty of estimating social costs and benefits (see Bitran [18], Candea et al. [23], and Cohon and Marks [27]). Another well-studied example where multiple criteria are used and uncertainty in the objective coefficients is present is the aircraft route guidance problem mentioned above (see Kuhn et al. [95]).

The remainder of this chapter is organized as follows. In Section 1.1, we review the current literature on robust optimization (both single-objective and multiobjective). In particular, we provide a short overview of methodologies for treating uncertainty in single-objective problems (SOPs) in Section 1.1.1, focusing on the robust optimization approach of Ben-Tal and Nemirovski [7]. Next, in Section 1.1.2, we thoroughly review the emerging field of robust multiobjective optimization, which has developed to treat uncertainty in multiobjective programs (MOPs). Within this section, we review the sources of uncertainty that are reflected in uncertain MOPs (UMOPs) in Section 1.1.2.1, the different models that have been considered and their associated solution concepts in Section 1.1.2.2, solution methods to solve these problems in Section 1.1.2.3, and applications that have benefited from being modeled as UMOPs in Section 1.1.2.4. Then in Section 1.1.2.5, we consider two particular models found within robust multiobjective optimization and concentrate on one solution concept, highly robust efficiency, that is addressed in the literature and is the focus of this dissertation. Finally, we discuss the research goals of our work to satisfy aspects of highly robust efficient solutions that have not been addressed yet in the literature in Section 1.2, and give an overview of the dissertation in Section 1.3.

1.1 Literature Review

In order to address the difficulty of solving problems involving uncertainty, the field of *robust optimization* developed in the late 1990s (e.g., Ben-Tal and Nemirovski [7]). Initially, researchers focused on SOPs establishing a varied field of theory and solution methods (refer to Ben-Tal et al. [5]). More recently, however, a separate field concentrating on MOPs has emerged.

1.1.1 Robust Single-Objective Optimization

As mentioned, many real-world optimization and decision-making problems involve uncertainty. The uncertainty can result from a crude or limited knowledge of the data at the time the decision is being made (as is the case when data is only estimated), or from the data being completely unknown due to possible changes in the future, or any number of other possibilities. To address this challenging issue, several methodologies generally employing any of three classical mathematical modeling perspectives, probabilistic, possibilistic, and deterministic, have been developed (see Liu [102]). The probabilistic type relies on distributions to evaluate the event probability (e.g., Schneider and Kirkpatrick [123]); the possibilistic type uses fuzzy sets and membership functions to assess the event plausibility (e.g., Lodwick and Kacprzyk [104]); and the deterministic type uses crisp sets to define domains within which uncertainties vary.

The latter perspective has been exploited by Ben-Tal and Nemirovski [7, 8, 9] who developed robust optimization for uncertain SOPs (USOPs) and initially focused on optimizing over worst-case realizations of uncertain data. In subsequent studies, e.g., Ben-Tal et al. [5] and Bertsimas et al. [13], other concepts of robust solutions have been developed and have led to a variety of robust optimization approaches. As a result, the way in which the concept of robust solutions or “worst case” is defined determines the specific robust optimization approach.

Practically, robust optimization may be preferred over possibilisitc or probabilistic approaches because both of these perspectives commonly allow constraints to be violated (see Ben-Tal and Nemirovski [7]). For example, in engineering contexts, the violation of constraints may be unacceptable. In addition, the probabilistic case requires knowledge of the distribution of the uncertain data, which is not guaranteed to be known or easily estimated. On the other hand, crisp sets can often times be provided by experienced decision makers, an advantage for robust optimization. As a result, the deterministic approach of robust optimization may be used rather than possibilistic or probabilistic approaches.

In the study of robust optimization, uncertainty can exist in both the objective and constraint function coefficients. However, it is not necessary to consider the variety of resulting combinations because it is possible to reformulate a USOP with uncertainty in any of the data, e.g., objective coefficients, constraint coefficients, or right-hand side (RHS) values. In particular, we may assume without loss of generality (WLOG) that the uncertainty is in the constraint coefficients, i.e., the left-hand side (LHS) values of the constraints (see Ben-Tal and Nemirovski [8]). If the uncertainty is in the objective function coefficients, an auxiliary variable may be used to move the objective function involving uncertainty down into the constraints. Similarly, if the uncertainty is in the RHS of the constraints, a variable that does not influence the objective function value (i.e., contributes a value of zero) may be used to move the unknown RHS values into the LHS by multiplying the new variable and the RHS and adding a constraint forcing the new variable to be 1. In either case, the uncertainty can easily be restricted to only the LHS of the constraints. Refer to Appendix A for complete derivations of the aforementioned transformations.

As with any optimization or decision-making problem, the main question of interest is how to identify "best" solutions. The classical approach of Ben-Tal and Nemirovski [7] is that of forming and solving a *robust counterpart* (RC). Since the uncertain problem itself has no well-defined solution concept, the RC is used. The RC is a deterministic problem, i.e., a problem whose data is determined, known, or certain, whose solutions are the solutions to the original uncertain problem. Given an *uncertainty set* or *set of scenarios* (of which the particular structure is not yet important), the RC is the problem that has all instances of the constraint functions as its set of constraints. Feasible solutions to the RC are considered *robust feasible solutions* to the original USOP, and optimal solutions to the RC are called *robust optimal solutions*, i.e., "best" solutions to the original USOP. The fact that a deterministic problem is used to compute the solutions to the USOP, along with the use of deterministic or crisp uncertainty sets, is why robust optimization is referred to as a deterministic approach.

Of important note is that the RC in this setting is in fact a semi-infinite program (refer to Goberna and Lopez [59]), i.e., under the reasonable assumption that the uncertainty set is infinite, the RC has an infinite number of constraints. One might expect then that the RC is always intractable, but this is not the case. For specific structures or geometries of the uncertainty set, the RC is explicitly known and computationally solvable. A specific example in which this is the case is ellipsoidal uncertainty (see Ben-Tal and Nemirovski [8]). For more in-depth studies of robust (single-objective) optimization, the interested reader may reference Ben-Tal et al. [5].

We also recognize the beginning efforts relating multiobjective optimization to USOPs in which the former is used to solve the latter. Steuer [127] is perhaps the first to do so by applying multiobjective linear programming to single-objective linear programs with interval objective function coefficients. However, this does not fall under robust optimization and is instead found in literature in the field of *interval programming*. Refer to Ishibuchi and Tanaka [86] and Chanas and Kuchta [25] for a similar line of investigation. The first efforts to specifically relate multiobjective and robust optimization were undertaken by Kouvelis and Yu [93] who use the former for the benefit of the latter. A general perspective is laid down by Hites et al. [68]. Assuming a finite number of realizations of uncertainty in the USOP objective function, the USOP is reformulated into a deterministic MOP, which is further explored by Köbis and Tammer [92], Klamroth et al. [89], Iancu and Trichakis [76], and Köbis [91]. A reverse effort is undertaken by Gorissen and Den Hertog [60] who make use of robust optimization as a tool to approximate the set of solutions to multiobjective linear programs (MOLPs).

1.1.2 Robust Multibjective Optimization

In addition to uncertainty, many real-world optimization and decision-making problems involve multiple conflicting criteria. As in the single-objective context, problems incorporating both uncertainty and multiple competing goals may be viewed from either probabilistic, possibilistic, or deterministic perspectives. Following the previous discussion on these perspectives, we focus on deterministic approaches, and in particular, robust multiobjective optimization. However, it is important here to note that robust optimization is not the only deterministic approach that has been applied to UMOPs. Specifically, Dellnitz and Witting [32] and Witting et al. [142] employ *parametric optimization* (refer to, e.g., Fiacco [51] and Domínguez et al. [34]) to analyze UMOPs with uncertain objective data. The advantage of taking such an approach is that a solution provides a mapping of the full range of optimal decision and function values before knowing the exact conditions represented by the (uncertain) parameters, and thus may provide a more full picture to the decision maker. However, the use of this approach in the literature is currently still limited, perhaps because parametric multiobjective optimization is not yet well developed (refer to Wiecek et al. [141]).

Robust multiobjective optimization emerged as an independent research field more than a decade ago starting with (deterministic) concepts of robust (efficient) solutions in engineering design (e.g., Deb and Gupta [31] and Li and Azarm [100]). In their work, robustness finds solutions with respect to a mean representation of the objective function values over its vicinity rather than the original objective functions or with respect to the original objective functions but only in a neighborhood determined by a user. Acceptable variation regions (AVR) for constraint and objective functions are proposed by Gunawan and Azarm [64] and Li et al. [101], and the solutions that remain feasible for each AVR are referred to as feasibly and objectively robust, respectively. In Besharati and Azarm [16], the concepts of absolute regret and dispersion are defined as measures of robustness. The engineering interest in developing robustness measures is only recently again undertaken by Wang et al. [137] who define the robustness of efficient solutions in terms of their performance with respect to problem specific indices that are different from the objective functions.

In the literature, different types of UMOPs are formulated based on various sources of uncertainty, which are discussed below. Additionally, uncertainty can be modeled with infinite or finite sets depending on the real-life context. In any case, the goal of robust multiobjective optimization is to solve the UMOP for *robust efficient* solutions, i.e., solutions that are feasible for every realization of uncertainty and that may be efficient for some or all realizations. Depending on the formulation resulting from modeling one or more of the sources of uncertainty, a variety of robustness concepts are defined and studied with respect to RCs, existence properties, solution methods, and applications.

1.1.2.1 Sources of Uncertainty

Uncertainty may be associated with MOPs in several different ways yielding a UMOP. A natural tactic is to perform a mathematical scrutiny of the assumed general formulation and ask what elements can be made uncertain. The answer is facilitated by the system view of Beyer and Sendhof [17] who integrate uncertainty into single-objective optimization for the purpose of enhancing the realism of modeling the process of system design. In this setting, the decision variables play the role of design variables that assume values constrained by a set of feasible designs. The objective function is optimized over the set of feasible designs so that the design variables are naturally the optimization variables. Four sources of uncertainty are discussed:

(i) *Endogenous* perturbations such as tolerances affecting manufacturing processes and systems. They are represented by parameters, which influence the design variables and indirectly the objective and/or constraint functions, and are not optimization variables.

(ii) *Feasibility* uncertainties affecting the set of feasible designs and the fulfillment of constraints the design variables must obey. They are modeled as parameters that along with the design variables directly affect the constraint functions but are not optimization variables.

(iii) *Exogenous* factors such as temperature, pressure, and material properties originating from the environment in which the system operates. They are modeled as parameters that along with the design variables directly affect the objective function but are not optimization variables.

(iv) *System output* uncertainties reflecting imprecision in evaluating system performance or errors due to the use of models instead of real physical objects. They are accounted for by one (or more) uncertain objective function(s).

Since multiobjective optimization is an extension of single-objective optimization, we directly adopt these four types of uncertainty into the former. However, since multiobjective optimization plays a broader role in decision making than single-objective optimization, we also propose two other sources of uncertainty that are characteristic for multiobjective settings.

(v) *Scalarization parameters* transforming an MOP into an SOP that are needed when single-objective optimization methods are used to solve an MOP (e.g., Wiecek et al. [141]). The challenge in the use of these methods results from the choice of the actual method, which may not be obvious, and once the method has been selected, also from the choice of the scalarizing parameters' values. In general, these values may be unknown and the decision maker faces a difficult situation of making a choice possibly under a great deal of uncertainty. For example, choosing weights as scalarizing parameters is discussed extensively from a psychological perspective by Eckenrode [41] and from an engineering point of view by Marler and Arora [108]. In any case, a scalarized MOP becomes a USOP and could benefit from being treated as such.

(vi) *Human preferences* that determine the solution concept for an MOP. Uncertainty in preferences may result from the different backgrounds and expectations of the decision makers representing the various parties engaged in the decision-making process (see Keeney and Raiffa [88] and Weber et al. [138]), as well as a decision maker's inability to articulate a preference. This type of uncertainty may be embedded in the convex cone representation used to model preferences. In the particular case of polyhedral convex cones, the entries of the matrices describing these cones can fulfill that role.

Depending on the source (or sources) of uncertainty that are taken into account, the formulation changes to reflect the specific situation.

1.1.2.2 Models and Solution Concepts

The models that have been developed in the literature to account for uncertainty and multiple conflicting criteria can be categorized as any of seven types depending on the sources of uncertainty that are being considered as previously described. The seven types of models, which we explore below along with their solution concepts, assume uncertainty in different aspects of the problem as follows: (i) uncertainty in only the constraint function coefficients, (ii) uncertainty in only the objective function coefficients, (iii) objective and constraint function coefficient uncertainty, (iv) uncertainty in the decision variables, (v) uncertain objective functions (where the criteria are treated as uncertain, not just the coefficients), (vi) uncertainty in scalarization parameters, and (vii) preference uncertainties.

First, UMOPs only accounting for feasibility uncertainties in the form of uncertain constraint coefficients are considered by Doolittle et al. [37], as well as Goberna et al. [57] who only consider linear problems. Such problems are treated by Doolittle et al. [37] in the same manner as USOPs are by Ben-Tal and Nemirovski [7], i.e., robustness is considered with respect to the worst-case realizations of the constraints, since they only differ from them by the vector-valued objective function. As a result, solutions to these problems are considered conservative, and methodologies for treating these problems rely upon the robust single-objective paradigm.

On the other hand, UMOPs modeling only exogenous uncertainties, which cause objective coefficients to be uncertain, are considered in numerous studies and do not follow the same line of research as in the single-objective setting. The reason for the different directions of investigation is the result of several factors. First, although in robust (single-objective) optimization uncertainty in the objective coefficients has not been considered as much due to Ben-Tal and Nemirovski (as mentioned in the previous section), the multiple objective functions present in UMOPs provide many opportunities for introducing new concepts of robustness beyond the scope of USOPs. In particular, solution concepts combining efficiency and robustness can be proposed. As MOPs have solution sets with many or infinitely many elements, the possibility exists for some efficient solutions to remain efficient and be robust so that their efficiency is not lost due to robustness. This is in contrast to SOPs, which typically have unique optimal solutions that are very unlikely to also be robust, and their optimality is sacrificed for robustness. Second, since efficiency and robustness may be combined in different ways, various concepts of robustness may be defined with attention to meaningful concepts in application. Even though UMOPs modeled in this way may be reformulated using auxiliary variables to move the uncertain objectives into the constraints resulting in a UMOP with only feasibility uncertainties, the aforementioned factors allow for researchers to provide information to decision makers that would be unavailable otherwise. In the following paragraphs, we review the solution concepts for this type of model that have been studied, highlighting three (flimsily, highly, and set-based min-max robust) in particular.

A permissive concept of robustness only requires efficiency with respect to *at least one* instance of the objective function over the feasible set. Such solutions, referred to as *flimsily robust efficient*, are defined by Ide and Schöbel [82] and Kuhn et al. [95]. This concept is first mentioned in 1980 by Bitran [18] in the context of *interval multiobjective programming* in which the uncertain objective coefficients fall within a closed interval that is assumed to be known. The solutions are referred to as *possibly efficient* solutions, a term that is borrowed from modal logic (see Inuiguchi and Kume [83]). Further studies of possibly efficient solutions can be found in Inuiguchi and Kume [84], Ida [77], Inuiguchi and Sakawa [85], and Oliveira and Antunes [116]. Due to their permissive nature, no RC is needed; in fact, any instance of the problem can serve as an RC (since its solutions are immediately robust).

On the other extreme, a restrictive concept of robustness requiring efficiency with respect to *every* instance of the objective function simultaneously over a common feasible set, referred to as *highly robust efficiency*, is provided by Ide and Schöbel [82] and Kuhn et al. [95]. A more detailed analysis of this solution concept is given in Section 1.1.2.5 because this concept is of special interest to our work.

As a compromise between these two extreme concepts, *set-based min-max* (*objective-wise worst-case*, *strict*) *robustness*, which incorporates the concept of *set domination* from set-valued optimization (see Eichfelder and Jahn [46]), is explored by Ehrgott et al. [45], Bokrantz and Fredriksson [19], and several others. Similar to the case of uncertainty only in the constraint coefficients, robustness is considered with respect to the worst-case instance of the objective function and is thus still conservative. It is also worth noting that set-based min-max robust solutions may be interior points, and the set of these solutions need not be connected, even in the linear case (see Majewski [106] and Ehrgott et al. [45]). These two properties are potential downsides as they suggest solving UMOPs for set-based min-max robust solutions is a global optimization task. Not only that, the properties are also in stark contrast to the deterministic linear case in which solutions (to nontrivial problems) are on the boundary of the feasible region and the solution set is connected, which lead directly to the effectiveness and applicability of multiobjective simplex methods.

Other concepts of robustness have also been proposed. For example, Kuhn et al. [94] propose **ε**-*(representative) lightly* robust solutions that are obtained as set-based min-max robust solutions in a neighborhood of a nominal scenario. Other set-dominance relations are also adopted from set-valued optimization in order to define several more concepts of set-based robustness as in Ide et al. [81], Ide and Köbis [80], and Wang et al. [136], but in the interest of brevity, we do not go into detail here. More recently, Sigler [125] has proposed ordering relations in order to define Pareto optimality under uncertainty directly as is done in deterministic multiobjective optimization. For a comprehensive survey of ten concepts of robust efficiency and their numerous relationships, refer to Ide and Schöbel [82].

Another modeling approach combines exogenous and feasibility uncertainties and is undertaken by Fliege and Werner [52], Kuroiwa and Lee [96], Wang et al. [137], and Goberna et al. [58] (who study uncertain MOLPs (UMOLPs) and highly robust efficient solutions). In Fliege and Werner [52], the authors first apply two scalarizations, the weighted-sum scalarization and the epsilon-costraint scalarization, to the deterministic MOP associated with a single scenario, and then they "robustify" the resulting SOPs. Second, they apply the same scalarizations to the RC, which assumes the worst-case constraints (as in Dooltittle et al. [37]) and the worst-case objectives (as in Ehrgott et al. [45]), that is, after the "robustification." Doing so, they examine whether the scalarization and the robustification are commutative operations.

UMOPs modeling endogenous uncertainty that require the decision variables to be uncertain are covered by Eichfelder et al. [47]. Motivation for this situation includes the design of a magnetic system in which the implementation of a decision is inexact, thereby leading to the uncertainty that must be accommodated. Robustness is defined in terms of set dominance in the objective space, and point-to-set maps to model the decision uncertainty are used. The work is a general extension of the single-objective approach of regularization robustness (see Lewis [98] and Lewis and

Pang [99]).

Doolittle et al. [36] address system output uncertainty that is reflected in the objective functions themselves being uncertain (not simply the coefficients). Motivation for such considerations include the design of a four-bar plane truss structure found in Engau and Wiecek [49] in which the weight of the truss and its displacement due to different loading conditions remain in conflict while the truss geometry that minimizes both is sought. Treating the displacement as an uncertain objective function would make the model more realistic. Other problems involving infinitely many objective functions that are encountered in control, game theory, and statistics, as mentioned by Engau [48], may also benefit since a finite number of uncertain functions could be used instead to account for the infinite criteria.

Uncertainty present in scalarization parameters involved in transforming MOPs into SOPs is important to examine due to the widespread use of scalarization methods to solve MOPs (see Wiecek et al. [141]). Studies on the uncertain weights in the weighted-sum method are reported by Palma and Nelson [117] and Hu and Mehrotra [73]. Other investigations regarding six scalarizations of MOLPs are provided in Doolittle et al. [35].

Finally, modeling human preferences using cones has been shown to be beneficial by Sawaragi et al. [122] in terms of gaining new mathematical insight, and by Hunt et al. [74], Klimova and Noghin [90], Noghin [115], and Wiecek [139] in the context of providing a tool for modeling decision makers' preferences. However, uncertainty in the preferences defining the cones may exist. The uncertainty may arise due to differences in human preferences and can be assumed into the cones that model these preferences. Under conditions of pointedness (of the cone) and rank (of cone defining matrices), a RC that takes the form of a UMOP modeling exogenous uncertainties is obtained. As a result, the robust concepts defined in that situation are directly applicable to the current discussion (see Wiecek and Dranichak [140]).

1.1.2.3 Solution Methods

Solution methods to solve UMOPs modeled in any of the seven ways described directly above largely do not exist in the literature. The ones that do either rely on brute-force attacks or directly follow from other research areas (e.g., deterministic multiobjective optimization or robust optimization). In other words, the solution methods currently available are not specially designed to solve UMOPs (or even UMOLPs). We review four solution methods that have been proposed to solve UMOPs with uncertainty in the constraint coefficients, in the objective coefficients, and in both. The methods either borrow from robust (single-objective) optimization, depend on scalarizations from deterministic multiobjective optimization, or rely on two-stage processes in which supersets of the desired solution set are first enumerated.

In the case of uncertainty in the constraint coefficients, a semi-infinite MOP, in which the set of constraints contains the constraints associated with each realization of uncertainty, is presented by Doolittle et al. [37] as the RC. As in the single-objective setting, this RC must be reformulated twice under several assumptions (including that Lagrangian duality holds) before being solved. The resulting MOP is easily solvable and generates the desired robust solutions.

When uncertainty is considered only in the objective function coefficients, the solution method depends upon the desired solution concept. If the concept is set-based min-max robustness, for example, then one approach is the following. First, an RC, which is a bilevel MOP that optimizes with respect to the worst-case instance of the objective function, is formulated. Second, the inner MOP is scalarized and solved for its efficient solutions. In Bokrantz and Fredriksson [19], necessary and sufficient conditions that rely on the existence of a scalarizing function for the inner problem are developed, but no methodology for obtaining these functions is given. Nevertheless, Ehrgott et al. [45] provide sufficient conditions for two common scalarizations, the weighted-sum and epsilon constraint methods, which can be used as methods to obtain solutions, but are clearly not guaranteed to find all solutions.

Alternatively, if the problem is the special case in which there are two objective functions with one deterministic and the other uncertain, then a method for obtaining highly and flimsily robust efficient solutions, as well as a method for $\boldsymbol{\varepsilon}$-(representative) lightly and (a second method for) set-based min-max robust solutions, is demonstrated by Kuhn et al. [95] provided that the uncertainty set is finite or can be considered as such due to its special structure. Refined subsets of highly and flimsily robust efficient solutions are computed by first solving the deterministic MOP associated with every scenario, while subsets of set-based min-max and $\boldsymbol{\varepsilon}$-(representative) lightly robust solutions are found by first obtaining the efficient set associated with the deterministic problem whose objectives are taken to be every instance of the objective functions. In either case, the second step is a filtering process that is applied to reduce the obtained sets to the desired robust solutions. Based on the algorithms, the authors also provide complexity results specific to each type of solution.

If the model takes into account uncertainty in both the objective and constraint coefficients, then a method that does not fully exercise robust optimization is given by Wang et al. [137]. In this context, a two-stage post-optimality approach is taken to obtain robust solutions. A deterministic MOP is solved under a nominal scenario yielding an efficient set, and then robust solutions are selected depending on their performance with respect to the chosen index (or measure of robustness).

1.1.2.4 Applications

Many of the above models, solution concepts, and solution methods have been exploited for the benefit of various applications including portfolio management (e.g., Fliege and Werner [52]), routing and transportation (e.g., Kuhn et al. [95]), the wood-cutting industry (e.g., Ide [79]) proton therapy for cancer treatment (e.g., Chen et al. [26], and Bokrantz and Fredriksson [19]), wind turbine design (Wang et al. [137]), and Internet routing (e.g., Doolittle et al. [37]). Other applications include forest management (e.g., Palma and Nelson [117]), irrigation strategy (e.g., Crespo et al. [28]), and multiobjective games (e.g., Yu and Liu [144]).

In particular, a model with only uncertain constraint coefficients is applied by Doolittle et al. [37] to an Internet routing problem that is modeled as an uncertain biobjective multicommodity flow problem on an Internet network. The uncertainty originates from an unknown amount of traffic for each commodity, which is caused by the cost of data collection and the complexity of data analysis, and is modeled by polyhedral sets. Robust efficient paths between all nodes in the network are computed along with their performance with respect to two conflicting criteria given by maximum and mean link utilizations.

A model with only uncertain objective coefficients is used by Ide [79] who considers an uncertain multiobjective wood-cutting industry problem. The uncertainty is the result of the unknown quality of the wood at the time cutting occurs and is modeled by finite sets. Set-based min-max robust cutting patterns with respect to quality are computed using the weighted-sum method proposed by Ehrgott et al. [45], and their performance is compared to existing manual cutting plans. Similarly, in the study of aircraft route guidance and hazardous materials transportation, Kuhn et al. [95] consider biobjective problems in which only one objective function involves uncertainty that results from the unpredictability of weather and travel times, respectively.

Utilizing a model that accounts for both constraint and objective coefficient uncertainty, Fliege and Werner [52] analyze an uncertain portfolio management problem in which revenue (to be maximized) and risk (to be minimized) are in conflict, and uncertainty in the objective and constraint function coefficients enters through estimates of expected returns and covariances. In contrast, Wang et al. [137] take an engineering approach to robust optimization in the examination of a multiobjective wind turbine design problem in which the uncertainty results from, for example, wind speed and temperature fluctuations over short periods of time. The authors instead consider robustness in terms of one of two different indices (or measures) that are quite problem specific, but provide a measure of a solution's ability to be efficient in different design environments.

1.1.2.5 Highly Robust Efficient Solutions

In the literature on robust multiobjective optimization, as mentioned in Section 1.1.2.2, various models and solution concepts have been proposed. Among the models, one that offers insightful study involves uncertain objective coefficients (with and without uncertain constraint coefficients) since in this case concepts combining efficiency and robustness in which efficiency is not lost due to robustness may be introduced. Further, among the many concepts available for problems modeled in this way, a restrictive concept referred to as *highly robust efficiency* has not been exhaustively analyzed. As a result, we focus on highly robust efficient solutions to UMOPs with uncertain objective coefficient data by examining the current literature and determining promising research directions.

Highly robust efficiency is a conservative concept of robustness requiring efficiency with respect to *every* instance of the objective function coefficient data simultaneously over a common feasible set. This definition of robustness is provided by Ide [79], Ide and Schöbel [82], and Kuhn et al. [94, 95] in the context of UMOPs with only uncertain objective coefficients (and a deterministic feasible set), and also by Goberna et al. [58] in terms of uncertain objective and constraint coefficients (where the feasible set is thus uncertain so feasibility is considered with respect to every realization of the constraints). Moreover, as with flimsily robust efficient solutions, highly robust efficient solutions are first defined in the context of interval multiobjective programming by Bitran [18] and are called *necessarily efficient* solutions. For this type of problem, solution methods are presented by Bitran [18], Benson [12], Inuiguchi and Kume [84], Ida [77], Inuiguchi and Sakawa [85], Oliveira and Antunes [116], and Hladík [69], while complexity analysis is studied by Hladík [70]. Additionally, Ida [78] computes necessarily efficient solutions to an uncertain biobjective quadratic portfolio selection problem.

Although highly robust efficient solutions have not been the focus of major research, some results are known. In particular, there is an existence result, various relationships and features, a computational method, and at least two applications.

In terms of existence, assume the UMOP has only uncertain objective function coefficients. Ide [79] and Ide and Schöbel [82] show that if one of the objective functions of the UMOP is deterministic (certain), i.e., does not contain any uncertain parameters, and if the optimization (minimization or maximization) of this objective yields a unique optimal solution, then this solution is also a highly robust efficient solution to the UMOP. That is, for this class of problems, the existence of highly robust efficient solutions is guaranteed and may be explicitly found by solving a deterministic SOP. It is important to note that the existence of an objective function which does not contain any uncertainty is not completely unrealistic in practice. For example, if the captain of a ship wants to minimize the length and the travel time of a trip, then the length of any path is exactly known while the travel time may depend, e.g., on weather conditions and ocean currents.

The various relationships between highly robust efficient solutions and other solution concepts are considered both when no assumptions about the uncertainty set are made and when it is assumed to be *objective-wise*, i.e., the uncertainties in the conflicting objective functions are independent of each other. On the other hand, a reduction result is given when the UMOP has only objective-wise uncertainty in the objective coefficient data (refer to Ide [79], Ide and Schöbel [82], and Kuhn et al. [94, 95]), while several features are provided when the UMOP has uncertain data in both the objective and constraint coefficients (see Goberna et al. [58]).

First, when no assumptions about the structure of the uncertainty set are made, Ide [79] and Ide and Schöbel [82] demonstrate a variety of relationships with other solution concepts such as flimsily robust efficiency and set-based min-max robustness. The most evident relationship is that if a solution is highly robust efficient, then it is also flimsily robust efficient (by definition). Additionally, they demonstrate that if the uncertainty set is a singleton (i.e., if there is actually no uncertainty in the data), then highly and flimsily robust efficient solutions coincide. On the other hand, highly robust efficiency does *not* imply (either set-based or point-based) min-max robustness. Similarly, if a solution is $\boldsymbol{\varepsilon}$-lightly robust for all nonnegative (but nonzero) epsilon, then it is *not* necessarily also highly robust efficient; and vice versa, if a solution is highly robust efficient, then it is *not* guaranteed to be $\boldsymbol{\varepsilon}$-lightly robust for some nonnegative (but nonzero) epsilon. Ide and Schöbel [82] also show relationships between highly robust efficient solutions and two other set-based concepts.

If the uncertainty set is assumed to be objective-wise, then new relationships emerge as shown by Ide [79], Ide and Schöbel [82], and Kuhn et al. [94, 95] (for a special case). Under this assumption, it is now the case that highly robust efficiency *does* imply min-max robustness (both set-based and point-based, which are equivalent under this assumption), as well as $\boldsymbol{\varepsilon}$-lightly robustness for all nonnegative (but nonzero) epsilon. Also, Ide and Schöbel [82] show that if a solution is highly robust efficient, then it is robust with respect to three other set-based concepts. However, it is still the case that if a solution is $\boldsymbol{\varepsilon}$-lightly robust for all nonnegative (but nonzero) epsilon, then it is *not* necessarily also highly robust efficient. A final relationship is given by Kuhn et al. [94]. If a solution is highly robust efficient, then it is efficient with respect to the (deterministic) MOP whose objective functions are the single deterministic objective along with every instance of the objective functions, which we later refer to as the *all-in-one* problem.

Finally, again under the assumption of objective-wise uncertainty, a reduction of the uncertainty set is shown by Ide [79], Ide and Schöbel [82], and Kuhn et al. [94, 95] (for a special case). If the uncertainty set is also the convex hull of a finite set of points, i.e., a bounded polyhedron or polytope, and the objective functions are affine with respect to the uncertainty, then a solution is highly robust efficient with respect to the entire uncertainty set if and only if it is highly robust efficient with respect to the finite set of points. In other words, rather than having to solve the UMOP with respect to the infinite number of scenarios of the (polytopal) uncertainty set, we only have to solve it with respect to the finite number of points from which the convex hull is formed. The importance of this result is, as in the case of robust single-objective optimization, that the reduction indicates that the UMOP is tractable.

In addition to the aforementioned, highly robust efficient solutions are also examined in the more general setting of uncertainty in the coefficients of both the objectives and the constraints, but less general case of UMOLPs. Under the assumption of objective-wise uncertainty and *constraint-wise* uncertainty (similarly defined), three results are identified by Goberna et al. [58]: a relationship, the radius of highly robust efficiency, i.e., the greatest value of a particular parameter associated with two families of the objective coefficient uncertainty sets such that the corresponding UMOLPs have highly robust weakly efficient solutions, and necessary and sufficient conditions for several types of constraint uncertainty sets. First, as in both of the above situations, it is still true that highly robust efficiency implies min-max robustness. Second, the radius of highly robust efficiency is bounded under both affine and radial objective data perturbations. Finally, necessary and sufficient conditions are provided for the case of radial objective coefficient uncertainty under general (no additional structure), convex, box, norm, and ellipsoidal constraint uncertainty sets.

A special case of UMOPs is also studied in the literature by Kuhn et al. [94, 95]. The UMOP is taken to be an uncertain biobjective problem (UBOP) in which the coefficients of one objective are deterministic, the other uncertain. In the context of this model, it is clear that the uncertainty set is necessarily objective-wise. Under the additional assumption that the uncertainty set is finite, the authors propose a solution method, which is applied to problems within the study of aircraft route guidance and hazardous materials transportation, to compute refined subsets of highly robust efficient solutions in a two-step procedure. First, the deterministic MOP associated with each scenario is solved. Then, a filtering step is applied to reduce the obtained sets to the desired highly robust efficient solutions. Based on the algorithm, the authors also provide a complexity result.

1.2 Completed Research Objectives

As evidenced by the above discussion, the current literature on highly robust efficient solutions to UMOPs with objective-wise uncertainty in the criteria coefficient data is lacking several key aspects. In order to address these incomplete aspects, as well as to provide meaningful theoretical and methodological tools for decision makers and practitioners, the research goals of this dissertation are to:

(i) develop properties of the highly robust efficient set including those regarding closedness, convexity, and connectedness;

(ii) provide a characterization of the highly robust efficient set;

(iii) verify the highly robust efficiency (or lack thereof) of a given feasible solution;

(iv) compute highly robust efficient solutions.

Properties of the highly robust efficient set, similar to any solution set, are important to study from both a theoretical and methodological perspective because different characteristics may provide revealing insights. For example, in the case of UMOLPs, the connectedness of the highly robust efficient set is an important feature to determine. If the highly robust efficient set is always connected, then a simplex algorithm approach to computing highly robust efficient points is advantageous to pursue. Otherwise, if the highly robust efficient set may be disconnected, then the task of obtaining highly robust efficient solutions is reserved for global optimization methods.

Likewise, characterizing the highly robust efficient set is valuable to pursue because a more complete understanding of the solution set is realized. In working toward a characterization, we provide not only necessary and/or sufficient conditions for the highly robust efficiency of feasible solutions, but also bound sets on the highly robust efficient set, an RC for a class of problems, and related existence conditions.

Methodologically, verifying the highly robust efficiency of a given feasible solution also serves as a meaningful tool for decision makers for several reasons. Primarily, it allows them to test a point deemed desirable a priori without having to actually solve the UMOP or compute a representation of the highly robust efficient set. Additionally, in the case of UMOLPs, the ability to check whether or not a feasible point is highly robust efficient allows decision makers to determine the highly robust efficiency of points in the relative interior of a face, which in turn may indicate that the entire face itself is highly robust efficient.

Finally, the importance of computing highly robust efficient solutions is obvious. Currently, the only existing solution method is a brute-force attack that involves solving every instance of a given UMOP and has only been applied to a special class of biobjective problems. As such, it is necessary to develop other approaches that do not require a collection of problems to be solved but rather allow for an individual problem to be solved.

With regard to the four stated research objectives, further focus throughout the dissertation is given to UMOLPs, however, more general UMOPs are also considered.

1.3 Overview

In view of the aforementioned research goals, the remainder of the dissertation is organized as follows. We provide the notation used herein and relevant mathematical preliminary results in Chapter 2. In Chapter 3, the theory of cones is explored. Existing definitions and results on cones, namely (polyhedral) convex, dual, polar, and strict polar cones, are given in Section 3.1, while new results are derived in Section 3.2. The new results mainly concern the polar and strict polar cones of three interrelated convex cones, include computational approaches to determine the acuteness of a cone, and provide the means with which to subsequently offer characterizations of the efficient and highly robust efficient sets in Chapters 4 and 6, respectively.

An overview of deterministic multiobjective programming is given in Chapter 4, with a brief look at MOPs in Section 4.1 and a more thorough examination of MOLPs in Section 4.2. Within Section 4.1, scalarization methods (such as the weighted-sum method) to compute efficient solutions are reviewed in Section 4.1.1, while methods with which to verify the efficiency (or lack thereof) of a given feasible point and to generate efficient points distinct from a given feasible solution are summarized in Section 4.1.2. Moreover, within Section 4.2, properties of the efficient set are reviewed in Section 4.2.1, characterizing the efficient set is studied in Section 4.2.2, and the computation of efficient solutions is covered in Section 4.2.3. Aside from the known results regarding deterministic MOLPs studied therein, a new result is derived in Section 4.2.4 that provides a different and useful perspective on an existing characterization of the efficient set, which is found in Section 4.2.2.3.

The main contributions of the dissertation begin in Chapter 5 (and continue in Chapter 6) in which highly robust efficient solutions to UMOPs are explored. In Section 5.1, the formulation of the UMOP under consideration is introduced, while a theoretical robust counterpart is developed in Section 5.2. Additionally, a naive approach to compute highly robust efficient solutions to the UMOP in question is given in Section 5.3, and several methods with which to determine whether or not a given feasible point is highly robust efficient or otherwise possibly generate a new highly robust efficient solution are established in Section 5.4.

Similarly, in Chapter 6, highly robust efficient solutions to UMOLPs are investigated. In Section 6.1, the formulation of the UMOLP under consideration is presented. An uncertainty set reduction for a class of UMOLPs is then given in Section 6.2, which allows for highly robust efficient solutions to be studied with respect to only UMOLPs whose uncertainty sets are finite. Within Section 6.3, properties (such as those regarding closedness, convexity, and connectedness) and characterizations of the highly robust efficient set are presented in Sections 6.3.1 and 6.3.2, respectively. Moreover, bound sets on the highly robust efficient set (i.e., sets that contain or are contained in the highly robust efficient set) are derived in Section 6.3.3, while a theoretical RC, as well as a classical RC that may be used to obtain the highly robust efficient set for a special class of UMOLPs, is determined in Section 6.3.4. Since the acuteness of various cones is necessary to know for results in Section 6.3.2, this property is discussed in more detail in Section 6.3.5 and methods with which to identify it are revisited. Finally, in Section 6.4, the computation of highly robust efficient solutions to UMOLPs is addressed. Within this section, approaches to computationally identify whether or not a given feasible solution is indeed highly robust efficient, possibly generate a different highly robust efficient point, or determine that the highly robust efficient set is empty are developed in Sections 6.4.1 and 6.4.2. Additionally, in Sections 6.4.3 and 6.4.4, solution methods to compute highly robust efficient points are obtained. In the former section a straightforward approach is given, while in the latter section a more sophisticated method is provided. An application in the area of bank balance-sheet management is also included in Section 6.4.5, and highly robust efficient solutions to the resulting UMOLP are computed. Concluding remarks are given in Chapter 7.

Chapter 2

Mathematical Preliminaries

Throughout the dissertation, a variety of mathematical concepts and results from different fields of study are necessary in later discussions. We provide many of these definitions and results here for convenience. The general notation employed is introduced in Section 2.1, and mathematical background results from several fields of mathematics including linear algebra and real analysis are presented in Sections 2.2, 2.3, and 2.4. Finally, two fundamental single-objective linear programming results are given in Section 2.5, while several relevant theorems of the alternative are provided in Section 2.6.

2.1 Notation

We use the following notation throughout the dissertation. Lower case letters in bold are used to denote vectors, and other lower case letters describe indices or scalars. Matrices are denoted by upper case bold letters, and sets are denoted by upper case letters. Subscripts differentiate matrices, vectors, and scalars, as well as indicate the components of a vector. Superscripts are used when necessary to differentiate vectors, but are otherwise treated as exponents. Euclidean vector spaces of a given dimension are denoted using $\mathbb{R}$. Note that vectors are written using typical vector notation as well as ordered pair notation.

Moreover, the $n \times n$ identity matrix is given by $\mathbf{I}_n$, the vector of all ones is denoted by $\mathbf{1}$, the vector of all zeros is denoted by $\mathbf{0}$, and the origin is written as $\{\mathbf{0}\}$. For all $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^p$, we write

$$\mathbf{y}_1 \leqq \mathbf{y}_2 \text{ if } y_{1k} \leq y_{2k} \text{ for all } k = 1, \dots, p;$$

$$\mathbf{y}_1 \leq \mathbf{y}_2 \text{ if } y_{1k} \leq y_{2k} \text{ for all } k = 1, \dots, p, \text{ and } \mathbf{y}_1 \neq \mathbf{y}_2;$$

$$\mathbf{y}_1 < \mathbf{y}_2 \text{ if } y_{1k} < y_{2k} \text{ for all } k = 1, \dots, p.$$

When $p = 1$, the symbols $\leqq$ and $\leq$ coincide. The inequalities $\geqq, \geq, >$ are used similarly. Additionally, the nonnegative orthant of dimension p is denoted by $\mathbb{R}^p_{\geqq} := \{\mathbf{y} \in \mathbb{R}^p : \mathbf{y} \geqq \mathbf{0}\}$. The semipositive, positive, nonpositive, seminegative, and negative orthants, denoted $\mathbb{R}^p_{\geq}, \mathbb{R}^p_{>}, \mathbb{R}^p_{\leqq}, \mathbb{R}^p_{\leq}$, and $\mathbb{R}^p_{<}$, respectively, are defined similarly.

2.2 Linear and Convex Functions

Linear, bilinear, and convex functions are widely studied (see, e.g., Rockafellar [121] and Lang [97]) in mathematics and are used throughout optimization. For example, entire branches of mathematics such as linear algebra rely on properties of linear functions, and whole fields of optimization such as linear programming depend on the objectives and constraints being either linear or convex. As a result, linear and convex functions are often considered fundamental functions in mathematics. Due to their wider significance and use in our current work, we provide the definition of each function, as well as a short discussion.

Definition 2.2.1. A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be a *linear function* if for any two vectors $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$, the function φ satisfies $\varphi(\mu_1\mathbf{z}_1 + \mu_2\mathbf{z}_2) = \mu_1\varphi(\mathbf{z}_1) + \mu_2\varphi(\mathbf{z}_2)$ for every $\mu_1, \mu_2 \in \mathbb{R}$.

In other words, a linear function is one that preserves addition and scalar multiplication. As such, it is possible to show (see Theorem 2.1, Lang [97]) that every linear function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{z} \mapsto \varphi(\mathbf{z})$ may be written as $\varphi(\mathbf{z}) = \mathbf{M}\mathbf{z}$ for some matrix $\mathbf{M} \in \mathbb{R}^{m\times n}$. When the function (or mapping) is of two variables, the concept of a linear function is extended to that of a bilinear function (refer to Section 5.4, Lang [97]). Note that the definition below is presented in terms of a vector-valued function (even though this is not necessary in general) due to our specific needs in this dissertation.

Definition 2.2.2. A function $\boldsymbol{\varphi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ of two variables, $\mathbf{z}$ and $\bar{\mathbf{z}}$, is *bilinear* if it is linear with respect to each of its variables. That is, $\boldsymbol{\varphi}$ is bilinear if it satisfies

$$\boldsymbol{\varphi}(\mu_1\mathbf{z}_1 + \mu_2\mathbf{z}_2, \bar{\mathbf{z}}) = \mu_1\boldsymbol{\varphi}(\mathbf{z}_1, \bar{\mathbf{z}}) + \mu_2\boldsymbol{\varphi}(\mathbf{z}_2, \bar{\mathbf{z}}) \tag{2.1}$$

for every $\mu_1, \mu_2 \in \mathbb{R}$ and $\mathbf{z}_1, \mathbf{z}_2, \bar{\mathbf{z}} \in \mathbb{R}^n$, and

$$\boldsymbol{\varphi}(\mathbf{z}, \mu_1\bar{\mathbf{z}}_1 + \mu_2\bar{\mathbf{z}}_2) = \mu_1\boldsymbol{\varphi}(\mathbf{z}, \bar{\mathbf{z}}_1) + \mu_2\boldsymbol{\varphi}(\mathbf{z}, \bar{\mathbf{z}}_2) \tag{2.2}$$

for every $\mu_1, \mu_2 \in \mathbb{R}$ and $\mathbf{z}, \bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2 \in \mathbb{R}^n$.

That is, a function of two variables is bilinear if when either the first variable or the second is fixed, the function is linear. Hence, for $\boldsymbol{\varphi}$, a function of two variables, it suffices to show that $\boldsymbol{\varphi}(\mu_1\mathbf{z}_1 + \mathbf{z}_2, \mu_2\bar{\mathbf{z}}_1 + \bar{\mathbf{z}}_2) = \mu_1\mu_2\boldsymbol{\varphi}(\mathbf{z}_1, \bar{\mathbf{z}}_1) + \mu_1\boldsymbol{\varphi}(\mathbf{z}_1, \bar{\mathbf{z}}_2) + \mu_2\boldsymbol{\varphi}(\mathbf{z}_2, \bar{\mathbf{z}}_1) + \boldsymbol{\varphi}(\mathbf{z}_2, \bar{\mathbf{z}}_2)$ for all $\mathbf{z}_1, \mathbf{z}_2, \bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2 \in \mathbb{R}^n$, and all scalars μ_1, μ_2 in order to show bilinearity, which is equivalent to showing $\boldsymbol{\varphi}$ satisfies (2.1) and (2.2). In addition to linear and bilinear functions, we study another fundamental type of function known as a convex function.

Definition 2.2.3. A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be a *convex function* if for any two vectors $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$, the inequality

$$\varphi(\alpha \mathbf{z}_1 + (1-\alpha)\mathbf{z}_2) \leq \alpha\varphi(\mathbf{z}_1) + (1-\alpha)\varphi(\mathbf{z}_2)$$

holds for all $\alpha \in [0, 1]$.

Geometrically, the convexity of a function may be interpreted similarly to that of a set (see Section 2.4). For every pair of vectors $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$ and $\alpha \in [0, 1]$, the chord or secant line joining $(\mathbf{z}_1, f(\mathbf{z}_1))$ and $(\mathbf{z}_2, f(\mathbf{z}_2))$ at the point $\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2$ must lie at or above the function. As a result, it is clear that linear functions are also convex.

2.3 Linear Algebra

In linear programming, the relevance of solving systems of equations is apparent. When solving a system of linear equations, the notion of the rank of a matrix is necessary. We define linear independence, as well as the rank of a matrix, and give a set of related basic results. Throughout, we use the superscript T to denote the transpose of a vector or matrix.

Definition 2.3.1. The vectors $\mathbf{z}_\ell \in \mathbb{R}^n, \ell = 1, \ldots, t$, are said to be *linearly independent* if $\sum_{\ell=1}^{t} \mu_\ell \mathbf{z}_\ell = \mathbf{0}$ implies $\mu_\ell = 0$ for all $\ell = 1, \ldots, t$.

Definition 2.3.2. The *rank* of a matrix $\mathbf{M} \in \mathbb{R}^{m \times n}$, denoted $\text{rank}(\mathbf{M})$, is the maximum number of linearly independent rows or columns of $\mathbf{M}$.

Proposition 2.3.3. *[134, p. 8] Let* $\mathbf{M} \in \mathbb{R}^{m\times n}$*. Then*

(i) $\operatorname{rank}(\mathbf{M}) \leq \min\{m, n\}$ *and* $\operatorname{rank}(\mathbf{M}) = \operatorname{rank}(\mathbf{M}^T)$*;*

(ii) when $m = n$*, we have* $\mathbf{M}^{-1}$ *(and* $(\mathbf{M}^T)^{-1}$*) exists if and only if* $\operatorname{rank}(\mathbf{M}) = n$*.*

The existence of solutions to systems of linear equations depends on the rank of the defining matrix. In particular, when the system is homogeneous (i.e., every equation in the system is equal to 0), we have the following result.

Theorem 2.3.4. *[22, Theorem 2.3] Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$*. The homogeneous linear system* $\mathbf{Ax} = \mathbf{0}$ *has a nontrivial solution* $\mathbf{x} \neq \mathbf{0}$ *if and only if* $\operatorname{rank}(\mathbf{A}) \neq n$*. When* $m = n$*, the system* $\mathbf{Ax} = \mathbf{0}$ *has a solution* $\mathbf{x} \neq \mathbf{0}$ *if and only if* $\operatorname{rank}(\mathbf{A}) < n$*.*

In other words, if $\operatorname{rank}(\mathbf{A}) = n$, then the only solution to the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$.

2.4 Set Theory

Set theory is one of the most fundamental fields within mathematics and is used extensively in our present work. We first define several basic set operations. Throughout, we use $\subseteq$ to denote *set containment* or *inclusion* and $\subset$ to denote *proper containment*.

Definition 2.4.1. Let $S, S_1, S_2 \subseteq \mathbb{R}^n$ be sets.

(i) The *negative* of S is defined to be $-S = \{-\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \in S\}$.

(ii) The *union* of S_1 and S_2 is defined to be $S_1 \cup S_2 := \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \in S_1 \text{ or } \mathbf{z} \in S_2\}$.

(iii) The *intersection* of S_1 and S_2 is defined to be $S_1 \cap S_2 := \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \in S_1 \text{ and } \mathbf{z} \in S_2\}$.

(iv) The *relative complement* of S_1 and S_2 is defined to be $S_1 \setminus S_2 := \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \in S_1$ and $\mathbf{z} \notin S_2\}$.

(v) The *complement* of S is defined to be $S^{\complement} := \mathbb{R}^n \setminus S$.

(vi) The *Minkowski sum* of S_1 and S_2 is defined to be $S_1 \oplus S_2 := \{\mathbf{z}_1 + \mathbf{z}_2 \in \mathbb{R}^n : \mathbf{z}_1 \in S_1, \mathbf{z}_2 \in S_2\}$.

(vii) The *Cartesian product* of S_1 and S_2 is defined to be $S_1 \times S_2 := \{(\mathbf{z}_1, \mathbf{z}_2) \in \mathbb{R}^{2n} : \mathbf{z}_1 \in S_1, \mathbf{z}_2 \in S_2\}$.

In the above definition, the relative complement $S_1 \setminus S_2$ may be understood as the removal of the elements from S_1 that are also in S_2, and is thus sometimes considered to be set subtraction or the difference of sets. Similarly, the Cartesian product may be considered to be set multiplication. With this in mind, we may write the Cartesian product of a set $S \subseteq \mathbb{R}^n$ with itself as $S \times S = S^2$. (This is precisely why the Euclidean vector space of dimension n is denoted $\mathbb{R}^n$.)

Remark 2.4.2. Based on Definition 2.4.1, numerous identities for each set operation may be given. Of those, the following regarding the Minkowski sum are needed. For the set $S \subseteq \mathbb{R}^n$ and the vector $\mathbf{z} \in \mathbb{R}^n$, it is apparent that $S \oplus \emptyset = \emptyset$ (which implies that $\emptyset \oplus \{\mathbf{z}\} = \emptyset$) and $S \oplus \{\mathbf{0}\} = S$ (see p. 16, Matheron [110]).

The different set operations in the previous definition may be combined in a wide variety of ways. In particular, the following result describes several distributive laws for set intersection, set union, and the Minkowski sum.

Theorem 2.4.3 (Distributive Laws)**.** *Let* $S_1, S_2, S_3 \subseteq \mathbb{R}^n$ *be sets. Then*

(i) [129, Theorem 5.1(3′)] $S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$*;*

(ii) [129, Theorem 5.1(3)] $S_1 \cup (S_2 \cap S_3) = (S_1 \cup S_2) \cap (S_1 \cup S_3)$*;*

(iii) [110, Formula 1-5-5] $S_1 \oplus (S_2 \cup S_3) = (S_1 \oplus S_2) \cup (S_1 \oplus S_3)$;

(iv) [110, Formula 1-5-5] $S_1 \oplus (S_2 \cap S_3) \subseteq (S_1 \oplus S_2) \cap (S_1 \oplus S_3)$.

In addition to these set operations that define how sets may be combined or changed by other sets, it is important to classify sets by the properties that they exhibit. Important classifications that we consider include boundedness, openness, closedness, connectedness, and convexity.

The first classification we introduce is that of boundedness. Intuitively, sets may either "extend to" infinity or may be "restricted" to finite regions.

Definition 2.4.4. A set $S \subseteq \mathbb{R}^n$ is said to be *bounded* if there exists a constant κ such that the absolute value of every component of every element of S is less than or equal to κ. Otherwise, S is said to be *unbounded*.

Open and closed sets are widely studied in real analysis and topology, and may be considered as generalizations of open and closed intervals on the real line, respectively. Before defining these sets, we need the notion of an open ball. Recall that the *Euclidean distance* between any two points $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$ is defined to be

$$|\mathbf{z}_1 - \mathbf{z}_2| := \sqrt{\sum_{i=1}^{n} (z_{1i} - z_{2i})}.$$

Using the Euclidean distance, we define an open ball, which we shall see is appropriately termed even though we have yet to define an open set.

Definition 2.4.5. Given $\bar{\mathbf{z}} \in \mathbb{R}^n$ and $\varepsilon > 0$, the *open ball* about $\bar{\mathbf{z}}$ of radius ε is defined to be $B_\varepsilon(\bar{\mathbf{z}}) := \{\mathbf{z} \in \mathbb{R}^n : |\bar{\mathbf{z}} - \mathbf{z}| < \varepsilon\}$.

With the notion of an open ball in mind, we now define both open and closed sets.

Definition 2.4.6. The set $S \subseteq \mathbb{R}^n$ is said to be

(i) an *open set* if for every $\mathbf{z} \in S$, there is some $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{z}) \subseteq S$;

(ii) a *closed set* if its complement $S^{\complement}$ is open.

Remark 2.4.7. Note that the empty set, as well as the whole space $\mathbb{R}^n$, is both open (see Example 4.1(a), Carothers [24]) and closed (see Example 4.8(a), Carothers [24]). Moreover, note that a set may be neither open nor closed. For example, the interval $[5, 6)$ is neither open nor closed.

An important question regarding open and closed sets is whether or not unions and intersections of open (respectively, closed) sets remain open (or closed).

Theorem 2.4.8. *(i) [24, Theorem 4.3] An arbitrary union of open sets is open.*

(ii) [24, Theorem 4.4] A finite intersection of open sets is open.

(iii) [24, Example 4.8(b)] An arbitrary intersection of closed sets is closed.

(iv) [24, Example 4.8(b)] A finite union of closed sets is closed.

Even though sets may be neither open nor closed, we may describe the so-called open and closed portions of a set, as well as the portion in between.

Definition 2.4.9. Let $S \subseteq \mathbb{R}^n$ be a set. The

(i) *interior* of S, denoted $\mathrm{int}(S)$, is defined to be the largest open set contained in S;

(ii) *closure* of S, denoted $\mathrm{cl}(S)$, is defined to be the smallest closed set containing S;

(iii) *boundary* of S is defined to be $\mathrm{bd}(S) := \mathrm{cl}(S) \setminus \mathrm{int}(S)$.

Remark 2.4.10. It is obvious by definition that the interior of a set is open and the closure of a set is closed (see p. 56, Carothers [24]). Hence, the interior of an open set, as well as the closure of a closed set, is the set itself. Similarly, it is clear by definition that any set contains its interior and is contained in its closure (see p. 6, Steen and Seebach, Jr. [126]).

As with open and closed sets, it is important to know the behavior of the interior of an intersection of sets.

Proposition 2.4.11. *[126, p. 6] The interior of a finite intersection of sets is the finite intersection of the interiors.*

Although it may not be obvious based on the definition of the interior of a set, it is quite often the case that the interior of a set is empty. For example, a line segment in $\mathbb{R}^2$ has an empty interior, while a square in $\mathbb{R}^3$ has an empty interior also. Since the interior of a set may likely be empty, the concept of the relative interior is widely used (refer to Section 6 in Rockafellar [121] for a thorough discussion). Before defining the relative interior, we discuss affine sets and the related affine hull.

Definition 2.4.12. The set $A \subseteq \mathbb{R}^n$ is said to be an *affine set* if $\mu\mathbf{z}_1 + (1-\mu)\mathbf{z}_2 \in A$ for every $\mathbf{z}_1, \mathbf{z}_2 \in A$ and $\mu \in \mathbb{R}^n$.

The empty set, any singleton set, and the whole space $\mathbb{R}^n$ are considered extreme examples of affine sets. Intuitively, with the exception of the empty set or a singleton set, an affine set must contain the entire line through any pair of points. As a result, the basic visual notion is that an affine set is an endless uncurved object such as a plane.

Definition 2.4.13. The *affine hull* of a set $S \subseteq \mathbb{R}^n$, denoted $\text{aff}(S)$, is defined to be the smallest affine set containing S.

Remark 2.4.14. Based on the intuitive notion of an affine set, it is clear and can be shown (Theorem 1.4, Rockafellar [121]) that every affine set $A \subseteq \mathbb{R}^n$ may be written in the form

$$A = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}, \tag{2.3}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Moreover, it is easy to show that the affine hull of S consists of every vector of the form $\sum_{j=1}^{m} \mu_j \mathbf{a}_j$ such that $\mathbf{a}_j \in S$ and $\sum_{j=1}^{m} \mu_j = 1$ (see p. 6, Rockafellar [121]).

We are now ready to define the relative interior of a set.

Definition 2.4.15. The *relative interior* of a set $S \subseteq \mathbb{R}^n$, denoted $\operatorname{rel\,int}(S)$, is defined to be the interior that results when S is considered as a subset of its affine hull. That is,

$$\operatorname{rel\,int}(S) := \{\mathbf{x} \in \operatorname{aff}(S) : \exists \varepsilon > 0, B_\varepsilon(\mathbf{x}) \cap \operatorname{aff}(S) \subseteq S\}.$$

Using the affine hull of a set, we may also define that set's dimension. The dimension of a set, especially a convex set, is regarded as an important feature (e.g., Eckhardt [43]).

Definition 2.4.16. The *(affine) dimension* of a set $S \subseteq \mathbb{R}^n$, denoted $\dim(S)$, is defined to be $\dim(S) := \dim(\operatorname{aff}(S))$, where the dimension of $\operatorname{aff}(S)$ is the dimension of the subspace $S \oplus (-S)$.

Remark 2.4.17. The dimension of the emptyset is considered to be -1 by convention, and sets of dimension 0 are naturally referred to as points (see p. 4, Rockafellar [121]). Moreover, when the dimension of a set $S \subseteq \mathbb{R}^n$ is equal to n, the set is said to be *full-dimensional*.

The next type of set we discuss is a connected set. The definition of a connected set has a wide variety of meanings throughout the literature. We focus on the following definition that may be found on p. 78, Carothers [24].

Definition 2.4.18. A set $S \subseteq \mathbb{R}^n$ is said to be *disconnected* if S can be written as $S = S_1 \cup S_2$, where $S_1, S_2 \subseteq \mathbb{R}^n$ are nontrivial open sets such that $S_1 \cap S_2 = \emptyset$. Otherwise, the set S is said to be *connected*.

Remark 2.4.19. [24, Example 6.2(c)] The empty set, as well as any singleton set, is vacuously connected.

Another important type of set is a convex set.

Definition 2.4.20. The set $C \subseteq \mathbb{R}^n$ is said to be a *convex set* if given any two points $\mathbf{z}_1, \mathbf{z}_2 \in C$, then $\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2 \in C$ for all $\alpha \in [0,1]$.

Geometrically, the convexity of a set may be interpreted as follows: For every pair of points $\mathbf{z}_1, \mathbf{z}_2 \in C$, the line segment joining them (i.e., $\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2, \alpha \in [0,1]$) must be contained in C as well.

Remark 2.4.21. The whole space $\mathbb{R}^n$ is clearly convex by definition, while the empty set is vacuously convex.

Since the interior of a set may be empty while its relative interior is nonempty, it is of interest to know when the concepts of interior and relative interior coincide, i.e., when the the interior and relative interior of a set are equal. For convex sets, we refer to the following proposition.

Proposition 2.4.22. *Let $C \subseteq \mathbb{R}^n$ be a convex set. If*

(i) [121, p. 44] $\dim(C) = n$, *then* $\text{int}(C) = \text{rel}\,\text{int}(C)$;

(ii) [30, Formula (14)] $\text{int}(C) \neq \emptyset$, *then* $\text{int}(C) = \text{rel}\,\text{int}(C)$.

In addition, with respect to their relative interior, convex sets exhibit a special property.

Theorem 2.4.23. *[121, Theorem 6.2] Let $C \subseteq \mathbb{R}^n$ be a nonempty convex set. Then* $\operatorname{rel\,int}(C) \neq \emptyset$.

A specific type of convex set is known as a polyhedral (convex) set or polyhedron, which may be defined constructively in terms of half-spaces.

Definition 2.4.24. Let $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The set

(i) $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \leq b\}$ is said to be a *(closed) half-space*;

(ii) $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\}$ is said to be a *hyperplane*.

We often refer to the (closed) half-space defined above simply as a half-space since it is clearly a closed set, and we choose the half-space to be defined by $\mathbf{a}^T\mathbf{x} \leq b$ instead of the alternative $\mathbf{a}^T\mathbf{x} \geq b$ for consistency later in the dissertation. Moreover, the boundary of a half-space is its corresponding hyperplane, and the vector $\mathbf{a}$ in the definition of a hyperplane is perpendicular to that hyperplane. In referencing half-spaces and their related hyperplanes, the hyperplane is usually said to *generate* the half-space. In view of Definition 2.4.24, we define a polyhedron as follows.

Definition 2.4.25. A set $\widetilde{P} \subset \mathbb{R}^n$ is said to be a *polyhedral set* or *polyhedron* if it is the intersection of a finite number of half-spaces. When the polyhedron is also bounded, it is referred to as a *polytope*.

In other words, a polyhedral set may be written algebraically as

$$\widetilde{P} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_1^T\mathbf{x} \leq b_1, \ldots, \mathbf{a}_m^T\mathbf{x} \leq b_m\},$$

where $\mathbf{a}_j \in \mathbb{R}^n, j = 1, \ldots, m$, are the normals to the generating hyperplanes passing through b_j whose associated closed half-spaces form $\widetilde{P}$. Taking the vectors $\mathbf{a}_j \in \mathbb{R}^n$ to be the rows of the matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and the scalars $b_j \in \mathbb{R}$ to be the elements of the vector $\mathbf{b} \in \mathbb{R}^m$, we may write

$$\widetilde{P} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leqq \mathbf{b}\}. \tag{2.4}$$

Two fundamental components of polyhedral sets, which we define below, are extreme points and extreme directions. Although both may be defined more generally for convex sets, we only consider each with respect to polyhedral sets due to our specific needs.

Definition 2.4.26. A vector $\mathbf{x} \in \widetilde{P}$ is said to be an *extreme point* of $\widetilde{P}$ if $\mathbf{x} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$, with $\alpha \in (0,1)$ and $\mathbf{x}_1, \mathbf{x}_2 \in \widetilde{P}$, implies that $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2$.

An extreme point is sometimes referred to as a corner point or vertex, and derives its name from the fact that it occurs at the intersection of the extreme or outer edges of the set. In other words, $\mathbf{x} \in \widetilde{P}$ is an extreme point if there are n constraints from $\{\mathbf{a}_1^T\mathbf{x} \leq b_1, \ldots, \mathbf{a}_m^T\mathbf{x} \leq b_m\}$ satisfied at equality, i.e., that are *active* or *binding*, whose corresponding vectors $\mathbf{a}_j$ are linearly independent. Additionally, in linear programming, the terms basic feasible solution and extreme point are often used interchangeably. In this context, if a linear program has an optimal solution then there exists an extreme point (alternate) optimal solution to the problem (cf. Theorem 2.8, Bertsimas and Tsitsiklis [15]).

The existence of extreme points, as it turns out, depends on whether or not the polyhedron $\widetilde{P}$ contains a(n) (infinite) line, where $\widetilde{P}$ *contains a line* if there exists an $\mathbf{x} \in \widetilde{P}$ and a nonzero vector $\mathbf{d} \in \mathbb{R}^n$ such that $\mathbf{x} + \mu\mathbf{d} \in \widetilde{P}$ for all $\mu \in \mathbb{R}$.

Theorem 2.4.27. *[15, Theorem 2.6] Suppose that $\widetilde{P} \neq \emptyset$. Then $\widetilde{P}$ has at least one extreme point if and only if $\widetilde{P}$ does not contain a line.*

Remark 2.4.28. In view of Theorem 2.4.27, a nonempty polytope (which is bounded and thus does not contain a line) immediately has at least one extreme point.

Similar to extreme points, polyhedral sets (and more generally, convex sets) may also contain extreme directions, where a nonzero vector $\mathbf{d} \in \mathbb{R}^n$ is said to be a *(recession) direction* of $\widetilde{P}$ (or any convex set) if for each $\mathbf{x} \in \widetilde{P}$, the *ray* $\{\mathbf{x} + \beta\mathbf{d} : \beta \geq 0\}$ is also in $\widetilde{P}$. In other words, $\mathbf{d} \in \mathbb{R}^n$ is a direction if for any step length $\beta \geq 0$, one can travel along $\mathbf{d}$ from $\mathbf{x}$ and remain feasible. It is clear that any positive multiple of a direction of $\widetilde{P}$ is likewise a direction. As a result, the idea of *distinct* directions becomes important, where two directions $\mathbf{d}_1$ and $\mathbf{d}_2$ are said to be distinct provided that $\mathbf{d}_1$ cannot be represented as a positive multiple of $\mathbf{d}_2$. Certainly, if $\widetilde{P}$ is bounded (i.e., a polytope), then it has no directions. Otherwise, any unbounded polyhedron has at least one direction. With the definition of a direction in mind, an extreme direction is defined as follows.

Definition 2.4.29. A direction $\mathbf{d}$ of the polyhedral set $\widetilde{P}$ is said to be an *extreme direction* provided there do not exist distinct directions $\mathbf{d}_1$ and $\mathbf{d}_2$ of $\widetilde{P}$, where $\mathbf{d}_1, \mathbf{d}_2 \neq \mathbf{d}$, and scalars $\mu_1, \mu_2 > 0$ such that $\mathbf{d} = \mu_1\mathbf{d}_1 + \mu_2\mathbf{d}_2$.

Intuitively, extreme directions are directions associated with the extreme or outer edges of the polyhedron that "extend to" infinity. That being said, it is evident that extreme directions need not be elements of the polyhedron (as is the case when the polyhedron is in the (strictly) positive orthant) and that a positive multiple of an extreme direction is also an extreme direction, which leads to so-called *equivalence classes*. In particular, two extreme directions are said to be *equivalent* if one is a positive multiple of the other. Due to possible confusion resulting from equivalent extreme directions, we refer to a finite collection of extreme directions as *complete* if it contains only one member from each equivalence class. Throughout the dissertation, the set of extreme directions of a polyhedron is assumed to be a complete set of extreme directions.

Remark 2.4.30. As $\widetilde{P}$ is formed by the intersection of finitely many closed half-spaces, it is evident that its sets of extreme points and extreme directions are both finite (cf. Corollary 2.1 and p. 176, Bertsimas and Tsitsiklis [15], respectively).

In addition to these fundamental building blocks of a polyhedral set, we may also represent a polyhedron via its *faces*.

Definition 2.4.31. A polyhedral subset $\overline{P}$ of $\widetilde{P}$ is said to be a *face* of $\widetilde{P}$ if every closed line segment in $\widetilde{P}$ with a relative interior point in $\overline{P}$ has both endpoints in $\overline{P}$.

Remark 2.4.32. The empty set and $\widetilde{P}$ are trivially considered faces of $\widetilde{P}$, while the extreme points of $\widetilde{P}$ may be regarded as zero-dimensional faces.

An important feature of a polyhedral set is that its extreme points and extreme directions may provide an internal characterization of the set (as opposed to the external characterization given by the definition as an intersection of a finite number of closed half-spaces). In particular, any point in a polyhedral set that has at least one extreme point can be represented by a convex combination of the set's extreme points and a conical combination of the set's extreme directions. If the set is a polytope, then any point can be represented as only a convex combination of the extreme points of the polytope. This characterization is known as the Representation (Resolution, or Caratheodory Characterization) Theorem (for polyhedral sets).

Theorem 2.4.33. *[15, Representation Theorem, Theorem 4.15] Let $\widetilde{P} \neq \emptyset$ with at least one extreme point. A vector $\mathbf{x}$ is in $\widetilde{P}$ if and only if it can be represented as* *a convex combination of the extreme points of* $\widetilde{P}$ *plus a nonnegative linear (conical) combination of the extreme directions of* $\widetilde{P}$*. That is,* $\mathbf{x}$ *is in* $\widetilde{P}$ *if and only if it can be represented as*

$$\mathbf{x} = \sum_{k=1}^{\eta} \alpha_k \mathbf{x}_k + \sum_{\ell=\eta+1}^{\eta+\tau} \beta_\ell \mathbf{x}_\ell,$$

where $\{\mathbf{x}_1, \ldots, \mathbf{x}_\eta\}$ *and* $\{\mathbf{x}_{\eta+1}, \ldots, \mathbf{x}_{\eta+\tau}\}$ *are the sets of extreme points and extreme directions of* $\widetilde{P}$*, respectively, and* $\alpha_1 + \cdots + \alpha_\eta = 1, \alpha_k \geq 0$ *for all* $k = 1, \ldots, \eta$*, and* $\beta_\ell \geq 0$ *for all* $\ell = \eta + 1, \ldots, \eta + \tau$*.*

What if $\widetilde{P}$ does not contain any extreme points? For example, consider the polyhedron that is a line given by $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = 2\}$. By Theorem 2.4.27, the polyhedron does not contain any extreme points (which is also clear graphically). In this case, the Representation Theorem is not applicable, so it is worthwhile to have another, more general, way of expressing elements of a polyhedron.

Theorem 2.4.34. *[124, Corollary 7.1b, Decomposition Theorem] The set* $\widetilde{P}$ *is a polyhedron if and only if*

$$\widetilde{P} = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{k=1}^{\pi} \alpha_k \mathbf{x}_k + \sum_{\ell=\pi+1}^{\pi+\sigma} \beta_\ell \mathbf{x}_\ell \right\}$$

for some sets of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_\pi\}$ *and* $\{\mathbf{x}_{\pi+1}, \ldots, \mathbf{x}_{\pi+\sigma}\}$*, where* $\alpha_1 + \cdots + \alpha_\pi = 1, \alpha_k \geq 0$ *for all* $k = 1, \ldots, \pi$*, and* $\beta_\ell \geq 0$ *for all* $\ell = \pi + 1, \ldots, \pi + \sigma$*.*

A direct consequence of the Decomposition Theorem is that any element of $\widetilde{P}$ may be represented as a convex combination of the points $\mathbf{x}_1, \ldots, \mathbf{x}_\pi$ plus a conical combination of the vectors $\mathbf{x}_{\pi+1}, \ldots, \mathbf{x}_{\pi+\sigma}$. However, a key difference between the Representation Theorem and the Decomposition Theorem is that the sets of extreme points and directions needed in the former are known in general, whereas the sets $\{\mathbf{x}_1, \ldots, \mathbf{x}_\pi\}$ and $\{\mathbf{x}_{\pi+1}, \ldots, \mathbf{x}_{\pi+\sigma}\}$ needed in the latter may not be. In particular, the extreme points and directions of $\widetilde{P}$ may be computed algorithmically, e.g., by the Double Description Method (refer to Motzkin et al. [113], Matheiss and Rubin [109], and Dandurand [29]), as well as by software including SageMath [132]. Accordingly, when available, the preferred representation of elements of $\widetilde{P}$ is with respect to the Representation Theorem. When not available, for examples similar to the line $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = 2\}$, the Decomposition Theorem may be used. In fact, one possible representation of the aforementioned line is given by

$$\{\mathbf{x} \in \mathbb{R}^2 : x_2 = 2\} = \left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\},$$

where $\beta_2, \beta_3 \geq 0$. In addition, the Representation Theorem takes on added significance in terms of linear programming (especially within the scope of the Simplex Method) since the feasible set of a linear program is polyhedral.

2.5 Linear Programming

Two fundamental aspects of (deterministic) single-objective linear programming are duality and the Karush-Kuhn-Tucker (KKT) conditions. First, associated with each linear program, there is another linear program called the *dual*. In this context, the original linear program is known as the *primal*. The dual linear program possesses many important properties relative to the primal problem, which result from the idea that when we are solving the original linear program, we are simultaneously solving the dual (refer to Chapter 6, Bazaraa et al. [3]).

Suppose that the *primal linear program* (LP) is given in the canonical form:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0}, \end{aligned} \tag{2.5}$$

where $\mathbf{c} \in \mathbb{R}^n$ is the *cost vector*, $\mathbf{x} \in \mathbb{R}^n$ is the *decision vector*, $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the *constraint matrix*, and $\mathbf{b} \in \mathbb{R}^m$ is the vector of right-hand side (RHS) values. The feasible set of LP (2.5) is a polyhedron denoted by

$$P := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}\}, \tag{2.6}$$

which may be equivalently written in the form of (2.4) by letting the constraint matrix be $\begin{bmatrix}\mathbf{A} & -\mathbf{I}_n\end{bmatrix}^T$ and the RHS be $\begin{bmatrix}\mathbf{b} & \mathbf{0}\end{bmatrix}^T$. Since P lies in the nonnegative orthant, which clearly does not contain a line, the feasible set P has at least one extreme point by Theorem 2.4.27, a fact helps motivate why an optimal solution (if it exists) to LP (2.5) occurs at an extreme point.

The *dual linear program* (DP) associated with LP (2.5) is then given by

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathbf{b}^T\mathbf{w} \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{w} \leqq \mathbf{c} \\ & \mathbf{w} \leqq \mathbf{0}, \end{aligned} \tag{2.7}$$

where $\mathbf{w} \in \mathbb{R}^m$ is the vector of *dual variables*. Note that there is exactly one dual variable w_j for each constraint in the primal problem, and there is exactly one constraint in the dual for each primal variable x_i. Relating the primal LP and its dual, we have the following result.

Theorem 2.5.1. *[3, Fundamental Theorem of Duality, Theorem 6.1] Consider* LP (2.5) *and its associated dual* DP (2.7)*. Exactly one of the following statements is true:*

(i) Both possess optimal solutions $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{v}}$ *with* $\mathbf{c}^T\hat{\mathbf{x}} = \mathbf{b}^T\hat{\mathbf{v}}$*;*

(ii) One problem has an unbounded optimal objective value, in which case the other problem must be infeasible; or

(iii) Both problems are infeasible.

Statement (i) in the Fundamental Theorem of Duality is often referred to as *Strong Duality.* Along with Strong Duality, it clear in the Fundamental Theorem of Duality that solving the primal problem is equivalent to solving the dual problem. Alternatively, it is apparent that while solving one (either the primal or dual), we are simultaneously solving the other.

Second, in addition to duality, the KKT conditions form the foundation of (continuous) optimization including both linear and nonlinear programming. In the context of LP (2.5), the KKT conditions are both necessary and sufficient optimality conditions as we see in the following result.

Theorem 2.5.2. *[3, KKT Conditions, pp. 238–239] The vector* $\hat{\mathbf{x}} \in \mathbb{R}^n$ *is an optimal solution to LP* (2.5) *if and only if there exists a* $\hat{\mathbf{w}} \in \mathbb{R}^m$ *such that* $(\hat{\mathbf{x}}, \hat{\mathbf{w}})$ *satisfy*

$$\mathbf{A}\mathbf{x} \leqq \mathbf{b} \qquad \mathbf{x} \geqq \mathbf{0} \tag{2.8}$$

$$\mathbf{A}^T\mathbf{w} \leqq \mathbf{c} \qquad \mathbf{w} \leqq \mathbf{0} \tag{2.9}$$

$$\mathbf{w}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = 0 \qquad \mathbf{x}^T(\mathbf{c} - \mathbf{A}^T\mathbf{w}) = 0. \tag{2.10}$$

The first condition (2.8) is typically called *primal feasibility* since it simply requires that $\hat{\mathbf{x}}$ be a feasible solution to LP (2.5). Similarly, the second condition (2.9) is usually referred to as *dual feasibility* since it forces $\hat{\mathbf{w}}$ to be a feasible solution to DP (2.7). Finally, the third condition (2.10) is typically called *complementary slackness*. The reason for this terminology is because $\mathbf{w}^T(\mathbf{Ax} - \mathbf{b}) = 0$ if and only if for every $j = 1, \ldots, m$, either w_j is 0 or the j-th *slack* variable associated with $\mathbf{Ax} \leqq \mathbf{b}$ is 0. Likewise, $\mathbf{x}^T(\mathbf{c} - \mathbf{A}^T\mathbf{w}) = 0$ if and only if for every $i = 1, \ldots, n$, either x_i is 0 or the i-th slack variable associated with $\mathbf{A}^T\mathbf{w} \leqq \mathbf{c}$ is 0.

As mentioned, the KKT conditions form the foundation of optimization including linear programming. With this in mind, the Simplex Method to solve LP (2.5) may be viewed as a systematic approach to finding the optimal extreme point solution that satisfies the KKT conditions. At each iteration, primal feasibility and complementary slackness are satisfied, while dual feasibility is partially violated until an optimal solution is reached. In addition to the clear fundamental nature of the KKT conditions in linear programming, we later use these optimality conditions to also solve optimization problems with uncertainty in Chapter 6.

2.6 Theorems of the Alternative

Theorems of the alternative, such as the classical Farkas' Lemma (refer to Lemma 5.1, Bazaraa et al. [3]), relate the occurrence of two mutually exclusive events represented as systems of linear inequalities and/or equations. Numerous versions of such theorems can be found in the literature, and several relevant theorems are quoted below.

Theorem 2.6.1 (Gale's Theorem). *Let* $\mathbf{A} \in \mathbb{R}^{m \times n}$ *be a matrix, and* $\mathbf{b} \in \mathbb{R}^m$ *a vector.*

(i) [56, Theorem 2.7] Either

$$\mathbf{A}\mathbf{x} \leqq \mathbf{b}$$

has a solution $\mathbf{x} \in \mathbb{R}^n$*, or*

$$\mathbf{A}^T\mathbf{w} = \mathbf{0}, \mathbf{b}^T\mathbf{w} = -1, \mathbf{w} \geqq \mathbf{0}$$

has a solution $\mathbf{w} \in \mathbb{R}^m$*, but never both.*

(ii) [56, Theorem 2.8] Either

$$\mathbf{A}\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}$$

has a solution $\mathbf{x} \in \mathbb{R}^n$*, or*

$$\mathbf{A}^T\mathbf{w} \geqq \mathbf{0}, \mathbf{b}^T\mathbf{w} < 0, \mathbf{w} \geqq \mathbf{0}$$

has a solution $\mathbf{w} \in \mathbb{R}^m$*, but never both.*

Theorem 2.6.2. *[107, Gordan's Theorem, Theorem 5] Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *be a matrix. Either*

$$\mathbf{A}\mathbf{x} > \mathbf{0}$$

has a solution $\mathbf{x} \in \mathbb{R}^n$*, or*

$$\mathbf{A}^T\mathbf{w} = \mathbf{0}, \mathbf{w} \geq \mathbf{0}$$

has a solution $\mathbf{w} \in \mathbb{R}^m$*, but never both.*

Theorem 2.6.3. *[105, Stiemke's Theorem, p. 19] Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *be a matrix. Either*

$$\mathbf{A}\mathbf{x} = \mathbf{0}, \mathbf{x} > \mathbf{0}$$

has a solution $\mathbf{x} \in \mathbb{R}^n$*, or*

$$\mathbf{A}^T\mathbf{w} \geq \mathbf{0}$$

has a solution $\mathbf{w} \in \mathbb{R}^m$*, but never both.*

Theorem 2.6.4. *[107, Theorem 11, p. 35] Let* $\mathbf{A} \in \mathbb{R}^{m \times n}$ *be a matrix, and* $\mathbf{b} \in \mathbb{R}^m$ *a vector. Either*

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$

has a solution $\mathbf{x} \in \mathbb{R}^n$*, or*

$$\left\{ \begin{array}{c} \mathbf{A}^T\mathbf{w} = \mathbf{0}, \mathbf{b}^T\mathbf{w} = -1, \mathbf{w} \geqq \mathbf{0} \\ \textit{or} \\ \mathbf{A}^T\mathbf{w} = \mathbf{0}, \mathbf{b}^T\mathbf{w} \leqq 0, \mathbf{w} > \mathbf{0} \end{array} \right\}$$

has a solution $\mathbf{w} \in \mathbb{R}^m$*, but never both.*

One of the main uses of theorems of the alternative (in the literature, as well as our work) is to provide additional existence results for linear systems. In particular, if we have some result occur when a specific linear system has a solution, then a theorem of the alternative may allow us to equivalently state that the result occurs when the alternative system has no solution. Likewise, if the result occurs when the linear system has no solution, then we may equivalently state that the result occurs when the alternative system does have a solution.

Chapter 3

Cones

Within convex analysis, convex cones have been well-studied (e.g., Rockafellar [121], Panik [118], and Borwein and Lewis [20]). Convex cones have particular importance in multiobjective (linear) optimization in terms of defining domination structures and ordering relationships (see Yu [145], Hartley [65], and Sawaragi et al. [122]), as well as with respect to defining the structure of the set of improving directions associated with a multiobjective linear program (refer to Thoai [133]) which is our main interest in studying them here.

We first quote from the literature definitions and existing results regarding (convex) cones relevant to the topic of multiobjective linear programming in Section 3.1. Then in Section 3.2, we present results that have not (to our knowledge) been given before in the literature regarding three interrelated convex cones of interest: polyhedral convex cones and the related convex cones obtained by removing either part or all of the boundary. In particular, we present properties and develop algebraic representations of the polar cones (respectively, strict polar cones) of these three convex cones, as well as of the unions of cones in collections associated with each of the three cones, in Sections 3.2.2 and 3.2.3. As the acuteness of the three convex cones and their associated unions emerges as an important characteristic, we also propose two methods to verify this property in Section 3.2.4.

3.1 Existing Results

We begin with the definition of a cone and definitions of several types (or characteristics) of cones.

Definition 3.1.1. A set $K \subseteq \mathbb{R}^n$ is said to be a *cone* if $\mathbf{z} \in K$ implies that $\lambda\mathbf{z} \in K$ for all $\lambda > 0$.

For our purposes, cones do not necessarily have to contain the origin, which is reflected in the above definition. Cones may exhibit various properties such as pointedness, acuteness, and convexity.

Definition 3.1.2. A cone $K \subseteq \mathbb{R}^n$ is said to be

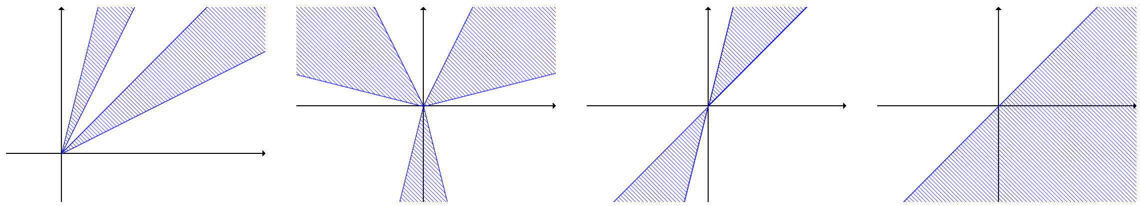
(i) *pointed* if $\mathbf{z} \in K$ and $\mathbf{z} \neq \mathbf{0}$ implies that $-\mathbf{z} \notin K$;

(ii) *acute* if $\text{cl}(K) \subseteq H \cup \{\mathbf{0}\}$, where H is an open half-space whose generating hyperplane passes through the origin;

(iii) *convex* if for any two points $\mathbf{z}_1, \mathbf{z}_2 \in K$, then $\mathbf{z}_1 + \mathbf{z}_2 \in K$.

It is important to note that not every cone is convex and that pointedness and acuteness are not equivalent (even in two-dimensions) which is clearly illustrated in Figure 3.1. That being said, if the convex cone K is closed, then pointedness and acuteness are equivalent as in the following.

Proposition 3.1.3. *[122, Proposition 2.1.4] Let $K \subseteq \mathbb{R}^n$ be a convex cone. Then K is acute if and only if $\text{cl}(K)$ is pointed.*



(a) Nonconvex, acute, and pointed
(b) Nonconvex, not acute, and pointed
(c) Nonconvex, not acute, and not pointed
(d) Convex, not acute, and not pointed

Figure 3.1: Nonconvex and convex cones in two-dimensions

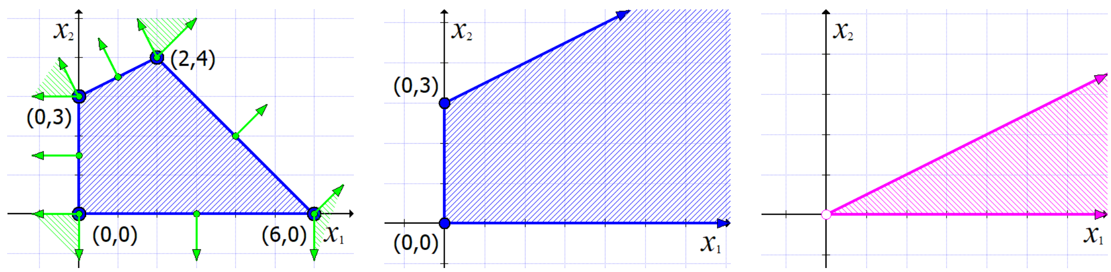
Two specific and relevant convex cones to our study are the normal cone (the set of all normal directions) and the recession cone (the set of all recession directions), which we associate with the polyhedral feasible set P (2.6). The normal cone may more generally be associated with the polyhedral set $\widetilde{P}$ (2.4), while the recession cone may be associated with a convex set.

Definition 3.1.4. The *normal cone* to the polyhedron P at $\bar{\mathbf{x}} \in P$ is a convex cone defined to be $N_P(\bar{\mathbf{x}}) := \{\mathbf{p} \in \mathbb{R}^n : \mathbf{p}^T(\mathbf{x} - \bar{\mathbf{x}}) \leq 0 \text{ for all } \mathbf{x} \in P\}$.

The normal cone $N_P(\mathbf{x})$ contains $\mathbf{0}$ for all $\mathbf{x} \in P$, and is thus always nonempty. When $\mathbf{x}$ is an interior point of P, the normal cone is necessarily $\{\mathbf{0}\}$. Otherwise, when $\mathbf{x}$ is a boundary point, we compute the normal cone as in Theorem 2.3.24, Luc [105]. See Figure 3.2a for an example of a bounded polyhedron and its corresponding normal cones for various points around the boundary.

Definition 3.1.5. The *recession cone* of the polyhedron P is a convex cone defined to be $R_P := \{\mathbf{d} \in \mathbb{R}^n : \mathbf{A}\mathbf{d} \leqq \mathbf{0}, \mathbf{d} \geq \mathbf{0}\}$.

A recession direction (an element of the recession cone) is thus a direction along which feasibility to P is always maintained. Hence, if P is bounded, then R_P is necessarily empty. Refer to Figures 3.2b and 3.2c for an example of an unbounded polyhedron and its associated recession cone.



(a) The polyhedron P_1 (4.10) (blue) with normal cones (green)

(b) An unbounded polyhedron

(c) The recession cone of the unbounded polyhedron

Figure 3.2: An example illustrating normal cones and the recession cone

Two other fundamental types of convex cones (that turn out to be equivalent) are finite and polyhedral convex cones.

Definition 3.1.6. A nonempty convex cone $K \subseteq \mathbb{R}^n$ is said to be

(i) *finite* if it consists of the set of all nonnegative linear combinations of a finite set of vectors $\{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_\rho\}$; K is said to be *spanned* or *generated* by the finite *set of generators* $\{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_\rho\}$, where $\boldsymbol{\gamma}_\ell \in \mathbb{R}^n, \ell = 1, \ldots, \rho$;

(ii) *polyhedral convex* if it is the intersection of a finite number of closed half-spaces whose generating hyperplanes pass through the origin.

Equivalently, K is finite if $K := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{\ell=1}^{\rho} \lambda_\ell \boldsymbol{\gamma}_\ell, \lambda_\ell \geq 0, \ell = 1, \ldots, \rho\}$, where $\{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_\rho\}$ is a finite set of generators of K and $\boldsymbol{\gamma}_\ell \in \mathbb{R}^n, \ell = 1, \ldots, \rho$. Unless K is the trivial cone $\{\mathbf{0}\}$, it is assumed that $\mathbf{0}$ is not a generator (an assumption that is maintained throughout the dissertation). Similarly, K is polyhedral convex if $K := \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\mu}_1^T\mathbf{x} \leq 0, \ldots, \boldsymbol{\mu}_m^T\mathbf{x} \leq 0\}$, where $\boldsymbol{\mu}_j \in \mathbb{R}^n, j = 1, \ldots, m$, are the normals to the generating hyperplanes passing through the origin whose associated closed half-spaces form K. Here, we have intentionally chosen to define a polyhedral convex cone using nonpositive (instead of nonnegative) half-spaces for consistency.

The well-known *Minkowski-Weyl Theorem* relates finite cones and polyhedral convex cones.

Theorem 3.1.7. *[118, Theorem 4.7.2] A nonempty cone $K \subseteq \mathbb{R}^n$ is polyhedral convex if and only if it is finite.*

In view of the Minkowski-Weyl Theorem, every polyhedral convex cone has two representations: (i) *generator form* $K(\mathbf{G}^T) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{G}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$, where $\mathbf{G}^T = \begin{bmatrix}\boldsymbol{\gamma}_1 & \cdots & \boldsymbol{\gamma}_\rho\end{bmatrix} \in \mathbb{R}^{n\times\rho}$ and $\{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_\rho\}$ is a finite set of generators of the cone (namely, nonzero generators unless the cone is the origin), and (ii) *inequality form* $K_{\leqq}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M}\mathbf{x} \leqq \mathbf{0}\}$, where $\mathbf{M} \in \mathbb{R}^{m\times n}$ and the rows of $\mathbf{M}$ are the normals to the generating hyperplanes whose half-spaces form the cone. We may convert between each form using various algorithms; see, e.g., Dobler [33] for theoretical work, and SageMath's [132] `polyhedron base class` for a software implementation. As is clear in the inequality form representation of a polyhedral convex cone, $K_{\leqq}(\mathbf{M})$ (equivalently, $K(\mathbf{G}^T)$) is always nonempty since it contains (at least) $\mathbf{0}$ for all $\mathbf{M}$. Additionally, $K_{\leqq}(\mathbf{M})$ is always closed since it is the intersection of a finite number of closed half-spaces whose generating hyperplanes pass through the origin (recalling that an arbitrary intersection of closed sets is closed as in Theorem 2.4.8(iii)).

Example 3.1.8. Consider the finite (polyhedral convex) cone, which is illustrated in Figure 3.3, generated by the vectors $\boldsymbol{\gamma}_1 = \begin{bmatrix}2 & 3\end{bmatrix}^T$ and $\boldsymbol{\gamma}_2 = \begin{bmatrix}-1 & 2\end{bmatrix}^T$. The generator form of the cone is

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\right\},$$

while the inequality form of the cone is

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix}3 & -2\\ -2 & -1\end{bmatrix}\mathbf{x} \leqq \mathbf{0}\right\}.$$

Another characterization of a polyhedral convex cone is as a cone that is also a polyhedron. Since extreme points and extreme directions are important pieces of any polyhedron, we discuss these features with respect to polyhedral convex cones here.

Theorem 3.1.9. *[15, Theorem 4.12] The zero vector is an extreme point of the polyhedral convex cone $K \subseteq \mathbb{R}^n$ if and only if K does not contain a line, i.e., is pointed.*

In fact, since a polyhedral convex cone is formed by half-spaces whose generating hyperplanes pass through the origin, the origin is the only possible extreme point. As a result, if a polyhedral convex cone is not pointed, then it has no extreme points. On the other hand, from the definition of a cone, it is clear that a nontrivial polyhedral convex cone has at least one extreme direction (which is an element of the cone unlike with a general polyhedron).

Theorem 3.1.10. *[15, Definition 4.2(a)] A nonzero element $\mathbf{x}$ of the polyhedral convex cone $K \subseteq \mathbb{R}^n$ is an extreme direction of K if and only if there are $n-1$ constraints from $\{\boldsymbol{\mu}_1^T\mathbf{x} \leq 0, \ldots, \boldsymbol{\mu}_m^T\mathbf{x} \leq 0\}$ active at $\mathbf{x}$ whose corresponding vectors $\boldsymbol{\mu}_j$ are linearly independent.*

In view of Theorems 3.1.9 and 3.1.10, the Representation Theorem 2.4.33 for a polyhedral convex cone (when applicable) reduces to only a conic combination of the cone's extreme directions. Meanwhile, in view of the generator form of a polyhedral convex cone, the Decomposition Theorem 2.4.34 reduces to only a conic combination of the vectors $\mathbf{x}_{\pi+1}, \ldots, \mathbf{x}_{\pi+\sigma}$.

Although the polyhedrality of convex cones is not necessarily preserved under their union as convexity may be lost, it is in fact preserved under their intersection.

Proposition 3.1.11. *[118, p. 84] Let* $\mathbf{M}_1, \mathbf{M}_2 \in \mathbb{R}^{m\times n}$*. Then* $K_{\leqq}(\mathbf{M}_1) \cap K_{\leqq}(\mathbf{M}_2)$ *is a polyhedral convex cone.*

For a polyhedral convex cone, pointedness is determined by the matrix $\mathbf{M}$.

Theorem 3.1.12. *[74, Theorem 3.1] Let* $\mathbf{M} \in \mathbb{R}^{m\times n}$*. Then* $K_{\leqq}(\mathbf{M})$ *is pointed if and only if* $\operatorname{rank}(\mathbf{M}) = n$*.*

The interior of a polyhedral convex cone in inequality form (with no rows of $\mathbf{M}$ all zero) is clear (as $\leqq$ becomes $<$), but it is not as clear when the cone is in generator form. One might expect that $\boldsymbol{\lambda} \geqq \mathbf{0}$ would become $\boldsymbol{\lambda} > \mathbf{0}$, yet this is not the case in part because the interior may be empty. (In particular, since the product $\mathbf{G}^T\boldsymbol{\lambda}$ always produces a result, it is impossible for the set $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{G}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$ to be empty even though the interior might be.) Instead, this expected result of the interior of the cone in generator form is given by the relative interior.

Theorem 3.1.13. *[62, Theorem 2.3.37] The relative interior of the finite cone* $K(\mathbf{G}^T)$ *is given by* $\operatorname{rel\,int}(K(\mathbf{G}^T)) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{G}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$*.*

An important question (as we see in Sections 3.2) is whether or not this relative interior contains the origin. The answer is directly related to Stiemke's Theorem 2.6.3, as well as pointedness, and is stated in the following.

Theorem 3.1.14. *[62, Theorem 2.3.38] Consider the finite cone* $K(\mathbf{G}^T)$*. The following statements are equivalent:*

(i) $K(\mathbf{G}^T) = \operatorname{Lin}(K(\mathbf{G}^T))$*;*

(ii) $\mathbf{0} \in \operatorname{rel\,int}(K(\mathbf{G}^T))$*;*

(iii) there does not exist an $\mathbf{x} \in \mathbb{R}^n$ *such that* $\mathbf{G}\mathbf{x} \geq \mathbf{0}$*.*

Here, $\mathrm{Lin}(K(\mathbf{G}^T))$ denotes the *lineality space* (see Definition 2.1.28, Greer [62]) of $K(\mathbf{G}^T)$. Since $K(\mathbf{G}^T)$ is convex, we have that $\mathrm{Lin}(K(\mathbf{G}^T)) = K(\mathbf{G}^T) \cap (-K(\mathbf{G}^T))$ by Theorem 2.1.32, Greer [62]. With this in mind, (i) and (ii) reveal that if $\mathbf{0} \in \mathrm{rel\,int}(K(\mathbf{G}^T))$ and $K(\mathbf{G}^T) \neq \{\mathbf{0}\}$, then $K(\mathbf{G}^T)$ is not pointed. (Note that an equivalent definition to ours of pointedness provided on p. 213 in Hartley [65] is that a pointed cone K satisfies the property that $K \cap (-K) = \{\mathbf{0}\}$.) In addition, since $\mathrm{rel\,int}(K(\mathbf{G}^T) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{G}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$ by Theorem 3.1.13, (ii) and (iii) may be obtained directly by Stiemke's Theorem 2.6.3.

Example 3.1.15. As an example of a finite cone whose relative interior contains the origin, let $\boldsymbol{\gamma}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\boldsymbol{\gamma}_2 = \begin{bmatrix} -1 & -1 \end{bmatrix}^T$ be the columns of $\mathbf{G}^T$. The resulting finite cone $K(\mathbf{G}^T)$ is then the line with slope 1 passing through the origin. In this case, we intuitively consider the relative interior of $K(\mathbf{G}^T)$ as its interior in $\mathbb{R}$. Hence, $\mathrm{rel\,int}(K(\mathbf{G}^T)) = K(\mathbf{G}^T)$ so that $\mathbf{0}$ is clearly in the relative interior.

Note that similar examples can be constructed in higher dimensions by ensuring that the cone is of a lower dimension than the space and that the cone passes through the origin.

In order to relate the relative interior of a polyhedral convex (finite) cone to its interior, it is necessary to compute the cone's dimension (cf. Proposition 2.4.22(i)). The dimension (refer to Definition 2.4.16) of a convex cone may be regarded as the maximum number of linearly independent vectors contained in the cone (see p. 79, Panik [118]). As a result, the dimension of a finite cone (i.e., a polyhedral convex cone in generator form) is clearly related to the rank of the defining matrix.

Proposition 3.1.16. *[118, p. 86] The equality* $\dim(K(\mathbf{G}^T)) = \mathrm{rank}(\mathbf{G})$ *holds.*

If the polyhedral convex cone is instead given in inequality (rather than generator) form, then various software, including SageMath's `polyhedron base class`, can readily provide the dimension.

Two of the fundamental operations on (convex) cones, and ones that we exploit, are those of duality and polarity. Although Definition 3.1.17 (as well as subsequent results, e.g., Proposition 3.1.19) is presented in the context of cones, it may be given for general sets.

Definition 3.1.17. Let $K \subseteq \mathbb{R}^n$ be a cone. Then

(i) its *dual cone* (or negative polar) is the set $K^* := \{\bar{\mathbf{z}} \in \mathbb{R}^n : \mathbf{z}^T\bar{\mathbf{z}} \leq 0 \text{ for all } \mathbf{z} \in K\}$;

(ii) its *polar cone* (or positive polar) is the set $K^+ := \{\bar{\mathbf{z}} \in \mathbb{R}^n : \mathbf{z}^T\bar{\mathbf{z}} \geq 0 \text{ for all } \mathbf{z} \in K\}$;

(iii) its *strict polar cone* (or strict positive polar) is the set $K^{\mathrm{s}+} := \{\bar{\mathbf{z}} \in \mathbb{R}^n : \mathbf{z}^T\bar{\mathbf{z}} > 0 \text{ for all } \mathbf{z} \in K \setminus \{\mathbf{0}\}\}$.

The dual cone K^* of a cone K consists of all vectors making a non-acute angle ($\geq \pi/2$ or $\leq -\pi/2$) with every vector of K. Similarly, the polar cone K^+ of a cone K consists of all vectors making a non-obtuse angle ($\leq \pi/2$ or $\geq -\pi/2$) with every vector of K, while the strict polar consists of all vectors making an acute angle with every vector of K. Observe that the polar cone is the negative of the dual cone (or vice versa), which gives rise to the use of the alternative terminology positive and negative polar. The definitions, as well as Figure 3.3, make this fact clear.

Several pertinent results regarding the dual, polar, and strict polar cones are given in the following collection of results.

Theorem 3.1.18. *[63, Dubovitskii-Milyutin Theorem, Theorem 6.23] Let $K_1, \ldots, K_{\zeta-1} \subseteq \mathbb{R}^n$ be open convex cones and $K_\zeta \subseteq \mathbb{R}^n$ be a convex cone. Then $\bigcap_{\ell=1}^{\zeta} K_\ell = \emptyset$*

if and only if there exists a $\mathbf{z}_\ell \in K_\ell^*$ *for each* $\ell = 1, \dots, \zeta, \mathbf{z}_1, \dots, \mathbf{z}_\zeta$ *not all zero, such that* $\mathbf{z}_1 + \cdots + \mathbf{z}_\zeta = \mathbf{0}$.

Proposition 3.1.19. *[122, Proposition 2.1.5] Let* $K, K_1, K_2 \subseteq \mathbb{R}^n$ *be cones. Then*

(i) $K^+ = [\mathrm{cl}(K)]^+$;

(ii) K^+ *is a closed convex cone and* $K^{\mathrm{s}+}$ *is a convex cone;*

(iii) if K *is open,* $K^{\mathrm{s}+} \cup \{\mathbf{0}\} = K^+$;

(iv) $K_1 \subseteq K_2$ *implies* $K_2^+ \subseteq K_1^+$ *and* $K_2^{\mathrm{s}+} \subseteq K_1^{\mathrm{s}+}$.

Proposition 3.1.20. *[122, Proposition 2.1.6(i)] Let* $K_1, K_2 \subseteq \mathbb{R}^n$ *be nonempty cones. Then* $(K_1 \cup K_2)^+ = K_1^+ \cap K_2^+$.

Theorem 3.1.21. *[145, Theorem 2.1] Let* $K \subseteq \mathbb{R}^n$ *be a nonempty cone. Then*

(i) $\mathrm{int}(K^+) \neq \emptyset$ *if and only if* K *is acute;*

(ii) if K *is acute,* $\mathrm{int}(K^+) = [\mathrm{cl}(K)]^{\mathrm{s}+}$.

It is worth noting that the results of Propositions 3.1.19(i), (ii), (iv), and 3.1.20, as well as Theorem 3.1.21(i), apply similarly to the dual since it is simply the negative of the polar cone. When the given cone is polyhedral, we obtain more specific results on duality and polarity as in the following two propositions.

Proposition 3.1.22. *[118, p. 88] The dual of the finite cone* $K(\mathbf{G}^T)$ *is* $[K(\mathbf{G}^T)]^* = K_{\leqq}(\mathbf{G})$.

Proposition 3.1.23. *[122, Proposition 2.1.13] The polar of the polyhedral convex cone* $K_{\leqq}(\mathbf{M})$ *is* $[K_{\leqq}(\mathbf{M})]^+ = K(-\mathbf{M}^T)$.

In particular, this means that the dual and polar of a polyhedral convex cone are also polyhedral convex cones. Note also that even though we state Proposition 3.1.22 (similarly for Proposition 3.1.23) beginning with the cone in generator form and obtain the dual in inequality form, we may present it conversely as in the following.

Proposition 3.1.24. *(i) [118, p. 89] The dual of the polyhedral convex cone $K_{\leqq}(\mathbf{M})$ is $[K_{\leqq}(\mathbf{M})]^* = K(\mathbf{M}^T)$.*

(ii) [118, p. 90] The polar of the finite cone $K(\mathbf{G}^T)$ is $[K(\mathbf{G}^T)]^+ = K_{\leqq}(-\mathbf{G})$.

Proposition 3.1.24 is clear based on the Duality Theorem for Finite Cones (refer to Theorem 4.2.1, Panik [118]), which states that the dual of the dual (equivalently, the polar of the polar) of a polyhedral convex (finite) cone is the original cone. For example, $[K(\mathbf{G}^T)]^* = K_{\leqq}(\mathbf{G})$ gives $[K(\mathbf{G}^T)]^{**} = [K_{\leqq}(\mathbf{G})]^*$, i.e., $K(\mathbf{G}^T) = [K_{\leqq}(\mathbf{G})]^*$ as desired.

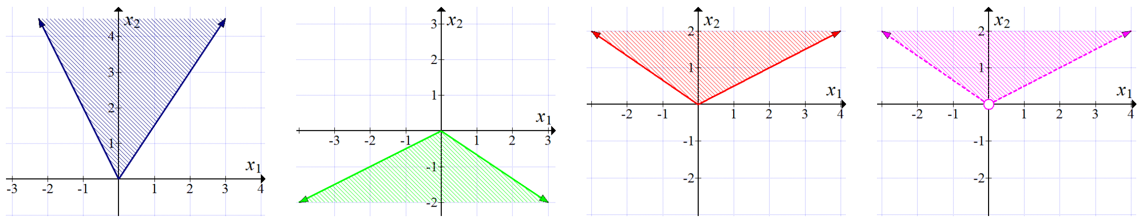
Example 3.1.25. Consider the polyhedral convex cone of Example 3.1.8. Its dual and polar cones are given by

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \mathbf{x} \leqq \mathbf{0}\right\} \text{ and } \left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = -\begin{bmatrix} 3 & -2 \\ -2 & -1 \end{bmatrix} \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\right\},$$

respectively. The dual and polar cones are shown in Figure 3.3, along with the strict polar cone (whose closed form representation we derive in Section 3.2).

Another operation on (convex) cones that we study is that of *translation*, which is denoted with the Minkowski sum. Although the following definition is presented in the context of cones, it may be given for general sets as well.

Definition 3.1.26. For $\mathbf{z}_0 \in \mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ a nonempty cone, $K \oplus \{\mathbf{z}_0\} := \{\mathbf{z} + \mathbf{z}_0 : \mathbf{z} \in K\}$ is the *translate* of K by the *translation* $\mathbf{z}_0$.



(a) The original cone (blue) (b) The dual cone (green) (c) The polar cone (red) (d) The strict polar cone (purple)

Figure 3.3: The polyhedral convex cone of Example 3.1.8, as well as its dual, polar, and strict polar cones

In the following section, we study the polar, strict polar, and translation operations on polyhedral convex cones and unions of polyhedral convex cones, as well as on two related convex cones and their associated unions, in more detail.

3.2 New Results

Although convex cones are well studied in convex analysis, the properties of three specific types of convex cones (polyhedral convex cones and two related convex cones) of interest and significance to multiobjective optimization have not been fully developed. In particular, the literature lacks descriptions of the polars, strict polars, and translations of these cones and of their respective unions, as well as the conditions under which the aforementioned polar and strict polar cones are expected to be nonempty. Here, the importance of studying unions of cones in collections associated with each of these convex cones rather than intersections is two-fold: (i) the union is not guaranteed to be convex while the intersection is, and (ii) the unions appear in our later research on multiobjective optimization problems with uncertainty (as in Chapter 6). Hence, the conditions that allow us to still demonstrate certain properties are attractive.

In Section 3.2.1, algebraic formulas for the translations of the three convex cones of interest and of their associated unions are presented and discussed. Properties and closed form representations of the polar and strict polar cones corresponding to the three convex cones and to their respective unions are then proposed in Sections 3.2.2 and 3.2.3. Since the acuteness of the cones examined in these two sections is often assumed, methods with which to verify this property are developed in Section 3.2.4.

Let $\mathbf{M}, \mathbf{M}_1, \ldots, \mathbf{M}_r$ be real $m \times n$ matrices. We study the polyhedral convex cone $K_{\leqq}(\mathbf{M})$ and two related cones given in inequality form:

$$K_{\leq}(\mathbf{M}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M}\mathbf{x} \leq \mathbf{0}\}, \text{ and } K_{<}(\mathbf{M}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M}\mathbf{x} < \mathbf{0}\},$$

as well as the unions of cones in collections associated with each of these three cones that are constructed using the matrices $\mathbf{M}_1, \ldots, \mathbf{M}_r$.

Observe that $K_{\leq}(\mathbf{M})$ and $K_{<}(\mathbf{M})$ are convex cones. It is clear (see Theorem 2.4.8(ii)) that $K_{<}(\mathbf{M})$ is open as $K_{<}(\mathbf{M})$ is the intersection of open half-spaces whose generating hyperplanes pass through the origin. On the other hand, $K_{\leq}(\mathbf{M})$ may be open, closed, or neither. When $m = 1$, we have that $K_{\leq}(\mathbf{M}) = K_{\leqq}(\mathbf{M})$ so that $K_{\leq}(\mathbf{M})$ is closed since $\leq$ and $\leqq$ coincide. When $m \geq 2$ and $\text{rank}(\mathbf{M}) = 1$, we have that $K_{\leq}(\mathbf{M}) = K_{<}(\mathbf{M})$, and $K_{\leq}(\mathbf{M})$ is thus open. When $m \geq 2$ and $\text{rank}(\mathbf{M}) = n$, the only vector $\mathbf{x}$ excluded by $\mathbf{M}\mathbf{x} \leq \mathbf{0}$ versus $\mathbf{M}\mathbf{x} \leqq \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$ (by Theorem 2.3, Bronson [22]). Hence, $K_{\leq}(\mathbf{M}) = K_{\leqq}(\mathbf{M}) \setminus \{\mathbf{0}\}$, i.e., $K_{\leq}(\mathbf{M})$ is the intersection of closed half-spaces whose generating hyperplanes pass through the origin, with the origin then removed, and is thus neither open nor closed (unless $K_{\leqq}(\mathbf{M}) = \{\mathbf{0}\}$, in which case $K_{\leq}(\mathbf{M}) = \emptyset$ and is thus both open and closed). Otherwise, when $m \geq 2$ and $1 < \text{rank}(\mathbf{M}) < n$, it may be that $K_{\leq}(\mathbf{M})$ is neither open nor closed.

Example 3.2.1. To illustrate the differences between the three cases when $p \geq 2$, consider the following examples.

(i) Let $\mathbf{M} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$. Hence, rank($\mathbf{M}$) = 1, and

$$K_{\leqq}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + 2x_2 = 0\}, K_{\leq}(\mathbf{M}) = K_{<}(\mathbf{M}) = \emptyset.$$

(ii) Let $\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Hence, rank($\mathbf{M}$) = 1, and

$$K_{\leqq}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^2 : x_1+2x_2 \leq 0\}, K_{\leq}(\mathbf{M}) = K_{<}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^2 : x_1+2x_2 < 0\}.$$

(iii) Let $\mathbf{M} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \\ 1 & 1 \end{bmatrix}$. Hence, rank($\mathbf{M}$) = 2 = n, and

$$\begin{aligned} K_{\leqq}(\mathbf{M}) &= \{\mathbf{x} \in \mathbb{R}^2 : 5x_1 + 2x_2 \leq 0, -x_1 + 3x_2 \leq 0, x_1 + x_2 \leq 0\}, \\ K_{\leq}(\mathbf{M}) &= \{\mathbf{x} \in \mathbb{R}^2 : 5x_1 + 2x_2 \leq 0, -x_1 + 3x_2 \leq 0, x_1 + x_2 \leq 0, \\ &\qquad \text{at least one strict}\}, \\ K_{<}(\mathbf{M}) &= \{\mathbf{x} \in \mathbb{R}^2 : 5x_1 + 2x_2 < 0, -x_1 + 3x_2 < 0, x_1 + x_2 < 0\}. \end{aligned}$$

Notice that $K_{\leq}(\mathbf{M})$ is $K_{\leqq}(\mathbf{M})$ with the origin removed, and is thus neither open nor closed.

(iv) Let $\mathbf{M} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$. Hence, $1 <$ rank($\mathbf{M}$) $= 2 < n$, and

$$\begin{aligned} K_{\leqq}(\mathbf{M}) &= \{\mathbf{x} \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0\}, \\ K_{\leq}(\mathbf{M}) &= \{\mathbf{x} \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, \text{ at least one strict}\}, \\ K_{<}(\mathbf{M}) &= \{\mathbf{x} \in \mathbb{R}^3 : x_1 > 0, x_2 > 0\}. \end{aligned}$$

Notice that $K_{\leq}(\mathbf{M})$ is $K_{\leqq}(\mathbf{M})$ with the x_3-axis (not just the origin) removed, and is thus neither open nor closed.

Moreover, note that unlike the polyhedral convex cone $K_{\leqq}(\mathbf{M})$ that may be represented in either inequality form or generator form, a generator form representation of the convex cones $K_{<}(\mathbf{M})$ and $K_{\leq}(\mathbf{M})$ may not be available. One obvious reason for this is that a cone represented in generator form is always nonempty, but both $K_{<}(\mathbf{M})$ and $K_{\leq}(\mathbf{M})$ may be empty, as in Example 3.2.1(i).

A second example illustrates this for the case when $m = 3$ and also shows that the polyhedral convex cone $K_{\leqq}(\mathbf{M})$ is always nonempty as it is at least the origin.

Example 3.2.2. Let $\mathbf{M} = \begin{bmatrix} 1 & 2 \\ -5 & -2 \\ 1 & -1 \end{bmatrix}$. Hence, $\text{rank}(\mathbf{M}) = 2 = n$, and

$$K_{\leqq}(\mathbf{M}) = \{\mathbf{0}\}, K_{\leq}(\mathbf{M}) = K_{<}(\mathbf{M}) = \emptyset.$$

As it is important to know the relationship between acuteness and pointedness in a general setting (cf. Proposition 3.1.3) in many of the proceeding results, we have the following proposition (which is based on a remark on p. 8, Sawaragi et al. [122]).

Proposition 3.2.3. *Let $K \subseteq \mathbb{R}^n$ be a cone. If K is acute, then it is also pointed.*

Proof. Let K be acute. By Definition 3.1.2(ii), there is an open half-space H generated by the hyperplane passing through the origin, $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = 0\}$, where $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{a} \neq \mathbf{0}$, such that $\text{cl}(K) \subseteq H \cup \{\mathbf{0}\}$. Without loss of generality, we have $H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} > 0\}$, and $\text{cl}(K) \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} > 0\} \cup \{\mathbf{0}\}$.

Now, assume for the sake of contradiction that K is not pointed. By Definition 3.1.2(i), there exists a $\mathbf{z} \in K, \mathbf{z} \neq \mathbf{0}$, such that $-\mathbf{z} \in K$. Since $\mathbf{z} \in K \subseteq \text{cl}(K)$ and $\mathbf{z} \neq \mathbf{0}$, we know that $\mathbf{a}^T\mathbf{z} > 0$. Similarly, since $-\mathbf{z} \in K \subseteq \text{cl}(K)$ and $-\mathbf{z} \neq \mathbf{0}$, we know that $\mathbf{a}^T(-\mathbf{z}) > 0$, which gives $\mathbf{a}^T\mathbf{z} < 0$, a contradiction. Hence, it must be that K is pointed as desired. □

Thus, it is apparent that acuteness is a stronger concept than pointedness.

3.2.1 Translations

We derive the algebraic representations of the translations of the convex cones $K_{\leqq}(\mathbf{M})$, $K_{\leq}(\mathbf{M})$, and $K_{<}(\mathbf{M})$. In the case of polyhedral convex cones in either generator or inequality form, we have the following result.

Proposition 3.2.4. *Let* $\mathbf{x}_0 \in \mathbb{R}^n$ *be given.*

(i) The finite cone $K(\mathbf{G}^T)$ *translated by the translate* $\mathbf{x}_0$ *is given by* $K(\mathbf{G}^T) \oplus \{\mathbf{x}_0\} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{G}^T\boldsymbol{\lambda} + \mathbf{x}_0, \boldsymbol{\lambda} \geqq \mathbf{0}\}$.

(ii) The polyhedral convex cone $K_{\leqq}(\mathbf{M})$ *translated by the translate* $\mathbf{x}_0$ *is given by* $K_{\leqq}(\mathbf{M}) \oplus \{\mathbf{x}_0\} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M}\mathbf{x} \leqq \mathbf{M}\mathbf{x}_0\}$.

Proof. (i) Since $K(\mathbf{G}^T) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{G}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$, the result follows from Definition 3.1.26.

(ii) Let $\bar{\mathbf{x}} \in K_{\leqq}(\mathbf{M}) \oplus \{\mathbf{x}_0\}$. Equivalently, $\bar{\mathbf{x}} - \mathbf{x}_0 \in K_{\leqq}(\mathbf{M})$, i.e., $\mathbf{M}(\bar{\mathbf{x}} - \mathbf{x}_0) \leqq \mathbf{0}$. Therefore, $\mathbf{M}\bar{\mathbf{x}} \leqq \mathbf{M}\mathbf{x}_0$, which gives the result. □

Proposition 3.2.5. *Let* $\mathbf{x}_0 \in \mathbb{R}^n$ *be given.*

(i) The convex cone $K_{\leq}(\mathbf{M})$ *translated by the translate* $\mathbf{x}_0$ *is given by* $K_{\leq}(\mathbf{M}) \oplus \{\mathbf{x}_0\} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M}\mathbf{x} \leq \mathbf{M}\mathbf{x}_0\}$.

(ii) The convex cone $K_{<}(\mathbf{M})$ *translated by the translate* $\mathbf{x}_0$ *is given by* $K_{<}(\mathbf{M}) \oplus \{\mathbf{x}_0\} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M}\mathbf{x} < \mathbf{M}\mathbf{x}_0\}$.

Proof. (i)-(ii) Follows similarly to the proof of Proposition 3.2.4(ii). □

Using the above propositions, we may also obtain clear formulas for translations of the unions of the three convex cones obtained by means of the matrices

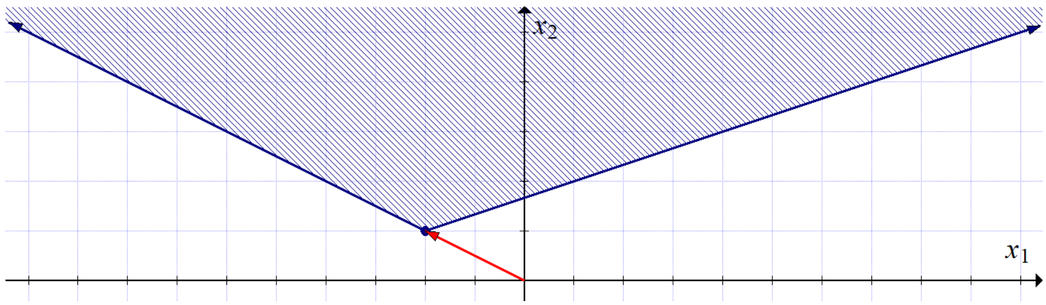


Figure 3.4: The translated cone (blue) by the translate $\mathbf{x}_0$ (red)

$M_\ell, \ell = 1, \ldots, r$. For example, we have that

$$\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell) \oplus \{\mathbf{x}_0\} = (K_{\leqq}(\mathbf{M}_1) \oplus \{\mathbf{x}_0\}) \cup \cdots \cup (K_{\leqq}(\mathbf{M}_r) \oplus \{\mathbf{x}_0\})$$

by Theorem 2.4.3(iii).

Example 3.2.6. Consider the following polyhedral convex cone given in generator and inequality form:

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\right\} = \left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} 1 & -3 \\ -1 & -2 \end{bmatrix} \mathbf{x} \leqq \mathbf{0}\right\}$$

and take the vector $\mathbf{x}_0 = (-2, 1)$. The translated cone by the translate $\mathbf{x}_0$, which is shown in Figure 3.4, may be written as

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \boldsymbol{\lambda} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \boldsymbol{\lambda} \geqq \mathbf{0}\right\}$$

if the cone is given in generator form, and

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} 1 & -3 \\ -1 & -2 \end{bmatrix} \mathbf{x} \leqq \begin{bmatrix} 1 & -3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$$

if the cone is given in inequality form.

Conceptually, Propositions 3.2.4 and 3.2.5 allow us to represent the action of translating the cones $K_{\leqq}(\mathbf{M}), K_{\leq}(\mathbf{M})$, and $K_{<}(\mathbf{M})$ around a region. In other words, the translated cone is the cone "attached" at the vector $\mathbf{x}_0$ (see Figure 3.4).

3.2.2 Polar Cones

Given the cones $K_{\leqq}(\mathbf{M})$, $K_{\leq}(\mathbf{M})$, and $K_{<}(\mathbf{M})$, we denote their polars (refer to Definition 3.1.17(ii)) by $K^{+}_{\leqq}(\mathbf{M})$, $K^{+}_{\leq}(\mathbf{M})$, and $K^{+}_{<}(\mathbf{M})$, respectively. We first derive the algebraic representation of the polars of the three convex cones of interest, and subsequently address their nonemptiness.

Proposition 3.2.7. *(i) The equality* $K^{+}_{\leqq}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{M}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ *holds.*

(ii) Let $\mathrm{cl}(K_{\leq}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$*. Then* $K^{+}_{\leq}(\mathbf{M}) = K^{+}_{\leqq}(\mathbf{M})$.

(iii) Let $\mathrm{cl}(K_{<}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$*. Then* $K^{+}_{<}(\mathbf{M}) = K^{+}_{\leqq}(\mathbf{M})$.

Proof. (i) Given by Proposition 3.1.23.

(ii)-(iii) Follow directly from Proposition 3.1.19(i). □

In view of the preceding proposition, several observations regarding the nonemptiness of the polars $K^{+}_{<}(\mathbf{M}), K^{+}_{\leq}(\mathbf{M})$, and $K^{+}_{\leqq}(\mathbf{M})$ are pertinent. First, it is clear that the polars are in fact nonempty since $K^{+}_{<}(\mathbf{M}) = K^{+}_{\leq}(\mathbf{M}) = K^{+}_{\leqq}(\mathbf{M}) = K(-\mathbf{M}^T)$ is a polyhedral convex cone (in generator form) and is therefore nonempty (as discussed previously). Second, since $K_{\leqq}(\mathbf{M}) \neq \emptyset$ (as it always contains $\mathbf{0}$) and the empty set is closed, the assumptions that $\mathrm{cl}(K_{\leq}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$ and $\mathrm{cl}(K_{<}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$ imply that $K_{\leq}(\mathbf{M})$ and $K_{<}(\mathbf{M})$ are nonempty as well. As a result, the above polars are thus nonempty since the polar of any nonempty set is at least the origin. Finally, the interior of each polar is nonempty (and thereby the polar itself is nonempty as well) when the cone is acute (cf. Theorem 3.1.21(i)).

We now consider collections associated with each of the three types of cones obtained by means of the matrices $\mathbf{M}_\ell, \ell = 1, \ldots, r$, and derive algebraic formulas for the polars of the unions of cones in each collection. Interestingly, the three polars have the same algebraic representation.

Proposition 3.2.8. *(i) The equality* $\left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^+ = \bigcap_{\ell=1}^r K_{\leqq}^+(\mathbf{M}_\ell)$ *holds.*

(ii) Let $\mathrm{cl}(K_{\leq}(\mathbf{M}_\ell)) = K_{\leqq}(\mathbf{M}_\ell)$ *for all* $\ell = 1, \ldots, r$. *Then* $[\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)]^+ = \left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^+$.

(iii) Let $\mathrm{cl}(K_{<}(\mathbf{M}_\ell)) = K_{\leqq}(\mathbf{M}_\ell)$ *for all* $\ell = 1, \ldots, r$. *Then* $[\bigcup_{\ell=1}^r K_{<}(\mathbf{M}_\ell)]^+ = \left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^+$.

Proof. (i) Follows directly from Proposition 3.1.20.

(ii)-(iii) Since $\mathrm{cl}(K_{\leq}(\mathbf{M}_\ell)) = K_{\leqq}(\mathbf{M}_\ell)$ and $\mathrm{cl}(K_{<}(\mathbf{M}_\ell)) = K_{\leqq}(\mathbf{M}_\ell)$ imply that $K_{\leq}(\mathbf{M}_\ell) \neq \emptyset$ and $K_{<}(\mathbf{M}_\ell) \neq \emptyset$ for all $\ell = 1, \ldots, r$, the result follows from Proposition 3.1.20, Propositions 3.2.7(ii) and (iii), and part (i), respectively. □

As in the previous discussion, these polars are always nonempty since the polar of any nonempty set is at least the origin, and the polars have nonempty interiors when the unions are acute (cf. Theorem 3.1.21(i)). Moreover, since each polar is the intersection of polyhedral convex cones (in generator form), existing algorithms may be used to compute the intersection and provide an algebraic representation of it (e.g., Hertel et al. [67] and SageMath's [132] `polyhedron base class`). In particular, since the intersection of polyhedral convex cones is still a polyhedral convex cone, each polar may be represented as in the following result.

Proposition 3.2.9. *(i) The polar* $\left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^+$ *is a polyhedral convex cone given by* $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ *for some suitable matrix* $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\widetilde{m}}$.

(ii) Let $\mathrm{cl}(K_{\leq}(\mathbf{M}_\ell)) = K_{\leqq}(\mathbf{M}_\ell)$ *for all* $\ell = 1, \ldots, r$. *Then* $[\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)]^+$ *is a polyhedral convex cone given by* $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ *for some suitable matrix* $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\widetilde{m}}$.

(iii) Let $\mathrm{cl}(K_{<}(\mathbf{M}_\ell)) = K_{\leqq}(\mathbf{M}_\ell)$ *for all* $\ell = 1, \ldots, r$. *Then* $[\bigcup_{\ell=1}^r K_{<}(\mathbf{M}_\ell)]^+$ *is a polyhedral convex cone given by* $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ *for some suitable matrix* $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\widetilde{m}}$.

Proof. (i) Since $\left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^+ = \bigcap_{\ell=1}^r K_{\leqq}^+(\mathbf{M}_\ell)$ by Proposition 3.2.8(i), and $K_{\leqq}^+(\mathbf{M}_\ell)$ is a polyhedral convex cone for each $\ell = 1, \ldots, r$, by Proposition 3.2.7(i), we conclude $\left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^+$ is also a polyhedral convex cone by Proposition 3.1.11. Therefore, by definition, we may express it in generator form for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\widetilde{m}}$.

(ii)-(iii) Follow from part (i) and Propositions 3.2.8(ii) and(iii), respectively. □

Remark 3.2.10. In each instance above, the phrase "for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\widetilde{m}}$" means "where the columns of $-\widetilde{\mathbf{M}}^T$ are a finite set of generators of $\left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^+$", a notion that is maintained throughout the remainder of the dissertation. Moreover, since the polar of any nonempty cone is always nonempty (as it is at least the origin), this matrix is guaranteed to exist. In order to compute $-\widetilde{\mathbf{M}}^T$, the intersection specified in Proposition 3.2.8 must be determined, which may be done (as previously mentioned) by using available software such as SageMath's `polyhedron base class`. The proceeding example demonstrates how to use SageMath along with Propositions 3.2.7 and 3.2.8 in a systematic procedure to generate $-\widetilde{\mathbf{M}}^T$. The corresponding SageMath code is also provided in Appendix B.

Example 3.2.11. Consider the two polyhedral convex (finite) cones $K_{\leqq}(\mathbf{M}_1)$ and $K_{\leqq}(\mathbf{M}_2)$ (in inequality form) given by

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{x} \leqq \mathbf{0}\right\} \text{ and } \left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} 4 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x} \leqq \mathbf{0}\right\},$$

respectively. By Proposition 3.2.7(i), the polars $K^+_{\leqq}(\mathbf{M}_1)$ and $K^+_{\leqq}(\mathbf{M}_2)$ are

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\right\} \text{ and } \left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \begin{bmatrix} -4 & 1 \\ -1 & 2 \end{bmatrix} \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\right\},$$

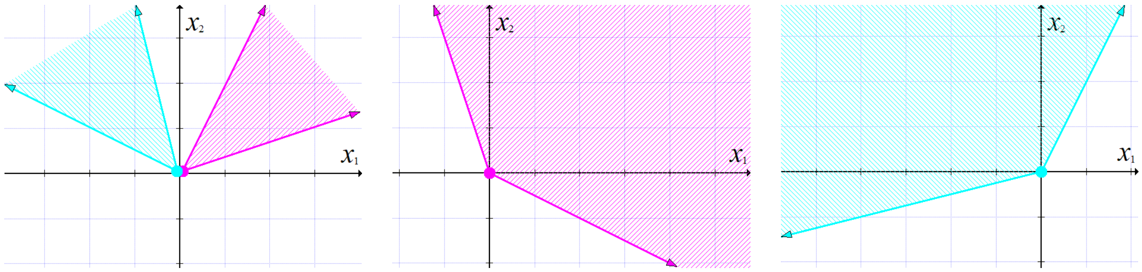
respectively. Applying Proposition 3.2.8(i) and utilizing SageMath's `polyhedron base class` functions `polyhedron.intersection()` to compute the intersection and `polyhedron.Vrepresentation()` to produce the generator form representation of the resulting polyhedral convex cone, the polar $\left[K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)\right]^+ = K^+_{\leqq}(\mathbf{M}_1) \cap K^+_{\leqq}(\mathbf{M}_2)$ is given by

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = -\widetilde{\mathbf{M}}^T \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\right\} = \left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\right\}. \tag{3.1}$$

Inspecting Figures 3.5b and 3.5c, the intersection of the two polar cones is clearly given by (3.1) since the generators of the resulting polyhedral convex cone, $\begin{bmatrix} 1 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 3 \end{bmatrix}^T$, are in fact the columns of $-\widetilde{\mathbf{M}}^T$ as expected.

3.2.3 Strict Polar Cones

Given the cones $K_{\leqq}(\mathbf{M}), K_{\leq}(\mathbf{M})$, and $K_{<}(\mathbf{M})$, we denote their strict polars by $K^{s+}_{\leqq}(\mathbf{M}), K^{s+}_{\leq}(\mathbf{M})$, and $K^{s+}_{<}(\mathbf{M})$, respectively. We explore the strict polars of the three convex cones of interest. In particular, we examine their nonemptiness and structure, as well as derive their algebraic representations, which interestingly do



(a) $K_{\leqq}(\mathbf{M}_1)$ (purple) and $K_{\leqq}(\mathbf{M}_2)$ (teal)   (b) $K^{+}_{\leqq}(\mathbf{M}_1)$ (purple)   (c) $K^{+}_{\leqq}(\mathbf{M}_2)$ (teal)

Figure 3.5: The polyhedral convex cones and their polars for Example 3.2.11

not follow the same pattern as for their polar cones (cf. Proposition 3.2.7). We first address the nonemptiness of the strict polar cones.

Proposition 3.2.12. *(i) Let $K_{\leqq}(\mathbf{M})$ be acute. Then $K^{s+}_{\leqq}(\mathbf{M}) \neq \emptyset$.*

(ii) Let $K_{\leq}(\mathbf{M})$ be nonempty and acute. Then $K^{s+}_{\leq}(\mathbf{M}) \neq \emptyset$.

(iii) Let $K_{<}(\mathbf{M})$ be nonempty and acute. Then $K^{s+}_{<}(\mathbf{M}) \neq \emptyset$.

Proof. (i) Since $K_{\leqq}(\mathbf{M})$ is nonempty, acute, and closed, Theorems 3.1.21(i) and (ii) yield $\emptyset \neq \text{int}(K^{+}_{\leqq}(\mathbf{M})) = K^{s+}_{\leqq}(\mathbf{M})$.

(ii) As $K_{\leq}(\mathbf{M})$ is nonempty and acute, Theorems 3.1.21(i) and (ii) yield $\emptyset \neq \text{int}(K^{+}_{\leq}(\mathbf{M})) = [\text{cl}(K_{\leq}(\mathbf{M}))]^{s+}$. Also, since $K_{\leq}(\mathbf{M}) \subseteq \text{cl}(K_{\leq}(\mathbf{M}))$, Proposition 3.1.19(iv) yields $[\text{cl}(K_{\leq}(\mathbf{M}))]^{s+} \subseteq K^{s+}_{\leq}(\mathbf{M})$. As $[\text{cl}(K_{\leq}(\mathbf{M}))]^{s+}$ is nonempty, this containment implies that $K^{s+}_{\leq}(\mathbf{M})$ is nonempty.

(iii) Follows similarly to the proof of part (ii). □

Having established nonemptiness, we now derive the algebraic formulas for the three convex cones of interest.

Theorem 3.2.13. *(i) Let $K_{\leqq}(\mathbf{M})$ be acute. Then $K^{s+}_{\leqq}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{M}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$.*

(ii) Let $K_{\leqq}(\mathbf{M})$ be acute. Then $K_{\leq}^{\mathrm{s}+}(\mathbf{M}) = K_{\leqq}^{\mathrm{s}+}(\mathbf{M})$.

(iii) Let $\mathrm{cl}(K_{<}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$. Then $K_{<}^{\mathrm{s}+}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{M}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}$.

Proof. (i) Since $K_{\leqq}(\mathbf{M})$ is closed, nonempty, and acute, we have $K_{\leqq}^{\mathrm{s}+}(\mathbf{M}) = \mathrm{int}(K_{\leqq}^{+}(\mathbf{M})) \neq \emptyset$ by Theorems 3.1.21(i) and (ii). As a result, the interior and relative interior coincide by Proposition 2.4.22(ii). Hence, Proposition 3.2.7(i) and Theorem 3.1.13 give the result.

(ii) Note that for $m = 1$ we have $K_{\leq}(\mathbf{M}) = K_{\leqq}(\mathbf{M})$ since $\leqq$ and $\leq$ coincide, so the result is immediate. Otherwise, $m \geq 2$. By assumption, $K_{\leqq}(\mathbf{M})$ is acute, which implies that $K_{\leqq}(\mathbf{M})$ is pointed by Proposition 3.2.3. As a result, $\mathrm{rank}(\mathbf{M}) = n$ by Theorem 3.1.12, which implies that $\mathbf{x} = \mathbf{0}$ is the only solution to $\mathbf{M}\mathbf{x} = \mathbf{0}$ by Theorem 2.3.4. Hence, the only vector $\mathbf{x}$ excluded by $\mathbf{M}\mathbf{x} \leq \mathbf{0}$ versus $\mathbf{M}\mathbf{x} \leqq \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, i.e., $K_{\leq}(\mathbf{M}) = K_{\leqq}(\mathbf{M}) \setminus \{\mathbf{0}\}$. Thus, by definition, we have that $K_{\leqq}^{\mathrm{s}+}(\mathbf{M}) = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{z} > 0 \text{ for all } \mathbf{x} \in K_{\leqq}(\mathbf{M}) \setminus \{\mathbf{0}\}\}$, which is equal to $\{\mathbf{z} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{z} > 0 \text{ for all } \mathbf{x} \in K_{\leq}(\mathbf{M})\}$. Since $\mathbf{0} \notin K_{\leq}(\mathbf{M})$, we obtain $K_{\leq}^{\mathrm{s}+}(\mathbf{M}) = K_{\leqq}^{\mathrm{s}+}(\mathbf{M})$.

(iii) Since $K_{<}(\mathbf{M})$ is open, we know by Proposition 3.1.19(iii) that $K_{<}^{\mathrm{s}+}(\mathbf{M}) \cup \{\mathbf{0}\} = K_{<}^{+}(\mathbf{M})$, which implies that $K_{<}^{\mathrm{s}+}(\mathbf{M}) = K_{<}^{+}(\mathbf{M}) \setminus \{\mathbf{0}\}$. As $K_{<}^{+}(\mathbf{M}) = K_{\leqq}^{+}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{M}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ by Propositions 3.2.7(iii) and (i), respectively, and $\boldsymbol{\lambda} = \mathbf{0}$ forces $\mathbf{x} = \mathbf{0}$, we obtain $K_{<}^{\mathrm{s}+}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0} : \mathbf{x} = -\mathbf{M}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}$. Since $K_{<}(\mathbf{M}) \neq \emptyset$ (which is implied by the assumption that $\mathrm{cl}(K_{<}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$), we that $\mathbf{M}\mathbf{x} < \mathbf{0}$ has a solution. Equivalently, by Gordan's Theorem 2.6.2, the system $-\mathbf{M}^T\boldsymbol{\lambda} = \mathbf{0}, \boldsymbol{\lambda} \geq \mathbf{0}$ has no solution, which gives the result. □

Remark 3.2.14. Observe that the strict polars of $K_{\leqq}(\mathbf{M})$, $K_{\leq}(\mathbf{M})$, and $K_{<}(\mathbf{M})$ derived in Theorem 3.2.13 are all clearly nonempty. As a result, we obtain that $K_{\leq}^{\mathrm{s}+}(\mathbf{M})$

and $K^{\mathrm{s}+}_{<}(\mathbf{M})$ are both nonempty under less restrictive assumptions than those of Propositions 3.2.12(ii) and (iii) since, for example, we only need to assume that $K_{\leq}(\mathbf{M})$ is acute instead of both acute and nonempty.

The strict polar cones are always convex (see Proposition 3.1.19(ii)), but may be open or neither open nor closed.

Proposition 3.2.15. *(i) Let $K_{\leqq}(\mathbf{M})$ be acute. Then $K^{\mathrm{s}+}_{\leqq}(\mathbf{M})$ is an open convex cone.*

(ii) Let $K_{\leqq}(\mathbf{M})$ be acute. Then $K^{\mathrm{s}+}_{\leq}(\mathbf{M})$ is an open convex cone.

(iii) Let $K_{<}(\mathbf{M})$ be acute and $\mathrm{cl}(K_{<}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$. Then $K^{\mathrm{s}+}_{<}(\mathbf{M})$ is a convex cone that is neither open nor closed.

Proof. (i) Since the strict polar of a cone is convex by Proposition 3.1.19(ii), $K^{\mathrm{s}+}_{\leqq}(\mathbf{M}) = \mathrm{int}(K^{+}_{\leqq}(\mathbf{M}))$ by Theorem 3.1.21(ii), and the interior of a set is open, the result follows.

(ii) Part (i) and Theorem 3.2.13(ii) yield the result.

(iii) Convexity holds as in part (i). Since $\mathrm{cl}(K_{<}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$, Proposition 3.2.7(iii) and Proposition 3.1.19(iii) yield $K^{\mathrm{s}+}_{<}(\mathbf{M}) = K^{+}_{\leqq}(\mathbf{M}) \setminus \{\mathbf{0}\}$. Moreover, since $K^{\mathrm{s}+}_{<}(\mathbf{M}) \neq \emptyset$ by Proposition 3.2.12 (where $K_{<}(\mathbf{M}) \neq \emptyset$ by the closure assumption and $K_{<}(\mathbf{M})$ is assumed to be acute) and $K^{+}_{\leqq}(\mathbf{M})$ is closed (as it is a polyhedral convex cone), it is clear that $K^{\mathrm{s}+}_{<}(\mathbf{M})$ is neither open nor closed. □

In the last part of this section, we characterize the strict polars of the unions obtained from the same three collections as in Section 3.2.2. We first address their nonemptiness.

Theorem 3.2.16. *(i) Let $\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_{\ell})$ be acute. Then $\left[\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_{\ell})\right]^{\mathrm{s}+} \neq \emptyset$.*

(ii) Let $\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)$ *be nonempty and acute. Then* $\left[\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)\right]^{\mathrm{s}+} \neq \emptyset$*.*

(iii) Let $\bigcup_{\ell=1}^r K_{<}(\mathbf{M}_\ell)$ *be nonempty and acute. Then* $\left[\bigcup_{\ell=1}^r K_{<}(\mathbf{M}_\ell)\right]^{\mathrm{s}+} \neq \emptyset$*.*

Proof. (i) Since $\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)$ is nonempty and acute, we have

$$\emptyset \neq \operatorname{int}\left(\left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^+\right) = \left[\operatorname{cl}\left(\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right)\right]^{\mathrm{s}+}$$

by Theorems 3.1.21(i) and (ii), respectively. Hence, as $K_{\leqq}(\mathbf{M}_\ell)$ is closed for each $\ell = 1, \dots, r$, and the union of closed sets is closed, the result follows.

(ii) Since $\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)$ is nonempty and acute, we have

$$\emptyset \neq \operatorname{int}\left(\left[\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)\right]^+\right) = \left[\operatorname{cl}\left(\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)\right)\right]^{\mathrm{s}+} \tag{3.2}$$

by Theorems 3.1.21(i) and (ii), respectively. Also, since $\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell) \subseteq \operatorname{cl}\left(\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)\right)$, Proposition 3.1.19(iv) yields

$$\left[\operatorname{cl}\left(\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)\right)\right]^{\mathrm{s}+} \subseteq \left[\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)\right]^{\mathrm{s}+}.$$

Hence, the result follows from (3.2).

(iii) Follows similarly to the proof of part (ii). □

Note in Theorem 3.2.16 that it is not enough to assume that each cone $K_{\leqq}(\mathbf{M}_\ell), \ell = 1, \dots, r$, (respectively, $K_{\leq}(\mathbf{M}_\ell)$ or $K_{<}(\mathbf{M}_\ell)$) is acute in order to guarantee that the union is also acute. For example, let $K_{\leqq}(\mathbf{M}_1) = \mathbb{R}^2_{\geqq}$ (the first quadrant) and $K_{\leqq}(\mathbf{M}_2) = \mathbb{R}^2_{\leqq}$ (the third quadrant). Clearly, the union of these two cones is not acute. As a result, it is necessary to make the stronger assumption that the union is acute explicitly. On the other hand, the assumption that the union is nonempty may be relaxed since we only need $K_{\leqq}(\mathbf{M}_\ell)$ (respectively, $K_{\leq}(\mathbf{M}_\ell)$ or $K_{<}(\mathbf{M}_\ell)$) to be nonempty for one $\ell \in \{1, \ldots, r\}$ in order to guarantee that the union is nonempty.

The following lemma relates the intersection of the strict polars of two cones to the strict polar of the union of the two cones (cf. Proposition 3.1.20) and is needed in determining the behavior of the strict polars of the unions of cones in collections associated with each of the three convex cones of interest (cf. Proposition 3.2.8).

Lemma 3.2.17. *Let $K_1, K_2 \subseteq \mathbb{R}^n$ be nonempty cones. Then $(K_1 \cup K_2)^{\mathrm{s}+} = K_1^{\mathrm{s}+} \cap K_2^{\mathrm{s}+}$.*

Proof. Let $\bar{\mathbf{z}} \in K_1^{\mathrm{s}+} \cap K_2^{\mathrm{s}+}$, or equivalently, $\bar{\mathbf{z}} \in K_1^{\mathrm{s}+}$ and $\bar{\mathbf{z}} \in K_2^{\mathrm{s}+}$. By definition, $\mathbf{z}^T\bar{\mathbf{z}} > 0$ for any $\mathbf{z} \in K_1 \setminus \{\mathbf{0}\}$ and $\mathbf{z}^T\bar{\mathbf{z}} > 0$ for any $\mathbf{z} \in K_2 \setminus \{\mathbf{0}\}$. Equivalently, $\mathbf{z}^T\bar{\mathbf{z}} > 0$ for any $\mathbf{z} \in (K_1 \cup K_2) \setminus \{\mathbf{0}\}$, i.e., $\bar{\mathbf{z}} \in (K_1 \cup K_2)^{\mathrm{s}+}$ as desired. □

Using Lemma 3.2.17, and under different assumptions than Proposition 3.2.8, we identify the behavior of the strict polars of the aforementioned unions.

Proposition 3.2.18. *(i) The equality $\left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^{\mathrm{s}+} = \bigcap_{\ell=1}^r K_{\leqq}^{\mathrm{s}+}(\mathbf{M}_\ell)$ holds.*

(ii) Let $K_{\leq}(\mathbf{M}_\ell) \neq \emptyset$ for all $\ell = 1, \ldots, r$. Then $[\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)]^{\mathrm{s}+} = \bigcap_{\ell=1}^r K_{\leq}^{\mathrm{s}+}(\mathbf{M}_\ell)$.

(iii) Let $K_{<}(\mathbf{M}_\ell) \neq \emptyset$ for all $\ell = 1, \ldots, r$. Then $[\bigcup_{\ell=1}^r K_{<}(\mathbf{M}_\ell)]^{\mathrm{s}+} = \bigcap_{\ell=1}^r K_{<}^{\mathrm{s}+}(\mathbf{M}_\ell)$.

Proof. (i)-(iii) Follow immediately from Lemma 3.2.17. □

Observe that if we make the additional assumption in part (ii) that $K_{\leq}(\mathbf{M}_\ell)$ is acute for all $\ell = 1, \ldots, r$, then we may use Theorem 3.2.13(ii) to obtain $[\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)]^{\mathrm{s}+} = \left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^{\mathrm{s}+}$.

With the previous result and Theorem 3.1.18 in mind, we may give a second condition for the nonemptiness of the strict polars of $\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)$ and $\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)$.

We may only provide this condition for two of the unions since only $K^{s+}_{\leqq}(\mathbf{M}_\ell)$ and $K^{s+}_{\leq}(\mathbf{M}_\ell)$ are guaranteed to be open (see Proposition 3.2.15).

Proposition 3.2.19. *(i) Let $K_{\leqq}(\mathbf{M}_\ell)$ be acute for all $\ell = 1, \ldots, r$. Then $\left[\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)\right]^{s+} = \emptyset$ if and only if there exists a $\mathbf{z}_\ell \in \left[K^{s+}_{\leqq}(\mathbf{M}_\ell)\right]^{*}$ for each $\ell = 1, \ldots, r, \mathbf{z}_1, \ldots, \mathbf{z}_r$ not all zero, such that $\mathbf{z}_1 + \cdots + \mathbf{z}_r = \mathbf{0}$;*

(ii) Let $K_{\leq}(\mathbf{M}_\ell)$ be nonempty and acute for all $\ell = 1, \ldots, r$. Then $[\bigcup_{\ell=1}^{r} K_{\leq}(\mathbf{M}_\ell)]^{s+} = \emptyset$ if and only if there exists a $\mathbf{z}_\ell \in \left[K^{s+}_{\leq}(\mathbf{M}_\ell)\right]^{*}$ for each $\ell = 1, \ldots, r, \mathbf{z}_1, \ldots, \mathbf{z}_r$ not all zero, such that $\mathbf{z}_1 + \cdots + \mathbf{z}_r = \mathbf{0}$.

Proof. (i) By Proposition 3.2.18(i), we have $\left[\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)\right]^{s+} = \bigcap_{\ell=1}^{r} K^{s+}_{\leqq}(\mathbf{M}_\ell)$. Moreover, we know that $K^{s+}_{\leqq}(\mathbf{M}_\ell)$ is an open convex cone for each $\ell = 1, \ldots, r$, by Proposition 3.2.15(i). Hence, Theorem 3.1.18 yields the result.

(ii) Follows similarly to the proof of (i). □

Similarly to Propositions 3.2.8 and 3.2.9, we may extend Proposition 3.2.18 to an explicit formula as in the following result.

Proposition 3.2.20. *(i) Let $\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)$ be acute. Then $\left[\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)\right]^{s+} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$ for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\widetilde{m}}$.*

(ii) Let $\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)$ be acute. Then $[\bigcup_{\ell=1}^{r} K_{\leq}(\mathbf{M}_\ell)]^{s+} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$ for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\widetilde{m}}$.

(iii) Let $\mathrm{cl}(K_{<}(\mathbf{M}_\ell)) = K_{\leqq}(\mathbf{M}_\ell)$ for all $\ell = 1, \ldots, r$. Then $[\bigcup_{\ell=1}^{r} K_{<}(\mathbf{M}_\ell)]^{s+} = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0} : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}$ for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\widetilde{m}}$. Moreover, if $K_{<}(\widetilde{\mathbf{M}}) \neq \emptyset$, then $[\bigcup_{\ell=1}^{r} K_{<}(\mathbf{M}_\ell)]^{s+} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}$.

Proof. (i) Since $\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)$ is nonempty, acute, and closed (as a finite union of closed sets is closed), Theorem 3.1.21 and Proposition 3.2.9(i) yield

$$\left[\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)\right]^{s+} = \operatorname{int}\left(\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}\right) \neq \emptyset \qquad (3.3)$$

for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\widetilde{m}}$. Thus, (3.3) and Theorem 3.1.13 yield the result.

(ii) Since $\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)$ is acute, we have that $K_{\leqq}(\mathbf{M}_\ell)$ is acute as well for all $\ell = 1,\dots,r$. Hence, we obtain $\bigcup_{\ell=1}^{r} K_{\leq}(\mathbf{M}_\ell) = \left[\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)\right]\setminus\{\mathbf{0}\}$ and the desired result as in Theorem 3.2.13(ii).

(iii) Since $K_{<}(\mathbf{M}_\ell)$ is open for each $\ell = 1,\dots,r$, and an arbitrary union of open sets is open, we know by Proposition 3.1.19(iii) that

$$[\bigcup_{\ell=1}^{r} K_{<}(\mathbf{M}_\ell)]^{s+} = [\bigcup_{\ell=1}^{r} K_{<}(\mathbf{M}_\ell)]^{+} \setminus \{\mathbf{0}\}.$$

Hence, Proposition 3.2.9(iii) yields

$$[\bigcup_{\ell=1}^{r} K_{<}(\mathbf{M}_\ell)]^{s+} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\} \setminus \{\mathbf{0}\}$$

for some matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\widetilde{m}}$. Since $\boldsymbol{\lambda} = \mathbf{0}$ forces $\mathbf{x} = \mathbf{0}$, the first part of the result follows.

Now, for the second part of the result, let $K_{<}(\widetilde{\mathbf{M}}) \neq \emptyset$. Thus, Gordan's Theorem 2.6.2 yields the result. □

As in Remark 3.2.10 and Example 3.2.11, the suitable matrix $\widetilde{\mathbf{M}}^T$ may be computed using existing software such as SageMath's `polyhedron base class`. In fact, as evidenced in each of the above proofs, the matrix $\widetilde{\mathbf{M}}^T$ in Proposition 3.2.20 is simply the matrix obtained in computing the corresponding polar in Proposition 3.2.8 and may thus be determined in the same manner as previously shown. For instance, since the the union $K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)$ in Example 3.2.11 is acute, the suitable matrix $\widetilde{\mathbf{M}}^T$ in $\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\} = \left[K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)\right]^{s+}$ from Proposition 3.2.20(ii) is as given by (3.1).

In addition, Proposition 3.2.20 reveals two important observations regarding the nonemptiness of the strict polars $\left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^{s+}$, $[\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)]^{s+}$, and $[\bigcup_{\ell=1}^r K_{<}(\mathbf{M}_\ell)]^{s+}$. First, as with Theorem 3.2.13 and discussed in Remark 3.2.14, these strict polars are clearly nonempty (excluding the representation obtained in the first part of (iii)). Second, as a result, the assumptions of Proposition 3.2.20 provide different (and possibly less restrictive) conditions in comparison to the earlier results of Theorem 3.2.16 and Proposition 3.2.19 under which the strict polars are nonempty. For example, the assumptions of Proposition 3.2.20(ii) are certainly less restrictive than those of Theorem 3.2.16(ii).

The final result of this section shows that the strict polar cones of these unions follow the behavior (under certain assumptions) of the strict polar cones of the three individual cones (see Proposition 3.2.15) in terms of convexity and openness.

Proposition 3.2.21. *(i) Let $\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)$ be acute. Then $\left[\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)\right]^{s+}$ is an open convex cone.*

(ii) Let $\bigcup_{\ell=1}^r K_{\leqq}(\mathbf{M}_\ell)$ be acute. Then $[\bigcup_{\ell=1}^r K_{\leq}(\mathbf{M}_\ell)]^{s+}$ is an open convex cone.

(iii) Let $\mathrm{cl}(K_{<}(\mathbf{M}_\ell)) = K_{\leqq}(\mathbf{M}_\ell)$ for all $\ell = 1, \ldots, r$, and $\bigcup_{\ell=1}^r K_{<}(\mathbf{M}_\ell)$ be acute. Then $[\bigcup_{\ell=1}^r K_{<}(\mathbf{M}_\ell)]^{s+}$ is a convex cone that is neither open nor closed.

Proof. (i) Since the strict polar is convex by Proposition 3.1.19(ii), $K_{\leqq}(\mathbf{M}_\ell)$ is closed for all $\ell = 1, \ldots, r$, and the interior of a set is open, Theorem 3.1.21(ii) yields the result.

(ii) Follows from part (i) and Proposition 3.2.20(ii).

(iii) Convexity holds as in part (i). Since $[\bigcup_{\ell=1}^{r} K_{<}(\mathbf{M}_\ell)]^{+} = \left[\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)\right]^{+}$ by Proposition 3.2.8(iii), we have that $[\bigcup_{\ell=1}^{r} K_{<}(\mathbf{M}_\ell)]^{s+} = \left[\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)\right]^{+} \setminus \{\mathbf{0}\}$ by Proposition 3.1.19(iii). Moreover, since $[\bigcup_{\ell=1}^{r} K_{<}(\mathbf{M}_\ell)]^{s+} \neq \emptyset$ by Theorem 3.2.16(iii) and $\left[\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)\right]^{+}$ is closed by Proposition 3.1.19(ii), the result follows. □

In the result directly above, the assumption of acuteness is key (at least in parts (i) and (ii)). Moreover, several results throughout this section have also relied on the assumption that the given cone is acute including Proposition 3.2.12, Theorem 3.2.13, and Proposition 3.2.20. Hence, it is clearly important to be able to identify this property, specifically for the (closed) polyhedral convex cone $K_{\leqq}(\mathbf{M})$, as well as for $\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)$.

3.2.4 Acuteness Recognition

Although algorithms are available to recognize polyhedrality (see Bemporad et al. [4]), such methods have not been presented in the literature for recognizing the acuteness of a cone. It is worth noting that an acute cone need not be polyhedral (as it may not even be convex), and a polyhedral convex cone need not be acute. Refer to Figures 3.1a and 3.1d, respectively, for an example of each situation. Hence, recognizing acuteness is a much different task than recognizing polyhedrality, which we address in this section. We specifically examine an acuteness recognition method for the (closed) polyhedral convex cone $K_{\leqq}(\mathbf{M})$, as well as for $\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)$, that relies upon the polar cone.

The polyhedral convex cone $K_{\leqq}(\mathbf{M})$ may be written in either of two forms, inequality or generator. When $K_{\leqq}(\mathbf{M})$ is given in inequality form, the algebraic formula of $K_{\leqq}^{+}(\mathbf{M})$ is explicitly given by Proposition 3.2.7(i). On the other hand, when $K_{\leqq}(\mathbf{M})$ is given in generator form, say $K(\mathbf{G}^T)$, where $\mathbf{G}^T$ is an $n \times \rho$ matrix whose columns are a finite set of generators and are nonzero unless $K_{\leqq}(\mathbf{M}) = \{\mathbf{0}\}$ (refer to pp. 54–55), we obtain a different algebraic representation as in Proposition 3.1.24(ii). Specifically, the polar of the polyhedral convex cone in generator form is given by

$$[K(\mathbf{G}^T)]^{+} = K_{\leqq}(-\mathbf{G}) = \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{G}\mathbf{x} \leqq \mathbf{0}\}. \tag{3.4}$$

With this in mind, we have the following method for recognizing the acuteness of (nontrivial) $K_{\leqq}(\mathbf{M})$. (Note that we do not need to consider the acuteness of $K_{\leqq}(\mathbf{M}) = \{\mathbf{0}\}$ as it is obviously acute and that it is explicitly known if $K_{\leqq}(\mathbf{M}) = \{\mathbf{0}\}$ based on whether or not $\mathbf{0}$ is a generator.)

Theorem 3.2.22. *Let $K_{\leqq}(\mathbf{M}) \neq \{\mathbf{0}\}$ be given in generator form. Then $K_{\leqq}(\mathbf{M})$ is acute if and only if the system*

$$-\mathbf{G}\mathbf{x} < \mathbf{0} \tag{3.5}$$

is consistent.

Proof. Since $K_{\leqq}(\mathbf{M})$ is nonempty, we know that $K_{\leqq}(\mathbf{M})$ is acute if and only if $\mathrm{int}(K_{\leqq}^{+}(\mathbf{M})) \neq \emptyset$ by Theorem 3.1.21(i). As $\mathbf{G}$ has no rows that are all zero, the interior of $\{\mathbf{x} \in \mathbb{R}^n : -\mathbf{G}\mathbf{x} \leqq \mathbf{0}\}$ is $\{\mathbf{x} \in \mathbb{R}^n : -\mathbf{G}\mathbf{x} < \mathbf{0}\}$ so that the result follows from (3.4). □

As a recognition method for acuteness, Theorem 3.2.22 allows one to equivalently verify that (3.5) has a feasible solution computationally, which may be done using a variety of software. A natural question to ask now is: Do we have a similar system if $K_{\leqq}(\mathbf{M})$ is given in inequality form? The answer is no since we do not, in general, have an algebraic formula for the interior of a polyhedral convex (finite) cone in generator form. That being said, we do have an explicit representation of the relative interior as in Theorem 3.1.13. Nevertheless, we have the following (more general) theorem.

Theorem 3.2.23. *If* $\dim(K^{+}_{\leqq}(\mathbf{M})) = n$*, then* $K_{\leqq}(\mathbf{M})$ *is acute.*

Proof. Let $\dim(K^{+}_{\leqq}(\mathbf{M})) = n$. Hence,

$$\operatorname{int}(K^{+}_{\leqq}(\mathbf{M})) = \operatorname{rel\,int}(K^{+}_{\leqq}(\mathbf{M})) \tag{3.6}$$

by Proposition 2.4.22(i). Moreover, since $K^{+}_{\leqq}(\mathbf{M})$ is nonempty (as discussed earlier) and convex (by Proposition 3.1.19(ii)), we obtain that

$$\operatorname{rel\,int}(K^{+}_{\leqq}(\mathbf{M})) \neq \emptyset \tag{3.7}$$

by Theorem 2.4.23. Thus, (3.6) and (3.7) yield that $K_{\leqq}(\mathbf{M}) \neq \emptyset$ is acute by Theorem 3.1.21(i). □

Observe that Theorem 3.2.23 does not depend on the form, inequality or generator, of $K_{\leqq}(\mathbf{M})$ despite our motivation. Even though we do not have a system to solve as in Theorem 3.2.22, we do have a condition to verify, namely that $\dim(K^{+}_{\leqq}(\mathbf{M})) = n$, which may be done computationally in several ways. In particular, if $K^{+}_{\leqq}(\mathbf{M}))$ is in generator form (as it is when $K_{\leqq}(\mathbf{M})$ is in inequality form), then $\dim(K^{+}_{\leqq}(\mathbf{M})) = \operatorname{rank}(\mathbf{M})$ by Proposition 3.1.16. Otherwise, various software, including SageMath's [132] `polyhedron base class`, can readily provide the dimension. Moreover, it is worth noting that Theorem 3.2.23 is applicable for any nonempty cone $K \subseteq \mathbb{R}^n$, while Theorem 3.2.22 is not because we do not generally have an explicit algebraic formula for the resulting polar cone. In particular, since Theorem 3.2.23 may be applied for any nonempty cone, the theorem may be used in the context of, e.g., Propositions 3.2.12(i) and (ii).

Using Theorems 3.2.22 and 3.2.23, we may similarly verify the acuteness of (nontrivial) $\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)$.

Corollary 3.2.24. *Let $K_{\leqq}(\mathbf{M}_\ell) \neq \{\mathbf{0}\}$ be given in generator form for each $\ell = 1, \ldots, r$. Then $\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)$ is acute if and only if the system*

$$-\mathbf{G}_\ell \mathbf{x} < \mathbf{0} \text{ for all } \ell = 1, \ldots, r \tag{3.8}$$

is consistent.

Proof. Follows from Theorem 3.2.22, Proposition 3.2.8(i), and Proposition 2.4.11. □

Likewise, we have the following extension of Theorem 3.2.23.

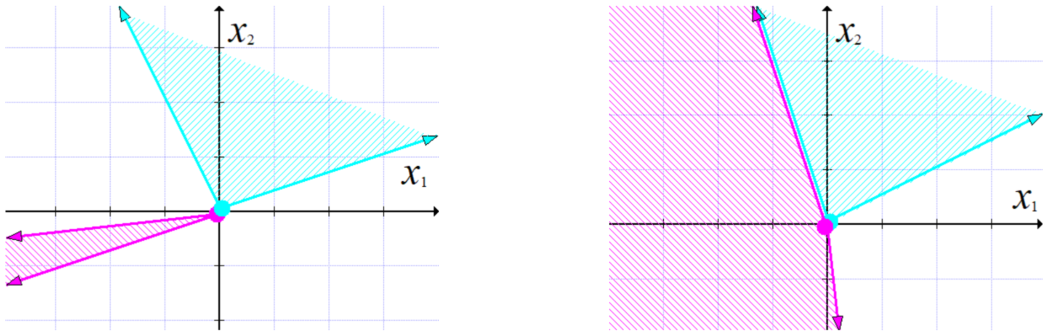
Proposition 3.2.25. *If $\dim([\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)]^+) = n$, then $\bigcup_{\ell=1}^{r} K_{\leqq}(\mathbf{M}_\ell)$ is acute.*

Proof. Follows similarly to the proof of Theorem 3.2.23. □

With respect to, e.g., Proposition 3.2.12, Theorem 3.2.13, and Proposition 3.2.20, we now have systematic approaches to verify the acuteness required to apply each result. As an illustration of two of the recognition methods, specifically Corollary 3.2.24 and Proposition 3.2.25, consider the following example.

Example 3.2.26. Consider the two polyhedral convex (finite) cones $K_{\leqq}(\mathbf{M}_1)$ and $K_{\leqq}(\mathbf{M}_2)$ given in generator form

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \begin{bmatrix} -3 & -9 \\ -1 & -1 \end{bmatrix} \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\right\} \text{ and } \left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\right\}, \tag{3.9}$$



(a) $K_{\leqq}(\mathbf{M}_1)$ (purple) and $K_{\leqq}(\mathbf{M}_2)$ (teal) (b) $K^+_{\leqq}(\mathbf{M}_1)$ (purple) and $K^+_{\leqq}(\mathbf{M}_2)$ (teal)

Figure 3.6: The polyhedral convex cones and their polars for Example 3.2.26

respectively. Hence, the polars $K^+_{\leqq}(\mathbf{M}_1)$ and $K^+_{\leqq}(\mathbf{M}_2)$ are

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} 3 & 1 \\ 9 & 1 \end{bmatrix} \mathbf{x} \leqq \mathbf{0}\right\} \text{ and } \left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{x} \leqq \mathbf{0}\right\}, \tag{3.10}$$

respectively. Therefore, by Corollary 3.2.24, $K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)$ is acute if and only if the system

$$\begin{aligned} 3x_1 + \ \ x_2 &< 0 \\ 9x_1 + \ \ x_2 &< 0 \\ -3x_1 - \ \ x_2 &< 0 \\ x_1 - 2x_2 &< 0 \end{aligned} \tag{3.11}$$

is consistent. Here, it is clear that the system is inconsistent as the first and third inequalities are inconsistent. Hence, as confirmed in Figure 3.6a, $K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)$ is not acute.

Moreover, note that with respect to Proposition 3.2.25, $[K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)]^+$ is the ray in the second quadrant emanating from the origin with slope -3 (see Figure 3.6b). Hence, $[K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)]^+$ is one-dimensional (and therefore not full-dimensional), and $\text{int}([K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)]^+) = \emptyset$. With Proposition 3.2.25 in mind, this means that we should not expect $K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)$ to be acute.

The results of this section are worthwhile in their own right, but take on added significance in the context of (highly robust) efficiency and (robust) multiobjective linear programming, which we address in Section 4.2 and Chapter 6.

Chapter 4

Deterministic Multiobjective Optimization

We turn our attention to an overview of deterministic multiobjective programming, which addresses optimization problems involving multiple conflicting criteria. Problems in which the decision-making process includes opposing goals are not easily handled by deterministic single-objective optimization since a unique optimal solution to the problem generally does not exist in the presence of these competing interests. Instead, deterministic multiobjective optimization exploits the optimization paradigm to resolve conflict by achieving and revealing a compromise in the form of an associated solution set of alternatives. Here, the use of the word deterministic refers to the notion that all of the data in the model/program is *determined* or *known*. Note that throughout the dissertation we oftentimes drop the use of 'deterministic' for the ease of exposition.

In Section 4.1, deterministic multiobjective programs (MOPs) are discussed. We introduce the model formulation, which is referenced throughout, as well as the natural solution concept attributed to Pareto [119]. The classical scalarizing approach of the weighted-sum method for computing efficient solutions is then reviewed in Section 4.1.1, while Benson's method for efficiency recognition and solution generation is studied in Section 4.1.2. In Section 4.2, multiobjective linear programs (MOLPs) are examined in more detail. The model formulation and solution concept are restated. In Section 4.2.1, several well-known properties of the set of efficient solutions (including those regarding closedness, convexity, and connectedness) are presented. A characterization of the efficient set by means of the cones of improving directions, normal cone, and recession cone is also provided in Section 4.2.2. Computational methods to obtain efficient solutions, determine the efficiency of a given feasible point, and identify whether or not efficient solutions exist, meanwhile, are presented in Section 4.2.3. Finally, in Section 4.2.4, a new result is proposed that provides a valuable perspective on one of the aforementioned characterizations.

4.1 Multiobjective Programs

In the interest of discussing (deterministic) MOPs, the basic formulation and commonly used solution concept are provided. A (deterministic) MOP is a problem of the form:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) & \cdots & f_p(\mathbf{x}) \end{bmatrix}^T \\ \text{s.t.} \quad & \mathbf{x} \in X, \end{aligned} \tag{4.1}$$

where $\mathbf{f} : X \to \mathbb{R}^p, p \geq 2$, is the vector-valued *objective function*, $\mathbf{x} \in \mathbb{R}^n, n \geq 1$, is the *decision vector*, and $X \subset \mathbb{R}^n$ is the *feasible region (set)*. An *outcome* or *criterion (objective) vector* $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^p$ is associated with every feasible decision $\mathbf{x} \in X$. The set of all outcomes for all feasible decisions is referred to as the *attainable set* or *set of criterion* $Y_{\mathbf{f},X} := \mathbf{f}(X) = \{\mathbf{y} \in \mathbb{R}^p : \mathbf{y} = \mathbf{f}(\mathbf{x}) \text{ for some } \mathbf{x} \in X\}$. The spaces $\mathbb{R}^n$ and $\mathbb{R}^p$ are referred to as the *decision (solution) space* and the *objective (criterion,* *outcome) space*, respectively. The feasible region X of MOP (4.1) is generally defined as

$$X := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{g}(\mathbf{x}) \leqq \mathbf{0}\}, \tag{4.2}$$

where $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ is the vector-valued *constraint function* representing inequality constraints including upper and lower bounds on the decision variables. Since equality constraints may be represented by a pair of related inequalities, $\mathbf{g}(\mathbf{x}) \leqq \mathbf{0}$ is understood to include equality constraints as well.

The solution concept for MOP (4.1) is typically based on the component-wise comparison of two outcomes $\mathbf{y}_1$ and $\mathbf{y}_2$ in $\mathbb{R}^p$ for which the three *ordering relations* given by $\leqq, \leq, <$ are used. These relations determine (*partial*) *orders* on $\mathbb{R}^p$ and are used to define *Pareto dominance* in $Y_{\mathbf{f},X}$. For two outcomes $\mathbf{y}_1$ and $\mathbf{y}_2$ in $Y_{\mathbf{f},X}$, outcome $\mathbf{y}_1$ is said to *(strictly) dominate* outcome $\mathbf{y}_2$ if $\mathbf{y}_1$ $(<) \leq$ $\mathbf{y}_2$. The solution set of MOP (4.1) contains the feasible decisions $\mathbf{x} \in X$ whose objective vectors cannot be (strictly) Pareto dominated or improved by other objective vectors.

Definition 4.1.1. A feasible solution $\hat{\mathbf{x}} \in X$ to MOP (4.1) is said to be *(weakly) Pareto-efficient* provided there does not exist an $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x})$ $(<) \leq$ $\mathbf{f}(\hat{\mathbf{x}})$. The set of all (weakly) efficient solutions $\hat{\mathbf{x}} \in X$ is denoted by $(\mathrm{wE}(X, \mathbf{f}))$ $\mathrm{E}(X, \mathbf{f})$ and is called the *(weakly) Pareto-efficient set.*

Note that throughout the dissertation, Pareto efficiency is simply referred to as efficiency since other more general concepts are not discussed. Moreover, it is important to recognize the practical interpretation of efficiency: An efficient solution is a decision that cannot be improved in at least one objective without negatively affecting the other objective(s). That is, if $\hat{\mathbf{x}} \in X$ is an efficient solution, then there does not exist an $\mathbf{x} \in X$ that is at least as good as $\hat{\mathbf{x}}$ in every objective and better in at least one. In order to guarantee the existence of (weakly) efficient solutions to

MOP (4.1), the standard conditions involving compactness and semicontinuity may be assumed (see Corollary 2.26 and Theorem 2.19, Ehrgott [44], respectively).

Remark 4.1.2. [44, Formula (2.17)] From the definition of (weak) efficiency, it is clear that $\mathrm{E}(X, \mathbf{f}) \subseteq \mathrm{wE}(X, \mathbf{f})$ since weak efficiency is more permissive than efficiency.

In many contexts, the efficient set $\mathrm{E}(X, \mathbf{f})$ is taken to be the solution set of MOP (4.1) and efficiency is thus the multiobjective analogue to optimality in single-objective optimization. However, since weakly efficient solutions may be easier to find, the set $\mathrm{wE}(X, \mathbf{f})$ is sometimes taken to be the solution set of MOP (4.1) instead. That being said, we are mainly concerned with efficient solutions in this work, but still discuss weakly efficient solutions when convenient. This decision is revisited in Chapter 6 in the context of problems with uncertainty.

The above definition of efficiency describes solutions exclusively in the decision space. However, in multiobjective programming, as opposed to single-objective programming, we are able to study solutions in the objective space $\mathbb{R}^p$ as well. In particular, points in the attainable set $Y_{\mathbf{f},X}$ are of interest.

Definition 4.1.3. The image $\hat{\mathbf{y}} := \mathbf{f}(\hat{\mathbf{x}}) \in Y_{\mathbf{f},X}$ is said to be *(weakly) Pareto nondominated* if the feasible solution $\hat{\mathbf{x}} \in X$ is (weakly) efficient. The set of all (weakly) Pareto-nondominated solutions $\hat{\mathbf{y}} \in Y_{\mathbf{f},X}$ is denoted by $(\mathrm{wP}(X, \mathbf{f}))$ $\mathrm{P}(X, \mathbf{f})$ and is called the *(weakly) Pareto-nondominated set*.

The Pareto-nondominated set is the image of the efficient set, and similarly, $\mathrm{wP}(X, \mathbf{f})$ is the image of $\mathrm{wE}(X, \mathbf{f})$. As a result, the Pareto-nondominated set and the efficient set are often used interchangeably when discussing solutions to MOP (4.1). The efficient points in the solution space and their images in the objective space reveal the available options and their performances, while the associated tradeoffs carry additional information to support the decision-making process. Whereas the single-objective paradigm rigorously exercises optimization, the multiobjective strategy, while having the same rigor but requiring more computational power, offers a broader perspective by providing these tradeoffs. The latter provides the user or decision maker with various alternatives, a compelling quality when making operations research methodologies attractive for customers (see Greco et al. [61]).

Remark 4.1.4. [44, Formula (2.16)] As in Remark 4.1.2, it is clear that $\mathrm{P}(X, \mathbf{f}) \subseteq \mathrm{wP}(X, \mathbf{f})$ by definition.

When studied in a more rigorous mathematical framework (see Tammer and Göpfert [130]), the solution concept of Pareto nondominance (and thereby efficiency) is implied by general binary relations, which, yielding partial orders on $\mathbb{R}^p$, determine preference relations between the outcomes in $Y_{\mathbf{f},X}$. Under some conditions, a binary relation on $\mathbb{R}^p$ may be associated with a pointed and convex cone in $\mathbb{R}^p$ (see Yu [145]). For example, the binary relation defining the Pareto preference in the minimization case considered in (4.1) is associated with the cone $-\mathbb{R}^p_{\geq}$ (or $\mathbb{R}^p_{\leq}$). When more general cones K in $\mathbb{R}^p$ are used, the term "Pareto nondominated" is replaced with "K-nondominated." Since only Pareto nondominated solutions are considered throughout this dissertation, the terminology is shortened by simply referring to such solutions as either Pareto or nondominated.

Example 4.1.5. [44, Example 1.3] Consider the following MOP:

$$\begin{aligned} \min_{x} \quad & \mathbf{f}(x) = \left[f_1(x) = \sqrt{1+x} \quad f_2(x) = x^2 - 4x + 5 \right]^T \\ \text{s.t.} \quad & x \geq 0. \end{aligned} \tag{4.3}$$

In this particular problem, examining the feasible region in the decision space is relatively uninteresting as it is simply the nonnegative halfline in $\mathbb{R}$. Instead, we investigate the attainable set in the objective space. To obtain the set of criterion,

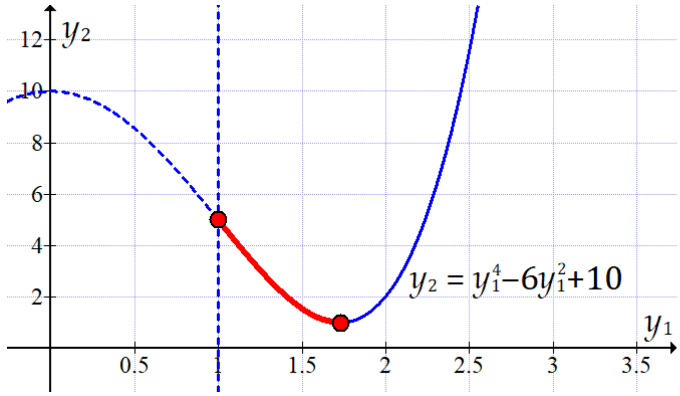


Figure 4.1: Attainable set (solid line) and Pareto set (red) of MOP (4.3)

we first let $f_1(x) = y_1$ and $f_2(x) = y_2$, which yields

$$y_1 = \sqrt{1+x} \text{ and } y_2 = x^2 - 4x + 5.$$

Solving for x in the first equation then gives

$$y_1^2 = 1 + x \implies x = y_1^2 - 1.$$

Now, substituting x into the second equation yields

$$y_2 = (y_1^2 - 1)^2 - 4(y_1^2 - 1) + 5 = y_1^4 - 6y_1^2 + 10.$$

The graph of this function would be the image of the feasible set if x were free. However, since x is constrained as $x \geq 0$, it follows that $y_1 \geq 1$ must also be taken into account. Therefore, the image of the feasible region is given by

$$Y_{\mathbf{f},X} = \{\mathbf{y} \in \mathbb{R}^2 : y_2 = y_1^4 - 6y_1^2 + 10, y_1 \geq 1\}$$

and is shown in Figure 4.1 as the solid line.

Even though we have not yet discussed how to compute the Pareto set, it (or equivalently the weakly Pareto set since the two solutions sets are equal in this example) is shown in Figure 4.1 as the red highlighted portion of the attainable set.

4.1.1 Weighted-Sum Method

A natural question to ask now is: How are such Pareto optimal or efficient solutions computed? Various techniques exist in the literature, e.g., weighted-sum methods, epsilon-constraint methods, weighted-norm methods, the weighted t-th power approach, etc. For a discussion of these methods and others, the reader is directed to Ehgrott [44] and Wiecek et al. [141]. We study one of these methods, the weighted-sum or weighted-sum scalarization method, in more detail since we apply this technique later to MOPs involving uncertainty.

Given MOP (4.1) and a vector of scalars $\boldsymbol{\lambda} \in \mathbb{R}^p$, the *weighted-sum method* combines the elements of the objective function $\mathbf{f}$ in a weighted sum:

$$\sum_{k=1}^{p} \lambda_k f_k(\mathbf{x}),$$

and then replaces the objective with this sum. The resulting problem is a deterministic single-objective problem, denoted WSP($\boldsymbol{\lambda}$):

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{k=1}^{p} \lambda_k f_k(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in X, \end{aligned} \tag{4.4}$$

called the *weighted-sum problem*. The intent is that solving this single-objective problem gives efficient solutions to MOP (4.1). However, two questions arise:

(i) Does the weighted-sum method always yield (weakly) efficient solutions?

(ii) Can all (weakly) efficient solutions be found in this way?

Both questions are answered by the following two propositions.

Proposition 4.1.6. *[44, Proposition 3.9] Suppose that* $\hat{\mathbf{x}}$ *is an optimal solution to* $\mathrm{WSP}(\boldsymbol{\lambda})$ *with* $\boldsymbol{\lambda} \in \mathbb{R}^p$.

(i) If $\boldsymbol{\lambda} \in \mathbb{R}^p_>$, *then* $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{f})$.

(ii) If $\boldsymbol{\lambda} \in \mathbb{R}^p_\geq$, *then* $\hat{\mathbf{x}} \in \mathrm{wE}(X, \mathbf{f})$.

Proof. (i) Let $\boldsymbol{\lambda} \in \mathbb{R}^p_>$, and let $\hat{\mathbf{x}} \in X$ be an optimal solution to $\mathrm{WSP}(\boldsymbol{\lambda})$. Assume for the sake of contradiction that $\hat{\mathbf{x}} \notin \mathrm{E}(X, \mathbf{f})$. By Definition 4.1.1, there exists an $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\hat{\mathbf{x}})$, i.e., such that $f_k(\mathbf{x}) \leq f_k(\hat{\mathbf{x}})$ for all $k = 1, \ldots, p$ with at least one strict. Equivalently, since $\lambda_k > 0$ for all $k = 1, \ldots, p$, it follows that $\lambda_k f_k(\mathbf{x}) \leq \lambda_k f_k(\hat{\mathbf{x}})$ for all $k = 1, \ldots, p$ with at least one strict, which implies

$$\sum_{k=1}^{p} \lambda_k f_k(\mathbf{x}) < \sum_{k=1}^{p} \lambda_k f_k(\hat{\mathbf{x}}).$$

By definition, $\hat{\mathbf{x}}$ is not an optimal solution to $\mathrm{WSP}(\boldsymbol{\lambda})$, which is a contradiction. Thus, it must be that $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{f})$ as desired.

(ii) Let $\boldsymbol{\lambda} \in \mathbb{R}^p_\geq$, and let $\hat{\mathbf{x}} \in X$ be an optimal solution to $\mathrm{WSP}(\boldsymbol{\lambda})$. Assume for the sake of contradiction that $\hat{\mathbf{x}} \notin \mathrm{wE}(X, \mathbf{f})$. By Definition 4.1.1, there exists an $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x}) < \mathbf{f}(\hat{\mathbf{x}})$. Equivalently, $f_k(\mathbf{x}) < f_k(\hat{\mathbf{x}})$ for all $k = 1, \ldots, p$, which implies

$$\sum_{k=1}^{p} \lambda_k f_k(\mathbf{x}) < \sum_{k=1}^{p} \lambda_k f_k(\hat{\mathbf{x}}).$$

By definition, $\hat{\mathbf{x}}$ is not an optimal solution to $\mathrm{WSP}(\boldsymbol{\lambda})$, which is a contradiction. Thus, it must be that $\hat{\mathbf{x}} \in \mathrm{wE}(X, \mathbf{f})$ as desired. □

Proposition 4.1.7. *[44, Proposition 3.10] Let the feasible set X be convex, and let $f_k, k = 1, \ldots, p$, be convex functions. If $\hat{\mathbf{x}} \in \mathrm{wE}(X, \mathbf{f})$, then there is some $\boldsymbol{\lambda} \in \mathbb{R}^p_{\geq}$ such that $\hat{\mathbf{x}}$ is an optimal solution to* $\mathrm{WSP}(\boldsymbol{\lambda})$*.*

Based on these two propositions, we see that the answer to our two questions is yes, under certain conditions. In particular, for all positive vectors $\boldsymbol{\lambda} \in \mathbb{R}^p$, we identify efficient (or equivalently, weakly efficient) solutions. Further, under convexity assumptions, all weakly efficient solutions, and therefore all efficient solutions by Remark 4.1.2, may be found as optimal solutions to $\mathrm{WSP}(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in \mathbb{R}^p_{\geq}$.

Example 4.1.8. Consider MOP (4.3). The associated weighted-sum problem is

$$\begin{aligned} \min_{x} \quad & \lambda_1\sqrt{1+x} + \lambda_2(x^2 - 4x + 5) \\ \text{s.t.} \quad & x \geq 0. \end{aligned}$$

If we take $\boldsymbol{\lambda} = (0, 1) \in \mathbb{R}^2_{\geq}$, then the optimal solution to $\mathrm{WSP}((0,1))$ is $\hat{x} = 2$, which corresponds to the right endpoint, $(y_1, y_2) = (\sqrt{3}, 1)$, of the Pareto set shown in Figure 4.1. As stated in Proposition 4.1.6(ii), since $\boldsymbol{\lambda} = (0, 1) \in \mathbb{R}^2_{\geq}, \hat{x} = 2$ is weakly efficient. But, as noted earlier, the Pareto and weakly Pareto sets are in fact equal here, so $\hat{x} = 2$ is also efficient.

Similarly, if we take $\boldsymbol{\lambda} = (1, 1) \in \mathbb{R}^2_{>}$, then the optimal solution to $\mathrm{WSP}((1,1))$ is $\hat{x} \approx 1.852$, which corresponds to the point $(y_1, y_2) \approx (1.689, 1.185)$. Since $\boldsymbol{\lambda} = (1, 1) \in \mathbb{R}^2_{>}$, it is guaranteed that $\hat{x} \approx 1.852$ is efficient (as shown in Figure 4.1) by Proposition 4.1.6(i).

4.1.2 Benson's Method

In addition to computing efficient points with the weighted-sum method, one may also check whether or not a given feasible solution $\mathbf{x}_0 \in X$ is efficient or generate one that is if $\mathbf{x}_0$ is not efficient. The importance of doing this is that checking a given solution $\mathbf{x}_0 \in X$ for efficiency allows for added input and control from the decision maker. In order to accomplish this task of efficient solution recognition or generation, an auxiliary single-objective problem called *Benson's problem* is solved. For a given feasible decision $\mathbf{x}_0 \in X$, Benson's problem, denoted $\mathrm{BP}(\mathbf{x}_0)$, is given by

$$\begin{aligned} \max_{\mathbf{x}, \boldsymbol{l}} \quad & \sum_{k=1}^{p} l_k \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}) + \mathbf{I}_p \boldsymbol{l} = \mathbf{f}(\mathbf{x}_0) \\ & \boldsymbol{l} \geqq \mathbf{0} \\ & \mathbf{x} \in X, \end{aligned} \tag{4.5}$$

where $\boldsymbol{l} \in \mathbb{R}^p$ is a so-called *deviation variable.* The following set of results first addresses the feasibility of $\mathrm{BP}(\mathbf{x}_0)$ and then the recognition and generation of efficient solutions to MOP (4.1).

Proposition 4.1.9. *Let* $\mathbf{x}_0 \in X$ *be given. Then* $\mathrm{BP}(\mathbf{x}_0)$ *is feasible.*

Proof. It is clear that $\mathrm{BP}(\mathbf{x}_0)$ is feasible since $\boldsymbol{l} = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_0$ satisfy the constraints. □

Theorem 4.1.10. *[44, Theorem 4.14] Let* $\mathbf{x}_0 \in X$ *be given. Then* $\mathbf{x}_0$ *is an efficient solution to* MOP (4.1) *if and only if* $\mathrm{BP}(\mathbf{x}_0)$ *has an optimal solution* $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ *with* $\hat{\boldsymbol{l}} = \mathbf{0}$.

Proof. ($\Longrightarrow$) Assume $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f})$. By Definition 4.1.1, there does not exist an $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\mathbf{x}_0)$. Accordingly, there does not exist an $\mathbf{x} \in X$ such that

$$\mathbf{f}(\mathbf{x}_0) - \mathbf{f}(\mathbf{x}) \geq \mathbf{0}, \tag{4.6}$$

which is obtained by subtracting $\mathbf{f}(\mathbf{x})$ from both sides of the former inequality.

Now, let $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ be an optimal solution to $\mathrm{BP}(\mathbf{x}_0)$. Hence,

$$\hat{\mathbf{x}} \in X \text{ and } \mathbf{I}_p \hat{\boldsymbol{l}} = \mathbf{f}(\mathbf{x}_0) - \mathbf{f}(\hat{\mathbf{x}}) \geqq \mathbf{0}. \tag{4.7}$$

Combining (4.6) and (4.7), it must be that $\hat{\boldsymbol{l}} = \mathbf{0}$. Thus, $\mathrm{BP}(\mathbf{x}_0)$ has an optimal solution $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ with $\hat{\boldsymbol{l}} = \mathbf{0}$.

($\Longleftarrow$) Let $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ with $\hat{\boldsymbol{l}} = \mathbf{0}$ be an optimal solution to $\mathrm{BP}(\mathbf{x}_0)$, and assume for the sake of contradiction that $\mathbf{x}_0$ is not an efficient solution to MOP (4.1). By Definition 4.1.1, there exists an $\bar{\mathbf{x}} \in X$ such that $\mathbf{f}(\bar{\mathbf{x}}) \leq \mathbf{f}(\mathbf{x}_0)$. Subtracting $\mathbf{f}(\bar{\mathbf{x}})$ from both sides of the inequality and letting $\bar{\boldsymbol{l}} = \mathbf{f}(\mathbf{x}_0) - \mathbf{f}(\bar{\mathbf{x}}) \in \mathbb{R}^p$, we have that there exists an $\bar{\boldsymbol{l}} \in \mathbb{R}^p$ such that $\bar{\boldsymbol{l}} \geq \mathbf{0}$.

Now, observe that $\sum_{k=1}^{p} \bar{l}_k > 0$ and that $(\bar{\mathbf{x}}, \bar{\boldsymbol{l}})$ is a feasible solution to $\mathrm{BP}(\mathbf{x}_0)$. Since $\hat{\boldsymbol{l}} = \mathbf{0}$ by assumption, we have constructed a solution that has an objective value greater than the optimal solution, which is a contradiction. Hence, it must be that $\mathbf{x}_0$ is an efficient solution to MOP (4.1). □

Proposition 4.1.11. *[44, Proposition 4.15] Let $\mathbf{x}_0 \in X$ be given. If $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ is an optimal solution to* $\mathrm{BP}(\mathbf{x}_0)$ *(such that the optimal objective value is finite), then $\hat{\mathbf{x}}$ is an efficient solution to* MOP (4.1)*.*

Proof. Suppose $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ is an optimal solution to $\mathrm{BP}(\mathbf{x}_0)$, and assume for the sake of contradiction that $\hat{\mathbf{x}} \notin \mathrm{E}(X, \mathbf{f})$. By Definition 4.1.1, there is some $\bar{\mathbf{x}} \in X$ such that $\mathbf{f}(\bar{\mathbf{x}}) \leq \mathbf{f}(\hat{\mathbf{x}})$. Define $\bar{\boldsymbol{l}} = \mathbf{f}(\mathbf{x}_0) - \mathbf{f}(\bar{\mathbf{x}})$. Hence, $(\bar{\mathbf{x}}, \bar{\boldsymbol{l}})$ is a feasible solution to $\mathrm{BP}(\mathbf{x}_0)$ since $\bar{\boldsymbol{l}} = \mathbf{f}(\mathbf{x}_0) - \mathbf{f}(\bar{\mathbf{x}}) \geq \mathbf{f}(\mathbf{x}_0) - \mathbf{f}(\hat{\mathbf{x}}) = \hat{\boldsymbol{l}} \geq \mathbf{0}$ and $\bar{\mathbf{x}} \in X$. Moreover, since $\bar{\boldsymbol{l}} \geq \hat{\boldsymbol{l}}$, we have that

$$\sum_{k=1}^{p} \bar{l}_k > \sum_{k=1}^{p} \hat{l}_k.$$

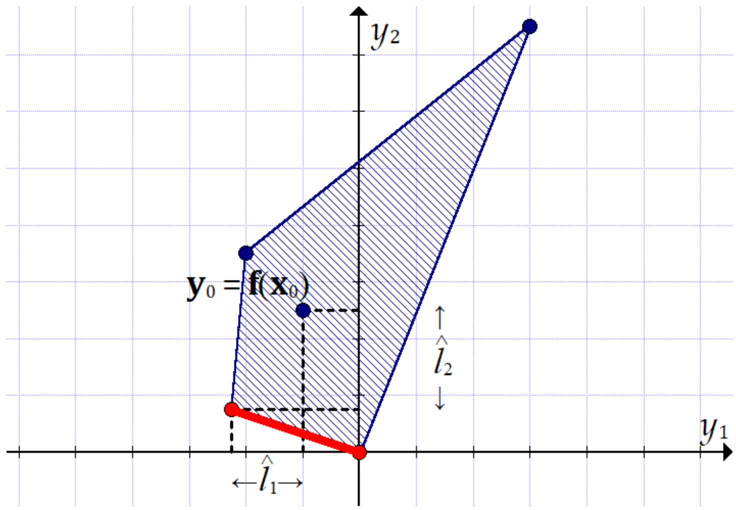


Figure 4.2: Illustration of Benson's problem with the attainable set shaded (blue) and the Pareto set highlighted (red)

Therefore, we have constructed a feasible solution that is better than the optimal solution $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$, which is a contradiction. Thus, it must be that $\hat{\mathbf{x}}$ is an efficient solution to MOP (4.1). $\square$

The idea of Benson's method (or in solving Benson's problem) begins with choosing some initial feasible solution $\mathbf{x}_0 \in X$. If $\mathbf{x}_0$ is not itself efficient, then a solution that is efficient is produced by maximizing the sum of nonnegative deviation variables $l_k = f_k(\mathbf{x}_0) - f_k(\mathbf{x})$. As a result, not only can Benson's method provide an approach to verify the efficiency of $\mathbf{x}_0 \in X$, but it can also generate efficient solutions. An illustration is provided in Figure 4.2.

Example 4.1.12. Consider MOP (4.3) of Example 4.1.5. For $x_0 \in X$, we have $\mathrm{BP}(x_0)$ is given by

$$\begin{aligned} \max_{x,\boldsymbol{l}} \quad & l_1 + l_2 \\ \text{s.t.} \quad & \sqrt{1+x} + l_1 = \sqrt{1+x_0} \\ & x^2 - 4x + 5 + l_2 = x_0^2 - 4x_0 + 5 \\ & x,\ l_1,\ l_2 \geq 0. \end{aligned}$$

In particular, if we take $x_0 = 2$, which we know is efficient due to Example 4.1.8, then BP(2) is given by

$$\begin{aligned} \max_{x,\boldsymbol{l}} \quad & l_1 + l_2 \\ \text{s.t.} \quad & \sqrt{1+x} + l_1 = \sqrt{3} \\ & x^2 - 4x + 5 + l_2 = 1 \\ & x,\ l_1,\ l_2 \geq 0. \end{aligned}$$

Solving BP(2), we are guaranteed that the optimal values of l_1 and l_2 are both zero since x_0 is efficient. On the other hand, if we started with an inefficient x_0, then we know that the resulting optimal x-solution would be efficient.

4.2 Multiobjective Linear Programs

As its own field within (deterministic) multiobjective programming, we now present an overview of relevant concepts and results on (deterministic) multiobjective linear programming, i.e., on MOPs where each component of the vector-valued objective function is linear and each constraint is also linear. That is, in the context of MOP (4.1), the p-dimensional objective function $\mathbf{f}$ is linear, and the feasible region is polyhedral. Certain results from the previous section are restated in the linear case, and different results unique to the linear case are also presented.

Directly below, we restate the model under consideration and its solution concept. In Section 4.2.1, we present several well-known properties of the set of efficient solutions. The properties are not necessarily true only for MOLPs, but we do not cover this is more detail. In Section 4.2.2, we then review several characterizations of the efficient set by means of the cones of improving directions, normal cone, and recession cone. Computational approaches to obtain efficient solutions, verify the efficiency of a given feasible decision, and determine whether or not the efficient set is empty are provided in Section 4.2.3. Within this part, the weighted-sum method is discussed in Section 4.2.3.1, Ecker and Kouada's method is considered in Section 4.2.3.2, Benson's method is presented in Section 4.2.3.3, and Isermann's Theorem is examined in Section 4.2.3.4. Finally, in Section 4.2.4, a new result that provides a useful point of view on one of the aforementioned characterizations is proposed.

A (deterministic) MOLP is a problem of the form:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{Cx} = \begin{bmatrix} \mathbf{c}_1\mathbf{x} & \cdots & \mathbf{c}_p\mathbf{x} \end{bmatrix}^T \\ \text{s.t.} \quad & \mathbf{x} \in P, \end{aligned} \tag{4.8}$$

where $\mathbf{c}_k, k = 1, \ldots, p$, is the k-th row of the $p \times n$ *cost (objective) matrix* $\mathbf{C}, p \geq 2, n \geq 1$, $\mathbf{x} \in \mathbb{R}^n$ is the *decision vector*, and $P \subset \mathbb{R}^n$ is the polyhedral feasible region given by (2.6), i.e.,

$$P := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}\},$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. An *outcome* or *criterion (objective) vector* $\mathbf{Cx} \in \mathbb{R}^p$ is associated with every feasible decision $\mathbf{x} \in P$. The set of all outcomes for all feasible decisions is referred to as the *attainable set* or *set of criterion* $Y_{\mathbf{C},P} := \mathbf{C}(P) = \{\mathbf{y} \in \mathbb{R}^p : \mathbf{y} = \mathbf{Cx} \text{ for some } \mathbf{x} \in P\}$.

At this point, we restate the basic definitions from the previous section in the linear context.

Definition 4.2.1. A feasible solution $\hat{\mathbf{x}} \in P$ to MOLP (4.8) is said to be *(weakly) efficient* provided there does not exist an $\mathbf{x} \in P$ such that $\mathbf{Cx}$ $(<) \leq \mathbf{C}\hat{\mathbf{x}}$. The set of all (weakly) efficient solutions $\hat{\mathbf{x}} \in P$ is denoted by (wE$(P, \mathbf{C})$) E$(P, \mathbf{C})$ and is called the *(weakly) efficient set.*

As in Section 4.1, this definition of efficiency describes solutions exclusively in the decision space. However, it is convenient for several reasons involving characterizing the efficient set and computing points in the efficient set to consider solutions in the criterion or objective space as well. Practically, it is only possible to graphically represent the objective space when $p = 2$ or $p = 3$. In any case, it is relatively easy to find the image of P under the objective $\mathbf{C}$. If the feasible region P is bounded, then the set $Y_{\mathbf{C},P}$ is also a bounded polyhedral set in the criterion space. The extreme points of P are mapped to $Y_{\mathbf{C},P}$ by $\mathbf{C}$, and are the extreme points of $Y_{\mathbf{C},P}$. In the unbounded instance, $Y_{\mathbf{C},P}$ is also unbounded, but remains a polyhedral subset of $\mathbb{R}^p$. As with the extreme points of P, the extreme directions of P are mapped to extreme directions of $Y_{\mathbf{C},P}$ by $\mathbf{C}$. In this context, efficient solutions to MOLP (4.8) in the objective space are defined as follows.

Definition 4.2.2. The outcome $\hat{\mathbf{y}} := \mathbf{C}\hat{\mathbf{x}} \in Y_{\mathbf{C},P}$ is said to be *(weakly) Pareto* if the feasible solution $\hat{\mathbf{x}} \in P$ is (weakly) efficient. The set of all (weakly) Pareto solutions $\hat{\mathbf{y}} \in Y_{\mathbf{C},P}$ is denoted by $(\mathrm{wP}(P, \mathbf{C}))$ $\mathrm{P}(P, \mathbf{C})$ and is called the *(weakly) Pareto set*.

Remark 4.2.3. As mentioned in Section 4.1, the (weakly) Pareto set is simply the image of the (weakly) efficient set. Similarly, as mentioned in Remarks 4.1.2 and 4.1.4, the containments $\mathrm{E}(P, \mathbf{C}) \subseteq \mathrm{wE}(P, \mathbf{C})$ and $\mathrm{P}(P, \mathbf{C}) \subseteq \mathrm{wP}(P, \mathbf{C})$ clearly hold.

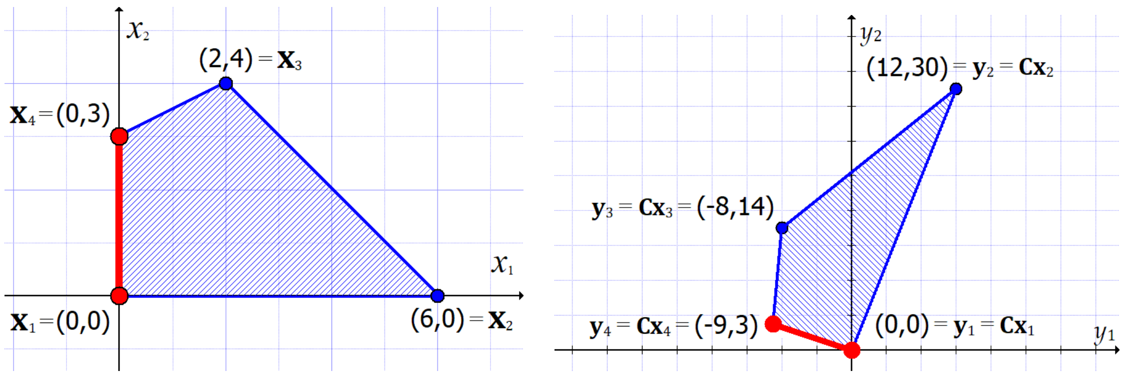
Example 4.2.4. Consider the following MOLP given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P_1, \end{aligned} \tag{4.9}$$

where

$$P_1 = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 \leq 6, -x_1 + 2x_2 \leq 6, x_1 \geq 0, x_2 \geq 0\}. \tag{4.10}$$

The efficient set of MOLP (4.9) is the (closed) line segment joining the extreme points $\mathbf{x}_1 = (0, 0)$ and $\mathbf{x}_4 = (0, 3)$, which is shown in Figure 4.3a. In this example, the



(a) The feasible region shaded (blue) and the efficient set (red) of MOLP (4.9)

(b) The attainable set shaded (blue) and the Pareto set (red) of MOLP (4.9)

Figure 4.3: The feasible and attainable sets of MOLP (4.9)

efficient and weakly efficient sets are equivalent. Similarly, the Pareto set of MOLP (4.9) is the (closed) line segment joining the extreme points $\mathbf{y}_1$ and $\mathbf{y}_4$ in the objective space (shown in Figure 4.3b), which are mapped to from the extreme points $\mathbf{x}_1$ and $\mathbf{x}_4$. As in the decision space, the Pareto and weakly Pareto sets are equivalent in this example.

Generally, the standard condition guaranteeing that (weakly) efficient solutions to MOLP (4.8) exist (cf. Corollary 2.26 and Theorem 2.19, Ehrgott [44], respectively) is that P is bounded. However, in the interest of providing a variety of other relevant existence results with natural extensions in Chapter 6, the assumption that P is bounded is not made in general.

4.2.1 Properties of the Efficient Set

Various properties of the efficient set of MOLP (4.8) are known in the literature including those regarding closedness, convexity, and connectedness. These properties offer insight into solving MOLPs and provide a better understanding of the overall structure of the efficient set.

Proposition 4.2.5. *(i) [105, Theorem 4.1.20]* $\mathrm{E}(P, \mathbf{C})$ *is closed.*

(ii) [44, Example 7.24] $\mathrm{E}(P, \mathbf{C})$ *is not necessarily convex.*

(iii) [44, Lemma 7.17] If $\mathrm{E}(P, \mathbf{C}) \neq \emptyset$*, then it is either the entire set* P *or is on the boundary of* P*.*

(iv) [44, Theorem 7.20] If $\mathrm{E}(P, \mathbf{C}) \neq \emptyset$ *and a point on the relative interior of a face of* P *is efficient, then so is the entire face.*

(v) [44, Lemma 7.1] If $\mathrm{E}(P, \mathbf{C}) \neq \emptyset$*, then there exists an efficient extreme point.*

(vi) [44, Theorem 7.23] $\mathrm{E}(P, \mathbf{C})$ *is connected.*

Each property sheds some light on how to approach solving MOLPs for efficient solutions. In particular, property (iii) may be exploited in graphical approaches to solve small (two- or three-dimensional) problems, while property (iv) suggests that it is enough to enumerate the efficient extreme points because if a relative interior point is efficient, then so is the entire face. The connectedness of the efficient set is also significant. Since the efficient set is connected, it is possible to begin with an efficient extreme point and explore only other efficient extreme points, which suggests that a simplex method approach to solving MOLP (4.8) is viable (refer to Chapter 7, Ehrgott [44]).

4.2.2 Characterizing the Efficient Set

In addition to providing properties of the efficient set, we give its characterization by means of various cones (refer to Chapter 3 for the relevant theory on cones). Using so-called *Pareto cones* (which for minimization problems are the nonpositive, nonpositive without the origin, and negative orthants), a characterization of efficient solutions in the objective space (i.e., Pareto solutions) is given in Section 4.2.2.1. Otherwise, in the decision space, the efficient set is characterized using the cone of improving directions in Section 4.2.2.2, normal cone in Section 4.2.2.3, and recession cone in Section 4.2.2.4. Each characterization offers a graphical approach to identifying the efficient set in two- or three-dimensions, which is shown in various examples. The first approach principally identifies the Pareto set in the criterion space, while the other three directly operate on the efficient set in the decision space.

4.2.2.1 Objective Space

Characterizing the Pareto set (equivalently, the efficient set) in the objective space is oftentimes the basic definition of efficiency (see p. 24, Ehrgott [44]). As such, the following characterization using Pareto cones (i.e., the orthants $\mathbb{R}^p_{\leqq}$, $\mathbb{R}^p_{\leq}$, and $\mathbb{R}^p_{<}$) is stated as a definition.

Definition 4.2.6. A feasible solution $\hat{\mathbf{x}} \in P$ is said to be

(i) *efficient* provided $\mathbf{C}(P) \cap (\{\mathbf{C}\hat{\mathbf{x}}\} \oplus \mathbb{R}^p_{\leqq}) = \{\mathbf{C}\hat{\mathbf{x}}\}$;

(ii) *efficient* provided $\mathbf{C}(P) \cap (\{\mathbf{C}\hat{\mathbf{x}}\} \oplus \mathbb{R}^p_{\leq}) = \emptyset$;

(iii) *weakly efficient* provided $\mathbf{C}(P) \cap (\{\mathbf{C}\hat{\mathbf{x}}\} \oplus \mathbb{R}^p_{<}) = \emptyset$.

Remark 4.2.7. Note that the above description is also true in the context of the previous section. However, it is only presented here because the set of criterion of MOP (4.1) may be difficult or impossible to find explicitly, while the attainable set is much easier to describe in the case of MOLPs. Moreover, since Definition 4.2.6 is in the context of the criterion space, Pareto solutions are directly identified. Nevertheless, efficient solutions are simultaneously obtained since an efficient solution is simply the preimage (using the inverse of $\mathbf{C}$ if it exists) of a Pareto solution.

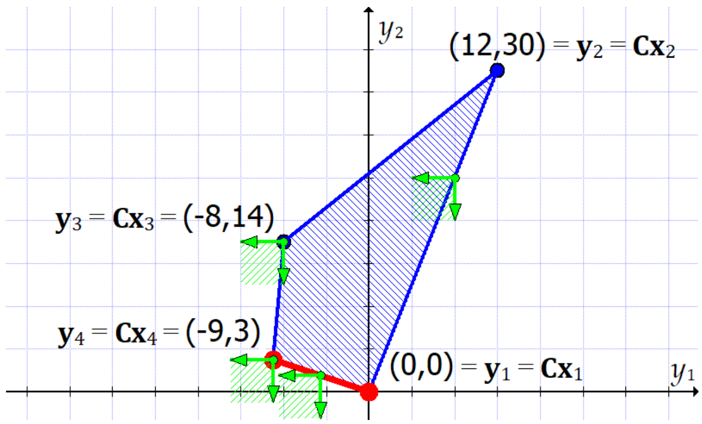


Figure 4.4: The region $\{\mathbf{C}\hat{\mathbf{x}}\} \oplus \mathbb{R}^2_{\leqq}$ (green) for several $\hat{\mathbf{x}} \in P_1$, as well as the Pareto set (red)

Intuitively, we see in Definition 4.2.6(i) that if the intersection $\mathbf{C}(P) \cap (\{\mathbf{C}\hat{\mathbf{x}}\} \oplus \mathbb{R}^p_{\leqq})$ is more than the singleton $\{\mathbf{C}\hat{\mathbf{x}}\}$, then there exists an outcome that is better than the outcome $\mathbf{C}\hat{\mathbf{x}}$ in every objective. In this case, $\hat{\mathbf{x}}$ is clearly not efficient. Similarly, if the intersection $\mathbf{C}(P) \cap (\{\mathbf{C}\hat{\mathbf{x}}\} \oplus \mathbb{R}^p_{\leqq}) = \{\mathbf{C}\hat{\mathbf{x}}\}$, then there is no outcome that is at least as good in every objective and better in at least one compared to $\mathbf{C}\hat{\mathbf{x}}$, which indicates that $\hat{\mathbf{x}}$ is efficient. An illustration of this intuition and of Definition 4.2.6(i) in general applied to MOLP (4.9) of Example 4.2.4 is provided in Figure 4.4.

As demonstrated in this figure, not only does Definition 4.2.6(i) provide a characterization of efficient solutions, but it may also be used as a graphical method in the criterion space in two-dimensions. In particular, the graphical approach (in two- or three-dimensions) involves translating the nonpositive orthant attached at the image of $\hat{\mathbf{x}}$, e.g., $\{\mathbf{C}\hat{\mathbf{x}}\} \oplus \mathbb{R}^2_{\leqq}$, around the boundary of $\mathbf{C}(P)$ (equivalently, $Y_{\mathbf{C},P}$) and examining the intersection $\mathbf{C}(P) \cap (\{\mathbf{C}\hat{\mathbf{x}}\} \oplus \mathbb{R}^p_{\leqq})$ for various $\hat{\mathbf{x}} \in P$. By performing this action, it is possible to identify the entire Pareto set. Although Definition 4.2.6(i) is true for general p, as a graphical method, it is only applicable when $p = 2$ or $p = 3$, i.e., when MOLP (4.8) has two or three criteria (since the criterion space cannot be depicted otherwise). In addition, we observe that Definition 4.2.6(i) reveals that only the boundary of $\mathbf{C}(P)$ needs to be considered when searching for Pareto (efficient) solutions since $\mathbf{C}(P) \cap (\{\mathbf{C}\hat{\mathbf{x}}\} \oplus \mathbb{R}^p_{\leqq}) \neq \{\mathbf{C}\hat{\mathbf{x}}\}$ necessarily for any $\mathbf{C}\hat{\mathbf{x}} \in \text{int}(\mathbf{C}(P))$. This supports the fact that the efficient set is either on the boundary of or is the entire feasible set (in which case $\text{int}(\mathbf{C}(P)) = \emptyset$) as in Proposition 4.2.5(iii).

In view of the preceding discussion, we now have a characterization in the criterion space that, in lower dimensions, allows efficient solutions to be identified graphically. Nonetheless, it is still pertinent to develop a characterization of efficient solutions directly in the decision space.

4.2.2.2 Cone of Improving Directions

The first characterization of efficient solutions directly in the decision space relies on the cone of improving directions. Before we define this cone, we give the definition of an improving direction.

Definition 4.2.8. The vector $\mathbf{d} \in \mathbb{R}^n$ is said to be an *improving direction* of MOLP (4.8) provided that $\mathbf{C}\mathbf{d} \leq \mathbf{0}$.

An improving direction may be understood as a direction along which we improve (i.e., decrease in the case of a minimization problem) in at least one component of the objective and do not deteriorate in the other components.

Definition 4.2.9. (i) The *open cone of improving directions* of MOLP (4.8) is defined to be $D_<(\mathbf{C}) := \{\mathbf{d} \in \mathbb{R}^n : \mathbf{C}\mathbf{d} < \mathbf{0}\}$.

(ii) The *cone of improving directions* of MOLP (4.8) is defined to be $D_{\leq}(\mathbf{C}) := \{\mathbf{d} \in \mathbb{R}^n : \mathbf{C}\mathbf{d} \leq \mathbf{0}\}$.

(iii) The *closed cone of improving directions* of MOLP (4.8) is defined to be $D_{\leqq}(\mathbf{C}) := \{\mathbf{d} \in \mathbb{R}^n : \mathbf{C}\mathbf{d} \leqq \mathbf{0}\}$.

Note that the cones of improving directions are equivalent to the cones $K_<(\mathbf{C})$, $K_\leq(\mathbf{C})$, and $K_\leqq(\mathbf{C})$, respectively. As such, $D_<(\mathbf{C})$ is an open convex cone that may be empty, $D_\leq(\mathbf{C})$ is a possibly empty convex cone that may be open, closed, or neither, and $D_\leqq(\mathbf{C})$ is a (closed) polyhedral convex cone, which is always nonempty since it contains $\mathbf{0}$.

Based on the definitions of efficiency and improving directions, we want to travel along these directions as far as possible within the feasible region in order to identify efficient solutions. This thought process leads to the following result, which is analogous to Proposition 1, Thoai [133]. We include the proofs for completeness.

Proposition 4.2.10. *Let $\hat{\mathbf{x}} \in P$. Then*

(i) $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$ *if and only if* $(D_\leq(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P = \emptyset$*;*

(ii) $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$ *if* $(D_\leqq(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P = \{\hat{\mathbf{x}}\}$*;*

(iii) $\hat{\mathbf{x}} \in \mathrm{wE}(P, \mathbf{C})$ *if and only if* $(D_<(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P = \emptyset$*.*

Proof. (i) ($\Longrightarrow$) Let $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$, and assume for the sake of contradiction that $(D_\leq(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P \neq \emptyset$. Equivalently, there exists an $\mathbf{x} \in (D_\leq(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P$. Accordingly, $\mathbf{x} \in D_\leq(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}$ and $\mathbf{x} \in P$. Hence, $\mathbf{x} - \hat{\mathbf{x}} \in D_\leq(\mathbf{C})$, which implies $\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) \leq \mathbf{0}$ by Definition 4.2.9(ii). As a result, $\mathbf{C}\mathbf{x} \leq \mathbf{C}\hat{\mathbf{x}}$ so that $\mathrm{E}(P, \mathbf{C})$ by Definition 4.2.1. However, this is a contradiction, so it must be that $(D_\leq(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P = \emptyset$.

($\Longleftarrow$) Let $(D_\leq(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P = \emptyset$, and assume for the sake of contradiction that $\hat{\mathbf{x}} \notin \mathrm{E}(P, \mathbf{C})$. Equivalently, there exists an $\mathbf{x} \in P$ such that $\mathbf{C}\mathbf{x} \leq \mathbf{C}\hat{\mathbf{x}}$ by Definition 4.2.1, which implies $\mathbf{C}\mathbf{x} - \mathbf{C}\hat{\mathbf{x}} \leq \mathbf{0}$. Hence, $\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) \leq \mathbf{0}$ so that $\mathbf{x} - \hat{\mathbf{x}} \in D_\leq(\mathbf{C})$ by Definition 4.2.9(ii). As a result, $\mathbf{x} \in D_\leq(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}$. Since

$\mathbf{x} \in D_{\leq}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}$ and $\mathbf{x} \in P$, it follows that $\mathbf{x} \in (D_{\leq}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P$ also. However, this implies that $(D_{\leq}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P \neq \emptyset$, which is a contradiction. Thus, it must be that $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$.

(ii) Follows the same as the backward direction of the proof of part (i).

(iii) Follows similarly to the proof of part (i). □

Intuitively, the intersection in Proposition 4.2.10(i) being empty indicates that there is no other $\mathbf{x} \in P$ that improves upon $\hat{\mathbf{x}}$ in at least one objective without deteriorating the other objectives. More simply, this means that there is no feasible direction that is also improving. The proceeding example provides an illustration.

Example 4.2.11. Consider Example 4.2.4. We have

$$D_{\leq}(\mathbf{C}) = \left\{ \mathbf{d} \in \mathbb{R}^2 : \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \mathbf{d} \leq \mathbf{0} \right\}.$$

Referring to Figure 4.5, it is clear by Proposition 4.2.10(i) that $\mathbf{x}_1 = \mathbf{0}$ is efficient, while $\mathbf{x}_3 = (2, 4)$ is not. Likewise, we see that only the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_4$ (including) is efficient.

Similarly to Definition 4.2.6(i), provided $D_{\leq}(\mathbf{C}) \neq \emptyset$, Proposition 4.2.10(i) implies that only points $\hat{\mathbf{x}}$ on the boundary of P need to be considered for efficiency since the intersection $(D_{\leq}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P$ is necessarily nonempty otherwise. Hence, the graphical method in the decision space in two- or three-dimensions (where the number of decision variables is $n = 2$ or 3) would have us translate the cone of improving directions around the boundary of P and examine the intersection. Unlike the characterization of efficient solutions in the objective space given by Definition 4.2.6(i), however, the above characterization in the decision space applies only when

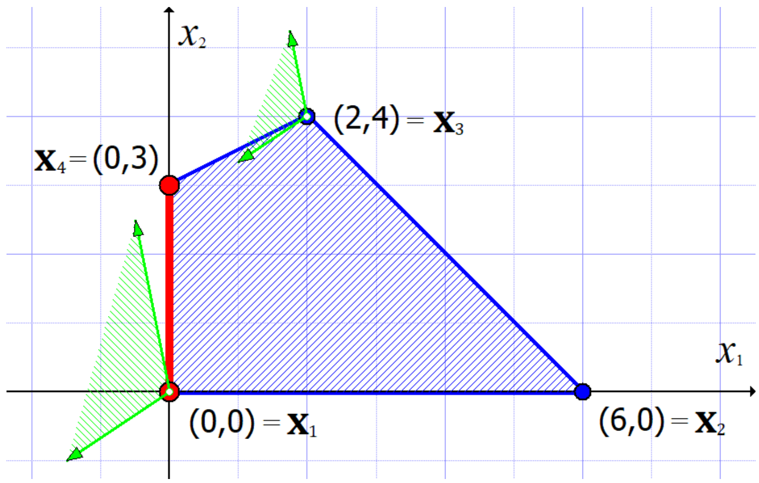


Figure 4.5: The region $D_{\leq}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}$ (green) for two feasible solutions $\hat{\mathbf{x}} \in P_1$ to MOLP (4.9)

the objective functions are linear (since the cone of improving directions is not easily representable otherwise).

Remark 4.2.12. It is worth noting that if $D_{\leq}(\mathbf{C}) = \emptyset$, then $\mathrm{E}(P, \mathbf{C}) = P$ since $\emptyset \oplus \{\hat{\mathbf{x}}\} = \emptyset$ (see Remark 2.4.2) so that the intersection in Proposition 4.2.10(i) holds trivially for all $\hat{\mathbf{x}} \in P$. Similarly, if $D_{<}(\mathbf{C}) = \emptyset$, then $\mathrm{wE}(P, \mathbf{C}) = P$. These observations along with the above discussion about the boundary lead to a potential proof of Proposition 4.2.5(iii). Moreover, it is clear that if P is unbounded, then $(D_{\leq}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P = \emptyset$ may not hold for any $\hat{\mathbf{x}} \in P$. Otherwise, if P is bounded, then the intersection must be empty for at least one $\hat{\mathbf{x}} \in P$ so that $\mathrm{E}(P, \mathbf{C}) \neq \emptyset$ (as we know is already guaranteed).

As mentioned above, the cone of improving directions, not the closed cone of improving directions, is used in terms of a (graphical) method to identify efficient solutions. This is because Proposition 4.2.10(ii) is not both necessary and sufficient. In particular, Proposition 4.2.10(ii) is only a sufficient condition for efficiency; the converse of the statement is not necessarily true. The final step from the proof of Proposition 4.2.10(i) does not yield a contradiction in all cases under the antecedent posed in (ii). Indeed, specific results depend on the characteristics of the objective matrix $\mathbf{C}$. For example, consider the trivial biobjective problem with cost matrix $\mathbf{C} = \mathbf{0}_{2\times 2}$, where $\mathbf{0}_{2\times 2}$ is the 2×2 matrix of all zeros. Given the feasible region P_1, we obtain $\mathrm{E}(P_1, \mathbf{0}_{p\times n}) = P_1$, yet $(D_{\leqq}(\mathbf{0}_{2\times 2}) \oplus \{\hat{\mathbf{x}}\}) \cap P_1 \neq \{\hat{\mathbf{x}}\}$ for all $\hat{\mathbf{x}} \in P_1$ since $D_{\leqq}(\mathbf{0}_{2\times 2}) = \mathbb{R}^2$. Alternatively, consider the nontrivial biobjective problem with $\mathbf{c}_1 = \begin{bmatrix}1 & 0\end{bmatrix}$ and $\mathbf{c}_2 = \begin{bmatrix}2 & 0\end{bmatrix}$. We have that $D_{\leq}(\mathbf{C}) = \{\mathbf{d} \in \mathbb{R}^2 : d_1 < 0\}$ (where $D_{\leq}(\mathbf{C}) = D_{<}(\mathbf{C})$ since $\mathrm{rank}(\mathbf{C}) = 1$ as in the general discussion of $K_{\leq}(\mathbf{M})$ and $K_{<}(\mathbf{M})$ in Section 3.2), while $D_{\leqq}(\mathbf{C}) = \{\mathbf{d} \in \mathbb{R}^2 : d_1 \leq 0\}$. Hence, given the feasible region P_1 we find that the efficient set is the whole line segment joining $(0, 0)$ and $(0, 3)$, yet $(D_{\leqq}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P_1 \neq \{\hat{\mathbf{x}}\}$ for all $\hat{\mathbf{x}} \in \mathrm{E}(P_1, \mathbf{C})$.

Proposition 4.2.10 may also be expressed algebraically, which allows computational methods to be used and gives the characterization added utility beyond the graphical approach in two- or three-dimensions.

Proposition 4.2.13. *Let* $\hat{\mathbf{x}} \in P$ *be a feasible solution to* MOLP (4.8)*. Then*

(i) $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$ *if and only if the system*

$$\begin{aligned} \mathbf{C}\mathbf{x} &\leq \mathbf{C}\hat{\mathbf{x}} \\ \mathbf{A}\mathbf{x} &\leqq \mathbf{b} \\ \mathbf{x} &\geqq \mathbf{0} \end{aligned} \tag{4.11}$$

has no solution;

(ii) $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$ *if the system*

$$\begin{aligned} \mathbf{C}\mathbf{x} &\leqq \mathbf{C}\hat{\mathbf{x}} \\ \mathbf{A}\mathbf{x} &\leqq \mathbf{b} \end{aligned} \tag{4.12}$$

$$\mathbf{x} \geqq \mathbf{0}$$

has $\hat{\mathbf{x}}$ *as its unique solution;*

(iii) $\hat{\mathbf{x}} \in \mathrm{wE}(P, \mathbf{C})$ *if and only if the system*

$$\begin{aligned} \mathbf{C}\mathbf{x} &< \mathbf{C}\hat{\mathbf{x}} \\ \mathbf{A}\mathbf{x} &\leqq \mathbf{b} \\ \mathbf{x} &\geqq \mathbf{0} \end{aligned} \tag{4.13}$$

has no solution.

Proof. (i) Suppose (4.11) has no solution. Equivalently, there does not exist an $\mathbf{x} \in P$ such that $\mathbf{C}\mathbf{x} \leq \mathbf{C}\hat{\mathbf{x}}$. That is, $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$ by Definition 4.2.1.

(ii) Suppose $\hat{\mathbf{x}} \in P$ is the unique solution to (4.12). Equivalently, $\hat{\mathbf{x}}$ is the only $\mathbf{x} \in P$ such that $\mathbf{C}\mathbf{x} \leqq \mathbf{C}\hat{\mathbf{x}}$. That is, $(D_{\leqq}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap P = \{\hat{\mathbf{x}}\}$ by Proposition 3.2.4(ii). Hence, $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$ by Proposition 4.2.10(ii).

(iii) Suppose (4.13) has no solution. Equivalently, there does not exist an $\mathbf{x} \in P$ such that $\mathbf{C}\mathbf{x} < \mathbf{C}\hat{\mathbf{x}}$. That is, $\hat{\mathbf{x}} \in \mathrm{wE}(P, \mathbf{C})$ by Definition 4.2.1. □

Even though each of these three systems gives an algebraic description of (weakly) efficient solutions, (4.11) does not have a simple implementation computationally as $\mathbf{C}\mathbf{x} \leq \mathbf{C}\hat{\mathbf{x}}$ means that $\mathbf{c}_k\mathbf{x} \leq \mathbf{c}_k\hat{\mathbf{x}}$ for all $k = 1, \ldots, p$, with at least one strict. As a result, either (4.12) or (4.13) should be used to determine (weak) efficiency. In addition, Gale's Theorem 2.6.1 and Theorem 2.6.4 may be used to give alternative systems whose solutions indicate the existence of (weakly) efficient solutions.

Further characterizing the efficient set of MOLP (4.8), the cone of improving directions may be used to provide a bound set on the efficient set of one MOLP with respect to the efficient set of another. Intuitively, the following lemma utilizes the idea that fewer improving directions should lead to a larger efficient set.

Lemma 4.2.14. *Let the (deterministic) MOLPs:*

$$\begin{array}{lll} \min\limits_{\mathbf{x}} & \mathbf{C}_1\mathbf{x} \\ \text{s.t.} & \mathbf{x} \in P, \end{array} \quad \textit{and} \quad \begin{array}{lll} \min\limits_{\mathbf{x}} & \mathbf{C}_2\mathbf{x} \\ \text{s.t.} & \mathbf{x} \in P \end{array}$$

be given. If $D_{\leq}(\mathbf{C}_1) \subseteq D_{\leq}(\mathbf{C}_2)$*, then* $\mathrm{E}(P, \mathbf{C}_2) \subseteq \mathrm{E}(P, \mathbf{C}_1)$*.*

Proof. Suppose $D_{\leq}(\mathbf{C}_1) \subseteq D_{\leq}(\mathbf{C}_2)$, and assume for the sake of contradiction that $\mathrm{E}(P, \mathbf{C}_2 \not\subseteq \mathrm{E}(P, \mathbf{C}_1)$, i.e., there exists an $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C}_2)$ such that $\hat{\mathbf{x}} \notin \mathrm{E}(P, \mathbf{C}_1)$. The former implies that $D_{\leq}(\mathbf{C}_1) \oplus \{\mathbf{x}\} \subseteq D_{\leq}(\mathbf{C}_2) \oplus \{\mathbf{x}\}$ for all $\mathbf{x} \in P$, while the latter yields $[D_{\leq}(\mathbf{C}_2) \oplus \{\hat{\mathbf{x}}\}] \cap P = \emptyset$, but $[D_{\leq}(\mathbf{C}_1) \oplus \{\hat{\mathbf{x}}\}] \cap P \neq \emptyset$ by Proposition 4.2.10(i). Hence,

$$\emptyset \neq [D_{\leq}(\mathbf{C}_1) \oplus \{\hat{\mathbf{x}}\}] \cap P \subseteq [D_{\leq}(\mathbf{C}_2) \oplus \{\hat{\mathbf{x}}\}] \cap P = \emptyset,$$

which is a contradiction. Thus, it must be that $\mathrm{E}(P, \mathbf{C}_2) \subseteq \mathrm{E}(P, \mathbf{C}_1)$ as desired. □

4.2.2.3 Normal Cones

Another important cone in characterizing the efficient set of MOLP (4.8) is the normal cone (see Definition 3.1.4). Using the normal cone, Luc [105] gives a necessary and sufficient condition for the efficiency of solutions to MOLP (4.8).

Theorem 4.2.15. *[105, Theorem 4.2.6] Let* $\hat{\mathbf{x}} \in P$*. Then*

(i) $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$ *if and only if* $N_P(\hat{\mathbf{x}})$ *contains some vector* $-\mathbf{C}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \in \mathbb{R}^p_>$*;*

(ii) $\hat{\mathbf{x}} \in \mathrm{wE}(P, \mathbf{C})$ *if and only if* $N_P(\hat{\mathbf{x}})$ *contains some vector* $-\mathbf{C}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \in \mathbb{R}^p_{\geq}$.

A different perspective on Theorem 4.2.15, involving the strict polars of the cones of improving directions, is also offered in Section 4.2.4. The new outlook provides not only another graphical approach to determine efficient solutions in two- or three-dimensions, but also additional insight with respect to highly robust efficient solutions to uncertain MOLPs in Chapter 6.

4.2.2.4 Recession Cones

A third cone that is used in characterizing the efficient set of MOLP (4.8) is the recession cone (see Definition 3.1.5). Using the recession cone of P, the following proposition relies on the intuition that if a direction along which feasibility is retained is also an improving direction, then no efficient solutions should exist.

Proposition 4.2.16. *If* $D_{\leq}(\mathbf{C}) \cap R_P \neq \emptyset$*, then* $\mathrm{E}(P, \mathbf{C}) = \emptyset$.

Proof. Suppose $D_{\leq}(\mathbf{C}) \cap R_P \neq \emptyset$, which implies that $(D_{\leq}(\mathbf{C}) \cap R_P) \oplus \{\mathbf{x}\} \neq \emptyset$ for all $\mathbf{x} \in P$. Hence,

$$(D_{\leq}(\mathbf{C}) \oplus \{\mathbf{x}\}) \cap (R_P \oplus \{\mathbf{x}\}) \neq \emptyset \tag{4.14}$$

by Theorem 2.4.3(iv). Additionally, by definition, it is clear that

$$R_P \oplus \{\mathbf{x}\} \subseteq P \tag{4.15}$$

for all $\mathbf{x} \in P$. Together, (4.14) and (4.15) yield $(D_{\leq}(\mathbf{C}) \oplus \{\mathbf{x}\}) \cap P \neq \emptyset$. Since this is true for all $\mathbf{x} \in P$, it must be that $\mathrm{E}(P, \mathbf{C}) = \emptyset$ by Proposition 4.2.10(i). □

Remark 4.2.17. Observe that this proposition is only relevant if P is unbounded since $R_P = \emptyset$, which forces $D_{\leq}(\mathbf{C}) \cap R_P = \emptyset$, if P is instead bounded (see the comment after Definition 3.1.5).

As a graphical approach in two- or three-dimensions, the recession cone allows for the existence of efficient solutions to be identified simply by looking at the intersection of the cone of improving directions and the recession cone of P, which is illustrated in the ensuing example.

Example 4.2.18. Consider the following two MOLPs:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \mathbf{x} \\ \text{s.t.} \quad & -x_1 + 2x_2 \leq 6 \\ & x_1 , \quad x_2 \geq 0, \end{aligned} \tag{4.16}$$

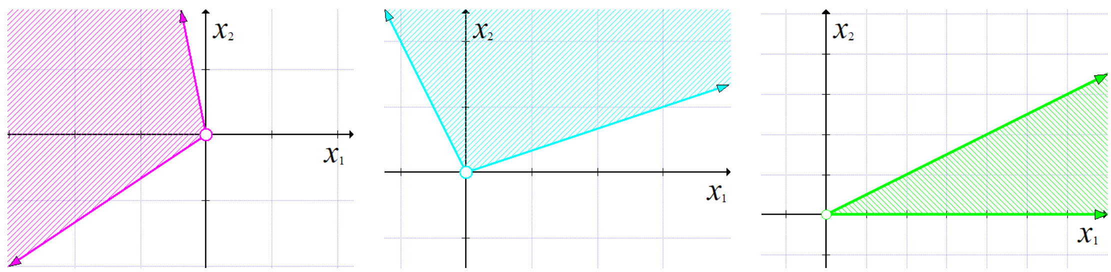
$$\begin{aligned} \min_{\mathbf{x}} \quad & \begin{bmatrix} 3 & -9 \\ -2 & -1 \end{bmatrix} \mathbf{x} \\ \text{s.t.} \quad & -x_1 + 2x_2 \leq 6 \\ & x_1 , \quad x_2 \geq 0. \end{aligned} \tag{4.17}$$

Since both MOLPs have the same feasible region (shown previously in Figure 3.2b), the two MOLPs share the same recession cone, which is shown in Figure 4.6c (and previously in Figure 3.2c) and whose closed-form representation is given by Definition 3.1.5. In addition, the cones of improving directions of MOLPs (4.16) and (4.17) are shown in Figures 4.6a and 4.6b, respectively.

As illustrated in Figure 4.6, it is clear that the antecedent of Proposition 4.2.16 is not satisfied for MOLP (4.16) and so no conclusion may be made. On the other hand, the antecedent is satisfied for MOLP (4.17), which implies that its efficient set is therefore empty.

4.2.3 Computing Efficient Solutions

The natural task now is the computation of efficient solutions to MOLP (4.8). The classical scalarization approach of the weighted-sum method to compute efficient solutions is examined again in Section 4.2.3.1 and revisited with Isermann's Theorem in Section 4.2.3.4. In addition, a method due to Ecker and Kouada using an auxiliary problem that determines whether a given feasible solution of interest is efficient,



(a) The cone of improving directions of MOLP (4.16) (purple)

(b) The cone of improving directions of MOLP (4.17) (teal)

(c) The recession cone (green)

Figure 4.6: The cones of improving directions of MOLPs (4.16) and (4.17), as well as their recession cone

generates an efficient point if the given feasible solution is not itself efficient, or identifies that no efficient solutions exist is reviewed in Section 4.2.3.2. Finally, a method using a second auxiliary problem due to Benson that identifies whether or not the efficient set is empty and generates an efficient extreme point if the efficient set is in fact nonempty is given in Section 4.2.3.3.

4.2.3.1 Weighted-Sum Method

As in the more general case of MOP (4.1), numerous computational approaches for determining the efficient set of MOLP (4.8) rely on scalarizing the multiple objective functions in order to obtain a single-objective linear program (LP). The weighted-sum method is an example of one of these scalarization methods. In particular, given MOLP (4.8) and a vector of scalars $\boldsymbol{\lambda} \in \mathbb{R}^p$, the weighted-sum method combines elements of the cost matrix $\mathbf{C}$ in a weighted sum:

$$\sum_{k=1}^{p} \lambda_k \mathbf{c}_k \mathbf{x},$$

and then replaces the objective with this sum. The resulting problem is known as the weighted-sum LP, denoted WSLP($\boldsymbol{\lambda}$), which is a (deterministic) single-objective

LP given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \boldsymbol{\lambda}^T \mathbf{C}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P. \end{aligned} \tag{4.18}$$

The intent is that solving the weighted-sum LP gives efficient solutions to MOLP (4.8). In fact, we know from Propositions 4.1.6 and 4.1.7 that we are indeed able to obtain (weakly) efficient solutions to MOLP (4.8). Formally, we restate both propositions as the following corollary.

Corollary 4.2.19. *Suppose that* $\hat{\mathbf{x}} \in P$.

(i) [44, Theorem 6.6] If $\hat{\mathbf{x}}$ *is an optimal solution to* $\mathrm{WSLP}(\boldsymbol{\lambda})$ *with* $\boldsymbol{\lambda} \in \mathbb{R}^p_>$, *then* $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$.

(ii) [44, Theorem 6.6] If $\hat{\mathbf{x}}$ *is an optimal solution to* $\mathrm{WSLP}(\boldsymbol{\lambda})$ *with* $\boldsymbol{\lambda} \in \mathbb{R}^p_\geq$, *then* $\hat{\mathbf{x}} \in \mathrm{wE}(P, \mathbf{C})$.

(iii) [44, Proposition 3.10] If $\hat{\mathbf{x}} \in \mathrm{wE}(P, \mathbf{C})$, *then there is some* $\boldsymbol{\lambda} \in \mathbb{R}^p_\geq$ *such that* $\hat{\mathbf{x}}$ *is an optimal solution to* $\mathrm{WSLP}(\boldsymbol{\lambda})$.

Proof. Statements (i) and (ii) are given by Proposition 4.1.6, while (iii) follows directly from Proposition 4.1.7 as P is convex (since it is polyhedral) and the component functions $\mathbf{c}_k\mathbf{x}, k = 1, \ldots, p$, are convex (since they are linear). □

As a result, we are guaranteed by conditions (ii) and (iii) to find all weakly efficient, and thereby all efficient, solutions to MOLP (4.8) by solving $\mathrm{WSLP}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \mathbb{R}^p_\geq$. However, this is not to say that we explicitly know the efficient solutions from the solutions that are weakly efficient but not also efficient, which is a potential downside for decision makers since we may be providing an overabundance of solutions as well as giving potentially worse solutions. We see in Section 4.2.3.4, though, that this may be avoided as all efficient solutions may be obtained directly via the weighted-sum LP with $\boldsymbol{\lambda} \in \mathbb{R}^p_>$.

4.2.3.2 Ecker and Kouada's Method

As in the more general case of MOPs, we may use Benson's method/problem in the context of MOLPs in order to check the efficiency of a given solution $\mathbf{x}_0 \in P$ of interest or generate an efficient point if $\mathbf{x}_0$ is not itself efficient. Additionally, in the context of MOLPs, this method allows for the determiniation of whether or not efficient solutions exist. However, in the current setting, we refer to Benson's problem (or Benson's method) as *Ecker and Kouada's problem* (or method), owing to the fact that Ecker and Kouada [42] presented the same results for MOLPs in 1975 that Benson [10] would present in 1978 for MOPs.

Our motivation here in studying Ecker and Kouada's problem, in addition to the reasons already stated for MOPs, is that the auxiliary problem is an LP so we can consider its dual and obtain further results. We restate the results regarding Benson's problem, which is associated with MOP (4.1), for Ecker and Kouada's problem and the linear case that we are considering, as well as provide new results based on the corresponding dual problem.

Given $\mathbf{x}_0 \in P$, Ecker and Kouada's problem, denoted EKLP($\mathbf{x}_0$), is given by

$$\begin{aligned}
\max_{\mathbf{x},\boldsymbol{l}} \quad & \sum_{k=1}^{p} l_k \\
\text{s.t.} \quad & \mathbf{C}\mathbf{x} + \mathbf{I}_p\boldsymbol{l} = \mathbf{C}\mathbf{x}_0 \\
& \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\
& \mathbf{x} \geqq \mathbf{0} \\
& \boldsymbol{l} \geqq \mathbf{0},
\end{aligned} \tag{4.19}$$

where $\boldsymbol{l} \in \mathbb{R}^p$ is a deviation variable. The dual of EKLP($\mathbf{x}_0$), denoted EKDP($\mathbf{x}_0$), is thus given by

$$\begin{array}{lll} \min\limits_{\mathbf{v},\mathbf{w}} & (\mathbf{C}\mathbf{x}_0)^T\mathbf{v} + \mathbf{b}^T\mathbf{w} & \\ \text{s.t.} & \mathbf{C}^T\mathbf{v} + \mathbf{A}^T\mathbf{w} \geqq \mathbf{0} & \\ & \mathbf{I}_p\mathbf{v} \geqq \mathbf{1} & \\ & \mathbf{w} \geqq \mathbf{0}, & \end{array} \tag{4.20}$$

where $\mathbf{v} \in \mathbb{R}^p, \mathbf{w} \in \mathbb{R}^m$ are dual variables.

We first address the feasibility of EKLP($\mathbf{x}_0$), which is needed when proving results regarding the dual EKDP($\mathbf{x}_0$).

Proposition 4.2.20. *Let $\mathbf{x}_0 \in P$ be given. Then* EKLP($\mathbf{x}_0$) *is feasible.*

Proof. It is clear that EKLP($\mathbf{x}_0$) is feasible since $\boldsymbol{l} = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_0$ satisfy the constraints. □

Using both EKLP($\mathbf{x}_0$) and EKDP($\mathbf{x}_0$), we may recognize the efficiency of a given feasible solution $\mathbf{x}_0 \in P$ as in the following.

Proposition 4.2.21. *Let $\mathbf{x}_0 \in P$ be given.*

(i) [44, Lemma 6.9] The point $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C})$ if and only if EKLP($\mathbf{x}_0$) *has an optimal solution $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ with $\hat{\boldsymbol{l}} = \mathbf{0}$.*

(ii) [44, Lemma 6.10] The point $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C})$ if and only if EKDP($\mathbf{x}_0$) *has an optimal solution $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$ with $(\mathbf{C}\mathbf{x}_0)^T\hat{\mathbf{v}} + \mathbf{b}^T\hat{\mathbf{w}} = 0$*

Proof. (i) Follows the same as the proof of Theorem 4.1.10.

(ii) Since EKDP($\mathbf{x}_0$) is the dual of EKLP($\mathbf{x}_0$), it follows by Strong Duality 2.5.1(i) that $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ is an optimal solution to EKLP($\mathbf{x}_0$) if and only if EKDP($\mathbf{x}_0$) has an optimal solution $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$ with

$$\mathbf{1}^T\hat{\boldsymbol{l}} = (\mathbf{C}\mathbf{x}_0)^T\hat{\mathbf{v}} + \mathbf{b}^T\hat{\mathbf{w}}.$$

Therefore, since $\mathbf{x}_0 \in P$ is efficient if and only if $\text{EKLP}(\mathbf{x}_0)$ has an optimal solution $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ with $\hat{\boldsymbol{l}} = \mathbf{0}$ by part (i), we obtain the result

$$0 = \mathbf{1}^T\hat{\boldsymbol{l}} = (\mathbf{C}\mathbf{x}_0)^T\hat{\mathbf{v}} + \mathbf{b}^T\hat{\mathbf{w}}$$

as desired. □

Not only can Ecker and Kouada's problem $\text{EKLP}(\mathbf{x}_0)$ be used as a method for checking whether or not a given $\mathbf{x}_0 \in P$ is efficient, but it can also generate efficient solutions.

Proposition 4.2.22. *[44, Proposition 6.12(1)] Let $\mathbf{x}_0 \in P$ be given. If $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ is an optimal solution to* $\text{EKLP}(\mathbf{x}_0)$ *(such that the optimal objective value is finite), then $\hat{\mathbf{x}}$ is an efficient solution to* MOLP (4.8)*.*

Proof. Follows the same as the proof of Proposition 4.1.11. □

As already mentioned, using both $\text{EKLP}(\mathbf{x}_0)$ and its dual $\text{EKDP}(\mathbf{x}_0)$, we may also obtain results regarding the emptiness of the efficient set. Since the efficient set is guaranteed to be nonempty if P is bounded (as previously discussed), the following proposition is of more practical importance when the feasible set is unbounded, which is apparent in part (i).

Proposition 4.2.23. *Let $\mathbf{x}_0 \in P$ be given.*

(i) [44, Proposition 6.12] If $\text{EKLP}(\mathbf{x}_0)$ *is unbounded, then* $\text{E}(P, \mathbf{C}) = \emptyset$*.*

(ii) If EKDP($\mathbf{x}_0$) *is infeasible, then* E($P, \mathbf{C}$) $= \emptyset$*.*

Proof. (i) Suppose EKLP($\mathbf{x}_0$) is unbounded, and assume for the sake of contradiction that E($P, \mathbf{C}$) $\neq \emptyset$. Since EKLP($\mathbf{x}_0$) is unbounded, we know that its dual EKDP($\mathbf{x}_0$) must be infeasible by the Fundamental Theorem of Duality 2.5.1. Further, since E($P, \mathbf{C}$) $\neq \emptyset$, there exists an efficient (and therefore weakly efficient) solution, say $\bar{\mathbf{x}} \in P$. Hence, by Corollary 4.2.19(iii), there exists a $\bar{\boldsymbol{\lambda}} \in \mathbb{R}^p_{\geq}$ such that $\bar{\mathbf{x}}$ is an optimal solution to WSLP($\bar{\boldsymbol{\lambda}}$), which is given by

$$\begin{array}{lrcl} \min\limits_{\mathbf{x}} & (\bar{\boldsymbol{\lambda}})^T\mathbf{C}\mathbf{x} & & \max\limits_{\mathbf{w}} & \mathbf{b}^T\mathbf{w} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leqq \mathbf{b} & \overset{\text{dual}}{\Longleftrightarrow} & \text{s.t.} & \mathbf{A}^T\mathbf{w} \leqq \mathbf{C}^T\bar{\boldsymbol{\lambda}} \\ & \mathbf{x} \geqq \mathbf{0} & & & \mathbf{w} \leqq \mathbf{0}. \end{array}$$

Since WSLP($\bar{\boldsymbol{\lambda}}$) is feasible, its dual must also be feasible. In other words, there exists a $\mathbf{w} \in \mathbb{R}^m$ such that $\mathbf{A}^T\mathbf{w} \leqq \mathbf{C}^T\bar{\boldsymbol{\lambda}}$, or equivalently, such that

$$-\mathbf{A}^T\mathbf{w} + \mathbf{C}^T\bar{\boldsymbol{\lambda}} \geqq \mathbf{0}. \tag{4.21}$$

In order to obtain the necessary contradiction, we construct a solution to EKDP($\mathbf{x}_0$) from $\mathbf{w}$ and $\bar{\boldsymbol{\lambda}}$. Let $\theta > 0$ be the smallest component in $\bar{\boldsymbol{\lambda}}$ and set $\boldsymbol{\lambda} = \bar{\boldsymbol{\lambda}}/\theta \geqq \mathbf{1}$. Accordingly, (4.21) gives $\mathbf{A}^T(-\mathbf{w}/\theta) + \mathbf{C}^T\boldsymbol{\lambda} \geqq \mathbf{0}$. Letting $\bar{\mathbf{v}} = \boldsymbol{\lambda} \geqq \mathbf{1}$ and $\bar{\mathbf{w}} = -\mathbf{w}/\theta \geqq \mathbf{0}$, we obtain a feasible solution $(\bar{\mathbf{v}}, \bar{\mathbf{w}})$ to EKDP($\mathbf{x}_0$), which is a contradiction. Thus, it must be that E($P, \mathbf{C}$) $= \emptyset$.

(ii) Suppose EKDP($\mathbf{x}_0$) is infeasible. Hence, EKLP($\mathbf{x}_0$) must be unbounded by the Fundamental Theorem of Duality 2.5.1 and Proposition 4.2.20. Therefore, E($P, \mathbf{C}$) $= \emptyset$ by part (i). $\square$

The proceeding example provides an illustration of Propositions 4.2.21, 4.2.22, and 4.2.23, which comprise Ecker and Kouada's method.

Example 4.2.24. Consider MOLP (4.9) of Example 4.2.4. The corresponding auxiliary problem or Ecker and Kouada's problem for any given $\mathbf{x}_0 \in P_1$ is

$$\begin{array}{lrcl} \max\limits_{\mathbf{x},\boldsymbol{l}} & & l_1 + l_2 & \\ \text{s.t.} & 2x_1 - 3x_2 + l_1 & & = 2x_{01} - 3x_{02} \\ & 5x_1 + \ \ x_2 + & l_2 & = 5x_{01} + x_{02} \\ & & l_1 \ , \ \ l_2 & \geq 0 \\ & (x_1 \ , \ \ x_2) & & \in P_1. \end{array}$$

In particular, if we take $\mathbf{x}_0 = (0, 3)$, which we know is efficient, then EKLP$((0, 3)$ is given by

$$\begin{array}{lrcl} \max\limits_{\mathbf{x},\boldsymbol{l}} & & l_1 + l_2 & \\ \text{s.t.} & 2x_1 - 3x_2 + l_1 & & = -9 \\ & 5x_1 + \ \ x_2 + & l_2 & = 3 \\ & & l_1 \ , \ \ l_2 & \geq 0 \\ & (x_1 \ , \ \ x_2) & & \in P_1. \end{array}$$

and the optimal solution is guaranteed to be $\boldsymbol{l} = \mathbf{0}$ by Proposition 4.2.21(i). Alternatively, if we take $\mathbf{x}_0 = (2, 4)$, for example, which is not efficient, then solving EKLP$((2, 4))$ yields some other solution $\hat{\mathbf{x}} \in P_1$ that is efficient due to Proposition 4.2.22.

With Propositions 4.2.21, 4.2.22, and 4.2.23, as well as Example 4.2.24, in mind, the impact of Ecker and Kouada's method is two-fold: the decision maker may (i) determine whether or not the efficient set of MOLP (4.8) is empty prior to attempting to solve the full problem, and (ii) check if a specific feasible solution deemed desirable is also efficient a priori. Importantly, similar results may be developed regarding solutions to MOLPs with uncertain objective coefficient data in Chapter 6 that still provide decision makers with these same two tools.

4.2.3.3 Benson's Method for MOLPs

The downside of Ecker and Kouada's method is that when it generates an efficient point, there is no guarantee that the point will be an extreme point of the feasible set (see Steur [128]). In particular, this is a negative characteristic when trying to implement a simplex algorithm to solve an MOLP since the algorithm moves from one extreme point to the next. If we do not have one from which to begin the algorithm, then we cannot apply the method at all. As an attempt to correct this flaw, Benson [11] demonstrates a second class of auxiliary problems that can be used to provide not only efficiency results, but also guarantee an extreme point.

For any given $\mathbf{x}_0 \in P$, this second class of auxiliary problems, referred to as *Benson's LP* and denoted BLP($\mathbf{x}_0$), is given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{1}^T\mathbf{C}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{C}\mathbf{x} \leqq \mathbf{C}\mathbf{x}_0 \\ & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0}, \end{aligned} \tag{4.22}$$

where $\mathbf{1} \in \mathbb{R}^p$ is the p-dimensional vector of all ones. The dual of BLP($\mathbf{x}_0$), denoted

BDP($\mathbf{x}_0$), is thus given by

$$\begin{aligned}\min_{\mathbf{v},\mathbf{w}} \quad & (\mathbf{C}\mathbf{x}_0)^T\mathbf{v} + \mathbf{b}^T\mathbf{w} \\ \text{s.t.} \quad & \mathbf{C}^T\mathbf{v} + \mathbf{A}^T\mathbf{w} \leqq \mathbf{C}^T\mathbf{1} \\ & \mathbf{v} \leqq \mathbf{0} \\ & \mathbf{w} \leqq \mathbf{0},\end{aligned}$$

or equivalently,

$$\begin{aligned}\min_{\mathbf{v},\mathbf{w}} \quad & (\mathbf{C}\mathbf{x}_0)^T\mathbf{v} + \mathbf{b}^T\mathbf{w} \\ \text{s.t.} \quad & \mathbf{C}^T\mathbf{v} + \mathbf{A}^T\mathbf{w} \geqq -\mathbf{C}^T\mathbf{1} \\ & \mathbf{v} \geqq \mathbf{0} \\ & \mathbf{w} \geqq \mathbf{0},\end{aligned} \tag{4.23}$$

where $\mathbf{v} \in \mathbb{R}^p$ and $\mathbf{w} \in \mathbb{R}^m$ are dual variables. The second representation of the dual is the primary form we utilize.

Theorem 4.2.25. *Let* $\mathbf{x}_0 \in P$ *be given.*

(i) [11, Proof of Theorem 3] BLP($\mathbf{x}_0$) *has an optimal solution if and only if* MOLP (4.8) *has an efficient solution.*

(ii) [11, Theorem 3] BDP($\mathbf{x}_0$) *has an optimal solution if and only if* MOLP (4.8) *has an efficient solution.*

Proof. (i) ($\Longrightarrow$) Assume that $\hat{\mathbf{x}} \in P$ is an efficient solution to MOLP (4.8). By Definition 4.2.1, there does not exist an $\mathbf{x} \in P$ such that $\mathbf{C}\mathbf{x} \leq \mathbf{C}\hat{\mathbf{x}}$, i.e., such that $\mathbf{c}_k\mathbf{x} \leq \mathbf{c}_k\hat{\mathbf{x}}$ for all $k = 1, \ldots, p$ with at least one strict. Hence,

$$\sum_{k=1}^{p} \mathbf{c}_k\mathbf{x} < \sum_{k=1}^{p} \mathbf{c}_k\hat{\mathbf{x}},$$

or equivalently,

$$\mathbf{1}^T\mathbf{C}\mathbf{x} < \mathbf{1}^T\mathbf{C}\hat{\mathbf{x}}.$$

Therefore, in terms of the objective function of BLP($\mathbf{x}_0$), there does not exist an $\mathbf{x} \in P$ that is better than $\hat{\mathbf{x}}$. Since the feasible region of BLP($\mathbf{x}_0$) is a restriction of P, it is clear that $\hat{\mathbf{x}}$ is also an optimal solution to BLP($\mathbf{x}_0$).

($\Longleftarrow$) Suppose $\hat{\mathbf{x}}$ is an optimal solution to BLP($\mathbf{x}_0$), and assume for the sake of contradiction that $\hat{\mathbf{x}}$ is not an efficient solution to MOLP (4.8). As a result,

$$\mathbf{C}\hat{\mathbf{x}} \leqq \mathbf{C}\mathbf{x}_0, \tag{4.24}$$

and there exists an $\bar{\mathbf{x}} \in P$ by Definition 4.2.1 such that

$$\mathbf{C}\bar{\mathbf{x}} \leq \mathbf{C}\hat{\mathbf{x}}, \tag{4.25}$$

respectively. Hence, (4.24) and (4.25) yield $\mathbf{C}\bar{\mathbf{x}} \leq \mathbf{C}\mathbf{x}_0$, i.e., $\mathbf{C}\bar{\mathbf{x}} \leqq \mathbf{C}\mathbf{x}_0$. Since $\bar{\mathbf{x}} \in P$, it follows that $\bar{\mathbf{x}}$ is a feasible solution to BLP($\mathbf{x}_0$). In addition, as in the converse direction, (4.25) implies that $\mathbf{1}^T\mathbf{C}\bar{\mathbf{x}} < \mathbf{1}^T\mathbf{C}\hat{\mathbf{x}}$. Thus, we have a feasible solution $\bar{\mathbf{x}}$ to BLP($\mathbf{x}_0$) that is better in terms of the objective function of BLP($\mathbf{x}_0$) than the optimal solution $\hat{\mathbf{x}}$, which is a contradiction. Therefore, it must be that $\hat{\mathbf{x}}$ is an efficient solution to MOLP (4.8).

(ii) Since BDP($\mathbf{x}_0$) is the dual of BLP($\mathbf{x}_0$), Strong Duality 2.5.1(i) yields BDP($\mathbf{x}_0$) has an optimal solution if and only if BLP($\mathbf{x}_0$) has an optimal solution. Therefore, since BLP($\mathbf{x}_0$) has an optimal solution if and only if MOLP (4.8) has an efficient solution by part (i), we obtain the desired result. □

Theorem 4.2.26. *[11, Theorem 2] Let* $\mathbf{x}_0 \in P$ *be given. If* $\mathrm{BDP}(\mathbf{x}_0)$ *has an optimal solution* $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$*, then* $\mathrm{WSLP}(\boldsymbol{\lambda})$ *with* $\boldsymbol{\lambda} = \hat{\mathbf{v}} + \mathbf{1}$ *has an extreme point optimal solution.*

Proof. Let $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$ be an optimal solution to $\mathrm{BDP}(\mathbf{x}_0)$. With $\boldsymbol{\lambda} = \hat{\mathbf{v}} + \mathbf{1}$, the dual of $\mathrm{WSLP}(\boldsymbol{\lambda})$ is given by

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathbf{b}^T\mathbf{w} \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{w} \leqq \mathbf{C}^T(\hat{\mathbf{v}} + \mathbf{1}) \\ & \mathbf{w} \leqq \mathbf{0} \end{aligned} \iff \begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{b}^T\mathbf{w} \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{w} \geqq -\mathbf{C}^T(\hat{\mathbf{v}} + \mathbf{1}) \\ & \mathbf{w} \geqq \mathbf{0}. \end{aligned} \tag{4.26}$$

Now, since $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$ is an optimal (and hence feasible) solution to $\mathrm{BDP}(\mathbf{x}_0)$, $\hat{\mathbf{w}}$ is also clearly a solution to LP (4.26), that is, LP (4.26) is feasible.

Moreover, since P is nonempty, $\mathrm{WSLP}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = \hat{\mathbf{v}} + \mathbf{1}$ is feasible. Because both the primal and dual problems are feasible, the Fundamental Theorem of Duality 2.5.1 guarantees that $\mathrm{WSLP}(\boldsymbol{\lambda})$ has a finite optimal solution. In particular, $\mathrm{WSLP}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = \hat{\mathbf{v}} + \mathbf{1}$ has an optimal extreme point solution (by Theorem 2.8, Bertsimas and Tsitsiklis [15]) as desired. □

Corollary 4.2.27. *[11, p. 497] Let* $\mathbf{x}_0 \in P$ *be given. If* $\mathrm{BDP}(\mathbf{x}_0)$ *has an optimal solution* $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$*, then* MOLP (4.8) *has an efficient extreme point efficient solution.*

Proof. Let $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$ be an optimal solution to $\mathrm{BDP}(\mathbf{x}_0)$. By Theorem 4.2.26, $\mathrm{WSLP}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = \hat{\mathbf{v}} + \mathbf{1}$ has an optimal extreme point solution, say $\hat{\mathbf{x}} \in P$. Since $\boldsymbol{\lambda} = \hat{\mathbf{v}} + \mathbf{1} \in \mathbb{R}^p_>$, Corollary 4.2.19(i) yields $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$. Therefore, we have obtained an efficient extreme point solution to MOLP (4.8) as desired. □

Considering Corollary 4.2.27, in order to obtain an (initial) efficient extreme point solution to MOLP (4.8), we simply need to solve $\mathrm{BDP}(\mathbf{x}_0)$, which yields the optimal solution $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$, and then solve $\mathrm{WSLP}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = \hat{\mathbf{v}} + \mathbf{1}$. The optimal solution to $\mathrm{WSLP}(\boldsymbol{\lambda})$ is the desired efficient extreme point solution.

4.2.3.4 Isermann's Theorem

Using the efficiency results for both EKLP($\mathbf{x}_0$) and EKDP($\mathbf{x}_0$), we now prove that all efficient solutions to MOLP (4.8) may be obtained by solving a weighted-sum LP (cf. Corollary 4.2.19). In other words, we are ready to prove that each efficient solution to an MOLP may be obtained as the optimal solution to a weighted-sum LP.

Theorem 4.2.28. *[44, Isermann's Theorem, Theorem 6.11] Let $\mathbf{x}_0 \in P$ be given. Then $\mathbf{x}_0$ is an efficient solution to* MOLP (4.8) *if and only if there exists a $\boldsymbol{\lambda} \in \mathbb{R}^p_>$ such that*

$$\boldsymbol{\lambda}^T\mathbf{C}\mathbf{x}_0 \leq \boldsymbol{\lambda}^T\mathbf{C}\mathbf{x}$$

for all $\mathbf{x} \in P$.

Proof. ($\Longleftarrow$) Suppose there exists a $\boldsymbol{\lambda} \in \mathbb{R}^p_>$ such that $\boldsymbol{\lambda}^T\mathbf{C}\mathbf{x}_0 \leq \boldsymbol{\lambda}^T\mathbf{C}\mathbf{x}$ for all $\mathbf{x} \in P$, and assume for the sake of contradiction that $\mathbf{x}_0 \notin \mathrm{E}(P, \mathbf{C})$. By Definition 4.2.1, there exists an $\mathbf{x} \in P$ such that $\mathbf{C}\mathbf{x} \leq \mathbf{C}\mathbf{x}_0$, i.e., such that $\mathbf{c}_k\mathbf{x} \leq \mathbf{c}_k\mathbf{x}_0$ for all $k = 1, \ldots, p$ with at least one strict. Equivalently, since $\lambda_k > 0$ for all $k = 1, \ldots, p$, it follows that $\lambda_k\mathbf{c}_k\mathbf{x} \leq \lambda_k\mathbf{c}_k\mathbf{x}_0$ for all $k = 1, \ldots, p$ with at least one strict, which implies

$$\sum_{k=1}^{p} \lambda_k\mathbf{c}_k\mathbf{x} < \sum_{k=1}^{p} \lambda_k\mathbf{c}_k\mathbf{x}_0.$$

By definition,

$$\boldsymbol{\lambda}^T\mathbf{C}\mathbf{x} < \boldsymbol{\lambda}^T\mathbf{C}\mathbf{x}_0,$$

which is a contradiction. Thus, it must be that $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C})$.

($\Longrightarrow$) Suppose $\mathbf{x}_0 \in P$ is an efficient solution to MOLP (4.8). By Proposition 4.2.21(ii), we equivalently know that EKDP($\mathbf{x}_0$) has an optimal solution $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$ with

$(\mathbf{Cx}_0)^T\hat{\mathbf{v}} + \mathbf{b}^T\hat{\mathbf{w}} = 0$, i.e., $\mathbf{b}^T\hat{\mathbf{w}} = -\hat{\mathbf{v}}^T\mathbf{Cx}_0$. Taking $\mathbf{v} = \hat{\mathbf{v}}$ in EKDP($\mathbf{x}_0$), we obtain

$$\begin{array}{lll} \min\limits_{\mathbf{v},\mathbf{w}} & (\mathbf{Cx}_0)^T\hat{\mathbf{v}} + \mathbf{b}^T\mathbf{w} & \\ \text{s.t.} & \mathbf{C}^T\hat{\mathbf{v}} + \mathbf{A}^T\mathbf{w} \geqq \mathbf{0} & \\ & \mathbf{I}_p\hat{\mathbf{v}} \qquad\quad \geqq \mathbf{1} & \\ & \qquad\qquad\quad \mathbf{w} \geqq \mathbf{0} & \end{array} \implies \begin{array}{ll} \min\limits_{\mathbf{w}} & \mathbf{b}^T\mathbf{w} \\ \text{s.t.} & \mathbf{A}^T\mathbf{w} \geqq -\mathbf{C}^T\hat{\mathbf{v}} \\ & \mathbf{w} \geqq \mathbf{0}. \end{array} \tag{4.27}$$

Observe that $\hat{\mathbf{w}}$ is an optimal solution to LP (4.27), and consider the dual of LP (4.27) given by

$$\begin{array}{ll} \max\limits_{\mathbf{x}} & -\hat{\mathbf{v}}^T\mathbf{Cx} \\ \text{s.t.} & \mathbf{Ax} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0} \end{array} \iff \begin{array}{ll} \min\limits_{\mathbf{x}} & \hat{\mathbf{v}}^T\mathbf{Cx} \\ \text{s.t.} & \mathbf{Ax} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0}. \end{array} \tag{4.28}$$

Hence, for an optimal solution $\hat{\mathbf{x}}$ to LP (4.28), we obtain

$$\mathbf{b}^T\hat{\mathbf{w}} = -\hat{\mathbf{v}}^T\mathbf{C}\hat{\mathbf{x}} \tag{4.29}$$

by Strong Duality 2.5.1(i). Since (4.29) is also satisfied by $\mathbf{x}_0$, it follows that $\mathbf{x}_0$ is an optimal solution to LP (4.28), i.e.,

$$\hat{\mathbf{v}}^T\mathbf{Cx}_0 \leq \hat{\mathbf{v}}^T\mathbf{Cx} \text{ for all } \mathbf{x} \in P.$$

Letting $\boldsymbol{\lambda} = \hat{\mathbf{v}} \geqq \mathbf{1} > \mathbf{0}$, we obtain the result. □

Recognizing $\boldsymbol{\lambda}^T\mathbf{Cx}$ as the objective function of WSLP($\boldsymbol{\lambda}$), it is thus possible based on Isermann's Theorem to obtain all efficient solutions to MOLP (4.8) by solving a weighted-sum LP. The only remaining challenge is finding the appropriate weights to obtain each efficient solution (see Wiecek et al. [141]).

4.2.4 New Result

As previously mentioned, in order to provide a different point of view on Theorem 4.2.15, the polars and strict polars of the cones of improving directions are needed. Given the cones of improving directions $D_{\leqq}(\mathbf{C}), D_{\leq}(\mathbf{C})$, and $D_{<}(\mathbf{C})$, we denote their polars by $D^{+}_{\leqq}(\mathbf{C}), D^{+}_{\leq}(\mathbf{C})$, and $D^{+}_{<}(\mathbf{C})$, and their strict polars by $D^{s+}_{\leqq}(\mathbf{C}), D^{s+}_{\leq}(\mathbf{C})$, and $D^{s+}_{<}(\mathbf{C})$, respectively. Under certain assumptions such as the acuteness or closure of the cones of improving directions, their polars and strict polars are given by Proposition 3.2.7 and Theorem 3.2.13, respectively.

The main motivation for reframing Theorem 4.2.15, which is given by Theorem 4.2.6, Luc [105], through a connection between the strict polar of the closed (or open) cone of improving directions and the set of all vectors $-\mathbf{C}^T\boldsymbol{\lambda}$ such that $\boldsymbol{\lambda} > \mathbf{0}$ (or $\boldsymbol{\lambda} \geq \mathbf{0}$) is that added insight in Chapter 6 is available.

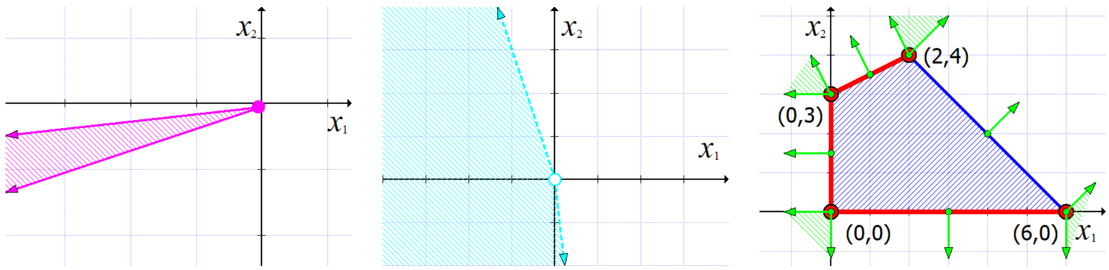
Theorem 4.2.29. *Let $\hat{\mathbf{x}} \in P$.*

(i) Assume $D_{\leqq}(\mathbf{C})$ is acute. Then $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C})$ if and only if $N_P(\hat{\mathbf{x}}) \cap D^{s+}_{\leqq}(\mathbf{C}) \neq \emptyset$.

(ii) Assume $\mathrm{cl}(D_{<}(\mathbf{C})) = D_{\leqq}(\mathbf{C})$. Then $\hat{\mathbf{x}} \in \mathrm{wE}(P, \mathbf{C})$ if and only if $N_P(\hat{\mathbf{x}}) \cap D^{s+}_{<}(\mathbf{C}) \neq \emptyset$.

Proof. (i) Since $D_{\leqq}(\mathbf{C})$ is acute, we know that $D^{s+}_{\leqq}(\mathbf{C}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{C}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$ by Theorem 3.2.13(i). As $D^{s+}_{\leqq}(\mathbf{C})$ is the set of all vectors $-\mathbf{C}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}$, the result follows from Theorem 4.2.15(i).

(ii) Since $\mathrm{cl}(D_{<}(\mathbf{C})) = D_{\leqq}(\mathbf{C})$, we know that $D^{s+}_{<}(\mathbf{C}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{C}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}$ by Theorem 3.2.13(iii). As $D^{s+}_{<}(\mathbf{C})$ is the set of all vectors $-\mathbf{C}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}$, the result follows from Theorem 4.2.15(ii). □



(a) $D_{\leqq}(\mathbf{C})$ (purple) (b) $D^{s+}_{\leqq}(\mathbf{C})$ (teal) (c) The efficient set (red) and normal cones (green)

Figure 4.7: Normal cones to P_1, as well as the closed cone of improving directions and its strict polar for MOLP (4.30)

The proceeding example is given as an illustration of Theorem 4.2.29.

Example 4.2.30. Consider the following MOLP:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \begin{bmatrix} 3 & -9 \\ -1 & 9 \end{bmatrix} \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P_1. \end{aligned} \tag{4.30}$$

The closed cone of improving directions is shown in Figure 4.7a. Since $D_{\leqq}(\mathbf{C})$ is clearly acute, we know that its strict polar is given by Theorem 3.2.13(i), which is shown in Figure 4.7b. As illustrated in Figure 4.7, the only points $\hat{\mathbf{x}} \in P_1$ at which the intersection $N_{P_1}(\hat{\mathbf{x}}) \cap D^{s+}_{\leqq}(\mathbf{C})$ in Theorem 4.2.29(i) hold are those that we know are efficient (by using, e.g., $D_{\leq}(\mathbf{C})$ and the graphical approach of Proposition 4.2.10(i)).

We note several important observations regarding Theorem 4.2.29 and Example 4.2.30. First, although Theorem 4.2.29 is weaker than Theorem 4.2.15 as a result of the additional assumptions about the cones of improving directions, the advantage, as already mentioned, is added insight. Second, Theorem 4.2.29(i) may be equivalently stated with $D^{s+}_{\leq}(\mathbf{C})$ instead of $D^{s+}_{\leqq}(\mathbf{C})$ since $D^{s+}_{\leq}(\mathbf{C}) = D^{s+}_{\leqq}(\mathbf{C})$ when $D_{\leqq}(\mathbf{C})$ is acute by Theorem 3.2.13(ii). In either case, the acuteness of $D_{\leqq}(\mathbf{C})$ may be verified as in Section 3.2.4. Next, observe that if $-\mathbf{C}^T\boldsymbol{\lambda} = \mathbf{0}$ for some $\boldsymbol{\lambda} > \mathbf{0}$, then the entire feasible set is efficient since $N_P(\hat{\mathbf{x}})$ necessarily contains $\mathbf{0}$. The same line of thought may be followed for $\boldsymbol{\lambda} \geq \mathbf{0}$ and the weakly efficient set. Additionally, as we want the intersection $N_P(\hat{\mathbf{x}}) \cap D^{\mathrm{s+}}_{\leqq}(\mathbf{C})$ to be nonempty, it is important to know if and when $D^{\mathrm{s+}}_{\leqq}(\mathbf{C})$ is nonempty since if it is not, then the result never holds. (We are only concerned with the nonemptiness of $D^{\mathrm{s+}}_{\leqq}(\mathbf{C})$ since $N_P(\hat{\mathbf{x}})$ is always nonempty as it always contains the origin.) To this end, since we assume that $D_{\leqq}(\mathbf{C})$ is acute, we know that $D^{\mathrm{s+}}_{\leqq}(\mathbf{C})$ is nonempty by Proposition 3.2.12(i).

It is also worth addressing the impact of the assumption that $D_{\leqq}(\mathbf{C})$ is acute. Since $D_{\leqq}(\mathbf{C})$ is closed, being acute is equivalent to $D_{\leqq}(\mathbf{C})$ being pointed by Proposition 3.1.3. Hence, we implicitly assume that $\mathrm{rank}(\mathbf{C}) = n$ by Theorem 3.1.12. Moreover, since $\mathrm{rank}(\mathbf{C}) \leq \min\{p, n\}$, we obtain that $p \geq n$. The consequence of this is that the number of criteria is greater than or equal to the number of decision variables. As a result, models that incorporate the numerous preferences of multiple decision makers explicitly through many criteria may be used. In the literature, *many-objective* problems that incorporate 4 or more criteria (see, e.g., Ishibuchi et al. [87]) may be relevant to this situation.

Chapter 5

Uncertain Multiobjective Programs

Assuming uncertainty in the objective coefficient data of a multiobjective program (MOP), we obtain an uncertain MOP (UMOP). We give consideration only to uncertainty in the objective function coefficients, as in, e.g., Ehrgott et al. [45], Ide [79], Ide and Schöbel [82], and Kuhn et al. [94]. This is for two reasons. First, the importance of multiobjective optimization is the inclusion of multiple criteria. Second, as in classical robust (single-objective) optimization, a solution is only considered feasible to the uncertain problem if it is feasible for every realization of the uncertain data. To this end, Ide and Schöbel [82] note that if the feasible region is considered to be uncertain as well, then a different feasible set results for each uncertainty so that any robust solution must be in their intersection. Redefining the original feasible region to be this intersection, it is possible to restrict uncertainty to only the objective function coefficients.

We introduce the formulation of the UMOP that we consider, as well as define and discuss the solution concept of interest, highly robust efficiency, with respect to general uncertainty sets in Section 5.1. In the remaining sections, we examine various results, namely computational methods, with respect to finite uncertainty sets. We first present an atypical or theoretical robust counterpart (RC) for highly robust efficient solutions to the UMOP under consideration and examine another possible counterpart problem in Section 5.2. Second, in Section 5.3, we propose a naive extension of the deterministic weighted-sum method (refer to Section 4.1.1) that may be used to compute highly robust efficient solutions. Finally, in Section 5.4, we extend the results due to Benson (see Section 4.1.2) from the deterministic setting to the uncertain one in order to develop recognition and generation results for highly robust efficient solutions. As a result of our problem formulation and solution concept, this extension results in (at least) three separate Benson-type problems of concern.

Some results in the following sections may include two proofs as they are extensions of results from deterministic multiobjective optimization. The first proof typically follows a similar format to that from the deterministic case, but is done to illustrate that the proof may be done independently of the original result. The second proof, on the other hand, typically utilizes the definition of highly robust efficiency along with the corresponding result from deterministic multiobjective optimization.

5.1 Problem Formulation and Solution Concept

Considering uncertain input data in the objective function coefficients of MOP (4.1), we obtain a UMOP, denoted MOP(U). The UMOP is a collection or family of MOPs, with each member denoted MOP($\mathbf{u}$), indexed by the (uncertain) parameter

$\mathbf{u}$. In particular, MOP(U) is given by

$$\left\{\begin{array}{ll}\min\limits_{\mathbf{x}} & \mathbf{f}(\mathbf{x},\mathbf{u}) \\ \text{s.t.} & \mathbf{x}\in X\end{array}\right\}_{\mathbf{u}\in U,} \tag{5.1}$$

where $U \subset \mathbb{R}^q$ is a nonempty set modeling the uncertainty referred to as the *uncertainty set* or *set of scenarios*, $X \subset \mathbb{R}^n$ is the (deterministic) feasible region given by (4.2), and $\mathbf{f} : X \times U \to \mathbb{R}^p$ is the vector-valued objective function carrying uncertain coefficients. Every problem MOP($\mathbf{u}$) in the collection, which is called an *instance* of MOP(U), is associated with a particular value of $\mathbf{u} \in U$ that is referred to as an *uncertainty*, *realization*, or *scenario*.

While the solution concept for MOP(U) is not obvious, the concept for each instance is clear since MOP($\mathbf{u}$) is a deterministic MOP given the scenario $\mathbf{u} \in U$. Accordingly, (wE($X, \mathbf{f}(\cdot, \mathbf{u})$)) E($X, \mathbf{f}(\cdot, \mathbf{u})$) denotes the *(weakly) efficient set* of MOP($\mathbf{u}$) for some realization $\mathbf{u} \in U$, and the uncertain problem (5.1) reduces to the deterministic problem (4.1) if the set of scenarios U is a singleton. As with MOP (4.1), in order to guarantee the existence of (weakly) efficient solutions to MOP($\mathbf{u}$) for each $\mathbf{u} \in U$, the standard conditions involving compactness and semicontinuity may be assumed (see Corollary 2.26 and Theorem 2.19, Ehrgott [44], respectively).

In practical problems, conflicting objective functions are unlikely to depend on the same uncertainties or scenarios. To accommodate this reality, we assume that the uncertainties of the objective functions $f_1, \ldots, f_p$ are independent of each other, which is a concept first introduced by Ehrgott et al. [45] known as objective-wise uncertainty. In particular, MOP(U) is said to be of *objective-wise uncertainty* if $U = U_1 \times \cdots \times U_p$, where $U_k \subset \mathbb{R}^{q_k}, k = 1, \ldots, p$, is referred to as a *partial uncertainty* *set*, such that

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} f_1(\mathbf{x},\mathbf{u}_1) & \cdots & f_p(\mathbf{x},\mathbf{u}_p) \end{bmatrix}^T \tag{5.2}$$

with $\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_p \end{bmatrix}^T \in U$, and $\mathbf{u}_k \in U_k, k = 1, \ldots, p$.

To solve UMOPs with objective-wise uncertainty in the objective function coefficients, a variety of possible solution concepts may be chosen. For a comprehensive survey of ten different concepts of robust efficiency for this type of problem and their numerous relationships, refer to Ide and Schöbel [82]. We choose to adopt the conservative concept of *necessary efficiency* (see Inuiguchi and Kume [83]) that is first mentioned in 1980 by Bitran [18] in the context of solutions to interval multiobjective linear programs. Such solutions are efficient with respect to every realization of the uncertain data. However, in keeping with the recent literature on robust multiobjective optimization, we refer to these solutions as *highly robust efficient*.

Definition 5.1.1. A solution $\mathbf{x}^* \in X$ to $\text{MOP}(U)$ is said to be *highly robust (weakly) efficient solution* provided for every $\mathbf{u} \in U$ there does not exist an $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x},\mathbf{u})\ (<) \leq \mathbf{f}(\mathbf{x}^*,\mathbf{u})$. The *highly robust (weakly) efficient set* of $\text{MOP}(U)$ is denoted by $(\text{wE}(X,\mathbf{f}(\cdot,\mathbf{u}),U))\ \text{E}(X,\mathbf{f}(\cdot,\mathbf{u}),U)$.

Based on the definition of highly robust (weak) efficiency, we have an immediate result.

Proposition 5.1.2. *[82, p. 242] A solution $\mathbf{x}^* \in X$ to $\text{MOP}(U)$ is highly robust (weakly) efficient if and only if* $(\mathbf{x}^* \in \bigcap_{\mathbf{u}\in U} \text{wE}(X,\mathbf{f}(\cdot,\mathbf{u})))\ \mathbf{x}^* \in \bigcap_{\mathbf{u}\in U} \text{E}(X,\mathbf{f}(\cdot,\mathbf{u}))$.

Proof. Suppose $\mathbf{x}^* \in X$ is a highly robust efficient solution to $\text{MOP}(U)$. By definition, for every $\mathbf{u} \in U$ there does not exist an $\mathbf{x} \in X$ (for that particular realization of $\mathbf{u} \in U$) such that $\mathbf{f}(\mathbf{x},\mathbf{u}) \leq \mathbf{f}(\mathbf{x}^*,\mathbf{u})$. Equivalently, $\mathbf{x}^* \in \text{E}(X,\mathbf{f}(\cdot,\mathbf{u}))$ for every $\mathbf{u} \in U$. That is, $\mathbf{x}^* \in \bigcap_{\mathbf{u}\in U} \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}))$. Therefore, $\mathbf{x}^* \in X$ is highly robust efficient if and only if $\mathbf{x}^* \in \bigcap_{\mathbf{u}\in U} \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}))$.

The proof follows similarly for weak solutions. □

With Proposition 5.1.2 in mind, we recognize that highly robust efficient solutions are those decisions $\mathbf{x} \in X$ that are efficient with respect to every instance MOP($\mathbf{u}$). As a result, although we generally assume that the (weakly) efficient set associated with MOP($\mathbf{u}$) is nonempty, it is apparent that highly robust efficient solutions may not exist. We do not provide existence results for highly robust efficient solutions to MOP(U) here, but do so in the next chapter for UMOLPs.

Remark 5.1.3. From Proposition 5.1.2 and Remark 4.1.2 (applied to the efficient and weakly efficient sets of MOP($\mathbf{u}$) for each $\mathbf{u} \in U$), it is clear that $\mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U) \subseteq \mathrm{wE}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$.

In general, when the uncertainty set contains infinitely many elements, Proposition 5.1.2 also indicates that finding highly robust (weakly) efficient solutions to MOP(U) is impractical. That is, if we wanted to find all highly robust (weakly) efficient points, then we would need to find the (weakly) efficient set of an infinite number of instances, which is unrealistic. Due to this nature, we need to explore other avenues. In particular, we restrict our attention to finite uncertainty sets or infinite uncertainty sets that may be considered as finite due to special properties. The latter situation is often referred to as an *uncertainty set reduction* or the two UMOPs, one with respect to the original uncertainty set and the other with respect to the finite set of scenarios, are said to be *equivalent* since the highly robust efficient set of the UMOP with respect to the original uncertainty set is equivalent to the highly robust efficient set of the UMOP with respect to the (reduced) finite set of scenarios. One such reduction or equivalence is provided by Theorem 46, Ide and Schöbel [82]).

Since Proposition 5.1.2 reveals that solving MOP(U) for highly robust efficient solutions is unrealistic unless the uncertainty set is finite or may be considered as such due to its special structure, we restrict our attention to finite uncertainty sets for the remainder of this chapter. Although some results may also be true for infinite uncertainty sets, we do not address this in more detail. Throughout, the finite set of scenarios is defined to be

$$U := \{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^s\} \subset \mathbb{R}^q, \tag{5.3}$$

where we assume WLOG that each scenario is distinct.

Under the assumption of a finite set of scenarios, we present a theoretical RC, develop a naive weighted-sum method, and extend the deterministic multiobjective programming results due to Benson to the current setting of MOP(U) and highly robust efficiency.

5.2 A Theoretical Robust Counterpart

In robust (single-objective) optimization, the formulation of an RC, which is a deterministic (scalar or vector) optimization problem associated with the original uncertain problem, is integral to providing a solution concept for and facilitating solution methods to the uncertain problem. As a result, we explore what the formulation of an RC of MOP(U) that generates highly robust efficient solutions might be, and what insight this counterpart provides. Intuitively, any RC of MOP(U) asks for a solution (optimal or efficient) that is a highly robust efficient solution to MOP(U) (see p. 420, Kuhn et al. [95]). We present two counterpart problems: an atypical (or theoretical) RC, and a classical counterpart (but not RC) whose solution set contains the highly robust efficient set and is thus only a bound set.

Due to the multiobjective nature of our problem, our solution concept, and the fact that the uncertainty is only considered to be in the objective function coefficients, the theoretical RC of MOP(U), which we present here, takes on a different form than possibly expected based upon classical robust optimization. In fact, the RC is a so-called *conjunctive multiobjective program* (CMOP). Before we can consider the RC, however, we must develop the idea of a CMOP.

Definition 5.2.1. A *conjunctive multiobjective program* is a problem of the form:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \bigwedge_{i=1}^{s} \mathbf{f}_i(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in X, \end{aligned} \tag{5.4}$$

where "$\bigwedge$" denotes *conjunction*, $\mathbf{f}_i : X \to \mathbb{R}^p, i = 1, \ldots, s$, is a vector-valued objective function, and $X \subset \mathbb{R}^n$ is the feasible region given by (4.2).

This problem is an MOP where the objective is given as the conjunction of vector-valued objective functions $\mathbf{f}_i, i = 1, \ldots, s$. The conjunction here forces the consideration of all objectives simultaneously over a common feasible region. What does this suggest about the solution concept for a CMOP? An intuitive solution concept for CMOP (5.4) states that a feasible decision is preferred if there is no index such that the associated objective can be improved (or at least equaled) in every component. Formally, we state this in the following definition. Let $\mathcal{S} := \{1, \ldots, s\}$ be the index set associated with the finite set of scenarios given by (5.3).

Definition 5.2.2. A feasible solution $\hat{\mathbf{x}} \in X$ to CMOP (5.4) is said to be *conjunctive (weakly) efficient* provided there does not exist an $\mathbf{x} \in X$ such that for at least one index $i \in \mathcal{S}$, $\mathbf{f}_i(\mathbf{x})\ (<) \leq \mathbf{f}_i(\hat{\mathbf{x}})$. The set of all conjunctive (weakly) efficient solutions

$\hat{\mathbf{x}} \in X$ is denoted by ($\mathrm{cwE}(X, \mathbf{f}_i, \mathcal{S})$) $\mathrm{cE}(X, \mathbf{f}_i, \mathcal{S})$ and is called the *conjunctive (weakly) efficient set*.

Since the objective function $\mathbf{f}(\mathbf{x}, \mathbf{u})$ becomes deterministic given the scenario $\mathbf{u} \in U$, the theoretical RC of MOP(U) that generates highly robust efficient solutions may immediately be written in the form of a CMOP. In particular, this RC is the CMOP given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \bigwedge_{\mathbf{u} \in U} \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \text{s.t.} \quad & \mathbf{x} \in X. \end{aligned} \tag{5.5}$$

Accordingly, the conjunctive (weakly) efficient set of RC (5.5) is denoted ($\mathrm{cwE}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$) $\mathrm{cE}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$. Using predicate logic, we are able to show that conjunctive (weakly) efficient solutions to RC (5.5) are in fact highly robust (weakly) efficient solutions to MOP(U).

Theorem 5.2.3. *A feasible solution* $\mathbf{x}^* \in X$ *to* MOP(U) *is highly robust (weakly) efficient if and only if it is a conjunctive (weakly) efficient solution to* RC (5.5)*.*

Proof. In the context of the proof, we adopt the logic notation "$\neg$" to denote "not" or negation. Consider that

$$\begin{aligned} \mathbf{x}^* \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U) &\overset{5.1.1}{\Longleftrightarrow} \forall \mathbf{u} \in U,\ \nexists \mathbf{x} \in X \text{ such that } \mathbf{f}(\mathbf{x}, \mathbf{u}) \leq \mathbf{f}(\mathbf{x}^*, \mathbf{u}) \\ &\Longleftrightarrow \forall \mathbf{u} \in U,\ \neg\big[\exists \mathbf{x} \in X \text{ such that } \mathbf{f}(\mathbf{x}, \mathbf{u}) \leq \mathbf{f}(\mathbf{x}^*, \mathbf{u})\big] \\ &\Longleftrightarrow \forall \mathbf{u} \in U \text{ and } \forall \mathbf{x} \in X,\ \neg\big[\mathbf{f}(\mathbf{x}, \mathbf{u}) \leq \mathbf{f}(\mathbf{x}^*, \mathbf{u})\big] \\ &\Longleftrightarrow \forall \mathbf{x} \in X \text{ and } \forall \mathbf{u} \in U,\ \neg\big[\mathbf{f}(\mathbf{x}, \mathbf{u}) \leq \mathbf{f}(\mathbf{x}^*, \mathbf{u})\big] \\ &\Longleftrightarrow \forall \mathbf{x} \in X,\ \Big[\forall \mathbf{u} \in U,\ \neg\big[\mathbf{f}(\mathbf{x}, \mathbf{u}) \leq \mathbf{f}(\mathbf{x}^*, \mathbf{u})\big]\Big] \\ &\Longleftrightarrow \forall \mathbf{x} \in X,\ \neg\big[\exists \mathbf{u} \in U \text{ such that } \mathbf{f}(\mathbf{x}, \mathbf{u}) \leq \mathbf{f}(\mathbf{x}^*, \mathbf{u})\big] \\ &\Longleftrightarrow \neg\big[\exists \mathbf{x} \in X \text{ such that } \exists \mathbf{u} \in U \text{ for which } \mathbf{f}(\mathbf{x}, \mathbf{u}) \leq \mathbf{f}(\mathbf{x}^*, \mathbf{u})\big] \end{aligned}$$

$$\iff \nexists \mathbf{x} \in X \text{ such that } \exists \mathbf{u} \in U \text{ for which } \mathbf{f}(\mathbf{x}, \mathbf{u}) \leq \mathbf{f}(\mathbf{x}^*, \mathbf{u})$$

$$\overset{5.2.2}{\iff} \mathbf{x}^* \in \mathrm{cE}(X, \mathbf{f}(\cdot, \mathbf{u}), U),$$

which gives the result.

The proof follows similarly for highly robust and conjunctive weak efficiency.

□

Corollary 5.2.4. *The highly robust (weakly) efficient set of* MOP(U) *and the conjunctive (weakly) efficient set of* RC (5.5) *are equal.*

Proof. The result follows immediately from Theorem 5.2.3. □

A natural question is: Are there any other RCs, specifically ones that assume the form of a classical optimization problem? One possible alternative RC is the so-called *all-in-one problem*, which is denoted AIOMOP(U) and given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{f}(\mathbf{x}, U) := \begin{bmatrix} \mathbf{f}(\mathbf{x}, \mathbf{u}^1) & \cdots & \mathbf{f}(\mathbf{x}, \mathbf{u}^s) \end{bmatrix}^T \\ \text{s.t.} \quad & \mathbf{x} \in X, \end{aligned} \tag{5.6}$$

where $\mathbf{f} : X \times U \to \mathbb{R}^{ps}$ is a vector-valued function. Given the uncertainty set U, it is clear that AIOMOP(U) is a deterministic problem. Accordingly, its (weakly) efficient set is denoted (wE($X, \mathbf{f}(\cdot, U)$)) E($X, \mathbf{f}(\cdot, U)$).

Although it is reasonable to expect that the all-in-one problem is an RC of MOP(U), we demonstrate that it is not. First, since AIOMOP(U) is a deterministic MOP given U whose efficient solutions are determined by ps criteria, we immediately obtain that highly robust efficient solutions to MOP(U) are at least weakly efficient solutions to AIOMOP(U) based on Proposition 1, Engau and Wiecek [50]. Moreover, in the following proposition, we are able to show that highly robust efficient solutions to MOP(U) are in fact efficient solutions to AIOMOP(U), but not vice versa. (In Example 6.3.20, it is explicitly shown that the opposite containment, i.e., $\mathrm{E}(X, \mathbf{f}(\cdot, U)) \subseteq \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$, does not hold.) The result is an extension of Lemma 17, Kuhn et al. [94], in which $p = 2$ and one (component) objective function is deterministic (while the other component is uncertain).

Proposition 5.2.5. *[140, Proposition 8] The containment* $\mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U) \subseteq \mathrm{E}(X, \mathbf{f}(\cdot, U))$ *holds.*

Proof. We prove the result via the contrapositive. Suppose $\mathbf{x}^* \notin \mathrm{E}(X, \mathbf{f}(\cdot, U))$. By Definition 4.1.1, there exists an $\mathbf{x} \in X$ such that

$$\begin{bmatrix}\mathbf{f}(\mathbf{x}, \mathbf{u}^1) & \cdots & \mathbf{f}(\mathbf{x}, \mathbf{u}^s)\end{bmatrix}^T \leq \begin{bmatrix}\mathbf{f}(\mathbf{x}^*, \mathbf{u}^1) & \cdots & \mathbf{f}(\mathbf{x}^*, \mathbf{u}^s)\end{bmatrix}^T,$$

which implies that there exists an $i \in \{1, \ldots, s\}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{u}^i) \leq \mathbf{f}(\mathbf{x}^*, \mathbf{u}^i)$. Equivalently, by Definition 4.1.1, $\mathbf{x}^* \notin \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}^i))$. Thus, $\mathbf{x}^* \notin \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$ by Proposition 5.1.2. □

Although we have presented a formulation of an RC for MOP(U), its usefulness is somewhat restricted since it is not classical or well-known. Despite this downside, the formulation reiterates the nature of highly robust efficient solutions indicating that any proposed solution methods must consider all of the objectives simultaneously over the common feasible set, yet not as a single vector-valued function.

5.3 Extension of the Weighted-Sum Method

In order to solve deterministic MOPs, scalarization methods are commonly used. As mentioned in Section 4.1.1, one such approach is the weighted-sum method. For the case of MOP(U) and highly robust efficiency, we may extend this deterministic approach to the uncertain setting by considering a family of weighted-sum problems.

For each $i = 1, \ldots, s$, the weighted-sum problem with respect to scenario $\mathbf{u}^i \in U$ and weight $\boldsymbol{\lambda}_i \in \mathbb{R}^p$, denoted $\mathrm{WSP}(\boldsymbol{\lambda}_i, \mathbf{u}^i)$, is given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{k=1}^{p} \lambda_{ik} f_k(\mathbf{x}, \mathbf{u}^i) \\ \text{s.t.} \quad & \mathbf{x} \in X. \end{aligned} \tag{5.7}$$

Given an arbitrary scenario $\mathbf{u}^i \in U$, it is clear that $\mathrm{WSP}(\boldsymbol{\lambda}_i, \mathbf{u}^i)$ is deterministic and is in fact the weighted-sum problem (4.4) associated with the instance $\mathrm{MOP}(\mathbf{u}^i)$. For the purposes of the following result and proof, a feasible solution to $\mathrm{WSP}(\boldsymbol{\lambda}_i, \mathbf{u}^i)$ for any $i \in \{1, \ldots, s\}$ is given by the point $\mathbf{x}(\mathbf{u}^i)$, where $\mathbf{x}(\mathbf{u}^i)$ explicitly indicates the dependence of the variable $\mathbf{x}$ on the scenario $\mathbf{u}^i$.

Proposition 5.3.1. *Let $\mathbf{x}(\mathbf{u}^i)$ be an optimal solution to $\mathrm{WSP}(\boldsymbol{\lambda}_i, \mathbf{u}^i)$ for each $i = 1, \ldots, s$, and let $\hat{\mathbf{x}} := \hat{\mathbf{x}}(\mathbf{u}^1) = \cdots = \hat{\mathbf{x}}(\mathbf{u}^s)$.*

(i) If $\boldsymbol{\lambda}_i \in \mathbb{R}^p_>$ for all $i = 1, \ldots, p$, then $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$.

(ii) If $\boldsymbol{\lambda}_i \in \mathbb{R}^p_\geq$ for all $i = 1, \ldots, p$, then $\hat{\mathbf{x}} \in \mathrm{wE}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$.

Proof. (i) Let $\boldsymbol{\lambda}_i \in \mathbb{R}^p_>$ for all $i = 1, \ldots, s$. Hence, $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}^i))$ for each $i = 1, \ldots, s$ by Proposition 4.1.6(i). Applying Proposition 5.1.2, we obtain the result.

(ii) The proof follows similarly to the proof of part (i). □

Although Proposition 5.3.1 is very similar to Proposition 4.1.6 in that we may obtain preferred solutions by using positive or semipositve weights, there is one noticeable difference: the extra assumption that $\hat{\mathbf{x}}(\mathbf{u}^1) = \cdots = \hat{\mathbf{x}}(\mathbf{u}^s)$. This assumption is clearly required, however, since the weighted-sum problems in the family act independently of one another. That is, when solving the weighted-sum problems, there is no guarantee that the particular set of weights chosen produce the same $\mathbf{x}$-solution in each individual problem. When considering the weighted-sum approach as a solution method for highly robust (weakly) efficient solutions to MOP(U), this discussion indicates a notable downside. That being said, parametric optimization may provide a means to identify the appropriate set of weights to produce a common $\mathbf{x}$-solution, which we explore in more detail in the following chapter.

In addition, it is still necessary to address whether or not all highly robust (weakly) efficient solutions may be computed.

Proposition 5.3.2. *Let the feasible set X be convex, and let $f_k(\cdot, \mathbf{u}^i), k = 1, \ldots, p$, be convex functions (in $\mathbf{x}$) for each $i = 1, \ldots, s$. If $\hat{\mathbf{x}} \in \mathrm{wE}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$, then there is some $\boldsymbol{\lambda}_i \in \mathbb{R}^p_{\geq}$ such that $\hat{\mathbf{x}}$ is an optimal solution to $\mathrm{WSP}(\boldsymbol{\lambda}_i, \mathbf{u}^i)$ for all $i = 1, \ldots, s$.*

Proof. Let $\hat{\mathbf{x}} \in \mathrm{wE}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$, or equivalently, $\hat{\mathbf{x}} \in \mathrm{wE}(X, \mathbf{f}(\cdot, \mathbf{u}^i))$ for all $i = 1, \ldots, s$. Applying Proposition 4.1.7, we obtain the result. □

Based on Proposition 5.3.2, all highly robust weakly efficient solutions may be obtained (under some convexity assumptions) with a collection of weights $\boldsymbol{\lambda}_i \in \mathbb{R}^p_{\geq}, i = 1, \ldots, s$. Equivalently, since highly robust efficient solutions are also weak by Remark 5.1.3, all highly robust efficient solutions to MOP(U) may be computed. Nevertheless, with Proposition 5.3.1 and the corresponding discussion in mind, we know that not every set of weights $\boldsymbol{\lambda}_i \in \mathbb{R}^p_{\geq}, i = 1, \ldots, s$, produces a highly robust (weakly) efficient solution.

5.4 Extension of Benson's Method

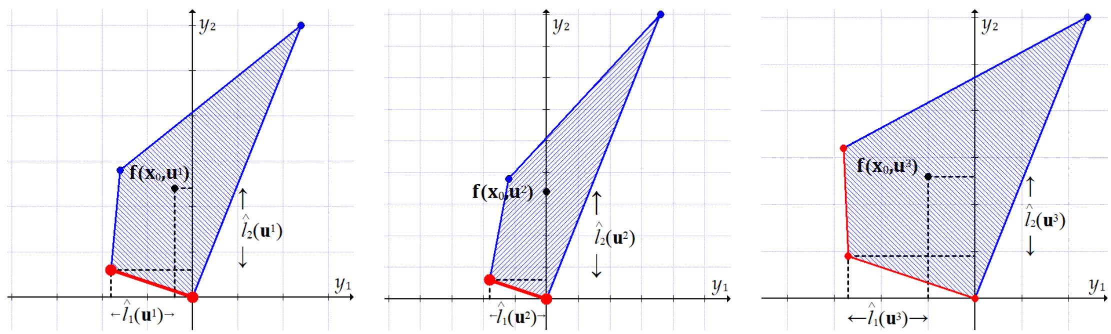
In the deterministic setting, it is well-known that an auxiliary single-objective problem, i.e., Benson's problem, may be used to give the decision maker an opportunity to check whether or not a given solution $\mathbf{x}_0 \in X$ to an MOP is efficient or generate a solution that is. We may extend the existing results from the deterministic case (refer to Section 4.1.2) to the case of uncertainty as well. Based on the formulation of MOP(U) and the definition of highly robust efficiency, the extension allows for (at least) three different Benson-type auxiliary problems. One is a family of problems, while the other two are single/individual problems. Regardless of the auxiliary problem, recognition and generation results are obtained, although some are not guaranteed to be necessary and sufficient.

We first examine the family of Benson-type auxiliary problems. For a given feasible solution $\mathbf{x}_0 \in X$ and an arbitrary $\mathbf{u} \in U$, the following problem, denoted BP($\mathbf{x}_0, \mathbf{u}$), is a representative member of the family of auxiliary problems and is given by

$$\begin{aligned} \max_{\mathbf{x},\boldsymbol{l}} \quad & \sum_{k=1}^{p} l_k \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x},\mathbf{u}) + \mathbf{I}_p \boldsymbol{l} = \mathbf{f}(\mathbf{x}_0,\mathbf{u}) \\ & \boldsymbol{l} \geqq \mathbf{0} \\ & \mathbf{x} \in X, \end{aligned} \tag{5.8}$$

where $\boldsymbol{l} \in \mathbb{R}^p$ is a so-called *deviation variable*. Given $\mathbf{u} \in U$, it is apparent that BP($\mathbf{x}_0, \mathbf{u}$) is deterministic and is simply Benson's problem (4.5) associated with the instance MOLP($\mathbf{u}$). For the purposes of the following results and proofs, a feasible solution to BP($\mathbf{x}_0, \mathbf{u}$) for an arbitrary $\mathbf{u} \in \{\mathbf{u}^1, \ldots, \mathbf{u}^s\}$ is given by the point $(\mathbf{x}(\mathbf{u}), \boldsymbol{l}(\mathbf{u}))$, where $\mathbf{x}(\mathbf{u})$ and $\boldsymbol{l}(\mathbf{u})$ explicitly indicate the dependence of the variables $\mathbf{x}$ and $\boldsymbol{l}$ on the scenario $\mathbf{u}$.

The idea of BP($\mathbf{x}_0, \mathbf{u}$), like in the deterministic setting, is that we first choose some initial feasible solution $\mathbf{x}_0 \in X$. If $\mathbf{x}_0$ is not itself highly robust efficient, then we try to produce a solution that is, which is accomplished by maximizing the sum



(a) The attainable set under $\mathbf{f}(\cdot, \mathbf{u}^1)$ shaded (blue) and the Pareto set (red)

(b) The attainable set under $\mathbf{f}(\cdot, \mathbf{u}^2)$ shaded (blue) and the Pareto set (red)

(c) The attainable set under $\mathbf{f}(\cdot, \mathbf{u}^3)$ shaded (blue) and the Pareto set (red)

Figure 5.1: Illustration of Benson's method for MOP(U)

of nonnegative deviation variables $l_k(\mathbf{u}) = f_k(\mathbf{x}_0, \mathbf{u}) - f_k(\mathbf{x}(\mathbf{u}), \mathbf{u}), k = 1, \ldots, p$, for each $\mathbf{u} \in U$. A key difference between this framework and the deterministic setting, however, is that we are not guaranteed to obtain a highly robust efficient solution like we are guaranteed to produce an efficient solution because the individual auxiliary problems, similarly to the members of the family of weighted-sum problems, act independently of one another. As a result, we only obtain a highly robust efficient solution if BP($\mathbf{x}_0, \mathbf{u}$) has the same optimal $\mathbf{x}$-solution for each $\mathbf{u} \in U$. That being said, not only can the family of Benson's problems provide us with a method for checking whether or not a given $\mathbf{x}_0 \in X$ is highly robust efficient, but it can also generate highly robust efficient solutions. An illustration is provided in Figure 5.1. In order to recognize whether or not a given feasible decision is highly robust efficient, the following result is used.

Proposition 5.4.1. *Let* $\mathbf{x}_0 \in X$ *be given. Then* $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$ *if and only if* BP($\mathbf{x}_0, \mathbf{u}$) *has an optimal solution* $(\hat{\mathbf{x}}(\mathbf{u}), \hat{\boldsymbol{l}}(\mathbf{u}))$ *with* $\hat{\boldsymbol{l}}(\mathbf{u}) = \mathbf{0}$ *for every* $\mathbf{u} \in U$*.*

Proof 1. ($\Longrightarrow$) Assume $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$, or equivalently, $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}))$ for all $\mathbf{u} \in U$. By Definition 4.1.1, for each $\mathbf{u} \in U$, there does not exist an $\mathbf{x}(\mathbf{u}) \in X$

such that $\mathbf{f}(\mathbf{x}(\mathbf{u}), \mathbf{u}) \leq \mathbf{f}(\mathbf{x}_0, \mathbf{u})$. Accordingly, for each $\mathbf{u} \in U$, there does not exist an $\mathbf{x}(\mathbf{u}) \in X$ such that

$$\mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{f}(\mathbf{x}(\mathbf{u}), \mathbf{u}) \geq \mathbf{0}, \tag{5.9}$$

which is obtained by subtracting $\mathbf{f}(\mathbf{x}(\mathbf{u}), \mathbf{u})$ from both sides of the former inequality. Now, for each $\mathbf{u} \in U$, let $(\hat{\mathbf{x}}(\mathbf{u}), \hat{\boldsymbol{l}}(\mathbf{u}))$ be an optimal solution to $\mathrm{BP}(\mathbf{x}_0, \mathbf{u})$. As a result,

$$\hat{\mathbf{x}}(\mathbf{u}) \in X \text{ and } \mathbf{I}_p \hat{\boldsymbol{l}}(\mathbf{u}) = \mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{f}(\hat{\mathbf{x}}(\mathbf{u}), \mathbf{u}) \geqq \mathbf{0} \tag{5.10}$$

for each $\mathbf{u} \in U$. Combining (5.9) and (5.10), it must be that $\hat{\boldsymbol{l}}(\mathbf{u}) = \mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{f}(\hat{\mathbf{x}}(\mathbf{u}), \mathbf{u}) = \mathbf{0}$ for every $\mathbf{u} \in U$. Therefore, $\mathrm{BP}(\mathbf{x}_0, \mathbf{u})$ has an optimal solution $(\hat{\mathbf{x}}(\mathbf{u}), \hat{\boldsymbol{l}}(\mathbf{u}))$ with $\hat{\boldsymbol{l}}(\mathbf{u}) = \mathbf{0}$ for every $\mathbf{u} \in U$.

($\Longleftarrow$) For each $\mathbf{u} \in U$, let $(\hat{\mathbf{x}}(\mathbf{u}), \hat{\boldsymbol{l}}(\mathbf{u}))$ be an optimal solution to $\mathrm{BP}(\mathbf{x}_0, \mathbf{u})$ with $\hat{\boldsymbol{l}}(\mathbf{u}) = \mathbf{0}$. Assume for the sake of contradiction that $\mathbf{x}_0 \notin \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$, or equivalently, there exists a $\bar{\mathbf{u}} \in U$ such that $\mathbf{x}_0 \notin \mathrm{E}(X, \mathbf{f}(\cdot, \bar{\mathbf{u}}))$. Hence, by Definition 4.1.1, there exists an $\mathbf{x}(\bar{\mathbf{u}}) \in X$ such that $\mathbf{f}(\mathbf{x}(\bar{\mathbf{u}}), \bar{\mathbf{u}}) \leq \mathbf{f}(\mathbf{x}_0, \bar{\mathbf{u}})$. Subtracting $\mathbf{f}(\mathbf{x}(\bar{\mathbf{u}}), \bar{\mathbf{u}})$ from both sides of the inequality and letting $\boldsymbol{l}(\bar{\mathbf{u}}) = \mathbf{f}(\mathbf{x}_0, \bar{\mathbf{u}}) - \mathbf{f}(\mathbf{x}(\bar{\mathbf{u}}), \bar{\mathbf{u}}) \in \mathbb{R}^p$, we have that there exists an $\boldsymbol{l}(\bar{\mathbf{u}}) \in \mathbb{R}^p$ such that $\boldsymbol{l}(\bar{\mathbf{u}}) \geq \mathbf{0}$.

Now, observe that $\sum_{k=1}^{p} l_k(\bar{\mathbf{u}}) > 0$ and that $(\mathbf{x}(\bar{\mathbf{u}}), \boldsymbol{l}(\bar{\mathbf{u}}))$ is a feasible solution to $\mathrm{BP}(\mathbf{x}_0, \bar{\mathbf{u}})$. Since $\hat{\boldsymbol{l}}(\bar{\mathbf{u}}) = \mathbf{0}$ by assumption (as $\bar{\mathbf{u}} \in U$), we have constructed a solution that has an objective value greater than the optimal solution, which is a contradiction. Thus, it must be that $\mathbf{x}_0$ is a highly robust efficient solution to $\mathrm{MOP}(U)$. □

Proof 2. An alternative proof utilizes the fact that $\mathrm{BP}(\mathbf{x}_0, \mathbf{u})$ is the deterministic Benson's problem associated with the instance $\mathrm{MOLP}(\mathbf{u})$ for each $\mathbf{u} \in U$.

Let an arbitrary $\mathbf{u} \in U$ be given. By Theorem 4.1.10, $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}))$

if and only if $\mathrm{BP}(\mathbf{x}_0, \mathbf{u})$ has an optimal solution $(\hat{\mathbf{x}}(\mathbf{u}), \hat{\boldsymbol{l}}(\mathbf{u}))$ with $\hat{\boldsymbol{l}}(\mathbf{u}) = \mathbf{0}$. Since $\mathbf{u} \in U$ is arbitrary and $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$ if and only if $\mathbf{x}_0 \in \bigcap_{\mathbf{u} \in U} \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}))$ by Proposition 5.1.2, the result follows. □

In the deterministic setting, it is expected that if $\mathbf{x}_0$ is not efficient, then another $\hat{\mathbf{x}} \in X$ that is efficient is generated by Benson's problem. Even though Proposition 5.4.1 mirrors the existing result on deterministic efficiency, as mentioned, this is not the case in the uncertain setting and an additional condition is required as the next proposition reveals.

Proposition 5.4.2. *Let* $\mathbf{x}_0 \in X$ *be given, and suppose* $(\hat{\mathbf{x}}(\mathbf{u}^i), \hat{\boldsymbol{l}}(\mathbf{u}^i))$ *is an optimal solution to* $\mathrm{BP}(\mathbf{x}_0, \mathbf{u}^i)$ *for each* $i = 1, \ldots, s$. *If* $\hat{\mathbf{x}} := \hat{\mathbf{x}}(\mathbf{u}^1) = \cdots = \hat{\mathbf{x}}(\mathbf{u}^s) \neq \mathbf{x}_0$ *and* $\hat{\boldsymbol{l}}(\mathbf{u}^i)$ *is finite for all* $i = 1, \ldots, s$, *then* $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$.

Proof. Let $\hat{\mathbf{x}} := \hat{\mathbf{x}}(\mathbf{u}^1) = \cdots = \hat{\mathbf{x}}(\mathbf{u}^s) \neq \mathbf{x}_0$, and let $\hat{\boldsymbol{l}}(\mathbf{u}^i)$ be finite for all $i = 1, \ldots, s$. Hence, $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}^i))$ for each $i = 1, \ldots, s$ by Proposition 4.1.11. Applying Proposition 5.1.2 gives the result. □

Instead of considering a possibly large (but finite) number of single-objective problems as in Propositions 5.4.1 and 5.4.2, it is of interest to have a single auxiliary problem. One such problem, denoted $\mathrm{BP1}(\mathbf{x}_0, U)$, is a block-style problem (see problem (15), Wiecek and Dranichak [140]) given by

$$\begin{array}{llll} \max\limits_{\mathbf{x}_1, \ldots, \mathbf{x}_s, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_s} & \displaystyle\sum_{i=1}^{s} \sum_{k=1}^{p} l_{ik} & & \\ \text{s.t.} & \mathbf{f}(\mathbf{x}_i, \mathbf{u}^i) + \mathbf{I}_p \boldsymbol{l}_i = \mathbf{f}(\mathbf{x}_0, \mathbf{u}^i) & \text{for all } i = 1, \ldots, s & \\ & \boldsymbol{l}_i \geqq \mathbf{0} & \text{for all } i = 1, \ldots, s & \\ & \mathbf{x}_i \in X & \text{for all } i = 1, \ldots, s, & \end{array} \tag{5.11}$$

where $\boldsymbol{l}_i \in \mathbb{R}^p$ for all $i = 1, \ldots, s$. Based on the block-style structure of problem (5.11), it is clear that $\mathrm{BP1}(\mathbf{x}_0, U)$ is equivalent to the family of auxiliary problems given by $\{\mathrm{BP}(\mathbf{x}_0, \mathbf{u})\}_{\mathbf{u} \in U}$. That being said, we may similarly propose solution recognition and generation methods.

Proposition 5.4.3. *[140, Proposition 7] Let* $\mathbf{x}_0 \in X$ *be given. Then* $\mathbf{x}_0$ *is a highly robust efficient solution to* $\mathrm{MOP}(U)$ *if and only if* $\mathrm{BP1}(\mathbf{x}_0, U)$ *has an optimal solution* $(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_s, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ *with* $\hat{\boldsymbol{l}}_i = \mathbf{0}$ *for all* $i = 1, \ldots, s$.

Proof. ($\Longrightarrow$) Assume $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$, or equivalently, $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}^i))$ for all $i = 1, \ldots, s$. By Definition 4.1.1, for each $i = 1, \ldots, s$, there does not exist an $\mathbf{x}(\mathbf{u}^i) \in X$ such that $\mathbf{f}(\mathbf{x}(\mathbf{u}^i), \mathbf{u}^i) \leq \mathbf{f}(\mathbf{x}_0, \mathbf{u}^i)$. Accordingly, for each $i = 1, \ldots, s$, there does not exist an $\mathbf{x}(\mathbf{u}^i) \in X$ such that

$$\mathbf{f}(\mathbf{x}_0, \mathbf{u}^i) - \mathbf{f}(\mathbf{x}(\mathbf{u}^i), \mathbf{u}^i) \geq \mathbf{0}, \tag{5.12}$$

which is obtained by subtracting $\mathbf{f}(\mathbf{x}(\mathbf{u}^i), \mathbf{u}^i)$ from both sides of the former inequality. Now, let $(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_s, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ be an optimal solution to $\mathrm{BP1}(\mathbf{x}_0, U)$. As a result,

$$\hat{\mathbf{x}}_i \in X \text{ and } \mathbf{I}_p \hat{\boldsymbol{l}}_i = \mathbf{f}(\mathbf{x}_0, \mathbf{u}^i) - \mathbf{f}(\hat{\mathbf{x}}_i, \mathbf{u}^i) \geqq \mathbf{0} \tag{5.13}$$

for all $i = 1, \ldots, s$. Combining (5.12) and (5.13), it must be that $\hat{\boldsymbol{l}}_i = \mathbf{f}(\mathbf{x}_0, \mathbf{u}^i) - \mathbf{f}(\hat{\mathbf{x}}_i, \mathbf{u}^i) = \mathbf{0}$ for all $i = 1, \ldots, s$. Therefore, $\mathrm{BP1}(\mathbf{x}_0, U)$ has an optimal solution $(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_s, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ with $\hat{\boldsymbol{l}}_i = \mathbf{0}$ for all $i = 1, \ldots, s$.

($\Longleftarrow$) Let $(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_s, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ be an optimal solution to $\mathrm{BP1}(\mathbf{x}_0, U)$ with $\hat{\boldsymbol{l}}_i = \mathbf{0}$ for all $i = 1, \ldots, s$. Assume for the sake of contradiction that $\mathbf{x}_0 \notin \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$, or equivalently, there exists an $i \in \{1, \ldots, s\}$ such that $\mathbf{x}_0 \notin \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}^i))$. Hence, by

Definition 4.1.1, there exists an $\mathbf{x}(\mathbf{u}^i) \in X$ such that $\mathbf{f}(\mathbf{x}(\mathbf{u}^i), \mathbf{u}^i) \leq \mathbf{f}(\mathbf{x}_0, \mathbf{u}^i)$. Subtracting $\mathbf{f}(\mathbf{x}(\mathbf{u}^i), \mathbf{u}^i)$ from both sides of the inequality and letting $\boldsymbol{l}(\mathbf{u}^i) = \mathbf{f}(\mathbf{x}_0, \mathbf{u}^i) - \mathbf{f}(\mathbf{x}(\mathbf{u}^i), \mathbf{u}^i)$, we have that there exists an $\boldsymbol{l}(\mathbf{u}^i) \in \mathbb{R}^p$ such that $\boldsymbol{l}(\mathbf{u}^i) \geq \mathbf{0}$.

Now, observe that $\sum_{k=1}^p l_k(\mathbf{u}^i) > 0$ and that $(\hat{\mathbf{x}}_1, \ldots, \mathbf{x}(\mathbf{u}^i), \ldots, \hat{\mathbf{x}}_s, \hat{\boldsymbol{l}}_1, \ldots, \boldsymbol{l}(\mathbf{u}^i), \ldots, \hat{\boldsymbol{l}}_s)$ is a feasible solution to BP1$(\mathbf{x}_0, U)$. Since $\hat{\boldsymbol{l}}_i = \mathbf{0}$ for all $i = 1, \ldots, s$ by assumption, we have constructed a solution with an objective value greater than the optimal solution, which is a contradiction. Thus, it must be that $\mathbf{x}_0$ is a highly robust efficient solution to MOP(U). □

Proposition 5.4.4. *Let* $\mathbf{x}_0 \in X$ *be given, and suppose* $(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_s, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ *is an optimal solution to* BP1$(\mathbf{x}_0, U)$. *If* $\hat{\mathbf{x}} := \hat{\mathbf{x}}_1 = \cdots = \hat{\mathbf{x}}_s \neq \mathbf{x}_0$ *and* $\hat{\boldsymbol{l}}_i$ *is finite for all* $i = 1, \ldots, s$, *then* $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$.

Proof. Let $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1 = \cdots = \hat{\mathbf{x}}_s \neq \mathbf{x}_0$, and let $\hat{\boldsymbol{l}}_i$ be finite for all $i = 1, \ldots, s$. Assume for the sake of contradiction that $\hat{\mathbf{x}} \notin \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$. Hence, there is some $\bar{\imath} \in \{1, \ldots, s\}$ such that $\hat{\mathbf{x}} \notin \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}^{\bar{\imath}}))$. By definition, there exists an $\bar{\mathbf{x}} \in X$ such that $\mathbf{f}(\bar{\mathbf{x}}, \mathbf{u}^{\bar{\imath}}) \leq \mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}^{\bar{\imath}})$. Define $\bar{\boldsymbol{l}} = \mathbf{f}(\mathbf{x}_0, \mathbf{u}^{\bar{\imath}}) - \mathbf{f}(\bar{\mathbf{x}}, \mathbf{u}^{\bar{\imath}})$. Hence, $(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_{\bar{\imath}-1}, \bar{\mathbf{x}}, \hat{\mathbf{x}}_{\bar{\imath}+1}, \ldots, \hat{\mathbf{x}}_s, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_{\bar{\imath}-1}, \bar{\boldsymbol{l}}, \hat{\boldsymbol{l}}_{\bar{\imath}+1}, \ldots, \hat{\boldsymbol{l}}_s)$ is a feasible solution to BP1$(\mathbf{x}_0, U)$ since $\bar{\boldsymbol{l}} = \mathbf{f}(\mathbf{x}_0, \mathbf{u}^{\bar{\imath}}) - \mathbf{f}(\bar{\mathbf{x}}, \mathbf{u}^{\bar{\imath}}) \geq \mathbf{f}(\mathbf{x}_0, \mathbf{u}^{\bar{\imath}}) - \mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}^{\bar{\imath}}) = \hat{\boldsymbol{l}}_{\bar{\imath}} \geqq \mathbf{0}$ and $\bar{\mathbf{x}} \in X$. Moreover, since $\bar{\boldsymbol{l}} \geq \hat{\boldsymbol{l}}_{\bar{\imath}}$, we have that

$$\sum_{k=1}^{p} (\hat{l}_{1k} + \cdots + \hat{l}_{\bar{\imath}-1,k} + \bar{l}_k + \hat{l}_{\bar{\imath}+1,k} + \cdots + \hat{l}_{sk}) > \sum_{i=1}^{s} \sum_{k=1}^{p} \hat{l}_{ik}.$$

Therefore, we have constructed a feasible solution to BP1$(\mathbf{x}_0, U)$ with an objective value greater than the optimal solution, which is a contradiction. Thus, it must be that $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$. □

A third Benson-type problem is associated with AIOMOP (5.6). The corresponding Benson's problem, denoted BP2$(\mathbf{x}_0, U)$, is given by

$$\begin{aligned}
\max_{\mathbf{x},\boldsymbol{l}_1,\ldots,\boldsymbol{l}_s} \quad & \sum_{i=1}^{s}\sum_{k=1}^{p} l_{ik} \\
\text{s.t.} \quad & \mathbf{f}(\mathbf{x},\mathbf{u}^i) + \mathbf{I}_p \boldsymbol{l}_i = \mathbf{f}(\mathbf{x}_0,\mathbf{u}^i) \quad \text{for all } i = 1,\ldots,s \\
& \boldsymbol{l}_i \geqq \mathbf{0} \quad \text{for all } i = 1,\ldots,s \\
& \mathbf{x} \in X,
\end{aligned} \tag{5.14}$$

where $\boldsymbol{l}_i \in \mathbb{R}^p$ for all $i = 1,\ldots,s$. Given the set of scenarios U, it is clear that BP2$(\mathbf{x}_0, U)$ is the deterministic Benson's problem associated with AIOMOP(U). Since we know that $\mathrm{E}(X, \mathbf{f}(\cdot,\mathbf{u}), U) \subseteq \mathrm{E}(X, \mathbf{f}(\cdot, U))$ (see Proposition 5.2.5), the Benson-type method utilizing BP2$(\mathbf{x}_0, U)$ does not provide necessary and sufficient conditions for highly robust efficiency recognition even though BP1$(\mathbf{x}_0, U)$ does. That being said, the advantage of BP2$(\mathbf{x}_0, U)$ is due to its reduced number of variables.

Proposition 5.4.5. *Let* $\mathbf{x}_0 \in X$ *be given. If* $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot,\mathbf{u}), U)$*, then* BP2$(\mathbf{x}_0, U)$ *has an optimal solution* $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ *with* $\hat{\boldsymbol{l}}_i = \mathbf{0}$ *for all* $i = 1,\ldots,s$.

Proof 1. Assume $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot,\mathbf{u}), U)$, or equivalently, $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot,\mathbf{u}^i))$ for all $i = 1,\ldots,s$. By Definition 4.1.1, for each $i = 1,\ldots,s$, there does not exist an $\mathbf{x}(\mathbf{u}^i) \in X$ such that $\mathbf{f}(\mathbf{x}(\mathbf{u}^i),\mathbf{u}^i) \leq \mathbf{f}(\mathbf{x}_0,\mathbf{u}^i)$. Accordingly, for each $i = 1,\ldots,s$, there does not exist an $\mathbf{x}(\mathbf{u}^i) \in X$ such that

$$\mathbf{f}(\mathbf{x}_0,\mathbf{u}^i) - \mathbf{f}(\mathbf{x}(\mathbf{u}^i),\mathbf{u}^i) \geq \mathbf{0}, \tag{5.15}$$

which is obtained by subtracting $\mathbf{f}(\mathbf{x}(\mathbf{u}^i),\mathbf{u}^i)$ from both sides of the former inequality. Now, let $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ be an optimal solution to BP2$(\mathbf{x}_0, U)$. As a result,

$$\mathbf{I}_p \hat{\boldsymbol{l}}^i = \mathbf{f}(\mathbf{x}_0,\mathbf{u}^i) - \mathbf{f}(\hat{\mathbf{x}},\mathbf{u}^i) \geqq \mathbf{0} \text{ for all } i = 1,\ldots,s \text{ and } \hat{\mathbf{x}} \in X. \tag{5.16}$$

Combining (5.15) and (5.16), it must be that $\hat{\boldsymbol{l}}_i = \mathbf{f}(\mathbf{x}_0, \mathbf{u}^i) - \mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}^i) = \mathbf{0}$ for all $i = 1, \ldots, s$. Therefore, BP2$(\mathbf{x}_0, U)$ has an optimal solution $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ with $\hat{\boldsymbol{l}}_i = \mathbf{0}$ for all $i = 1, \ldots, s$. □

Proof 2. Assume $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$. Hence, $\mathbf{x}_0 \in \mathrm{E}(X, \mathbf{f}(\cdot, U))$ also by Proposition 5.2.5. Since BP2$(\mathbf{x}_0, U)$ is the deterministic Benson's problem associated with AIOMOP(U), the result follows immediately from Theorem 4.1.10. □

Remark 5.4.6. We can easily observe why the above recognition condition is not also sufficient. When we try to construct a feasible solution that has a better objective value as in the proof involving BP1$(\mathbf{x}_0, U)$, we are not able to do so. In particular, consider the following:

> If we take $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ to be an optimal solution to BP2$(\mathbf{x}_0, U)$ with $\hat{\boldsymbol{l}}_i = \mathbf{0}$ for all $i = 1, \ldots, s$, and assume for the sake of contradiction that $\mathbf{x}_0 \notin \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}), U)$, then we obtain that there exists an $\bar{\imath} \in \{1, \ldots, s\}$ such that $\mathbf{x}_0 \notin \mathrm{E}(X, \mathbf{f}(\cdot, \mathbf{u}^{\bar{\imath}}))$. Hence, by Definition 4.1.1, there exists an $\mathbf{x}(\mathbf{u}^{\bar{\imath}}) \in X$ such that $\mathbf{f}(\mathbf{x}(\mathbf{u}^{\bar{\imath}}), \mathbf{u}^{\bar{\imath}}) \leq \mathbf{f}(\mathbf{x}_0, \mathbf{u}^{\bar{\imath}})$. Subtracting $\mathbf{f}(\mathbf{x}(\mathbf{u}^{\bar{\imath}}), \mathbf{u}^{\bar{\imath}})$ from both sides of the inequality and letting $\boldsymbol{l}(\mathbf{u}^{\bar{\imath}}) = \mathbf{f}(\mathbf{x}_0, \mathbf{u}^{\bar{\imath}}) - \mathbf{f}(\mathbf{x}(\mathbf{u}^{\bar{\imath}}), \mathbf{u}^{\bar{\imath}})$, we have that there exists an $\boldsymbol{l}(\mathbf{u}^{\bar{\imath}}) \in \mathbb{R}^p$ such that $\boldsymbol{l}(\mathbf{u}^{\bar{\imath}}) \geq \mathbf{0}$. Thus, we have a point $(\mathbf{x}(\mathbf{u}^{\bar{\imath}}), \hat{\boldsymbol{l}}_1, \ldots, \boldsymbol{l}(\mathbf{u}^{\bar{\imath}}), \ldots, \hat{\boldsymbol{l}}_s)$ such that
>
> $$\mathbf{f}(\mathbf{x}(\mathbf{u}^{\bar{\imath}}), \mathbf{u}^{\bar{\imath}}) + \mathbf{I}_p \boldsymbol{l}(\mathbf{u}^{\bar{\imath}}) = \mathbf{f}(\mathbf{x}_0, \mathbf{u}^{\bar{\imath}}).$$
>
> However, we cannot claim that the point $(\mathbf{x}(\mathbf{u}^{\bar{\imath}}), \hat{\boldsymbol{l}}_1, \ldots, \boldsymbol{l}(\mathbf{u}^{\bar{\imath}}), \ldots, \hat{\boldsymbol{l}}_s)$ is feasible to BP2$(\mathbf{x}_0, U)$ because it is unknown whether or not
>
> $$\mathbf{f}(\mathbf{x}(\mathbf{u}^i), \mathbf{u}^i) + \mathbf{I}_p \hat{\boldsymbol{l}}_i \stackrel{?}{=} \mathbf{f}(\mathbf{x}_0, \mathbf{u}^i) \text{ for all } i = 1, \ldots, s, i \neq \bar{\imath}.$$
>
> As a result, we are not able to construct a feasible solution that has an objective

value greater than the optimal solution, which is needed in order to obtain the required contradiction.

In addition, it is clear that Proposition 5.4.5 is not both necessary and sufficient since BP2$(\mathbf{x}_0, U)$ is the deterministic Benson's problem associated with AIOMOP(U) and the efficient set of AIOMOP(U) contains the highly robust efficient set (as already mentioned).

A natural question now is, do optimal solutions to BP2$(\mathbf{x}_0, U)$ lead to highly robust efficient solutions as is the case for BP1$(\mathbf{x}_0, U)$? That is, if the given solution $\mathbf{x}_0$ is not highly robust efficient, then does solving BP2$(\mathbf{x}_0, U)$ generate a solution that is? The obvious answer is no since our initial result, Proposition 5.4.5, is not necessary and sufficient. In particular, it is important to realize here that conditions similar to those stated in Proposition 5.4.4 for BP1$(\mathbf{x}_0, U)$ are not readily available for BP2$(\mathbf{x}_0, U)$ since an optimal solution to BP2$(\mathbf{x}_0, U)$ is foremost an efficient solution to the all-in-one problem. As a result, it need not also be a highly robust efficient solution to MOP(U).

In any case, the Benson-type results give an important tool to decision makers by providing methods with which any solution that is deemed desirable a priori may be verified as highly robust efficient without having to solve the entire UMOP.

Chapter 6

Uncertain Multiobjective Linear Programs

Assuming uncertainty in the cost matrix coefficient data of a multiobjective linear program (MOLP), i.e., MOLP (4.8), or equivalently assuming that the feasible set X of an uncertain multiobjective program (UMOP), i.e., UMOP (5.1), is polyhedral and the objective functions $\mathbf{f}(\cdot, \mathbf{u})$ are linear with respect to $\mathbf{x}$, we obtain an uncertain MOLP (UMOLP). Since UMOPs are a generalization of this case, all of the results of the previous chapter still hold but some may be restated for completeness. That being said, the interest in studying UMOLPs is developing additional meaningful results.

We present the problem formulation and restate the solution concept of interest, highly robust efficiency, in Section 6.1. In Section 6.2, an uncertainty set reduction for a class of UMOLPs is then given, which along with an existing reduction result allows for highly robust efficient solutions to be studied with respect to only UMOLPs whose uncertainty sets are finite. Under the assumption that the uncertainty set is finite, the highly robust efficient set is examined in Section 6.3 and methods with which to compute highly robust efficient solutions are proposed in Section 6.4. Within Section 6.3, properties and characterizations of the highly robust efficient set are presented in Sections 6.3.1 and 6.3.2, respectively, bound sets on the highly robust efficient set are derived in Section 6.3.3, and a theoretical robust counterpart (RC) of the same form as in Section 5.2, as well as a classical RC, is proposed in Section 6.3.4. Moreover, as the acuteness of various cones emerges as an important property during the course of Section 6.3.2, this feature is discussed in more detail and methods with which to identify it are revisited in Section 6.3.5. Within Section 6.4, approaches to identify whether or not a given feasible solution of interest is also highly robust efficient, possibly generate a different highly robust efficient point, or determine that no highly robust efficient solutions exist are developed in Sections 6.4.1 and 6.4.2, while solution methods to compute highly robust efficient points are proposed in Sections 6.4.3 and 6.4.4. Using the approach prescribed in Section 6.4.4, an application problem in the area of bank balance-sheet management is solved for its highly robust efficient solutions.

As in the previous part, results throughout this section may include two proofs as they are extensions of results from deterministic multiobjective linear programming. The first proof typically follows a similar format to that from the deterministic case, but is done to illustrate that the proof may be done independently of the original result. The second proof, on the other hand, typically utilizes the definition of highly robust efficiency along with the corresponding result from deterministic multiobjective linear programming. Generally speaking, some results are more straightforward extensions of existing results from deterministic multiobjective linear programming, such as properties of the highly robust efficient set, while others are more sophisticated and significant, including the bilevel approach to compute highly robust efficient solutions.

6.1 Problem Formulation and Solution Concept

Considering uncertain input data in the cost matrix coefficients of MOLP (4.8), we obtain a UMOLP, denoted MOLP(U), which is defined to be a collection or family of MOLPs indexed by the (uncertain) parameter $\mathbf{u}$. In particular, MOLP(U) is given by

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \mathbf{C}(\mathbf{u})\mathbf{x} \\ \text{s.t.} & \quad\quad \mathbf{x} \in P \end{array} \right\}_{\mathbf{u}\in U,} \tag{6.1}$$

where $U \subset \mathbb{R}^q$ is a nonempty set modeling the uncertainty, $P \subset \mathbb{R}^n$ is the (deterministic) polyhedral feasible region given by (2.6), and $\mathbf{C}(\mathbf{u})$ is the $p \times n$ cost matrix under uncertainty $\mathbf{u} \in U$. As previously mentioned, U is the *uncertainty set* or *set of scenarios*, $\mathbf{u}$ is referred to as an *uncertainty*, *realization*, or *scenario*, and MOLP($\mathbf{u}$) is an *instance* of MOLP(U). Since each instance MOLP($\mathbf{u}$) is a deterministic MOLP given the realization $\mathbf{u} \in U$, we let (wE($P, \mathbf{C}(\mathbf{u})$)) E($P, \mathbf{C}(\mathbf{u})$) denote the *(weakly) efficient set* of MOLP($\mathbf{u}$) for some realization $\mathbf{u} \in U$, and note that the uncertain problem (6.1) reduces to the deterministic problem (4.8) if U is a singleton. As with MOLP (4.8), in order to guarantee (weakly) efficient solutions to MOLP($\mathbf{u}$) exist for each $\mathbf{u} \in U$, the standard condition that P is bounded (cf. Corollary 2.26 and Theorem 2.19, Ehrgott [44], respectively) may be assumed. However, in the interest of providing various pertinent existence results, the assumption that P is bounded is not made in general.

In any multiobjective optimization problem, the multiple criteria are assumed to be in conflict. Hence, it is reasonable to expect that the conflicting objective functions are unlikely to depend on the same uncertainties. To accommodate this situation, as previously mentioned, we assume that the UMOLP is of objective-wise uncertainty. In particular, UMOLP (6.1) is said to be of *objective-wise uncertainty* if

$U = U_1 \times \cdots \times U_p$, where $U_k \subset \mathbb{R}^{q_k}, k = 1, \ldots, p$, is referred to as a *partial uncertainty set*, such that $\mathbf{C}(\mathbf{u}) = \begin{bmatrix}\mathbf{c}_1(\mathbf{u}_1) & \cdots & \mathbf{c}_p(\mathbf{u}_p)\end{bmatrix}^T$ with $\mathbf{u} = \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_p\end{bmatrix}^T \in U$ and $\mathbf{u}_k \in U_k, k = 1, \ldots, p$. For our purposes, we only consider the UMOLP of objective-wise uncertainty with $U = U_1 \times \cdots U_p$ such that $U_k \subseteq \mathbb{R}^n, k = 1, \ldots, p$, and

$$\mathbf{C}(\mathbf{u}) = \begin{bmatrix}\mathbf{c}_1(\mathbf{u}_1) \\ \vdots \\ \mathbf{c}_p(\mathbf{u}_p)\end{bmatrix} = \begin{bmatrix} c_{11}u_{11} & \cdots & c_{1n}u_{1n} \\ \vdots & \ddots & \vdots \\ c_{p1}u_{p1} & \cdots & c_{pn}u_{pn}\end{bmatrix}, \tag{6.2}$$

where $\mathbf{u} = \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_p\end{bmatrix}^T \in U$ and $\mathbf{u}_k \in U_k, k = 1, \ldots, p$. Based on (6.2), it is easy to see that $\mathbf{C}(\mathbf{u})\mathbf{x}$ is bilinear with respect to $\mathbf{x} \in P$ and $\mathbf{u} \in U$, a fact that is important in later results.

Perhaps the first objective-wise UMOLPs with only uncertain objective coefficients encountered in the literature are from the field of interval multiobjective programming (see Bitran [18]) in which every cost matrix coefficients fall within a closed interval that is assumed to be known. Bitran [18] defines an interval MOLP (IMOLP) to be the collection of MOLPs indexed by the cost matrix $\mathbf{C}$ given by

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \mathbf{C}\mathbf{x} \\ \text{s.t.} & \mathbf{x} \in P \end{array} \right\}_{\mathbf{C}\in\Phi}, \tag{6.3}$$

where $\Phi \subseteq \mathbb{R}^{p\times n}$ is the nonempty set of $p \times n$ matrices with elements $c_{ki} \in [c^{\mathrm{L}}_{ki}, c^{\mathrm{U}}_{ki}], k = 1, \ldots, p, i = 1, \ldots, n$. The lower bounds c^{L}_{ki} and upper bounds c^{U}_{ki} are assumed to be known. Although IMOLP (6.3) and UMOLP (6.1) with cost matrix (6.2) appear to be nearly identical, we show that the latter is in fact more general than the former.

It is clear that all IMOLPs can be reformulated as objective-wise UMOLPs by taking $c_{kj} = 1$ in (6.2) for all $k = 1, \ldots, p, j = 1, \ldots, n$, and $U_k = \{\mathbf{u}_k \in \mathbb{R}^n : c^{\mathrm{L}}_{k1} \le u_{k1} \le c^{\mathrm{U}}_{k1}, \ldots, c^{\mathrm{L}}_{kn} \le u_{kn} \le c^{\mathrm{U}}_{kn}\}, k = 1, \ldots, p$, which is often referred to as a *box uncertainty set*. On the other hand, it is equally clear that not all objective-wise UMOLPs can be reformulated as IMOLPs, which is the case, for instance, when U is finite. As an example, consider

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \mathbf{x} \\ \text{s.t.} & \qquad\qquad \mathbf{x} \in P \end{array} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2,} \tag{6.4}$$

where $U_1 = \{(1,1),(2,3)\}$, and $U_2 = \{(1,2)\}$. Here, we have that $U = \{(1,1,1,2),$ $(2,3,1,2)\}$. As UMOLP (6.4) is a collection of two MOLPs, it cannot possibly be reformulated as an IMOLP that is necessarily an infinite collection of MOLPs or a singleton (if $c^{\text{L}}_{ki} = c^{\text{U}}_{ki}$ for all k and i). Since all IMOLPs can be reformulated as objective-wise UMOLPs with box uncertainty sets, which is only one of many possible types of uncertainty sets, and UMOLPs with finite uncertainty sets cannot be reformulated as IMOLPs, it is evident that UMOLP (6.1) is more general than IMOLP (6.3) and permits a wider variety of problems to study. Therefore, we instead investigate UMOLP (6.1) with cost matrix (6.2) and do not study IMOLP (6.3) any further.

As already mentioned, the desired solution to objective-wise UMOLPs with uncertain objective function coefficients is not immediately obvious. Although a wide variety of possible solution concepts have been proposed, we choose to adopt the conservative concept of *highly robust efficiency*, as it is referred to in the more recent robust multiobjective optimization literature, in which solutions are efficient with respect to every realization of the uncertain data.

Definition 6.1.1. A solution $\mathbf{x}^* \in P$ to MOLP(U) is said to be *highly robust (weakly) efficient* provided for every $\mathbf{u} \in U$ there does not exist an $\mathbf{x} \in P$ such that $\mathbf{C}(\mathbf{u})\mathbf{x}\ (<) \leq \mathbf{C}(\mathbf{u})\mathbf{x}^*$. The *highly robust (weakly) efficient set* of MOLP(U) is denoted by $(\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U))$ $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$.

Based on the definition of highly robust (weak) efficiency, we have – as in the previous chapter – an immediate result.

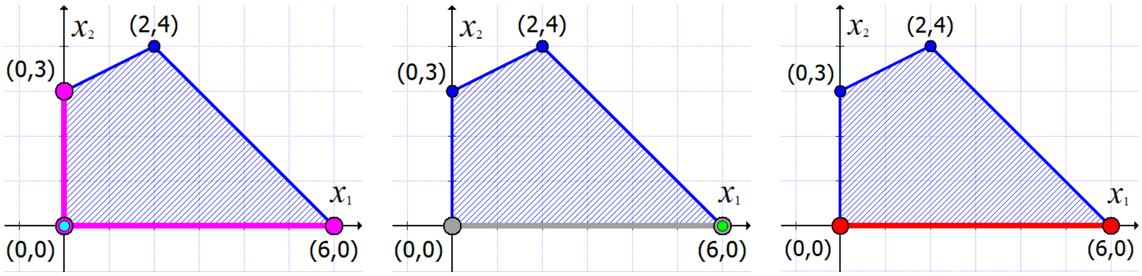
Proposition 6.1.2. *[82, p. 242] A point $\mathbf{x}^* \in P$ is a highly robust (weakly) efficient solution to* MOLP(U) *if and only if* $(\mathbf{x}^* \in \bigcap_{\mathbf{u}\in U} \mathrm{wE}(P, \mathbf{C}(\mathbf{u})))$ $\mathbf{x}^* \in \bigcap_{\mathbf{u}\in U} \mathrm{E}(P, \mathbf{C}(\mathbf{u}))$.

Proof. The proof follows the same as the proof of Proposition 5.1.2. □

More simply, Proposition 6.1.2 indicates that highly robust efficient solutions are those decisions $\mathbf{x} \in P$ that are efficient with respect to every instance MOLP$(\mathbf{u})$. As a result, although we generally assume that the (weakly) efficient set associated with MOLP$(\mathbf{u})$ is nonempty, it is apparent that highly robust efficient solutions may not exist. We provide several existence results for highly robust efficient solutions to MOLP(U) in this chapter, and discuss the ramifications of these results.

Remark 6.1.3. From Proposition 6.1.2 and Remark 4.2.3 (applied to the efficient and weakly efficient sets of MOLP$(\mathbf{u})$ for each $\mathbf{u} \in U$), it is clear that $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \subseteq \mathrm{wE}(P, \mathbf{C}(\mathbf{u}), U)$. That being said, it is important to recognize a key difference between the solutions to deterministic and uncertain MOLPs. In the deterministic case, provided that P is bounded, the weakly efficient and efficient sets of MOLP (4.8) are nonempty (cf. Corollary 2.26 and Theorem 2.19, Ehrgott [44]). On the other hand, in the uncertain case, the highly robust weakly efficient set of UMOLP (6.1) may be nonempty while the highly robust efficient set is empty even if P is bounded. For example, consider the UMOLP given by

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \mathbf{x} \\ \text{s.t.} & \qquad\qquad \mathbf{x} \in P_1 \end{array} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2,} \tag{6.5}$$



(a) The weakly efficient (purple) and efficient (teal) sets under $\mathbf{u}^1 = (1, 0, 0, 1)$

(b) The weakly efficient (grey) and efficient (green) sets under $\mathbf{u}^2 = (-1, 0, 0, 1)$

(c) The highly robust weakly efficient set (red)

Figure 6.1: Weakly efficient, efficient, and highly robust weakly efficient points of UMOLP (6.5), and feasible set P_1

where $U_1 = \{(1, 0), (-1, 0)\}, U_2 = \{(0, 1)\}$, and P_1 is given by (4.10). We have, as shown in Figure 6.1c, that the highly robust weakly efficient set is nonempty while the highly robust efficient set is empty. With this in mind, we only address highly robust weakly efficient solutions in certain cases and focus our attention on highly robust efficient solutions.

As previously mentioned in the context of Proposition 5.1.2, when the uncertainty set U contains infinitely many elements, Proposition 6.1.2 also indicates that finding highly robust (weakly) efficient solutions to MOLP(U) is impractical since we would need to find the (weakly) efficient set of an infinite number of instances, which is unrealistic. Due to this inefficacy, we need to explore other avenues, namely restricting our attention to finite uncertainty sets as a consequence of infinite uncertainty sets that may be considered as finite due to special properties. A more thorough discussion is provided in the following section.

6.2 Uncertainty Set Reductions

Since Proposition 6.1.2 reveals that solving MOLP(U) for highly robust efficient solutions is unrealistic unless the uncertainty set is finite, we explore infinite uncertainty sets that may be considered as or reduced to finite sets of scenarios with respect to MOLP(U) due to their special geometry or structure. In other words, we consider UMOLPs that are infinite collections of MOLPs, which results when the associated uncertainty set is infinite, but may be reduced to equivalent UMOLPs that are only finite collections.

The first result, which is special case of Theorem 46, Ide and Schöbel [82], reduces a polytopal uncertainty set to the finite set of its extreme points.

Theorem 6.2.1. *Let U be a nonempty polytope, and let $U^{\text{pts}} := \{\mathbf{u}^1, \ldots, \mathbf{u}^\eta\}$, where $\mathbf{u}^k = \begin{bmatrix}\mathbf{u}_1^k & \cdots & \mathbf{u}_p^k\end{bmatrix}$ for all $k = 1, \ldots, \eta$, be the set of extreme points of U. Then $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U^{\text{pts}})$.*

Proof. Follows immediately from Theorem 46, Ide and Schöbel [82], since MOLP(U) is assumed to be of objective-wise uncertainty and $\mathbf{C}(\mathbf{u})$ is clearly linear (and therefore affine) with respect to $\mathbf{u} \in U$. $\square$

A second result, which is true for a special class of UMOLPs, allows for the reduction of an unbounded infinite uncertainty set. In particular, a reduction similar to Theorem 6.2.1 is possible even when the polyhedron is allowed to be unbounded for a generalization of the model used by Kuhn et al. [95]. While this result pertains to a very specific class of problems, it is unique since in the robust optimization literature the uncertainty set is typically assumed to be bounded.

Consider the UMOLP obtained by accounting for uncertainty only in the input data of the cost vector $\mathbf{c}_1$ of MOLP (4.8), i.e., the UMOLP in which one objective is uncertain and the other $p-1$ objectives are certain or deterministic. This modified UMOLP is given by

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \widehat{\mathbf{C}}(\mathbf{u})\mathbf{x} := \begin{bmatrix} \mathbf{c}_1(\mathbf{u}) & \mathbf{c}_2 & \cdots & \mathbf{c}_p \end{bmatrix}^T \mathbf{x} \\ \text{s.t.} & \quad \mathbf{x} \in P \end{array} \right\}_{\mathbf{u} \in \widehat{U},} \tag{6.6}$$

where $\widehat{U} \subseteq \mathbb{R}^n$ is the nonempty set modeling the uncertainty. Equivalently, UMOLP (6.6) may be obtained from UMOLP (6.1) by letting $\widehat{U} = U_1$ and $U_2 = \cdots = U_p = \{\mathbf{1}\}$. As such, UMOLP (6.6) is trivially objective-wise, and the terminology and notation that we have already introduced transfers to the current context.

Theorem 6.2.2. *Let $\widehat{U}$ be a nonempty polyhedron with at least one extreme point, and let $\widehat{U}^{\text{ext}} := \{\mathbf{u}_1, \ldots, \mathbf{u}_\eta\} \cup \{\mathbf{u}_{\eta+1}, \ldots, \mathbf{u}_{\eta+\tau}\}$, where $\mathbf{u}_k = \begin{bmatrix} u_{k1} & \cdots & u_{kn} \end{bmatrix}$ for all $k = 1, \ldots, \eta+\tau$, be the union of the finite sets of extreme points and extreme directions of $\widehat{U}$, respectively, such that $\widehat{U}^{\text{ext}} \subseteq \widehat{U}$. Then* $\mathrm{E}(P, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}) = \mathrm{E}(P, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}^{\text{ext}})$.

Proof. ($\Longrightarrow$) Since $\widehat{U}^{\text{ext}} \subseteq \widehat{U}$, it follows that

$$\mathrm{E}(P, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}) = \textstyle\bigcap_{\mathbf{u} \in \widehat{U}} \mathrm{E}(P, \widehat{\mathbf{C}}(\mathbf{u})) \subseteq \textstyle\bigcap_{\mathbf{u} \in \widehat{U}^{\text{ext}}} \mathrm{E}(P, \widehat{\mathbf{C}}(\mathbf{u})) = \mathrm{E}(P, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}^{\text{ext}}).$$

($\Longleftarrow$) Let $\mathbf{x}^* \in \mathrm{E}(P, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}^{\text{ext}})$, but $\mathbf{x}^* \notin \mathrm{E}(P, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U})$. That is, there exists a $\bar{\mathbf{u}} \in \widehat{U} \setminus \widehat{U}^{\text{ext}}$ such that $\mathbf{x}^* \notin \mathrm{E}(P, \widehat{\mathbf{C}}(\bar{\mathbf{u}}))$. Hence, there exists an $\bar{\mathbf{x}} \in P$ such that

$$\widehat{\mathbf{C}}(\bar{\mathbf{u}})\bar{\mathbf{x}} \leq \widehat{\mathbf{C}}(\bar{\mathbf{u}})\mathbf{x}^* \tag{6.7}$$

by definition, and we may write

$$\bar{\mathbf{u}} = \sum_{j=1}^{\eta} \alpha_j \mathbf{u}_j + \sum_{k=\eta+1}^{\eta+\tau} \beta_k \mathbf{u}_k, \tag{6.8}$$

where $\alpha_j \geq 0$ for all $j = 1, \ldots, \eta$, $\sum_{j=1}^{\eta} \alpha_j = 1$, and $\beta_k \geq 0$ for all $k = \eta+1, \ldots, \eta+\tau$, by the Representation Theorem 2.4.33. Since $\widehat{\mathbf{C}}(\mathbf{u})$ is (clearly) linear with respect to $\mathbf{u}$, (6.7) and (6.8) yield

$$\begin{aligned} \sum_{j=1}^{\eta} \alpha_j \mathbf{c}_1(\mathbf{u}_j)\bar{\mathbf{x}} + \sum_{k=\eta+1}^{\eta+\tau} \beta_k \mathbf{c}_1(\mathbf{u}_k)\bar{\mathbf{x}} \leq \sum_{j=1}^{\eta} \alpha_j \mathbf{c}_1(\mathbf{u}_j)\mathbf{x}^* + \sum_{k=\eta+1}^{\eta+\tau} \beta_k \mathbf{c}_1(\mathbf{u}_k)\mathbf{x}^* \\ \mathbf{c}_2\bar{\mathbf{x}} \leq \mathbf{c}_2\mathbf{x}^* \\ \vdots \\ \mathbf{c}_p\bar{\mathbf{x}} \leq \mathbf{c}_p\mathbf{x}^* \end{aligned} \tag{6.9}$$

with at least one inequality strict.

Note that $\alpha_j > 0$ for at least one j, and that if $\alpha_j > 0$ for one j and $\beta_k = 0$ for all k, then (6.8) gives $\bar{\mathbf{u}} = \mathbf{u}_j$, which is a contradiction since $\bar{\mathbf{u}} \in \widehat{U} \setminus \widehat{U}^{\text{ext}}$. Regardless, we obtain

$$\begin{aligned} \sum_{j=1}^{\eta} \alpha_j [\mathbf{c}_1(\mathbf{u}_j)\bar{\mathbf{x}} - \mathbf{c}_1(\mathbf{u}_j)\mathbf{x}^*] + \sum_{k=\eta+1}^{\eta+\tau} \beta_k [\mathbf{c}_1(\mathbf{u}_k)\bar{\mathbf{x}} - \mathbf{c}_1(\mathbf{u}_k)\mathbf{x}^*] \leq 0 \\ \mathbf{c}_2\bar{\mathbf{x}} \leq \mathbf{c}_2\mathbf{x}^* \\ \vdots \\ \mathbf{c}_p\bar{\mathbf{x}} \leq \mathbf{c}_p\mathbf{x}^* \end{aligned} \tag{6.10}$$

with at least one inequality strict from (6.9). Since $\alpha_j \geq 0$ for all j and $\beta_k \geq 0$ for all k, there exists a $\varsigma \in \{1, \ldots, \eta+\tau\}$ such that $\mathbf{c}_1(\mathbf{u}_\varsigma)\bar{\mathbf{x}} - \mathbf{c}_1(\mathbf{u}_\varsigma)\mathbf{x}^* \leq 0$, or equivalently,

$\mathbf{c}_1(\mathbf{u}_\varsigma)\bar{\mathbf{x}} \leq \mathbf{c}_1(\mathbf{u}_\varsigma)\mathbf{x}^*$. Hence, we obtain that there exists a $\varsigma \in \{1, \ldots, \eta+\tau\}$ such that

$$\begin{aligned} \mathbf{c}_1(\mathbf{u}_\varsigma)\bar{\mathbf{x}} &\leq \mathbf{c}_1(\mathbf{u}_\varsigma)\mathbf{x}^* \\ \mathbf{c}_2\bar{\mathbf{x}} &\leq \mathbf{c}_2\mathbf{x}^* \\ &\vdots \\ \mathbf{c}_p\bar{\mathbf{x}} &\leq \mathbf{c}_p\mathbf{x}^* \end{aligned}$$

with at least one inequality strict from (6.10), which implies that $\mathbf{x}^* \notin \mathrm{E}(P, \widehat{\mathbf{C}}(\mathbf{u}_\varsigma))$. Since $\mathbf{u}_\varsigma \in \widehat{U}^{\mathrm{ext}}$ and $\mathbf{x}^* \in \cap_{\mathbf{u}\in\widehat{U}^{\mathrm{ext}}} \mathrm{E}(P, \widehat{\mathbf{C}}(\mathbf{u}))$, we obtain a contradiction and therefore the result. □

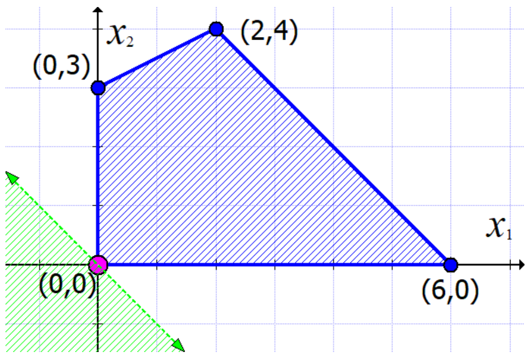
The following example provides an illustration of Theorem 6.2.2.

Example 6.2.3. Consider the biobjective UMOLP with the first objective uncertain and the second deterministic given by

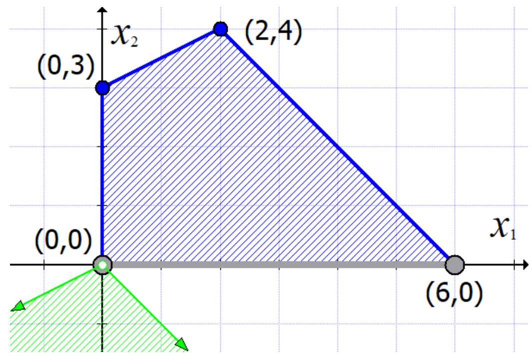
$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \begin{bmatrix} u_1 & u_2 \\ 1 & 1 \end{bmatrix}\mathbf{x} \\ \text{s.t.} & \mathbf{x} \in P_1 \end{array} \right\}_{\mathbf{u}\in\widehat{U},} \tag{6.11}$$

where $\widehat{U} = \{\mathbf{u} \in \mathbb{R}^2 : -2u_1 - u_2 \leq 0, u_1 - 3u_2 \leq 0\}$ is a polyhedral convex cone, and P_1 is given by (4.10). Note that $\widehat{U}$ has one extreme point, namely $(0, 0)$, the set of extreme points and directions of $\widehat{U}$ is given by $\widehat{U}^{\mathrm{ext}} = \{\mathbf{u}_1 = (0, 0), \mathbf{u}_2 = (-1, 2), \mathbf{u}_3 = (3, 1)\}$, and $\widehat{U}^{\mathrm{ext}} \subseteq \widehat{U}$ clearly holds. As such, Theorem 6.2.2 is applicable.

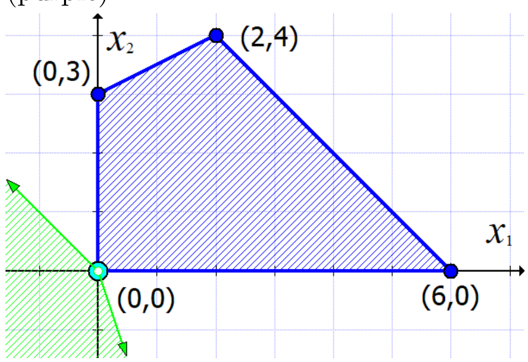
With this in mind, we first examine $\mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}^{\mathrm{ext}})$. Observe that the cones of improving directions associated with the extreme point/direction scenarios $\mathbf{u}_1, \mathbf{u}_2$,



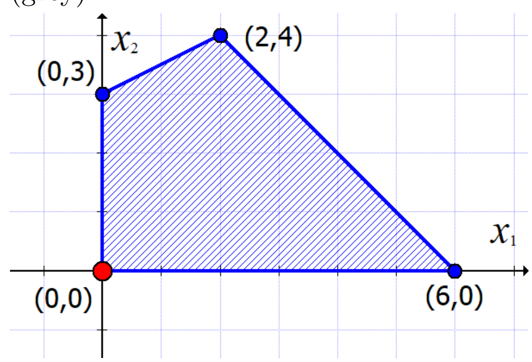
(a) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_1))$ (green) and $\mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}_1))$ (purple)



(b) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_2))$ (green) and $\mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}_2))$ (grey)



(c) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_3))$ (green) and $\mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}_3))$ (teal)



(d) $\mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}^{\mathrm{ext}})$ (red)

Figure 6.2: Cones of improving directions and efficient points associated with the scenarios $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$, as well as the highly robust efficient set $\mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}^{\mathrm{ext}})$

and $\mathbf{u}_3$ are given by

$$D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_1)) = \{\mathbf{d} \in \mathbb{R}^2 : d_1 + d_2 < 0\},$$
$$D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_2)) = \{\mathbf{d} \in \mathbb{R}^2 : -d_1 + 2d_2 \leq 0, d_1 + d_2 \leq 0, \text{ at least one strict}\},$$
$$D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_3)) = \{\mathbf{d} \in \mathbb{R}^2 : 3d_1 + d_2 \leq 0, d_1 + d_2 \leq 0, \text{ at least one strict}\}.$$

Solving each instance using the corresponding cone of improving directions as in Section 4.2.2.2, we obtain the efficient sets as shown in Figures 6.2a–6.2c. Applying Proposition 6.1.2, the intersection then yields that $\mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}^{\mathrm{ext}}) = \{(0,0)\}$, which is shown in Figure 6.2d.

We now examine $\mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U})$ in a similar manner. Observe that each $\mathbf{u} \in \widehat{U}$ may be expressed as

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \beta_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \beta_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

where $\beta_1, \beta_2 \geq 0$ by the Representation Theorem 2.4.33. Hence, the cone of improving directions of UMOLP (6.11) is given by

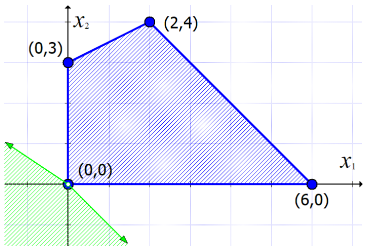
$$D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u})) = \left\{ \mathbf{d} \in \mathbb{R}^2 : \begin{bmatrix} (-\beta_1 + 3\beta_2)d_1 + (2\beta_1 + \beta_2)d_2 \\ d_1 + d_2 \end{bmatrix} \leq \mathbf{0} \right\}.$$

In order to show that $\mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}) = \mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}^{\text{ext}})$ holds, we analyze $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ for various values of β_1 and β_2. For example, if $\beta_1 = \beta_2 = 0$, then $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u})) = D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_1))$. Similarly, if $\beta_1 = 0$, then $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u})) = D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_2))$ for all $\beta_2 > 0$, while if $\beta_2 = 0$, then $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u})) = D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_3))$ for all $\beta_1 > 0$. Otherwise, $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ ranges between the extreme directions $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 1 \end{bmatrix}^T$ of $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_2))$ and $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_3))$, respectively, i.e., within the halfspace $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}_1))$, as shown in Figures 6.3a–6.3i. Regardless, for any values of $\beta_1, \beta_2 \geq 0$, it is thus clear that the efficient set of each instance is either the singleton $\{(0,0)\}$ or the line segment joining $(0,0)$ and $(6,0)$, which yields that $\mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u}), \widehat{U}) = \bigcap_{\mathbf{u} \in \widehat{U}} \mathrm{E}(P_1, \widehat{\mathbf{C}}(\mathbf{u})) = \{(0,0)\}$ as expected.

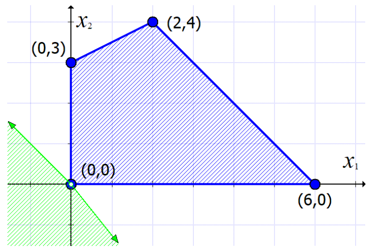
In view of Theorems 6.2.1 and 6.2.2, we restrict our attention to finite uncertainty sets throughout the remainder of this chapter. The finite set of scenarios is given by (5.3), i.e.,

$$U := \{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^s\} \subset \mathbb{R}^q,$$

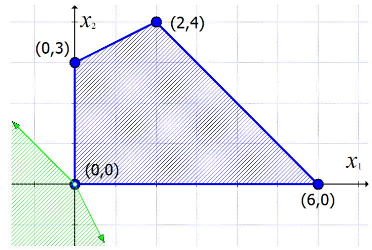
where we assume WLOG that each scenario is distinct. As mentioned, under the assumption that the uncertainty set is finite, the highly robust efficient set of UMOLP (6.1) with cost matrix (6.2) is studied in Section 6.3 and the computation of highly robust efficient solutions is addressed in Section 6.4. Although certain results may



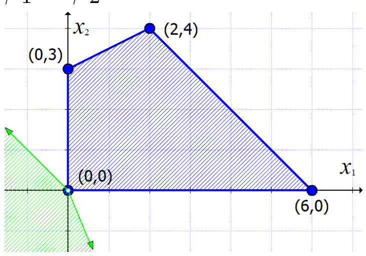
(a) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ (green) with $\beta_1 = \beta_2 = 1$



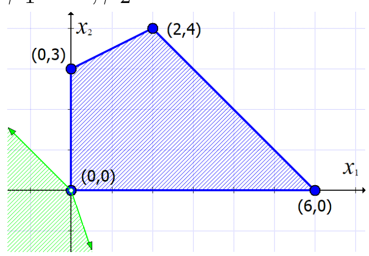
(b) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ (green) with $\beta_1 = 1, \beta_2 = 2$



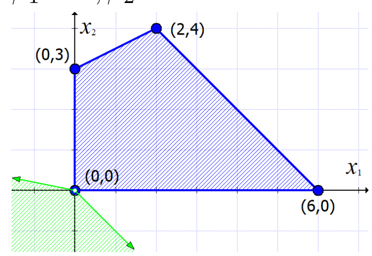
(c) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ (green) with $\beta_1 = 1, \beta_2 = 5$



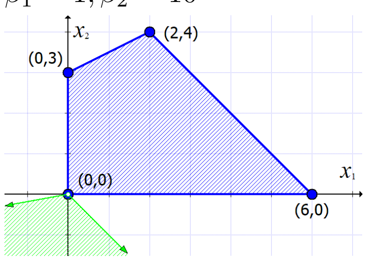
(d) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ (green) with $\beta_1 = 1, \beta_2 = 10$



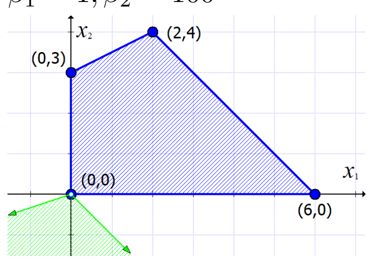
(e) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ (green) with $\beta_1 = 1, \beta_2 = 100$



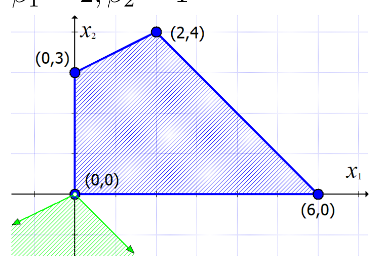
(f) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ (green) with $\beta_1 = 2, \beta_2 = 1$



(g) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ (green) with $\beta_1 = 5, \beta_2 = 1$



(h) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ (green) with $\beta_1 = 10, \beta_2 = 1$



(i) $D_{\leq}(\widehat{\mathbf{C}}(\mathbf{u}))$ (green) with $\beta_1 = 100, \beta_2 = 1$

Figure 6.3: Cones of improving directions associated with varying values of β_1 and β_2

also be true for infinite uncertainty sets, as in the previous chapter, we do not address this in more detail.

6.3 Regarding the Highly Robust Efficient Set

In this section, we explore properties of the highly robust efficient set such as those that we extend from deterministic efficiency, as well as those specific to UMOLPs and the definition of highly robust efficiency. The former are examined directly below, while the latter, including a characterization of the highly robust efficient set and bound sets on the highly robust efficient set, are presented in the subsequent subsections. As acuteness emerges as an important element in the course of this analysis, we address this property in more detail as well.

6.3.1 Properties

Various properties of the efficient set of MOLP (4.8) are known in the literature including those regarding closedness, convexity, and connectedness. We examine how some of these properties extend from efficient solutions in the deterministic case to highly robust efficient solutions in the uncertain case. In particular, we provide five properties of the efficient set of MOLP (4.8) that directly extend to the highly robust efficient set of MOLP(U), as well as one property that does not.

Proposition 6.3.1. *(i)* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ *is closed.*

(ii) $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ *is not necessarily convex.*

(iii) If $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \neq \emptyset$*, then it is either the entire set* P *or on the boundary of* P*.*

(iv) If $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \neq \emptyset$ *and a point in the relative interior of a face of* P *is highly robust efficient, then so is the entire face.*

(v) If $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \neq \emptyset$*, then there exists a highly robust efficient extreme point.*

(vi) $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ *is not necessarily connected.*

Proof. (i) Since MOLP$(\mathbf{u})$ is a deterministic MOLP for each $\mathbf{u} \in U$, it follows that $\mathrm{E}(P, \mathbf{C}(\mathbf{u}))$ is closed for each $\mathbf{u} \in U$ by Proposition 4.2.5(i). Hence, as an arbitrary intersection of closed sets is closed by Theorem 2.4.8(iii), the result follows from Proposition 6.1.2.

(ii)–(iv) Similarly, (ii)–(iv) follow from Proposition 6.1.2 combined with Propositions 4.2.5(ii), (iii), and (iv), respectively.

(v) Follows from parts (iii) and (iv).

(vi) As $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ is the intersection of possibly nonconvex sets, it may be disconnected. Refer to Example 6.3.2. □

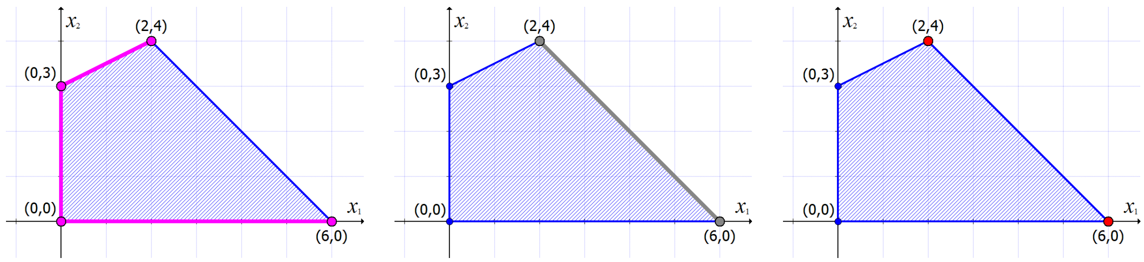
Although the first five properties immediately extend from the deterministic to uncertain setting, the same cannot be said of connectedness. Since the efficient set of MOLP (4.8) is connected as in Proposition 4.2.5(vi), it might be expected that the highly robust efficient set of MOLP(U) is also connected but this is not the case. As an illustration, consider the following example.

Example 6.3.2. Consider the UMOLP given by

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \begin{bmatrix} 3u_{11} & -9u_{12} \\ -u_{21} & 9u_{22} \end{bmatrix} \mathbf{x} \\ \text{s.t.} & \qquad\qquad \mathbf{x} \in P_1 \end{array} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2,} \tag{6.12}$$

where $U_1 = \{(1,1)\}, U_2 = \{(1,1), (2, -1/9)\}$, and P_1 is given by (4.10). Solving each of the two instances and taking the intersection of their efficient sets, we observe that the highly robust efficient set is disconnected, as shown in Figure 6.4c.

Remark 6.3.3. The fact that the highly robust efficient set is not necessarily connected suggests that an algorithm to obtain highly robust efficient solutions to MOLP(U) similar to the multiobjective simplex method is not advantageous to pursue since the effectiveness of this simplex algorithm relies on the connectedness of efficient bases associated with extreme points, which we do not have in general for highly robust efficient solutions. Nevertheless, Bitran [18] and Benson [12] both implement an extension of the multiobjective simplex method in order to compute all necessarily



(a) The efficient set under scenario $(1, 1, 1, 1)$ (purple)

(b) The efficient set under scenario $(1, 1, 2, -1/9)$ (grey)

(c) The highly robust efficient set of UMOLP (6.12) (red)

Figure 6.4: Efficient and highly robust efficient points for Example 6.3.2

efficient extreme point solutions of interval multiobjective linear programs. Each method works by solving a nominal problem for its extreme points and then reducing the obtained set of extreme points to the desired set by solving a subproblem for each extreme point.

6.3.2 Characterization

Similarly to properties of the highly robust efficient set, we extend known results about the efficient set of MOLP (4.8) that use convex cones (such as the cone of improving directions and the normal cone) to those regarding the highly robust efficient set of MOLP(U).

We first examine the objective space. As each instance of MOLP(U) is a deterministic MOLP, we may define the *attainable set* or *set of criterion* of MOLP($\mathbf{u}$) for each scenario $\mathbf{u} \in U$ as in the deterministic setting (see Section 4.2), where $\mathbf{C}$ is replaced by $\mathbf{C}(\mathbf{u})$. Namely, the attainable set of MOLP($\mathbf{u}$) is given by $Y_{\mathbf{C}(\mathbf{u}),P} = \{\mathbf{y} \in \mathbb{R}^p : \mathbf{y} = \mathbf{C}(\mathbf{u})\mathbf{x} \text{ for some } \mathbf{x} \in P\}$. With this in mind, we provide a definition for highly robust (weak) efficiency in the objective space that is equivalent to Definition 6.1.1

Definition 6.3.4. A feasible solution $\mathbf{x}^* \in P$ is said to be

(i) *highly robust efficient* provided $Y_{\mathbf{C}(\mathbf{u}),P} \cap (\{\mathbf{C}(\mathbf{u})\mathbf{x}^*\} \oplus \mathbb{R}^p_{\leqq}) = \{\mathbf{C}(\mathbf{u})\mathbf{x}^*\}$ for all $\mathbf{u} \in U$;

(ii) *highly robust efficient* provided $Y_{\mathbf{C}(\mathbf{u}),P} \cap (\{\mathbf{C}(\mathbf{u})\mathbf{x}^*\} \oplus \mathbb{R}^p_{\leq}) = \emptyset$ for all $\mathbf{u} \in U$;

(iii) *highly robust weakly efficient* provided $Y_{\mathbf{C}(\mathbf{u}),P} \cap (\{\mathbf{C}(\mathbf{u})\mathbf{x}^*\} \oplus \mathbb{R}^p_{<}) = \emptyset$ for all $\mathbf{u} \in U$.

As should be clear based on Definition 4.2.6, the above definition simply states that $\mathbf{x}^*$ is highly robust (weakly) efficient provided that it is efficient with respect to every scenario, and is thus equivalent to the original Definition 6.1.1. Moreover, as in the deterministic setting, Definitions 6.3.4(i) and (ii) clearly imply that, for each $\mathbf{u} \in U$, only outcomes on the boundary of $Y_{\mathbf{C}(\mathbf{u}),P}$ need to be considered when searching for highly robust efficient solutions since the intersections are necessarily more than the singleton $\{\mathbf{C}(\mathbf{u})\mathbf{x}^*\}$ or the empty set, respectively, for any $\mathbf{C}(\mathbf{u})\mathbf{x}^* \in \text{int}(Y_{\mathbf{C}(\mathbf{u}),P})$. This interpretation supports the fact that the highly robust efficient set is either on the boundary of or is the entire feasible set, cf. Proposition 6.3.1(iii).

We next consider various characterizations in the decision space, the first of which involves the cone of improving directions. As above, since each instance of MOLP(U) is a deterministic MOLP, we may denote the cones of improving directions of MOLP($\mathbf{u}$) for each scenario $\mathbf{u} \in U$ as in the deterministic setting, where $\mathbf{C}$ in Definition 4.2.9 is replaced by $\mathbf{C}(\mathbf{u})$. In addition, we may define the cones of improving directions of MOLP(U) by accounting for the improving directions associated with every scenario $\mathbf{u} \in U$.

Definition 6.3.5. (i) The *open cone of improving directions* of MOLP(U) is defined to be $D_<(\mathbf{C}(\mathbf{u}), U) := \bigcup_{\mathbf{u} \in U} D_<(\mathbf{C}(\mathbf{u}))$.

(ii) The *cone of improving directions* of MOLP(U) is defined to be $D_{\leq}(\mathbf{C}(\mathbf{u}), U) := \bigcup_{\mathbf{u}\in U} D_{\leq}(\mathbf{C}(\mathbf{u}))$.

(iii) The *closed cone of improving directions* of MOLP(U) is defined to be $D_{\leqq}(\mathbf{C}(\mathbf{u}), U) := \bigcup_{\mathbf{u}\in U} D_{\leqq}(\mathbf{C}(\mathbf{u}))$.

In the deterministic setting, the cones of improving directions of MOLP (4.8) may be used to characterize the (weak) efficiency of solutions as in Proposition 4.2.10. Analogously to the deterministic case, we may characterize the highly robust (weak) efficiency of solutions to MOLP(U) using the cones of improving directions given in the above definition.

Theorem 6.3.6. *Let* $\mathbf{x}^* \in P$. *Then*

(i) $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ *if and only if* $(D_{\leq}(\mathbf{C}(\mathbf{u}), U) \oplus \{\mathbf{x}^*\}) \cap P = \emptyset$;

(ii) $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ *if* $\left(D_{\leqq}(\mathbf{C}(\mathbf{u}), U) \oplus \{\mathbf{x}^*\}\right) \cap P = \{\mathbf{x}^*\}$;

(iii) $\mathbf{x}^* \in \mathrm{wE}(P, \mathbf{C}(\mathbf{u}), U)$ *if and only if* $(D_{<}(\mathbf{C}(\mathbf{u}), U) \oplus \{\mathbf{x}^*\}) \cap P = \emptyset$.

Proof. (i) Since $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}))$ if and only if $(D_{\leq}(\mathbf{C}(\mathbf{u})) \oplus \{\mathbf{x}^*\}) \cap P = \emptyset$ by Proposition 4.2.10(i), it likewise follows that $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ if and only if $(D_{\leq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}) \cap P = \emptyset$ for all $i = 1, \ldots, s$. Equivalently, the latter becomes $[(D_{\leq}(\mathbf{C}(\mathbf{u}^1)) \oplus \{\mathbf{x}^*\}) \cap P] \cup \cdots \cup [(D_{\leq}(\mathbf{C}(\mathbf{u}^s)) \oplus \{\mathbf{x}^*\}) \cap P] = \emptyset$, i.e.,

$$\left[(D_{\leq}(\mathbf{C}(\mathbf{u}^1)) \oplus \{\mathbf{x}^*\}) \cup \cdots \cup (D_{\leq}(\mathbf{C}(\mathbf{u}^s)) \oplus \{\mathbf{x}^*\})\right] \cap P = \emptyset \tag{6.13}$$

by the Distributive Law of Intersections 2.4.3(i). Hence, (6.13) equivalently becomes

$$\left[\left(\bigcup_{\mathbf{u}\in U} D_{\leq}(\mathbf{C}(\mathbf{u}))\right) \oplus \{\mathbf{x}^*\}\right] \cap P = \emptyset$$

by Theorem 2.4.3(iii). Applying Definition 6.3.5(i), the result follows.

(ii) Let $(D_{\leqq}(\mathbf{C}(\mathbf{u}), U) \oplus \{\mathbf{x}^*\}) \cap P = \{\mathbf{x}^*\}$, i.e.,

$$\left[(D_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \cup \cdots \cup D_{\leqq}(\mathbf{C}(\mathbf{u}^s))) \oplus \{\mathbf{x}^*\}\right] \cap P = \{\mathbf{x}^*\}$$

by Definition 6.3.5(iii). Equivalently,

$$\left[(D_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \oplus \{\mathbf{x}^*\}) \cap P\right] \cup \cdots \cup \left[(D_{\leqq}(\mathbf{C}(\mathbf{u}^s)) \oplus \{\mathbf{x}^*\}) \cap P\right] = \{\mathbf{x}^*\}$$

by Theorem 2.4.3(iii) and the Distributive Law of Intersections 2.4.3(i), respectively. That is, either $(D_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}) \cap P = \{\mathbf{x}^*\}$ or $(D_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}) \cap P = \emptyset$ for each $i = 1, \ldots, s$, with at least one equal to $\{\mathbf{x}^*\}$. However, it is clear that $(D_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}) \cap P \neq \emptyset$ for each $i \in \{1, \ldots, s\}$ since $D_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}$ must contain at least $\mathbf{x}^* \in P$. Hence, $(D_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}) \cap P = \{\mathbf{x}^*\}$ for all $i = 1, \ldots, s$, which implies that $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^i))$ by Proposition 4.2.10(ii) for all $i = 1, \ldots, s$. Thus, $\mathbf{x}^*$ is highly robust efficient by definition.

(iii) The proof follows similarly to the proof of part (i). □

Intuitively, the above result states that a point is highly robust efficient if and only if there does not exist a feasible direction that is also improving in *any* scenario at that point, which is indicated by the intersection being empty. Moreover, as with Definition 6.3.4, the first part of the above theorem implies that, provided $D_{\leq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$, only points $\mathbf{x}^*$ on the boundary of P need to be considered for highly robust efficiency since the intersection $(D_{\leq}(\mathbf{C}(\mathbf{u}), U) \oplus \{\mathbf{x}^*\}) \cap P$ is necessarily nonempty otherwise. This observation, as in the objective space, supports the fact that the highly robust efficient set is either on the boundary or is the entire feasible set, cf. Proposition 6.3.1(iii).

Remark 6.3.7. It is worth noting that if $D_{\leq}(\mathbf{C}(\mathbf{u}), U) = \emptyset$, then $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = P$ since $\emptyset \oplus \{\mathbf{x}^*\} = \emptyset$ (refer to Remark 2.4.2) so that the intersection in Theorem 6.3.6(i) holds trivially for all $\mathbf{x}^* \in P$. Similarly, if $D_{<}(\mathbf{C}(\mathbf{u}), U) = \emptyset$, then $\mathrm{wE}(P, \mathbf{C}(\mathbf{u}), U) = P$. In view of the previous discussion, the former observation reveals that the cone of improving directions of MOLP(U) may be used to prove Proposition 6.3.1(iii) directly without using the corresponding result regarding deterministic efficiency.

In addition to the more geometric interpretation of highly robust (weak) efficiency offered by Theorem 6.3.6, an algebraic perspective may also be derived as in the following.

Corollary 6.3.8. *Let* $\mathbf{x}^* \in P$ *be a feasible solution to* MOLP(U). *Then*

(i) $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ *if and only if the system*

$$\begin{aligned} \mathbf{C}(\mathbf{u})\mathbf{x} &\leq \mathbf{C}(\mathbf{u})\mathbf{x}^* \\ \mathbf{A}\mathbf{x} &\leqq \mathbf{b} \\ \mathbf{x} &\geqq \mathbf{0} \end{aligned} \tag{6.14}$$

has no solution for each $\mathbf{u} \in U$*;*

(ii) $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ *if* $\mathbf{x}^*$ *is the unique solution to the system*

$$\begin{aligned} \mathbf{C}(\mathbf{u})\mathbf{x} &\leqq \mathbf{C}(\mathbf{u})\mathbf{x}^* \\ \mathbf{A}\mathbf{x} &\leqq \mathbf{b} \\ \mathbf{x} &\geqq \mathbf{0} \end{aligned} \tag{6.15}$$

for each $\mathbf{u} \in U$*;*

(iii) $\mathbf{x}^* \in \mathrm{wE}(P, \mathbf{C}(\mathbf{u}), U)$ *if and only if the system*

$$\begin{aligned} \mathbf{C}(\mathbf{u})\mathbf{x} &< \mathbf{C}(\mathbf{u})\mathbf{x}^* \\ \mathbf{A}\mathbf{x} &\leqq \mathbf{b} \\ \mathbf{x} &\geqq \mathbf{0} \end{aligned} \tag{6.16}$$

has no solution for each $\mathbf{u} \in U$.

Proof. Applying Proposition 4.2.13 for each scenario $\mathbf{u} \in U$ yields the result. □

Similar to the deterministic setting, each of the three systems above provides an algebraic description of highly robust (weak) efficiency. That being said, it is not simple computationally to determine whether or not (6.14) has a solution since the vector inequality $\mathbf{C}(\mathbf{u})\mathbf{x} \leq \mathbf{C}(\mathbf{u})\mathbf{x}^*$ requires that *at least* one component is strict but it is unknown precisely which component(s) or how many. As a result, (6.15) may be used to determine the highly robust efficiency of a solution $\mathbf{x}^* \in P$, while (6.16) may be employed to identify its highly robust weak efficiency.

In addition to Corollary 6.3.8, several similar results may be obtained by applying different theorems of the alternative, such as Gale's Theorem 2.6.1 and Theorem 2.6.4, to systems (6.14) and (6.16). (Note that system (6.15) is not used since the available theorems of the alternative do not account for the requirement that $\mathbf{x}^*$ must be the unique solution.) To keep the notation compact in the following corollaries, for each $\mathbf{u} \in U$ and some $\mathbf{x}^* \in P$, let

$$\mathbf{A}_1(\mathbf{u}) := \begin{bmatrix} \mathbf{C}(\mathbf{u}) \\ \mathbf{A} \\ -\mathbf{I}_n \end{bmatrix} \in \mathbb{R}^{(p+m+n)\times n}, \mathbf{b}_1(\mathbf{u}, \mathbf{x}^*) := \begin{bmatrix} \mathbf{C}(\mathbf{u})\mathbf{x}^* \\ \mathbf{b} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{p+m+n}$$

and

$$\mathbf{A}_2(\mathbf{u}) := \begin{bmatrix} \mathbf{C}(\mathbf{u}) \\ \mathbf{A} \end{bmatrix} \in \mathbb{R}^{(p+m)\times n}, \mathbf{b}_2(\mathbf{u}, \mathbf{x}^*) := \begin{bmatrix} \mathbf{C}(\mathbf{u})\mathbf{x}^* \\ \mathbf{b} \end{bmatrix} \in \mathbb{R}^{p+m}.$$

We first consider system (6.14) and Theorem 2.6.4.

Corollary 6.3.9. *If* $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$*, then*

$$\left\{ \begin{array}{c} \mathbf{A}_1(\mathbf{u})^T\mathbf{v} = \mathbf{0}, \mathbf{b}_1(\mathbf{u}, \mathbf{x}^*)^T\mathbf{v} = -1, \mathbf{v} \geqq \mathbf{0} \\ \textit{or} \\ \mathbf{A}_1(\mathbf{u})^T\mathbf{v} = \mathbf{0}, \mathbf{b}_1(\mathbf{u}, \mathbf{x}^*)^T\mathbf{v} \leqq 0, \mathbf{v} > \mathbf{0} \end{array} \right\} \tag{6.17}$$

has a solution $\mathbf{v} \in \mathbb{R}^{p+m+n}$ *for each* $\mathbf{u} \in U$*.*

Proof. If $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$, then (6.14) has no solution for each $\mathbf{u} \in U$ by Corollary 6.3.8(i), which implies that $\mathbf{A}_1(\mathbf{u})\mathbf{x} \leq \mathbf{b}_1(\mathbf{u}, \mathbf{x}^*)$ has no solution for each $\mathbf{u} \in U$. By Theorem 2.6.4, it must be that (6.17) has a solution $\mathbf{v} \in \mathbb{R}^{p+m+n}$ for each $\mathbf{u} \in U$ as claimed. □

Considering Corollary 6.3.9 follows from Corollary 6.3.8(i), which is both a necessary and sufficient condition for highly robust efficiency, it is natural to ask why the former is only a necessary condition. The reason, as it turns out, lies in the fact that systems (6.14) and $\mathbf{A}_1(\mathbf{u})\mathbf{x} \leq \mathbf{b}_1(\mathbf{u}, \mathbf{x}^*)$ are not equivalent. Although this may not be immediately obvious, consider for example that $\begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}^T \leq \begin{bmatrix} 5 & 6 & 7 & 9 \end{bmatrix}^T$, yet $\begin{bmatrix} 5 & 6 \end{bmatrix}^T \nleq \begin{bmatrix} 5 & 6 \end{bmatrix}^T$.

Next, we obtain a sufficient condition for highly robust (weak) efficiency by way of systems (6.14) and (6.16), respectively, combined with Gale's Theorem 2.6.1(i). Similarly to the preceding discussion, the following corollary does not provide necessary and sufficient conditions since the vector inequalities do not permit direct equivalences between the associated systems.

Corollary 6.3.10. *Let* $\mathbf{x}^* \in P$ *be given, and assume that*

$$\mathbf{A}_1(\mathbf{u})^T\mathbf{v} = \mathbf{0}, \mathbf{b}_1(\mathbf{u}, \mathbf{x}^*)^T\mathbf{v} = -1, \mathbf{v} \geqq \mathbf{0} \tag{6.18}$$

has a solution $\mathbf{v} \in \mathbb{R}^{p+m+n}$ *for each* $\mathbf{u} \in U$*.*

(i) The point $\mathbf{x}^*$ *is highly robust efficient.*

(ii) The point $\mathbf{x}^*$ *is highly robust weakly efficient.*

Proof. (i) Assume that (6.18) has a solution $\mathbf{v} \in \mathbb{R}^{p+m+n}$ for each $\mathbf{u} \in U$. Equivalently, by Gale's Theorem 2.6.1(i), $\mathbf{A}_1(\mathbf{u})\mathbf{x} \leqq \mathbf{b}_1(\mathbf{u}, \mathbf{x}^*)$ has no solution for each $\mathbf{u} \in U$. Since the previous system is a relaxation of (6.14), the result follows from Corollary 6.3.8(i).

(ii) The proof follows similarly to the proof of part (i). □

Finally, a second sufficient condition is derived using systems (6.14) and (6.16), respectively, combined with Gale's Theorem 2.6.1(ii).

Corollary 6.3.11. *Let* $\mathbf{x}^* \in P$ *be given, and assume that*

$$\mathbf{A}_2(\mathbf{u})^T\mathbf{v} \geqq \mathbf{0}, \mathbf{b}_2(\mathbf{u}, \mathbf{x}^*)^T\mathbf{v} < 0, \mathbf{v} \geqq \mathbf{0} \tag{6.19}$$

has a solution $\mathbf{v} \in \mathbb{R}^{p+m}$ *for each* $\mathbf{u} \in U$*.*

(i) The point $\mathbf{x}^*$ *is highly robust efficient.*

(ii) The point $\mathbf{x}^*$ *is highly robust weakly efficient.*

Proof. (i) Assume that (6.19) has a solution $\mathbf{v} \in \mathbb{R}^{p+m}$ for each $\mathbf{u} \in U$. Equivalently, by Gale's Theorem 2.6.1(ii), $\mathbf{A}_2(\mathbf{u})\mathbf{x} \leqq \mathbf{b}_2(\mathbf{u}, \mathbf{x}^*), \mathbf{x} \geqq \mathbf{0}$ has no solution for each $\mathbf{u} \in U$. Since the previous system is a relaxation of (6.14), applying Corollary 6.3.8(i) yields the result.

(ii) The proof follows similarly to the proof of part (i). □

The second characterization of the highly robust efficient set in the decision space we examine concerns the normal cone (refer to Definition 3.1.4). In particular, Theorems 4.2.15 and 4.2.29, which involve the normal cone, may be extended from the deterministic to uncertain setting (in a manner similar to results regarding the cone of improving directions). As mentioned before Theorem 4.2.29, by reframing the theorem due to Luc in the context of the strict polars of the cones of improving directions, we achieve a different perpective that leads to further insight in the form of conditions on highly robust (weak) efficiency. Recasting this theorem also allows us to exploit properties of cones. To this end, as each instance of MOLP(U) is a deterministic MOLP, the strict polars of the cones of improving directions of MOLP($\mathbf{u}$) for each scenario $\mathbf{u} \in U$ are given by (under the specified assumptions) Theorem 3.2.13, where $\mathbf{M}$ is replaced by $\mathbf{C}(\mathbf{u})$.

Remark 6.3.12. (i) We extend Theorem 4.2.15 as follows. For a solution $\mathbf{x}^* \in P$, it is highly robust (weakly) efficient if and only if $N_P(\mathbf{x}^*)$ contains some vector $-\mathbf{C}(\mathbf{u})^T\boldsymbol{\lambda}, \boldsymbol{\lambda}\ (\geq) > \mathbf{0}$, for all $\mathbf{u} \in U$. It is worth noting that if $-\mathbf{C}(\mathbf{u})^T\boldsymbol{\lambda} = \mathbf{0}$ for some $\mathbf{u} \in U$ and some $\boldsymbol{\lambda} > \mathbf{0}$, then the entire feasible set is efficient in that scenario since $N_P(\mathbf{x}^*)$ necessarily contains $\mathbf{0}$. Similarly, if for all $\mathbf{u} \in U$ there exists a $\boldsymbol{\lambda} > \mathbf{0}$ such that $-\mathbf{C}(\mathbf{u})^T\boldsymbol{\lambda} = \mathbf{0}$, then the entire feasible set is in fact highly robust efficient. (The same line of thought may be followed for $\boldsymbol{\lambda} \geq \mathbf{0}$ and the highly robust weakly efficient set.)

(ii) Similarly, we extend Theorem 4.2.29 (under the same assumptions, but for all $\mathbf{u} \in U$) by saying that $\mathbf{x}^* \in P$ is highly robust (weakly) efficient if and only if $(N_P(\mathbf{x}^*) \cap D^{\mathrm{s}+}_{<}(\mathbf{C}(\mathbf{u})) \neq \emptyset)$ $N_P(\mathbf{x}^*) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ for all $\mathbf{u} \in U$. As in Theorem 4.2.29, we may equivalently use $D^{\mathrm{s}+}_{\leq}(\mathbf{C}(\mathbf{u}))$ since $D^{\mathrm{s}+}_{\leq}(\mathbf{C}(\mathbf{u})) = D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}))$ when $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute by Theorem 3.2.13(ii). Moreover, as we need $N_P(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$, it is important to know when $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ since if it is not, the result never holds. (We are only concerned with the nonemptiness of $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}))$ since $N_P(\mathbf{x}^*) \neq \emptyset$.) To this end, it is clear that $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ for all $\mathbf{u} \in U$ by Theorem 3.2.13(i).

In order to obtain a result that, unlike the extensions in Remark 6.3.12, does not require checking the necessary and sufficient conditions of Theorems 4.2.15 and 4.2.29 for every scenario $\mathbf{u} \in U$, we use the strict polars of the cones of improving directions of MOLP(U) (cf. Proposition 3.2.18, where $\mathbf{M}_\ell$ is replaced by $\mathbf{C}(\mathbf{u})$). Given the cones of improving directions $D_{\leqq}(\mathbf{C}(\mathbf{u}), U), D_{\leq}(\mathbf{C}(\mathbf{u}), U)$, and $D_{<}(\mathbf{C}(\mathbf{u}), U)$ of MOLP(U), we denote their strict polars by $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}), U), D^{s+}_{\leq}(\mathbf{C}(\mathbf{u}), U)$, and $D^{s+}_{<}(\mathbf{C}(\mathbf{u}), U)$, respectively.

Theorem 6.3.13. *Let* $\mathbf{x}^* \in P$.

(i) Let $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ *be acute for all* $\mathbf{u} \in U$. *If* $N_P(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$, *then* $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$.

(ii) Let $\mathrm{cl}(D_{<}(\mathbf{C}(\mathbf{u}))) = D_{\leqq}(\mathbf{C}(\mathbf{u}))$ *for all* $\mathbf{u} \in U$. *If* $N_P(\mathbf{x}^*) \cap D^{s+}_{<}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$, *then* $\mathbf{x}^* \in \mathrm{wE}(P, \mathbf{C}(\mathbf{u}), U)$.

Proof. (i) Let $N_P(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$. Equivalently, by Proposition 3.2.18(i), $N_P(\mathbf{x}^*) \cap \bigcap_{\mathbf{u} \in U} D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$. That is, $\left[N_P(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}^1))\right] \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}^2)) \cap \cdots \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}^s)) \neq \emptyset$ by the Associative Law of Intersections. Accordingly, the associative law yields $N_P(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \neq \emptyset$ for all $i = 1, \ldots, s$. Thus, the result follows from Theorem 4.2.29(i).

(ii) Follows similarly to the proof of part (i), where $\mathrm{cl}(D_{<}(\mathbf{C}(\mathbf{u}))) = D_{\leqq}(\mathbf{C}(\mathbf{u}))$ implies that $D_{<}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ for all $\mathbf{u} \in U$ so that we may use Proposition 3.2.18(iii). □

Remark 6.3.14. As in Remark 6.3.12(ii), it is of interest to know when $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$ since if it is not, then $N_P(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$ never holds. To this end, since $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is a closed cone, we know that $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$ when $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is acute due to Theorem 3.2.16(i). Moreover, with the additional assumption that $D_{\leq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ for all $\mathbf{u} \in U$ (which is needed for Proposition 3.2.18(ii)), we may rewrite Theorem 6.3.13(i) using $D^{s+}_{\leq}(\mathbf{C}(\mathbf{u}), U)$.

For an illustration of Theorem 6.3.13(i), as well as the extension of Theorem 4.2.29(i) described in Remark 6.3.12(ii), consider the following example.

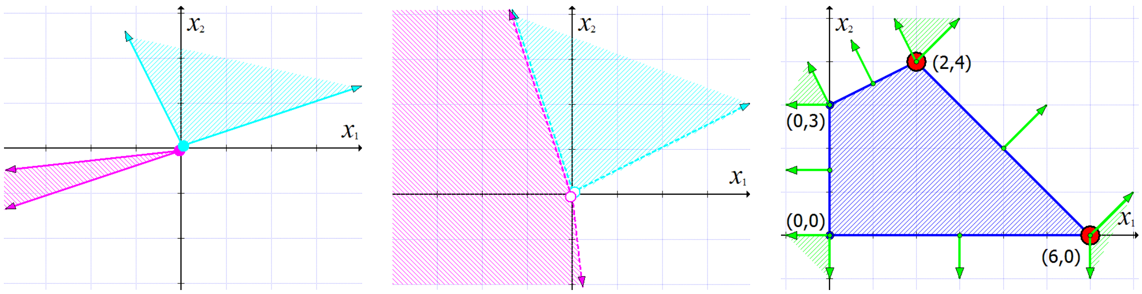
Example 6.3.15. Consider UMOLP (6.12) in Example 6.3.2. We have two scenarios $\mathbf{u}^1 = (1, 1, 1, 1)$ and $\mathbf{u}^2 = (1, 1, 2, -1/9)$. The closed cones of improving directions $D(\mathbf{C}(\mathbf{u}^1))$ and $D(\mathbf{C}(\mathbf{u}^2))$ are shown in Figure 6.5a, while their strict polars are shown in Figure 6.5b. Since $D_{\leqq}(\mathbf{C}(\mathbf{u}^i))$ is acute for $i = 1, 2$, the assumptions of Theorems 4.2.29(i) (for each $\mathbf{u} \in U$) and 6.3.13(i) hold. As illustrated in Figure 6.5, the only points at which Theorem 4.2.29(i) holds for each $\mathbf{u} \in U$ are the two highly robust efficient points $(2, 4)$ and $(6, 0)$. However, as $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}^2)) = \emptyset$ (clearly shown in Figure 6.5b), the sufficient condition of Theorem 6.3.13(i) does not hold (trivially) at either highly robust efficient point, so we are unable to identify either point via this theorem.

Similarly, using the union of strict polars rather than the intersection, we obtain a necessary condition for highly robust (weak) efficiency.

Theorem 6.3.16. *Let* $\mathbf{x}^* \in X$.

(i) Assume $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ *is acute for all* $\mathbf{u} \in U$. *If* $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$, *then* $N_P(\mathbf{x}^*) \cap \bigcup_{\mathbf{u} \in U} D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$.

(ii) Assume $\mathrm{cl}(D_{<}(\mathbf{C}(\mathbf{u}))) = D_{\leqq}(\mathbf{C}(\mathbf{u}))$ *for all* $\mathbf{u} \in U$. *If* $\mathbf{x}^* \in \mathrm{wE}(P, \mathbf{C}(\mathbf{u}), U)$, *then* $N_P(\mathbf{x}^*) \cap \bigcup_{\mathbf{u} \in U} D^{s+}_{<}(\mathbf{C}(\mathbf{u})) \neq \emptyset$.



(a) $D_{\leqq}(\mathbf{C}(\mathbf{u}^1))$ (purple) and $D_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ (teal)

(b) $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}^1))$ (purple) and $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ (teal)

(c) The highly robust efficient set (red) and normal cones (green)

Figure 6.5: Normal cones to P_1, and the closed cones of improving directions and their strict polars for Example 6.3.15

Proof. (i) Let $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$. Equivalently, $N_P(\mathbf{x}^*) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ for all $\mathbf{u} \in U$ by Theorem 4.2.29(i). Since

$$N_P(\mathbf{x}^*) \cap \bigcup_{\mathbf{u}\in U} D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u})) = \bigcup_{\mathbf{u}\in U} \left[N_P(\mathbf{x}^*) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}))\right]$$

by the Distributive Law of Intersections 2.4.3(i), the result follows.

(ii) The proof follows similarly to the proof of part (i). □

It is important to note that since Theorem 4.2.29, which is both necessary and sufficient, is split into two separate theorems, Theorems 6.3.13 and 6.3.16, one that is sufficient and the other that is necessary, respectively, we lose the strength of the original theorem. This is supported by Example 6.3.15 in which applying Theorem 4.2.29(i) for each scenario yields the entire highly robust efficient set, while applying Theorem 6.3.13(i) does not yield any highly robust efficient solutions and the entire boundary satisfies the consequent of Theorem 6.3.16(i) even though the entire boundary is not highly robust efficient.

The final characterization in the decision space we investigate involves the recession cone (refer to Definition 3.1.5) and is an extension of a result mentioned on p. 698, Bitran [18].

Proposition 6.3.17. *If* $D_{\leq}(\mathbf{C}(\mathbf{u})) \cap R_P \neq \emptyset$ *for some* $\mathbf{u} \in U$*, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

Proof 1. Suppose $D_{\leq}(\mathbf{C}(\mathbf{u})) \cap R_P \neq \emptyset$ for some $\mathbf{u} \in U$, which implies that $(D_{\leq}(\mathbf{C}(\mathbf{u})) \cap R_P) \oplus \{\mathbf{x}\} \neq \emptyset$ for all $\mathbf{x} \in P$. Hence,

$$(D_{\leq}(\mathbf{C}(\mathbf{u})) \oplus \{\mathbf{x}\}) \cap (R_P \oplus \{\mathbf{x}\}) \neq \emptyset \tag{6.20}$$

by Theorem 2.4.3(iv). Additionally, by definition, it is clear that

$$R_P \oplus \{\mathbf{x}\} \subseteq P \tag{6.21}$$

for all $\mathbf{x} \in P$. Together, (6.20) and (6.21) yield $(D_{\leq}(\mathbf{C}(\mathbf{u})) \oplus \{\mathbf{x}\}) \cap P \neq \emptyset$. Since this is true for all $\mathbf{x} \in P$, it must be that $\mathrm{E}(P, \mathbf{C}(\mathbf{u})) = \emptyset$ by Proposition 4.2.10(i). Since $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \bigcap_{\mathbf{u} \in U} \mathrm{E}(P, \mathbf{C}(\mathbf{u}))$, we conclude that $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$ also. □

Proof 2. Suppose $D_{\leq}(\mathbf{C}(\mathbf{u})) \cap R_P \neq \emptyset$ for some $\mathbf{u} \in U$. As the corresponding instance MOLP($\mathbf{u}$) is deterministic given $\mathbf{u} \in U$, Proposition 4.2.16 yields $\mathrm{E}(P, \mathbf{C}(\mathbf{u})) = \emptyset$. Since $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \bigcap_{\mathbf{u} \in U} \mathrm{E}(P, \mathbf{C}(\mathbf{u}))$, the result follows. □

As in the deterministic setting, the above proposition relies on the intuition that if a recession direction along which feasibility is retained is also an improving direction, then no highly robust efficient solutions exist since there is always a "better" solution. Since recession directions necessarily do not exist when P is bounded, i.e., $R_P = \emptyset$ so that $D_{\leqq}(\mathbf{C}(\mathbf{u})) \cap R_P = \emptyset$ for all $\mathbf{u} \in U$, Proposition 6.3.17 is only relevant in the case that P is unbounded. Accordingly, further note that this proposition indicates that the highly robust efficient set is empty because the efficient set associated with at least one instance MOLP($\mathbf{u}$) is empty, which is only possible when P is unbounded. However, as should be clear, the highly robust efficient set may be empty even when P is bounded (refer to Remark 6.1.3 and UMOLP (6.5)). A more general method to identify whether or not the highly robust efficient set is empty is addressed in Section 6.4.4.

6.3.3 Bound Sets

In robust optimization, an RC, which is a deterministic (scalar or vector) optimization problem associated with the original uncertain optimization problem whose solutions are the desired robust solutions, is commonly used. The solution set of an RC may be interpreted as both an upper and lower bound set on the set of robust solutions to the original uncertain problem. Working toward an RC to obtain highly robust efficient solutions to MOLP(U), in this section, we develop several bound sets on the highly robust efficient set, and then present an RC for a special class of UMOLPs in Section 6.3.4.

First, we know that, in general, the efficient set of any instance MOLP($\mathbf{u}$) is an upper bound set on the highly robust efficient set of MOLP(U).

Proposition 6.3.18. *The containment* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \subseteq \mathrm{E}(P, \mathbf{C}(\mathbf{u}))$ *holds for every* $\mathbf{u} \in U$.

Proof. Immediate since $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \bigcap_{\mathbf{u}\in U} \mathrm{E}(P, \mathbf{C}(\mathbf{u}))$. □

Another upper bound set on the highly robust efficient set is given by the efficient set of the so-called *all-in-one problem* (refer to Section 5.2 and Proposition

5.2.5). The all-in-one MOLP, denoted AIOMOLP(U), is given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{C}(U)\mathbf{x} := \begin{bmatrix}\mathbf{C}(\mathbf{u}^1) & \cdots & \mathbf{C}(\mathbf{u}^s)\end{bmatrix}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P, \end{aligned} \tag{6.22}$$

where $\mathbf{C}(U) \in \mathbb{R}^{ps \times n}$ is a deterministic cost matrix given U. Immediately, since AIOMOLP(U) is a deterministic MOLP whose efficient solutions are determined by ps criteria, we know that highly robust efficient solutions to MOLP(U) are at least weakly efficient solutions to AIOMOLP(U) based on Proposition 1, Engau and Wiecek [50]. Even more, as shown in Proposition 5.2.5, the highly robust efficient set is contained in the efficient set of AIOMOLP(U), which is denoted $\mathrm{E}(P, \mathbf{C}(U))$.

Proposition 6.3.19. *The containment* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \subseteq \mathrm{E}(P, \mathbf{C}(U))$ *holds.*

Proof. The proof follows the same as the proof of Proposition 5.2.5. □

In general, however, the opposite containment does not hold as demonstrated in the proceeding example.

Example 6.3.20. Consider the UMOLP given by

$$\left\{ \begin{aligned} \min_{\mathbf{x}} \quad & \begin{bmatrix} 2u_{11} & -3u_{12} \\ 5u_{21} & u_{22} \end{bmatrix} \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P_1 \end{aligned} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2}, \tag{6.23}$$

where $U_1 = \{(-1, 2)\}$ and $U_2 = \{(-1, 2), (2, 3)\}$, yielding two scenarios $\mathbf{u}^1 = (-1, 2, -1, 2)$ and $\mathbf{u}^2 = (-1, 2, 2, 3)$. The associated all-in-one problem is given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \begin{bmatrix} -2 & -6 \\ -5 & 2 \\ -2 & -6 \\ 10 & 3 \end{bmatrix} \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P_1 \end{aligned} \tag{6.24}$$

Solving each instance separately, we obtain that $\mathrm{E}(P_1, \mathbf{C}(\mathbf{u}^1))$ is the line segment joining the extreme points $(2,4)$ and $(6,0)$, while $\mathrm{E}(P_1, \mathbf{C}(\mathbf{u}^2))$ is the portion of the boundary joining the extreme points $(0,0), (0,3)$, and $(2,4)$ (cf. Figure 4.5). Hence, $\mathrm{E}(P_1, \mathbf{C}(\mathbf{u}), U) = \{(2,4)\}$ by Proposition 6.1.2. On the other hand, $\mathrm{E}(P_1, \mathbf{C}(U)) = P_1$ since the cone of improving directions of AIOMOP (6.24) is empty (cf. Remark 4.2.12). As a result, it is clear that the opposite containment $\mathrm{E}(P, \mathbf{C}(U)) \nsubseteq \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ in Proposition 6.3.19 does not hold.

Third, for two related special classes of UMOLPs, we may obtain additional upper bound sets with the use of Lemma 4.2.14. The following proposition is an extension of Proposition 3.1, Bitran [18].

Proposition 6.3.21. *Suppose each column of* $\mathbf{C}(\mathbf{u})$ *is nonnegative for all* $\mathbf{u} \in U$ *with no column all* 0. *For the (deterministic) MOLP given by*

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{I}_n \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P, \end{aligned} \tag{6.25}$$

the containment $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \subseteq \mathrm{E}(P, \mathbf{I}_n)$ *holds.*

Proof. The cones of improving directions associated with MOLP (6.25) and an instance MOLP($\mathbf{u}$) are given by

$$\begin{aligned} D_{\leq}(\mathbf{I}_n) &= \{\mathbf{d} \in \mathbb{R}^n : \mathbf{I}_n \mathbf{d} \leq \mathbf{0}\} \\ &= \{\mathbf{d} \in \mathbb{R}^n : d_1 \leq 0, d_2 \leq 0, \ldots, d_n \leq 0, \text{ at least one strict}\}, \end{aligned}$$

and

$$\begin{aligned} D_{\leq}(\mathbf{C}(\mathbf{u})) &= \{\mathbf{d} \in \mathbb{R}^n : \mathbf{C}(\mathbf{u})\mathbf{d} \leq \mathbf{0}\} \\ &= \{\mathbf{d} \in \mathbb{R}^n : c_{11}u_{11}d_1 + \cdots + c_{1n}u_{1n}d_n \leq 0, \\ &\qquad\qquad\qquad\qquad \vdots \\ &\qquad\qquad c_{p1}u_{p1}d_1 + \cdots + c_{pn}u_{pn}d_n \leq 0, \text{ at least one strict}\}, \end{aligned}$$

respectively, where $c_{ki}u_{ki} \geq 0$ for all $k = 1, \ldots, p, i = 1, \ldots, n$, by assumption. If $\mathbf{d} \in D_{\leq}(\mathbf{I}_n)$, then $d_i \leq 0, i = 1, \ldots, n$, with at least one strict. Since $c_{ki}u_{ki} \geq 0$ for all $i = 1, \ldots, n$, clearly $\mathbf{d} \in D_{\leq}(\mathbf{C}(\mathbf{u}))$ also (which is not true, however, without the assumption that no column is entirely 0). Hence, $D_{\leq}(\mathbf{I}_n) \subseteq D_{\leq}(\mathbf{C}(\mathbf{u}))$ for all $\mathbf{u} \in U$, which implies that $\mathrm{E}(P, \mathbf{C}(\mathbf{u})) \subseteq \mathrm{E}(P, \mathbf{I}_n)$ for all $\mathbf{u} \in U$ by Lemma 4.2.14. Thus, $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \cap_{\mathbf{u} \in U} \mathrm{E}(P, \mathbf{C}(\mathbf{u})) \subseteq \mathrm{E}(P, \mathbf{I}_n)$ as desired. □

Two observations regarding the previous proposition are worth considering. First, the assumption that no column is all zero is needed. For instance, if a column of $\mathbf{C}(\mathbf{u})$ is all 0, say the first column, then the direction $\mathbf{d} = \begin{bmatrix} -1 & 0 & \cdots & 0 \end{bmatrix}^T \in D_{\leq}(\mathbf{I}_n)$ is not also an element of $D_{\leq}(\mathbf{C}(\mathbf{u}))$ since $\mathbf{C}(\mathbf{u})\mathbf{d} = \mathbf{0} \not\leq \mathbf{0}$, i.e., *none* of the inequalities are strict.

Second, it is important to note that the opposite containment does not hold. For example, consider the case when $p = n = 2, U_1 = U_2 = \{(1, 1), (2, 4)\}$, and the cost matrix under uncertainty is given by

$$\mathbf{C}(\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & 2u_{22} \end{bmatrix}$$

for $\mathbf{u} \in U$. Here,

$$D_{\leq}(\mathbf{I}_2) = \{\mathbf{d} \in \mathbb{R}^2 : \mathbf{I}_2\mathbf{d} \leq \mathbf{0}\},$$

and

$$D_{\leq}(\mathbf{C}((1,1,1,1))) = \left\{\mathbf{d} \in \mathbb{R}^2 : \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{d} \leq \mathbf{0} \right\}.$$

The direction $\mathbf{d} = \begin{bmatrix} 1 & -2 \end{bmatrix}^T \in D_{\leq}(\mathbf{C}((1,1,1,1)))$ since $\mathbf{C}((1,1,1,1))\mathbf{d} = \begin{bmatrix} -1 & -3 \end{bmatrix}^T \leq \mathbf{0}^T$, but $\mathbf{d} \notin D_{\leq}(\mathbf{I}_2)$ since $d_1 = 1 \nleq 0$. Hence, in general, we conclude that $D_{\leq}(\mathbf{I}_n) \nsubseteq D_{\leq}(\mathbf{C}(\mathbf{u}))$ for $\mathbf{u} \in U$.

A more general upper bound set may be given by accounting for both nonnegative and nonpositive columns, as in the following theorem.

Theorem 6.3.22. *Suppose each column of* $\mathbf{C}(\mathbf{u})$ *is either nonnegative for all* $\mathbf{u} \in U$ *or nonpositive for all* $\mathbf{u} \in U$ *with no column all 0. Let* $\mathbf{I}$ *be the diagonal matrix with a 1 corresponding to the nonnegative columns of* $\mathbf{C}(\mathbf{u})$ *and a* -1 *for the nonpositive columns. For the (deterministic) MOLP given by*

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{I}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P, \end{aligned} \tag{6.26}$$

the containment $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \subseteq \mathrm{E}(P, \mathbf{I})$ *holds.*

Proof. Let I and J be the subsets of the index set $\{1, \ldots, n\}$ for which the columns of $\mathbf{C}(\mathbf{u})$ are nonnegative for all $\mathbf{u} \in U$ and nonpositive for all $\mathbf{u} \in U$, respectively. The cones of improving directions associated with MOLP (6.26) and an instance MOLP($\mathbf{u}$) are given by

$$\begin{aligned} D_{\leq}(\mathbf{I}) &= \{\mathbf{d} \in \mathbb{R}^n : \mathbf{I}\mathbf{d} \leq \mathbf{0}\} \\ &= \{\mathbf{d} \in \mathbb{R}^n : d_i \leq 0, i \in I, d_j \geq 0, j \in J, \text{ at least one strict}\}, \end{aligned}$$

and

$$\begin{aligned} D_{\leq}(\mathbf{C}(\mathbf{u})) &= \{\mathbf{d} \in \mathbb{R}^n : \mathbf{C}(\mathbf{u})\mathbf{d} \leq \mathbf{0}\} \\ &= \{\mathbf{d} \in \mathbb{R}^n : c_{11}u_{11}d_1 + \cdots + c_{1n}u_{1n}d_n \leq 0, \\ &\qquad\qquad \vdots \\ &\qquad c_{p1}u_{p1}d_1 + \cdots + c_{pn}u_{pn}d_n \leq 0, \text{ at least one strict}\}, \end{aligned}$$

respectively, where $c_{ki}u_{ki} \geq 0$ for all $k = 1, \ldots, p, i \in I$, and $c_{kj}u_{kj} \leq 0$ for all $k = 1, \ldots, p, j \in J$, by assumption. If $\mathbf{d} \in D_{\leq}(\mathbf{I})$, then $d_i \leq 0, i \in I$, and $d_j \geq 0, j \in J$, with at least one strict. Since $c_{ki}u_{ki} \geq 0$ for all $i \in I$ and $c_{kj}u_{kj} \leq 0$ for all $j \in J$, clearly $\mathbf{d} \in D_{\leq}(\mathbf{C}(\mathbf{u}))$ also. Hence, $D_{\leq}(\mathbf{I}) \subseteq D_{\leq}(\mathbf{C}(\mathbf{u}))$ for all $\mathbf{u} \in U$, which implies that $\mathrm{E}(P, \mathbf{C}(\mathbf{u})) \subseteq \mathrm{E}(P, \mathbf{I})$ for all $\mathbf{u} \in U$ by Lemma 4.2.14. Thus, $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \bigcap_{\mathbf{u} \in U} \mathrm{E}(P, \mathbf{C}(\mathbf{u})) \subseteq \mathrm{E}(P, \mathbf{I})$ as desired. $\square$

As with the previous proposition, we note that this theorem is not true without the assumption that no column is entirely 0, and the opposite containment does not necessarily hold. In addition, we recognize that the assumptions regarding the columns of $\mathbf{C}(\mathbf{u})$ in Proposition 6.3.21 and Theorem 6.3.22, although conspicuous, are realistic in practice. For example, problems in bank balance sheet management, portfolio management, and knapsack packing generally satisfy these assumptions.

Fourth, for MOLP(U) in general, we may obtain another bound set (either upper or lower) with a proposition similar to Lemma 4.2.14. As the proposition involves two different uncertainty sets, it can also be used to provide additional information to decision makers by presenting the effects of adding or removing scenarios from a given uncertainty set.

Proposition 6.3.23. *Let the following UMOLPs:*

$$\left\{\begin{array}{ll} \min\limits_{\mathbf{x}} & \mathbf{C}(\mathbf{u})\mathbf{x} \\ \text{s.t.} & \mathbf{x} \in P \end{array}\right\}_{\mathbf{u}\in U'} \quad \textit{and} \quad \left\{\begin{array}{ll} \min\limits_{\mathbf{x}} & \mathbf{C}(\mathbf{u})\mathbf{x} \\ \text{s.t.} & \mathbf{x} \in P \end{array}\right\}_{\mathbf{u}\in U''}.$$

be given. If $D_{\leq}(\mathbf{C}(\mathbf{u}), U') \subseteq D_{\leq}(\mathbf{C}(\mathbf{u}), U'')$*, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U'') \subseteq \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U')$*.*

Proof. Suppose $D_{\leq}(\mathbf{C}(\mathbf{u}), U') \subseteq D_{\leq}(\mathbf{C}(\mathbf{u}), U'')$, and assume for the sake of contradiction that $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U'') \nsubseteq \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U')$, i.e., there exists an $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U'')$ such that $\mathbf{x}^* \notin \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U')$. The former implies that $D_{\leq}(\mathbf{C}(\mathbf{u}), U') \oplus \{\mathbf{x}\} \subseteq D_{\leq}(\mathbf{C}(\mathbf{u}), U'') \oplus \{\mathbf{x}\}$ for all $\mathbf{x} \in P$, while the latter yields $[D_{\leq}(\mathbf{C}(\mathbf{u}), U'') \oplus \{\mathbf{x}^*\}] \cap P = \emptyset$, but $[D_{\leq}(\mathbf{C}(\mathbf{u}), U') \oplus \{\mathbf{x}^*\} \cap P \neq \emptyset$ by Theorem 6.3.6(i). Hence,

$$\emptyset \neq [D_{\leq}(\mathbf{C}(\mathbf{u}), U') \oplus \{\mathbf{x}^*\}] \cap P \subseteq [D_{\leq}(\mathbf{C}(\mathbf{u}), U'') \oplus \{\mathbf{x}^*\}] \cap P = \emptyset,$$

which is a contradiction. Thus, it must be that $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U'') \subseteq \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U')$ as desired. □

The intuition, similar to that of Lemma 4.2.14, is that fewer improving directions leads to a larger highly robust efficient set. However, unlike Proposition 6.3.21 and Theorem 6.3.22, no special assumption about the structure of the cost matrix is necessary.

Finally, in order to obtain a lower bound set on the highly robust efficient set, we utilize the sufficient condition of Theorem 6.3.13.

Theorem 6.3.24. *Assume* $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ *is acute. Then* $\mathrm{E}(P, \widetilde{\mathbf{C}}) \subseteq \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ *for some suitable matrix* $\widetilde{\mathbf{C}}^T \in \mathbb{R}^{n \times \tilde{p}}$*.*

Proof. We have that $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{C}}^T \boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$ for some suitable matrix $\widetilde{\mathbf{C}}^T \in \mathbb{R}^{n \times \tilde{p}}$ by Proposition 3.2.20(i). Hence, we may write $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) =$

$D^{s+}_{\leqq}(\widetilde{\mathbf{C}})$, where $D_{\leqq}(\widetilde{\mathbf{C}})$ is an acute cone as in Theorem 3.2.13(i) and is the cone of improving directions of the deterministic MOLP given by $\min_{\mathbf{x}\in P} \widetilde{\mathbf{C}}\mathbf{x}$. Equivalently, for $\mathbf{x}^* \in \mathrm{E}(P, \widetilde{\mathbf{C}})$, we have that

$$N_P(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$$

by Theorem 4.2.29(i). Consequently, since $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ being acute implies that $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute for all $\mathbf{u} \in U$, we have that $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ also by Theorem 6.3.13(i). Therefore, $\mathrm{E}(P, \widetilde{\mathbf{C}}) \subseteq \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ as desired. □

The suitable matrix $\widetilde{\mathbf{C}}^T \in \mathbb{R}^{n\times\tilde{p}}$ mentioned in the statement of Theorem 6.3.24, as revealed in the above proof, is guaranteed to exist by Proposition 3.2.20(i). Moreover, as detailed previously in Remark 3.2.10 and shown in Example 3.2.11, the matrix may be computed using readily available software such as SageMath's `polyhedron base class`.

Regardless, since the above theorem provides a lower bound set on the highly robust efficient set, it follows that $\mathrm{E}(P, \widetilde{\mathbf{C}})$ may be used to provide conditions under which the highly robust efficient set is nonempty.

Corollary 6.3.25. *Let $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ be acute and P be bounded. Then the highly robust efficient set is nonempty.*

Proof. Since P is bounded, the efficient set of any deterministic MOLP (with P as its feasible set) is nonempty by Theorem 2.19, Ehrgott [44]. Using Theorem 6.3.24, we obtain $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \neq \emptyset$ as desired. □

Even though the above corollary utilizes Theorem 6.3.24, note that $\widetilde{\mathbf{C}}$ does not need to be constructed. Instead, only the acuteness of $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ and the boundedness of P need to be verified. The former is addressed in Section 6.3.5, while the latter may be accomplished using software such as SageMath's [132] `polyhedron base class`.

6.3.4 Robust Counterpart

Having discussed various bound sets on the highly robust efficient set, we now address a theoretical RC of MOLP(U), as well as an RC for a special class of UMOLPs that may be used to obtain highly robust efficient solutions to MOLP(U). First, as in Section 5.2, the theoretical RC of MOLP(U) is the *conjunctive multiobjective linear program* (CMOLP) given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \bigwedge_{i=1}^{s} \mathbf{C}(\mathbf{u}^i)\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P. \end{aligned} \tag{6.27}$$

As is the case with MOP(U) and its theoretical RC, the solutions to CMOLP (6.27), referred to as conjunctive (weakly) efficient (see Definition 5.2.2), are in fact highly robust (weakly) efficient solutions to MOLP(U).

Corollary 6.3.26. *(i) A feasible solution* $\mathbf{x}^* \in P$ *to* MOLP(U) *is highly robust (weakly) efficient if and only if it is a conjunctive (weakly) efficient solution to* RC (6.27)*.*

(ii) The highly robust (weakly) efficient set of MOLP(U) *and the conjunctive (weakly) efficient set of* RC (6.27) *are equal.*

Proof. The proofs follow the same as the proofs of Theorem 5.2.3 and Corollary 5.2.4, respectively. □

In the form of a CMOLP, we have already discussed that the RC has limited use. However, it does reiterate that we must consider all instances of the objectives over a common feasible set yielding one solution.

Second, with regard to obtaining a classical RC, e.g., an MOLP whose efficient set is equal to the highly robust efficient set rather than a bound set as in the results of Section 6.3.3, we consider a special class of UMOLPs. The advantage of having an RC that is a (deterministic) MOLP is that MOLPs are well-known problems with numerous solution methods (refer to Wiecek et al. [141]) that may be exploited.

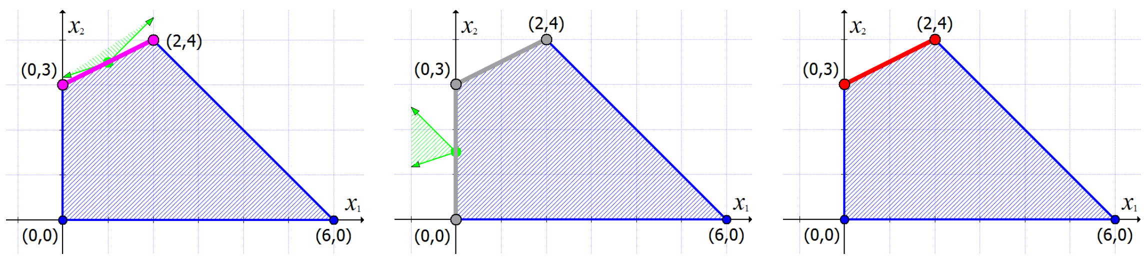
Theorem 6.3.27. *Assume* $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ *is a polyhedral convex cone. Then* $\mathrm{E}(P, \overline{\mathbf{C}}) = \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ *for some suitable matrix* $\overline{\mathbf{C}} \in \mathbb{R}^{\bar{p}\times n}$*.*

Proof. By assumption, we may write $D_{\leqq}(\mathbf{C}(\mathbf{u}), U) = \{\mathbf{d} \in \mathbb{R}^n : \overline{\mathbf{C}}\mathbf{d} \leqq \mathbf{0}\}$ for some suitable matrix $\overline{\mathbf{C}} \in \mathbb{R}^{\bar{p}\times n}$. Here, the suitability of $\overline{\mathbf{C}}$ means that the rows of $\overline{\mathbf{C}}$ are the normals to the generating hyperplanes whose half-spaces form $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$. Hence, $D_{\leq}(\mathbf{C}(\mathbf{u}), U) = \{\mathbf{d} \in \mathbb{R}^n : \overline{\mathbf{C}}\mathbf{d} \leq \mathbf{0}\} = D_{\leq}(\overline{\mathbf{C}})$, which is the cone of improving directions of the deterministic MOLP given by $\min_{\mathbf{x}\in P} \overline{\mathbf{C}}\mathbf{x}$. Since $D_{\leq}(\mathbf{C}(\mathbf{u}), U)$ is the cone of improving directions of both MOLP(U) and $\min_{\mathbf{x}\in P} \overline{\mathbf{C}}\mathbf{x}$, we obtain $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \mathrm{E}(P, \overline{\mathbf{C}})$ by Proposition 4.2.10(i) and Theorem 6.3.6(i). □

The deterministic MOLP implied by Theorem 6.3.27, which is given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \overline{\mathbf{C}}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P, \end{aligned} \tag{6.28}$$

is an RC of MOLP(U) since a solution to MOLP(U) is highly robust efficient if and only if it is an efficient solution to MOLP (6.28). As a direct consequence of this, MOLP (6.28) and Theorem 6.3.27 may be used to show conditions under which the highly robust efficient set is nonempty.



(a) $\mathrm{E}(P_1, \mathbf{C}(\mathbf{u}^1))$ (purple) and $D_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \oplus \{\mathbf{x}^*\}$ (green)

(b) $\mathrm{E}(P_1, \mathbf{C}(\mathbf{u}^2))$ (grey) and $D_{\leqq}(\mathbf{C}(\mathbf{u}^2)) \oplus \{\mathbf{x}^*\}$ (green)

(c) The highly robust efficient set of UMOLP (6.29) (red)

Figure 6.6: Efficient and highly robust efficient points for Example 6.3.29

Corollary 6.3.28. *Let* $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ *be a polyhedral convex cone, and let* P *be bounded. Then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ *is nonempty and connected.*

Proof. Since P is bounded, the efficient set of any deterministic MOLP (with P as its feasible set) is nonempty and connected by Theorem 2.19, Ehrgott [44], and Proposition 4.2.5(vi), respectively. Using Theorem 6.3.27, we obtain $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ is nonempty and connected as desired. □

As an illustration of both Theorem 6.3.27 and Corollary 6.3.28, including computing the associated RC, we present the following example.

Example 6.3.29. Consider the following UMOLP given by

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \begin{bmatrix} u_{11} & -3u_{12} \\ u_{21} & u_{22} \end{bmatrix} \mathbf{x} \\ \text{s.t.} & \qquad\qquad \mathbf{x} \in P_1 \end{array} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2,} \tag{6.29}$$

where $U_1 = \{(1,1)\}$ and $U_2 = \{(1,-1),(1,1)\}$. For scenarios $\mathbf{u}^1 = (1,1,1,-1)$ and $\mathbf{u}^2 = (1,1,1,1)$, it is clear that $D_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \cup D_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ is a polyhedral convex cone (as the union is simply $D_{\leqq}(\mathbf{C}(\mathbf{u}^1))$), which is shown in Figure 6.6. Hence, we have that the cost matrix of the RC is

$$\overline{\mathbf{C}} = \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix},$$

where the rows are the normals to the generating hyperplanes whose half-spaces form $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ as previously mentioned. Moreover, since P_1 is bounded, Corollary 6.3.28 guarantees that the highly robust efficient set is nonempty and connected, which is confirmed in Figure 6.6c.

While Theorem 6.3.27 and Corollary 6.3.28 address the special case that $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is polyhedral convex, in general, this cone is nonconvex since it is a union (rather than an intersection). Hence, we may not always be able to formulate an RC that is a deterministic MOLP as in Theorem 6.3.27. In particular, when the highly robust efficient set is disconnected, any RC would have at least one nonconvex objective (cf. Theorem 3.40, Ehrgott [44]). Despite these facts, as shown in Theorem 6.3.27, there exists a class of UMOLPs, those that have $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ being polyhedral convex, whose RC is a deterministic MOLP. Since MOLPs are readily solvable and their solution sets have desirable properties such as connectedness, it is of interest to identify UMOLPs that have this characteristic. Consequently, recognizing the polyhedrality of $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ and computing its representation in order to obtain the cost matrix $\overline{\mathbf{C}}$ of RC (6.28) become important tasks. An algorithm to accomplish these two tasks is available in, e.g., Bemporad et al. [4].

6.3.5 Acuteness Recognition and Discussion

Since the assumption of acuteness is key to several of the results we have already presented in Sections 6.3.2 and 6.3.3, it is important to examine this property in more detail. A similar discussion and set of results is given in Section 3.2.4, as well as following Theorem 4.2.29, but are reiterated here for completeness.

We first discuss the algebraic implication of the assumption that $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute for at least one $\mathbf{u} \in U$. (Note that this analysis also encompasses the situation that $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is acute since $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute for each $\mathbf{u} \in U$ if $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is acute.) Since $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is closed, being acute is equivalent to $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ being pointed by Proposition 3.1.3. Hence, by assuming that $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute, we implicitly assume that $\text{rank}(\mathbf{C}(\mathbf{u})) = n$ by Theorem 3.1.12. Moreover, since $\text{rank}(\mathbf{C}(\mathbf{u})) \leq \min\{p, n\}$, we obtain that the number of criteria p is greater than or equal to the number of decision variables n. The consequence of this is that models that incorporate the numerous preferences of multiple decision makers explicitly through many criteria may be used.

We next investigate the recognition of the acuteness of a cone. Given the cone $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ for some $\mathbf{u} \in U$, we know that it may be expressed in both inequality form $\{\mathbf{d} \in \mathbb{R}^n : \mathbf{C}(\mathbf{u})\mathbf{d} \leqq \mathbf{0}\}$ (which is the form immediately available) and generator form $\{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} = \mathbf{G}(\mathbf{u})^T \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$, where $\mathbf{G}(\mathbf{u})^T$ is an $n \times \phi$ matrix whose columns are a finite set of generators of $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ and are nonzero unless $D_{\leqq}(\mathbf{C}(\mathbf{u})) = \{\mathbf{0}\}$ (see pp. 54–55). If $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is given in inequality form, then its polar is explicitly given in generator form as in Proposition 3.2.7(i). Similarly, if $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is given in generator form, then its polar is given in inequality form as in Proposition 3.1.24(ii). Namely,

$$\{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} = \mathbf{G}(\mathbf{u})^T \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}^+ = \{\mathbf{d} \in \mathbb{R}^n : -\mathbf{G}(\mathbf{u})\mathbf{d} \leqq \mathbf{0}\}. \tag{6.30}$$

With this in mind, we have the following method for recognizing the acuteness of (nontrivial) $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ for some $\mathbf{u} \in U$.

Theorem 6.3.30. *Let* $\mathbf{u} \in U$ *be given, and let* $D_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \{\mathbf{0}\}$ *be given in generator* *form. Then* $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ *is acute if and only if the system* $-\mathbf{G}(\mathbf{u})\mathbf{d} < \mathbf{0}$ *is consistent.*

Proof. Since $D_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$, we know that $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute if and only if $\operatorname{int}(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) \neq \emptyset$ by Theorem 3.1.21(i). As $\mathbf{G}(\mathbf{u})$ has no rows that are all zero, $\operatorname{int}(\{\mathbf{d} \in \mathbb{R}^n : -\mathbf{G}(\mathbf{u})\mathbf{d} \leqq \mathbf{0}\}) = \{\mathbf{d} \in \mathbb{R}^n : -\mathbf{G}(\mathbf{u})\mathbf{d} < \mathbf{0}\}$ so that the result follows from (6.30). □

More generally, we have a second recognition method given by the following theorem.

Theorem 6.3.31. *Let* $\mathbf{u} \in U$ *be given. If* $\dim(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) = n$*, then* $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ *is acute.*

Proof. Let $\dim(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) = n$. Hence,

$$\operatorname{int}(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) = \operatorname{rel\,int}(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) \tag{6.31}$$

by Proposition 2.4.22(i). Moreover, since $D^+_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ (refer to the discussion following Proposition 3.2.7) and convex (by Proposition 3.1.19(ii)), we obtain that

$$\operatorname{rel\,int}(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) \neq \emptyset \tag{6.32}$$

by Theorem 2.4.23. Thus, (6.31) and (6.32) yield that $D_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ is acute by Theorem 3.1.21(i). □

Observe that Theorem 6.3.31 does not depend on the form, inequality or generator, of $D_{\leqq}(\mathbf{C}(\mathbf{u}))$, but instead relies on $\dim(D^+_{\leqq}(\mathbf{C}(\mathbf{u})))$. Even though we do not have a system to solve as in Theorem 6.3.30, we do have a condition to verify, namely that $\dim(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) = n$. In particular, if $D^+_{\leqq}(\mathbf{C}(\mathbf{u}))$ is in generator form (as it is when $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is in inequality form), then $\dim(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) = \text{rank}(\mathbf{C}(\mathbf{u}))$ by Proposition 3.1.16. Otherwise, software such as SageMath's `polyhedron base class` can readily provide the dimension. We also note that Theorem 6.3.31 is applicable to any nonempty cone, which is relevant if the acuteness of $D_{\leq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ is needed as is the case in Remark 6.3.14 for example, while Theorem 6.3.30 is not. Using Theorems 6.3.30 and 6.3.31, we may similarly verify the acuteness of (nontrivial) $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$.

Corollary 6.3.32. *Let $D_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \{\mathbf{0}\}$ be given in generator form for each $\mathbf{u} \in U$. Then $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is acute if and only if the system $-\mathbf{G}(\mathbf{u}^i)\mathbf{d} < \mathbf{0}, i = 1, \ldots, s$, is consistent.*

Proof. Follows from Theorem 6.3.30, Proposition 3.2.8(i), where $\mathbf{M}_\ell$ is replaced by $\mathbf{C}(\mathbf{u}^i)$, and Proposition 2.4.11. □

Likewise, we have the following extension of Theorem 6.3.31.

Proposition 6.3.33. *If $\dim(D^+_{\leqq}(\mathbf{C}(\mathbf{u}), U)) = n$, then $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is acute.*

Proof. Follows similarly to the proof of Theorem 6.3.31. □

With respect to, e.g., Theorems 6.3.13, 6.3.16, and 6.3.24, we now have systematic approaches to verify the acuteness required to apply each result. That being said, it is important to note that when the proposed methods are used to verify the acuteness of $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ in Theorem 6.3.24, for example, they do not necessarily compute the cost matrix $\widetilde{\mathbf{C}}$.

6.4 Computing Highly Robust Efficient Solutions

In this section, we address the computation of highly robust efficient solutions to MOLP(U). First, in Section 6.4.1, Ecker and Kouada's problem/method (refer to Section 4.2.3.2) is extended from the deterministic setting to the uncertain one, providing methods to determine whether or not a given feasible solution of interest is highly robust efficient, possibly generate a new highly robust efficient point if the given feasible decision is not itself highly robust efficient, and possibly identify whether the highly robust efficient set is empty. Similarly, in Section 6.4.2, Benson's problem/method (see Section 4.2.3.3) is extended from the deterministic to uncertain context. Although Benson's method identifies an efficient extreme point in the deterministic case (cf. Corollary 4.2.27), the derived extensions do not provide a highly robust efficient extreme point but instead give several avenues to identify that the highly robust efficient set is empty. Again similarly to the aforementioned results, in Section 6.4.3, a naive extension of the weighted-sum method and Isermann's Theorem (refer to Section 4.2.3.4) is given. Finally, a novel two-step bilevel procedure is derived in Section 6.4.4, and an application problem from bank balance-sheet management is solved in Section 6.4.5.

6.4.1 Extension of Ecker and Kouada's Method

In the deterministic setting, it is well-known that the auxiliary single-objective linear program (LP) referred to as Ecker and Kouada's problem and its associated dual may be used to give the decision maker the opportunity to verify whether or not a given solution $\mathbf{x}_0 \in P$ to an MOLP is efficient, generate a solution that is, or determine that no efficient solutions exist. Ecker and Kouada's problem/method may be extended to (at least) four different Ecker-and-Kouada-type auxiliary problems in the uncertain setting. One is a family of problems, while the other three are single/individual problems. Regardless of the auxiliary problem, results on the recognition, generation, and/or existence of highly robust efficient solutions are obtained through the primal and dual formulations. Note that since MOLP(U) is a special case of the more general problem MOP(U), many of the results here follow directly from the results of Section 5.4. For the sake of completeness, the results are explicitly given here as well.

We first examine the family of Ecker-and-Kouada-type auxiliary LPs. For a given feasible solution $\mathbf{x}_0 \in P$ and an arbitrary $\mathbf{u} \in U$, the following problem, denoted EKLP($\mathbf{x}_0, \mathbf{u}$), is a representative member of the family of auxiliary problems and is given by

$$\begin{aligned} \max_{\mathbf{x},\boldsymbol{l}} \quad & \sum_{k=1}^{p} l_k \\ \text{s.t.} \quad & \mathbf{C}(\mathbf{u})\mathbf{x} + \mathbf{I}_p \boldsymbol{l} = \mathbf{C}(\mathbf{u})\mathbf{x}_0 \\ & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0} \\ & \boldsymbol{l} \geqq \mathbf{0}, \end{aligned} \tag{6.33}$$

where $\boldsymbol{l} \in \mathbb{R}^p$ is a deviation variable. The corresponding dual of EKLP($\mathbf{x}_0, \mathbf{u}$), denoted EKDP($\mathbf{x}_0, \mathbf{u}$), is thus given by

$$\begin{aligned} \min_{\mathbf{v},\mathbf{w}} \quad & [\mathbf{C}(\mathbf{u})\mathbf{x}_0]^T \mathbf{v} + \mathbf{b}^T \mathbf{w} \\ \text{s.t.} \quad & \mathbf{C}(\mathbf{u})^T \mathbf{v} + \mathbf{A}^T \mathbf{w} \geqq \mathbf{0} \\ & \mathbf{v} \geqq \mathbf{1} \\ & \mathbf{w} \geqq \mathbf{0}, \end{aligned} \tag{6.34}$$

where $\mathbf{1} \in \mathbb{R}^p$ is the p-dimensional vector of ones, and $\mathbf{v} \in \mathbb{R}^p$ and $\mathbf{w} \in \mathbb{R}^m$ are dual variables.

Given $\mathbf{u} \in U$, it is apparent that EKLP($\mathbf{x}_0, \mathbf{u}$) and EKDP($\mathbf{x}_0, \mathbf{u}$) are deterministic and are simply Ecker and Kouada's LP (4.19) and DP (4.20), respectively, associated with the instance MOLP($\mathbf{u}$). For the purposes of the following results and proofs, a feasible solution to $\mathrm{EKLP}(\mathbf{x}_0, \mathbf{u})$ for an arbitrary $\mathbf{u} \in \{\mathbf{u}^1, \ldots, \mathbf{u}^s\}$ is given by the point $(\mathbf{x}(\mathbf{u}), \boldsymbol{l}(\mathbf{u}))$, where $\mathbf{x}(\mathbf{u})$ and $\boldsymbol{l}(\mathbf{u})$ explicitly indicate the dependence of the variables $\mathbf{x}$ and $\boldsymbol{l}$ on the scenario $\mathbf{u}$. Similarly, a feasible solution to $\mathrm{EKDP}(\mathbf{x}_0, \mathbf{u})$ for an arbitrary $\mathbf{u} \in \{\mathbf{u}^1, \ldots, \mathbf{u}^s\}$ is given by the point $(\mathbf{v}(\mathbf{u}), \mathbf{w}(\mathbf{u}))$.

The idea of $\mathrm{EKLP}(\mathbf{x}_0, \mathbf{u})$, like in the deterministic context, is that we first choose some initial feasible solution $\mathbf{x}_0 \in P$. If $\mathbf{x}_0$ is not itself highly robust efficient, then we try to produce a solution that is or identify that the highly robust efficient set is empty, which is accomplished by maximizing the sum of nonnegative deviation variables $l_k(\mathbf{u}) = \mathbf{c}_k(\mathbf{u})\mathbf{x}_0 - \mathbf{c}_k(\mathbf{u})\mathbf{x}$, $k = 1, \ldots, p$, for each $\mathbf{u} \in U$. We first demonstrate that $\mathrm{EKLP}(\mathbf{x}_0, \mathbf{u})$ is feasible for all $\mathbf{u} \in U$.

Lemma 6.4.1. *Let $\mathbf{x}_0 \in P$ and $\mathbf{u} \in U$ be given. Then $\mathrm{EKLP}(\mathbf{x}_0, \mathbf{u})$ is feasible.*

Proof. It is clear that $\mathrm{EKLP}(\mathbf{x}_0, \mathbf{u})$ is feasible since $\boldsymbol{l}(\mathbf{u}) = \mathbf{0}$ and $\mathbf{x}(\mathbf{u}) = \mathbf{x}_0$ satisfy the constraints. □

Given a feasible decision $\mathbf{x}_0 \in P$, whether or not it is highly robust efficient may be verified using either the family of LPs given by $\mathrm{EKLP}(\mathbf{x}_0, \mathbf{u})$ or of DPs given by $\mathrm{EKDP}(\mathbf{x}_0, \mathbf{u})$.

Proposition 6.4.2. *Let $\mathbf{x}_0 \in P$ be given.*

(i) The point $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ if and only if $\mathrm{EKLP}(\mathbf{x}_0, \mathbf{u})$ has an optimal solution $(\hat{\mathbf{x}}(\mathbf{u}), \hat{\boldsymbol{l}}(\mathbf{u}))$ with $\hat{\boldsymbol{l}}(\mathbf{u}) = \mathbf{0}$ for every $\mathbf{u} \in U$.

(ii) The point $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ if and only if $\mathrm{EKDP}(\mathbf{x}_0, \mathbf{u})$ has an optimal solution $(\hat{\mathbf{v}}(\mathbf{u}), \hat{\mathbf{w}}(\mathbf{u}))$ with $[\mathbf{C}(\mathbf{u})\mathbf{x}_0]^T \hat{\mathbf{v}}(\mathbf{u}) + \mathbf{b}^T \hat{\mathbf{w}}(\mathbf{u}) = 0$ for every $\mathbf{u} \in U$.

Proof. (i) The proof follows the same as the proof of Proposition 5.4.1.

(ii) Since EKDP($\mathbf{x}_0, \mathbf{u}$) is the dual of EKLP($\mathbf{x}_0, \mathbf{u}$) for each $\mathbf{u} \in U$, it follows by Strong Duality 2.5.1(i) that $(\hat{\mathbf{x}}(\mathbf{u}), \hat{\boldsymbol{l}}(\mathbf{u}))$ is an optimal solution to EKLP($\mathbf{x}_0, \mathbf{u}$) if and only if $(\hat{\mathbf{v}}(\mathbf{u}), \hat{\mathbf{w}}(\mathbf{u}))$ is an optimal solution to EKDP($\mathbf{x}_0, \mathbf{u}$) with

$$\sum_{k=1}^{p} l_k(\mathbf{u}) = [\mathbf{C}(\mathbf{u})\mathbf{x}_0]^T \hat{\mathbf{v}}(\mathbf{u}) + \mathbf{b}^T \hat{\mathbf{w}}(\mathbf{u})$$

for each $\mathbf{u} \in U$. Therefore, part (i) yields the result. □

Note that Proposition 6.4.2(ii) may be proven alternatively by utilizing the fact that EKDP($\mathbf{x}_0, \mathbf{u}$) is the deterministic Ecker and Kouada dual associated with the instance MOLP($\mathbf{u}$) for each $\mathbf{u} \in U$ along with Proposition 4.2.21(ii) regarding deterministic efficiency. Further note that in solving the family of LPs given by EKLP($\mathbf{x}_0, \mathbf{u}$) with $\mathbf{x}_0 \in P$, it is expected that (provided the highly robust efficient set is nonempty) if $\mathbf{x}_0$ is not highly robust efficient itself, then another feasible decision that is highly robust efficient is generated. While this property is guaranteed in the deterministic setting with efficiency (cf. Proposition 4.2.22), this is not the case in the uncertain context and an additional condition is required as the next proposition reveals.

Proposition 6.4.3. *Let* $\mathbf{x}_0 \in P$ *be given, and suppose* $(\hat{\mathbf{x}}(\mathbf{u}^i), \hat{\boldsymbol{l}}(\mathbf{u}^i))$ *is an optimal solution to* EKLP($\mathbf{x}_0, \mathbf{u}^i$) *for each* $i = 1, \ldots, s$. *If* $\hat{\mathbf{x}} := \hat{\mathbf{x}}(\mathbf{u}^1) = \cdots = \hat{\mathbf{x}}(\mathbf{u}^s) \neq \mathbf{x}_0$ *and* $\hat{\boldsymbol{l}}(\mathbf{u}^i)$ *is finite for all* $i = 1, \ldots, s$, *then* $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$.

Proof. Let $\hat{\mathbf{x}} := \hat{\mathbf{x}}(\mathbf{u}^1) = \cdots = \hat{\mathbf{x}}(\mathbf{u}^s) \neq \mathbf{x}_0$, and let $\hat{\boldsymbol{l}}(\mathbf{u}^i)$ is finite for all $i = 1, \ldots, s$. Hence, $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^i))$ for each $i = 1, \ldots, s$ by Proposition 4.2.22. Applying Proposition 6.1.2 yields the result. □

In addition to using the family of LPs given by EKLP($\mathbf{x}_0, \mathbf{u}$) and of DPs given by EKDP($\mathbf{x}_0, \mathbf{u}$) to obtain solution recognition and generation methods, we may also propose conditions under which the highly robust efficient set is empty.

Proposition 6.4.4. *Let* $\mathbf{x}_0 \in P$ *be given.*

(i) If $\mathrm{EKLP}(\mathbf{x}_0, \mathbf{u})$ *has an unbounded optimal objective value for at least one* $\mathbf{u} \in U$*, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$*.*

(ii) IF $\mathrm{EKDP}(\mathbf{x}_0, \mathbf{u})$ *is infeasible for at least one* $\mathbf{u} \in U$*, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$*.*

Proof. (i) Suppose the optimal objective value of $\mathrm{EKLP}(\mathbf{x}_0, \mathbf{u})$ is unbounded for at least one $\mathbf{u} \in U$, say $\mathbf{u}^1$. Hence, $\mathrm{E}(P, \mathbf{C}(\mathbf{u}^1)) = \emptyset$ by Proposition 4.2.23(i). Applying Proposition 6.1.2 gives the result.

(ii) The proof follows similarly to the proof of part (i). □

Note that Proposition 6.4.4(ii) may be proven alternatively and more directly by utilizing part (i) along with Lemma 6.4.1 in a fashion similar to the deterministic proof of Proposition 4.2.23(ii). It is also important to note that Proposition 6.4.4 indicates that the highly robust efficient set is empty because the efficient set associated with at least one instance $\mathrm{MOLP}(\mathbf{u})$ is empty, which is only possible when P is unbounded. However, as should be clear, the highly robust efficient set may be empty even when P is bounded (cf. UMOLP (6.5) and Figure 6.1). That being said, the identification of whether or not the highly robust efficient set is empty is addressed in general in Section 6.4.4.

Second, we consider the individual Ecker-and-Kouada-type auxiliary LP, denoted EKLP1($\mathbf{x}_0, U$), which is a block-style problem given by

$$\begin{aligned}
\max_{\mathbf{x}_1,\ldots,\mathbf{x}_s,\boldsymbol{l}_1,\ldots,\boldsymbol{l}_s} \quad & \sum_{i=1}^{s}\sum_{k=1}^{p} l_{ik} \\
\text{s.t.} \quad & \mathbf{C}(\mathbf{u}^i)\mathbf{x}_i + \mathbf{I}_p\boldsymbol{l}_i = \mathbf{C}(\mathbf{u}^i)\mathbf{x}_0 \quad \text{for all } i = 1,\ldots,s \\
& \mathbf{A}\mathbf{x}_i \leqq \mathbf{b} \quad \text{for all } i = 1,\ldots,s \\
& \mathbf{x}_i \geqq \mathbf{0} \quad \text{for all } i = 1,\ldots,s \\
& \boldsymbol{l}_i \geqq \mathbf{0} \quad \text{for all } i = 1,\ldots,s,
\end{aligned} \tag{6.35}$$

where $\boldsymbol{l}_i \in \mathbb{R}^p$ for all $i = 1,\ldots,s$. The corresponding dual of EKLP1($\mathbf{x}_0, U$), denoted EKDP1($\mathbf{x}_0, U$), is similarly given by

$$\begin{aligned}
\min_{\mathbf{v}_1,\ldots,\mathbf{v}_s,\mathbf{w}_1,\ldots,\mathbf{w}_s} \quad & \sum_{i=1}^{s}\left[\mathbf{C}(\mathbf{u}^i)\mathbf{x}_0\right]^T\mathbf{v}_i + \sum_{i=1}^{s}\mathbf{b}^T\mathbf{w}_i \\
\text{s.t.} \quad & \mathbf{C}(\mathbf{u}^i)^T\mathbf{v}_i + \mathbf{A}^T\mathbf{w}_i \geqq \mathbf{0} \quad \text{for all } i = 1,\ldots,s \\
& \mathbf{v}_i \geqq \mathbf{1} \quad \text{for all } i = 1,\ldots,s \\
& \mathbf{w}_i \geqq \mathbf{0} \quad \text{for all } i = 1,\ldots,s,
\end{aligned} \tag{6.36}$$

where $\mathbf{v}_i \in \mathbb{R}^p$ and $\mathbf{w}_i \in \mathbb{R}^m$ are dual variables for all $i = 1,\ldots,s$.

As with the family of auxiliary LPs, the idea is that we choose some initial feasible solution $\mathbf{x}_0 \in P$. If $\mathbf{x}_0$ is not itself highly robust efficient, then we try to produce a solution that is or identify that the highly robust efficient set is empty, which is accomplished by maximizing the sum of nonnegative deviation variables $\boldsymbol{l}_i = \mathbf{C}(\mathbf{u}^i)\mathbf{x}_0 - \mathbf{C}(\mathbf{u})\mathbf{x}$ for each scenario $\mathbf{u}^i \in U, i = 1,\ldots,s$. We first show that EKLP1($\mathbf{x}_0, U$) is feasible.

Lemma 6.4.5. *Let* $\mathbf{x}_0 \in P$ *be given. Then* EKLP1($\mathbf{x}_0, U$) *is feasible.*

Proof. It is clear that EKLP1($\mathbf{x}_0, U$) is feasible since $\boldsymbol{l}_i = \mathbf{0}$ and $\mathbf{x}_i = \mathbf{x}_0$ for all $i = 1,\ldots,s$, satisfy the constraints. □

Given a feasible solution $\mathbf{x}_0 \in P$, its highly robust efficiency may be verified by using either EKLP1$(\mathbf{x}_0, U)$ or EDKP1$(\mathbf{x}_0, U)$ as in the following proposition.

Proposition 6.4.6. *Let $\mathbf{x}_0 \in P$ be given.*

(i) The point $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ if and only if EKLP1$(\mathbf{x}_0, U)$ *has an optimal solution $(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_s, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ with $\hat{\boldsymbol{l}}_i = \mathbf{0}$ for all $i = 1, \ldots, s$*

(ii) The point $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ if and only if EKDP1$(\mathbf{x}_0, U)$ *has an optimal solution $(\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_s, \hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_s)$ with $\sum_{i=1}^{s}\left(\left[\mathbf{C}(\mathbf{u}^i)\mathbf{x}_0\right]^T \hat{\mathbf{v}}_i + \mathbf{b}^T\hat{\mathbf{w}}_i\right) = 0$.*

Proof. (i) The proof follows the same as the proof of Proposition 5.4.3.

(ii) Since EKDP1$(\mathbf{x}_0, U)$ is the dual of EKLP1$(\mathbf{x}_0, U)$, it follows by Strong Duality 2.5.1(i) that $(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_s, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ is an optimal solution to EKLP1$(\mathbf{x}_0, U)$ if and only if $(\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_s, \hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_s)$ is an optimal solution to EKDP1$(\mathbf{x}_0, U)$ with

$$\sum_{i=1}^{s}\sum_{k=1}^{p} \hat{l}_{ik} = \sum_{i=1}^{s}\left[\mathbf{C}(\mathbf{u}^i)\mathbf{x}_0\right]^T \hat{\mathbf{v}}_i + \sum_{i=1}^{s}\mathbf{b}^T\hat{\mathbf{w}}_i.$$

Therefore, part (i) yields the result. □

In solving EKLP1$(\mathbf{x}_0, U)$ with $\mathbf{x}_0 \in P$, it is expected (as with the family of auxiliary LPs) that if $\mathbf{x}_0$ is not highly robust efficient, then another feasible solution that is highly robust efficient is generated provided the highly robust efficient is nonempty. Although this property is guaranteed in the deterministic setting, this is not the case in the uncertain context with highly robust efficiency as the next proposition reveals.

Proposition 6.4.7. *Let $\mathbf{x}_0 \in P$ be given, and suppose $(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_s, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ is an optimal solution to* EKLP1$(\mathbf{x}_0, U)$. *If $\hat{\mathbf{x}} := \hat{\mathbf{x}}_1 = \cdots = \hat{\mathbf{x}}_s \neq \mathbf{x}_0$ and $\hat{\boldsymbol{l}}_i$ is finite for all $i = 1, \ldots, s$, then $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$.*

Proof. The proof follows the same as the proof of Proposition 5.4.4. □

If $\mathbf{x}_0$ is not found to be highly robust efficient and another feasible solution that is itself highly robust efficient is not generated, then the highly robust efficient set may be identified as empty as in the proceeding proposition.

Proposition 6.4.8. *Let* $\mathbf{x}_0 \in P$ *be given.*

(i) If $\mathrm{EKLP1}(\mathbf{x}_0, U)$ *has an unbounded optimal objective value, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

(ii) If $\mathrm{EKDP1}(\mathbf{x}_0, U)$ *is infeasible, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

Proof. (i) Suppose the optimal objective value of $\mathrm{EKLP1}(\mathbf{x}_0, U)$ is unbounded. Hence, there exists an $\bar{\imath} \in \{1, \ldots, s\}$ such that at least one component of $\hat{\boldsymbol{l}}_{\bar{\imath}}$ is unbounded. Due to the block structure of $\mathrm{EKLP1}(\mathbf{x}_0, U)$, this implies that the optimal objective value of the deterministic Ecker and Kouada LP associated with $\mathrm{MOLP}(\mathbf{u}^{\bar{\imath}})$ is unbounded. Thus, $\mathrm{E}(P, \mathbf{C}(\mathbf{u}^{\bar{\imath}})) = \emptyset$ by Proposition 4.2.23(i), which implies that $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$ by Proposition 6.1.2.

(ii) Suppose $\mathrm{EKDP1}(\mathbf{x}_0, U)$ is infeasible. Hence, $\mathrm{EKLP1}(\mathbf{x}_0, U)$ must be unbounded by the Fundamental Theorem of Duality 2.5.1 and Lemma 6.4.5. Therefore, $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$ by part (i). □

Comparing the family of Ecker and Kouada problems with the individual Ecker-and-Kouada-type problem given by $\mathrm{EKLP1}(\mathbf{x}_0, U)$, we observe that the same set of the three results regarding recognition, generation, and existence is available. However, the former requires solving a finite number of problems, which may be done in parallel, while the latter only requires solving a single problem.

Third, another individual Ecker-and-Kouada-type auxiliary LP, denoted EKLP2($\mathbf{x}_0, U$), is the deterministic Ecker and Kouada LP associated with AIOMOLP (6.22) given by

$$\begin{aligned} \max_{\mathbf{x}, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_s} \quad & \sum_{i=1}^{s} \sum_{k=1}^{p} l_{ik} \\ \text{s.t.} \quad & \mathbf{C}(\mathbf{u}^i)\mathbf{x} + \mathbf{I}_p \boldsymbol{l}_i = \mathbf{C}(\mathbf{u}^i)\mathbf{x}_0 \quad \text{for all } i = 1, \ldots, s \\ & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0} \\ & \boldsymbol{l}_i \geqq \mathbf{0} \quad \text{for all } i = 1, \ldots, s, \end{aligned} \tag{6.37}$$

where $\boldsymbol{l}_i \in \mathbb{R}^p$ for all $i = 1, \ldots, s$. Moreover, the corresponding dual of EKLP2($\mathbf{x}_0, U$), denoted EKDP2($\mathbf{x}_0, U$), is given by

$$\begin{aligned} \min_{\mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{w}} \quad & \sum_{i=1}^{s} \left[\mathbf{C}(\mathbf{u}^i)\mathbf{x}_0\right]^T \mathbf{v}_i + \mathbf{b}^T \mathbf{w} \\ \text{s.t.} \quad & \sum_{i=1}^{s} \mathbf{C}(\mathbf{u}^i)^T \mathbf{v}_i + \mathbf{A}^T \mathbf{w} \geqq \mathbf{0} \\ & \mathbf{v}_i \geqq \mathbf{1} \quad \text{for all } i = 1, \ldots, s \\ & \mathbf{w} \geqq \mathbf{0}, \end{aligned} \tag{6.38}$$

where $\mathbf{v}_i \in \mathbb{R}^p$, $i = 1, \ldots, s$, and $\mathbf{w} \in \mathbb{R}^m$ are dual variables.

By Proposition 6.3.19, we know that the highly robust efficient set is a subset of the efficient set of AIOMOLP (6.22). As a result of this relationship, the Ecker-and-Kouada-type method utilizing EKLP2($\mathbf{x}_0, U$) does not provide necessary and sufficient conditions for highly robust efficiency recognition whereas EKLP1($\mathbf{x}_0, U$) and the family of EKLPs do. Nevertheless, the advantage of EKLP2($\mathbf{x}_0, U$) (and its dual) is due to its reduced number of variables. We first establish that EKLP2($\mathbf{x}_0, U$) is feasible.

Lemma 6.4.9. *Let* $\mathbf{x}_0 \in P$ *be given. Then* $\text{EKLP2}(\mathbf{x}_0, U)$ *is feasible.*

Proof. It is clear that $\text{EKLP2}(\mathbf{x}_0, U)$ is feasible since $\mathbf{x} = \mathbf{x}_0$ and $\boldsymbol{l}_i = \mathbf{0}$ for all $i = 1, \ldots, s$ satisfy the constraints. □

As discussed directly above, we obtain in the following proposition necessary (not necessary and sufficient) conditions for the highly robust efficiency of a feasible decision $\mathbf{x}_0$ by examining both $\text{EKLP2}(\mathbf{x}_0, U)$ and its dual.

Proposition 6.4.10. *Let* $\mathbf{x}_0 \in P$ *be given. If* $\mathbf{x}_0 \in \text{E}(P, \mathbf{C}(\mathbf{u}), U)$*, then*

(i) $\text{EKLP2}(\mathbf{x}_0, U)$ *has an optimal solution* $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ *with* $\hat{\boldsymbol{l}}_i = \mathbf{0}$ *for all* $i = 1, \ldots, s$*;*

(ii) $\text{EKDP2}(\mathbf{x}_0, U)$ *has an optimal solution* $(\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_s, \hat{\mathbf{w}})$ *with* $\sum_{i=1}^{s} \left[\mathbf{C}(\mathbf{u}^i)\mathbf{x}_0\right]^T \hat{\mathbf{v}}_i + \mathbf{b}^T\hat{\mathbf{w}} = 0$*.*

Proof. (i) The proof follows the same as the proof of Proposition 5.4.5.

(ii) Assume $\mathbf{x}_0 \in \text{E}(P, \mathbf{C}(\mathbf{u}), U)$. By part (i), $\text{EKLP2}(\mathbf{x}_0, U)$ has an optimal solution $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}}_1, \ldots, \hat{\boldsymbol{l}}_s)$ with $\hat{\boldsymbol{l}}_i = \mathbf{0}$ for all $i = 1, \ldots, s$. Hence, it follows by Strong Duality 2.5.1(i) that $\text{EKDP2}(\mathbf{x}_0, U)$ must have an optimal solution $(\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_s, \hat{\mathbf{w}})$ with

$$\sum_{i=1}^{s} \left[\mathbf{C}(\mathbf{u}^i)\mathbf{x}_0\right]^T \hat{\mathbf{v}}_i + \mathbf{b}^T\hat{\mathbf{w}} = \sum_{i=1}^{s} \sum_{k=1}^{p} \hat{l}_{ik} = 0$$

as desired. □

Since the efficient set of AIOMOLP (6.22) only contains the highly robust efficient set, it is clear that Proposition 6.4.10 cannot be both necessary and sufficient. Similarly, a result comparable to Proposition 6.4.7 to generate a highly robust efficiet point is not available because although the optimal solution to $\text{EKLP2}(\mathbf{x}_0, U)$ is guaranteed to be efficient to AIMOLP (6.22), it may lie outside of the highly robust efficient set. Nonetheless, sufficient conditions for the emptiness of the highly robust efficient set may still be presented as in the proceeding proposition.

Proposition 6.4.11. *Let* $\mathbf{x}_0 \in P$ *be given.*

(i) If $\text{EKLP2}(\mathbf{x}_0, U)$ *has an unbounded optimal objective value, then* $\text{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

(ii) If $\text{EKDP2}(\mathbf{x}_0, U)$ *is infeasible, then* $\text{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

Proof. (i) Suppose the objective value of $\text{EKLP2}(\mathbf{x}_0, U)$ is unbounded. Hence, $\text{E}(P, \mathbf{C}(U)) = \emptyset$ by Proposition 4.2.23(i), where $\text{EKLP2}(\mathbf{x}_0, U)$ is the Ecker and Kouada LP associated with the deterministic problem AIOMOLP (6.22). Since $\text{E}(P, \mathbf{C}(\mathbf{u}), U) \subseteq \text{E}(P, \mathbf{C}(U))$ by Proposition 6.3.19, the result follows.

(ii) Suppose $\text{EKDP2}(\mathbf{x}_0, U)$ is infeasible. Hence, $\text{EKLP2}(\mathbf{x}_0, U)$ must be unbounded by the Fundamental Theorem of Duality 2.5.1 and Lemma 6.4.9. Therefore, $\text{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$ by part (i). □

Note that Proposition 6.4.11 indicates that the highly robust efficient set is empty because the efficient set of the all-in-one problem is empty and, as mentioned in the preceding discussion, the efficient set of AIOMOLP (6.22) contains the highly robust efficient set. However, as with Proposition 6.4.8, this result is not both necessary and sufficient, and so it is possible that the highly robust efficient set is empty even when the efficient set of AIOMOLP (6.22) is nonempty.

Fourth, the final individual Ecker-and-Kouada-type auxiliary LP we present, denoted $\text{EKLP3}(\mathbf{x}_0, U)$, is the deterministic Ecker and Kouada LP associated with MOLP (6.26). As such, in the following setup and results, suppose each column of

$\mathbf{C}(\mathbf{u})$ is either nonnegative for all $\mathbf{u} \in U$ or nonpositive for all $\mathbf{u} \in U$ with no column all 0, and define $\mathbf{I}$ to be the diagonal matrix with a 1 corresponding to the nonnegative columns of $\mathbf{C}(\mathbf{u})$ and a -1 for the nonpositive columns. Hence, EKLP3$(\mathbf{x}_0, U)$ is given by

$$\begin{aligned} \max_{\mathbf{x},\boldsymbol{l}} \quad & \sum_{k=1}^{p} l_k \\ \text{s.t.} \quad & \mathbf{I}\mathbf{x} + \mathbf{I}_p \boldsymbol{l} = \mathbf{I}\mathbf{x}_0 \\ & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0} \\ & \boldsymbol{l} \geqq \mathbf{0}, \end{aligned} \tag{6.39}$$

where $\boldsymbol{l} \in \mathbb{R}^p$. Moreover, the dual of EKLP3$(\mathbf{x}_0, U)$, denoted EKDP3$(\mathbf{x}_0, U)$, is given by

$$\begin{aligned} \min_{\mathbf{v},\mathbf{w}} \quad & (\mathbf{I}\mathbf{x}_0)^T \mathbf{v} + \mathbf{b}^T \mathbf{w} \\ \text{s.t.} \quad & \mathbf{I}^T \mathbf{v} + \mathbf{A}^T \mathbf{w} \geqq \mathbf{0} \\ & \mathbf{v} \geqq \mathbf{1} \\ & \mathbf{w} \geqq \mathbf{0}, \end{aligned} \tag{6.40}$$

where $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$ are dual variables.

By Theorem 6.3.22, we know that $\text{E}(P, \mathbf{C}(\mathbf{u}), U) \subseteq \text{E}(P, \mathbf{I})$. As a result of this relationship, the Ecker-and-Kouada-type method utilizing EKLP3$(\mathbf{x}_0, U)$, similarly to with EKLP2$(\mathbf{x}_0, U)$, does not provide necessary and sufficient conditions for highly robust efficiency recognition. That being said, the benefit of EKLP3$(\mathbf{x}_0, U)$ (and its dual) is that, even compared to EKLP2$(\mathbf{x}_0, U)$, the number of variables is significantly reduced. We first establish that EKLP3$(\mathbf{x}_0, U)$ is feasible.

Lemma 6.4.12. *Let* $\mathbf{x}_0 \in P$ *be given. Then* EKLP3$(\mathbf{x}_0, U)$ *is feasible.*

Proof. It is clear that EKLP3$(\mathbf{x}_0, U)$ is feasible since $\mathbf{x} = \mathbf{x}_0$ and $\boldsymbol{l} = \mathbf{0}$ satisfy the constraints. □

As mentioned directly above, necessary conditions (not necessary and sufficient) for identifying the highly robust efficiency of a given solution $\mathbf{x}_0$ are available for EKLP3$(\mathbf{x}_0, U)$ and its dual.

Proposition 6.4.13. *Let $\mathbf{x}_0 \in P$ be given. If $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$, then*

(i) EKLP3$(\mathbf{x}_0, U)$ *has an optimal solution* $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ *with* $\hat{\boldsymbol{l}} = \mathbf{0}$*;*

(ii) EKDP3$(\mathbf{x}_0, U)$ *has an optimal solution* $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$ *with* $(\mathbf{I}\mathbf{x}_0)^T \hat{\mathbf{v}} + \mathbf{b}^T \hat{\mathbf{w}} = 0$.

Proof. (i) Suppose $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$. Hence, $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{I})$ also by Theorem 6.3.22. Since EKLP3$(\mathbf{x}_0, U)$ is the Ecker and Kouada problem associated with the deterministic problem MOLP (6.26), the result follows immediately from Proposition 4.2.21(i).

(ii) Assume $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$. By part (i), EKLP3$(\mathbf{x}_0, U)$ has an optimal solution $(\hat{\mathbf{x}}, \hat{\boldsymbol{l}})$ with $\hat{\boldsymbol{l}} = \mathbf{0}$. Hence, it follows by Strong Duality 2.5.1(i) that EKDP3$(\mathbf{x}_0, U)$ must have an optimal solution $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$ with

$$(\mathbf{I}\mathbf{x}_0)^T \hat{\mathbf{v}} + \mathbf{b}^T \hat{\mathbf{w}} = \sum_{k=1}^{p} \hat{l}_k = 0$$

as desired. □

Similarly to EKLP2$(\mathbf{x}_0, U)$ and Proposition 6.4.10, since the efficient set of MOLP (6.26) only contains the highly robust efficient set, it is clear that Proposition 6.4.13 cannot be both necessary and sufficient. Moreover, as mentioned with EKLP2$(\mathbf{x}_0, U)$, a result comparable to Proposition 6.4.7 to generate a highly robust efficient point is not available because even though the optimal solution to EKLP3$(\mathbf{x}_0, U)$ is guaranteed to be efficient to MOLP (6.26), it may lie outside of the highly robust efficient set. Nevertheless, sufficient conditions for the emptiness of the highly robust efficient set may still be given as in the following proposition.

Proposition 6.4.14. *Let* $\mathbf{x}_0 \in P$ *be given.*

(i) If $\mathrm{EKLP3}(\mathbf{x}_0, U)$ *has an unbounded optimal objective value, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

(ii) IF $\mathrm{EKDP3}(\mathbf{x}_0, U)$ *is infeasible, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

Proof. (i) Suppose the objective value of $\mathrm{EKLP3}(\mathbf{x}_0, U)$ is unbounded. Since $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \subseteq \mathrm{E}(P, \mathbf{I})$ by Theorem 6.3.22 and $\mathrm{EKLP3}(\mathbf{x}_0, U)$ is the Ecker and Kouada problem associated with the deterministic problem MOLP (6.26), the result follows immediately from Proposition 4.2.23(i).

(ii) Suppose $\mathrm{EKDP3}(\mathbf{x}_0, U)$ is infeasible. Hence, $\mathrm{EKLP3}(\mathbf{x}_0, U)$ must be unbounded by the Fundamental Theorem of Duality 2.5.1 and Lemma 6.4.12. Therefore, $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$ by part (i). □

Regardless of whether the family of Ecker and Kouada problems or one of the individual Ecker-and-Kouada-type problems is utilized, the decision maker is able to verify whether a given feasible solution of interest is also highly robust efficient or possibly determine that the highly robust efficient set is empty. If the family of Ecker and Kouada problems or the individual Ecker-and-Kouada-type problem given by $\mathrm{EKLP1}(\mathbf{x}_0, U)$ is used, then the decision maker may also be able to generate another feasible solution that is in fact highly robust efficient. In any case, these extensions of Ecker and Kouada's method provide useful tools regarding highly robust efficient solutions to $\mathrm{MOLP}(U)$ to decision makers.

6.4.2 Extension of Benson's Method

As demonstrated in Section 4.2.3.3 with respect to efficient solutions to MOLP (4.8), a second class of auxiliary problems introduced by Benson [11] could be utilized (in much the same way as Ecker and Kouada's problem) to verify whether efficient solutions exist and more importantly (for the multiobjective simplex method) generate an initial efficient extreme point. Similarly, in the uncertain setting, Benson's problem may be extended to provide a second class of auxiliary problems to identify whether or not the highly robust efficient set is empty aside from the Ecker-and-Kouada-type problems derived in the previous section. In particular, Benson's problem/method may be extended to (at least) four different Benson-type auxiliary problems. One is a family of problems, while the other three are individual problems (including one that is an MOLP). Regardless of the formulation, necessary or sufficient conditions for the existence of highly robust efficient solutions are given. However, since none of the conditions are both necessary and sufficient, the ability to generate a highly robust efficient extreme point solution is not available.

We first examine the family of Benson-type auxiliary LPs. For a given feasible solution $\mathbf{x}_0 \in P$ and an arbitrary $\mathbf{u} \in U$, the following problem, denoted BLP$(\mathbf{x}_0, \mathbf{u})$, is a representative member of the family of auxiliary problems and is given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{1}^T \mathbf{C}(\mathbf{u})\mathbf{x} \\ \text{s.t.} \quad & \mathbf{C}(\mathbf{u})\mathbf{x} \leqq \mathbf{C}(\mathbf{u})\mathbf{x}_0 \\ & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0}. \end{aligned} \tag{6.41}$$

Given $\mathbf{u} \in U$, it is clear that BLP$(\mathbf{x}_0, \mathbf{u})$ is deterministic and is simply Benson's LP (4.22) associated with the instance MOLP$(\mathbf{u})$. A feasible solution to BLP$(\mathbf{x}_0, \mathbf{u})$

for an arbitrary $\mathbf{u} \in \{\mathbf{u}^1, \dots, \mathbf{u}^s\}$ is given by the point $\mathbf{x}(\mathbf{u})$, where $\mathbf{x}(\mathbf{u})$ explicitly indicates the dependence of the variable $\mathbf{x}$ on the scenario $\mathbf{u}$. Using the family of LPs, the existence of highly robust efficient solutions may be verified, and a highly robust efficient decision generated, as in the following proposition.

Proposition 6.4.15. *Let $\mathbf{x}_0 \in P$ be given. For each $i = 1, \dots, s$, suppose $\hat{\mathbf{x}}(\mathbf{u}^i)$ is an optimal solution to* $\mathrm{BLP}(\mathbf{x}_0, \mathbf{u}^i)$*. If $\hat{\mathbf{x}} := \hat{\mathbf{x}}(\mathbf{u}^1) = \cdots = \hat{\mathbf{x}}(\mathbf{u}^s)$, then $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$.*

Proof. Let $\hat{\mathbf{x}} = \hat{\mathbf{x}}(\mathbf{u}^1) = \cdots = \hat{\mathbf{x}}(\mathbf{u}^s)$. Hence, $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^i))$ for each $i = 1, \dots, s$ by Theorem 4.2.25(i). Applying Proposition 6.1.2 yields the desired result. □

Second, in addition to the family of LPs given by $\mathrm{BLP}(\mathbf{x}_0, \mathbf{u})$, we present the following individual Benson-type auxiliary MOLP. Due to the structure of Benson's problem with the decision variable $\mathbf{x}$ remaining in the objective, it is not possible to formulate a block-structured problem in the same way as done with $\mathrm{EKLP1}(\mathbf{x}_0, U)$. Instead, an MOLP is constructed such that each row of the cost matrix corresponds to the Benson-type objective associated with each instance of UMOLP (6.1). In order to keep the notation compact, this cost matrix is defined to be

$$\mathbf{C}_{\mathrm{B}}(U) := \begin{bmatrix} \mathbf{1}^T\mathbf{C}(\mathbf{u}^1) \\ \vdots \\ \mathbf{1}^T\mathbf{C}(\mathbf{u}^s) \end{bmatrix} \in \mathbb{R}^{s \times n}.$$

The Benson-type MOLP, denoted $\mathrm{BMOLP}(\mathbf{x}_0, U)$ is thus given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{C}_{\mathrm{B}}(U)\mathbf{x} \\ \text{s.t.} \quad & \mathbf{C}(\mathbf{u}^i)\mathbf{x} \leqq \mathbf{C}(\mathbf{u}^i)\mathbf{x}_0 \quad \text{for all } i = 1, \dots, s \\ & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0}. \end{aligned} \tag{6.42}$$

Rather than a sufficient condition for the existence of highly robust efficient solutions obtained with the family of Benson-type LPs above, BMOLP($\mathbf{x}_0, U$) provides a necessary condition as in the proceeding proposition.

Proposition 6.4.16. *Let* $\mathbf{x}_0 \in P$ *be given. If* MOLP(U) *has a highly robust efficient solution, then* BMOLP($\mathbf{x}_0, U$) *has a weakly efficient solution.*

Proof. Assume $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$. By Definition 6.1.1, for each $i = 1, \ldots, s$, there does not exist an $\mathbf{x} \in P$ such that $\mathbf{C}(\mathbf{u}^i)\mathbf{x} \leq \mathbf{C}(\mathbf{u}^i)\hat{\mathbf{x}}$, i.e., such that $\mathbf{c}_k(\mathbf{u}^i)\mathbf{x} \leq \mathbf{c}_k(\mathbf{u}^i)$ for all $k = 1, \ldots, p$ with at least one strict. As a result,

$$\sum_{k=1}^{p} \mathbf{c}_k(\mathbf{u}^i)\mathbf{x} < \sum_{k=1}^{p} \mathbf{c}_k(\mathbf{u}^i)\hat{\mathbf{x}}$$

for all $i = 1, \ldots, s$. Accordingly, $\mathbf{1}^T\mathbf{C}(\mathbf{u}^i)\mathbf{x} < \mathbf{1}^T\mathbf{C}(\mathbf{u}^i)\hat{\mathbf{x}}$ for all $i = 1, \ldots, s$, which equivalently yields

$$\mathbf{C}_{\mathrm{B}}\mathbf{x} < \mathbf{C}_{\mathrm{B}}\hat{\mathbf{x}}.$$

Therefore, in terms of the vector-valued objective function of BMOLP($\mathbf{x}_0, U$), there does not exist an $\mathbf{x} \in P$ that strictly dominates $\hat{\mathbf{x}}$. Since the feasible region of BMOLP($\mathbf{x}_0, U$) is a restriction of P, it follows that $\hat{\mathbf{x}}$ is a weakly efficient solution to BMOLP($\mathbf{x}_0, U$). □

In addition, as a direct consequence of Proposition 6.4.16, BMOLP($\mathbf{x}_0, U$) may be used to provide a sufficient condition for the emptiness of the highly robust efficient set as in the following corollary.

Corollary 6.4.17. *Let* $\mathbf{x}_0 \in P$ *be given. If the weakly efficient set of* BMOLP($\mathbf{x}_0, U$) *is empty, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

Proof. The result follows immediately from Proposition 6.4.16. □

Third, Benson's method may be extended by using AIOMOLP (6.22) in much the same way as done with Ecker and Kouada's method in the previous section. The Benson-type auxiliary LP associated with the all-in-one MOLP, denoted BLP1($\mathbf{x}_0, U$), is the deterministic Benson LP given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{1}^T \mathbf{C}(U)\mathbf{x} \\ \text{s.t.} \quad & \mathbf{C}(U)\mathbf{x} \leqq \mathbf{C}(U)\mathbf{x}_0 \\ & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0}. \end{aligned} \tag{6.43}$$

The corresponding dual of BLP1($\mathbf{x}_0, U$), denoted BDP1($\mathbf{x}_0, U$), is thus given by

$$\begin{aligned} \min_{\mathbf{v},\mathbf{w}} \quad & [\mathbf{C}(U)\mathbf{x}_0]^T \mathbf{v} + \mathbf{b}^T \mathbf{w} \\ \text{s.t.} \quad & \mathbf{C}(U)^T \mathbf{v} + \mathbf{A}^T \mathbf{w} \geqq -\mathbf{C}(U)^T \mathbf{1} \\ & \mathbf{v} \geqq \mathbf{0} \\ & \mathbf{w} \geqq \mathbf{0}, \end{aligned} \tag{6.44}$$

where $\mathbf{v} \in \mathbb{R}^{ps}$ and $\mathbf{w} \in \mathbb{R}^m$ are dual variables.

Similarly to EKLP2($\mathbf{x}_0, U$) in the previous subsection, BLP1($\mathbf{x}_0, U$) and its dual may be used in order to provide additional sufficient conditions for the emptiness of the highly robust efficient set.

Proposition 6.4.18. *Let* $\mathbf{x}_0 \in P$ *be given.*

(i) If BLP1($\mathbf{x}_0, U$) *has no optimal solution, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

(ii) If BDP1($\mathbf{x}_0, U$) *has no optimal solution, then* $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

Proof. (i) Suppose BLP1($\mathbf{x}_0, U$) has no optimal solution. Hence, $\mathrm{E}(P, \mathbf{C}(U)) = \emptyset$, i.e., the efficient set associated with AIOMOLP (6.22) is empty, by Theorem

4.2.25. Since $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) \subseteq \mathrm{E}(P, \mathbf{C}(U))$ by Proposition 6.3.19, it follows that $\mathrm{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$ as well.

(ii) Suppose BDP1($\mathbf{x}_0, U$) has no optimal solution. Since BDP1($\mathbf{x}_0, U$) is the dual of BLP1($\mathbf{x}_0, U$), the result follows from part (i) and Strong Duality 2.5.1(i). □

Finally, the fourth Benson-type auxiliary problem considered is the deterministic Benson LP, denoted BLP2($\mathbf{x}_0, U$), associated with MOLP (6.26). As a result, in the following setup and proposition, suppose each column of $\mathbf{C}(\mathbf{u})$ is either nonnegative for all $\mathbf{u} \in U$ or nonpositive for all $\mathbf{u} \in U$ with no column all 0, and define $\mathbf{I}$ to be the diagonal matrix with a 1 corresponding to the nonnegative columns of $\mathbf{C}(\mathbf{u})$ and a -1 for the nonpositive columns. Hence, BLP2($\mathbf{x}_0, U$) is given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{1}^T \mathbf{I}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{I}\mathbf{x} \leqq \mathbf{I}\mathbf{x}_0 \\ & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0}. \end{aligned} \tag{6.45}$$

The corresponding dual of BLP2($\mathbf{x}_0, U$), denoted BDP2($\mathbf{x}_0, U$), is thus given by

$$\begin{aligned} \min_{\mathbf{v},\mathbf{w}} \quad & (\mathbf{I}\mathbf{x}_0)^T \mathbf{v} + \mathbf{b}^T \mathbf{w} \\ \text{s.t.} \quad & \mathbf{I}^T \mathbf{v} + \mathbf{A}^T \mathbf{w} \geqq -\mathbf{I}^T \mathbf{1} \\ & \mathbf{v} \geqq \mathbf{0} \\ & \mathbf{w} \geqq \mathbf{0}, \end{aligned} \tag{6.46}$$

where $\mathbf{v} \in \mathbb{R}^{ps}$ and $\mathbf{w} \in \mathbb{R}^m$ are dual variables.

As with BLP1($\mathbf{x}_0, U$) above, BLP2($\mathbf{x}_0, U$) and its dual may also be utilized in order to provide further sufficient conditions for the emptiness of the highly robust efficient set.

Proposition 6.4.19. *Let* $\mathbf{x}_0 \in P$ *be given.*

(i) If $\text{BLP2}(\mathbf{x}_0, U)$ *has no optimal solution, then* $\text{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$*.*

(ii) If $\text{BDP2}(\mathbf{x}_0, U)$ *has no optimal solution, then* $\text{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$*.*

Proof. (i) Suppose $\text{BLP2}(\mathbf{x}_0, U)$ has no optimal solution. Hence, $\text{E}(P, \mathbf{I}) = \emptyset$, i.e., the efficient set associated with MOLP (6.26) is empty, by Theorem 4.2.25. Since $\text{E}(P, \mathbf{C}(\mathbf{u}), U) \subseteq \text{E}(P, \mathbf{I})$ by Theorem 6.3.22, it follows that $\text{E}(P, \mathbf{C}(\mathbf{u}), U) = \emptyset$ as well.

(ii) Suppose $\text{BDP2}(\mathbf{x}_0, U)$ has no optimal solution. Since $\text{BDP2}(\mathbf{x}_0, U)$ is the dual of $\text{BLP2}(\mathbf{x}_0, U)$, the result follows from part (i) and Strong Duality 2.5.1(i). □

Although none of the above Benson-type conditions is necessary and sufficient, each provides an additional avenue for determining the existence of highly robust efficient solutions.

6.4.3 Extension of the Weighted-Sum Method

As in Section 5.3 and the more general case of $\text{MOP}(U)$, the weighted-sum scalarization method may be extended in order to solve for highly robust efficient solutions to $\text{MOLP}(U)$ by considering a family of weighted-sum LPs. In fact, similarly to the deterministic setting, it is possible to compute every highly robust efficient point, which is shown by extending Isermann's Theorem 4.2.28.

For each $i = 1, \ldots, s$, the weighted-sum LP with respect to scenario $\mathbf{u}^i \in U$

and weight $\boldsymbol{\lambda}_i \in \mathbb{R}^p$, denoted WSLP$(\boldsymbol{\lambda}_i, \mathbf{u}^i)$, is given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \boldsymbol{\lambda}_i^T \mathbf{C}(\mathbf{u}^i)\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in P. \end{aligned} \tag{6.47}$$

Given an arbitrary scenario $\mathbf{u}^i \in U$, it is clear that WSLP$(\boldsymbol{\lambda}_i, \mathbf{u}^i)$ is deterministic and is indeed the weighted-sum LP (4.18) associated with the instance MOLP$(\mathbf{u}^i)$. For the purposes of the following results and proofs, a feasible solution to WSLP$(\boldsymbol{\lambda}_i, \mathbf{u}^i)$ for any $i \in \{1, \dots, s\}$ is given by the point $\mathbf{x}(\mathbf{u}^i)$, where $\mathbf{x}(\mathbf{u}^i)$ explicitly indicates the dependence of the variable $\mathbf{x}$ on the scenario $\mathbf{u}^i$.

As in the following proposition, a highly robust efficient solution may be obtained from the family of weighted-sum LPs provided that solving each member yields the same optimal solution.

Proposition 6.4.20. *Suppose $\hat{\mathbf{x}}(\mathbf{u}^i)$ is an optimal solution to* WSLP$(\boldsymbol{\lambda}_i, \mathbf{u}^i)$ *with $\boldsymbol{\lambda}_i \in \mathbb{R}^p$ for every $i = 1, \dots, s$ such that $\hat{\mathbf{x}} := \hat{\mathbf{x}}(\mathbf{u}^1) = \cdots = \hat{\mathbf{x}}(\mathbf{u}^s)$.*

(i) If $\boldsymbol{\lambda}_i \in \mathbb{R}^p_>$ for all $i = 1, \dots, s$, then $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$.

(ii) If $\boldsymbol{\lambda}_i \in \mathbb{R}^p_\geq$ for all $i = 1, \dots, s$, then $\hat{\mathbf{x}} \in \mathrm{wE}(P, \mathbf{C}(\mathbf{u}), U)$.

Proof. (i) Let $\boldsymbol{\lambda}_i \in \mathbb{R}^p_>$ for all $i = 1, \dots, s$. Hence, $\hat{\mathbf{x}} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^i))$ for each $i = 1, \dots, s$ by Corollary 4.2.19(i). Applying Proposition 6.1.2 yields the desired result.

(ii) The proof follows similarly to the proof of part (i). □

In order to show that every highly robust efficient solution may be obtained by solving the family of weighted-sum LPs in this manner with $\boldsymbol{\lambda}_i \in \mathbb{R}^p_>$, Isermann's Theorem 4.2.28 is extended as in the proceeding theorem.

Theorem 6.4.21. *Let* $\mathbf{x}_0 \in P$ *be given. Then* $\mathbf{x}_0$ *is a highly robust efficient solution to* MOLP(U) *if and only if there exists a* $\boldsymbol{\lambda}_i \in \mathbb{R}^p_>, i = 1, \ldots, s,$ *such that*

$$\boldsymbol{\lambda}_i^T \mathbf{C}(\mathbf{u}^i)\mathbf{x}_0 \leq \boldsymbol{\lambda}_i^T \mathbf{C}(\mathbf{u}^i)\mathbf{x}$$

for all $i = 1, \ldots, s$ *and for all* $\mathbf{x} \in P$.

Proof 1. ($\Longleftarrow$) Suppose there exists a $\boldsymbol{\lambda}_i \in \mathbb{R}^p_>$ such that $\boldsymbol{\lambda}_i^T \mathbf{C}(\mathbf{u}^i)\mathbf{x}_0 \leq \boldsymbol{\lambda}_i^T \mathbf{C}(\mathbf{u}^i)\mathbf{x}$ for all $i = 1, \ldots, s$ and for all $\mathbf{x} \in P$, and assume for the sake of contradiction that $\mathbf{x}_0 \notin \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$. By Proposition 6.1.2, $\mathbf{x}_0 \notin \bigcap_{i=1}^s \mathrm{E}(P, \mathbf{C}(\mathbf{u}^i))$, i.e., there exists an $\bar{\imath} \in \{1, \ldots, s\}$ such that $\mathbf{x}_0 \notin \mathrm{E}(P, \mathbf{C}(\mathbf{u}^{\bar{\imath}}))$. By Definition 4.2.1, there exists an $\mathbf{x}(\mathbf{u}^{\bar{\imath}}) \in P$ such that $\mathbf{C}(\mathbf{u}^{\bar{\imath}})\mathbf{x}(\mathbf{u}^{\bar{\imath}}) \leq \mathbf{C}(\mathbf{u}^{\bar{\imath}})\mathbf{x}_0$, or equivalently, $\mathbf{c}_k(\mathbf{u}^{\bar{\imath}})\mathbf{x}(\mathbf{u}^{\bar{\imath}}) \leq \mathbf{c}_k(\mathbf{u}^{\bar{\imath}})\mathbf{x}_0$ for all $k = 1, \ldots, p$ with at least one strict. Since $\lambda_{\bar{\imath}k} > 0$ for all $k = 1, \ldots, p$, it equivalently follows that $\lambda_{\bar{\imath}k}\mathbf{c}_k(\mathbf{u}^{\bar{\imath}})\mathbf{x}(\mathbf{u}^{\bar{\imath}}) \leq \lambda_{\bar{\imath}k}\mathbf{c}_k(\mathbf{u}^{\bar{\imath}})\mathbf{x}_0$ for all $k = 1, \ldots, p$ with at least one strict, which implies

$$\sum_{k=1}^{p} \lambda_{\bar{\imath}k}\mathbf{c}_k(\mathbf{u}^{\bar{\imath}})\mathbf{x}(\mathbf{u}^{i}) < \sum_{k=1}^{p} \lambda_{\bar{\imath}k}\mathbf{c}_k(\mathbf{u}^{\bar{\imath}})\mathbf{x}_0.$$

By definition,

$$\boldsymbol{\lambda}_{\bar{\imath}}^T \mathbf{C}(\mathbf{u}^{\bar{\imath}})\mathbf{x}(\mathbf{u}^{\bar{\imath}}) < \boldsymbol{\lambda}_i^T \mathbf{C}(\mathbf{u}^{\bar{\imath}})\mathbf{x}_0$$

which is a contradiction. Thus, it must be that $\mathbf{x}_0 \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$.

($\Longrightarrow$) Suppose $\mathbf{x}_0 \in P$ is a highly robust efficient solution to MOLP(U). By Proposition 6.4.2(ii), for each $i = 1, \ldots, s$, we equivalently know that EKDP$(\mathbf{x}_0, \mathbf{u}^i)$ has an optimal solution $(\hat{\mathbf{v}}(\mathbf{u}^i), \hat{\mathbf{w}}(\mathbf{u}^i))$ with $[\mathbf{C}(\mathbf{u}^i)\mathbf{x}_0]^T \hat{\mathbf{v}}(\mathbf{u}^i) + \mathbf{b}^T\hat{\mathbf{w}}(\mathbf{u}^i) = 0$, i.e., $\mathbf{b}^T\hat{\mathbf{w}}(\mathbf{u}^i) =$

$-\left[\mathbf{C}(\mathbf{u}^i)\mathbf{x}_0\right]^T \hat{\mathbf{v}}(\mathbf{u}^i)$. For every $i = 1, \ldots, s$, taking $\mathbf{v} = \hat{\mathbf{v}}(\mathbf{u}^i)$ in EKDP$(\mathbf{x}_0, \mathbf{u}^i)$ yields

$$\begin{array}{lrl} \min\limits_{\mathbf{v},\mathbf{w}} & \left[\mathbf{C}(\mathbf{u}^i)\mathbf{x}_0\right]^T \hat{\mathbf{v}}(\mathbf{u}^i) + \mathbf{b}^T\mathbf{w} & \\ \text{s.t.} & \mathbf{C}(\mathbf{u}^i)^T\hat{\mathbf{v}}(\mathbf{u}^i) + \mathbf{A}^T\mathbf{w} & \geqq \mathbf{0} \\ & \hat{\mathbf{v}}(\mathbf{u}^i) & \geqq \mathbf{1} \\ & \mathbf{w} & \geqq \mathbf{0} \end{array} \implies \begin{array}{ll} \min\limits_{\mathbf{w}} & \mathbf{b}^T\mathbf{w} \\ \text{s.t.} & \mathbf{A}^T\mathbf{w} \geqq -\mathbf{C}(\mathbf{u}^i)^T\hat{\mathbf{v}}(\mathbf{u}^i) \\ & \mathbf{w} \geqq \mathbf{0}. \end{array} \tag{6.48}$$

For each $i \in \{1, \ldots, s\}$, observe that $\hat{\mathbf{w}}(\mathbf{u}^i)$ is an optimal solution to LP (6.48), and consider the corresponding dual given by

$$\begin{array}{lr} \max\limits_{\mathbf{x}} & -\hat{\mathbf{v}}(\mathbf{u}^i)^T\mathbf{C}(\mathbf{u}^i)\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0} \end{array} \iff \begin{array}{lr} \min\limits_{\mathbf{x}} & \hat{\mathbf{v}}(\mathbf{u}^i)^T\mathbf{C}(\mathbf{u}^i)\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leqq \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0}. \end{array} \tag{6.49}$$

Hence, for each $i = 1, \ldots, s$ and $\hat{\mathbf{x}}(\mathbf{u}^i)$ an optimal solution to LP (6.49), we obtain

$$\mathbf{b}^T\hat{\mathbf{w}}(\mathbf{u}^i) = -\hat{\mathbf{v}}(\mathbf{u}^i)^T\mathbf{C}(\mathbf{u}^j)\hat{\mathbf{x}}(\mathbf{u}^i) \tag{6.50}$$

by Strong Duality 2.5.1(i). Since (6.50) is also satisfied by $\mathbf{x}_0$ for each $i = 1, \ldots, s$, it follows that $\mathbf{x}_0$ is an optimal solution to LP (6.49) as well, i.e.,

$$\hat{\mathbf{v}}(\mathbf{u}^i)^T\mathbf{C}(\mathbf{u}^i)\mathbf{x}_0 \leq \hat{\mathbf{v}}(\mathbf{u}^i)^T\mathbf{C}(\mathbf{u}^i)\mathbf{x}(\mathbf{u}^i)$$

for all $\mathbf{x}(\mathbf{u}^i) \in P, i = 1, \ldots, s$. Letting $\boldsymbol{\lambda}_i = \hat{\mathbf{v}}(\mathbf{u}^i) \geqq \mathbf{1} > \mathbf{0}$ for each $i = 1, \ldots, s$, we obtain the result. □

Proof 2. Suppose $\mathbf{x}_0 \in P$ is a highly robust efficient solution MOLP(U). Equivalently, $\mathbf{x}_0 \in \text{E}(P, \mathbf{C}(\mathbf{u}))$ for all $\mathbf{u} \in U$ by Proposition 6.1.2. Thus, the result follows from

Isermann's Theorem 4.2.28. □

Similarly to the deterministic case, the above theorem indicates that it is thus possible to compute all highly robust efficient solutions to MOLP(U) by solving a family of weighted-sum LPs. The main issue, however, is that the same optimal solution must be obtained from each weighted-sum LP in the family, which is not guaranteed for every set of weights $\boldsymbol{\lambda}_i \in \mathbb{R}^p_>, i = 1, \ldots, s$.

6.4.4 Two-Step Bilevel Approach

In the previous subsections, e.g., Sections 6.3.4, 6.4.1, 6.4.2, and 6.4.3, we have provided various results that indicate methods with which highly robust efficient solutions to MOLP(U) may be obtained. However, in each situation, some assumption is first required so that highly robust efficient solutions may not be generated in general. To address this issue and compute highly robust efficient solutions to MOLP(U) in general, we propose a two-step procedure. The first step is to determine whether or not the highly robsut efficient set is empty, and if it is nonempty, then the second step is to find other highly robust efficient points (if they exist). This second phase is accomplished using a bilevel approach in which a function is optimized over the highly robust efficient set, which is a natural extension of optimization over the deterministic efficient set (refer to Horst et al. [71]).

For the purposes of this subsection, we include slack variables in the polyhedral feasible set P. Hence, P is redefined to be

$$P := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqq \mathbf{0}\} \tag{6.51}$$

for the remainder of this part.

The bilevel problem is, in general, given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & F(\mathbf{x}, \mathbf{u}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^i)) \text{ for all } i = 1, \ldots, s, \end{aligned} \tag{6.52}$$

where $F : P \times U \to \mathbb{R}$, and the constraints enforce that a solution is highly robust efficient. Although any objective function F clearly produces highly robust efficient solutions, the choice of F is of practical significance. For example, if F is a utility function, a highly robust efficient solution with some desirable characteristic(s) may be obtained. Meanwhile, if F is a scalarizing function, such as a weighted sum, defined by means of scalarizing parameters, then the bilevel problem yields a highly robust efficient solution associated with a particular value of the parameters or a subset of the highly robust efficient set corresponding to a collection of selected parameter values.

Within the scope of this dissertation, we choose F to be a scalarizing function. In particular, we select the weighted-sum scalarization, where the weighted-sum LP with respect to weight vector $\boldsymbol{\lambda} \in \mathbb{R}^p$ and scenario $\mathbf{u} \in U$ is denoted WSLP$(\boldsymbol{\lambda}, \mathbf{u})$ and given by LP (6.47). When the weight $\boldsymbol{\lambda} \in \mathbb{R}^p$ is positive, solutions to WSLP$(\boldsymbol{\lambda}, \mathbf{u})$ are guaranteed to be efficient solutions to MOLP$(\mathbf{u})$ by Isermann's Theorem 4.2.28 as discussed in the previous section. That being said, the bilevel problem (6.52) becomes

$$\begin{aligned} \min_{\mathbf{x}} \quad & \boldsymbol{\lambda}_{\bar{\imath}}^T \mathbf{C}(\mathbf{u}^{\bar{\imath}}) \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^i)) \quad \text{for all } i = 1, \ldots, s, \end{aligned} \tag{6.53}$$

where $\bar{\imath} \in \{1, \ldots, s\}$ is the index corresponding to a nominal scenario that may be arbitrarily chosen, and $\boldsymbol{\lambda}_{\bar{\imath}} \in \mathbb{R}^p$ is a positive weight.

Since the highly robust efficient set is unknown a priori, it is necessary to reformulate the constraint $\mathbf{x} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^i))$ for all $i = 1, \ldots, s$. To that end, the constraint may be written equivalently as $\mathbf{x}$ in the set of minimizers of a weighted-sum problem yielding

$$\begin{array}{lll} \displaystyle\min_{\mathbf{x}, \boldsymbol{\lambda}_i, i=1,\ldots,s, i\neq \bar{\imath}} & \boldsymbol{\lambda}_{\bar{\imath}}^T \mathbf{C}(\mathbf{u}^{\bar{\imath}})\mathbf{x} & \\ \text{s.t.} & \mathbf{x} \in \left\{ \begin{array}{ll} \displaystyle\operatorname*{argmin}_{\mathbf{z}} & \boldsymbol{\lambda}_i^T \mathbf{C}(\mathbf{u}^i)\mathbf{z} \\ \text{s.t.} & \mathbf{z} \in P \end{array} \right\} & \text{for all } i = 1, \ldots, s, \end{array} \tag{6.54}$$

where $\boldsymbol{\lambda}_i > \mathbf{0}, i = 1, \ldots, s$. The upper-level problem is a weighted-sum scalarization associated with some nominal scenario, while the lower-level consists of a collection of weighted-sum problems that ensures efficiency with respect to every scenario. At the lower level, the weights $\boldsymbol{\lambda}_i, i = 1, \ldots, s$ are implicitly known as soon as an optimal $\mathbf{x}$ is known and are, therefore, not optimization variables. However, at the upper level, $\boldsymbol{\lambda}_i, i = 1, \ldots, s, i \neq \bar{\imath}$, are unknown and become optimization variables so that they may be determined. In addition, observe that in solving problem (6.54), the optimal weights $\boldsymbol{\lambda}_i, i = 1, \ldots, s, i \neq \bar{\imath}$, obtained and the nominal weight $\boldsymbol{\lambda}_{\bar{\imath}}$ selected are indeed the weights such that the optimal $\mathbf{x}$-solution is an optimal solution to WSLP$(\boldsymbol{\lambda}_i, \mathbf{u}^i)$ and WSLP$(\boldsymbol{\lambda}_{\bar{\imath}}, \mathbf{u}^{\bar{\imath}})$ as well.

In order to obtain solutions at the lower level, a final transformation is still needed. Applying the KKT conditions (refer to Theorem 2.5.2) to the lower level in

(6.54) yields

$$\begin{array}{llll}
\min\limits_{\substack{\mathbf{x},\boldsymbol{\lambda}_i, i=1,\ldots,s, i\neq\bar{\imath}\\ \mathbf{w}_i,\mathbf{v}_i, i=1,\ldots,s}} & \boldsymbol{\lambda}_{\bar{\imath}}^T\mathbf{C}(\mathbf{u}^{\bar{\imath}})\mathbf{x} & & \\
\text{s.t.} & \boldsymbol{\lambda}_i^T\mathbf{C}(\mathbf{u}^i)+\mathbf{w}_i^T\mathbf{A}-\mathbf{v}_i & =\mathbf{0} & \text{for all } i=1,\ldots,s \\
& \mathbf{v}_i^T\mathbf{x} & =0 & \text{for all } i=1,\ldots,s \\
& \mathbf{v}_i & \geqq \mathbf{0} & \text{for all } i=1,\ldots,s \\
& \boldsymbol{\lambda}_i & >\mathbf{0} & \text{for all } i=1,\ldots,s, i\neq\bar{\imath} \\
& \mathbf{x} & \in P, &
\end{array} \tag{6.55}$$

where $\boldsymbol{\lambda}_{\bar{\imath}} > 0$ is a vector of parameters, and $\mathbf{v}_i \in \mathbb{R}^n, i = 1,\ldots,s$, and $\mathbf{w}_i \in \mathbb{R}^m, i = 1,\ldots,s$, are the vectors of dual variables (Lagrange multipliers) associated with the inequality and equality constraints in P, respectively. Note that the weight $\boldsymbol{\lambda}_{\bar{\imath}}$ is not treated as a variable but rather as a vector of parameters, and the constraint $\mathbf{v}_i^T\mathbf{x} = 0$ is nonlinear but would be eliminated if the original problem did not require the nonnegativity of $\mathbf{x}$.

Before discussing how (6.55) may be used as part of a method to obtain highly robust efficient solutions, we address the feasibility of this problem. First, when the highly robust efficient set is empty, it is clear that (6.55) is infeasible. Otherwise, when the highly robust efficient set is nonempty, the feasibility of (6.55) depends on the nominal weight $\boldsymbol{\lambda}_{\bar{\imath}}$. In particular, once $\boldsymbol{\lambda}_{\bar{\imath}}$ is selected, the constraints associated with $\bar{\imath}$, as well as $\mathbf{x} \in P$, effectively determine the optimal $\mathbf{x}$-solution to (6.55) in the case that WSLP$(\boldsymbol{\lambda}_{\bar{\imath}}, \mathbf{u}^{\bar{\imath}})$ has a unique solution (or optimal $\mathbf{x}$-solutions in the case that alternate optimal solutions exist). Due to this interaction between the nominal scenario and the $\mathbf{x}$-solution to (6.55), it is possible that this problem is infeasible even if the highly robust efficient set is nonempty. If the nominal weight $\boldsymbol{\lambda}_{\bar{\imath}}$ is such that the corresponding $\mathbf{x} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^{\bar{\imath}}))$ is not efficient with respect to at least one other scenario, then (6.55) is infeasible. Additionally, if the nominal weight $\boldsymbol{\lambda}_{\bar{\imath}}$ is such that the associated weighted-sum problem WSLP$(\boldsymbol{\lambda}_{\bar{\imath}}, \mathbf{u}^{\bar{\imath}})$ is unbounded (i.e., an extreme direction is efficient), then the corresponding KKT constraints are inconsistent (as the dual problem is infeasible if the primal is unbounded) and so (6.55) is infeasible. On the other hand, if the nominal weight $\boldsymbol{\lambda}_{\bar{\imath}}$ is such that the corresponding $\mathbf{x} \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^{\bar{\imath}}))$ is efficient with respect to every other scenario, then (6.55) is feasible.

In view of the possibility that (6.55) is infeasible for a given weight $\boldsymbol{\lambda}_{\bar{\imath}}$ even when highly robust efficient solutions exist, it is desirable to determine whether or not the highly robust efficient set is empty prior to solving the bilevel problem. To accomplish this task, the following KKT system given by

$$\begin{aligned}
\boldsymbol{\lambda}_i^T \mathbf{C}(\mathbf{u}^i) + \mathbf{w}_i^T \mathbf{A} - \mathbf{v}_i &= \mathbf{0} \quad \text{for all } i = 1, \ldots, s \\
\mathbf{v}_i^T \mathbf{x} &= 0 \quad \text{for all } i = 1, \ldots, s \\
\mathbf{v}_i &\geqq \mathbf{0} \quad \text{for all } i = 1, \ldots, s \\
\boldsymbol{\lambda}_i &> \mathbf{0} \quad \text{for all } i = 1, \ldots, s \\
\mathbf{x} &\in P,
\end{aligned} \tag{6.56}$$

where $\boldsymbol{\lambda}_i \in \mathbb{R}^p, \mathbf{v}_i \in \mathbb{R}^n, \mathbf{w}_i \in \mathbb{R}^m, i = 1, \ldots, s$, and $\mathbf{x}$ are all treated as variables, may be used.

Theorem 6.4.22. *The highly robust efficient set is nonempty if and only if* (6.56) *is consistent.*

Proof. Let the highly robust efficient set be nonempty, i.e., there exists an $\mathbf{x}^* \in P$ such that $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^i))$ for all $i = 1, \ldots, s$. Equivalently, by Isermann's Theorem 4.2.28, there exists a $\bar{\boldsymbol{\lambda}}_i > \mathbf{0}$ such that $\mathbf{x}^*$ is an optimal solution to WSLP$(\bar{\boldsymbol{\lambda}}_i, \mathbf{u}^i)$ for all $i = 1, \ldots, s$. As WSLP$(\bar{\boldsymbol{\lambda}}_i, \mathbf{u}^i)$ is an LP for each $i = 1, \ldots, s$, $\mathbf{x}^*$ is an optimal solution to WSLP$(\bar{\boldsymbol{\lambda}}_i, \mathbf{u}^i)$ if and only if there exist $\bar{\mathbf{v}}_i \in \mathbb{R}^n$ and $\bar{\mathbf{w}}_i \in \mathbb{R}^m$ such that

$(\bar{\mathbf{v}}_i, \bar{\mathbf{w}}_i, \mathbf{x}^*)$ satisfy the KKT system given by

$$\begin{aligned} \bar{\boldsymbol{\lambda}}_i^T \mathbf{C}(\mathbf{u}^i) + \mathbf{w}_i^T \mathbf{A} - \mathbf{v}_i &= \mathbf{0} \\ \mathbf{v}_i^T \mathbf{x}^* &= 0 \\ \mathbf{v}_i &\geqq \mathbf{0} \\ \mathbf{x}^* &\in P, \end{aligned} \tag{6.57}$$

for each $i = 1, \dots, s$. Since $\bar{\boldsymbol{\lambda}}_i > \mathbf{0}$ and $(\bar{\mathbf{v}}_i, \bar{\mathbf{w}}_i, \mathbf{x}^*)$ is feasible to (6.57) for all $i = 1, \dots, s$, it follows that $(\mathbf{x}^*, \bar{\boldsymbol{\lambda}}_1, \dots, \bar{\boldsymbol{\lambda}}_s, \bar{\mathbf{w}}_1, \dots, \bar{\mathbf{w}}_s, \bar{\mathbf{v}}_1, \dots, \bar{\mathbf{v}}_s)$ is also a solution to (6.56) □

An immediate consequence of Theorem 6.4.22 is that a feasible solution $\mathbf{x}^* \in P$ is highly robust efficient if and only if $(\mathbf{x}^*, \bar{\boldsymbol{\lambda}}_1, \dots, \bar{\boldsymbol{\lambda}}_s, \bar{\mathbf{w}}_1, \dots, \bar{\mathbf{w}}_s, \bar{\mathbf{v}}_1, \dots, \bar{\mathbf{v}}_s)$ is a solution to (6.56). Hence, in using (6.56) to check whether or not the highly robust efficient set is nonempty, a highly robust efficient solution is generated along with a nominal weight $\boldsymbol{\lambda}_{\bar{\imath}}$ for which (6.55) is feasible. It is also worth noting that even if the highly robust efficient set is unbounded, there exists a highly robust efficient extreme point by Proposition 6.3.1(v) so that (6.56) has a feasible solution.

If the highly robust efficient set is determined to be nonempty by virtue of Theorem 6.4.22, then the bilevel problem is considered next in order to compute other highly robust efficient solutions (if they exist). The following result accounts for the feasibility of (6.55) and offers a means to compute highly robust efficient solutions by solving (6.55) with different weights $\boldsymbol{\lambda}_{\bar{\imath}} > \mathbf{0}$.

Theorem 6.4.23. *Let $\bar{\imath} \in \{1, \dots, s\}$ be a given nominal index. A feasible solution $\mathbf{x}^* \in P$ is a highly robust efficient solution to* MOLP(U) *if and only if there exists a $\bar{\boldsymbol{\lambda}}_{\bar{\imath}} > \mathbf{0}$ such that $(\mathbf{x}^*, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{\bar{\imath}-1}, \boldsymbol{\lambda}_{\bar{\imath}+1}, \dots, \boldsymbol{\lambda}_s, \mathbf{w}_1, \dots, \mathbf{w}_s, \mathbf{v}_1, \dots, \mathbf{v}_s)$ is an optimal solution to* (6.55).

Proof. Let $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}), U)$, i.e., $\mathbf{x}^* \in \mathrm{E}(P, \mathbf{C}(\mathbf{u}^i))$ for all $i = 1, \ldots, s$. Equivalently, by Isermann's Theorem, there exists a $\bar{\boldsymbol{\lambda}}_i > \mathbf{0}$ such that $\mathbf{x}^*$ is an optimal solution to WSLP$(\bar{\boldsymbol{\lambda}}_i, \mathbf{u}^i)$ for each $i = 1, \ldots, s$. Since WSLP$(\bar{\boldsymbol{\lambda}}_i, \mathbf{u}^i)$ is an LP for each $i = 1, \ldots, s$, $\mathbf{x}^*$ is an optimal solution to WSLP$(\bar{\boldsymbol{\lambda}}_i, \mathbf{u}^i)$ if and only if there exist $\bar{\mathbf{v}}_i \in \mathbb{R}^n$ and $\bar{\mathbf{w}}_i \in \mathbb{R}^m$ such that $(\bar{\mathbf{v}}_i, \bar{\mathbf{w}}_i, \mathbf{x}^*)$ satisfy the KKT system given by (6.57) for each $i = 1, \ldots, s$.

As $\bar{\boldsymbol{\lambda}}_i > \mathbf{0}$ for all $i = 1, \ldots, s$, and $(\bar{\mathbf{v}}_i, \bar{\mathbf{w}}_i, \mathbf{x}^*)$ is feasible to (6.57) for all $i = 1, \ldots, s$, $(\mathbf{x}^*, \bar{\boldsymbol{\lambda}}_1, \ldots, \bar{\boldsymbol{\lambda}}_{\bar{\imath}-1}, \bar{\boldsymbol{\lambda}}_{\bar{\imath}+1}, \ldots, \bar{\boldsymbol{\lambda}}_s, \bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_s, \bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_s)$ is also a feasible solution to (6.55). Moreover, since there exists a $\bar{\boldsymbol{\lambda}}_{\bar{\imath}} > \mathbf{0}$ such that $\bar{\boldsymbol{\lambda}}_{\bar{\imath}}^T \mathbf{C}(\mathbf{u}^{\bar{\imath}})\mathbf{x}^* \leq \bar{\boldsymbol{\lambda}}_{\bar{\imath}}^T \mathbf{C}(\mathbf{u}^{\bar{\imath}})\mathbf{x}$ for all $\mathbf{x} \in P$ by optimality to WSLP$(\bar{\boldsymbol{\lambda}}_{\bar{\imath}}, \mathbf{u}^{\bar{\imath}})$, the point $(\mathbf{x}^*, \bar{\boldsymbol{\lambda}}_1, \ldots, \bar{\boldsymbol{\lambda}}_{\bar{\imath}-1}, \bar{\boldsymbol{\lambda}}_{\bar{\imath}+1}, \ldots, \bar{\boldsymbol{\lambda}}_s,$ $\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_s, \bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_s)$ is a feasible and optimal solution to (6.55). □

Considering Theorems 6.4.22 and 6.4.23, the two-step procedure to compute highly robust efficient solutions to MOLP(U) involves first verifying the consistency of (6.56), and then solving the bilevel problem (6.55) as follows:

1. If $\boldsymbol{\lambda}_{\bar{\imath}}$ is a known vector of parameters (e.g., its value is provided by the decision maker, or it is chosen from within a neighborhood of the weight $\bar{\boldsymbol{\lambda}}_{\bar{\imath}}$ that is obtained during the first phase), then (6.55) is solved for $\mathbf{x}, \boldsymbol{\lambda}_i, i = 1, \ldots, s, i \neq \bar{\imath}, \mathbf{w}_i, \mathbf{v}_i, i = 1, \ldots, s$, where $\mathbf{x}$, if it exists, is a highly robust efficient solution to the UMOLP. This method generates a highly robust efficient solution for a given $\boldsymbol{\lambda}_{\bar{\imath}}$. To generate other highly robust efficient solutions, different weights must be selected. With the involvement of a decision maker in the process of selecting different nominal weights, this approach may be classified as an *interactive method* (see Miettinen et al. [111]).

2. If $\boldsymbol{\lambda}_{\bar{\imath}}$ is an unknown vector of parameters, then (6.55) is a multiparametric problem (refer to Domínguez et al. [34]) and parametric solutions may be obtained by:

(i) discretizing the parameter space $\Lambda_{\bar{\imath}} = \{\boldsymbol{\lambda}_{\bar{\imath}} \in \mathbb{R}^p : \sum_{k=1}^{p} \lambda_{\bar{\imath}k} = 1, \boldsymbol{\lambda}_{\bar{\imath}} > \mathbf{0}\}$ into a finite set of vectors $\{\boldsymbol{\lambda}_{\bar{\imath}}^1, \ldots, \boldsymbol{\lambda}_{\bar{\imath}}^\nu\}$ and solving (6.55) with $\boldsymbol{\lambda}_{\bar{\imath}}^\ell, \ell = 1, \ldots, \nu$, for $\mathbf{x}^\ell, \boldsymbol{\lambda}_i^\ell, i = 1, \ldots, s, i \neq \bar{\imath}, \mathbf{w}_i^\ell, \mathbf{v}_i^\ell, i = 1, \ldots, s$, where $\mathbf{x}^\ell$, if it exists, is a highly robust efficient solution to MOLP(U). This approach provides a collection of highly robust efficient solutions and may be referred to as a *discretized multiparametric method*.

(ii) using multiparameteric optimization and solving (6.55) for $\mathbf{x}(\boldsymbol{\lambda}_{\bar{\imath}}), \boldsymbol{\lambda}_i(\boldsymbol{\lambda}_{\bar{\imath}}), i = 1, \ldots, s, i \neq \bar{\imath}, \mathbf{w}_i(\boldsymbol{\lambda}_{\bar{\imath}}), \mathbf{v}_i(\boldsymbol{\lambda}_{\bar{\imath}}), i = 1, \ldots, s$, where $\mathbf{x}(\boldsymbol{\lambda}_{\bar{\imath}})$, if it exists, is a highly robust efficient solution function to MOLP(U). If the nonlinear constraints $\mathbf{v}_i^T\mathbf{x} = 0, i = 1, \ldots, s$ in (6.55) are eliminated, then the bilevel problem is a multiparametric LP (see Gal and Nedoma [55]) and may be solved using the Multi-Parametric Toolbox in MATLAB (refer to Herceg et al. [66]) or a two-phase algorithm proposed by Adelgren and Wiecek [1]. In any case, this approach yields highly robust efficient solutions as functions of the nominal weight $\boldsymbol{\lambda}_{\bar{\imath}}$ and treats (6.55) as a *(continuous) multiparametric optimization problem* (see Gal and Greenberg [54]).

We illustrate the discretized multiparametric approach on three small examples.

Example 6.4.24. (i) Consider the following UMOLP, which is a transformed version of UMOLP (6.12) obtained by adding slack variables x_3 and x_4, given by

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \begin{bmatrix} 3u_{11} & -9u_{12} & 0 & 0 \\ -u_{21} & 9u_{22} & 0 & 0 \end{bmatrix} \mathbf{x} \\ \text{s.t.} & \qquad\qquad\qquad\qquad \mathbf{x} \in P_1' \end{array} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2,} \tag{6.58}$$

where $U_1 = \{(1, 1, 0, 0)\}, U_2 = \{(1, 1, 0, 0), (2, -1/9, 0, 0)\}$, and P_1' is the bounded

λ_{11}	$\mathbf{x}^*$
$0.05, \ldots, 0.25$	$(6,0)$
$0.3, \ldots, 0.65$	infeasible
$0.7, \ldots, 0.95$	$(2,4)$

(a) UMOLP (6.58)

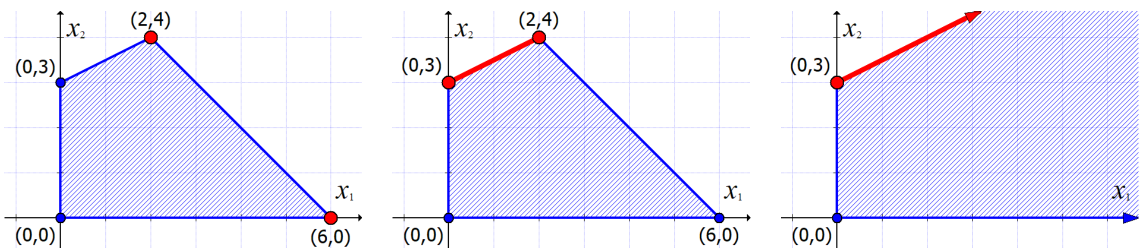
λ_{21}	$\mathbf{x}^*$
$0.05, \ldots, 0.75$	$(0,3)$
$0.8, \ldots, 0.95$	$(2,4)$

(b) UMOLP (6.60)

λ_{11}	$\mathbf{x}^*$
$0.05, \ldots, 0.5$	$(0,3)$
$0.55, \ldots, 0.95$	infeasible

(c) UMOLP (6.61)

Table 6.1: Optimal $\mathbf{x}$-solutions to (6.55) corresponding to UMOLPs (6.58), (6.60), and (6.61) with varying nominal weights as given



(a) UMOLP (6.58) (b) UMOLP (6.60) (c) UMOLP (6.61)

Figure 6.7: Feasible sets (blue) and highly robust efficient points (red) for Examples 6.4.24(i), (ii), and (iii)

feasible set given by

$$P'_1 := \{\mathbf{x} \in \mathbb{R}^4 : -x_1+2x_2+x_3 = 6, x_1+x_2+x_4 = 6, x_i \geq 0, i = 1, \ldots, 4\}. \quad (6.59)$$

As shown in Example 6.3.2, the highly robust efficient set (refer to Figure 6.7a) in the original decision space $\mathbb{R}^2$ is the disconnected set of isolated extreme points $(2,4)$ and $(6,0)$. In terms of Theorem 6.4.23 and problem (6.55), we choose $\bar{\imath} = 1$ and discretize the parameter space Λ_1 by letting $\Lambda_1 = \{\boldsymbol{\lambda}_1 \in \mathbb{R}^2 : \lambda_{11} + \lambda_{12} = 1, \lambda_{11} = 0.05\mu, \mu = 1, \ldots, 19\}$. The results of solving the subsequent collection of problems in AMPL [53] with the nonlinear solver MINOS 5.51 [114] are presented in Table 6.1a.

Inspecting Table 6.1a, we observe that nominal weights $\boldsymbol{\lambda}_1$ for which $\lambda_{11} = 0.3, \ldots, 0.65$, return that (6.55) is infeasible because an optimal solution to WSLP($\boldsymbol{\lambda}_1, \mathbf{u}^1$) for these weights is either the extreme point $(0, 0)$ or $(0, 3)$. These two extreme points, although efficient with respect to scenario $\mathbf{u}^1$, are not highly robust efficient and therefore, as discussed earlier, lead to the infeasibility of (6.55). Furthermore, we observe that the results provide no indication that the highly robust efficient set is disconnected.

(ii) Second, consider the UMOLP, which is a transformed version of UMOLP (6.29), given by

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \begin{bmatrix} u_{11} & -3u_{12} & 0 & 0 \\ u_{21} & u_{22} & 0 & 0 \end{bmatrix} \mathbf{x} \\ \text{s.t.} & \mathbf{x} \in P_1' \end{array} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2,} \tag{6.60}$$

where $U_1 = \{(1, 1, 0, 0)\}, U_2 = \{(1, -1, 0, 0), (1, 1, 0, 0)\}$, and P_1' is as in (6.59). As shown in Example 6.3.29, the highly robust efficient set (see Figure 6.7b) in the original decision space $\mathbb{R}^2$ is the connected set given by the line segment joining the extreme points $(0, 3)$ and $(2, 4)$. With respect to Theorem 6.4.23 and problem (6.55), we choose $\bar{\imath} = 2$ and discretize Λ_2 in the same manner as with UMOLP (6.58). The results of solving (6.55) with respect to Λ_2 in AMPL with MINOS 5.51 are summarized in Table 6.1b. Similar to the above discussion, the connectedness of the highly robust efficient set is not apparent based on the obtained solutions.

(iii) Finally, consider the UMOLP given by

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \begin{bmatrix} u_{11} & -3u_{12} & 0 \\ u_{21} & u_{22} & 0 \end{bmatrix} \mathbf{x} \\ \text{s.t.} & \mathbf{x} \in P_2 \end{array} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2,} \tag{6.61}$$

where $U_1 = \{(1, 1, 0)\}, U_2 = \{(1, -1, 0), (1, 1, 0)\}$, and P_2 is the unbounded feasible set (obtained by eliminating the second equality constraint from P_1 and adding a single slack variable x_3) given by

$$P_2 := \{\mathbf{x} \in \mathbb{R}^3 : -x_1 + 2x_2 + x_3 = 6, x_i \geq 0, i = 1, 2, 3\}. \tag{6.62}$$

It can be shown that the highly robust efficient set in the original decision space $\mathbb{R}^2$ is the vertex $(0, 3)$ and the ray with slope $1/2$ emanating from it (see Figure 6.7c). With respect to Theorem 6.4.23 and problem (6.55), we choose $\bar{\imath} = 1$ and discretize Λ_1 in the same manner as with the previous examples. The results of solving (6.55) with respect to Λ_1 in AMPL with MINOS 5.51 are summarized in Table 6.1c. In this case, not only is the connectedness of the highly robust efficient set not immediately obvious based on the obtained solutions, but also the unboundedness of the highly robust efficient set is not indicated.

In view of Examples 6.4.24(i), (ii), and (iii), we observe that the bilevel approach does not clearly identify the connectedness nor unboundedness of the highly robust efficient set. To address the former issue, the results in Section 6.4.1, e.g., the recognition method of Proposition 6.4.6(i), in conjunction with Proposition 6.3.1(iv) may be used to identify whether or not a face containing two or more of the efficient points obtained by the bilevel approach is itself highly robust efficient. If the face is in fact highly robust efficient, then it forms a connected subset of the highly robust efficient set. Otherwise, the highly robust efficient solutions may form isolated points within the highly robust efficient set. The application of Propositions 6.4.6(i) and 6.3.1(iv) to identify the connectedness of the highly robust efficient set is demonstrated in the following example.

Example 6.4.25. First, consider UMOLP (6.58). The highly robust efficiency of a point $\mathbf{x}_0$ in the relative interior of the line segment (face) joining $(2, 4)$ and $(6, 0)$ may be verified using EKLP1$(\mathbf{x}_0, U)$ and Proposition 6.4.6(i). For example, if $\mathbf{x}_0$ is chosen to be $(4, 2)$, then solving EKLP1($\mathbf{x}_0, U$) in AMPL with MINOS 5.51 yields an optimal objective value of $8 \neq 0$. Hence, $\mathbf{x}_0 = (4, 2)$ is not highly robust efficient by Proposition 6.4.6(i), which implies that the line segment joining the two highly robust efficient extreme points is not highly robust efficient as well, indicating that the highly robust efficient set is disconnected.

Similarly, consider UMOLP (6.60). The highly robust efficiency of a point $\mathbf{x}_0$ in the relative interior of the line segment (face) joining $(0, 3)$ and $(2, 4)$ may be confirmed using EKLP1($\mathbf{x}_0, U$) and Proposition 6.4.6(i). For instance, if $\mathbf{x}_0$ is selected to be $(1, 3.5)$, then solving EKLP1($\mathbf{x}_0, U$) in AMPL with MINOS 5.51 yields an optimal objective value of 0. Thus, $\mathbf{x}_0 = (1, 3.5)$ is indeed highly robust efficient, which implies that the line segment joining the two highly robust efficient extreme points is also highly robust efficient, indicating that the highly robust efficient set is connected.

Considering Example 6.4.25, not only do the Ecker-and-Kouada-type results of Section 6.4.1 give decision makers the ability to select any feasible solution that is deemed desirable a priori and verify whether or not it is also highly robust efficient, but they also provide a tool to identify whether highly robust efficient solutions obtained from the bilevel method form a connected set.

6.4.5 Application

To demonstrate the bilevel approach, we consider the deterministic triobjective linear program given in Eatman and Sealey [40] (and subsequently studied by Tayi and Leonard [131], Hwang et al. [75], and Doolittle et al. [35]) that models a commercial bank balance sheet management problem. The three criteria are the bank's (after-tax) profit to be maximized, the capital-adequacy ratio to be minimized, and the risk-asset to capital ratio to also be minimized, where the capital-adequacy and risk-asset to capital ratios are measures of the bank's liquidity/risk. In particular, the capital-adequacy ratio is the ratio of required to actual bank capital, while the risk-asset to capital ratio is a type of capital-adequacy ratio involving the bank's least liquid assets with the highest rates of default. The model involves 16 decision variables, the first 13 of which represent changes (with respect to balances at the beginning of the period) in the bank's assets and liabilities, and incorporates 12 context-specific constraints. Eatman and Sealey report a complete list of 11 efficient extreme points and examine the managerial utility performance of several solutions in order to choose a preferred efficient solution. When profit is considered more important than risk, the point yielding the most profit emerges as the preferred efficient extreme point. On the other hand, if the levels of importance (as dictated by the bank manager) change, then other efficient extreme points become preferred.

Since the model by Eatman and Sealey naturally exhibits uncertainty under dynamic economic conditions and the subjective judgments of decision makers (Hwang et al. [75]), we reformulate the problem as a UMOLP with objective-wise uncertainty. In particular, the UMOLP we consider is given by

$$\left\{\begin{array}{ll} \min\limits_{\mathbf{x}} & \begin{bmatrix} u_{12}x_2 + u_{13}x_3 + u_{14}x_4 + u_{15}x_5 + u_{16}x_6 + \\ \qquad u_{17}x_7 + u_{18}x_8 + u_{19}x_9 + u_{1,10}x_{10} + \\ \qquad u_{1,11}x_{11} + u_{1,12}x_{12} + u_{1,13}x_{13} \\ u_{22}x_2 + u_{23}x_3 + u_{24}x_4 + u_{25}x_5 + u_{26}x_6 + \\ \qquad u_{27}x_7 + u_{28}x_8 + u_{29}x_9 + u_{2,10}x_{10} + \\ \qquad u_{2,14}x_{14} + u_{2,15}x_{15} + u_{2,16}x_{16} \\ u_{37}x_7 + u_{38}x_8 + u_{39}x_9 + u_{3,10}x_{10} \end{bmatrix} \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - \\ & \qquad x_{11} - x_{12} - x_{13} = 12.2 \\ & x_1 - 0.04x_{13} - x_{17} = 6.4 \\ & x_1 + 0.995x_2 + 0.995x_3 + 0.96x_4 - x_{11} - x_{12} - \\ & \qquad x_{13} + x_{14} - x_{18} = 22.832 \\ & x_1 + 0.995x_2 + 0.995x_3 + 0.96x_4 + 0.9x_5 - x_{11} - \\ & \qquad x_{12} - x_{13} + x_{15} - x_{19} = 20.762 \\ & x_1 + 0.995x_2 + 0.995x_3 + 0.96x_4 + 0.9x_5 + 0.85x_6 - \\ & \qquad x_{11} - x_{12} - x_{13} + x_{16} - x_{20} = 13.877 \\ & x_2 - 0.4x_{11} - x_{21} = 0 \\ & x_3 - 0.4x_{13} - x_{22} = 2.4 \\ & x_{11} + x_{12} + x_{13} + x_{23} = 6.5 \\ & x_{11} + x_{24} = 3.9 \\ & x_{12} + x_{25} = 3.9 \\ & x_{13} + x_{26} = 3.9 \\ & x_8 - 0.25x_{11} - 0.25x_{12} - 0.25x_{13} - x_{27} = 1.45 \\ & x_i \geq 0 \text{ for all } i = 1, \ldots, 27 \end{array}\right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2, \mathbf{u}_3 \in U_3} \tag{6.63}$$

where slack variables $x_{17}, \ldots, x_{27}$ have been included so that the constraints are of the form $\mathbf{Ax} = \mathbf{b}$, and the partial uncertainty sets $U_1, U_2, U_3 \subset \mathbb{R}^{27}$ are given by polytopes. The extreme points of each polytope, as discussed below, take on an important meaning with respect to the original deterministic coefficients specified by Eatman and Sealey, as well as the lower and upper bounds given by Hwang et al.

The partial uncertainty sets each contain three extreme points. As such, the sets of extreme points of each partial uncertainty set are defined by $U_k^{\text{pts}} := \{\mathbf{u}_k^1, \mathbf{u}_k^2, \mathbf{u}_k^3\}, k = 1, 2, 3$, where $\mathbf{u}_k^j$ is referred to as a *partial scenario* in general for each $j = 1, 2, 3$. Since the convex hull of the Cartesian product of sets is the Cartesian product of the convex hulls, the uncertainty set $U = U_1 \times U_2 \times U_3 \subset \mathbb{R}^{81}$ contains 27 total extreme points, which are given by the triples $(\mathbf{u}_1^{j_1}, \mathbf{u}_2^{j_2}, \mathbf{u}_3^{j_3}), j_1, j_2, j_3 \in \{1, 2, 3\}$. In view of Theorem 6.2.1, the task of solving for the highly robust efficient set with respect to U thus reduces to finding the highly robust efficient set with respect to U^{pts}, the set of 27 extreme points. That is, in order to obtain highly robust efficient points of UMOLP (6.63), we may instead compute highly robust efficient solutions to the collection of 27 instances corresponding to the extreme points of U.

Regarding Table 6.2, each column represents one of the extreme points $\mathbf{u}_k^j \in U_k, j = 1, 2, 3$. Observe that partial scenario (extreme point) components corresponding to zero coefficients in the cost matrix of (6.63) are treated as zero since no uncertainty exists in these coefficients. Accordingly, since the slack variables $x_{17}, \ldots, x_{27}$ do not contribute to the objective functions of (6.63), the partial scenario components are all considered to be zero and are therefore omitted from Table 6.2. Note also that the first element in each extreme point set gives the original deterministic coefficients specified by Eatman and Sealey, while the second and third elements yield the lower and upper bounds, respectively, provided by Hwang et al. These partial scenarios produce, when combined to form (extreme point) scenarios in U, a variety of instances (27 total with one corresponding to each $\mathbf{u} \in U^{\text{pts}}$) whose cost matrices are combinations of the deterministic coefficients and the lower and upper bounds.

U_1^{pts}			U_2^{pts}			U_3^{pts}		
$\mathbf{u}_1^1$	$\mathbf{u}_1^2$	$\mathbf{u}_1^3$	$\mathbf{u}_2^1$	$\mathbf{u}_2^2$	$\mathbf{u}_2^3$	$\mathbf{u}_3^1$	$\mathbf{u}_3^2$	$\mathbf{u}_3^3$
0	0	0	0	0	0	0	0	0
−0.052	−0.072	−0.042	0.001	0	0.003	0	0	0
−0.053	−0.073	−0.043	0.001	0	0.003	0	0	0
−0.056	−0.076	−0.046	0.008	0.006	0.012	0	0	0
−0.058	−0.078	−0.048	0.008	0.006	0.012	0	0	0
−0.059	−0.079	−0.049	0.012	0.009	0.018	0	0	0
−0.062	−0.082	−0.052	0.02	0.015	0.03	0.2	0.1	0.4
−0.076	−0.096	−0.066	0.02	0.015	0.03	0.2	0.1	0.4
−0.071	−0.091	−0.061	0.02	0.015	0.03	0.2	0.1	0.4
−0.095	−0.115	−0.085	0.02	0.015	0.03	0.2	0.1	0.4
0.052	0.042	0.072	0	0	0	0	0	0
0.05	0.04	0.07	0	0	0	0	0	0
0.055	0.045	0.075	0	0	0	0	0	0
0	0	0	0.013	0.01	0.019	0	0	0
0	0	0	0.008	0.006	0.012	0	0	0
0	0	0	0.019	0.014	0.029	0	0	0

Table 6.2: The sets of extreme points of U_1, U_2, and U_3 associated with UMOLP (6.63), where the components corresponding to the slack variables are all treated as zero and are therefore omitted

Further note that the deterministic coefficients for the third objective (the first partial scenario $\mathbf{u}_3^1 \in U_3$ shown in Table 6.2) are those used by Eatman and Sealey, which differ from those used by Hwang et al., and the lower and upper bounds corresponding to the third cost coefficients are adjusted accordingly.

In order to obtain highly robust efficient solutions to UMOLP (6.63), we utilize the discretized multiparametric approach described in the previous section. We choose $\bar{\imath} = 1$, which corresponds to the scenario $\mathbf{u}^1 = (\mathbf{u}_1^1, \mathbf{u}_2^1, \mathbf{u}_3^1)$ yielding the deterministic model from Eatman and Sealey, and discretize the parameter space Λ_1 by using a mesh with an interval step size of 0.00625. The results of solving the subsequent collection of 12,720 problems in AMPL (refer to Appendix C for a sample of the AMPL files used) with the nonlinear solver MINOS 5.51 are presented in Table 6.3. As previously mentioned, Eatman and Sealey report 11 efficient extreme point solutions to the deterministic model. Of those points, as presented in Table 6.3, six (numbered 1, 2, 3, 7, 10, and 11) remain as highly robust efficient solutions to UMOLP (6.63). (Note that we may confirm that the other five efficient extreme points are in fact not highly robust efficient by applying Proposition 6.4.6(i) to each solution.)

The practical implications of solution 7 are addressed by Eatman and Sealey, while the utility of solutions 1, 2, 3, 10, and 11 is not discussed. Regarding solution 7, Eatman and Sealey comment that among the efficient extreme points it is the most profitable, least liquid, and most risky solution and may therefore be too risky for even the most profit-minded bank managers. Even though solutions 1, 2, 3, 10, and 11 are not examined further, as highly robust efficient solutions that remain efficient under a variety of cost matrix conditions, their relevance and practical importance is obvious. In particular, solution 11 emerges as an even more attractive decision when considering the findings of Doolittle et al., who obtain it as a min-max robust weakly efficient solution in the sense of their definition of robust efficiency (Definition 7, Doolittle et al. [35]).

	1	2	3	7	10	11
x_1	6.4	6.4	6.4	6.504	6.4	8.35
x_2	0	0	0	0	0	0
x_3	2.4	2.4	2.4	2.504	4.35	2.4
x_4	1.95	0	0	0	0	0
x_5	0	1.95	0	0	0	0
x_6	0	0	1.95	0	0	0
x_7	0	0	0	0	0	0
x_8	1.45	1.45	1.45	3.075	1.45	1.45
x_9	0	0	0	0	0	0
x_{10}	0	0	0	6.617	0	0
x_{11}	0	0	0	0	0	0
x_{12}	0	0	0	3.9	0	0
x_{13}	0	0	0	2.6	0	0
x_{14}	12.172	14.044	14.044	20.33652	12.10375	12.094
x_{15}	10.102	10.219	11.974	18.26652	10.03375	10.024
x_{16}	3.217	3.334	3.4315	11.38152	3.14875	3.139

Table 6.3: Highly robust efficient extreme point solutions to UMOLP (6.63), where the solutions are numbered as in Eatman and Sealey [40]

Chapter 7

Conclusions and Future Research

7.1 Contributions

In this dissertation, we have presented the first in-depth analysis of highly robust efficient solutions to objective-wise uncertain multiobjective linear programs (UMOLPs), as well as uncertain multiobjective programs (UMOPs), under finite sets of scenarios, while also addressing the unboundedness of the sets of feasible decisions and uncertainties. The assumed objective-wise uncertainty has three main benefits including that it permits (i) the model to incorporate the practical reality that conflicting criteria are unlikely to depend on the same uncertainty, (ii) interval multiobjective linear programming to be considered as a special case, and (iii) the application of an existing polytopal uncertainty set reduction, which consequently motivates the use of finite sets of scenarios. Although UMOLPs without objective-wise uncertainty are not considered herein, if the three aforementioned reasons or benefits of studying objective-wise uncertainty are not of concern, the theoretical and methodological results regarding highly robust efficient solutions are still applicable. Further theoretical and methodological contributions are summarized in Sections 7.1.1

and 7.1.2, respectively, and a unifying framework to obtain highly robust efficient solutions is outlined in Section 7.1.3.

7.1.1 Theoretical

During the course of the preceding chapter, we address various theoretical results regarding highly robust efficient solutions to UMOLPs such as an uncertainty set reduction, properties and characterizations of the highly robust efficient set, bound sets on (i.e., sets that contain or are contained in) the highly robust efficient set, and a robust counterpart (RC) for a class of UMOLPs.

We first derive an unbounded polyhedral uncertainty set reduction for a class of UMOLPs in which the highly robust efficient set of a UMOLP whose uncertainty set is an unbounded polyhedron is shown to be equal to the highly robust efficient set of the same UMOLP whose uncertainty set is instead the finite set of extreme points and directions. The reduction simultaneously illustrates that, at least under specific circumstances, unbounded uncertainty sets may be considered, and also gives added reason for the consideration of finite sets of scenarios. Although this reduction pertains to a very specific class of problems, it is unique in the robust optimization literature since the uncertainty set is typically assumed to be bounded.

In addition, we present a variety of properties of the highly robust efficient set including those regarding closedness, convexity, and connectedness. The properties of the highly robust efficient set highlight several key aspects of solving UMOLPs for highly robust efficient solutions. One such aspect is that since the highly robust efficient set is shown to be possibly disconnected, a simplex algorithm approach to computing highly robust efficient points is not advantageous to pursue and obtaining highly robust efficient solutions is in fact a global optimization task. Moreover, characterizations of the highly robust efficient set are provided by means of the cones of improving directions associated with the UMOLP, the normal cone (under certain acuteness assumptions), and the recession cone. The characterizations include necessary and/or sufficient conditions for the highly robust efficiency of feasible solutions, as well as conditions under which the highly robust efficient set is empty.

Following these characterizations, multiple bound sets on the highly robust efficient set are proposed. The existence of such bound sets is closely related to the above characterizations and two properties of the cone of improving directions of the UMOLP, acuteness and polyhedrality. In fact, several of the bound sets follow directly from the above characterizations while the acuteness of the closed cone of improving directions leads to a lower bound set on the highly robust efficient set that also guarantees the highly robust efficient set is nonempty provided that the feasible set is bounded. The acuteness of the cone may be checked by either of two proposed methods, solving a system of linear inequalities or computing the dimension of the cone, both of which are easily performed using readily available software. Furthermore, the polyhedrality of the closed cone of improving directions also leads to a deterministic multiobjective linear program (MOLP) that is an RC of the UMOLP. The computation of this RC is an important task since MOLPs are readily solvable and highly robust efficient solutions may thus be promptly obtained. The polyhedrality of the cone may be verified and its algebraic representation computed by an existing algorithm that immediately leads to a closed form representation of the aforementioned RC.

7.1.2 Methodological

Similarly, throughout the previous two chapters, we address several methodological approaches with respect to highly robust efficient solutions to UMOPs and UMOLPs including methods to identify whether or not highly robust efficient solutions exist, determine the highly robust efficiency of a given feasible decision, and generate highly robust efficient solutions.

The first approach involves determining whether or not the highly robust efficient set of a UMOLP is empty. As is clear, being able to determine whether or not highly robust efficient solutions exist before attempting to solve a UMOLP is important. If highly robust efficient solutions exist, then the UMOLP needs to be solved. Otherwise, highly robust efficient solutions do not exist and the decision maker needs to possibly consider other solution concepts or other uncertainty sets. Several methods are given to determine the emptiness of the highly robust efficient set, including various ones that are not guaranteed to identify this property and one that is. The former methods are extensions of both Ecker and Kouada's method and Benson's method for deterministic MOLPs, and generally indicate that the highly robust efficient set is empty because the efficient set of at least one instance of the UMOLP is empty or an upper bound set on the highly robust efficient set is empty. On the other hand, the latter method expresses highly robust efficiency in terms of the Karush-Kuhn-Tucker (KKT) conditions associated with the weighted-sum problem corresponding to each instance of the UMOLP. As such, this method is "foolproof" and identifies the emptiness of the highly robust efficient set in general.

Another approach concerns the recognition of the highly robust efficiency of a given feasible solution. This verification is a particularly meaningful tool for decision makers for two reasons: (i) a feasible solution that is deemed desirable a priori may be verified as highly robust efficient without computing (subsets of) the highly robust efficient set, and (ii) whether obtained highly robust efficient solutions to a UMOLP form a connected set may be determined by utilizing the property of the highly robust efficient set that a face is highly robust efficient provided a point in its relative interior is as well. Methods to perform this recognition task are proposed for both UMOPs and UMOLPs. In the case of the former problem, an extension of Benson's method for deterministic MOPs is derived, while an extension of Ecker and Kouada's method for deterministic MOLPs is provided in the case of the latter problem.

Moreover, a third approach is with regards to the computation of highly robust efficient points. Since highly robust efficient solutions are considered desirable under all realizations of the uncertain data, the ability to compute them is worthwhile. Assorted methods to compute highly robust efficient solutions to both UMOPs and UMOLPs are thus developed. One method is a straightforward extension of the weighted-sum method in which the family of instances defining the UMOP or UMOLP is solved by the corresponding family of weighted-sum problems. A second method is an extension of Benson's method for deterministic MOPs, respectively Ecker and Kouada's method for deterministic MOLPs, in which a new highly robust efficient point may be generated from given a current feasible solution by solving an associated auxiliary problem. Finally, highly robust efficient solutions to UMOLPs may also be computed (even when the feasible set is unbounded) using a two-step approach. The first step is determining whether or not the highly robust efficient set is empty, and if it is nonempty, then the second step is to solve a bilevel problem for highly robust efficient solutions. In order to implement the latter to generate multiple highly robust efficient points, three separate approaches are described: an interactive method, a discretized multiparametric method, and a (continuous) multiparametric optimization method. The discretized multiparametric bilevel approach is demonstrated on an application problem from bank balance-sheet management. Considering the subsequent discussion, a particular highly robust efficient solution emerges as a very attractive decision that the bank manager should examine in more detail.

7.1.3 Summary

In view of the above theoretical and methodological contributions, a systematic approach for the computation of highly robust efficient solutions is given in Figures 7.1 and 7.2.

The scheme begins by determining whether or not the cone of improving directions $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ of the input UMOLP (which is specified by the polyhedral feasible set P, the cost matrix under uncertainty $\mathbf{C}(\mathbf{u})$, and the uncertainty set U) is polyhedral. As previously discussed, the polyhedrality of the cone may be verified computationally using an existing algorithm. If the cone is in fact polyhedral, the algorithm also provides an algebraic representation of the cone that immediately leads to a closed form representation of the associated RC. Since the RC is a deterministic MOLP that may be readily solved using a variety of available methods, all highly robust efficient points may be efficiently computed.

Otherwise, if the cone of improving directions is not polyhedral, then its acuteness is verified by either of two proposed methods. If the cone of improving directions is indeed acute, then its strict polar is used to produce a deterministic MOLP whose efficient set is a lower bound set on the highly robust efficient set. Since the deterministic MOLP may be solved (similarly to the RC) using various efficient methods, some highly robust efficient points may be computed by solving this MOLP.

On the other hand, if the cone of improving directions is not acute, then the two-step approach to compute highly robust efficient decisions is used. First, the emptiness of the highly robust efficient set is determined in general using the "foolproof" method described earlier. If the highly robust efficient set is empty, then the procedure terminates and the decision maker should consider alternative uncertainty sets or possibly different robust solution concepts. Otherwise, a highly robust efficient solution is generated and additional points may be computed by completing the second step of solving the bilevel problem (which may be done using any of three suggested approaches).

7.2 Future Research

Our work immediately opens up several avenues for continued research. First and foremost, it is desirable to implement the above scheme and automate the computation of highly robust efficient points. By following the proposed strategy, highly robust efficient solutions are computed in the least computationally expensive manner currently available depending on the characteristics of the UMOLP being solved.

In addition, the only proposed MOLP whose efficient set is a lower bound set on the highly robust efficient set results from the fact that the closed cone of improving directions of the UMOLP is acute. However, as the assumption that this cone is acute limits the types of UMOLPs the bound set addresses, it is also desirable to relax this assumption.

Moreover, developing other upper or lower bound sets on the highly robust efficient set of a UMOLP is worthwhile to pursue. Foremost, the bound sets provide valuable information regarding the highly robust efficient set and may be used, in the case of lower bound sets for example, to prove that the highly robust efficient set is nonempty. Additionally, new recognition and existence conditions for highly robust efficient solutions to UMOLPs may be derived since these conditions may be obtained

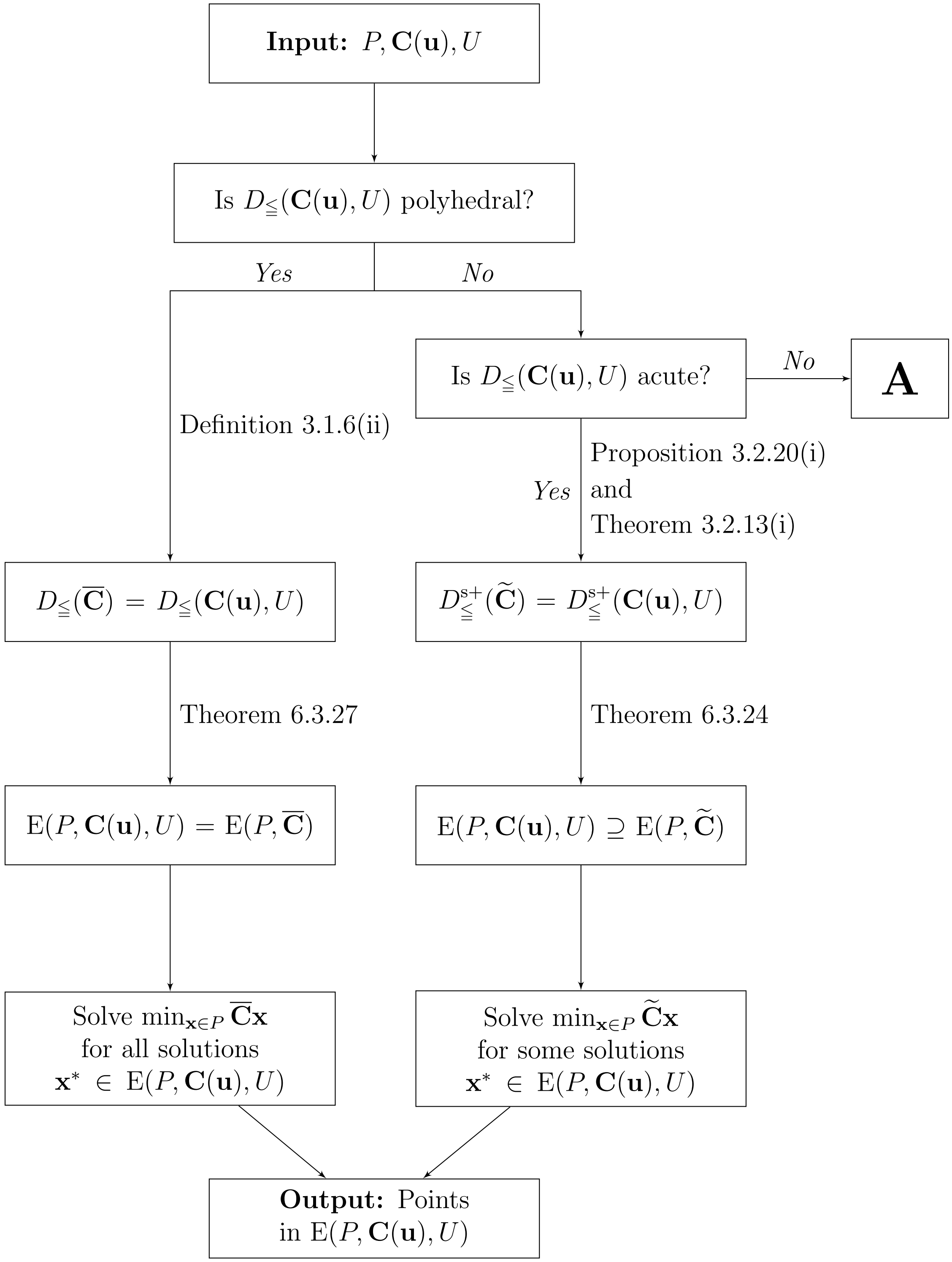


Figure 7.1: Flow-chart scheme for the computation of highly robust efficient solutions

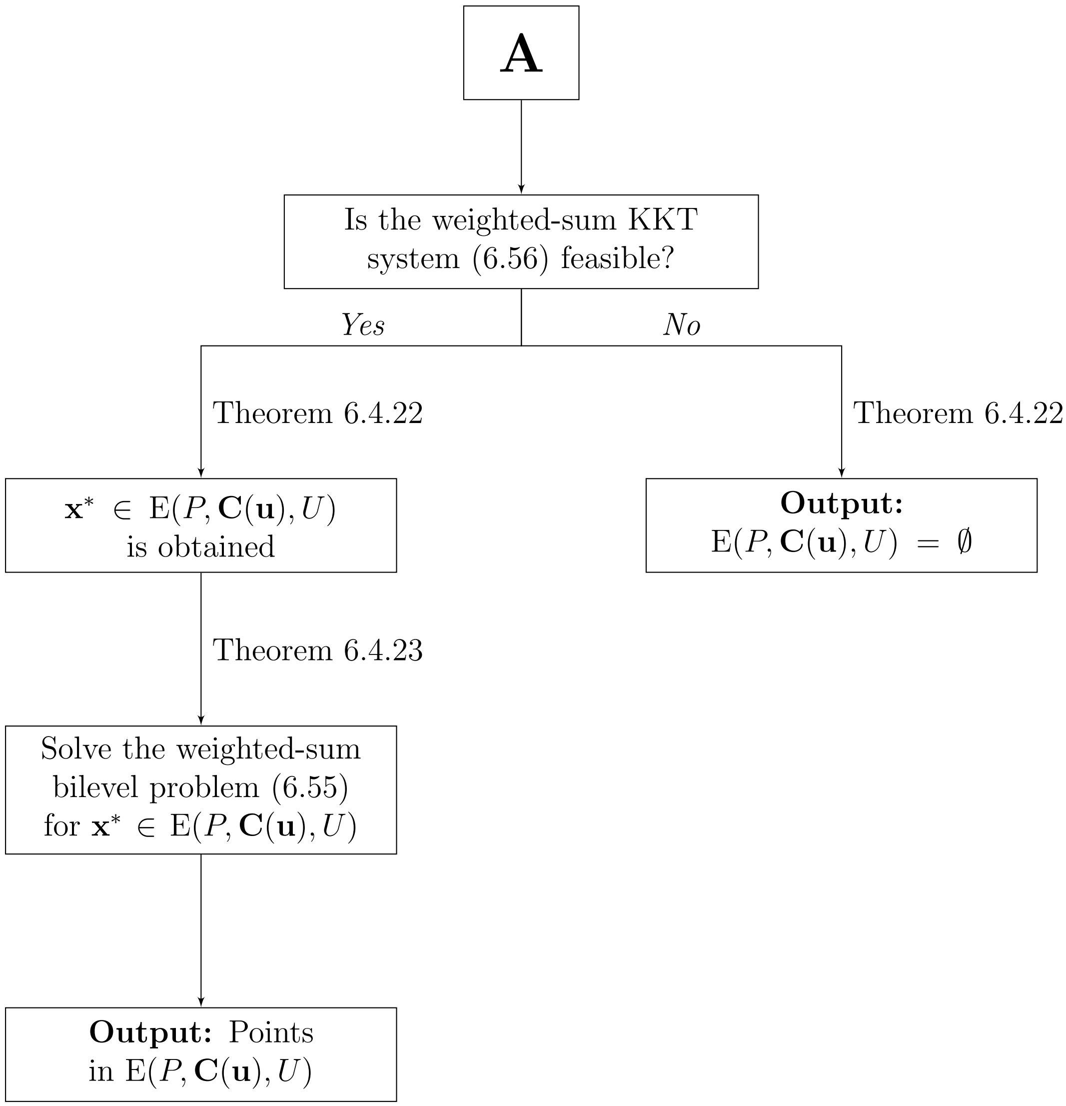


Figure 7.2: Continuation of the flow-chart scheme for the computation of highly robust efficient solutions

by relating Ecker and Kouada's or Benson's method to the MOLPs corresponding to the bound sets.

Finally, our work reveals that pursuing other means to identify highly robust efficient solutions is still advantageous. In particular, the computational expense of solving the bilevel problem may be prohibitive in some applications since each scenario dramatically increases the size of the problem. Further, even though our two-step procedure is able to accommodate the situation that the feasible or highly robust efficient set is unbounded, it does not directly identify that the highly robust efficient set is unbounded or compute highly robust efficient direction(s).

Appendices

Appendix A USOP Reformulation

A single-objective linear program (LP) is given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geqq \mathbf{b}, \end{aligned} \tag{1}$$

where $\mathbf{c} \in \mathbb{R}^n$ is the cost vector or vector of objective coefficients, $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the constraint matrix, and $\mathbf{b} \in \mathbb{R}^m$ is the vector of right-hand side (RHS) values. Considering uncertainty in any of the data $\mathbf{c}, \mathbf{A}$, or $\mathbf{b}$, we obtain an uncertain single-objective LP (USOLP). Although uncertainty may exist in any of the problem data, it may be assumed to be in the left-hand side (LHS) of the constraints without loss of generality (WLOG). In order to demonstrate this fact (first described by Ben-Tal and Nemirovski [8]), we consider USOLP (1) with uncertainty in either the cost vector $\mathbf{c}$ or the RHS vector $\mathbf{b}$.

If the uncertainty is in the objective coefficients $\mathbf{c}$, we use an auxiliary variable to transform USOLP (1) as follows:

$$\begin{aligned} \min_{\mathbf{x},\vartheta} \quad & \vartheta \\ \text{s.t.} \quad & \mathbf{c}^T\mathbf{x} \leq \vartheta \\ & \mathbf{A}\mathbf{x} \geqq \mathbf{b}. \end{aligned}$$

Now, the uncertainty is only in the LHS. On the other hand, if the uncertainty is in the vector of RHS values $\mathbf{b}$, we perform the following transformations:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geqq \mathbf{b} \end{aligned} \iff \begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} + 0 \cdot x_{n+1} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} - \mathbf{b} \cdot x_{n+1} \geqq \mathbf{0} \\ & \mathbf{0}^T\mathbf{x} + 1 \cdot x_{n+1} = 1 \end{aligned} \iff \begin{aligned} \min_{\mathbf{z}} \quad & (\mathbf{c}')^T\mathbf{z} \\ \text{s.t.} \quad & \mathbf{A}'\mathbf{z} \geqq \mathbf{0} \\ & \mathbf{a}_{m+1}^T\mathbf{z} = 1, \end{aligned}$$

where $(\mathbf{c}')^T = \begin{bmatrix}\mathbf{c}^T & 0\end{bmatrix} \in \mathbb{R}^{n+1}, \mathbf{z}^T = \begin{bmatrix}\mathbf{x}^T & x_{n+1}\end{bmatrix} \in \mathbb{R}^{n+1}, \mathbf{A}' = \begin{bmatrix}\mathbf{A} & -\mathbf{b}\end{bmatrix} \in \mathbb{R}^{m\times(n+1)}$, and $\mathbf{a}_{m+1}^T = \begin{bmatrix}\mathbf{0}^T & 1\end{bmatrix} \in \mathbb{R}^{n+1}$. Again, the uncertainty is now only in the LHS. Therefore, WLOG, the uncertainty may always be restricted to the LHS of the constraints.

Appendix B Computing $\widetilde{\mathbf{M}}^T$ in SageMath

Recall Example 3.2.11. The two cones $K_{\leqq}(\mathbf{M}_1)$ and $K_{\leqq}(\mathbf{M}_2)$, as well as their polars, are shown in Figure 3.5.

The goal in this example is to compute the matrix $\widetilde{\mathbf{M}}^T$ in $\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\} = \left[K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)\right]^+$ from Proposition 3.2.9(i). In order to do so, the following steps using SageMath's [132] `polyhedron base class` are used.

First, `polyhedron` objects corresponding to $K^+_{\leqq}(\mathbf{M}_1)$ and $K^+_{\leqq}(\mathbf{M}_2)$, whose generator form representations are given by Proposition 3.2.7(i), are created.

```
K1Polar = Polyhedron(rays = [[2, -1], [-1, 3]]); K2Polar = Polyhedron(rays = [[-4, ...
    -1], [1, 2]]);
```

Second, the intersection $K^+_{\leqq}(\mathbf{M}_1) \cap K^+_{\leqq}(\mathbf{M}_2) = \left[K_{\leqq}(\mathbf{M}_1) \cup K_{\leqq}(\mathbf{M}_2)\right]^+$ from Proposition 3.2.8(i) is computed.

```
UnionPolar = K1Polar.intersection(K2Polar);
```

Finally, the generator form representation of the intersection is obtained.

```
UnionPolar.Vrepresentation();
```

The resulting output from SageMath

```
(A vertex at (0, 0),
 A ray in the direction (1, 2),
 A ray in the direction (-1, 3))
```

yields that the columns of $-\widetilde{\mathbf{M}}^T$ are the generators or rays $\begin{bmatrix}1 & 2\end{bmatrix}^T$ and $\begin{bmatrix}-1 & 3\end{bmatrix}^T$.

Appendix C Application Problem AMPL Code

In order to compute highly robust efficient solutions to the bank balance-sheet management UMOLP (6.63), the discretized multiparametric bilevel approach described in Section 6.4.4 is used. As discussed in Section 6.4.5, the nominal index is chosen to be $\bar{\imath} = 1$, which corresponds to the scenario $\mathbf{u}^1 = (\mathbf{u}_1^1, \mathbf{u}_2^1, \mathbf{u}_3^1)$ yielding the deterministic model from Eatman and Sealey [40], and the parameter space Λ_1 is discretized by using a mesh with an interval step size of 0.00625. Since each value of the parameter $\boldsymbol{\lambda}_1 \in \Lambda_1$ is associated with an instance of the bilevel problem (6.55), the chosen discretization results in a collection of 12,720 problems to be solved.

A sample of the AMPL [53] files needed to solve the collection of problems just described is provided below and includes `.run`, `.mod`, and `.dat` files. Note that the sample files account for only four scenarios instead of the complete set of 27. Accordingly, the data file corresponds to four scenarios, specifically $(\mathbf{u}_1^1, \mathbf{u}_2^1, \mathbf{u}_3^1)$, $(\mathbf{u}_1^1, \mathbf{u}_2^3, \mathbf{u}_3^1)$, $(\mathbf{u}_1^3, \mathbf{u}_2^1, \mathbf{u}_3^1)$, $(\mathbf{u}_1^3, \mathbf{u}_2^3, \mathbf{u}_3^1)$, where $\mathbf{u}_k^j$ is as defined in Table 6.2.

```
## Run file
## BilevelApplication_0-00625_Mesh.run

model BilevelApplication_FourScenarios_Scenario1_Nominal.mod;
data BilevelApplication_FourScenarios.dat;

option solver_msg 0;    #suppresses solver messages
option send_statuses 0; #status information about variables returned by the ...
    previous solve will not be used as starting point for the next solve

set INDICES;                            #set of indices to count over
set XSOLUTIONS dimen 16 default {};     #set of x solutions
set LSOLUTIONS dimen 3 default {};      #set of lambda_1 solutions
set XTEST dimen 16 default {};

```

```
param numberofweights;                  #number of lambda_1 weights
param counter;
param infeasible_counter;

let numberofweights := 159;
let INDICES := 1..numberofweights by 1;
let l1[1] := 0;                         #where l1 corresponds to lambda_1
let counter := 0;
let infeasible_counter := 0;

for {i in INDICES} {
    let l1[1] := l1[1] + 0.00625;
    let l1[2] := 0;

    for {j in INDICES} {
        let l1[2] := l1[2] + 0.00625;
        let l1[3] := 1 - l1[1] - l1[2];

        if l1[3] > 0 then {
            solve;
            let counter := counter + 1;

            if match (solve_message, "infeasible") > 0 then {
                let infeasible_counter := infeasible_counter + 1;
                printf "\n\n--- infeasible at %d and %d ---\n\n", i, j;
            }
            else {
                let XTEST := XSOLUTIONS;

                for {k in COLUMNS} {
                    let x[k] := round(x[k],8);
                }

                let XSOLUTIONS := XSOLUTIONS union ...
                    {(x[1],x[2],x[3],x[4],x[5],x[6],x[7],x[8],x[9],x[10],x[11], ...
                    x[12],x[13],x[14],x[15],x[16])};

                if XSOLUTIONS not within XTEST then {
                    let LSOLUTIONS := LSOLUTIONS union ...
```

```
                        {(round(l1[1],8),round(l1[2],8),round(l1[3],8))};
                }
            }
        }
    }
}

printf "\n\n Test 11 \n\n";
display XSOLUTIONS;
display LSOLUTIONS;
display counter;
display infeasible_counter;

## Model file
## BilevelApplication_FourScenarios_Scenario1_Nominal.mod

param n > 0;    #number of variables
param m > 0;    #number of constraints
param p > 0;    #number of objectives

set COLUMNS := 1..n;
set ROWS := 1..m;
set OBJECTIVES := 1..p;

param A {ROWS,COLUMNS};
param C1 {OBJECTIVES,COLUMNS};      #C(u^1)
param C2 {OBJECTIVES,COLUMNS};      #C(u^2)
param C3 {OBJECTIVES,COLUMNS};      #C(u^3)
param C4 {OBJECTIVES,COLUMNS};      #C(u^4)
param b {ROWS};
param epsilon {OBJECTIVES};         #used to represent strict inequalities
param l1 {OBJECTIVES};              #lambda_1

var x {COLUMNS} >= 0;
var v1 {COLUMNS} >= 0;
var w1 {ROWS};
var l2 {OBJECTIVES};
var v2 {COLUMNS} >= 0;
var w2 {ROWS};
```

```
var l3 {OBJECTIVES};
var v3 {COLUMNS} >= 0;
var w3 {ROWS};
var l4 {OBJECTIVES};
var v4 {COLUMNS} >= 0;
var w4 {ROWS};

minimize Obj: sum {i in OBJECTIVES,j in COLUMNS} l1[i]*C1[i,j]*x[j];

subject to Feasibility {i in ROWS}: sum {j in COLUMNS} A[i,j]*x[j] == b[i];
subject to Gradient1 {i in COLUMNS}: sum {j in OBJECTIVES} C1[j,i]*l1[j] + sum {j ...
    in ROWS} A[j,i]*w1[j] - v1[i] == 0;
subject to Gradient2 {i in COLUMNS}: sum {j in OBJECTIVES} C2[j,i]*l2[j] + sum {j ...
    in ROWS} A[j,i]*w2[j] - v2[i] == 0;
subject to Gradient3 {i in COLUMNS}: sum {j in OBJECTIVES} C3[j,i]*l3[j] + sum {j ...
    in ROWS} A[j,i]*w3[j] - v3[i] == 0;
subject to Gradient4 {i in COLUMNS}: sum {j in OBJECTIVES} C4[j,i]*l4[j] + sum {j ...
    in ROWS} A[j,i]*w4[j] - v4[i] == 0;
subject to CompSlack1: sum {j in COLUMNS} v1[j]*x[j] == 0;
subject to CompSlack2: sum {j in COLUMNS} v2[j]*x[j] == 0;
subject to CompSlack3: sum {j in COLUMNS} v3[j]*x[j] == 0;
subject to CompSlack4: sum {j in COLUMNS} v4[j]*x[j] == 0;
subject to Positive2 {i in OBJECTIVES}: l2[i] >= epsilon[i];
subject to Positive3 {i in OBJECTIVES}: l3[i] >= epsilon[i];
subject to Positive4 {i in OBJECTIVES}: l4[i] >= epsilon[i];

## Data file
## BilevelApplication_FourScenarios.dat

param n := 27;
param m := 12;
param p := 3;

param A:
        1   2   3   4   5   6   7   8   9   10  11  12  ...
            13  14  15  16  17  18  19  20  21  22  23  24  25  26  27:=
    1   1   1   1   1   1   1   1   1   1   1   -1  -1  ...
        -1  0   0   0   0   0   0   0   0   0   0   0   0   0   0
    2   1   0   0   0   0   0   0   0   0   0   0   0   ...
```

```
      -0.04  0   0   0   -1  0   0   0   0   0   0   0   0   0   0
    3 1  0.995  0.995  0.96  0   0      0   0   0   0   -1      -1      ...
      -1     1   0   0   0   -1  0   0   0   0   0   0   0   0   0
    4 1  0.995  0.995  0.96  0.9 0      0   0   0   0   -1      -1      ...
      -1     0   1   0   0   0   -1  0   0   0   0   0   0   0   0
    5 1  0.995  0.995  0.96  0.9 0.85   0   0   0   0   -1      -1      ...
      -1     0   0   1   0   0   0   -1  0   0   0   0   0   0   0
    6 0  1      0      0     0   0      0   0   0   0   -0.4    0       ...
      0      0   0   0   0   0   0   0   -1  0   0   0   0   0   0
    7 0  0      1      0     0   0      0   0   0   0   0       0       ...
      -0.04  0   0   0   0   0   0   0   0   -1  0   0   0   0   0
    8 0  0      0      0     0   0      0   0   0   0   1       1       ...
      1      0   0   0   0   0   0   0   0   0   1   0   0   0   0
    9 0  0      0      0     0   0      0   0   0   0   1       0       ...
      0      0   0   0   0   0   0   0   0   0   0   1   0   0   0
   10 0  0      0      0     0   0      0   0   0   0   0       1       ...
      0      0   0   0   0   0   0   0   0   0   0   0   1   0   0
   11 0  0      0      0     0   0      0   0   0   0   0       0       ...
      1      0   0   0   0   0   0   0   0   0   0   0   0   1   0
   12 0  0      0      0     0   0      0   1   0   0   -0.25   -0.25   ...
      -0.25  0   0   0   0   0   0   0   0   0   0   0   0   0   -1;

param C1:
      1  2      3      4      5      6      7      8      9      10  ...
             11     12     13     14     15     16     17  18  19  20  ...
         21  22  23  24  25  26  27:=
    1 0  -0.052  -0.053  -0.056  -0.058  -0.059  -0.062  -0.076  -0.071  ...
      -0.095  0.052  0.05  0.055  0      0      0      0   0   0   0   ...
      0   0   0   0   0   0   0
    2 0  0.001   0.001   0.008   0.008   0.012   0.02    0.02    0.02    ...
      0.02   0      0      0      0.013  0.008  0.019  0   0   0   0   ...
      0   0   0   0   0   0   0
    3 0  0      0      0      0      0      0.2    0.2    0.2    0.2 ...
         0      0      0      0      0      0      0   0   0   0   0   ...
      0   0   0   0   0   0;

param C2:
      1  2      3      4      5      6      7      8      9      10  ...
             11     12     13     14     15     16     17  18  19  20  ...
```

```
             21  22  23  24  25  26  27:=
    1   0   −0.052  −0.053  −0.056  −0.058  −0.059  −0.062  −0.076  −0.071  ...
        −0.095  0.052   0.05    0.055   0       0       0       0   0   0   0   ...
        0   0   0   0   0   0   0
    2   0   0.003   0.003   0.012   0.012   0.018   0.03    0.03    0.03    ...
        0.03    0       0       0       0.019   0.012   0.029   0   0   0   0   ...
        0   0   0   0   0   0   0
    3   0   0       0       0       0       0       0.2     0.2     0.2     0.2 ...
            0       0       0       0       0       0       0   0   0   0   0   ...
        0   0   0   0   0   0;

param C3:
        1   2       3       4       5       6       7       8       9       10  ...
                11      12      13      14      15      16      17  18  19  20  ...
            21  22  23  24  25  26  27:=
    1   0   −0.042  −0.043  −0.046  −0.048  −0.049  −0.052  −0.066  −0.061  ...
        −0.085  0.072   0.07    0.075   0       0       0       0   0   0   0   ...
        0   0   0   0   0   0   0
    2   0   0.001   0.001   0.008   0.008   0.012   0.02    0.02    0.02    ...
        0.02    0       0       0       0.013   0.008   0.019   0   0   0   0   ...
        0   0   0   0   0   0   0
    3   0   0       0       0       0       0       0.2     0.2     0.2     0.2 ...
            0       0       0       0       0       0       0   0   0   0   0   ...
        0   0   0   0   0   0;

param C4:
        1   2       3       4       5       6       7       8       9       10  ...
                11      12      13      14      15      16      17  18  19  20  ...
            21  22  23  24  25  26  27:=
    1   0   −0.042  −0.043  −0.046  −0.048  −0.049  −0.052  −0.066  −0.061  ...
        −0.085  0.072   0.07    0.075   0       0       0       0   0   0   0   ...
        0   0   0   0   0   0   0
    2   0   0.003   0.003   0.012   0.012   0.018   0.03    0.03    0.03    ...
        0.03    0       0       0       0.019   0.012   0.029   0   0   0   0   ...
        0   0   0   0   0   0   0
    3   0   0       0       0       0       0       0.2     0.2     0.2     0.2 ...
            0       0       0       0       0       0       0   0   0   0   0   ...
        0   0   0   0   0   0;

```

```
param b :=
     1   12.2
     2   6.4
     3   22.832
     4   20.762
     5   13.877
     6   0
     7   2.4
     8   6.5
     9   3.9
     10  3.9
     11  3.9
     12  1.45;

param epsilon :=
     1   0.0001
     2   0.0001
     3   0.0001;
```

Bibliography

[1] N. Adelgren and M.M. Wiecek. A two-phase algorithm for the multiparametric linear complementarity problem. *European Journal of Operational Research*, 254(3):715–738, 2016.

[2] P.D. Barba. *Multiobjective Shape Design in Electricity and Magnetism.* Springer, New York, 2010.

[3] M.S. Bazaraa, J.J. Jarvis, and H.D. Sherali. *Linear Programming and Network Flows.* John Wiley & Sons, fourth edition, 2010.

[4] A. Bemporad, K. Fukuda, and F.D. Torrisi. Convexity recognition of the union of polyhedra. *Computational Geometry*, 18(3):141–154, 2001.

[5] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust Optimization.* Princeton University Press, 2009.

[6] A. Ben-Tal and A. Nemirovski. Robust truss topology design via semidefinite programming. *SIAM Journal on Optimization*, 7(4):991–1016, 1997.

[7] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

[8] A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25:1–13, 1999.

[9] A. Ben-Tal and A. Nemirovski. Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming*, 88(3):411–424, 2000.

[10] H.P. Benson. Existence of efficient solutions for vector maximization problems. *Journal of Optimization Theory and Applications*, 26(4):569–580, 1978.

[11] H.P. Benson. Finding an initial efficient extreme point for a linear multiple objective program. *Journal of the Operational Research Society*, pages 495–498, 1981.

[12] H.P. Benson. Multiple objective linear programming with parametric criteria coefficients. *Management Science*, 31(4):461–474, 1985.

[13] D. Bertsimas, D.B. Brown, and C. Caramanis. Theory and applications of robust optimization. *SIAM Review*, 53(3):464–501, 2011.

[14] D. Bertsimas and A. Thiele. A robust optimization approach to inventory theory. *Operations Research*, 54(1):150–168, 2006.

[15] D. Bertsimas and J.N. Tsitsiklis. *Introduction to Linear Optimization*. Optimization and Computation Series. Athena Scientific, Belmont, Massachusetts, 1997.

[16] B. Besharati and S. Azarm. Worst case deterministic feasibility and multiobjective robustness measures for engineering design optimisation. *International Journal of Reliability and Safety*, 1(1-2):40–58, 2006.

[17] H. Beyer and B. Sendhoff. Robust optimization–a comprehensive survey. *Computer Methods in Applied Mechanics and Engineering*, 196(33):3190–3218, 2007.

[18] G.R. Bitran. Linear multiple objective problems with interval coefficients. *Management Science*, 26(7):694–706, 1980.

[19] R. Bokrantz and A. Fredriksson. Necessary and sufficient conditions for Pareto efficiency in robust multiobjective optimization. *European Journal of Operational Research*, 262(2):682–692, 2017.

[20] J. Borwein and A.S. Lewis. *Convex Analysis and Nonlinear Optimization: Theory and Examples*. CMS Books in Mathematics. Springer Science & Business Media, second edition, 2006.

[21] S.P. Boyd, S. Kim, D.D. Patil, and M.A. Horowitz. Digital circuit optimization via geometric programming. *Operations Research*, 53(6):899–932, 2005.

[22] R. Bronson. *Schaum's Outline of Theory and Problems of Matrix Operations*. McGraw-Hill, 1989.

[23] D.I. Candea, A.C. Hax, and U.S. Karmarkar. Economic and social evaluation of capital investment decisions–an application. In A.C. Hax, editor, *Studies in Operations Management*, pages 205–252. North-Holland Publishing Company, 1978.

[24] N.L. Carothers. *Real Analysis*. Cambridge University Press, 2000.

[25] S. Chanas and D. Kuchta. Multiobjective programming in optimization of interval objective functions–a generalized approach. *European Journal of Operational Research*, 94(3):594–598, 1996.

[26] W. Chen, J. Unkelbach, A. Trofimov, T. Madden, H. Kooy, T. Bortfeld, and D. Craft. Including robustness in multi-criteria optimization for intensity-modulated proton therapy. *Physics in Medicine and Biology*, 57(3):591–608, 2012.

[27] J.L. Cohon and D.H. Marks. A review and evaluation of multiobjective programing techniques. *Water Resources Research*, 11(2):208–220, 1975.

[28] O. Crespo, J.E. Bergez, and F. Garcia. Multiobjective optimization subject to uncertainty: Application to irrigation strategy management. *Computers and Electronics in Agriculture*, 74(1):145–154, 2010.

[29] B. Dandurand. Enumeration of vertices and extreme rays of polyhedra: An implementation of the double description method for the COIN-OR C++ library, 2008. Master's project, Clemson University, Clemson, SC.

[30] J. Dattorro. *Convex Optimizatoin & Euclidean Distance Geometry*. Meboo Publishing USA, Palo Alto, CA, 2015.

[31] K. Deb and H. Gupta. Introducing robustness in multi-objective optimization. *Evolutionary Computation*, 14(4):463–494, 2006.

[32] M. Dellnitz and K. Witting. Computation of robust Pareto points. *International Journal of Computing Science and Mathematics*, 2(3):243–266, 2009.

[33] C.P. Dobler. A matrix approach to finding a set of generators and finding the polar (dual) of a class of polyhedral cones. *SIAM Journal on Matrix Analysis and Applications*, 15(3):796–803, 1994.

[34] L.F. Domínguez, D.A. Narciso, and E.N. Pistikopoulos. Recent advances in multiparametric nonlinear programming. *Computers & Chemical Engineering*, 34(5):707–716, 2010.

[35] E.K. Doolittle, G.M. Dranichak, K. Muir, and M.M. Wiecek. A note on robustness of the min-max solution to multi-objective linear programs. *International Journal of Multicriteria Decision Making*, 6(4):343–365, 2016.

[36] E.K. Doolittle, H.L.M. Kerivin, and M.M. Wiecek. Operators preserving efficiency for robust multiobjective problems. Working paper, Clemson University, Clemson, SC, 2016.

[37] E.K. Doolittle, H.L.M. Kerivin, and M.M. Wiecek. A robust multiobjective optimization problem with application to Internet routing. *Annals of Operations Research*, 2017. DOI: `https://doi.org/10.1007/s10479-017-2751-5`.

[38] G.M. Dranichak and M.M. Wiecek. On highly robust efficient solutions to uncertain multiobjective linear programs. Technical report TR2017–12–gd.mw, Department of Mathematical Sciences, Clemson University, Clemson, SC, 2017.

[39] G.M. Dranichak and M.M. Wiecek. On computing highly robust efficient solutions. Technical report TR2018–2–gd.mw, Department of Mathematical Sciences, Clemson University, Clemson, SC, 2018.

[40] J.L. Eatman and C.W. Sealey, Jr. A multiobjective linear programming model for commercial bank balance sheet management. *Journal of Bank Research*, 9:227–236, 1979.

[41] R.T. Eckenrode. Weighting multiple criteria. *Management Science*, 12(3):180–192, 1965.

[42] J.G. Ecker and I.A. Kouada. Finding efficient points for linear multiple objective programs. *Mathematical Programming*, 8(1):375–377, 1975.

[43] Ulrich Eckhardt. Theorems on the dimension of convex sets. *Linear Algebra and its Applications*, 12(1):63–76, 1975.

[44] M. Ehrgott. *Multicriteria Optimization*. Springer, second edition, 2005.

[45] M. Ehrgott, J. Ide, and A. Schöbel. Minmax robustness for multi-objective optimization problems. *European Journal of Operational Research*, 239:17–31, 2014.

[46] G. Eichfelder and J. Jahn. Vector optimization problems and their solution concepts. In Q.H. Ansari and J.C. Yao, editors, *Recent Developments in Vector Optimization*, volume 1 of *Vector Optimzation*, pages 1–27. Springer, Berlin, 2012.

[47] G. Eichfelder, C. Krüger, and A. Schöbel. Decision uncertainty in multiobjective optimization. *Journal of Global Optimization*, 69(2):485–510, 2017.

[48] A. Engau. Definition and characterization of Geoffrion proper efficiency for real vector optimization with infinitely many criteria. *Journal of Optimization Theory and Applications*, 165:439–457, 2015.

[49] A. Engau and M.M. Wiecek. 2D decision making for multi-criteria design optimization. *Structural and Multidisciplinary Optimization*, 34(4):301–315, 2007.

[50] A. Engau and M.M. Wiecek. Interactive coordination of objective decompositions in multiobjective programming. *Management Science*, 54(7):1350–1363, 2008.

[51] A.V. Fiacco. *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*. Elsevier, 1983.

[52] J. Fliege and R. Werner. Robust multiobjective optimization & applications in portfolio optimization. *European Journal of Operational Research*, 234:422–433, 2014.

[53] R. Fourer, D.M. Gay, and B.W. Kernighan. *AMPL: A Modeling Language for Mathematical Programming*. Scientific Press. Thomson/Brooks/Cole, second edition, 2003.

[54] T. Gal and H.J. Greenberg, editors. *Advances in Sensitivity Analysis and Parametric Programming*, volume 6 of *International Series in Operations Research & Management Science*. Springer, New York, 1997.

[55] T. Gal and J. Nedoma. Multiparametric linear programming. *Management Science*, 18(7):406–422, 1972.

[56] D. Gale. *The Theory of Linear Economic Models*. University of Chicago Press, 1960.

[57] M.A. Goberna, V. Jeyakumar, G. Li, and J. Vicente-Pérez. Robust solutions to multiobjective linear semi-infinite programs under constraint data uncertainty. *SIAM Journal of Optimization*, 24(3):1402–1419, 2014.

[58] M.A. Goberna, V. Jeyakumar, G. Li, and J. Vicente-Pérez. Robust solutions to multi-objective linear programs with uncertain data. *European Journal of Operational Research*, 242:730–743, 2015.

[59] M.A. Goberna and M.A. Lopez. Linear semi-infinite programming theory: An updated survey. *European Journal of Operational Research*, 143(2):390–405, 2002.

[60] B.L. Gorissen and D. Den Hertog. Approximating the Pareto set of multiobjective linear programs via robust optimization. *Operations Research Letters*, 40(5):319–324, 2012.

[61] S. Greco, M. Ehrgott, and J.R. Figueira. *Multiple Criteria Decision Analysis: State of the Art Surveys*. International Series in Operations Research & Management Science. Springer, New York, second edition, 2016.

[62] R. Greer. *Trees and Hills: Methodology for Maximizing Functions of Systems of Linear Relations*, volume 22 of *Annals of Discrete Mathematics*. Elsevier Science Publishers, Amsterdam, The Netherlands, 1984.

[63] O. Güler. *Foundations of Optimization*. Springer, 2010.

[64] S. Gunawan and S. Azarm. Multi-objective robust optimization using a sensitivity region concept. *Structural and Multidisciplinary Optimization*, 29(1):50–60, 2005.

[65] R. Hartley. On cone-efficiency, cone-convexity and cone-compactness. *SIAM Journal on Applied Mathematics*, 34(2):211–222, 1978.

[66] M. Herceg, M. Kvasnica, C.N. Jones, and M. Morari. Multi-parametric toolbox 3.0. In *European Control Conference*, pages 502–510, 2013. URL: `http://control.ee.ethz.ch/~mpt`, last accessed March 28, 2016.

[67] S. Hertel, M. Mäntylä, K. Mehlhorn, and J. Nievergelt. Space sweep solves intersection of convex polyhedra. *Acta Informatica*, 21(5):501–519, 1984.

[68] R. Hites, Y. De Smet, N. Risse, M. Salazar-Neumann, and P. Vincke. About the applicability of MCDA to some robustness problems. *European Journal of Operational Research*, 174(1):322–332, 2006.

[69] M. Hladík. On necessarily efficient solutions in interval multiobjective linear programming. In *Proceedings of the 25th Mini-EURO Conference on Uncertainty and Robustness in Planning and Decision Making*, pages 1–10, 2010.

[70] M. Hladík. Complexity of necessary efficiency in interval linear programming and multiobjective linear programming. *Optimization Letters*, 6(5):893–899, 2012.

[71] R. Horst, N.V. Thoai, Y. Yamamoto, and D. Zenke. On optimization over the efficient set in linear multicriteria programming. *Journal of Optimization Theory and Applications*, 134(3):433–443, 2007.

[72] K. Hsiung, S. Kim, and S.P. Boyd. Power control in lognormal fading wireless channels with uptime probability specifications via robust geometric programming. In *American Control Conference*, pages 3955–3959. IEEE, 2005.

[73] J. Hu and S. Mehrotra. Robust and stochastically weighted multiobjective optimization models and reformulations. *Operations Research*, 60(4):936–953, 2012.

[74] B.J. Hunt, M.M. Wiecek, and C.S. Hughes. Relative importance of criteria in multiobjective programming: A cone-based approach. *European Journal of Operational Research*, 207(2):936–945, 2010.

[75] C. Hwang, Y. Lai, and M. Ko. ISGP-II for multiobjective optimization with imprecise objective coefficients. *Computers & Operations Research*, 20(5):503–514, 1993.

[76] D.A. Iancu and N. Trichakis. Pareto efficiency in robust optimization. *Management Science*, 60(1):130–147, 2013.

[77] M. Ida. Generation of efficient solutions for multiobjective linear programming with interval coefficients. In *SICE'96. Proceedings of the 35th SICE Annual Conference. International Session Papers*, pages 1041–1044. IEEE, 1996.

[78] M. Ida. Portfolio selection problem with interval coefficients. *Applied Mathematics Letters*, 16(5):709–713, 2003.

[79] J. Ide. *Concepts of Robustness for Uncertain Multi-Objective Optimization*. PhD thesis, Georg-August-Universität Göttingen, Göttingen, Germany, 2014.

[80] J. Ide and E. Köbis. Concepts of efficiency for uncertain multi-objective optimization problems based on set order relations. *Mathematical Methods of Operations Research*, 80(1):99–127, 2014.

[81] J. Ide, E. Köbis, D. Kuroiwa, A. Schöbel, and C. Tammer. The relationship between multi-objective robustness concepts and set-valued optimization. *Fixed Point Theory and Applications*, 2014(1), 2014.

[82] J. Ide and A. Schöbel. Robustness for uncertain multi-objective optimization: A survey and analysis of different concepts. *OR Spectrum*, 38(1):235–271, 2015.

[83] M. Inuiguchi and Y. Kume. Dominance relations as bases for constructing solution concepts in linear programming with multiple interval objective functions. *Bulletin of the University of Osaka Prefecture*, 40(2):275–292, 1991.

[84] M. Inuiguchi and Y. Kume. Efficient solutions versus nondominated solutions in linear programming with multiple interval objective functions. *Bulletin of the University of Osaka Prefecture*, 40(2):293–298, 1992.

[85] M. Inuiguchi and M. Sakawa. Possible and necessary efficiency in possibilistic multiobjective linear programming problems and possible efficiency test. *Fuzzy Sets and Systems*, 78(2):231–241, 1996.

[86] H. Ishibuchi and H. Tanaka. Multiobjective programming in optimization of the interval objective function. *European Journal of Operational Research*, 48(2):219–225, 1990.

[87] H. Ishibuchi, N. Tsukamoto, and Y. Nojima. Evolutionary many-objective optimization: A short review. In *2008 IEEE Congress on Evolutionary Computation*, pages 2424–2431. IEEE, 2008.

[88] R.L. Keeney and H. Raiffa. *Decisions with Multiple Objectives: Preferences and Value Trade-offs*. John Wiley & Sons, New York, 1976.

[89] K. Klamroth, E. Köbis, A. Schöbel, and C. Tammer. A unified approach for different concepts of robustness and stochastic programming via non-linear scalarizing functionals. *Optimization*, 62(5):649–671, 2013.

[90] O.N. Klimova and V.D. Noghin. Using interdependent information on the relative importance of criteria in decision making. *Computational Mathematics and Mathematical Physics*, 46(12):2080–2091, 2006.

[91] E. Köbis. On robust optimization. *Journal of Optimization Theory and Applications*, 167(3):969–984, 2015.

[92] E. Köbis and C. Tammer. Relations between strictly robust optimization problems and a nonlinear scalarization method. In *International Conference of Numerical Analysis and Applied Mathematics*, volume 1479, pages 2371–2374, Kos, Greece, 2012. AIP Publishing.

[93] P. Kouvelis and G. Yu. *Robust Discrete Optimization and Its Applications*. Springer, 1997.

[94] K. Kuhn, A. Raith, M. Schmidt, and A. Schöbel. Bicriteria robust optimisation. Technical report, Georg-August-Universität Göttingen, Göttingen, Germany, 2013.

[95] K. Kuhn, A. Raith, M. Schmidt, and A. Schöbel. Bi-objective robust optimisation. *European Journal of Operational Research*, 252(2):418–431, 2016.

[96] D. Kuroiwa and G.M. Lee. On robust multiobjective optimization. *Vietnam Journal of Mathematics*, 40(2):305–317, 2012.

[97] S. Lang. *Linear Algebra*. Undergraduate Texts in Mathematics. Springer, New York, third edition, 2004.

[98] A.S. Lewis. Robust regularization. Technical report, Cornell University, Ithaca, NY, 2002.

[99] A.S. Lewis and C.H.J. Pang. Lipschitz behavior of the robust regularization. *SIAM Journal on Control and Optimization*, 48(5):3080–3104, 2010.

[100] M. Li and S. Azarm. Multiobjective collaborative robust optimization with interval uncertainty and interdisciplinary uncertainty propagation. *Journal of Mechanical Design*, 130(8):081402, 2008.

[101] M. Li, S. Azarm, and A. Boyars. A new deterministic approach using sensitivity region measures for multi-objective robust and feasibility robust design optimization. *Journal of Mechanical Design*, 128(4):874–883, 2006.

[102] B. Liu. *Theory and Practice of Uncertain Programming*. Springer, Berlin, 2002.

[103] M.S. Lobo and S.P. Boyd. The worst-case risk of a portfolio. Technical report, Stanford University, Stanford, CA, 2000.

[104] W.A. Lodwick and J. Kacprzyk, editors. *Fuzzy Optimization: Recent Advances and Applications*. Studies in Fuzziness and Soft Computing. Springer, 2010.

[105] D.T. Luc. *Multiobjective Linear Programming: An Introduction*. Springer International Publishing, 2016.

[106] D.E. Majewski. Robust bi-objective linear optimization. Masters thesis, Georg-August-Universität Göttingen, Göttingen, Germany, 2013.

[107] O.L. Mangasarian. *Nonlinear Programming*. Systems Science. McGraw-Hill, 1969.

[108] R.T. Marler and J.S. Arora. The weighted sum method for multi-objective optimization: New insights. *Structural and Multidisciplinary Optimization*, 41(6):853–862, 2010.

[109] T.H. Matheiss and D.S. Rubin. A survey and comparison of methods for finding all vertices of convex polyhedral sets. *Mathematics of Operations Research*, 5(2):167–185, 1980.

[110] G. Matheron. *Random Sets and Integral Geometry*. John Wiley & Sons, 1975.

[111] K. Miettinen, J. Hakanen, and D. Podkopaev. Interactive nonlinear multiobjective optimization methods. In S. Greco, M. Ehrgott, and J.R. Figueira, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, International Series in Operations Research & Management Science, pages 927–976. Springer, New York, second edition, 2016.

[112] A. Mostashari. *Collaborative Modeling and Decision-Making for Complex Energy Systems*. World Scientific, 2011.

[113] T.S. Motzkin, H. Raiffa, G.L. Thompson, and R.M. Thrall. The double description method. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games II*, volume 8 of *Annals of Mathematics Study 28*, pages 51–73. Princeton University Press, 1953.

[114] B.A. Murtaugh and M.A. Saunders. MINOS 5.51 user's guide. Technical report SOL 83–20R, Systems Optimization Laboratory, Department of Management Science and Engineering, Stanford University, Stanford, CA, 2003.

[115] V.D. Noghin. Relative importance of criteria: A quantitative approach. *Journal of Multi-Criteria Decision Analysis*, 6(6):355–363, 1997.

[116] C. Oliveira and C.H. Antunes. Multiple objective linear programming models with interval coefficients–an illustrated overview. *European Journal of Operational Research*, 181(3):1434–1463, 2007.

[117] C.D. Palma and J.D. Nelson. Bi-objective multi-period planning with uncertain weights: A robust optimization approach. *European Journal of Forest Research*, 129(6):1081–1091, 2010.

[118] M.J. Panik. *Fundamentals of Convex Analysis*. Kluwer Academic Publishers, 1993.

[119] V. Pareto. *Cours d'Economie Politique*. Lausanne, Switzerland, 1896.

[120] G.P. Rangaiah. *Multi-objective Optimization–Techniques and Applications in Chemical Engineering*. World Scientific, 2008.

[121] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[122] Y. Sawaragi, H. Nakayama, and T. Tanino. *Theory of Multiobjective Optimization*, volume 176 of *Mathematics in Science and Engineering*. Academic Press, 1985.

[123] J. Schneider and S. Kirkpatrick. *Stochastic Optimization*. Springer, 2006.

[124] A. Schrijver. *Theory of Linear and Integer Programming*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Chichester, England, 1998.

[125] D.P. Sigler. *Multi-Objective Optimization under Uncertainty*. PhD thesis, University of Colorado at Denver, Denver, CO, 2017.

[126] L.A. Steen and J.A. Seebach, Jr. *Counterexamples in Topology*. Holt, Rinehart and Winston, Inc., USA, 1970.

[127] R.E. Steuer. Algorithms for linear programming problems with interval objective function coefficients. *Mathematics of Operations Research*, 6(3):333–348, 1981.

[128] R.E. Steuer. *Multiple Criteria Optimization: Theory, Computation, and Applications*. John Wiley & Sons, 1986.

[129] R.R. Stoll. *Set Theory and Logic*. Dover, New York, 1979.

[130] C. Tammer and A. Göpfert. Theory of vector optimization. In M. Ehrgott and X. Gandibleux, editors, *Multiple Criteria Optimization: State of the Art Annotated Bibliographic Surveys*, volume 52 of *International Series in Operations Research & Management Science*, pages 1–70. Kluwer Academic Publishers, 2002.

[131] G.K. Tayi and P.A. Leonard. Bank balance-sheet management: An alternative multi-objective model. *Journal of the Operational Research Society*, 39(4):401–410, 1988.

[132] The Sage Developers. *SageMath, the Sage Mathematics Software System*, 2017. URL: `http://www.sagemath.org`.

[133] N.V. Thoai. Criteria and dimension reduction of linear multiple criteria optimization problems. *Journal of Global Optimization*, 52(3):499–508, 2012.

[134] L.N. Trefethen and D. Bau III. *Numerical Linear Algebra*. SIAM, 1997.

[135] R.H. Tütüncü and M. Koenig. Robust asset allocation. *Annals of Operations Research*, 132(1-4):157–187, 2004.

[136] F. Wang, S. Liu, and Y. Chai. Robust counterparts and robust efficient solutions in vector optimization under uncertainty. *Operations Research Letters*, 43(3):293–298, 2015.

[137] W. Wang, S. Caro, F. Bennis, R. Soto, and B. Crawford. Multi-objective robust optimization using a postoptimality sensitivity analysis technique: Application to a wind turbine design. *Journal of Mechanical Design*, 137(1):011403, 2015.

[138] E.U. Weber, J. Baron, and G. Loomes, editors. *Conflict and Tradeoffs in Decision Making*. Cambridge University Press, 2001.

[139] M.M. Wiecek. Advances in cone-based preference modeling for decision making with multiple criteria. *Decision Making in Manufacturing and Services*, 1:153–173, 2007.

[140] M.M. Wiecek and G.M. Dranichak. Robust multiobjective optimization for decision making under uncertainty and conflict. In A. Gupta, A. Capponi, and J.C. Smith, editors, *Optimization Challenges in Complex, Networked, and Risky Systems*, INFORMS TutORials in Operations Research, chapter 4, pages 84–114. The Sheridan Press, 2016.

[141] M.M. Wiecek, M. Ehrgott, and A. Engau. Continuous multiobjective programming. In S. Greco, M. Ehrgott, and J.R. Figueira, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, International Series in Operations Research & Management Science, pages 738–815. Springer, New York, second edition, 2016.

[142] K. Witting, S. Ober-Blöbaum, and M. Dellnitz. A variational approach to define robustness for parametric multiobjective optimization problems. *Journal of Global Optimization*, 57(2):331–345, 2013.

[143] P. Xidonas, G. Mavrotas, T. Krintas, J. Psarras, and C. Zopounidis. *Multicriteria Portfolio Management*. World Scientific, 2012.

[144] H. Yu and H.M. Liu. Robust multiple objective game theory. *Journal of Optimization Theory and Applications*, 159(1):272–280, 2013.

[145] P.L. Yu. Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. *Journal of Optimization Theory and Applications*, 14(3):319–377, 1974.