8-2008

A Novel Proof of the Heine-Borel Theorem

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Abstract. Every beginning real analysis student learns the classic Heine-Borel theorem, that the interval $[0, 1]$ is compact. In this article, we present a proof of this result that doesn't involve the standard techniques such as constructing a sequence and appealing to the completeness of the reals. We put a metric on the space of infinite binary sequences and prove that compactness of this space follows from a simple combinatorial lemma. The Heine-Borel theorem is an immediate corollary.

1. The Heine-Borel theorem

Think back to your first real analysis class. In the beginning, most of the definitions were fairly straightforward. Open and closed sets made sense, because of the common usage of open and closed intervals in previous math classes. It was a bit odd that open sets could also be closed, or that sets could be neither open nor closed. But this was "higher math," so you could let that one slide. Then came the definition of "compact." Completely out of nowhere. Why anyone would ever find themselves with an open cover, let alone try to extract a finite subcover, was beyond your wildest dreams. As you sat there in class trying to figure out what this really meant, the professor wrote the following two sentences on the board, with "Heine-Borel" preceding one of them.[1]

- The interval $[0, 1]$ is compact.
- A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

You might remember what came next. From an arbitrary infinite sequence contained in $[0, 1]$, a divide-and-conquer technique to construct a particular sequence of nested intervals, from this a subsequence of real numbers, and then a summon to the completeness of the reals, one of those blatantly obvious analysis facts that you had no idea how to prove (and likely still don't know). It felt a little unsatisfying, and almost seemed like cheating. About this time, it dawned on you that your roommate was right: mathematicians make a living saying the simplest things in the most difficult round-about way.

By now, you understand in ways you never could have imagined back then, how wise your old roommate was. But you also remember what attracted you to mathematics in the first place, those mysterious qualities that, like the silver bell in the *Polar Express*, could

2000 *Mathematics Subject Classification.* 54E45, 03C99, 03F03.

Key words and phrases. Binary sequences, ultrametric, model theory, Brouwer's fan theorem, Gödel compactness, König's infinity lemma.

[1]Textbooks vary as to which of these statements is called the Heine-Borel theorem and which one is a lemma or corollary. We will refer to the compactness of $[0, 1]$ as the Heine-Borel theorem. See McCleary (2006).

only be heard by a select few. Your friends shook their heads and exchanged private smiles when you marveled at the sheer beauty of mathematics, such as the surprising connections between seemingly unrelated topics, and the way that a basic result could be proven in vastly different ways. In fact, it is likely fore these reasons why you are reading this paper right now, and it is precisely for these reasons that drove the authors to write it. So jump aboard, and enjoy a quick but enlightening tour of diverse topics such as ultrametrics and model theory, and we'll drop you off back in your first real analysis course at the classic Heine-Borel theorem.

2. A Combinatorial Lemma

We begin by stating a combinatorial lemma due to Brouwer [1], and two proofs of it. These proofs are quite different; one follows from König's infinity lemma, and the other from Gödel compactness. We will briefly explain these concepts for those unfamiliar with them, but note the remarkable coincidence that both have an umlauted 'o' in their name.

Brouwer's fan theorem. *Let $\mathcal{B}$ be a collection of finite bitstrings (binary sequences) so that every infinite bitstring has an initial segment in $\mathcal{B}$. Then there is a finite subset $\mathcal{A} \subseteq \mathcal{B}$ so that every infinite bitstring has an initial segment in $\mathcal{A}$.*

We pause to recall König's infinity lemma, which says that an infinite tree where every vertex has finite degree contains an infinite path [4]. If the infinite tree is uncountable, as ours will be, the axiom of choice is required.

(The Königian Proof). Assume (to reach a contradiction) that the theorem is false. Recursively construct a tree $T_{\mathbb{F}_2}$ with the empty bitstring at the root so that the children of b are the bitstrings $b0$ and $b1$. Now remove all bitstrings from $T_{\mathbb{F}_2}$ that have an initial segment in $\mathcal{B}$ to get the tree T.

For every $n \geq 1$ there exists a length-n bitstring with no initial segment in $\mathcal{B}$ (if every bitstring of length n had an initial segment in $\mathcal{B}$, then the bitstrings from $\mathcal{B}$ of length at most n would work for $\mathcal{A}$). Thus, T is infinite.

Since every vertex of T has finite degree, we may apply König's infinity lemma to get an infinite path through T starting at the root. Hence we have a sequence $y_1, y_2, y_3, \ldots$, where y_i is a length-i bitstring, y_i is an initial segment of y_{i+1}, and none of the y_i's have initial segments in $\mathcal{B}$. Let y be the infinite bitstring with length-i initial segment y_i for each i. If y had an initial segment of length n in $\mathcal{B}$, then y_n would have an initial segment in $\mathcal{B}$, which is forbidden by construction. Hence y has no initial segment in $\mathcal{B}$, and this contradiction completes the proof. □

We can get a more transparent proof of the lemma using Gödel compactness. This is a classic result from model theory [5] which says that a set of boolean clauses is satisfiable if and only if every finite subset of clauses is satisfiable [3].

(The Gödelian Proof). Consider a set $\{a_1, a_2, a_3, \ldots\}$ of distinct boolean variables, and let K be the following set of clauses:

$$\{N(b) \mid b \in \mathcal{B}\} \ ,$$

where $N(b) = \neg\left[(b_1 = a_1) \wedge (b_2 = a_2) \wedge \cdots \wedge (b_k = a_k)\right]$ for any $b = b_1 b_2 \cdots b_k \in \mathcal{B}$. Note that $N(b)$ is satisfiable if and only if b is not an initial segment of $a_1 a_2 a_3 \cdots$.

Now, assume (to reach a contradiction) that the theorem is false. Then for any finite $\mathcal{A} \subseteq \mathcal{B}$ there exists a bitstring $a_1 a_2 a_3 \cdots$ that has no initial segment in $\mathcal{A}$. Hence, every finite subset of clauses of K is satisfiable, and by Gödel compactness, K is satisfiable. But by construction, this yields an infinite bitstring $a_1 a_2 a_3 \cdots$ with no initial segment in $\mathcal{B}$. This contradiction completes the proof. $\square$

3. The Bit-Metric

Equipped with our combinatorial lemma, we resume our tour in the land of bitstrings. Let $\mathbb{F}_2 = \{0, 1\}$, and let $\mathbb{F}_2^{\mathbb{N}}$ denote the set of infinite bitstrings. Again, we write a bitstring as $a = a_1 a_2 \cdots$, and call the individual a_i's *bits*. Define the function ι that sends an element of $\mathbb{F}_2^{\mathbb{N}}$ to the corresponding number in $[0, 1]$ written in binary, by

$$\iota \colon \mathbb{F}_2^{\mathbb{N}} \longrightarrow [0,1]\,, \qquad \iota(a_1 a_2 a_3 \cdots) = \sum_{i=1}^{\infty} a_i 2^{-i} = 0.a_1 a_2 a_3 \ldots\ .$$

At this point, it is easy to overlook the fact that decimals have a few subtle but pesky properties, such as the fact that

$$\iota(a_1 a_2 \cdots a_k 1000 \cdots) = \iota(a_1 a_2 \cdots a_k 0111 \cdots)\ .$$

Fortunately, ι is injective on bitstrings not of this form. With this in mind, we say that a binary decimal representation of $x \in [0, 1)$ is in *standard form* if there are infinitely many 0s. Since nobody would ever write infinitely many 1s instead of just one, when we speak of a number $x \in [0, 1]$, we shall assume that it is written in standard binary form. With this assumption, we can define the preimage of $x \in [0, 1)$ under ι to be the bitstring with infinitely many 0s, which is denoted by $\iota^{-1}(x)$. At this time it should be noted that the authors aren't analysts, and thus are prone to omit crucial but obvious details, such as what the standard binary form of 1 is, and how to define $\iota^{-1}(1)$.

Once you resolve these tiny missing details, we may proceed together, and put a metric on $\mathbb{F}_2^{\mathbb{N}}$ by saying that two distinct bitstrings a and b are a distance $\beta(a, b) = 2^{-k}$ apart, where k is the last bit at which a and b agree. It is straightforward to show that $(\mathbb{F}_2^{\mathbb{N}}, \beta)$ is an *ultrametric*, and we call it the *bit-metric* on $\mathbb{F}_2^{\mathbb{N}}$. An ultrametric is any metric that satisfies the strong triangle inequality: $\beta(a, c) \le \max\{\beta(a, b), \beta(b, c)\}$, and this gives it some extra special properties such as:

- *Russian doll property of balls*: If $B_r(a) \cap B_r(b) \ne \emptyset$, then either $B_r(a) \subseteq B_r(b)$ or $B_r(a) \supseteq B_r(b)$.
- *Center of the universe property*: If $|a - b| < r$, then $B_r(a) = B_r(b)$.

These properties are very useful when studying $(\mathbb{F}_2^{\mathbb{N}}, \beta)$, and we utilize them in papers that are much more difficult to read than this one. However, we will not need them for Heine-Borel, but we mention them for completeness (of the paper, not the reals).

At this point, you might be suspecting that the map ι, being so simple, is continuous. This is indeed correct since, by definition, $|\iota(a) - \iota(b)| \le \beta(a, b)$ for any $a, b \in \mathbb{F}_2^{\mathbb{N}}$.

Lemma 1. *The map ι is continuous under the bit-metric.*

You might also be suspecting that under the bit-metric, $\mathbb{F}_2^{\mathbb{N}}$, being a collection of infinite sequences, is not compact. This is incorrect.

Lemma 2. $(\mathbb{F}_2^{\mathbb{N}}, \beta)$ *is compact.*

Proof. Consider an open cover $\cup_{i\in I} B_{\epsilon_i}(a_i) = \mathbb{F}_2^{\mathbb{N}}$ of balls, where each $a_i \in \mathbb{F}_2^{\mathbb{N}}$. Let S_i be the first $\lfloor \log_2(\epsilon_i^{-1}) \rfloor + 1$ bits of a_i for $i \in I$. Then $b \in \mathbb{F}_2^{\mathbb{N}}$ is in $B_{\epsilon_i}(a_i)$ if and only if S_i is an initial segment of b. Put $\mathcal{B} = \{S_i \mid i \in I\}$ and apply Brouwer's fan theorem to get a finite $\mathcal{A} \subseteq \mathcal{B}$. By construction, the set

$$\bigcup_{a_i \in \mathcal{A}} B_{\epsilon_i}(a_i) = \mathbb{F}_2^{\mathbb{N}} ,$$

and thus we have a finite subcover of $\mathbb{F}_2^{\mathbb{N}}$. □

Equipped with Lemmas 1 and 2, we can now present The Shortest Proof of Heine-Borel Ever.

Theorem 3 (Heine-Borel)**.** *The interval* $[0, 1]$ *is compact.*

Proof. $\iota(\mathbb{F}_2^{\mathbb{N}}) = [0, 1]$ is the continuous image of a compact set. □

This concludes our tour, now that we have arrived back at your first real analysis class, on that special day when you first saw the Heine-Borel theorem proven. For those of you out there that have yet to take real analysis, but are advanced and motivated enough to be reading this article, pay attention. When you find yourself in an analysis class, and the professor draws that little box at the end of the proof of Heine-Borel, raise your hand, and inquire:

"*Doesn't that just follow from König's infinity lemma, and the standard ultrametric on the space of binary sequences?*"

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