5-2017

A Theoretical Foundation for the Development of Process Capability Indices and Process Parameters Optimization under Truncated and Censoring Schemes

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A THEORETICAL FOUNDATION FOR THE DEVELOPMENT OF PROCESS CAPABILITY INDICES AND PROCESS PARAMETERS OPTIMIZATION UNDER TRUNCATED AND CENSORING SCHEMES

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Industrial Engineering

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by
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May 2017

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ABSTRACT

Process capability indices (PCIs) provide a measure of the output of an in-control process that conforms to a set of specification limits. These measures, which assume that process output is approximately normally distributed, are intended for measuring process capability for manufacturing systems. After implementing inspections, however, non-conforming products are typically scrapped when units fail to meet the specification limits; hence, after inspections, the actual resulting distribution of shipped products that customers perceive is truncated. In this research, a *set* of customer-perceived PCIs is developed focused on the truncated normal distribution, as an extension of traditional manufacturer-based indices. Comparative studies and numerical examples reveal considerable differences among the traditional PCIs and the proposed PCIs. The comparison results suggest using the proposed PCIs for capability analyses when non-conforming products are scrapped prior to shipping to customers. The confidence interval approximations for the proposed PCIs are also developed. A simulation technique is implemented to compare the proposed PCIs with its traditional counterparts across multiple performance scenarios.

The robust parameter design (RPD), as a systematic method for determining the optimum operating conditions that achieve the quality improvement goals, is also studied within the realm of censored data. Data censoring occurs in time-oriented observations when some data is unmeasurable outside a predetermined study period. The underlying conceptual basis of the current RPD studies is the random sampling from a normal

distribution, assuming that all the data points are uncensored. However, censoring schemes are widely implemented in lifetime testing, survival analysis, and reliability studies. As such, this study develops the detailed guidelines for a new RPD method with the consideration of type I-right censoring concepts. The response functions are developed using nonparametric methods, including the Kaplan-Meier estimator, Greenwood's formula, and the Cox proportional hazards regression method. Various response-surface-based robust parameter design optimization models are proposed and are demonstrated through a numerical example. Further, the process capability index for type I-right censored data using the nonparametric methods is also developed for assessing the performance of a product based on its lifetime.

ACKNOWLEDGMENTS

First and foremost, I would like to express my sincere gratitude to my mentor and advisor, Dr. B. Rae Cho, for his guidance and encouragement throughout my doctoral study. It has been such a memorable and invaluable experience for me to work with and to learn from him. This dissertation would not have been possible without his support.

My gratitude also goes to my dissertation committee members, including Dr. Joel Greenstein, Dr. Tugce Isik, and Dr. David Neyens, for their insightful advice and constructive comments to improve my research, and for their considerations and support during the process of conducting this dissertation.

I would like to extend my gratitude to all the excellent professors at Clemson University, who have shaped my academic life in various ways. I am also very grateful to my friends at Clemson University for their guidance and friendship during my challenging journey. A special thank you goes to my friends in the advanced quality engineering laboratory, for their support, encouragement, and care.

Last but not least, I would like to express my deepest appreciation to my parents, my brother, my husband, and my daughter, for their endless love and support that give me strength to overcome this critical stage of my life.

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CHAPTER 1
INTRODUCTION

1.1 Introduction

Quality as a competitive advantage has become one of the keys to business success. The importance of quality is recognized from any link in a supply chain to a manufacturing process, and also to a conceptual product design. Any quality flaw along the line could lead to consequences that jeopardize the safety of end users and the negative reputation that impacts on the profitability of a company. Since 1700's, the empirical statistical methods for managing quality-related problems under different situations are continually developed and improved.

Due to variation that occurs during a production process, it is a fact that products are not identically produced. Quantifying the quality level using statistical parameters such as mean and variance, which are estimated from a probability distribution, are commonly used as primary measures. These parameters are then utilized in sequential quality control methods for measuring, controlling, and improving the overall quality of product. The traditional statistical quality control methods that are widely recognized among practitioners are developed on assumptions of the normal distribution, although data is non-normally distributed in several situations. In responding to this issue, numerous research efforts develop quality improvement approaches that suit various types of probability distributions, however, there remains a significant need for improvement.

1.2 Research motivations

The research scheme of this dissertation is twofold. The first scheme is the development of process capability indices based on the customer perception, which are developed using the statistical foundations of the truncated normal distribution. The second scheme is the development of robust parameter design optimization and process capability index for time-oriented quality characteristics, based on the nonparametric methods for censored data. The research motivations are presented as follows.

1.2.1 Development of process capability indices based on the customer perception

The process capability index (PCI) is a unitless measure used for indicating the production performance in meeting the customer requirements. In addition to being used for communicating between production and managerial levels or comparing between different processes, PCIs are also employed for selecting suppliers, redesigning systems, and optimizing operating conditions. In a case that a probability distribution of a product follows the normal distribution, the traditional PCIs are widely used to estimate the capability of a process. However, the assumptions of the normality may not be satisfied in some situations. Using the traditional PCIs for assessing the process capability of a non-normally distributed data may mislead the analysis, hence, numerous alternative PCIs based on the non-normal distributions have been proposed in the research community. Only few of them, however, consider a case where the information of observations are partially known, hereafter referred to as the incomplete data, which

impacts the accuracy of statistical calculations if an improper statistical foundation is employed.

The incomplete data occurs in analysis consists of truncated distributions and censored data. Chapters 2 and 3 focus on the truncated distribution. Truncation of a probability distribution appears when some set of values in the distribution are unknown beyond a specific point, also called the truncation point. For instance, assume that observations of a product are normally distributed. After performing a quality inspection and screening within a given set of specifications, units that fail to meet the specifications, known as the nonconforming products, are removed from the shipped units, also called the conforming products. As a result, the probability distribution of conforming products that a customer perceives is truncated within the specification limits, which is statistically defined as the truncated normal distribution.

Upon reviewing the literature, we found that the truncated normal distribution is reported to cause difficulties for assessing the process capability under several situations, for examples, measuring PCI of a multi-level inspection process, selecting suppliers based on a PCI value of products received, and assessing PCI based on a quality characteristic withnin a specific range, e.g., the gap tolerance between two assembled components. Since the statistical estimators of the truncated normal distribution are measured differently from the normal distribution, ignoring the effects of truncated distribution may cause inaccuracy in the process capability analysis and the subsequent decision making. Despite the practical importance of the role of truncated normal distributions, there has been little work on the theoretical foundation of PCIs associated

with truncated normal distributions. Therefore, the goal of Chapter 2 is to develop a set of process capability indices for truncated normal distribution based on three types of quality characteristics. These include the nominal-the-best type (NTB-type) with lower and upper specification limits, the smaller-the-better type (STB-type) with only upper specification limit, and the larger-the-better type (LTB-type) with only lower specification limit.

In Chapter 3, we extend the collection of the PCIs in Chapter 2 by developing the PCIs based on the truncated normal distribution with respect to the quality loss function of a product. Since selling only perfect items to a customer seems impossible in economic reality, the interval of the specification limits, or tolerance, is used during an inspection process to determine if a unit satisfies the customer's requirements. Although a customer may receive an item that has passed its inspection, there may be some level of risk incurred in producing a unit that fails to achieve its ideal target value, known as the quality loss. Therefore, measuring the process capability of a product with consideration of the quality loss may provide additional information for a comparison between processes.

Furthermore, it is important to note that the degree of accuracy in PCI calculation, which is a point estimate, can be affected by the statistical fluctuations occurred from estimating statistical parameters under a specific sample size. Therefore, the confidence intervals estimators for the proposed PCIs are also needed to be developed, so that various levels of accuracy associated with a sample size of calculating PCI may be assessed. Besides, since a PCI can be obtained from estimating only variance or both

mean and variance, each index needs a unique statistical approximation method for deriving its confidence interval estimators. Thus, the second goal of Chapter 3 is to develop the confidence interval estimators for the truncated normal distribution based PCIs. Nevertheless, a simulation study is also required for providing the details and insights of PCI development to facilitate comparison between the traditional PCIs and the proposed PCIs under various truncation schemes, which becomes the third research goals of Chapter 3.

1.2.2 Development of robust parameter design optimization and process capability index for censored data

The robust parameter design (RPD) is a sequential mathematical method used for designing and improving products or processes by optimizing its operating conditions. Despite numerous extensions reported in the research community, RPD has been found in various engineering applications. The three mathematical phases of RPD consist of performing a planned experiment to observe experimental responses with respect to input variables, developing regression models for indicating the effects of input variables on responses, and optimizing the input variables to seek for the optimum operating conditions. It is important to note that RPD is mostly employed for minimizing the quality loss occurred in a process based on non-time orientated quality characteristics such as the strength of materials. Despite being considered as a critical quality criterion, the time-oriented quality characteristics, e.g., a product's lifetime, has not been effectively utilized in the context of RPD.

The time-oriented quality characteristic, hereafter referred to as the survival time, is observed within a limited period; thus, the information about some of the observed survival times that last longer than a censoring time, e.g., a predetermined termination time of a study period, is partially known. For instance, if a unit fails within a study period, the survival time of a unit is recorded as its actual survival time. On the contrary, if a unit still survives at a termination time, the survival time is recorded as the termination time, known as the censored survival time, which implies that the survival time of a unit is longer than the termination time; however, its actual survival time is unknown. Thus, it is important to note that, first, each recorded survival time can be either an actual lifetime or a censored survival time. Second, there are several types of censoring and each type of censoring has its own set of statistical foundations. Third, since the survival time is non-negative, its probability distribution generally follows a non-normal distribution, e.g., the exponential distribution and the Weibull distribution. Fourth, the traditional RPD assumes the normal distribution as a default probability distribution. For these reasons, the traditional RPD requires an improvement for effectively obtaining the optimum operating condition when some observations are censored. Thus, the first goal of Chapter 4 is to fill the potential research gap stated above. Finally, upon investigating the practical situations of the time-oriented quality characteristics, we also found a significant research gap in the context of PCIs, which is the development of process capability index for censored data. Similar to the RPD, the problems of censored data in the PCI scheme appear when the quality characteristic of

interest is time-oriented type. Therefore, the development of PCI for censored data becomes the second goal of Chapter 4.

Subsequently, Table 1.1 presents goals and features of each chapter in responding to the research questions in Figure 1.1. Also, the research tools, contributions, and disseminations of the dissertation are summarized in Table 1.2.

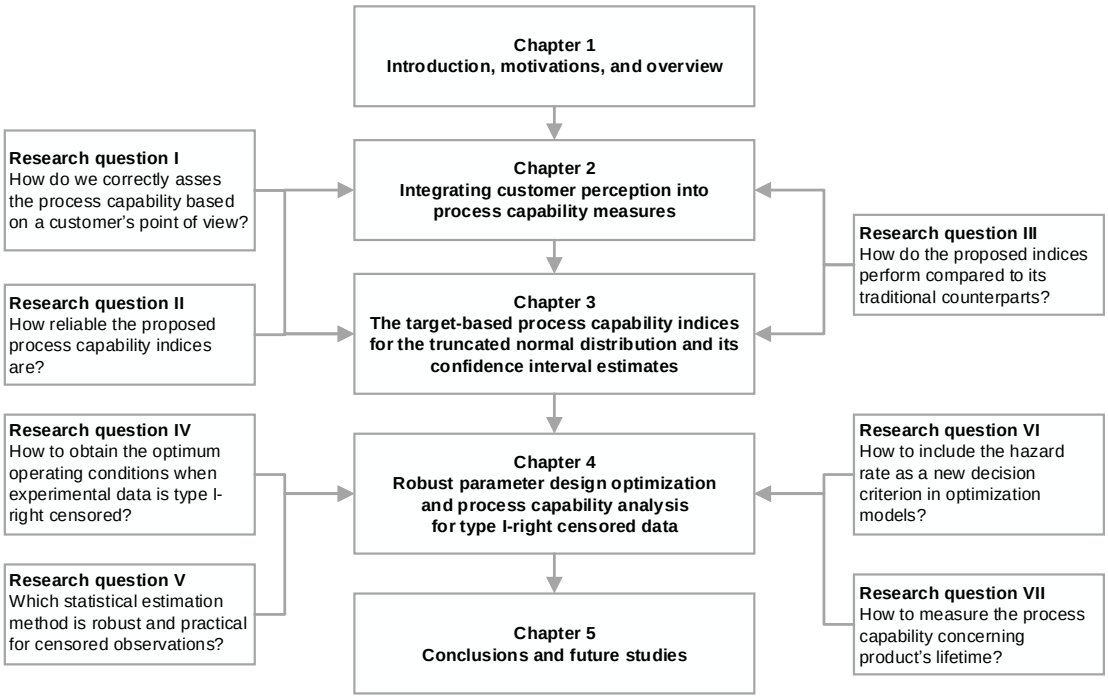


Figure 1.1 Dissertation structure with corresponding research questions

Table 1.1 Dissertation overview

Chapters and goals	Research features
Chapter 1: Introduction	Address research motivations, goals, and overview of the dissertation
Chapter 2: Integrating customer perception into process capability measures Goal: To develop a set of PCIs for assessing the capability of a process that follows the truncated normal distribution	• Develop $C_{TN\text{-}p}$, $C_{TN\text{-}pl}$, and $C_{TN\text{-}pu}$, for the NTB-type quality characteristic • Develop the C_{ST} for the STB-type quality characteristic • Develop the C_{LT} for the LTB-type quality characteristic • Derive the truncated normal estimators for two-sided truncations, left-sided truncation, and right-sided truncation • Conduct sensitivity study for the proposed PCIs under various ranges of specifications and levels of statistical parameters • Demonstrate the proposed PCIs through a case study of chemical product with multi-specification limits • Conduct comparison study among the proposed PCIs, the traditional PCIs, and the traditional PCIs with data transformation
Chapter 3: The target-based process capability indices for the truncated normal distribution and its confidence interval estimators Goals: • To develop target-based PCIs concerning the truncated normal distribution • To develop the confidence interval estimators for the truncated normal distribution based PCIs focusing on the NTB-type quality characteristics • To investigate the features of the proposed PCIs compared to its traditional counterparts	• Develop $C_{TN\text{-}pm}$ and $C_{TN\text{-}pkm}$ as the target-based PCIs with respect to the truncated normal distribution • Conduct sensitivity study for the proposed target-based PCIs • Develop the confidence interval estimators for the proposed PCIs, including $\left\{C^{L}_{TN-p}, C^{U}_{TN-p}\right\}$, $\left\{C^{L}_{TN-pk}, C^{U}_{TN-pk}\right\}$, $\left\{C^{L}_{TN-pm}, C^{U}_{TN-pm}\right\}$, and $\left\{C^{L}_{TN-pmk}, C^{U}_{TN-pmk}\right\}$. • Develop a simulation model to investigate characteristics of PCI values under various shapes of truncated normal distribution • Demonstrate the proposed PCIs and its confidence interval estimators through a case study of chemical product with multi-specification limits and a case study of supplier selection
Chapter 4: Robust parameter design optimization and process capability analysis for the type I-right censoring data Goals: • To optimize the process parameters using robust parameter design where data is time-oriented and is type I-right censored • To develop PCI for type I-right censored data	• Modify the central composite design for censored data • Incorporate nonparametric methods, the KM estimator, Greenwood's formula, and the Cox PH model, for constructing response functions of type I-right censored data • Develop optimization models for obtaining the optimum operating conditions with inclusion of the median survival time, the variance of median survival time, and the hazard rate • Demonstrate the developed method through a case study of drug degradation • Conduct a study to investigate the problems of using parametric approaches for censored data • Develop the PCI for type I-right censored data • Develop the confidence interval estimator for the type I-right censoring based PCI
Chapter 5: Conclusion and future studies	Summarize research works, discuss limitations, and suggest future studies

Table 1.2 Dissertation summary

Chapter	Potential contributions	Truncated distribution	Censored data	Performance measurement	Performance comparison	System redesign	Operating condition optimization	Conceptual design	Quality assurance	Continuous process improvement	Tools/Theorems used
2	The collection of customer-perceived PCIs for the three quality charaeristics	●		●	●					●	Truncated normal distribution, C_p, C_{pk}, C_{pu}, C_{pl}, Defective rate, Qulity characteristics (NTB-,LTB-, STB-type)
3	1) Customer-perceived PCIs based on the quality loss function 2) Confidence interval estimators for the customer-perceived PCIs	●		●	●					●	Qulity loss function, C_{pm}, C_{pmk} , Chi-square distribution, Wilson-Hilferty method, Bissell method, Taylor series expansion
4	1) Robust parameter design optimization for type-I right censored data		●			●	●	●	●	●	CCD, KM estimator, Greenwood formula, Cox PH model, Multiple linear regresion, Nonlinear optimization
4	2) PCI for type I-right censored data and its confidence interval estimator		●	●	●				●	●	Percentile-based PCI, KM estimator, Greenwood formula

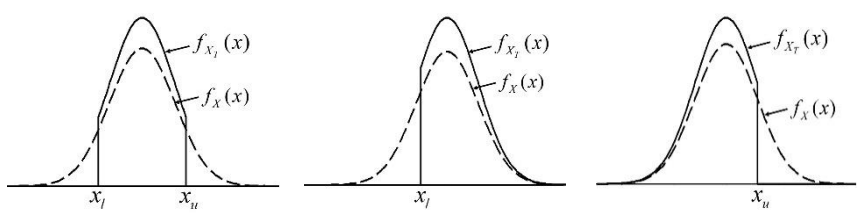
CHAPTER 2
INTEGRATING CUSTOMER PERCEPTION INTO PROCESS CAPABILITY MEASURES

2.1 Introduction

Quantifying the ability of a process to produce output that conforms to specifications is vital to understanding a performance baseline – it is an integral component of continuous process improvement. Process capability indices (PCIs) often serve as a tool and basis for comparing practices, redesigning systems, selecting suppliers, and optimizing operating conditions. A comprehensive review of PCIs can be found in the survey papers by Kane (1986), Kotz et al. (2002), Spiring et al. (2003), and Yum and Kim (2011), most of which assume that process output is normally distributed. However, there are practical situations where specification limits on a process are imposed externally, and the product is typically scrapped if its performance does not fall within the specification range. As such, the actual distribution that the customer perceives after the inspection is truncated. Despite the practical importance of the role of truncated distributions, there has been little work on the theoretical foundation of PCIs associated with truncated normal distributions.

The focus of this chapter is on the development of customer-perceived PCIs (i.e., a consumer versus manufacturer perspective) under three types of quality characteristics. These include the nominal-the-best type (NTB), the smaller-the-better type (STB), and the larger-the-better (LTB) type, which have not been fully explored in the literature. Figure 2.1 shows (a) a two-sided left and right truncated distribution at the lower

specification limit (*LSL*) and upper specification limit (*USL*) for an NTB-type characteristic, (b) a one-sided left truncated distribution at the *LSL* (x_l) for an LTB-type characteristic, and (c) a one-sided right truncated distribution at the *USL* (x_u) for an STB-type characteristic. Dotted and solid lines represent the original normal and truncated normal distributions, respectively. It is noted that the shape of a truncated normal distribution $f_x(x)$ varies based on the number of specification limits that are implemented and where they are located. It is also observed that the variance of the distribution, after implementing a truncation, will no longer be the same as the original variance associated with the untruncated normal distribution $f_x(x)$. Applications of the truncated normal distribution are found in Khasawneh et al. (2004, 2005), Hong and Cho (2007), Shin and Cho (2009), Cha et al. (2013), and Cha and Cho (2014).



(a) (b) (c)

Figure 2.1 Different types of truncated normal distributions

Regarding the field of process capability analysis within contemporary literature, this chapter offers several contributions that have not been previously explored. A new set of process capability indices, referred to as customer-perceived PCIs, are developed, which are based on the truncated normal distribution and are designed to provide improved accuracy when inspections are implemented. Our proposal accounts for the three different types of quality characteristics, using two-sided, left, and right truncations.

Upon illustrating these measures using a numerical example, a comparison study is performed to relate the customer-perceived PCIs to their traditional counterparts. Finally, the PCIs are further investigated using data transformation methods when the process output is not normally distributed. This chapter is organized into five remaining sections as follows. In Section 2.2, previous work with respect to process capability analysis is presented; then, in Section 2.3, the models for the customer-perceive PCIs are developed in detail. A numerical example is provided and comparison study is performed in Sections 2.4 and 2.5, respectively, with a summary of conclusions in Section 2.6.

2.2 Traditional process capability indices

The PCI is one of the most popular tools for measuring the performance of a process. The initial concept of the PCI was first introduced by Feigenbaum (1951) and Juran (1951) as a process measurement contained within six standard deviations, or 6σ, representative of the inherent variability of a process. Juran (1962) and Juran and Gryna (1980) examined PCI ratios on various tolerance intervals. C_p, one of the most basic PCIs, is given as $C_p = (USL - LSL)/6\sigma$. It is believed that Kane (1986) is credited with introducing C_p into the process capability literature. However, to correctly measure a process using C_p, it must be approximately normally distributed with the process mean centered between the *LSL* and *USL*. If these assumptions are not met, PCI values may incur serious error (Montgomery, 2007), since different processes with the same level of variation may provide the identical C_p value regardless of the mean location. It is precisely this issue with C_p for which C_{pk}, a PCI designed to relax the centered mean

assumption (see Kane, 1986), was developed. Given a process with an *LSL* and *USL*, mean (μ), and standard deviation (σ), C_{pk} is obtained as $C_{pk} = \min\left(C_{pu}, C_{pl}\right)$, where $C_{pl} = \left(\mu - LSL\right)/3\sigma$ and $C_{pu} = \left(USL - \mu\right)/3\sigma$. As an extension, multivariate PCIs are introduced by Chan et al. (1991), Chen (1994), Hubele et al. (1991), Wang and Chen (1998), Bothe (1992), and Goethals and Cho (2011).

In manufacturing processes, non-normal data is frequently found in several forms, as illustrated by Polansky et al. (1998), Sweet and Tu (2006), and Pearn et al. (2007). Measuring non-normal data with PCIs that are based upon a requisite normal distribution is one common real-world problem. Some prominent transformation methods are the Johnson transformation method (Johnson, 1949) and the Box-Cox power transformation method (Box and Cox, 1964). However, these data transformation methods are not appropriate for small sample sizes and can require excessive computing time (Tang and Than, 1999). For data that follows an unknown distribution, Clements (1989) developed percentile-based PCIs, whereby percentiles are estimated through the four data characteristics, i.e., mean, standard deviation, skewness and kurtosis. Tang and Than (1999) and Chang et al. (2002) concluded that, in order to validate the reliability and precision of skewness and kurtosis, a large sample size is required. In addition, Johnson et al. (1992) introduced the flexible PCI, C_{jkp} using C_{pm} as a basis that accounts for the associated difference in variability above and below the target, and hence incorporates the asymmetry of a non-normal process. Moreover, Wright (1995) proposed C_s as an index to account for skewness by incorporating a correction factor obtained from process data. Furthermore, Kotz and Lovelace (1998) proposed a heuristic weighted variance

method, a technique which divides a non-normal skewed distribution into two different distributions, resulting in a normally distributed outcome distribution with the same mean but a different standard deviation. Using the weighted variance method, Wu and Swain (2001) modified standard PCIs with the skewness and kurtosis of distribution. Also, a specific PCI for data that follows a log-normal distribution was proposed by Lovelace and Swain (2009).

For process capability analysis that is based on a truncated distribution, Sweet and Tu (2006) applied the truncated distribution concept to the tolerance of the assembled gap between a bore and a shaft. In order to obtain increased accuracy for analysing a truncated normal process, Pearn et al. (2007) derived the probability density function of a truncated normal distribution data – in doing so, they suggested obtaining the exact probability density and the distribution function using the Edgeworth expansion technique. Finally, Tao and Xinzhang (2012) studied the effect of a truncated distribution in measuring the yield and performance of semiconductor manufacturing.

2.3 Development of customer-perceived process capability indices

A typical NTB-type characteristic has the desired target value where two-sided specifications (*LSL* and *USL*) are accordingly implemented. Similarly, the respective target values of STB-type and LTB-type characteristics are zero and infinite; thus, one-sided specifications, such as the *USL* for the STB-type and the *LSL* for the LTB-type, are known useful. The well-known two underlying assumptions behind PCIs, which are a normally distributed quality characteristic and an in-control process, also hold here. In

this section, the customer-perceived PCI is derived from each type of quality characteristic.

2.3.1 The NTB-Type case

The quality characteristic of interest (X) is normally distributed, $X \sim N\left(\mu, \sigma^2\right)$, with two specification limits $\left[x_l, x_u\right]$. Let X_{TN} be the truncated normal random variable. The probability density function of X_{TN} is then given by $f\left(X_{TN}\right) = \dfrac{f(x)}{\int_{x_l}^{x_u} f_x(y)\,dy}$, and the cumulative probability density function of X_{TN} is defined as

$F\left(X_{TN}\right) = \int_{-\infty}^{x} \dfrac{f(x)}{\int_{x_l}^{x_u} f_x(y)\,dy}\,dx.$ Furthermore, the truncated mean and the truncated variance are obtained from $\mu_{TN} = \int_{-\infty}^{\infty} x f\left(X_{TN}\right) dx$ and $\sigma_{TN}^2 = \int_{-\infty}^{\infty} x^2 f\left(X_{TN}\right) dx - \left(\int_{-\infty}^{\infty} x f\left(X_{TN}\right) dx\right)^2$.

When $x_l \le x \le x_u$, the truncated mean is derived as

$$\mu_{TN} = E\left(X_{TN}\right) = \frac{\int_{x_l}^{x_u} x\left(\frac{1}{\sqrt{2\pi\sigma}} e\right)^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx}{\int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy}$$

Given $t = \int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$, $z_t = \dfrac{x-\mu}{\sigma}$, and $dx = \sigma dz$, we have

$$\mu_{TN} = \frac{1}{t}\left\{\mu \int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dx - \int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma dz\right\}$$ which can be expressed as

$$\mu_{TN} = \mu - \frac{\sigma}{t}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\Bigg|_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \tag{2.1}$$

Since $t = \int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy = \int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \frac{1}{\sqrt{2\sqrt{\pi}}} e^{-\frac{1}{2}z^2} dz = \Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)$,

the mean of two-sided truncated normal distribution is obtained as

$$\mu_{TN} = \mu + \sigma\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right) - \phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right) \tag{2.2}$$

For the variance of two-sided truncated normal distribution, we note

$$\sigma_{TN}^2 = V\left(X_{TN}\right) = E\left(X_{TN}^2\right) - E\left(X_{TN}\right)^2$$

where $E\left(X_{TN}^2\right) = \int_{-\infty}^{\infty} x^2 f\left(X_{TN}\right) dx = \int_{-\infty}^{\infty} \frac{x^2 \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy} dx$

Let $t = \int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$, $z = \frac{x-\mu}{\sigma}$, and $dx = \sigma dz$, we then have

$$E\left(X_{TN}^2\right) = \frac{\int_{x_l}^{x_u} x^2 \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx}{\int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy} \quad \text{or}$$

$$E\left(X_{TN}^2\right) = \frac{1}{t}\left(\int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} z^2 \frac{\sigma^2}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} dz + 2\mu \times \mu_T \times t - \mu^2 t\right) \tag{2.3}$$

From $\frac{d}{dz}\left(-\frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\right) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} + \frac{z^2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, we get

$$\frac{z^2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} = \frac{d}{dz}\left(-\frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$, thus

$$\int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz = -\frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \Bigg|_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} + \int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \tag{2.4}$$

By incorporating Equation (2.4) into Equation (2.3), we have

$$E\left(X_{TN}^2\right) = \frac{1}{t}\left\{ \sigma^2 \left(\frac{x_u-\mu}{\sigma}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_u-\mu}{\sigma}\right)^2} + \left(\frac{x_l-\mu}{\sigma}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_l-\mu}{\sigma}\right)^2} + \sigma^2 f' + 2\mu \times \mu_T \times t - \mu^2 t \right\}$$

Since $\mu_{TN} = \mu + \sigma \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right) - \phi\left(\frac{x_u-\mu}{\sigma}\right)}{t} \right)$, $V\left(X_{TN}\right)$ is expressed as

$$\sigma_{TN}^2 = \sigma^2 \left(1 + \frac{\left(\frac{x_l-\mu}{\sigma}\right) \times \phi\left(\frac{x_l-\mu}{\sigma}\right) - \left(\frac{x_u-\mu}{\sigma}\right) \times \phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right) - \phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)} \right)^2 \right) \tag{2.5}$$

Finally, the proposed PCIs for an NTB-type characteristic, C_{TN-p}, C_{TN-pl}, C_{TN-pu}, and C_{TN-pk}, are described as

$$C_{TN-p} = \frac{USL - LSL}{6\ \sigma_{TN}} = \frac{USL - LSL}{6\sqrt{\sigma^2 \left(1 + \frac{\left(z_{LSL}\right) \times \phi\left(z_{LSL}\right) - \left(z_{USL}\right) \times \phi\left(z_{USL}\right)}{\Phi\left(z_{USL}\right) - \Phi\left(z_{LSL}\right)} - \left(\frac{\phi\left(z_{LSL}\right) - \phi\left(z_{USL}\right)}{\Phi\left(z_{USL}\right) - \Phi\left(z_{LSL}\right)} \right)^2 \right)}} \tag{2.6}$$

$$C_{TN-pl} = \frac{\mu_{TN} - LSL}{3\ \sigma_{TN}} = \frac{\mu + \sigma \left(\frac{\phi\left(z_{LSL}\right) - \phi\left(z_{USL}\right)}{\Phi\left(z_{USL}\right) - \Phi\left(z_{LSL}\right)} \right) - LSL}{3\sqrt{\sigma^2 \left(1 + \frac{\left(z_{LSL}\right) \times \phi\left(z_{LSL}\right) - \left(z_{USL}\right) \times \phi\left(z_{USL}\right)}{\Phi\left(z_{USL}\right) - \Phi\left(z_{LSL}\right)} - \left(\frac{\phi\left(z_{LSL}\right) - \phi\left(z_{USL}\right)}{\Phi\left(z_{USL}\right) - \Phi\left(z_{LSL}\right)} \right)^2 \right)}} \tag{2.7}$$

$$C_{TN-pu} = \frac{USL - \mu_{TN}}{3\ \sigma_{TN}} = \frac{USL - \left[\mu + \sigma\left(\frac{\phi(z_{LSL}) - \phi(z_{USL})}{\Phi(z_{USL}) - \Phi(z_{LSL})}\right)\right]}{3\sqrt{\sigma^2\left(1 + \frac{(z_{LSL}) \times \phi(z_{LSL}) - (z_{USL}) \times \phi(z_{USL})}{\Phi(z_{USL}) - \Phi(z_{LSL})} - \left(\frac{\phi(z_{LSL}) - \phi(z_{USL})}{\Phi(z_{USL}) - \Phi(z_{LSL})}\right)^2\right)}} \tag{2.8}$$

$$C_{TN-pk} = min\left\{C_{TN-pu}, C_{TN-pl}\right\} \tag{2.9}$$

where $z_{LSL} = \frac{x_l - \mu}{\sigma}$ and $z_{USL} = \frac{x_u - \mu}{\sigma}$, and $\phi(\bullet)$ and $\Phi(\bullet)$ are the probability density function and its cumulative density function, respectively.

2.3.2 The STB-Type case

For the STB-type quality characteristic, only the upper specification limit is implemented. Note that with $X \sim N\left(\mu, \sigma^2\right)$ in the range $[-\infty, x_u]$, the distribution is considered as a right-sided, or STB-type, truncated normal distribution. Let X_{TS} represent the quality characteristic of the distribution, whereby the probability density function is given by $f\left(X_{TS}\right) = \frac{f(x)}{\int_{-\infty}^{x_u} f_x(y)\,dy}$, and the cumulative probability density function is written as $F\left(X_{TS}\right) = \int_{-\infty}^{x} \frac{f(x)}{\int_{-\infty}^{x_u} f_x(y)\,dy}\,dx$. The mean of the distribution is $\mu_{TS} = \int_{-\infty}^{\infty} x f\left(X_{TS}\right) dx$, and the variance is $\sigma_{TS}^2 = \int_{-\infty}^{\infty} x^2 f\left(X_{TS}\right) dx - \left(\int_{-\infty}^{\infty} x f\left(X_{TS}\right) dx\right)^2$.

Since X_{TS} conforms to the upper specification only, x_l is assumed to be negative

infinity ($-\infty$). Consequently, the terms $\left(\frac{x_l-\mu}{\sigma}\right), \phi\left(\frac{x_l-\mu}{\sigma}\right)$ and $\Phi\left(\frac{x_l-\mu}{\sigma}\right)$ become zero. Therefore, the mean and variance of this distribution can be modified as follows

Using Equation (2.2), $\mu_{TN}=\mu+\sigma\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)$, and by substituting $\left(\frac{x_l-\mu}{\sigma}\right)=0$, $\phi\left(\frac{x_l-\mu}{\sigma}\right)=0$, and $\Phi\left(\frac{x_l-\mu}{\sigma}\right)=0$, the mean for the STB-type truncated normal distribution is then expressed as

$$\mu_{TS}=\mu-\sigma\left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right) \tag{2.10}$$

Further, the variance for the STB-type truncated normal distribution is obtained as

$$\sigma_{TS}^2=\sigma^2\left(1-\frac{\left(\frac{x_u-\mu}{\sigma}\right)\times\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right)^2\right) \tag{2.11}$$

To evaluate the STB-type quality characteristic using the C_p index family, the C_{pu} is recommended. Therefore, the truncated normal distribution based index for STB-type characteristic (i.e., C_{TS}) is proposed as

$$C_{TS} = \frac{USL - \mu_{TS}}{3\,\sigma_{TS}} = \frac{USL - \left[\mu - \sigma\left(\frac{\phi\left(\frac{x_u - \mu}{\sigma}\right)}{\Phi\left(\frac{x_u - \mu}{\sigma}\right)}\right)\right]}{3\sqrt{\sigma^2\left(1 - \frac{\left(\frac{x_u - \mu}{\sigma}\right)\times\phi\left(\frac{x_u - \mu}{\sigma}\right)}{\Phi\left(\frac{x_u - \mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_u - \mu}{\sigma}\right)}{\Phi\left(\frac{x_u - \mu}{\sigma}\right)}\right)^2\right)}} \tag{2.12}$$

2.3.3 The LTB-Type case

The LTB-type truncation characteristic is modified from the NTB-type truncated normal distribution, with the *USL* being infinity. By letting $X \sim N\left(\mu, \sigma^2\right)$, $X \in \left(x_l, \infty\right)$, $-\infty < x_L$, X_{TL} is considered a quality characteristic with a left-sided truncated normal distribution.

For $x_l \le x \le \infty$, the probability density function is given as $f(X_{TL}) = \frac{f\left(x\right)}{\int_{x_l}^{\infty} f_x\left(y\right)dy}$,

and the cumulative density function is defined as $F(X_{TL}) = \int_{-\infty}^{x} \frac{f\left(x\right)}{\int_{x_l}^{\infty} f_x\left(y\right)dy}dx$.

The truncated mean is expressed as $\mu_{TL} = \int_{-\infty}^{\infty} x f(X_{TL})dx$, and the variance is obtained using $\sigma_{TL}^2 = \int_{-\infty}^{\infty} x^2 f\left(X_{TL}\right)dx - \left(\int_{-\infty}^{\infty} x f\left(X_{TL}\right)dx\right)^2$. Since the LTB-type truncated normal distribution has only a lower specification, x_u is assumed to be infinity. Hence, the term

$\left(\frac{x_u-\mu}{\sigma}\right)$ and $\phi\left(\frac{x_u-\mu}{\sigma}\right)$ become zero, while $\Phi\left(\frac{x_u-\mu}{\sigma}\right)$ equates to 1. The mean of the LTB-type truncated normal distribution (μ_{TL}) is derived as follows:

Since $\left(\frac{x_u-\mu}{\sigma}\right)=0$, $\left(\frac{x_u-\mu}{\sigma}\right)=0$, and $\Phi\left(\frac{x_u-\mu}{\sigma}\right)=1$, Equation (2.2) becomes

$\mu_{TL}=\mu+\sigma\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-0}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)$, and the mean for the LTB-type truncated normal distribution is expressed as

$$\mu_{TL}=\mu+\sigma\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right) \tag{2.13}$$

By assuming an infinite upper specification, Equation (2.5) then becomes

$$\sigma_{TL}^2=\sigma^2\left(1+\frac{\left(\frac{x_l-\mu}{\sigma}\right)\times\phi\left(\frac{x_l-\mu}{\sigma}\right)-0}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-0}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right)$$

Finally, the variance for the LTB-type truncated normal distribution is given as

$$\sigma_{TL}^2=\sigma^2\left(1+\frac{\left(\frac{x_l-\mu}{\sigma}\right)\times\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right) \tag{2.14}$$

Based on the C_{pl}, the proposed LTB-type truncated normal distribution based PCI, denoted by the C_{TL}, is given as

$$C_{TL} = \frac{\mu_{TL} - LSL}{3\,\sigma_{TL}} = \frac{\mu + \sigma\left(\dfrac{\phi\left(\dfrac{x_l - \mu}{\sigma}\right)}{1 - \Phi\left(\dfrac{x_l - \mu}{\sigma}\right)}\right) - LSL}{3\sqrt{\sigma^2\left(1 + \dfrac{\left(\dfrac{x_l - \mu}{\sigma}\right) \times \phi\left(\dfrac{x_l - \mu}{\sigma}\right)}{1 - \Phi\left(\dfrac{x_l - \mu}{\sigma}\right)} - \left(\dfrac{\phi\left(\dfrac{x_l - \mu}{\sigma}\right)}{1 - \Phi\left(\dfrac{x_l - \mu}{\sigma}\right)}\right)^2\right)}} \tag{2.15}$$

2.4 Comparisons and insights

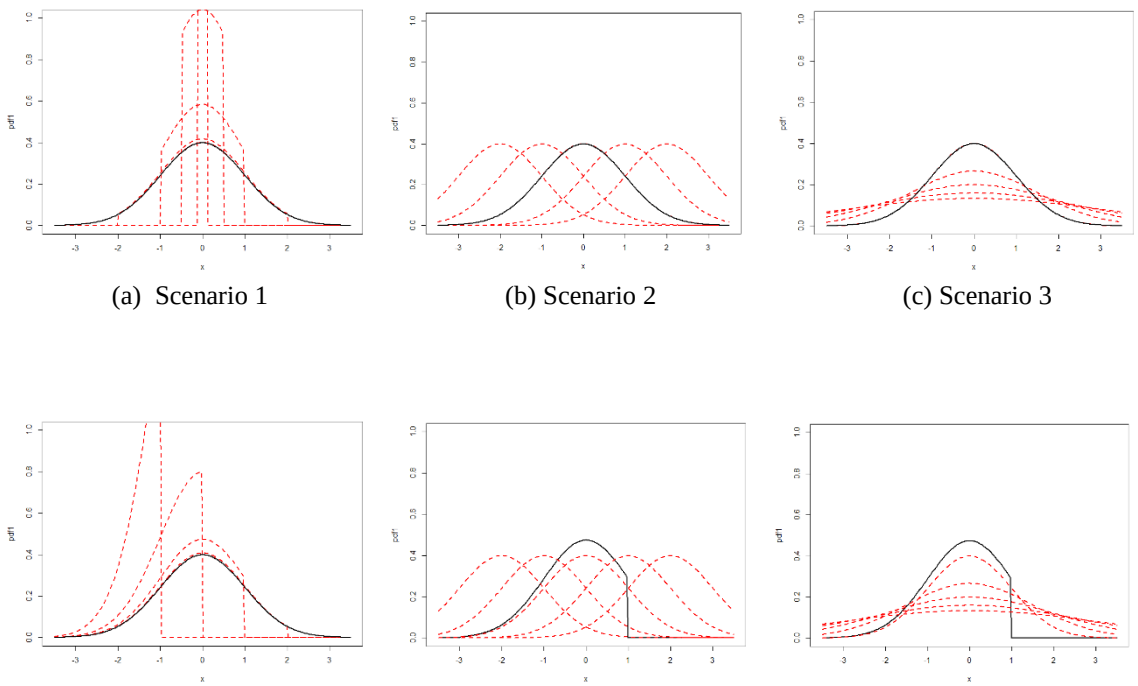
In this section, we examine and compare the customer-perceived PCIs based on a truncated normal distribution with traditional PCIs. A sensitivity analysis is performed to identify the effects of changing the truncation point and range, as well as altering the location of the mean and increasing variability. The proposed index, $C_{TN\text{-}p}$, will be compared with traditional PCIs, such as C_p and C_{pk}. Six scenarios were examined based on two different groups: the two-sided truncation-based and the one-sided truncation-based groups. The sensitivity analysis testing utilizes various factors to include the truncation points, range, mean, and variance. The scenario settings, whose results are illustrated in Figure 2.2(a)-(f), are summarized and shown in Table 2.1. Each graph consists of three different lines which represent the characteristic of the proposed truncation-based PCIs, the traditional PCIs, and the ratio between the compared PCIs ($r_p = C_p / C_{TN-p}$ and $r_{pk} = C_{pk} / C_{TN-pk}$). If the ratio is equal to one, both PCIs provide identical values. Given the settings $\mu = 0$, $\sigma^2 = 1$, $[a,b]_{\text{two-sided}} = [-3,3]$,

$[a,b]_{\text{one-sided}} = [-\infty,1]$, and $\delta = 6$, where *a* is the left truncation point (*LSL*), *b* is the right truncation point (*USL*), and $\delta = b - a$, the results are shown in Figures 2.3 and 2.4.

The first scenario aims to capture the effect of altering the truncation range. For the two-sided truncated normal distribution, when both truncation points are shifted away symmetrically from the specification midpoint $\left(c = \left(USL - LSL\right)/2\right)$, the effect of truncation is small, as shown in Figure 2.3(a)-(b). In Scenario 2, where the mean is relocated from the *LSL* to the *USL* (from *a* to *b*), we observe that the difference between the ratios is larger when the mean is shifted away from the midpoint, as shown in Figure 2.3(c)-(d). For the third scenario, when the variance is increased, both the customer-perceived and traditional PCIs demonstrate similar trends in their index values, as shown in Figure 2.3(e)-(f).

For the one-sided truncation-based scenario, the effect of decreasing the truncation range by moving the truncation point from *b* to *a* is a significant change in the increase of kurtosis for the distribution. While the C_p and the C_{pk} values decrease proportionally to the truncation range, the C_{TN-p} and the C_{TN-pk} index values decrease quadratically, as depicted in Figure 2.4(a)-(b). The location of the mean also affects the PCI value in the one-sided truncation - specifically, it is at the specification midpoint where we observe similar values among the different PCIs, with variation elsewhere, as shown in Figure 2.4(c)-(d). Furthermore, as variability increases, the truncation-based PCI values tend toward the traditional PCI values – the difference observed is greater for the one-sided truncation-based scenario than that of the two-sided truncation based

scenario, as shown in Figure 2.4(e)-(f). Based on these results, the customer-perceived PCIs are recommended when the mean is not located at the specification midpoint, and the right truncation point is less than 3.5 $\left(a < z_x = 3.5\right)$ or the left truncation point is larger than -3.5 $\left(b > z_x = -3.5\right)$, where $z_x = \left(x - \mu\right) / \sigma$.



(a) Scenario 1 (b) Scenario 2 (c) Scenario 3

(d) Scenario 4 (e) Scenario 5 (f) Scenario 6

Figure 2.2 Truncated distributions under different scenarios

Table 2.1 Sensitivity analysis scenarios

	Sensitivity analysis setting			
	μ	σ	[a, b]	δ
Scenario 1: Two-sided truncation with symmetric truncation points	Fixed (0)	Fixed (1)	Varied [-0.1, 0.1] to [-3.5, 3.5]	Varied (0.2 to 7)
Scenario 2: Symmetric two-sided truncated distribution with *varied mean*	Varied (-3 to 3)	Fixed (1)	Fixed [-3, 3]	Fixed (6)
Scenario 3: Symmetric two-sided truncated distribution with *varied variance*	Fixed (0)	Varied (1 to 3.5)	Fixed [-3, 3]	Fixed (6)
Scenario 4: One-sided truncation	Fixed (0)	Fixed (1)	Varied [-3.5, -3.4] to [-3.5, 3.5]	Varied (0.2 to 7)
Scenario 5: One-sided truncated distribution with *varied mean*	Varied (-∞ to 1)	Fixed (1)	Fixed [-∞, 1]	Fixed (6)
Scenario 6: One-sided truncated distribution with *varied variance*	Fixed (0)	Varied (1 to 3.5)	Fixed [-∞, 1]	Fixed (6)

2.5 Numerical example

A chemical company supplies a chemical solvent to three customers, companies A, B, and C, with three different specifications on the same product. Products are sorted according to each customer's specifications and the company ships the product to the customers within their specifications, as shown in Figure 2.5. This example was first published in Polansky et al. (1998). The process distribution is assumed normal with a process mean of 0 and a variance of 1. The customer specifications are given as follows:

(1) Customer A : LSL = -1.0 and USL = 1.0

(2) Customer B : LSL = 0.5 and USL = 2.0

(3) Customer C : LSL = 0 and USL = 1.5

The data for each customer is shown in Table 2.2, whereby the sequence of the data is read from top to bottom in each column.

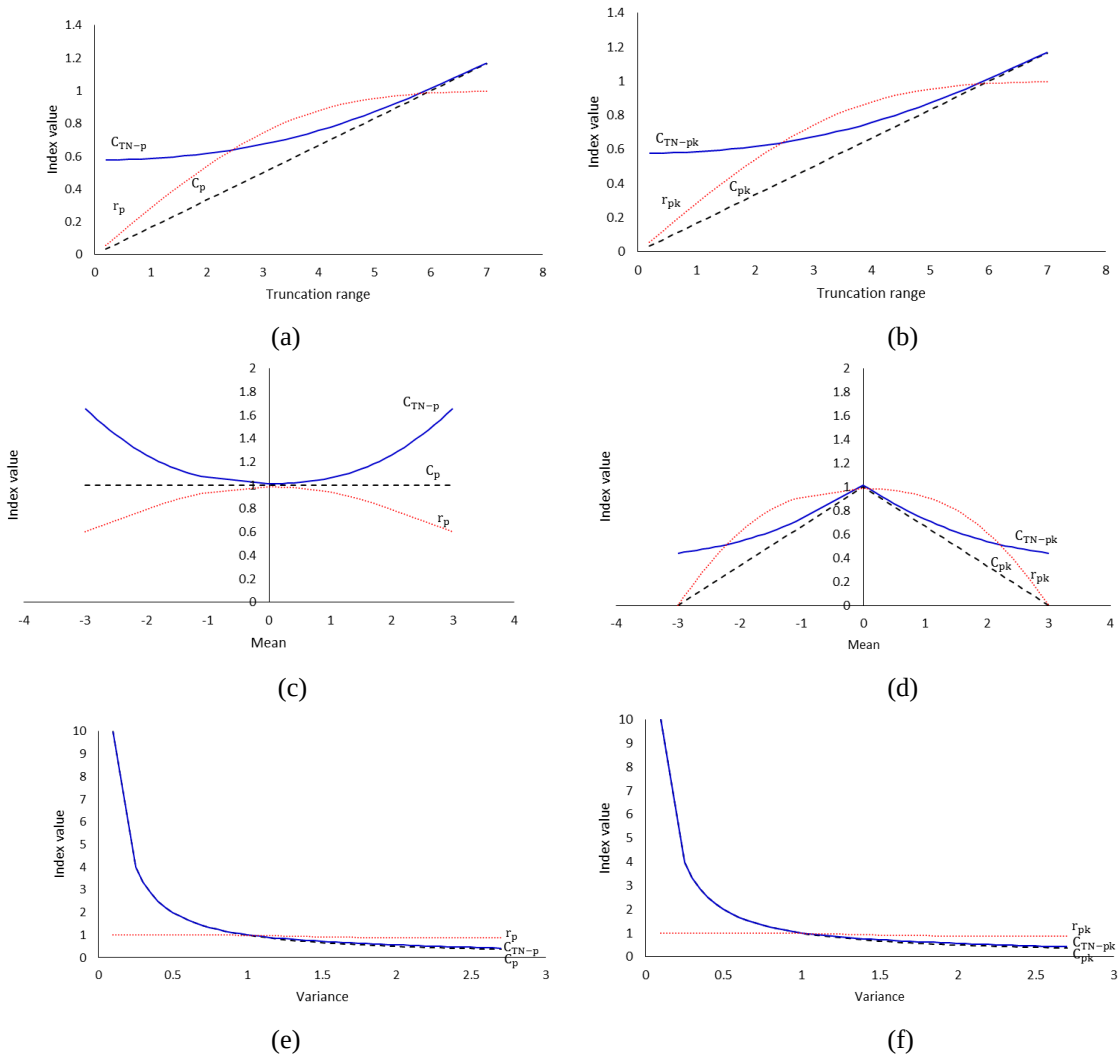


Figure 2.3 Sensitivity analysis results for scenario 1 in (a) and (b), scenario 2 in (c),(d), and scenario 3 in (e),(f).

a) The NTB-Type case

The data for customer A is chosen to illustrate the proposed NTB-type of the truncated normal distribution based PCI. Given that the LSL = -1, USL = 1, μ = 0, and σ =1, the truncated mean and variance can be calculated as

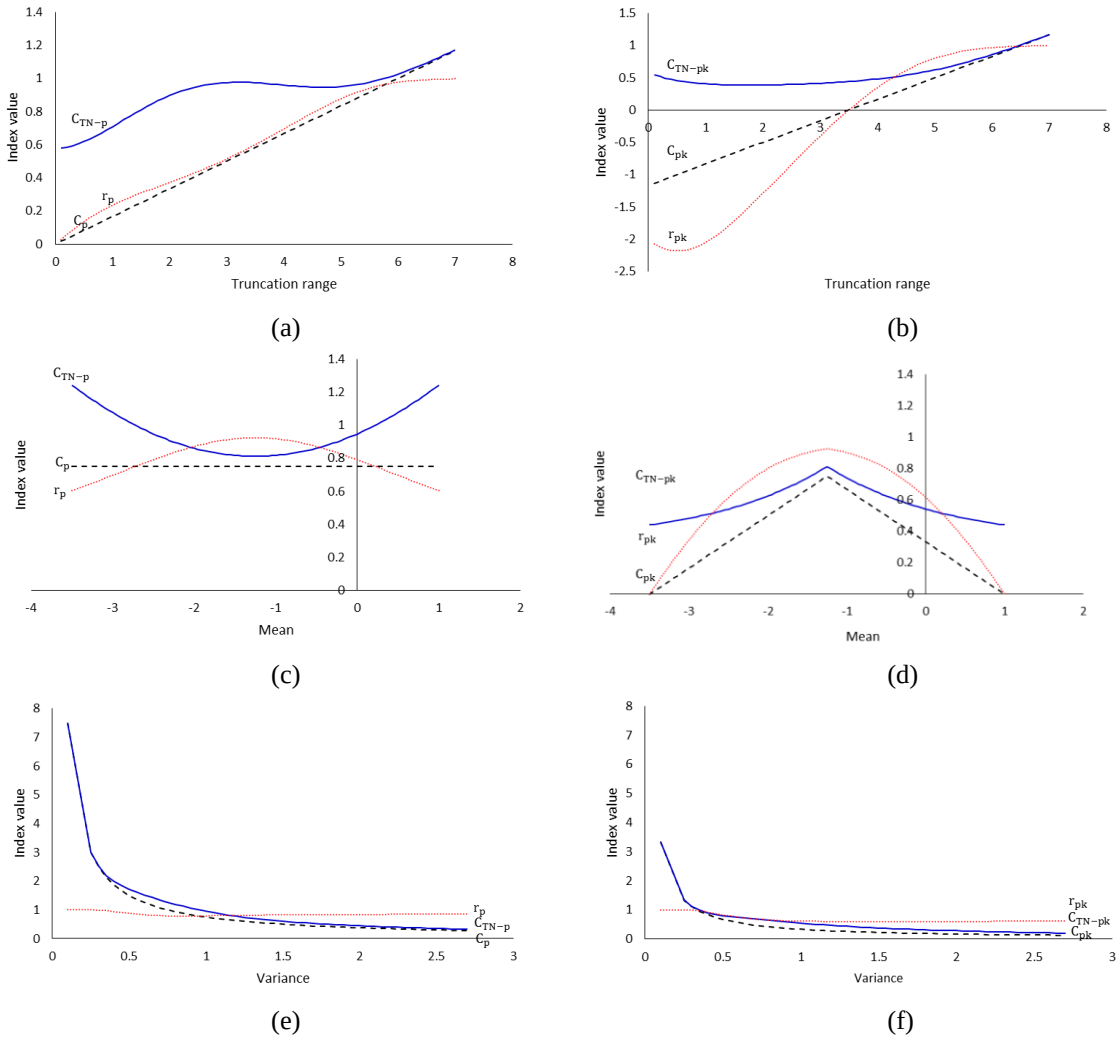
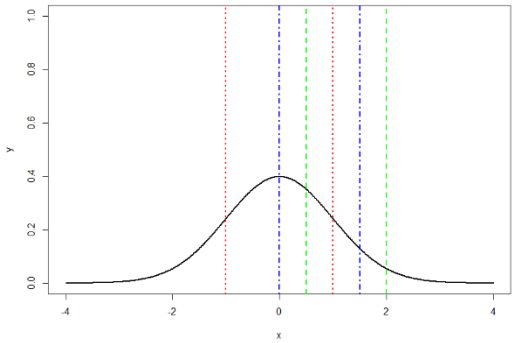


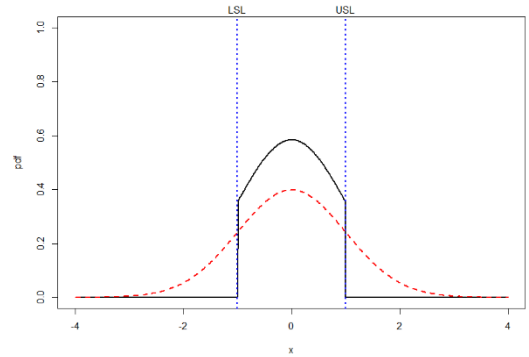
Figure 2.4 Sensitivity analysis results for scenario 4 in (a) and (b), scenario 5 in (c) and (d), and scenario 6 in (e) and (f).

$$\mu_{TN} = \mu + \sigma\left(\frac{\phi(z_{LSL}) - \phi(z_{USL})}{\Phi(z_{USL}) - \Phi(z_{LSL})}\right) = 0 + 1\left(\frac{\phi\left(\frac{-1-0}{1}\right) - \phi\left(\frac{1-0}{1}\right)}{\Phi\left(\frac{1-0}{1}\right) - \Phi\left(\frac{-1-0}{1}\right)}\right) = \frac{0.2420 - 0.2420}{0.8413 - 0.1587} = 0$$

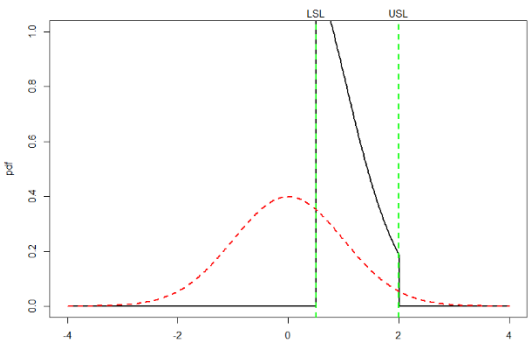
and $$\sigma_{TN} = \sqrt{\sigma^2\left(1 + \frac{(z_{LSL}) \times \phi(z_{LSL}) - (z_{USL}) \times \phi(z_{USL})}{\Phi(z_{USL}) - \Phi(z_{LSL})} - \left(\frac{\phi(z_{LSL}) - \phi(z_{USL})}{\Phi(z_{USL}) - \Phi(z_{LSL})}\right)^2\right)}$$



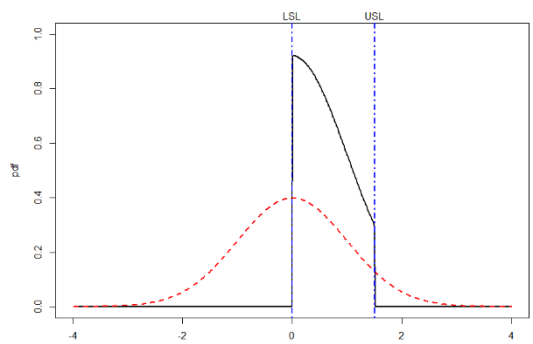
(a) The distributions with the specification limits for each customer



(b) The truncated distribution for customer A



(c) The truncated distribution for customer B



(d) The truncated distribution for customer C

Figure 2.5 Illustrative truncated distributions associated with three customers

$$= \sqrt{1^2\left(1+\frac{(-1)\times 0.2420-(1)\times 0.2420}{0.8413-0.1587}-\left(\frac{0.2420-0.2420}{0.8413-0.1587}\right)^2\right)} = 0.5396$$

The process capability indices are obtained from Equations (2.6) through (2.9) as follows

$$C_{TN-p} = \frac{USL-LSL}{6\ \sigma_{TN}} = \frac{1-(-1)}{6\times 0.5396} = 0.6178\,,\ C_{TN-pu} = \frac{USL-\mu_{TN}}{3\ \sigma_{TN}} = \frac{1-(0)}{3\times 0.5396} = 0.6178$$

$$C_{TN-pl} = \frac{\mu_{TN}-LSL}{3\ \sigma_{TN}} = \frac{0-(-1)}{3\times 0.5396} = 0.6178\ \ ,\ C_{TN-pk} = min\left\{C_{TN-pu}, C_{TN-pl}\right\} = 0.6178$$

Table 2.2 Data of the customers

Customer A					Customer B					Customer C				
-0.3314	0.3181	0.3995	-0.9821	-0.6882	1.1406	0.9594	0.5850	0.5406	0.7355	0.3467	1.2699	0.3117	0.4415	0.7530
0.9871	-0.9333	-0.0249	0.5218	0.8087	0.6138	1.7288	0.8751	0.7356	1.1082	1.0546	0.3202	0.0924	0.6491	0.9401
-0.0222	0.9424	0.2824	-0.3579	0.9668	1.0374	1.7490	0.7996	0.9462	0.5758	0.5673	1.3605	0.9591	0.5836	0.2313
-0.1130	-0.2151	0.7865	0.8831	-0.0197	1.3188	0.9031	0.8934	1.0329	0.5084	0.2269	0.6790	0.4091	1.1681	0.9829
0.0585	0.2964	-0.2800	0.0085	-0.3628	0.7905	1.5853	1.3405	1.8386	0.7622	0.1113	0.2993	0.1137	0.2129	0.2120
-0.4999	-0.5243	0.0741	-0.1524	-0.7510	0.8623	1.4824	1.1584	0.7914	0.7295	0.7628	1.3471	0.8687	0.0686	0.4409
-0.0676	0.0790	0.3984	0.7267	-0.2529	0.8770	0.6876	1.2322	0.5323	0.6744	1.1362	0.1241	1.1161	0.3874	0.1360
-0.1426	0.9448	0.5491	0.9173	0.8778	0.7615	0.9814	1.5953	0.8678	1.4955	0.3228	0.9451	0.8438	0.0676	0.0207
0.0519	0.5985	-0.1320	-0.6884	-0.3272	1.1473	0.5594	0.7797	1.5403	1.5533	0.7295	0.4653	0.5284	1.0303	0.3291
0.6074	0.9539	0.0857	-0.7616	0.6671	1.1369	1.5243	0.9859	0.7377	1.9755	0.0302	0.7422	0.5711	0.8494	0.7975
0.8159	0.2884	0.2524	0.4083	-0.9650	1.1099	1.5907	0.7457	1.8796	1.2574	0.8329	0.1912	0.6166	0.0965	0.2843
-0.7168	0.2427	0.2838	0.8312	0.6599	0.5236	0.6842	1.2462	1.0946	1.0963	0.7761	0.7005	0.1574	1.0634	0.0750
0.4865	0.2178	-0.1875	-0.1620	0.9484	1.3765	0.8826	0.7021	1.7426	1.3252	0.1942	0.9106	0.5871	0.2685	0.6785
0.7044	-0.9101	0.1434	0.6199	-0.4557	0.6238	1.0611	0.8859	0.6368	0.8379	0.0161	0.2735	1.1293	0.7615	0.5379
0.0026	0.2207	0.5125	0.6892	0.4990	1.2996	0.7719	0.5550	1.6381	0.8758	0.5083	0.6930	0.5776	0.0753	0.7954
0.7188	0.9643	-0.9541	0.0960	0.4609	1.1196	0.5555	1.3757	1.0401	1.3272	1.1085	0.1773	0.3160	0.0727	1.2133
0.6194	0.0359	0.5481	0.1275	0.3393	0.8620	0.9493	1.9274	1.1346	0.8109	0.5458	0.1725	0.5196	0.7597	1.0395
0.9915	0.0055	0.2320	0.9682	-0.8213	1.1570	1.8238	1.3392	1.6988	1.2029	0.3832	0.2601	0.0418	0.9121	0.8435
0.9688	0.3119	0.3708	0.6984	0.8547	1.5462	0.7122	1.2931	1.0386	1.1110	0.1489	0.0524	0.1468	0.3237	0.7018
0.2022	0.6260	0.3297	-0.7264	-0.0782	0.7522	0.5110	0.8669	1.8537	0.7924	0.1426	1.0624	0.5885	0.0725	1.1738

b) The STB-Type case

For the STB-type quality characteristic, the data for customer C is used to demonstrate this calculation. Given that the USL = 1.5, $\mu_{TS} = 0$ and $\sigma = 1$, C_{TS} is obtained as follows

From $\mu_{TS} = \mu - \sigma \left[\dfrac{\phi\left(\dfrac{x_u - \mu}{\sigma}\right)}{\Phi\left(\dfrac{x_u - \mu}{\sigma}\right)} \right] = 0 - 1 \times \left[\dfrac{0.1295}{0.9332} \right] = -0.1387$

and $\sigma_{TS} = \sqrt{\sigma^2 \left(1 - \frac{\left(\frac{x_u - \mu}{\sigma}\right) \times \phi\left(\frac{x_u - \mu}{\sigma}\right)}{\Phi\left(\frac{x_u - \mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_u - \mu}{\sigma}\right)}{\Phi\left(\frac{x_u - \mu}{\sigma}\right)} \right)^2 \right)}$

$$\sigma_{TS} = \sqrt{1 \times \left(1 - \frac{1.5 \times 0.1295}{0.9332} - \left(\frac{0.1295}{0.9332} \right)^2 \right)} = 0.9006$$

Using Equation (2.12), we then obtain $C_{TS} = \frac{USL - \mu_{TS}}{3\ \sigma_{TS}} = \frac{1.5 - (-0.1387)}{3 \times 0.9006} = 0.6065$

c) The LTB-Type case

The data for customer B is used to illustrate an LTB-type characteristic. The LSL is given as 0.5, with the process mean and variance remaining the same at 0 and 1, respectively. The LTB-type truncated normal distribution is used to measure this case as follows

Since $\mu_{TL} = \mu + \sigma \left(\frac{\phi\left(\frac{x_l - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{x_l - \mu}{\sigma}\right)} \right) = 0 + 1 \times \left(\frac{0.3521}{1 - 0.6915} \right) = 1.1413$

and $\sigma_{TL} = \sqrt{\sigma^2 \left(1 + \frac{\left(\frac{x_l - \mu}{\sigma}\right) \times \phi\left(\frac{x_l - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{x_l - \mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_l - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{x_l - \mu}{\sigma}\right)} \right)^2 \right)}$

$$= \sqrt{1 \times \left(1 + \frac{0.5 \times 0.3521}{1 - 0.6915} - \left(\frac{0.3521}{1 - 0.6915} \right)^2 \right)} = 0.2774$$

From Equation (2.15), we have that $C_{TL} = \frac{\mu_{TL} - LSL}{3\ \sigma_{TL}} = \frac{1.1413 - 0.5}{3 \times 0.2774} = 0.7706$

2.6 Comparison of proposed PCIs and traditional PCIs

2.6.1 Traditional PCIs

Using the numerical example in Section 2.5, the PCIs are calculated directly from the process mean, variance, and specification limits for each customer by assuming that the process is normally distributed. The capability indices are obtained through the following equations: $C_p = \frac{USL - LSL}{6\sigma}$, and $C_{pk} = \min\left(C_{pu}, C_{pl}\right)$ where $C_{pl} = \frac{\mu - LSL}{3\sigma}$ and $C_{pu} = \frac{USL - \mu}{3\sigma}$. The results are shown in Table 2.4.

2.6.2 The data transformation method

Considering the distributions depicted in Figures 2.5(a)-(d), the truncated normal distribution appears to be a more reasonable fit than that of the normal distribution. In this case, as suggested in Polansky et al. (1998) and Chou et al. (1998), the truncated normal data needs to be transformed to satisfy the normality assumption in order to obtain the true PCI values. After transforming the data using the Johnson data transformation method, the non-normal data and its corresponding specification limits then meet the normality requirement. Using Minitab®, the transforming equations for each customer were obtained; in Table 2.3, the transformation critical values, including the p-value, Z value, transformation type, and transformation function are presented. The transformed specification limits, mean, variance, and the calculated PCIs values are then

shown in Table 2.4. Using the equations shown in Section 2.6.1, the traditional PCIs $(C_p, C_{pl}, C_{pu}, C_{pk})$ are then obtained for the purpose of comparison. As part of the comparison, the probability plot of the original data (non-normal) and the transformed data (normal) are graphed in Figure 2.6(a)-(f). It is noted that the PCI values for Customer C could not be calculated – the logarithm term returned a complex number when the specification limits were substituted into the transformation function, a limitation that is generally found in some instances with the Johnson transformation application.

2.6.3 The proposed customer-perceived PCIs based on the truncated normal distribution concept

The same data set is used for testing our proposed models. Since the data has both *USL* and *LSL*, the NTB-type truncated normal distribution PCIs, C_{TN-pl}, C_{TN-pu}, C_{TN-pl}, and C_{TN-pk}, will be used in this comparison. The process mean and variance are substituted by the two-sided truncated normal distribution mean (μ_{TN}) and variance (σ_{TN}) using Equations (2.2) and (2.6), respectively, and the specification limits are used as the truncation points. Shown in Table 2.4 are the process capability index values obtained from Equations (2.6) through (2.9). Of note is the fact that the proposed PCI values are higher than the values obtained from the traditional PCIs method in Table 2.4, as illustrated in Figure 2.7(a)-(c).

Table 2.3 Critical values obtained using the Johnson transformation method

Data	P-value for the best fit	Z value for the best fit	Transformation type	Transformation function
Customer A	0.2641	0.54	S_b	$-0.851737 + 1.01175 \times \ln\left(\frac{x + 1.80074}{1.20522 - x}\right)$
Customer B	0.8054	0.82	S_b	$0.710664 + 0.932677 \times \ln\left(\frac{x - 0.452505}{2.19812 - x}\right)$
Customer C	0.8132	0.67	S_b	$0.413628 + 0.618295 \times \ln\left(\frac{x - 0.00053786}{1.38902 - x}\right)$

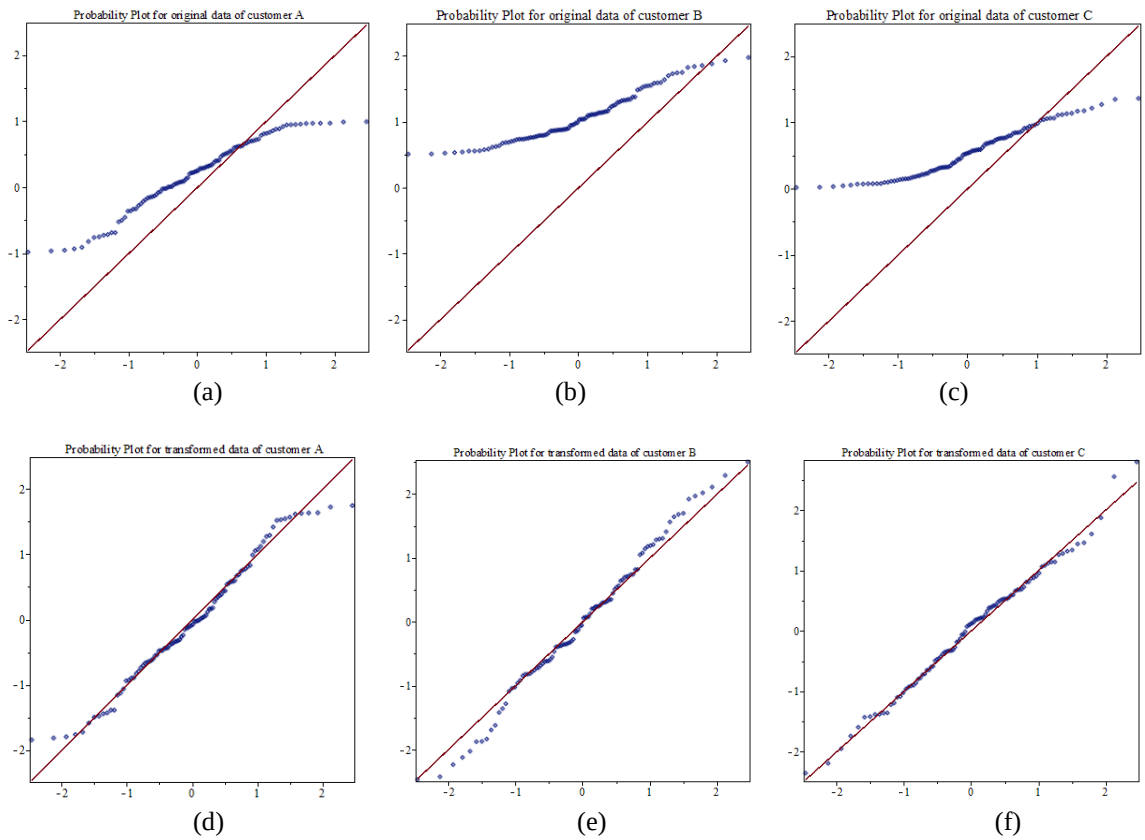


Figure 2.6 Probability plot of the original data of (a) customer A, (b) customer B, (c) customer C, and the probability plot of the transformed data of (d) customer A, (e) customer B, and (f) customer C.

2.6.4 Studies on linking the proposed PCIs to defective rates

A defective rate is another convenient indicator of measuring process performance. The defective rate is the number of defective parts per total inspected samples. Assuming that the quality characteristic X is normally distributed, $X \sim N\left(\mu, \sigma^2\right)$, the defective rate is expressed as:

$$\text{Defective rate} = 1 - \int_{LSL}^{USL} f\left(x\right) dx \tag{2.16}$$

The proportion of defective units is affected by the sigma level and C_p value, as shown in Table 2.5.

In this study, a customer's perception of the number of defective units is found to be zero, since all defective units are assumed to be eliminated during the inspection and screening processes. Thus, by using Equation (2.16), the C_{pk} values from Table 2.4, the defective rate is calculated for each of the indices and is shown in Table 2.6. The results present evidence that the proposed PCIs have a lower defective rate when compared to traditional PCIs; in Figure 2.7(d), a defective rate comparison is illustrated.

2.7 Conclusions

Process capability indices are a vital means of understanding and interpreting a process' ability to manufacture a product that meets specifications. Several process capability indices that apply a manufacturer's point of view, such as C_p, C_{pl}, and C_{pu}, have gained considerable popularity in many industries. As part of customer-driven continuous quality improvement, the proposed customer-perceived capability indices developed in this chapter fill this void. In this chapter, we observed that the proposed

process capability indices that the customer actually perceives offer higher ratios and lower parts per million than the manufacturer-focused traditional process capability indices. It is believed that the proposed capability indices can offer some valuable insights as a complementary system of measures for process performance. An analysis diagram of the customer-perceived process capability indices is depicted in Figure 2.8. New findings have the potential to impact a wide range of many other engineering and science problems such as those found in process improvement, allowing for a more accurate understanding of process capability analysis.

Table 2.4 Process capability indices obtained from the comparison study

	Customer	Customer specifications		Customer's mean and variance		The Traditional PCIs			
The traditional PCIs		LSL	USL	μ_C	σ_C	C_p	C_{pu}	C_{pl}	C_{pk}
	A	-1	1	0	0.54	0.6173	0.6173	0.6173	0.6173
	B	0.5	2	1.04	0.39	0.6410	0.8205	0.4615	0.4615
	C	0	1.5	0.62	0.41	0.6098	0.7155	0.5041	0.5041
	Customer	Transformed specification limits		Transformed mean and variance of each customer's distribution		The PCIs obtained from the data transformation method			
The data transformation-based PCIs		$\tilde{L}$	$\tilde{U}$	$\tilde{\mu}_C$	$\tilde{\sigma}_C$	C_p	C_{pu}	C_{pl}	C_{pk}
	A	-1.8767	1.7925	-0.0288	0.9508	0.6432	0.6385	0.6478	0.6385
	B	-2.6252	2.6278	-0.0080	1.1080	0.7902	0.7930	0.7874	0.7874
	C	n/a	n/a	0.0154	0.9853	n/a	n/a	n/a	n/a
	Customer	Customer specifications		Two-sided truncated normal distribution mean and variance		Customer-perceived PCIs based on the truncated normal distribution (proposed model)			
The customer-perceived PCIs (the proposed model)		LSL	USL	μ_{TN}	σ_{TN}	C_{TN-p}	C_{TN-pu}	C_{TN-pl}	C_{TN-pk}
	A	-1	1	0	0.3717	0.8969	0.8969	0.8969	0.8969
	B	0.5	2	1.1869	0.1393	1.7948	1.94578	1.64381	1.64381
	C	0	1.5	0.7150	0.1873	1.3346	1.3968	1.2723	1.2723

Table 2.5 Defective rate for the C_{pk}

C_{pk} Value	Sigma Level	% Defective	Defective Rate (ppm)
0.33	1	0.317310508	317,311
0.50	1.5	0.133614403	133,614
0.67	2	0.045500264	45,500
0.83	2.5	0.012419331	12,419
1.00	3	0.002699796	2,700
1.17	3.5	0.000465258	465
1.33	4	0.000063342	63
1.50	4.5	0.000006795	7
1.67	5	0.000000573	1
1.83	5.5	0.000000038	0
2.00	6	0.000000002	0

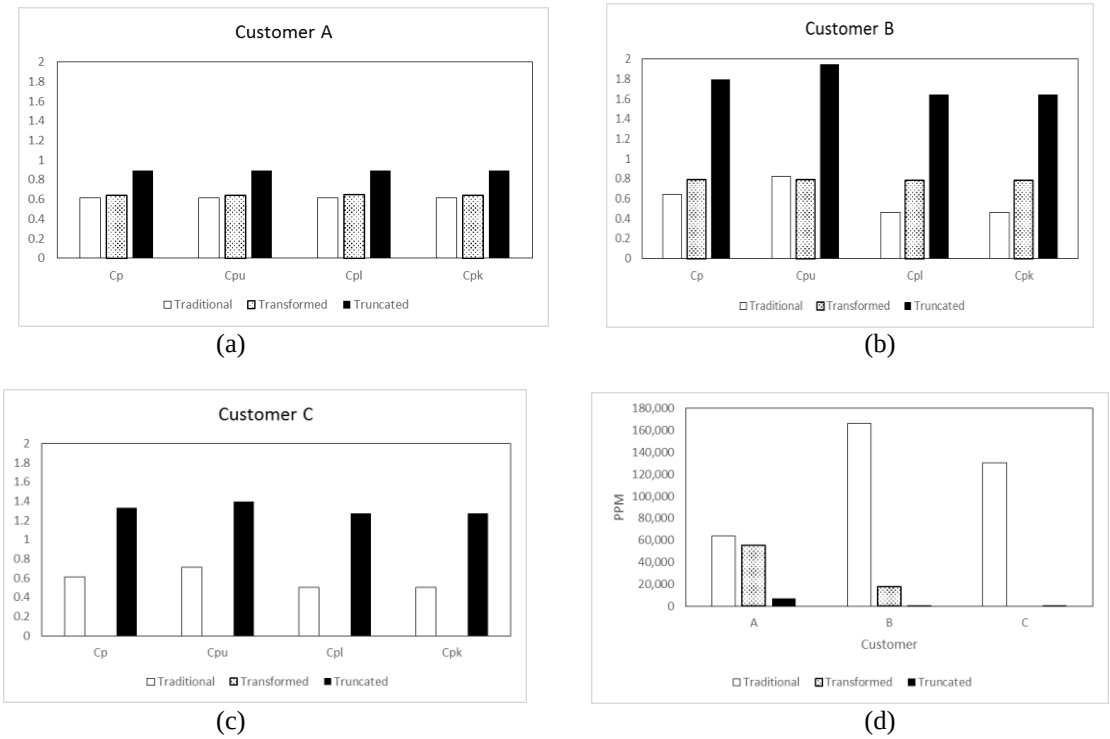


Figure 2.7 Comparison of the traditional PCI values, transformed data based PCI values, and the proposed truncated normal based PCI values for (a) customer A, (b) customer B, (c) customer C, and (d) the comparison of defective rates.

Table 2.6 The index and defective rate comparison

Customer	Method					
	The traditional PCIs		The data transformation based PCIs		Customer's perceived PCIs based on the truncated normal distribution	
	C_{pk}	Defective Rate (ppm)	C_{pk}	Defective Rate (ppm)	$C_{TN\text{-}pk}$	Defective Rate (ppm)
A	0.61728	64,049	0.6385	55,429	0.89687	7,132
B	0.46154	166,169	0.7874	18,167	1.64381	1
C	0.50407	130,480	*	*	1.27229	135

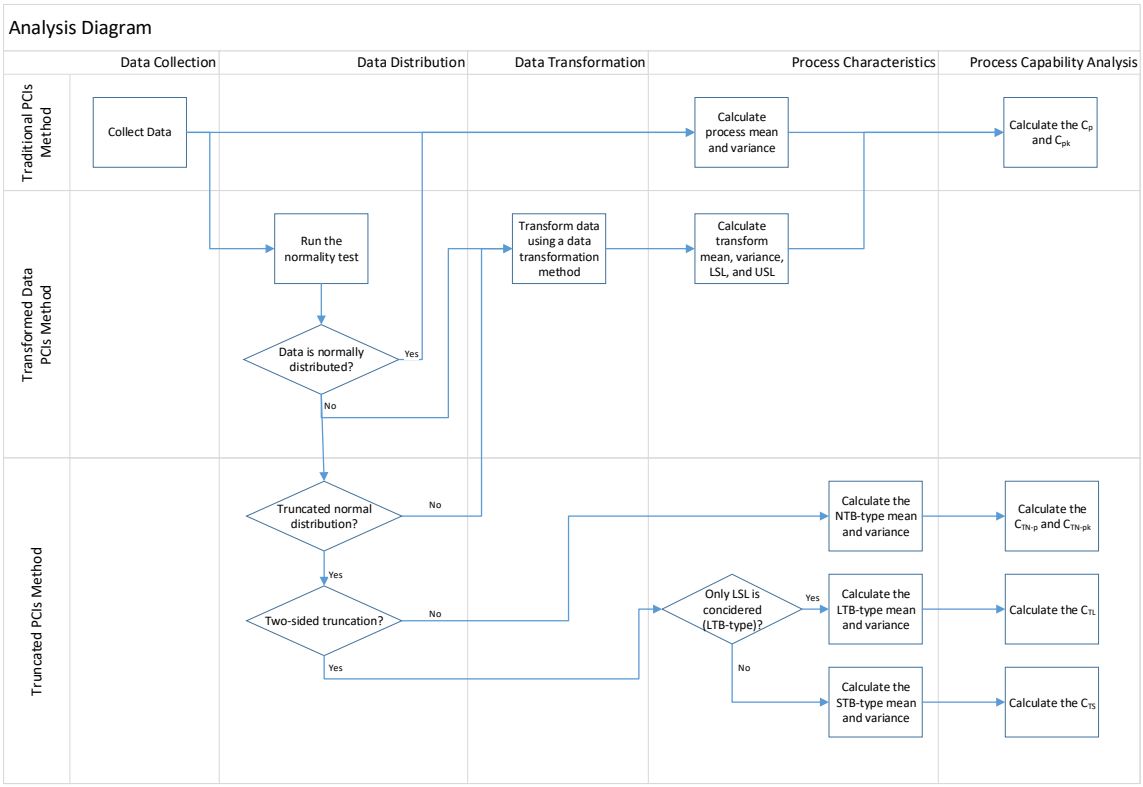


Figure 2.8 Analysis diagram for using customer-perceived PCIs

CHAPTER 3
THE TARGET-BASED PROCESS CAPABILITY INDICES FOR THE TRUNCATED NORMAL DISTRIBUTION AND THE CONFIDENCE INTERVAL ESTIMATORS

3.1 Introduction

Process capability analysis, which is a technique frequently used to measure process performance, involves a comparison of product output against its specifications. To do so, statistical parameters for quality control, such as the process mean and variance are investigated. The process mean for a particular product characteristic indicates the average value of its observations, while the process variance depicts the spread of these observations around the mean. To consider the position of the mean or the relationship between variance and a characteristic's specifications alone is usually not sufficient enough for comparing different processes. A benchmark for comparison such as the sigma level is needed, where processes that have natural output at higher sigma levels may be considered more capable. For this reason, process capability indices became very popular – a specific index value may be obtained, just by computing the ratio between the natural variability of an in-control process and its tolerance. Larger index values suggest that the number of product defects is small and the process is well-performed.

Recently, researchers have sought to relax the assumption of normality for processes in the development of PCIs. These efforts have considered either transforming non-normal data to satisfy using PCIs based on a normal distribution, or altering the distribution itself, such as indices developed with the log-normal distribution as its basis.

Another trend in the PCI research is the extension toward measuring process performance when multiple quality characteristics are considered. These multivariate PCIs included work with data transformations, new approaches in approximation, and even vector-valued results as a means to provide a greater sense of capability.

When observations on a product's characteristics fail to fall within the specification limits, the product is typically scrapped. As a result, the actual distribution of observations after inspection that is recognized (or perceived) by the customer, is truncated. If PCIs based on the assumption of normality are used to assess process performance where the underlying distribution is actually truncated, it is likely that the measurement will be significantly inaccurate. The study of PCIs based on a truncated normal distribution was the focus of several studies in the last twenty years. For instance, the distribution was used to model a concentrated product in a chemical process by Polansky et al. (1998), a supplier's screened lot by Asokan and Unnithan (1999), the tolerance of an assembled gap between a bore and a shaft by Sweet and Tu (2006), a light emitting diode production process by Pearn et al. (2007), the yield of a semiconductor manufacturing practice in Tao and Xinzhang (2012), and non-target based quality characteristics by Wu et al. (2015) and Khamkanya, et al. (2016).

To accurately evaluate and compare the capability among different processes, it is imperative that an appropriate model be chosen for the underlying distribution of characteristic observations. With little previous work done on the theoretical foundation of PCIs using the truncated normal distribution, this chapter proposes a set of indices that incorporate the distribution in their development. We refer to these indices as "posterior"

process capability indices, since they are formulated based upon the underlying distribution following product inspection. A loss function is also included to account for the diminished product quality when the process mean deviates from the ideal target value. After providing the posterior PCI development details and insights in Section 3.3, a simulation study is introduced in Section 3.4 to facilitate comparing traditional PCIs with the proposed posterior PCIs. Subsequently, numerical examples are presented in Section 3.5 to illustrate their use in practice, given an industrial context. Finally, prior to concluding the manuscript, the confidence interval bounds for the proposed indices are derived, so that various levels of accuracy associated with sample size may be calculated.

3.2 Traditional target-based process capability indices

In measuring the capability of a process, a PCI is designed to quantify the relationship between the actual observations of a quality characteristic and its specification limits. Several well-known PCIs, such as C_p, C_{pk}, and C_{pm}, are widely cited and continue to be studied. The first PCI to appear in the research literature, introduced by Kane (1986), was C_p. Assuming that the process mean is centered at its target value, C_p not only computed its ratio based upon a standard six sigma spread, but also provided an interpretation of its corresponding defective rate. For instance, a process with observations following a normal distribution and $C_p = 1.0$ would have 0.27 percent of its observations beyond the specification limits. Unfortunately, however, this index fails to account for both the mean and variance in its measurement of process performance. To account for the simultaneous position of the mean and the spread of observations, the index, C_{pk}, was introduced. While C_{pk} was adept at measuring process yield in terms of its

specification limits, it failed to adequately assess whether the process mean was centered on a target value. Hence, C_{pk} was often used in comparison with C_p, whereby an off-center process resulted in $C_{pk} < C_p$. The performance testing of C_{pk} is studied in Pearn and Lin (2004)

In order to ensure a product characteristic is achieving its highest level of quality, it is necessary to minimize the difference between the process mean and its ideal value, or target. Since selling only perfect items to the customer seems impossible in economic reality, the interval of the specification limits, or tolerance, is used during an inspection process to determine if an item is acceptable (or passing). Although a customer may receive an item that has passed its inspection, there may be some level of risk incurred in delivering an item that fails to achieve its ideal target value. This risk of receiving an imperfect item is frequently referred to as customer loss, and can be represented by linear, step, or quadratic loss functions in a model framework, see Cho and Leonard (1997). The quadratic loss function, also known as the Taguchi loss function, is well-known for approximating the loss to the customer. The earliest use of the loss function within a process capability index, whereby off-target production may be measured, was that of the target-based index, C_{pm}, proposed independently by Chan et al. (1988) and Boyles (1991). The index is defined as $C_{pm} = (USL - LSL) \Big/ \left(6\sqrt{\sigma^2 + (\mu - \tau)^2}\right)$, where LSL and USL are the lower and upper specification limits, respectively, μ is the process mean, σ^2 is the process variance, and τ represents the target value. Despite the improvements offered by C_{pm}, the index failed to serve as an adequate measure when the

position of the target is asymmetric with respect to the specification limits. Furthermore, C_{pm} index values may be identical for several different combinations of the process mean and variance. For example, consider two processes with the same set of specification limits, *A* and *B*, where process *A* has a mean of 50 and variance of 5, and process *B* has a mean and variance of 53 and 3.83, respectively. Given these settings, C_{pm} calculates the capability of both processes to be 1.0. To address this issue and differentiate among identical C_{pm} values, the C_{pmk} index was proposed by Pearn et al. (1992) to adequately treat this problem.

Several other variations of C_p, C_{pk}, and C_{pm} have been proposed by PCI researchers. Vännman (1995) proposed a unified class of indices to generalize the four well-known index forms (C_p, C_{pk}, C_{pm} and C_{pmk}), by using two non-negative parameters *u* and *v* in a composite index $C_p(u,v)$. Chen and Pearn (1997) modified $C_p(u,v)$ and proposed a quantile-based PCI, $C_{Np}(u,v)$, which relaxed the assumption of normality in the PCIs. Eslamipoor and Hosseini-nasab (2016) later incorporated the loss function into $C_p(u,v)$ to account for the deterioration in product quality when the mean diverges from its target value. An extended version of C_{pmk} to evaluate both process yield and a potential process loss of resubmitted lots is proposed in Wu et al. (2015). Moreover, to deal with the non-normal effect of a process distribution, Wang et al. (2016) proposed the inverse normalizing transformation method for the process capability index computation.

3.3 Development of target-based PCIs using the truncated normal distribution

Although observations of a quality characteristic for a product may conform to the specification limits, an off-target and on-target observation should not be considered the same. Taguchi (1986) recognized this and used the loss function to account for the target deviation. Given a target value, τ, for a nominal-the best type of quality characteristic *Y* assumed to have observations *y* that are normally distributed, the quadratic loss function is given by $L_N(Y) = k(y-\tau)^2$, where *k*, a proportionality coefficient, may be calculated by comparing the known function limit and the amount of loss caused by a defect. In this construct, the loss is zero at the target value and increases when *y* deviates from the target in any direction. The average loss of *n* products is

$$L_N(Y) = \frac{k(y_1-\tau)^2 + k(y_2-\tau)^2 + \cdots + k(y_n-\tau)^2}{n} = \frac{k\left[(y_1-\tau)^2 + (y_2-\tau)^2 + \cdots + (y_n-\tau)^2\right]}{n}$$

$$L_N(Y) = \frac{k}{n}\sum_{i=1}^{n}(y_i-\tau)^2 = \frac{k}{n}\sum_{i=1}^{n}(y_i-\mu)^2 + (\mu-\tau)^2$$

$$L_N(Y) = k\left(\frac{(y_1-\mu)^2 + (y_2-\mu)^2 + \cdots + (y_n-\mu)^2}{n} + (\mu-\tau)^2\right) \qquad (3.1)$$

Taking the expected value of the loss function, we obtain the NTB-type formulation: $E\left[L_N(Y)\right] = k\left(\sigma^2 + (\mu-\tau)^2\right)$, where μ is the process mean, and σ^2 is the process variance.

3.3.1 Model development

Suppose that the quality characteristic of interest, *Y*, is normally distributed with process mean and variance, μ and σ^2, respectively, and y_l and y_u represent the lower and upper specification limits for the process, respectively. Moreover, let Y_T represent the truncated normal random variable with observations y_T for this distribution. Then, the probability density function of Y_T is defined as $f\left(Y_T\right)=\dfrac{f\left(y_T\right)}{\int_{y_l}^{y_u} f_Y\left(y\right)dy}$, and the cumulative probability density function of Y_T is written as $F\left(Y_T\right)=\int_{-\infty}^{y_T}\dfrac{f\left(y_T\right)}{\int_{y_l}^{y_u} f_Y\left(y\right)dy}dy_T$.

The truncated mean and the truncated variance are obtained from $\mu_T=\int_{-\infty}^{\infty} y_T f\left(Y_T\right)dy_T$ and $\sigma_T^2=\int_{-\infty}^{\infty} y_T^2 f\left(Y_T\right)dy_T-\left(\int_{-\infty}^{\infty} y_T f\left(Y_T\right)dy_T\right)^2$. When $y_l \le y_T \le y_u$, the truncated mean and variance are formulated as

$$\mu_T=\mu+\sigma\left(\frac{\phi\left(z_l\right)-\phi\left(z_u\right)}{\Phi\left(z_u\right)-\Phi\left(z_l\right)}\right), \text{ and} \tag{3.2}$$

$$\sigma_T^2=\sigma^2\left(1+\frac{\left(z_l\right)\times\phi\left(z_l\right)-\left(z_u\right)\times\phi\left(z_u\right)}{\Phi\left(z_u\right)-\Phi\left(z_l\right)}-\left(\frac{\phi\left(z_l\right)-\phi\left(z_u\right)}{\Phi\left(z_u\right)-\Phi\left(z_l\right)}\right)^2\right) \tag{3.3}$$

where $z_l=\dfrac{y_l-\mu}{\sigma}$, $z_u=\dfrac{y_u-\mu}{\sigma}$, and $\phi(\bullet)$ and $\Phi(\bullet)$ represent the probability density function and cumulative density function for the distribution, respectively. In addition,

readers may find more information regarding the statistical inferences and applications of the truncated distribution in Cohen (1991), Cha and Cho (2015), and Krenek et al. (2016).

When Equations (3.1), (3.2), and (3.3) are integrated into the formulation of the C_{pm} index, a target-based PCI for a truncated normal distribution, C_{T-pm}, may be introduced as

$$C_{T-pm} = \frac{USL - LSL}{6\sqrt{\left[\sigma^2\left(1+\frac{(z_l)\times\phi(z_l)-(z_u)\times\phi(z_u)}{\Phi(z_u)-\Phi(z_l)}-\left(\frac{\phi(z_l)-\phi(z_u)}{\Phi(z_u)-\Phi(z_l)}\right)^2\right)\right]+\left[\left(\mu+\sigma\left(\frac{\phi(z_l)-\phi(z_u)}{\Phi(z_u)-\Phi(z_l)}\right)\right)-\tau\right]^2}}$$

$$C_{T-pm} = \frac{USL - LSL}{6\sqrt{\sigma_T^2 + (\mu_T - \tau)^2}} \tag{3.4}$$

In addition, to increase the sensitivity of measurements to the deviation of the process mean from the desired target, a truncated formulation based upon C_{pmk} may also be introduced. Pearn et al. (1992) defined C_{pmk} as

$$C_{pmk} = \frac{C_{pk}}{\sqrt{1+\left(\frac{\mu-\tau}{\sigma}\right)^2}}, \text{ where } C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\} \tag{3.5}$$

For the truncated version of this index, the C_{pk} term is replaced with $C_{T\text{-}pk}$, where

$$C_{T-pk} = \min\left\{C_{T-pu}, C_{T-pl}\right\}, \tag{3.6}$$

$$C_{T-pl} = \frac{\mu+\sigma\left(\frac{\phi(z_{LSL})-\phi(z_{USL})}{\Phi(z_{USL})-\Phi(z_{LSL})}\right)-LSL}{3\sqrt{\sigma^2\left(1+\frac{(z_{LSL})\times\phi(z_{LSL})-(z_{USL})\times\phi(z_{USL})}{\Phi(z_{USL})-\Phi(z_{LSL})}-\left(\frac{\phi(z_{LSL})-\phi(z_{USL})}{\Phi(z_{USL})-\Phi(z_{LSL})}\right)^2\right)}}$$

$$C_{T-pl} = \frac{\mu_T - LSL}{3\sigma_T} \tag{3.7}$$

$$C_{T-pu} = \frac{USL - \left[\mu + \sigma\left(\frac{\phi(z_{LSL}) - \phi(z_{USL})}{\Phi(z_{USL}) - \Phi(z_{LSL})}\right)\right]}{3\sqrt{\sigma^2\left(1 + \frac{(z_{LSL}) \times \phi(z_{LSL}) - (z_{USL}) \times \phi(z_{USL})}{\Phi(z_{USL}) - \Phi(z_{LSL})} - \left(\frac{\phi(z_{LSL}) - \phi(z_{USL})}{\Phi(z_{USL}) - \Phi(z_{LSL})}\right)^2\right)}}$$

$$C_{T-pu} = \frac{USL - \mu_T}{3\sigma_T} \tag{3.8}$$

where $z_{LSL} = \frac{y_l - \mu}{\sigma}$ and $z_{USL} = \frac{y_u - \mu}{\sigma}$.

Using Equations (3.5) and (3.6), we can then formulate C_{T-pmk} as a truncated version of C_{pmk}, which is expressed as

$$C_{T-pmk} = \frac{C_{T-pk}}{\sqrt{1 + \left[\left\{\left(\mu + \sigma\left(\frac{\phi(z_l) - \phi(z_u)}{\Phi(z_u) - \Phi(z_l)}\right)\right) - \tau\right\} \Big/ \sqrt{\sigma^2\left(1 + \frac{(z_l) \times \phi(z_l) - (z_u) \times \phi(z_u)}{\Phi(z_u) - \Phi(z_l)} - \left(\frac{\phi(z_l) - \phi(z_u)}{\Phi(z_u) - \Phi(z_l)}\right)^2\right)}\right]^2}}$$

$$C_{T-pkm} = \frac{C_{T-pk}}{\sqrt{1 + \left(\frac{\mu_T - \tau}{\sigma_T}\right)^2}} \tag{3.9}$$

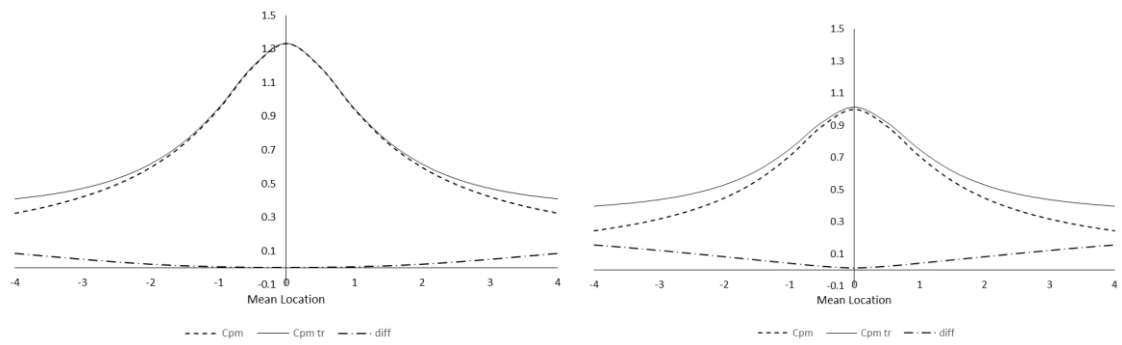
A similar procedure was performed to develop the truncated posterior index, C_{T-p} modelled from the traditional C_p.

3.3.2 Insights

To observe the behavior of the index, C_{T-pm}, we first examined the effect of the mean location within the truncation range on its measurements. Since one of the most important assumptions of C_{pm} is that the mean is located in the center within the interval of the specification limits, a specific objective of the study is to see if the proposed index can relax this assumption. In testing the indices, the variables are initially fixed, with the target set to zero, standard deviation to one, and the mean ranging from -4 to 4. In Figure 3.1(a), where [*LSL*, *USL*] = [-4, 4], it is observed that both C_{pm} and C_{T-pm} have similar trends in measurement. If the truncation points $[y_l, y_u]$ approach $(-\infty, \infty)$, it can be seen that the truncated normal distribution becomes the normal distribution, and hence C_{T-pm} approaches C_{pm}. When the specification range is tightened, both PCI values decrease, as depicted in Figures 3.1(b), 3.1(c), and 3.1(d). However, the C_{T-pm} index values are observed to be higher than C_{pm}, especially when the mean is not centered on the target.

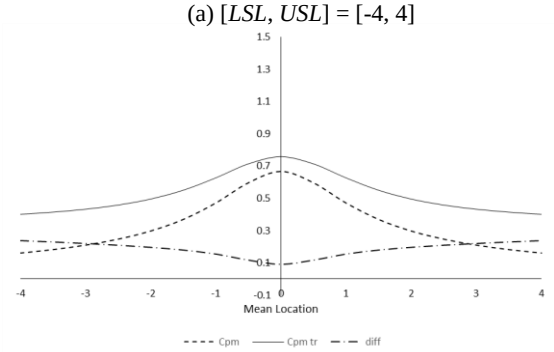
We also examined how the indices responded to an asymmetric tolerance setting as opposed to the target being positioned at the specification interval midpoint (*m*), $m = (USL - LSL)/2$. Given that $u = 0$, $\sigma = 1$, $\tau = m$, *LSL* = [-4, 4) and *USL* = 4; when the target is at the midpoint of the interval and the specification widens asymmetrically, both C_{pm} and C_{T-pm} increase quadratically. The difference between these indices decreases when the specification range is expanded, as depicted in Figure 3.2(a). As can be seen in Figure 3.2(b), we study a characteristic of C_{pm} when target is not centered with

asymmetric changed specifications by setting $u = \tau = 0$, $\sigma = 1$, *LSL* = [-4, 4) and *USL* = 4. We found that C_{pm} increases linearly because C_{pm} computes its specification range as *USL-LSL*, and so we cannot differentiate shifts in the range, with the intervals [4,0] and [-2, 2], and each has a specification range of 4. As a result, when the process mean and variance are fixed and the specification range is widened, C_{pm} exhibits a linear increase in measurement regardless of the type of specification range (symmetric or asymmetric). In contrast, since the truncated mean and variance are derived using the specific bounds for the probability density function and the cumulative density function, C_{T-pm} exhibits a quadratic increase in measurement with asymmetric adjustments in the specification limits. In Figure 3.2(c), we observe the effect of broadening the specification range around a centered target, as the settings change to $u = \tau = m$, $\sigma = 1$, *LSL* = [-4, 0) and *USL* = (0, 4]. In this case, the difference between C_{pm} and C_{T-pm} decreases as the specification interval width increases. In Figure 3.2(d), we use the same settings as Figure 3.2(c) except $\sigma = 1.05$ to study the characteristic of the indices when the process variance is increased. The result shows that C_{T-pm} is more responsive to the increased variance than C_{pm}. In conclusion, if the statement $\tau = m = (USL - LSL)/2$ is valid, C_{pm} is recommended for its ease of calculation; otherwise, C_{T-pm} is recommended as a viable alternative for evaluating process capability.

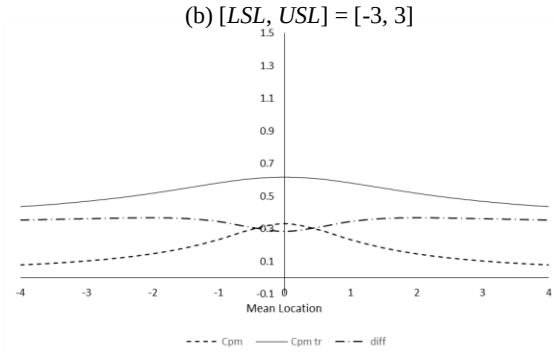


(a) [*LSL*, *USL*] = [-4, 4]

(b) [*LSL*, *USL*] = [-3, 3]



(c) [*LSL*, *USL*] = [-2, 2]

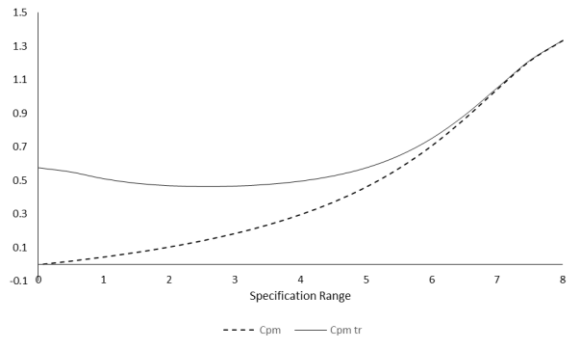


(d) [*LSL*, *USL*] = [-1, 1]

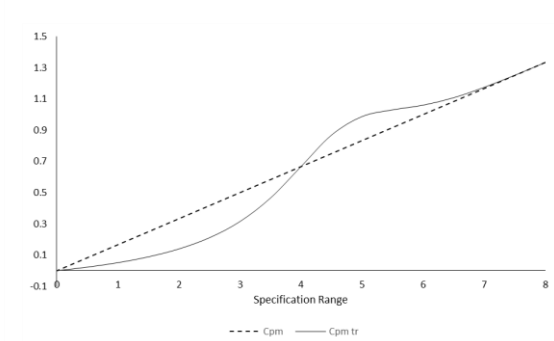
Figure 3.1 The index value of C_{T-pm} and C_{pm} under different specification limits

3.4 Comparison study of PCIs

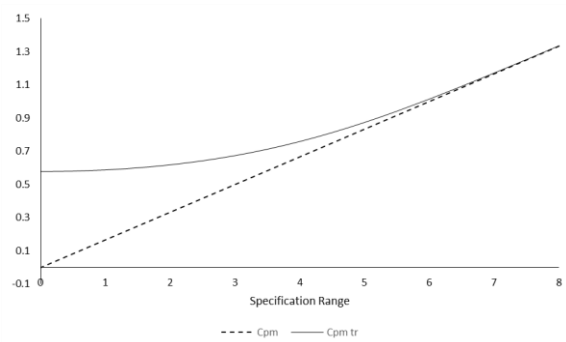
After gaining some insights on the behavior of truncated PCIs, a comparison study is performed between the proposed posterior indices and their complementary traditional formulation. Generally, simulation techniques are employed in the comparison of PCIs to assess the efficacy and accuracy of their measurement. For example, English and Taylor (1993) used fixed values of C_p and C_{pk} in their simulation model to investigate the robustness of the indices to non-normality, while Rivera et al. (1995), Tang and Than (1999), and Hosseinifard et al. (2009) utilized the actual number of non-conforming units to evaluate the measurement of their respective PCIs.



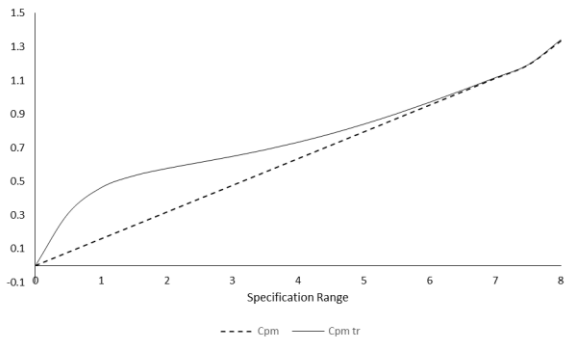
(a) $\tau = m,\ \mu = 0$ with asymmetric specification adjustments



(b) $\tau = \mu = 0$ with asymmetric specification adjustments



(c) $\tau = \mu = m$ with symmetric specification adjustments



(d) $\tau = \mu = m, \sigma = 1.05$ with symmetric specification

Figure 3.2 Index values of C_{T-pm} and C_{pm}

In order to evaluate the performance of traditional PCIs, C_p, C_{pk}, C_{pm} and C_{pmk}, against their truncated posterior counterparts, $C_{T\text{-}p}$, $C_{T\text{-}pk}$, $C_{T\text{-}pm}$ and $C_{T\text{-}pmk}$, respectively, the sample sizes are varied under a diverse array of settings. Furthermore, greater focus is given to two factors that may have a high impact on the difference in PCI values, the truncation range and the location of the mean. Since it is known that a narrow truncation range tends to create leptokurtic conditions (i.e. a positive kurtosis or high-peaked bell curve), two settings of the truncation range are implemented: narrow and wide truncation ranges. In addition, three different mean locations are applied to gain an understanding of

PCI measurement for the off-target case. In summary, a simulation model programmed in Matlab® was developed with six testing scenarios (see Matlab code in Appendix D), whose settings are shown in Table 3.1. Also, the simulation procedure is depicted in Figure 3.3.

Table 3.1 Scenario settings

Scenario	Truncation range	Mean	*LSL*	*USL*
1	Wide	0	-3	3
2	Wide	1	-3	3
3	Wide	-1	-3	3
4	Narrow	0	-2	2
5	Narrow	1	-2	2
6	Narrow	-1	-2	2

3.4.1 Simulation Steps

Step 1: Generate the population: A set of the population (Y) is generated that has C_p = 1.0, by assuming that $Y \sim N(\mu, \sigma^2)$, where $\mu = \{-1, 0, 1\}$, $\sigma = (USL - LSL)/6$, where $\mu = \{-1, 0, 1\}$, $\sigma = (USL - LSL)/6$, and [*LSL*, *USL*] = {[-2,2], [-3,3]}, depending on the specific scenario setting.

Step 2: Evaluate the process capability index of the population: The traditional PCIs $(C_p, C_{pk}, C_{pm}, C_{pmk})$, along with their complementary truncated indices $(C_{T-p}, C_{T-pk}, C_{T-pm}, C_{T-pmk})$, are calculated to further use as a baseline. The sample mean of the traditional PCIs is obtained from $\hat{\mu} = \sum_{i=1}^{n} x_i \Big/ n$ and the sample variance is

calculated from $\hat{\sigma}^2 = \sum_{i=1}^{n}\left(x_i - \bar{x}\right) \Big/ (n-1)$; for the truncated PCI, the mean and variance are obtained from Eqautions (3.2) and (3.3), respectively. For the target-based PCIs, the given target is $\tau = 1$ for all scenarios.

Step 3: Draw samples randomly. A sample is then taken with replacement among four different sample sizes from the generated population in Step 1, where *n* = 50, 100, 150, 250, or 500; as such, this represents a sample rate of 1%, 2%, 3%, 5%, or 10%, respectively.

Step 4: Calculate descriptive statistics and PCIs from samples. To compare index values, the sample mean and variance are considered. The PCIs are then computed similar to Step 2.

Step 5: Repeat Steps 2 to 4 until all of the index values are obtained for the entire settings of sample size.

Step 6: Repeat Steps 3 to 5 (10,000 replications). The average value of PCIs with different sample sizes are computed.

Step 7: Repeat Steps 1-6 for all scenarios.

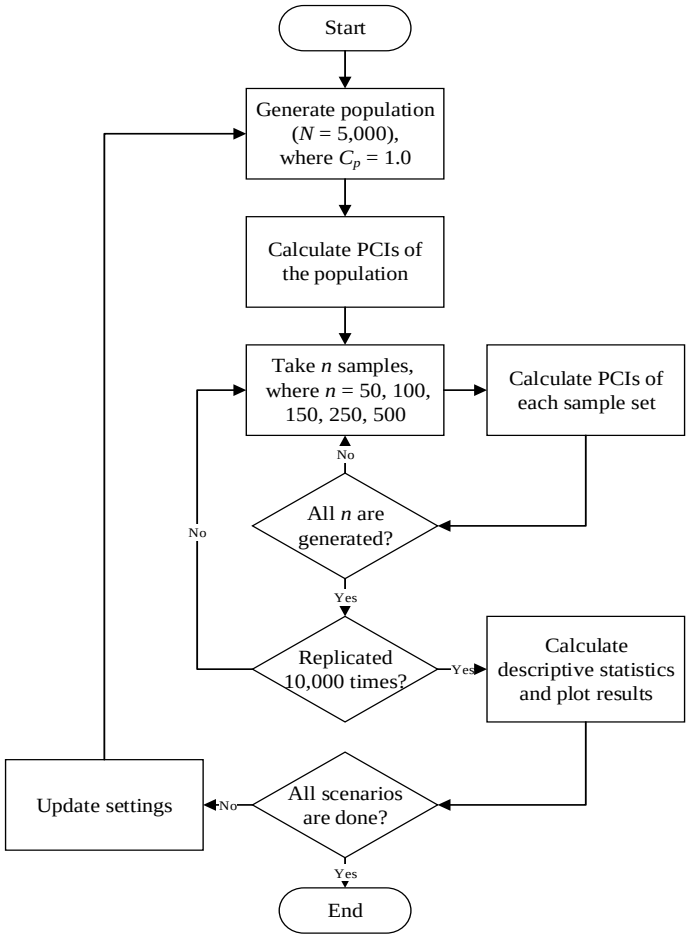


Figure 3.3 Simulation procedure

3.4.2 Results and discussions

The simulation study is conducted to enable a comparison of the characteristics among the truncated posterior indices and their traditional index counterparts. The test was designed in a manner to facilitate observing the effects of altering the truncation range, the location of the mean, or both, on the corresponding index values. Scenario *1* is used to validate the accuracy of the simulation model. For this scenario, the mean and

target are set to zero and the truncation range is set at the lower level (wide specification interval), where the *LSL* and *USL* are -3 and 3, respectively. At this setting, all of the index values are expected to be equivalent to 1.0; the results shown in Table 3.2 and Figure 3.4 confirm this expectation. The reason behind this result is that when the mean is located at the target ($\mu = \tau$), the bias term, $(\mu - \tau)^2$ equals zero, and the C_{pm} formulation reduces to C_p and C_{pmk} reduces to C_{pk} . Moreover, when the specification limits are symmetric about the target, we get $C_{pl} = C_{pu} = C_p$; hence $C_{pk} = \min\{C_{pl}, C_{pu}\} = C_p$.

Scenario 2 and 3 are designed to verify the effects of shifted mean, i.e. $\mu \neq \tau$, when the truncation range is wide, which facilitates a comparison of the characteristics in C_{pk}, C_{pm}, and C_{pmk}. In Scenario 2, the mean is set to the left-side of the target, $\mu < \tau$, and in Scenario 3, is set to the right-side of the target, $\mu > \tau$. In doing so, as can be seen in Scenario 2 in Table 3.3 and Figures 3.5 and Scenario 3 in Table 3.4 and Figure 3.6, the C_p values remain at 1.0 due to the index being insensitive to the location of the mean, while the values for C_{pk}, C_{pm}, and C_{pmk} drop below 1.0, suggesting the ability to measure a deviation from the target. Moreover, when the truncation range for these scenarios is wide, both traditional and truncated posterior PCI values are close in proximity, yet slightly higher for the truncated indices.

To examine the responsiveness of the indices when the truncation range is set at the higher level (or narrowing specification interval), the experiments are repeated for all of the scenarios with the [*LSL*, *USL*] set to [-2, 2]. In Scenario 4, where $\mu = \tau$, the C_p,

C_{pk}, C_{pm} and C_{pmk} values remain at 1.0, see Table 3.5 and Figure 3.7. For Scenarios 5 and 6, the observed trends are similar to that of Scenarios 2 and 3. With the narrow specification interval setting, however, the truncated posterior index values are significantly higher than that of the traditional PCIs, as shown in Table 3.6 and Figures 3.8 for Scenario 5, and in Table 3.7 and Figure 3.9 for Scenario 6.

In general, the results show that the truncated C_p-based and C_{pm}-based posterior indices provide higher estimated values when the specification interval is narrow, in comparison to the traditional PCIs. With this considerable differentiation in measurement, when it is clear that the underlying process distribution is truncated, the proposed posterior indices are the recommended alternative. When the mean is on-target for a process, $C_{T\text{-}p}$ is the recommended capability measurement, for its sensitivity, ease of use, and dual ability to provide an estimate of the defect rate. For these same reasons, when the mean is off-target for a process, $C_{T\text{-}pk}$ serves as a more viable measurement tool. However, if process bias is a concern, either the $C_{T\text{-}pm}$ or $C_{T\text{-}pmk}$ indices may be useful, based upon the user's preferences. Both indices provide a similar value; yet, $C_{T\text{-}pm}$ is easier to compute, while $C_{T\text{-}pmk}$ is more sensitive to the effects of narrowing or widening the truncation range.

Table 3.2 Average PCI values obtained from Scenario 1

n	C_p	$C_{T\text{-}p}$	C_{pl}	$C_{T\text{-}pl}$	C_{pu}	$C_{T\text{-}pu}$	C_{pk}	$C_{T\text{-}pk}$	C_{pm}	$C_{T\text{-}pm}$	C_{pmk}	$C_{T\text{-}pmk}$
50	1.0154	1.0303	1.0148	1.0298	1.0159	1.0309	0.9765	0.9919	1.0047	1.0197	0.9666	0.9819
100	1.0074	1.0217	1.0066	1.0209	1.0082	1.0225	0.9803	0.9950	1.0022	1.0166	0.9753	0.9900
150	1.0052	1.0193	1.0053	1.0194	1.0051	1.0192	0.9831	0.9975	1.0018	1.0159	0.9798	0.9942
250	1.0030	1.0170	1.0032	1.0171	1.0029	1.0169	0.9856	0.9997	1.0009	1.0148	0.9835	0.9976
500	1.0019	1.0157	1.0018	1.0156	1.0020	1.0158	0.9894	1.0034	1.0008	1.0146	0.9884	1.0023
5,000	1.0001	1.0137	1.0001	1.0137	1.0001	1.0137	0.9963	1.0099	1.0000	1.0136	0.9962	1.0098

Table 3.3 Average PCI values obtained from Scenario 2

n	C_p	$C_{T\text{-}p}$	C_{pl}	$C_{T\text{-}pl}$	C_{pu}	$C_{T\text{-}pu}$	C_{pk}	$C_{T\text{-}pk}$	C_{pm}	$C_{T\text{-}pm}$	C_{pmk}	$C_{T\text{-}pmk}$
50	1.0151	1.0771	1.3543	1.4176	0.6759	0.7365	0.6759	0.7365	0.7106	0.7523	0.4755	0.5179
100	1.0081	1.0700	1.3438	1.4071	0.6723	0.7329	0.6723	0.7329	0.7098	0.7518	0.4745	0.5172
150	1.0048	1.0669	1.3395	1.4029	0.6701	0.7309	0.6701	0.7309	0.7087	0.7510	0.4734	0.5163
250	1.0029	1.0653	1.3376	1.4013	0.6682	0.7293	0.6682	0.7293	0.7075	0.7500	0.4719	0.5150
500	1.0017	1.0640	1.3356	1.3993	0.6677	0.7288	0.6677	0.7288	0.7076	0.7501	0.4719	0.5150
5,000	1.0001	1.0625	1.3334	1.3971	0.6667	0.7279	0.6667	0.7279	0.7072	0.7498	0.4715	0.5147

Table 3.4 Average PCI values obtained from Scenario 3

n	C_p	$C_{T\text{-}p}$	C_{pl}	$C_{T\text{-}pl}$	C_{pu}	$C_{T\text{-}pu}$	C_{pk}	$C_{T\text{-}pk}$	C_{pm}	$C_{T\text{-}pm}$	C_{pmk}	$C_{T\text{-}pmk}$
50	1.0163	1.0782	0.6769	0.7373	1.3557	1.4190	0.6769	0.7373	0.7113	0.7528	0.4761	0.5184
100	1.0065	1.0686	0.6713	0.7321	1.3417	1.4050	0.6713	0.7321	0.7093	0.7515	0.4743	0.5171
150	1.0047	1.0670	0.6694	0.7304	1.3399	1.4035	0.6694	0.7304	0.7081	0.7505	0.4726	0.5156
250	1.0027	1.0648	0.6688	0.7297	1.3365	1.4000	0.6688	0.7297	0.7083	0.7507	0.4729	0.5159
500	1.0022	1.0644	0.6685	0.7294	1.3359	1.3994	0.6685	0.7294	0.7082	0.7506	0.4726	0.5156
5,000	1.0002	1.0626	0.6668	0.7279	1.3337	1.3974	0.6668	0.7279	0.7071	0.7498	0.4714	0.5146

Table 3.5 Average PCI values obtained from Scenario 4

n	C_p	$C_{T\text{-}p}$	C_{pl}	$C_{T\text{-}pl}$	C_{pu}	$C_{T\text{-}pu}$	C_{pk}	$C_{T\text{-}pk}$	C_{pm}	$C_{T\text{-}pm}$	C_{pmk}	$C_{T\text{-}pmk}$
50	1.0159	1.0387	1.0171	1.0399	1.0147	1.0375	0.9779	1.0014	1.0056	1.0286	0.9683	0.9917
100	1.0084	1.0301	1.0081	1.0298	1.0086	1.0303	0.9814	1.0036	1.0032	1.0250	0.9765	0.9986
150	1.0056	1.0270	1.0053	1.0267	1.0060	1.0274	0.9833	1.0051	1.0021	1.0235	0.9799	1.0017
250	1.0034	1.0245	1.0032	1.0243	1.0036	1.0247	0.9861	1.0076	1.0013	1.0225	0.9841	1.0055
500	1.0016	1.0225	1.0015	1.0224	1.0017	1.0226	0.9892	1.0103	1.0005	1.0214	0.9881	1.0092
5,000	1.0001	1.0207	1.0001	1.0207	1.0001	1.0207	0.9963	1.0170	1.0000	1.0206	0.9962	1.0169

Table 3.6 Average PCI values obtained from Scenario 5

n	C_p	$C_{T\text{-}p}$	C_{pl}	$C_{T\text{-}pl}$	C_{pu}	$C_{T\text{-}pu}$	C_{pk}	$C_{T\text{-}pk}$	C_{pm}	$C_{T\text{-}pm}$	C_{pmk}	$C_{T\text{-}pmk}$
50	1.0145	1.2609	1.5209	1.8062	0.5081	0.7155	0.5081	0.7155	0.5578	0.6583	0.2812	0.3951
100	1.0062	1.2517	1.5086	1.7913	0.5039	0.7120	0.5039	0.7120	0.5564	0.6581	0.2795	0.3945
150	1.0053	1.2514	1.5080	1.7915	0.5027	0.7114	0.5027	0.7114	0.5556	0.6576	0.2784	0.3937
250	1.0030	1.2493	1.5046	1.7881	0.5013	0.7105	0.5013	0.7105	0.5551	0.6575	0.2778	0.3936
500	1.0017	1.2473	1.5023	1.7845	0.5012	0.7101	0.5012	0.7101	0.5552	0.6577	0.2780	0.3937
5,000	1.0001	1.2459	1.5001	1.7824	0.5000	0.7094	0.5000	0.7094	0.5547	0.6575	0.2774	0.3935

Table 3.7 Average PCI values obtained from Scenario 6

n	C_p	$C_{T\text{-}p}$	C_{pl}	$C_{T\text{-}pl}$	C_{pu}	$C_{T\text{-}pu}$	C_{pk}	$C_{T\text{-}pk}$	C_{pm}	$C_{T\text{-}pm}$	C_{pmk}	$C_{T\text{-}pmk}$
50	1.0158	1.2629	0.5081	0.7159	1.5236	1.8098	0.5081	0.7159	0.5576	0.6579	0.2806	0.3945
100	1.0080	1.2546	0.5039	0.7125	1.5121	1.7966	0.5039	0.7125	0.5560	0.6576	0.2788	0.3938
150	1.0065	1.2528	0.5032	0.7119	1.5098	1.7938	0.5032	0.7119	0.5558	0.6575	0.2785	0.3936
250	1.0036	1.2495	0.5019	0.7107	1.5053	1.7883	0.5019	0.7107	0.5554	0.6576	0.2781	0.3937
500	1.0020	1.2480	0.5010	0.7101	1.5030	1.7858	0.5010	0.7101	0.5550	0.6575	0.2777	0.3935
5,000	1.0001	1.2460	0.5001	0.7094	1.5002	1.7826	0.5001	0.7094	0.5547	0.6575	0.2774	0.3935

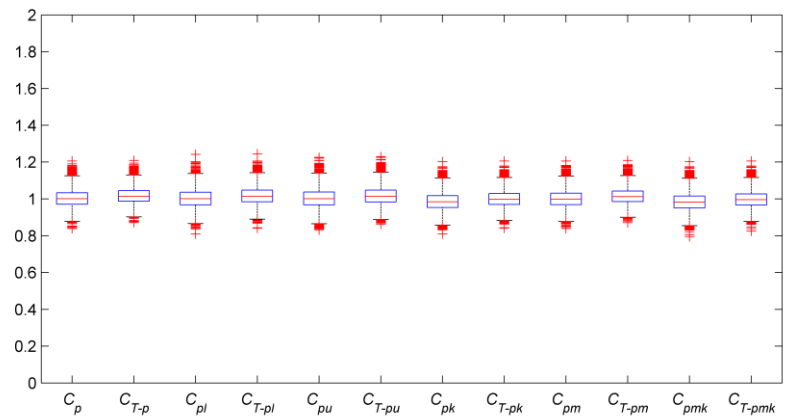


Figure 3.4 Results from Scenario 1 (n = 250)

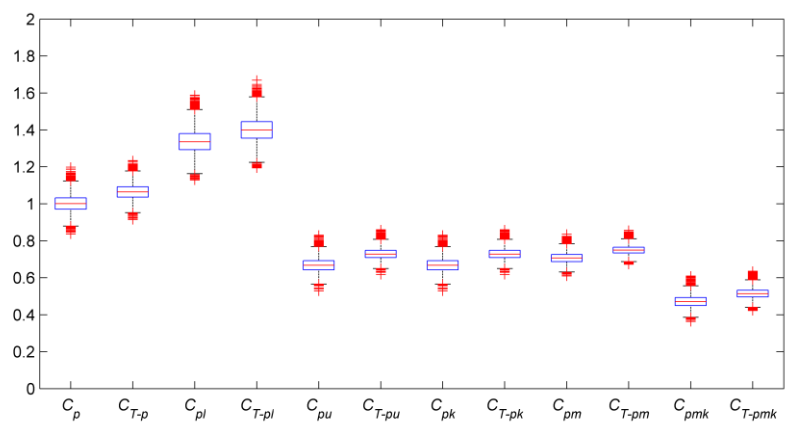


Figure 3.5 Results from Scenario 2 (n = 250)

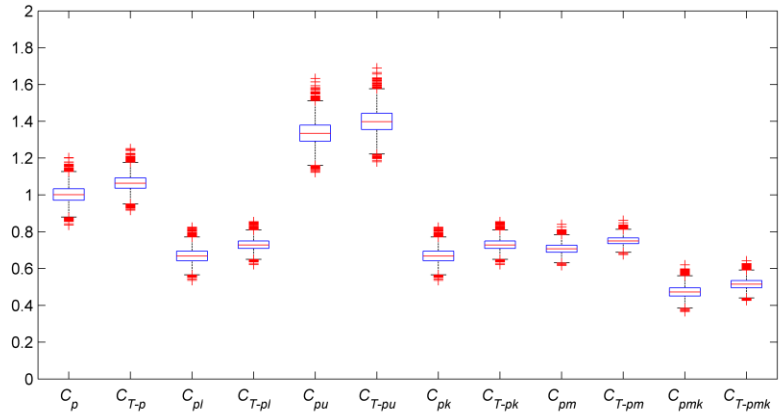


Figure 3.6 Results from Scenario 3 (n = 250)

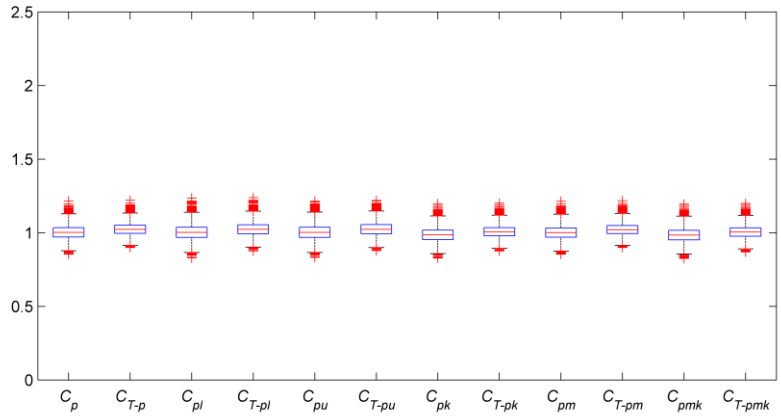


Figure 3.7 Results from Scenario 4 (n = 250)

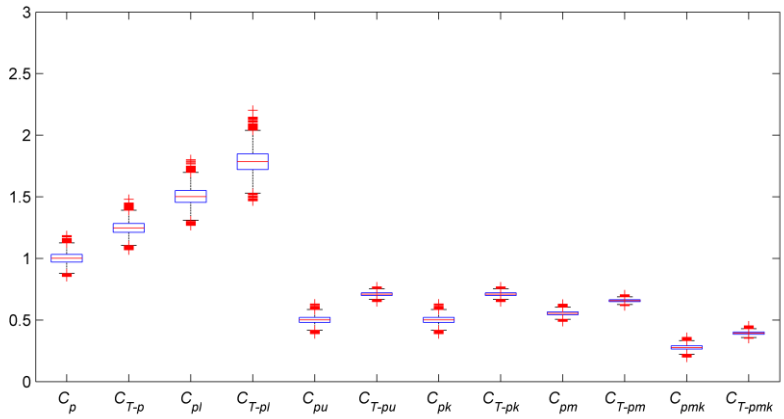


Figure 3.8 Results from Scenario 5 (n = 250)

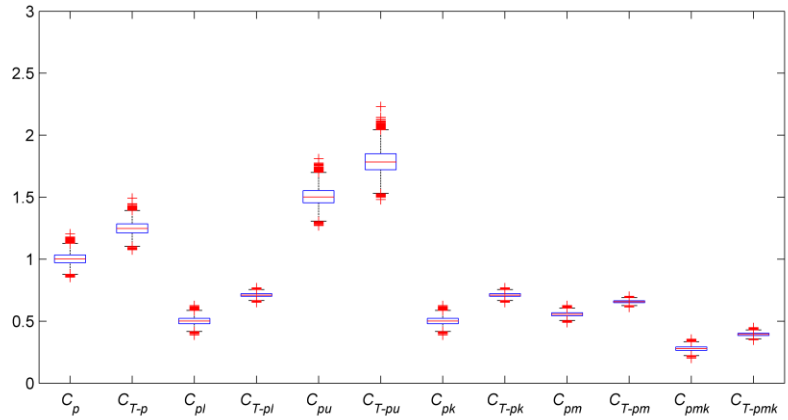
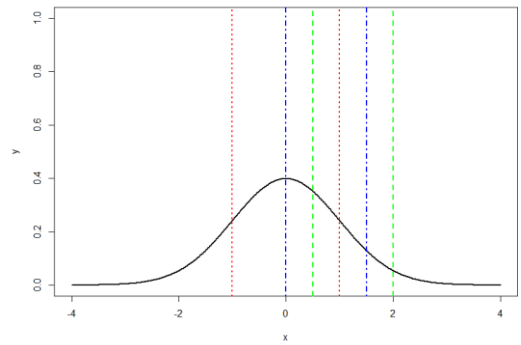


Figure 3.9 Results from Scenario 6 (n = 250)

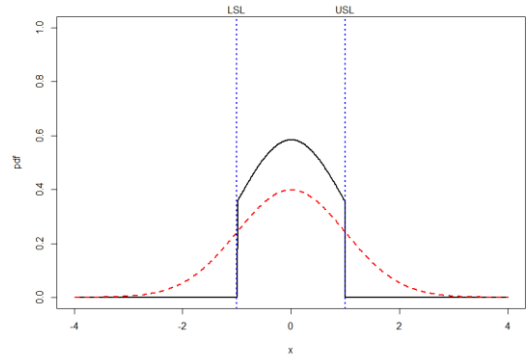
3.5 Numerical examples

3.5.1 Chemical concentrate case

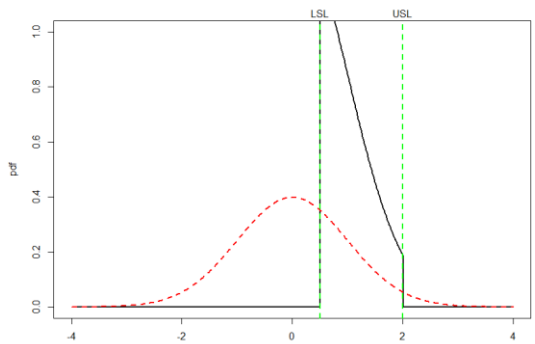
A chemical company supplies a chemical solvent to three customers, companies A, B, and C, with three different sets of specifications on the same product. Products are sorted in line with each customer's specifications and the company ships the product to the customers according to the truncated distributions shown in Figure 3.10. This example was first published in Polansky et al. (1998). The process distribution is assumed to be normally distributed with a process mean of 0 and variance of 1. The customer specifications are: LSL = -1.0 and USL = 1.0 for Customer A, LSL = 0.5 and USL = 2.0 for Customer B, and LSL = 0 and USL = 1.5 for Customer C.



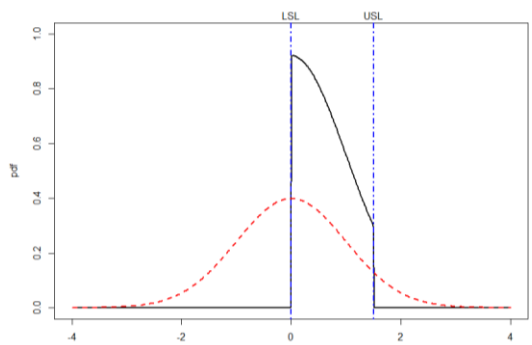
(b) The distributions with the specification limits for each customer



(b) The truncated distribution for customer A



(c) The truncated distribution for customer B



(d) The truncated distribution for customer C

Figure 3.10 Truncated distribution illustrations associated with three customers

For Customer A, given that the LSL_A = -1, USL_A = 1, μ_A = 0, σ_A =0.54, and τ_A =0, the truncated mean and variance can be calculated using Equations (3.2) and (3.3), we get $\mu_T = 0$, and $\sigma_T = 0.3717$. Consequently, the process capability indices are obtained from Equations (3.4) and (3.6) through (3.9) as follows: $C_{T-pm} = 0.8969$, $C_{T-pu} = 0.8969$, $C_{T-pl} = 0.8969$, $C_{T-pk} = 0.8969$, and $C_{T-pmk} = 0.8969$. In repeating the calculations above, from LSL_B = 0.5, USL_B = 2, μ_B = 1.04, σ_B = 0.39, and τ_B = 1.25, we obtain C_{T-pm} = 1.6349 and C_{T-pmk} = 1.4974 for Customer B; and, with LSL_C = 0,

USL_C = 1.5, μ_C = 0.725, σ_C = 0.41, and τ_C = 0.725, we have C_{T-pm} = 1.3327 and C_{T-pmk} = 1.2705 for Customer C.

3.5.2 Supplier's process capability problem

As noted by Asokan and Unnithan (2007), the submitted lot from a vendor for a supplier problem appears to follow a truncated normal distribution (shown by the histogram, Figure 3.10). The quality characteristic of interest for this problem is the width of a particular component, established at 20 ± 2 mm. The average $(\bar{y})$ and standard deviation of 100 samples is observed to be 20.0876 and 0.9393 mm, respectively.

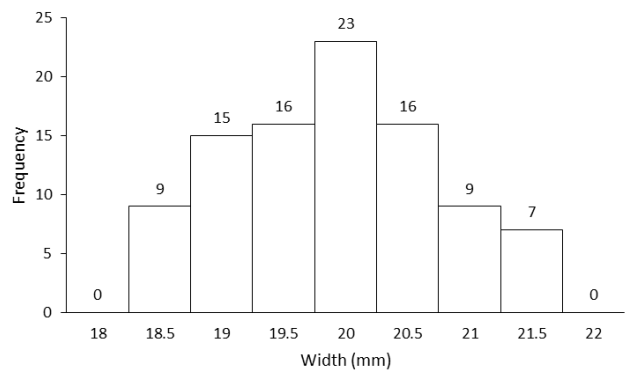


Figure 3.11 Histogram of the supplier's submitted lot

The problem specifies that LSL = 18, USL = 22, μ = 20.0876, σ =0.9393, and τ = 20. To find the PCI values for this study, we start by calculating μ_T and σ_T from Equations (3.2) and (3.3), followed by the formulations outlined at Equation (3.4) and Equations (3.6) through (3.9). The resulting measurements obtained are μ_T = 20.07056, σ_T = 0.8424, C_{T-p} = 0.7914, C_{T-pk} = 0.7635, C_{T-pm} = 0.7886, and C_{T-pmk} = 0.7608.

3.6 Confidence intervals for the truncated normal PCIs

Confidence intervals (CI) for process capability indices provide an alternative method for estimating capability, by relaxing the statistical fluctuation that can occur with various statistical parameters. Another purpose of using the CI for process capability indices is to evaluate the degree of accuracy in calculations pertaining to a specific sample size. Since the confidence limit of PCIs may be obtained from estimating only variance (for C_p), both mean and variance (for C_{pk}), and the process bias term (for C_{pm}), each index needs a unique statistical approximation method, as will be discussed later in this section. Several closed forms for obtaining the CI of the traditional PCIs have been proposed from researchers. Kushler and Hurley (1990) suggested a specific method for evaluating the lower confidence interval of C_p, C_{pk}, and C_{pm}. In addition, Chou et al. (1990) developed lower confidence limits for both C_p and C_{pk}, Boyles (1991) proposed an approximate confidence interval for C_{pm}, and both Johnson and Kotz (1993), as well as Pearn and Kotz (2006), provided a review of confidence interval estimates for the traditional indices.

3.6.1 The approximate confidence intervals for C_{T-p}

Given that $\hat{C}_{T-p}$ is an estimator for C_{T-p} and $\hat{C}_{T-p} = (USL - LSL)/6\hat{\sigma}_T$, where $\hat{\sigma}_T$ is the truncated form of the sample standard deviation, it is clear that an estimator for the standard deviation is the only parameter needed. The quality characteristic Y is assumed to follow a normal distribution, $Y = \{y_1, y_2, ..., y_n\}$, with mean μ and standard deviation σ , whereby $\hat{\mu}$ and $\hat{\sigma}$ represent the sample mean and sample standard deviation of Y.

Hence, $\hat{\sigma}$ is distributed according to the chi-square distribution, χ^2_{n-1}, with *n*-1 degrees of freedom, for which $\hat{\sigma}^2 = \frac{\sigma^2}{n-1}\chi^2_{n-1}$.

From Equation (3.3) in Section 3.3.3, we have that $\sigma_T^2 = \sigma^2 t_\sigma$, where t_σ is a constant term,

$t_\sigma = \left(1 + \frac{(z_l)\times\phi(z_l) - (z_u)\times\phi(z_u)}{\Phi(z_u) - \Phi(z_l)} - \left(\frac{\phi(z_l) - \phi(z_u)}{\Phi(z_u) - \Phi(z_l)}\right)^2\right)$, and $\hat{\sigma}_T^2$ is assumed to follow a chi-square distribution; hence, $\hat{\sigma}_T^2 = \frac{\sigma^2 t_\sigma}{n-1}\chi^2_{n-1}$.

To estimate the confidence interval for $\hat{C}_{T-p}$, from $\frac{C_{T-p}}{\hat{C}_{T-p}} = \frac{\hat{\sigma} t_\sigma}{\sigma t_\sigma} = \frac{\hat{\sigma}}{\sigma}$, we have that $\mathrm{P}\left[\frac{C_{T-p}}{\hat{C}_{T-p}} > c\right] = \mathrm{P}\left[\chi^2_{n-1} < \frac{n-1}{c^2}\right]$. With $\mathrm{P}\left[\chi^2_{n-1} < \chi^2_{n-1,\varepsilon}\right] = \varepsilon$, we can obtain the confidence interval for σ^2 from $\mathrm{P}\left[\frac{\sigma^2}{n-1}\chi^2_{n-1,\alpha/2} \le \hat{\sigma}^2 \le \frac{\sigma^2}{n-1}\chi^2_{n-1,1-\alpha/2}\right] = 1-\alpha$, or

$$\mathrm{P}\left[\frac{\hat{\sigma}^2(n-1)}{\chi^2_{n-1,1-\alpha/2}} \le \sigma^2 \le \frac{\hat{\sigma}^2(n-1)}{\chi^2_{n-1,\alpha/2}}\right] = 1-\alpha \tag{3.10}$$

When multiplying the inverse of the constant term for $\hat{C}_{T-p}$ in Equation (3.10), we have

$$\mathrm{P}\left[\frac{6}{USL-LSL}\hat{\sigma} t_\sigma \sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/2}}} \le \frac{6}{USL-LSL}\hat{\sigma} t_\sigma \le \frac{6}{USL-LSL}\hat{\sigma} t_\sigma \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/2}}}\right] = 1-\alpha \tag{3.11}$$

$$\mathrm{P}\left[\frac{USL-LSL}{6\hat{\sigma}_T}\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}} \le \frac{USL-LSL}{6\hat{\sigma}_T} \le \frac{USL-LSL}{6\hat{\sigma}_T}\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}}\right] = 1-\alpha$$

$$\mathrm{P}\left[\hat{C}_{T-p}\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}} \le C_{T-p} \le \hat{C}_{T-p}\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}}\right] = 1-\alpha$$

Finally, the $100(1-\alpha)\%$ confidence interval for $\hat{C}_{T-p}$ is derived as

$$\left\{\hat{C}_{T-p}\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}}\ ,\ \hat{C}_{T-p}\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}}\right\} \tag{3.12}$$

If the desired percentage is not found in the standard chi-square table, the Wilson-Hilferty method introduced by Johnson and Kotz (1970) can be used to approximate the chi-square percentile point, which is expressed as

$$\chi^2_{\nu,1-\alpha} \cong \nu\left(1-\frac{2}{9\nu}+z_{1-\alpha}\left(\frac{2}{9\nu}\right)^{\frac{1}{2}}\right)^3, \tag{3.13}$$

where ν represents the degrees of freedom for the chi-square, and z_α is the upper quartile of the standard normal distribution. If the sample size is less than ten, it is noted that this approximation may not offer a suitable level of precision.

By substituting formulation (3.13) in (3.12), the $100(1-\alpha)\%$ confidence intervals for $\hat{C}_{T-p}$ is derived as

$$\left\{\hat{C}_{T-p}\left(1-\frac{2}{9(n-1)}+z_{\alpha/2}\left(\frac{2}{9(n-1)}\right)^{\frac{1}{2}}\right)^{\frac{3}{2}},\ \hat{C}_{T-p}\left(1-\frac{2}{9(n-1)}+z_{1-\alpha/2}\left(\frac{2}{9(n-1)}\right)^{\frac{1}{2}}\right)^{\frac{3}{2}}\right\} \tag{3.14}$$

3.6.2 The approximate confidence intervals for C_{T-pk}

Estimating the confidence interval for C_{T-pk} involves not only the variance, but the mean as well. While several distributions have been suggested previously by researchers, such as the folded normal distribution, t distribution, or chi-square distribution, the most prevalent method is to use the normal distribution. The distribution is first reported in use by Zhang et al. (1990) for confidence interval estimation, with major modifications proposed by Kusher and Hurley (1992), and Nagata and Nagahata (1993).

For a normally distributed process with a large sample size, i.e., *kn* > 30, where *k* is the subgroup size and *n* is the sample size of each sub group, $\hat{C}_{T-pk}$ can be estimated using the Bissell method outlined by Bissell (1990). This particular method was shown by Kushler and Hurley (1992) to outperform other approaches when *n* is large.

Using the general form, $C_{T-pk} = \min\left\{\frac{USL-\mu_T}{3\sigma_T}, \frac{\mu_T - USL}{3\sigma_T}\right\}$, we use the coefficient of variation for C_{T-pk}, $CV(C_{T-pk})$ to estimate $\hat{C}_{T-pk}$, which is defined as

$$CV\left(C_{T-pk}\right) = \sqrt{CV\left(\frac{USL-\mu_T}{3\sigma_T}\right)} \quad \text{or} \quad \sqrt{CV\left(\frac{\mu_{TN}-USL}{3\sigma_{TN}}\right)}.$$

From the Taylor series expansion, it follows that $CV(a) = \sqrt{\left(CV(b)\right)^2 + \left(CV(c)\right)^2}$, where $a = b/c$ (see Stuart and Ord, 1987). Hence,

$$CV(\hat{C}_{T-pk}) = \sqrt{\left(CV(USL-\hat{\mu}_T)\right)^2 + \left(CV(3\hat{\sigma}_T)\right)^2} \tag{3.15}$$

From $\left(CV(USL-\hat{\mu}_T)\right)^2 = \dfrac{Var\left(USL-\hat{\mu}_T\right)}{\left(E\left[USL-\hat{\mu}_T\right]\right)^2} = \dfrac{Var\left(\hat{\mu}_T\right)}{\left(USL-\hat{\mu}_T\right)^2} = \dfrac{\hat{\sigma}^2}{kn\left(USL-\hat{\mu}_T\right)^2} = \dfrac{1}{9kn\hat{C}_{T-pk}^2}$

and $\left(CV(3\hat{\sigma}_T)\right)^2 = \dfrac{Var(3\hat{\sigma}_T)}{E\left(3\hat{\sigma}_T\right)^2} = \dfrac{9\hat{\sigma}_T^2}{2k\left(n-1\right)\times 9\hat{\sigma}_T^2} = \dfrac{1}{2k\left(n-1\right)}$, Equation (3.15) becomes

$$CV(\hat{C}_{T-pk}) = \sqrt{\frac{1}{9kn\hat{C}_{T-pk}^2} + \frac{1}{2k\left(n-1\right)}}$$

The standard error of $\hat{C}_{T-pk}$, $SE(\hat{C}_{T-pk})$, is obtained by multiplying $\hat{C}_{T-pk}$ and $CV(\hat{C}_{T-pk})$.

As a result, we get $SE(\hat{C}_{T-pk}) = \hat{C}_{T-pk}\sqrt{\dfrac{1}{9kn\hat{C}_{T-pk}^2} + \dfrac{1}{2k\left(n-1\right)}} = \sqrt{\dfrac{1}{9kn} + \dfrac{\hat{C}_{T-pk}^2}{2k\left(n-1\right)}}$

When n is large, i.e., $n > 30$, the term $\dfrac{1}{9kn}$ approaches zero, thus, with one subgroup ($k = 1$), $SE(\hat{C}_{T-pk})$ becomes $SE(\hat{C}_{T-pk}) = \sqrt{\dfrac{\hat{C}_{T-pk}^2}{2\left(n-1\right)}}$. Finally, the approximate $100(1-\alpha)\%$ confidence interval for C_{T-pk} is derived as

$\left\{\hat{C}_{T-pk} - z_{\alpha/2}\sqrt{\dfrac{\hat{C}_{T-pk}^2}{2\left(n-1\right)}}, \hat{C}_{T-pk} + z_{\alpha/2}\sqrt{\dfrac{\hat{C}_{T-pk}^2}{2\left(n-1\right)}}\right\}$, which can be further written as

$$\left\{\hat{C}_{T-pk}\left[1-\frac{z_{\alpha/2}}{\sqrt{2n-2}}\right], \hat{C}_{T-pk}\left[1+\frac{z_{\alpha/2}}{\sqrt{2n-2}}\right]\right\} \tag{3.16}$$

3.6.3 The approximate confidence intervals for C_{T-pm}

With $C_{T-pm} = \dfrac{USL-LSL}{6\sqrt{\sigma_T^2 + \left(\mu_T - \tau\right)^2}} = \dfrac{USL-LSL}{6\sqrt{\sigma^2 t_\sigma + \left(\mu + t_\mu - \tau\right)^2}}$,

we have $C_{T-pm} = \dfrac{d}{\sqrt{\sigma^2 t_\sigma + \left(\mu + t_\mu - \tau\right)^2}}$, where $d = \dfrac{(USL - LSL)}{6}$.

Therefore, $\hat{C}_{T-pm} = \dfrac{d}{\sqrt{\hat{\sigma}^2 t_\sigma + \left(\hat{\mu} + t_\mu - \tau\right)^2}} = \dfrac{d}{\sqrt{\dfrac{(n-1)}{n} s^2 t_\sigma + \left(\bar{y} + t_\mu - \tau\right)^2}}$,

where $s^2 = \sum_{i=1}^{n} \left(y_i - \bar{y}\right) \Big/ \left(n-1\right)$.

The process bias term, $\sum_{i=1}^{n} (y-\tau)^2$, can be estimated from $E\left[\sum_{i=1}^{n} (y-\tau)^2\right] = \sigma^2 + \left(\mu - \tau\right)^2$; see Taguchi (1985). As reported in Boyles (1991), Pearn et al. (1992), Vannman and Kotz (1995), and Zimmer et al. (2001), it is generally known that the bias term follows a non-central chi-square distribution, $\sigma^2 \chi^2_{n,\lambda}$, with *n* degrees of freedom and the non-centrality parameter, λ.

Given that $\beta^2 = \sigma^2 t_\sigma + \left(\mu + t_\mu - \tau\right)^2$, we have $\dfrac{n\hat{\beta}^2}{\sigma^2 t_\sigma} \sim \chi^2_{n,\lambda}$, thus $\dfrac{\hat{\beta}^2}{\sigma^2 t_\sigma} \sim \dfrac{\chi^2_{n,\lambda}}{n}$. See Johnson and Kotz (1970) for the proof.

Hence, with $C^2_{T-pm} = \dfrac{d^2}{\sigma^2 t_\sigma + \left(\mu + t_\mu - \tau\right)^2}$, we have $\hat{\beta}^2 = \dfrac{d^2}{\hat{C}^2_{T-pm}}$, $\sigma^2 t_\sigma = \dfrac{d^2}{C^2_{T-pm}\left(1 + \lambda/n\right)^2}$,

and $\dfrac{\hat{\beta}^2}{\sigma^2 t_\sigma} = \left(\dfrac{d^2}{\hat{C}^2_{T-pm}}\right)\left(\dfrac{C^2_{T-pm}\left(1+\lambda/n\right)^2}{d^2}\right) = \dfrac{C^2_{T-pm}}{\hat{C}^2_{T-pm}}\left(1 + \dfrac{\lambda}{n}\right)^2$.

Thus, with $C_{T-pm}^2\left(1+\frac{\lambda}{n}\right)^2=\frac{d^2}{\sigma^2 t_\sigma}$, we have that $\frac{\hat{\beta}^2}{\sigma^2 t_\sigma}=\frac{d^2}{\hat{C}_{T-pm}^2\sigma^2 t_\sigma}\sim\frac{\chi_{n,\lambda}^2}{n}$, or

$\hat{C}_{T-pm}^2\sim\frac{nd^2}{\sigma^2 t_\sigma \chi_{n,\lambda}^2}$. Finally, we get the cumulative density function of $\hat{C}_{T-pm}$ as

$$\hat{C}_{T-pm}=\frac{d}{\sigma t_\sigma}\sqrt{\frac{n}{\chi_{n,\lambda}^2}}\text{, where }\lambda=n\left(\frac{\mu+t_\mu-\tau}{\sigma t_\sigma}\right)^2$$

and from $C_{T-pm}\sqrt{\frac{\chi_{1-\alpha,\hat{\nu}}^2}{\hat{\nu}}}$, where $\hat{\nu}=\frac{n\left(1+\left(\frac{\hat{\mu}+t_\mu-\tau}{\hat{\sigma}t_\sigma}\right)^2\right)^2}{1+2\left(\frac{\hat{\mu}+t_\mu-\tau}{\hat{\sigma}t_\sigma}\right)^2}$. See Patniak (1949).

To approximate the $100(1-\alpha)\%$ lower confidence bound for C_{T-pm}, we have

$$P\left[C_{T-pm}\sqrt{1+\frac{\lambda}{n}}\sqrt{\frac{n}{\chi_{1-\alpha/2,n,\lambda}^2}}\le\hat{C}_{T-pm}\le C_{T-pm}\sqrt{1+\frac{\lambda}{n}}\sqrt{\frac{n}{\chi_{\alpha/2,n,\lambda}^2}}\right]=1-\alpha$$

Let $C_{T-pm}^L=C_{T-pm}\sqrt{1+\frac{\lambda}{n}}\sqrt{\frac{n}{\chi_{1-\alpha/2,n,\lambda}^2}}$ be the lower confidence limit and

$C_{T-pm}^U=C_{T-pm}\sqrt{1+\frac{\lambda}{n}}\sqrt{\frac{n}{\chi_{\alpha/2,n,\lambda}^2}}$ be the upper confidence limit, then

$$P\left[C_{T-pm}^L\le C_{T-pm}\le C_{T-pm}^U\right]\quad=1-\alpha$$

$$P\left[\hat{\nu}\left(\frac{C_{T-pm}^L}{\hat{C}_{T-pm}}\right)^2\le\hat{\nu}\left(\frac{C_{T-pm}}{\hat{C}_{T-pm}}\right)^2\le\hat{\nu}\left(\frac{C_{T-pm}^U}{\hat{C}_{T-pm}}\right)^2\right]=1-\alpha$$

$$P\left[\hat{\nu}\left(\frac{C_{T-pm}^L}{\hat{C}_{T-pm}}\right)^2\le\chi_{n,\hat{\nu}}^2\le\hat{\nu}\left(\frac{C_{T-pm}^U}{\hat{C}_{T-pm}}\right)^2\right]=1-\alpha$$

Since $\chi^2_{\alpha,n,\hat{\nu}}$ is the $100(1-\alpha)\%$ lower confidence interval for $\chi^2_{n,\hat{\nu}}$, the previous equation yields

$$P\left[\hat{C}_{T-pm}\sqrt{\frac{\chi^2_{\alpha/2,n,\hat{\nu}}}{\hat{\nu}}} \le \hat{C}_{T-pm} \le \hat{C}_{T-pm}\sqrt{\frac{\chi^2_{1-\alpha/2,n,\hat{\nu}}}{\hat{\nu}}}\right] = 1-\alpha$$

Finally, the $100(1-\alpha)\%$ interval approximation for $C_{T\text{-}pm}$ is written as

$$\left[\hat{C}_{T-pm}\sqrt{\frac{\chi^2_{\alpha/2,n,\hat{\nu}}}{\hat{\nu}}}, \hat{C}_{T-pm}\sqrt{\frac{\chi^2_{1-\alpha/2,n,\hat{\nu}}}{\hat{\nu}}}\right], \text{ where } \hat{\nu} = \frac{n\left(1+\left(\frac{\hat{\mu}_T-\tau}{\hat{\sigma}_T}\right)^2\right)^2}{1+2\left(\frac{\hat{\mu}_T-\tau}{\hat{\sigma}_T}\right)^2} \quad (3.17)$$

Equation (3.17) is similar to Equation (3.12), yet in this case, the non-central parameter is added to the calculation. Thus, by using Equation (3.13) and letting $t_d = \frac{\hat{\mu}_T-\tau}{\hat{\sigma}_T}$, we then obtain the form of the approximation in Equation (3.18) as

$$\left[\hat{C}_{T-pm}\left(1-\frac{2\left(1+2t_d^{\,2}\right)}{9n\left(1+t_d^{\,2}\right)^2}+z_{1-\alpha/2}\sqrt{\frac{2\left(1+2t_d^{\,2}\right)}{9n\left(1+t_d^{\,2}\right)^2}}\right)^{\frac{3}{2}},\ \hat{C}_{T-pm}\left(1-\frac{2\left(1+2t_d^{\,2}\right)}{9n\left(1+t_d^{\,2}\right)^2}+z_{\alpha/2}\sqrt{\frac{2\left(1+2t_d^{\,2}\right)}{9n\left(1+t_d^{\,2}\right)^2}}\right)^{\frac{3}{2}}\right] \quad (3.18)$$

3.6.4 The approximate confidence intervals for C_{T-pmk}

The index, C_{T-pmk}, is a combined function of C_{T-pk} and C_{T-pm}. Hence, from Sections 3.6.2 and 3.6.3, a natural estimator of C_{T-pmk} is distributed as a mixture of the chi-square and the non-central chi-square distribution. In this section, the approximate confidence interval of $\hat{C}_{T-pmk}$ will be modified from a derivation provided by previous

researchers. In particular, Chen and Hsu (1995) derived the sampling distribution of C_{pmk} and reported that $\hat{C}_{pmk}$ is asymptotically normal and consistent if the fourth moment of Y is finite. Therefore, under similar circumstances, since we will assume that $\hat{C}_{T-pmk}$ is distributed by a truncated normal distribution, the confidence interval is estimated by considering the variance of $\hat{C}_{T-pmk}$ at a certain confidence level from the standard normal distribution.

Staring with $C_{T-pk} = \min\left\{\frac{USL-\mu_T}{3\sigma_T}, \frac{\mu_T - USL}{3\sigma_T}\right\}$, and using the formula $\min(x,y) = \frac{1}{2}(x+y) - \frac{1}{2}|x-y|$, we obtain

$$C_{T-pk} = \frac{1}{2}\left(\frac{USL-\mu_T}{3\sigma_T} + \frac{\mu_T - USL}{3\sigma_T}\right) - \frac{1}{2}\left|\frac{USL-\mu_T}{3\sigma_T} - \frac{\mu_T + USL}{3\sigma_T}\right|$$

Thus. $C_{T-pk} = \frac{(USL-LSL) - |USL+LSL-2\mu_T|}{6\sigma_T}$. From $C_{T-pmk} = \frac{C_{T-pk}}{\sqrt{1+\left(\frac{\mu_T-\tau}{\sigma_T}\right)^2}}$, we get

$C_{T-pmk} = \frac{(USL-LSL) - |USL+LSL-2\mu_T|}{6\sqrt{\sigma_T^2 + (\mu_T-\tau)^2}}$.Similarly, the estimator for C_{T-pmk} is

$\hat{C}_{T-pmk} = \frac{(USL-LSL) - |USL+LSL-2\hat{\mu}_T|}{6\sqrt{\hat{\sigma}_T^2 + (\hat{\mu}_T-\tau)^2}}$. If the mean is centered at the midpoint between the specification limits, $\mu = \frac{USL-LSL}{2}$, then we have $\hat{C}_{T-pmk} = \hat{C}_{T-pm}$, so that $\hat{C}_{T-pmk}$ can be estimated using Equation (3.18). Otherwise, the $100(1-\alpha)\%$ interval approximation for $\hat{C}_{T-pmk}$ is obtained as

$$\left[\hat{C}_{T-pmk} - z_{\alpha/2}\frac{\hat{\sigma}_{T-pmk}}{\sqrt{n}}, \hat{C}_{T-pmk} + z_{\alpha/2}\frac{\hat{\sigma}_{T-pmk}}{\sqrt{n}}\right] \tag{3.19}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quartile of the standard normal distribution, and $\hat{\sigma}^2_{T-pmk}$ is the asymptotic estimator of $Var(\hat{C}_{T-pmk})$, which is formulated as

$$\hat{\sigma}^2_{T-pmk} = \hat{C}_{T-pmk}\left(\frac{1}{9\left(1+\delta^2\right)} + \frac{2\delta}{3\left(1+\delta^2\right)^{\frac{3}{2}}}\right) + \hat{C}^2_{T-pmk}\frac{24\delta^2 + d\left(\frac{m_4}{\sigma_T^4} - 1\right)}{24\left(1+\delta^2\right)^2} \tag{3.20}$$

where $d = \frac{(USL - LSL)}{6}$, $m_4 = \sum_{i=1}^{n}\frac{\left(y_i - \hat{\mu}_T\right)^4}{n}$, and $\delta = \frac{\hat{\mu}_T - \tau}{\hat{\sigma}_T}$. See the full derivation of $\hat{C}_{pmk}$ in Pearn et al. (1992) and Chen and Hsu (1995).

3.7 Numerical examples of confidence intervals for proposed posterior PCIs

Given that n = 30, [LSL, USL] = [-3, 3], α = 0.05, $\hat{\sigma}_T$ = 1, $\hat{\mu}_T$ = 0, and τ = 0, the confidence intervals of the posterior PCIs are shown as follows. Using Equation (3.13), the estimated 95% confidence interval for $\hat{C}_{T-p} = 1.0$ is obtained as $\left(0.7439 \le \hat{C}_{T-p} \le 1.2556\right)$. The estimated 95% confidence interval for $\hat{C}_{T-pk} = 1.0$ is computed as $\left(0.7426 \le \hat{C}_{T-pk} \le 1.2574\right)$ using Equation (3.15). Similarly, the estimated confidence interval of $\hat{C}_{T-pm}$ is obtained from Equation (3.17). With $t_d = \left(\hat{\mu}_T - \tau\right)/\hat{\sigma}_T = 0$, the 95% confidence interval of $\hat{C}_{T-pm} = 1.0$ is then $\left(0.7481 \le \hat{C}_{T-pm} \le 1.2514\right)$. Finally, the asymptotic 95% confidence interval for $\hat{C}_{T-pmk}$ is computed using Equation (3.19).

Prior to using Equation (3.19), the $\hat{\sigma}^2_{T-pmk}$ need to be obtained from Equation (3.20). Assuming $d = 1,\ m_4 = 0,$ and $\delta = 0$, we then have $\hat{\sigma}^2_{T-pmk} = 0.0694$. Using Equation (3.19), the confidence interval of $\hat{C}_{T-pmk} = 1.0$ then yields $\left(0.9057 \le \hat{C}_{T-pmk} \le 1.0943\right)$. The estimated confidence intervals of the proposed posterior PCIs with different sample size *n* and different confidence level α are presented in Table 3.8 and Figure 3.12, where the wider confidence intervals are observed as sample size or the confidence level decreases.

3.8 Concluding remarks

In summary, this chapter proposes a new set of posterior process capability indices for the situations where the underlying process follows a truncated normal distribution and a target-based framework is desired. The simulation results demonstrate that the proposed posterior indices have higher index values, compared to traditional PCI values. This is because smaller process variances are transmitted to the customer after implementing specifications on a process. As a result, a significant degree of difference may result from using traditional PCIs on processes where the observations clearly follow a truncated normal distribution which is the actual process distribution transmitted to the customers. The purpose of this chapter is twofold. First, the proposed customer-based posterior PCIs are derived and compared with traditional manufacturer-based PCIs. The confidence interval is a prominent tool providing an interval estimation of PCIs designed to increase the versatility of the process capability analysis. Accordingly, the confidence intervals for the proposed posterior PCIs are also developed.

Table 3.8 Confidence intervals of the proposed PCIs under different settings

$\hat{C}_{T-p}$	n	α	Lower $\hat{C}_{T-p}$	Upper $\hat{C}_{T-p}$	$\hat{C}_{T-p}$	n	α	Lower $\hat{C}_{T-p}$	Upper $\hat{C}_{T-p}$
1.0	10	0.05	0.5478	1.4538	1.0	30	0.1	0.7814	1.2114
1.0	20	0.05	0.6847	1.3149	1.0	30	0.05	0.7439	1.2556
1.0	30	0.05	0.7439	1.2556	1.0	30	0.025	0.7110	1.2955
1.0	50	0.05	0.8025	1.1971	1.0	30	0.010	0.6726	1.3434
1.0	100	0.05	0.8608	1.1389	1.0	30	0.005	0.6467	1.3767
1.0	250	0.05	0.9122	1.0877	1.0	30	0.0025	0.6228	1.4081
1.0	500	0.05	0.9380	1.0620	1.0	30	0.0010	0.5938	1.4472
$\hat{C}_{T-pk}$	n	α	Lower $\hat{C}_{T-pk}$	Upper $\hat{C}_{T-pk}$	$\hat{C}_{T-pk}$	n	α	Lower $\hat{C}_{T-pk}$	Upper $\hat{C}_{T-pk}$
1.0	10	0.05	0.5380	1.4620	1.0	30	0.1	0.7840	1.2160
1.0	20	0.05	0.6821	1.3179	1.0	30	0.05	0.7426	1.2574
1.0	30	0.05	0.7426	1.2574	1.0	30	0.025	0.7057	1.2943
1.0	50	0.05	0.8020	1.1980	1.0	30	0.010	0.6618	1.3382
1.0	100	0.05	0.8607	1.1393	1.0	30	0.005	0.6314	1.3686
1.0	250	0.05	0.9122	1.0878	1.0	30	0.0025	0.6030	1.3970
1.0	500	0.05	0.9380	1.0620	1.0	30	0.0010	0.5679	1.4321
$\hat{C}_{T-pm}$	n	α	Lower $\hat{C}_{T-pm}$	Upper $\hat{C}_{T-pm}$	$\hat{C}_{T-pm}$	n	α	Lower $\hat{C}_{T-pm}$	Upper $\hat{C}_{T-pm}$
1.0	10	0.05	0.5698	1.4312	1.0	30	0.1	0.8141	1.1807
1.0	20	0.05	0.6925	1.3071	1.0	30	0.05	0.7816	1.2180
1.0	30	0.05	0.7481	1.2514	1.0	30	0.025	0.7529	1.2516
1.0	50	0.05	0.8045	1.1952	1.0	30	0.010	0.7195	1.2920
1.0	100	0.05	0.8615	1.1382	1.0	30	0.005	0.6967	1.3200
1.0	250	0.05	0.9124	1.0875	1.0	30	0.0025	0.6757	1.3464
1.0	500	0.05	0.9380	1.0619	1.0	30	0.0010	0.6501	1.3793
$\hat{C}_{T-pmk}$	n	α	Lower $\hat{C}_{T-pmk}$	Upper $\hat{C}_{T-pmk}$	$\hat{C}_{T-pmk}$	n	α	Lower $\hat{C}_{T-pmk}$	Upper $\hat{C}_{T-pmk}$
1.0	10	0.05	0.8367	1.1633	1.0	30	0.1	0.9209	1.0791
1.0	20	0.05	0.8845	1.1155	1.0	30	0.05	0.9057	1.0943
1.0	30	0.05	0.9057	1.0943	1.0	30	0.025	0.8922	1.1078
1.0	50	0.05	0.9270	1.0730	1.0	30	0.010	0.8761	1.1239
1.0	100	0.05	0.9484	1.0516	1.0	30	0.005	0.8649	1.1351
1.0	250	0.05	0.9673	1.0327	1.0	30	0.0025	0.8545	1.1455
1.0	500	0.05	0.9769	1.0231	1.0	30	0.0010	0.8417	1.1583

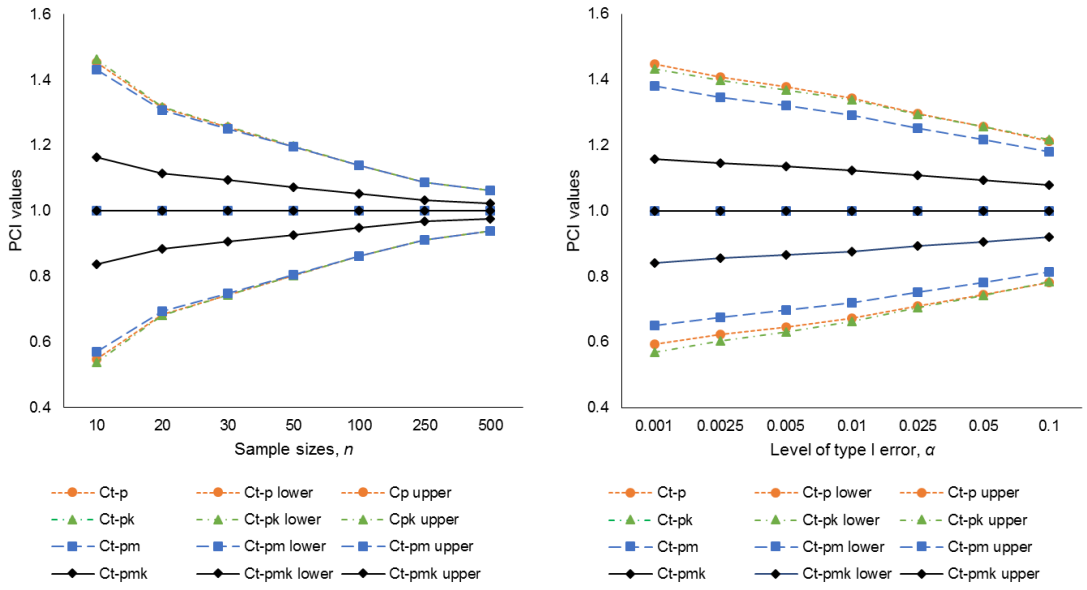


Figure 3.12 Estimated confidence intervals of the proposed posterior PCIs under various sample sizes (left) and various levels of type I error (right)

CHAPTER 4
ROBUST PARAMETER DESIGN OPTIMIZATION AND PROCESS CAPABILITY ANALYSIS FOR TYPE I-RIGHT CENSORED DATA

4.1 Introduction

Continuous process improvement is critical in maintaining a competitive advantage in the marketplace. It is also recognized that process improvement activities are most efficient and cost-effective when implemented during the early design stage. Based on this awareness, robust parameter design (RPD) was introduced as a systematic method for applying experimental design and optimization tools. The primary goal of RPD is to determine the best design factor settings, or the optimum operating conditions, that minimize performance variability and deviations from the target value of a product. Because of their practicability in reducing the inherent uncertainty associated with system performance, the widespread application of RPD techniques has resulted in significant improvements in product quality, manufacturability, and reliability at low cost (see Robinson *et al*., 2004, Jen, 2005, Hasenkamp *et al*., 2009, and Montgomery, 2013). Although a number of RPD methods have been developed, there is still ample room for improvement, particularly when censored data are under study. This chapter aims to develop new RDP methodologies that can be applied in survival analysis and reliability studies where censored data are common. In this section, applications of RPD that have not been widely explored by the research community are discussed, and specific objectives of this chapter are outlined.

4.1.1 Current research gaps and research objectives

The main purpose of this chapter is twofold. We first develop a series of RPD-based methodological models for a time-oriented quality characteristic, coupled with type I-right censoring. We then propose various RPD optimization models to determine the optimum operating conditions that maximize survival times and minimize proportional hazards in conjunction with variability. The specific goals of this chapter with supporting rationales are as follows.

Static non-time oriented quality characteristics have been the main focus in traditional RPD studies, even though time-oriented quality characteristics routinely appears in many engineering problems, particularly in survival analysis and reliability studies. The term survival analysis refers to statistical inferences for time-related data with censoring. A survival data set contains observations within a predetermined time or restricted values. On the other hand, unmeasured data outside the restricted time range or values are called censored data. By definition, censoring is a form of incomplete data similar to data truncation. In truncation, any data beyond the truncation points is neglected. However, for censoring, any data beyond the censoring points are censored but are still included in the study, although its actual value is unknown. For instance, censored observations t that survive longer than censoring time t_c (i.e., $t > t_c$) are entered as t_c. Therefore, the distribution of this incomplete data set is said to follow a censored distribution (see Cohen, 1991). There are four main types of censoring mechanisms: left censoring, right censoring, interval censoring, and progressive censoring. In addition, there are two types of censoring. Type I-censoring is a scheme based on a specific time,

and time-censored experiments take place when a test is terminated at a pre-specified time; hence, the number of data points censored is a random variable. In contrast, the basis of type II-censoring is a fixed number of data points being censored, and the censoring threshold becomes a random variable. While each censoring type has its own set of statistical foundations, type I-right censoring is the most common methodology in reliability studies. Despite the potential application areas and practical needs, there is little research work on the censoring-based RPD modeling and optimization. As such, this chapter develops detailed guidelines for the design of a new RPD with the consideration of type I-right censoring concepts.

It is expected that incorporating considerations for reliability in the early stages of production design leads to a higher level of product quality over time and less costs due to product recalls. However, little research has been done on how to achieve this when optimizing process parameters. The survival time used in reliability studies is restricted to be positive and often has a skewed distribution. In contrast, the standard statistical methods of the traditional RPD rely on the normal distribution, which may be unsuitable to fit the survival time into such models. As a result, the connection between the survival time and RPD requires further methodological developments. To this end, we propose the censoring-based RPD methods for assessing the optimum operating conditions using the response surface methodology. In addition, special methods that can estimate failure rates with censored data, such as the hazard function, the Kaplan-Meier (KM) estimator, and the maximum likelihood estimation, are routinely used for regression in survival analysis. However, the statistical inferences for survival time can be developed parametrically or

non-parametrically. Since the distribution of survival time is typically unknown prior to the actual data collection, nonparametric estimation methods for survival time are often preferred over the parametric counterpart. In particular, hazard functions, accompanied with median survival functions, may provide more insights into a failure mechanism. More specifically, survival times are often highly skewed, and the median is generally considered a better measure of central location than the mean. While the median survival function represents the survival probability of a subject at time *t*, the hazard function indicates the risk of failure of the subject at time *t*. The proportion of hazard rates, called a hazard ratio or proportional hazards, is useful in this particular research context. For a predicted response surface model, the semiparametric Cox proportional hazards (PH) regression model is often employed, as outlined in the literature review section. Thus, another purpose of this chapter is to propose a new set of response-surface-based optimization models by incorporating the nonparametric methods, such as the Cox PH model and the KM estimator, to investigate the effect of input variables under the type I-right censoring scheme.

4.1.2 Literature review

Traditionally, RPD and survival analysis have been treated as separate fields with no significant connections to each other. Previous works in those areas are abundant, but virtually no research work on potential links between the two fields has been reported. In this section, some key papers that are potentially aligned with this study are included.

The basic concepts of RPD were introduced by Taguchi (1986) and have been successfully used as an efficient tool for building quality into the design of processes to

improve the quality of products. The main purpose of the RPD is to minimize variability in the output response of a product around the target value. The comprehensive discussion of Taguchi's RPD concepts and tools was examined by Nair *et al.* (1992), Robinson *et al.* (2003), and Park *et al.* (2006). Vining and Myers (1990) were among the first to offer the response-surface-based RPD as an alternative approach for modeling process relationships. They used a well-established statistical method, the response surface methodology, while incorporating basic principles of Taguchi's original version of RPD that include minimizing process variability and deviations from the target value. Along those lines, Del Castillo and Montgomery (1993) proposed the use of the generalized reduced gradient method as an optimization technique in order to obtain RPD solutions more efficiently. Lin and Tu (1995) then showed that the RPD solutions obtained from the aforementioned models might potentially eliminate better solutions from consideration, since their models strictly force the process mean to be located at a specific target value. Accordingly, they used the mean squared error (MSE) model to include a bias allowance based on the deviations between the process mean and the target value. The tradeoff in balancing bias and variability is a significant research issue. Based on this awareness, Cho *et al*. (2000) conducted further modifications of the MSE model by incorporating priority concepts and developed a nonlinear goal programming RPD model. In addition, Kim and Cho (2002) introduced a priority-based preemptive RPD model.

Special optimization methods are necessary to optimize the multiple response processes when there are multiple quality characteristics under study. For instance,

Kovach and Cho (2009), He *et al.* (2012), Goethals and Cho (2012), and Brito *et al*. (2014) proposed various multi-criteria optimization models for studying tradeoffs in RPD problems. Also, Shin and Cho (2005) proposed a bias-specified bi-objective RPD model as a relaxed zero-bias approach while keeping variability at the minimum. As an extension, Shaibu and Cho (2009) proposed an RPD model by incorporating a higher-order polynomial function. Another challenge in solving RPD problems is that part of the data is often missing when conducting experiments. Cho and Park (2005) developed RPD models using the iterative expectation maximization algorithm when the data is unbalanced. Other RPD articles written by Hu *et al*. (2014), Fang *et al*. (2015), Bao *et al.* (2016), Brito *et al*. (2016), and Ouyang *et al*. (2016) illustrated a wide spectrum of application areas of RPD, including the hydrokinetic turbine system, the fatigue life of a product, and machine parts. Finally, there are many practical situations where some of the input variables are constrained to be integer values. Ozdemir and Cho (2016, 2017) developed mixed nonlinear integer programming RPD models and numerically solved the Karush-Khun-Tucker system of equations.

Survival time and hazard functions are well developed in the area of survival analysis (see Kleinbaum and Klein, 2006, and Kalbfleisch and Prentice, 2011). A few authors studied how to maximize the survivability using regression functions in the medical research (see Mead and Pike (1975) and Carter and Wamper (1986)). In particular, Carter et al. (1979) proposed a regression model using a hazard function to estimate drug interactions from chemotherapy experiments. In addition, Solana et al. (1987) evaluated biological interactions of three genotoxic agents by maximizing the

median survival time using a polynomial regression. Das (2009) also proposed a regression model for handling survival data using a modified least squares estimator, assuming that the data follows a Weibull distribution. Similarly, Kuhn et al. (2000) used regression models to find an optimal combination of two drugs that maximizes survival time of patients in tumor studies and used the maximum likelihood estimator to estimate the mean and variance of experimental data. On the other hand, Shaibu et al. (2009) proposed new optimization models for censored data and obtained optimal factor settings using the expectation maximization method. Finally, Li et al. (2012a, 2012b, 2013) developed censoring-embedded nonlinear optimization models to find the optimal formulations for new tablet drugs in dissolution and bioequivalence studies.

4.2 Model development

This study consists of three sequential phases: experimentation, estimation, and optimization. In Section 4.2.1, the modified central composite design (CCD) for type I-right censored data is developed since the experimental design schemes that are currently available may not be effective in handling censored data. Based on the proposed CCD, Section 4.2.2 develops the survival distribution to estimate the median survival time and standard error of the median survival time, followed by their fitted functions in Section 4.2.3. In parallel with the fitted functions for median survival time and standard error of the median survival time based on the KM estimator, the fitted function for proportional hazards rate using the Cox proportional hazards model is developed in Section 4.2.4. Based on the fitted functions developed in Sections 4.2.3 and 4.2.4, various RPD optimization models are proposed in Section 4.3, followed by the numerical examples in

Sections 4.4.1, 4.4.2, and 4.4.3, where optimum operating conditions are compared. Based on the numerical results, Section 4.5 presents additional insights on the advantage of the nonparametric KM estimator over the maximum likelihood estimator using other parametric distributions. Finally, conclusions and future studies are discussed in Section 4.6. The structure of the research is depicted in Figure 4.1.

4.2.1 Development of the modified central composite design under the type I-right censoring scheme

For many production processes, a model that incorporates linear and quadratic effects of input variables on the response variable of interest is often required to approximate the response (see Montgomery, 2013). The central composite design (CCD), developed by Box and Wilson (1951), is a useful design of experiments for building the second-order model for the input variables. The CCD is partitioned into three sets of design points, which are factorial points to estimate linear and interactions effects, axial points to capture curvature effects and maintain the design rotatability, and center points to maintain the orthogonality of a design with the parameters minimally correlated to each other. The CCD is rotatable in the sense that all equidistant points from the center point in any direction have approximately the same variance in prediction.

This rotatability is achieved by setting the axial points, ω, equal to $\left(n_f\right)^{1/4}$, where n_f represents the number of factorial points. The rotatability is a crucial design property because the optimum operating conditions determined through the CCD can maximize process yield in a consistent manner, particularly when making prototyping decisions about process models in the early stages of industrial research and development.

Under the type I-right censoring scheme, the traditional CCD needs to be modified for two reasons. First, when the data is censored, we assume that there are two types of observations: the actual survival time of a unit (i.e., uncensored survival time) and the survival time at the termination time of an experimental period (i.e., censored survival time). Let $\tilde{\mathbf{T}}$ denote the matrix of the observed survival times from r experimental runs with n units in the ith run where $\tilde{t}_{ij} \in \tilde{\mathbf{T}}$, $i = 1,...,r$, and $j = 1,...,n$. Then, each recorded survival time t_{ij} is determined as follows:

$$t_{ij} = \begin{cases} \tilde{t}_{ij} \text{ for } \tilde{t}_{ij} < t_c \ \forall i \text{ and } j \\ t_c \text{ for } \tilde{t}_{ij} \geq t_c \forall i \text{ and } j \end{cases}$$

where $t_{ij} \in \mathbf{T}$, $\mathbf{T}$ is an $r \times n$ matrix of recorded survival times, t_c is the termination time, known as censoring time, and $t_{ij} \geq 0 \ \forall i$ and j. Second, to appropriately estimate the statistical descriptions of the recorded survival data at each design point, the traditional statistical estimators used in CCD (i.e., the mean and the variance of the normal distribution) are then replaced with two survival analysis-based nonparametric estimators. These include the median survival time and the variance of the median survival time. As such, the erroneous results from assuming an improper underlying distribution of the recorded survival time can be prevented; see Section 4.5. Table 4.1 portrays the modified rotatable CCD in coded levels, along with center point replications, under the type I-right censoring scheme.

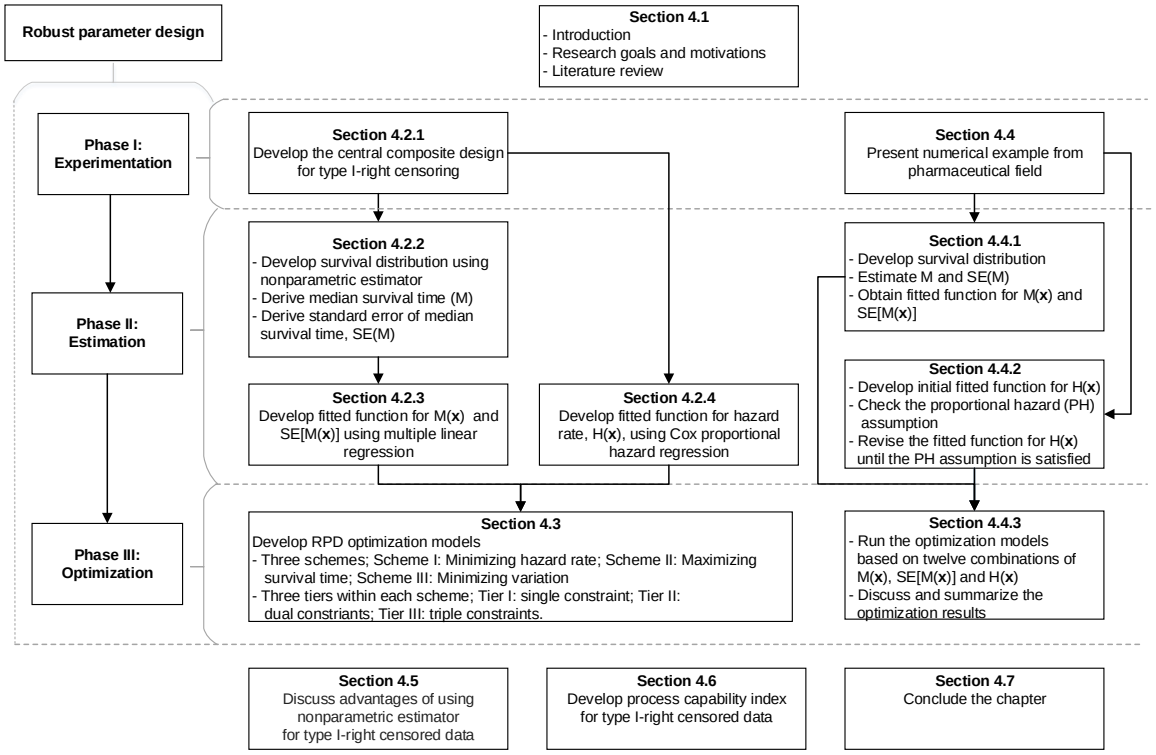


Figure 4.1 Structure of Chapter 4

Table 4.1 Experimental format for the modified central composite design for censored and uncensored data under the type I-right censoring scheme

	Input variables				Observations				M	$SE(M)$
	x_1	x_2	...	x_k						
Factorial points	-1	-1	…	-1	t_{11}	t_{12}	...	t_{in}	m_1	$SE(m_1)$
	1	-1	…	-1	t_{21}	t_{22}	...	t_{2n}	m_2	$SE(m_2)$
	⋮	⋮		⋮						
	1	1	…	1						
Axial points	$-\omega$	0	...	0						
	ω	0	...	0						
	0	$-\omega$	...	0						
	0	ω	...	0						
	⋮	⋮		⋮						
	0	0	...	$-\omega$						
	0	0	...	ω						
Center points	0	0	...	0						
	0	0	...	0						
	⋮	⋮		⋮						
	0	0	...	0	t_{r1}	t_{r2}	...	t_{rn}	m_r	$SE(m_r)$

4.2.2 Development of the fitted functions for median survival time and variance of the median survival time

4.2.2.1 Construction of the survival distribution for type I-right censored samples

A survival distribution is a probability distribution that links each outcome of an experiment to its probability of being survived. In this chapter, the Kaplan-Meier (KM) estimator developed by Kaplan and Meier (1958), one of the most-widely used nonparametric estimators for evaluating survival times with right censoring, is used. Initially, the observations in the experimental run *i* are ranked in increasing order; thus, we have $t_{ij} \le t_{ij+1} \forall i$ and j. The survival distribution, also known as a survival curve, at the ith design point consists of survival functions, $S(t_{ij}) \forall j$. Note that the survival function shows the probability that a unit survives longer than time t_{ij} and *T* denotes the random variable of observed survival times. Since the units that survive at time t_{ij} (i.e., $T > t_{ij}$) also survive at the time t_{ij-1} (i.e., $T > t_{ij-1}$), the survival function $S(t_{ij})$ is estimated from the joint probability density function of $T > t_{ij}$ and $T > t_{ij-1}$. Thus, it can be defined as $S(t_{ij}) = P(T > t_{ij}, T > t_{ij-1}) = P(T > t_{ij} \mid T > t_{ij-1}) \cdot P(T > t_{ij-1})$. Consequently, we have

$$S(t_{ij}) = \begin{Bmatrix} P(T > t_{ij} \mid T > t_{ij-1}) \cdot P(T > t_{ij-1} \mid T > t_{ij-2}) \cdot P(T > t_{ij-2} \mid T > t_{ij-3}) \cdots \\ \cdot P(T > t_{i2} \mid T > t_{i1}) \cdot P(T > t_{i1} \mid T > 0) P(T > 0) \end{Bmatrix}$$

Suppose that n_{ij} is the number of units at risk at time t_{ij} and d_{ij} is the censoring indicator associated with t_{ij} where $d_{ij} = \left\{ 0, \text{ for } t_{ij} = t_c \,;\, 1, \text{ for } t_{ij} = \tilde{t}_{ij} \right\}$. The conditional probability

function $P(T > t_{ij} \mid T > t_{ij-1})$ is then estimated as $P(T > t_{ij} \mid T > t_{ij-1}) = \dfrac{n_{ij} - d_{ij}}{n_{ij}}$ and

$P(T > 0) = 1$. Therefore, the estimated survival function at time t_{ij} is obtained as

$$S(t_{ij}) = \prod_{\kappa=1}^{j} \frac{n_{i\kappa} - d_{i\kappa}}{n_{i\kappa}} = \prod_{\kappa=1}^{j} \left(1 - \frac{d_{i\kappa}}{n_{i\kappa}}\right), \tag{4.1}$$

where $0 \le S(t_{ij}) \le 1$ and $S(t_{ij} = 0) = 1$. Note that if the survival time is censored $\left(d_{ij} = 0\right)$, we obtain $S(t_{ij}) = S(t_{ij-1})$. As a result, $S(t_{ij})\,\forall j$ can be plotted versus $t_{ij}\,\forall j$ to show the survival curve of the ith design point. An illustrative plot is shown in the numerical example section.

4.2.2.2 Estimation of the median survival time and the variance of the median survival time

To find the central value of the survival data and the variation of the central value, the median is used because it is more effective in capturing the characteristics of a skewed distribution that is often found in censored data. For details, see Kaplan and Meier (1958), and Lee and Wang (2003). The median is, by definition, the 50th percentile of the distribution, or the largest data point t_{ij} such that $S(t_{ij}) \le 0.5$. Therefore, the estimated median survival time of experimental observations of the ith design point, m_i, is obtained as

$$m_i = \max\left(t_{ij} \mid S(t_{ij}) \le 0.5\right) \forall i \tag{4.2}$$

For instance, given that $S(t_{i3}=5)=0.45$ and $S(t_{i4}=7)=0.51$, the estimated median survival time is $m_i = t_{i3} = 5$ since $\hat{S}(t_{i3}=5)=0.45$ is the closest survival function below 0.5.

Since the survival probability distribution is developed nonparametrically, the variance of the median survival time, $Var(m_i)$, is estimated from the variance of the survival function at the median survival time, $Var\left[S\left(t_{ij}\right)\right]$ where $t_{ij} = m_i$, using Greenwood's formula (see Greenwood, 1926, Kaplan and Meier, 1958, and Breslow and Crowley, 1974), which is expressed as

$$Var\left[S(t_{ij})\right] \cong \left(S(t_{ij})\right)^2 \sum_{\kappa=1}^{j} \frac{d_{i\kappa}}{n_{i\kappa}\left(n_{i\kappa}-d_{i\kappa}\right)} \tag{4.3}$$

The derivation of Equation (4.3) is presented in Appendix A. Consequently, $Var\left(m_i\right)$ can be approximated by using the estimation method of variance at the $(100p)^{\text{th}}$ percentile (see Collett, 2015 and Klein and Moeschberger, 2005). Let $f\left(t\left(p\right)\right)$ be the estimated probability density function of the survival time t at the $(100p)^{th}$ percentile where $f\left(t\left(p\right)\right) = -\Delta S\left(t\left(p\right)\right)/\Delta t\left(p\right)$; see the derivation in Appendix B. Further, suppose that p_m = 0.50 is the percentile at the median in probability terms and $f_i\left(t\left(p_m \pm \alpha\right)\right)$ is the estimated probability density function of the median survival time m_i under the type I error probability α. We then have

$$f_i\left(t\left(p_m \pm \alpha\right)\right) = -\frac{\min\left(S\left(t_{ij}\right) \mid S\left(t_{ij}\right) \geq p_m + \alpha\right) - \max\left(S\left(t_{ij}\right) \mid S\left(t_{ij}\right) \leq p_m - \alpha\right)}{\max\left(t_{ij} \mid S\left(t_{ij}\right) \geq p_m + \alpha\right) - \min\left(t_{ij} \mid S\left(t_{ij}\right) \leq p_m - \alpha\right)} \tag{4.4}$$

Finally, the variance of the median survival time of at the design point i^{th} is obtained as

$$Var\left(m_i\right) = \frac{Var[S\left(m_i\right)]}{\left[\, f_i\left(t\left(p_m \pm \alpha\right)\right)\right]^2} \tag{4.5}$$

where $Var[S\left(m_i\right)]$ is obtained from Equation (4.3), i.e., $t_{ij} = m_i$. Similarly, the standard error of the median survival time, $SE\left(m_i\right)$, is written as

$$SE(m_i) = \sqrt{\frac{Var[S\left(m_i\right)]}{\left[\, f_i\left(t\left(p_m \pm \alpha\right)\right)\right]^2}} \tag{4.6}$$

4.2.3 Estimation of fitted functions for median survival times and variance of median survival times

The fitted function, known as the response surface function, is a statistical model for estimating the relationship between input variables and their responses, which serves as a bridge between the experimentation phase and the optimization phase of RPD. The fitted function for a curvature is generally obtained using the multiple linear regression, which is defined as $\mathbf{y} = \mathbf{X\beta} + \mathbf{e}$, where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{r1} & x_{r2} & \cdots & x_{rk} \end{bmatrix}$, $\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$, and $\mathbf{e} = \begin{bmatrix} e_1 & e_2 & \cdots & e_r \end{bmatrix}^T$. Note that $\mathbf{y}$ is an $r \times 1$ vector of responses; $\mathbf{X}$ is an $r \times k$ vector of design matrix of *k* regression parameters; $\boldsymbol{\beta}$ is a $k \times 1$ vector of regression coefficient; and $\mathbf{e}$ is an $r \times 1$ vector of random errors, where $\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}}$. Using the least squares method, the vector of least squares estimates, $\mathbf{b}$, is then obtained from $\mathbf{b} = (\mathbf{X'X})^{-1}\mathbf{X'y}$. Therefore, $\hat{\mathbf{y}} = \mathbf{Xb}$. Letting $\mathbf{M}$ be a vector of the estimated median

survival time where $\mathbf{M} = \begin{bmatrix} m_1 & m_2 & \cdots & m_r \end{bmatrix}^T$, the fitted function of the median survival time is estimated as

$$\hat{M}(\mathbf{x}) = \mathbf{X}\mathbf{b}_M \tag{4.7}$$

where $\mathbf{b}_M = (\mathbf{X'X})^{-1}\mathbf{X'M}$. In a similar manner, the fitted function for the standard error of the median survival time is then obtained as

$$\hat{S}E\left[\hat{M}(\mathbf{x})\right] = \mathbf{X}\mathbf{b}_S\,, \tag{4.8}$$

where $\mathbf{b}_S = (\mathbf{X'X})^{-1}\mathbf{X'S}$ and $\mathbf{S}$ is a vector of the estimated standard error of the median survival time, $\boldsymbol{S} = \begin{bmatrix} SE(m_1) & SE(m_2) & \cdots & SE(m_r) \end{bmatrix}^T$.

4.2.4 Development of fitted function for proportional hazards using the Cox proportional hazards regression

In parallel with the fitted functions for median survival times and variance of median survival times as shown in Section 4.2.3, the fitted functions for proportional hazards rate are developed in this section. In order to strengthen the traditional RPD optimization model, one of this study's contributions is to incorporate the proportional hazards rate as an additional measure of the product's lifetime. Unlike the median survival time and its variance, the proportional hazards rate give better insights on the failure risk of a unit, which is known as the mortality rate in the medical research. To estimate the functional relationship between the input variables and their proportional hazards rate, the Cox proportional hazards (PH) model (Cox and Oakes, 1972) is recommended since it does not require the assumption of the underlying distribution of

censored data. This section presents the definition of the proportional hazards, as well as, a brief background of the Cox PH model.

The hazard rate $(h(t))$ represents another perspective of survivability, and it is defined as the risk (or probability) that an observed unit fails within [t, $t + \Delta t$] with $\Delta t \to 0$, or $h(t) = \lim_{\Delta t \to 0} \frac{P(t \le T < t + \Delta t)}{\Delta t \cdot P(T > t)}$. Denoting $f(t)$ and $F(t)$ as a probability density function and a cumulative density function of survival time T respectively, we have $f(t) = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$, $F(t) = \int_0^t f(u)\,du$, and $S(t) = P(T > t)$ where $S(t) = 1 - F(t)$.

The hazard rate is then defined as $h(t) = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t \cdot S(t)}$, yielding $h(t) = \frac{f(t)}{S(t)}$.

Suppose that $\mathbf{x}$ is a vector of regression parameters where $\mathbf{x} = (x_1, x_2, ..., x_k)$ and $h(t \mid \mathbf{x})$ is the hazard rate of $\mathbf{x}$ at time t where $0 \le h(t \mid \mathbf{x}) \le \infty$. Since $h(t \mid \mathbf{x})$ is non-negative, a logarithmic linear regression is used to predict the relationship between $\mathbf{x}$ and $h(t \mid \mathbf{x})$, which is written as $\log h(t \mid \mathbf{x}) = \boldsymbol{\beta}\mathbf{x} + \mathbf{e}$, where $\mathbf{e}$ is the vector of random errors. Since the logarithm is the inverse operation to exponentiation, the general form of the Cox PH model (Cox, 1975) is expressed as

$$h(t \mid \mathbf{x}) = h_0(t)\, e^{\boldsymbol{\beta}\mathbf{x}}$$

where $h_0(t)$ is the baseline hazard rate (i.e., the exponential error term) at time t, which does not depend on the input variables, and $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_k)^T$ is the estimated

coefficients of $\mathbf{x}$. We assume that $\mathbf{x}$ is time independent where the effect of the input variables is constant for all *t*. Thus, the previous equation reduces to

$$h(\mathbf{x}) = h_0\, e^{\boldsymbol{\beta}\mathbf{x}}$$

The proportional hazards rate indicate the ratio of the hazard rate at the level of the optimum operating conditions $0 \le h(\mathbf{x}) \le \infty$, $\hat{h}(\mathbf{x}^*)$, to the hazard rate of the baseline operating conditions or $\hat{h}(\mathbf{x})$ where $\mathbf{x} = \mathbf{0}$ in order to compare their predicted values. Thus, the fitted function for proportional hazards rate of $\mathbf{x}$, $\hat{H}(\mathbf{x})$, is expressed as

$$\hat{H}(\mathbf{x}) = \frac{\hat{h}(\mathbf{x}^*)}{\hat{h}(\mathbf{x})} = \frac{h_0\, e^{\boldsymbol{\beta}_H \mathbf{x}^*}}{h_0 e^{\boldsymbol{\beta}_H \mathbf{x}}} = e^{\left(\boldsymbol{\beta}_H (\mathbf{x}^* - \mathbf{x})\right)} \tag{4.9}$$

where $\boldsymbol{\beta}_H$ is a vector of regression coefficients obtained by maximizing the Cox's partial log-likelihood function; see Cox (1975). The Cox PH model does not require an assumption regarding the distribution shape of observations. However, since the input variables are assumed to be time independent, the Cox PH model assumes that the proportional hazards rate remains constant over time.

4.3 Development of type I-right censoring based RPD optimization models

One of the measures of product quality is a fraction of units that meets specifications. This is only a partly correct measure because, in fact, a product that meets the specifications will not necessarily be working properly over time. If the notion of such defects over time can be incorporated into the conceptual design of products, significant cost savings would be made by avoiding the likelihood of product recalls.

Unfortunately, there is little research work on this particular subject. As such, this section proposes various RPD optimization models for the optimum operating conditions **x*** that maximize the median survival time $\hat{M}(\mathbf{x})$, minimize the standard error of the median survival time $\hat{S}E(\mathbf{x})$, and minimize the proportional hazards rate $\hat{H}(\mathbf{x})$, as shown in Table 4.2.

Optimization criteria

a. *Survival time.* The fitted function for median survival time, $\hat{M}(\mathbf{x})$, is obtained using Equation (4.7) where $m_i = \max\left(t_{ij} \mid S(t_{ij}) \le 0.5\right)$.

b. *Variability of survival time.* The fitted function for the standard error of median survival time, $\hat{S}E\left(M(\mathbf{x})\right)$, is estimated from Equation (4.8). The standard error of median survival time at each design point under a significance level of α, or a type I error probability, is computed as $SE(m_i) = \sqrt{\dfrac{Var[S(m_i)]}{\left[f_i\left(t\left(p_m \pm \alpha\right)\right)\right]^2}}$, where

$$Var\left[S(m_i)\right] = \left[S(m_i)\right]^2 \sum_{\kappa=1}^{\max(j \mid S(t_{ij}) \le 0.5)} \frac{d_{i\kappa}}{n_{i\kappa}\left(n_{i\kappa} - d_{i\kappa}\right)}, \quad S(m_i) = \prod_{\kappa=1}^{\max(j \mid S(t_{ij}) \le 0.5)} \left(1 - \frac{d_{i\kappa}}{n_{i\kappa}}\right),$$

and $f_i\left(t\left(0.5 \pm 0.05\right)\right) = -\dfrac{\min\left(S\left(t_{ij}\right) \mid S\left(t_{ij}\right) \ge 0.55\right) - \max\left(S\left(t_{ij}\right) \mid S\left(t_{ij}\right) \le 0.45\right)}{\max\left(t_{ij} \mid S\left(t_{ij}\right) \ge 0.55\right) - \min\left(t_{ij} \mid S\left(t_{ij}\right) \le 0.45\right)}$.

See Section 4.2.4.

c. *Proportional hazards.* The fitted function for the proportional hazards, $\hat{H}(\mathbf{x})$, is obtained using the Cox PH model in Equation (4.9).

Constraints

a. Operability range of input variables associated with the modified central composite design, $\omega = \left(2^k\right)^{\frac{1}{4}}$ where k is the number of input variables

b. Desired lower bound of the median survival time, τ

c. Desired upper bound of standard error of the median survival time, ε

d. Desired minimum level of proportional hazards, $1-\lambda$

Table 4.2 Proposed type I-right censored RPD optimization models

I	Maximize $\hat{M}(\mathbf{x})$ subject to $\lvert\mathbf{x}\rvert \le \omega$	V	Minimize $\hat{S}E[M(\mathbf{x})]$ subject to $\lvert\mathbf{x}\rvert \le \omega$	IX	Minimize $\hat{H}(\mathbf{x})$ subject to $\lvert\mathbf{x}\rvert \le \omega$
II	Minimize $\hat{M}(\mathbf{x})$ subject to $\lvert\mathbf{x}\rvert \le \omega$ $\hat{S}E[M(\mathbf{x})] \le \varepsilon$	VI	Minimize $\hat{S}E[M(\mathbf{x})]$ subject to $\lvert\mathbf{x}\rvert \le \omega$ $\hat{M}(\mathbf{x}) \ge \tau$	X	Minimize $\hat{H}(\mathbf{x})$ subject to $\lvert\mathbf{x}\rvert \le \omega$ $\hat{M}(\mathbf{x}) \ge \tau$
III	Minimize $\hat{M}(\mathbf{x})$ subject to $\lvert\mathbf{x}\rvert \le \omega$ $\hat{H}(\mathbf{x}) \le 1-\lambda$	VII	Minimize $\hat{S}E[M(\mathbf{x})]$ subject to $\lvert\mathbf{x}\rvert \le \omega$ $\hat{H}(\mathbf{x}) \le 1-\lambda$	XI	Minimize $\hat{H}(\mathbf{x})$ subject to $\lvert\mathbf{x}\rvert \le \omega$ $\hat{S}E[M(\mathbf{x})] \le \varepsilon$
IV	Minimize $\hat{M}(\mathbf{x})$ subject to $\lvert\mathbf{x}\rvert \le \omega$ $\hat{S}E[M(\mathbf{x})] \le \varepsilon$ $\hat{H}(\mathbf{x}) \le 1-\lambda$	VIII	Minimize $\hat{S}E[M(\mathbf{x})]$ subject to $\lvert\mathbf{x}\rvert \le \omega$ $\hat{M}(\mathbf{x}) \ge \tau$ $\hat{H}(\mathbf{x}) \le 1-\lambda$	XII	Minimize $\hat{H}(\mathbf{x})$ subject to $\lvert\mathbf{x}\rvert \le \omega$ $\hat{M}(\mathbf{x}) \ge \tau$ $\hat{S}E[M(\mathbf{x})] \le \varepsilon$

4.4 Numerical example

To illustrate how the proposed models are applied, the actual data collected by Cho *et al*. (2010), shown in Table 4.3, is used. The selected experimental deign is a rotatable central composite design with three input variables (x_1, x_2, x_3), eight factorial points, six axial points with $\omega = \left(2^3\right)^{1/4} = 1.6818,$ and five center points. Each design point has thirteen observations, and the censoring time is set at 500 hours, meaning that any observations survived after 500 hours are censored and recorded as 500+.

4.4.1 Fitting the median survival time and its variance

The goal of this phase is to obtain the fitted functions for the median survival time and the standard error of median survival time. The computation steps are explained as follows.

Step 1: Developing the survival distribution. The survival data points are sorted in increasing order at each design point. For instance, the smallest survival time at the first design point is $t_{1,1}$ = 480.21. Furthermore, the survival time $t_{1,1}$ is uncensored; thus, $d_{1,1}$ *=1*. Since there is no failed observation occurred prior to time $t_{1,1}$, $n_{1,1}$ = *13*. Using Equation (4.1), the first survival function at the first design point (i.e., the probability that the observed units at the first design point survive longer than $(t_{1,1})$ is computed as $S(t_{1,1})$ = 0.9231 where $d_{1,1}$ = 1 and $n_{1,1}$ =13. Table 4.4 shows the survival functions for the first and second design points. The same procedure is repeated for the entire design points. As a result, the distribution of the survival function is plotted in Figure 4.2.

Step 2: Estimating the median survival time. The median survival time is obtained from Equation (4.2). Based on the survival distribution of the first design point in Table 4.4, the largest survival time under the 50th percentile is the seventh survival time (j = 7). Thus, $S(m_1) = S(t_{1,7}) = 0.4615$. As a result, the estimated median survival time at the first design point is m_1 = 490.76. Similarly, the median survival times at other design points can be computed.

Step 3: Estimating the standard error of the median survival time. At the first design point, the median locates at $\hat{S}(t_{1,7} = 490.76) = 0.4615$. By using Equation (4.3), we have $Var(S(m_1)) = Var(S(t_{1,7})) = 0.0191$. Given $\alpha = 0.05$, $f_1(t(p_m \pm \alpha))$ is then obtained using Equation (4.4). From Table 4.4, we have that $\min(S(t_{ij}) \mid S(t_{ij}) \geq 0.55) = S(t_{1,5})$ = 0.6154, $\max(S(t_{ij}) \mid S(t_{ij}) \leq 0.45) = S(t_{1,9}) =$ 0.3846, $\max(t_{ij} \mid S(t_{ij}) \geq 0.55) = t_{1,5} =$ 491.43, and $\min(t_{ij} \mid S(t_{ij}) \leq 0.45) = t_{1,9}$ = 494.94. Hence, $f_1(t(0.5 \pm 0.05))$ = 0.1789. Finally, using Equations (4.5) and (4.6), $Var(m_1)$ = 0.5974 and $SE(m_1)$ = 0.7729.

Step 4: *Repeating steps 1 – 3 for all other runs*. The median survival times and their standard errors are summarized in Table 4.5.

Step 5: Estimating the fitted function of median survival time and its standard error. Using the multiple linear regression (see Section 4.2.3), the fitted function for the median survival time is obtained as

$$\hat{M}(\mathbf{x}) = 488.38 + 0.392x_1 - 1.086x_2 - 0.716x_3 + 1.721x_1^2 - 0.084x_2^2 + 1.104x_3^2 \\ - 2.28x_1x_2 - 1.10x_1x_3 + 0.25x_2x_3$$

Similarly, the fitted function for the standard error of the median survival time is

$$\hat{S}E[M(\mathbf{x})] = 1.904 + 0.771x_1 - 0.438x_2 - 0.002x_3 + 0.324x_1^2 + 0.361x_2^2 - 0.286x_3^2 \\ - 0.601x_1x_2 - 0.575x_1x_3 - 0.252x_2x_3$$

4.4.2 Fitting the proportional hazards function using the Cox PH regression

In this section, the observed survival times in Table 4.3 are analyzed using the Cox PH model to predict the relationship between the input variables and their proportional hazards, as shown in Section 4.2.4. The evaluation of the fitted function of hazard and the testing of the PH assumption are conducted using Stata® (2016). The computational steps with Stata commands are presented as follows.

Step 1: Define the survival time and the censoring indicator using the *stset* command.

Step 2: Obtain the initial PH model. The coefficient of the variables (x_1, x_2, x_3), the quadratic terms (x_1^2, x_2^2, x_3^2), and the interaction terms (x_1x_2, x_1x_3, x_2x_3) are obtained by using the command *stcox.* As a result, the initial PH model is fitted as

$$\hat{H}(\mathbf{x}) = \exp\begin{pmatrix} -0.0395x_1 - 0.1023x_2 - 0.2619x_3 - 0.0992x_1^2 + 0.0279x_2^2 \\ +0.0516x_3^2 - 0.1780x_1x_2 + 0.0672x_1x_3 + 0.1786x_2x_3 \end{pmatrix}$$

The estimated coefficients of the initial Cox PH model are presented on the upper part of Table 4.6.

Table 4.3 Experimental data

Design point	x_1	x_2	x_3	Observed Survival time* (n_i = 13, N = 247)						
1	-1	-1	-1	488.77	484.37	490.76	491.34	486.22	494.56	494.56
				480.21	490.18	491.48	490.94	489.65	492.9	
2	-1	-1	1	490.04	500+	497.1	491.24	499.2	500+	495.78
				500+	500+	490.24	486.72	500+	490.98	
3	-1	1	-1	500+	481.27	494.05	499.71	500+	494.94	497.41
				479.66	480.62	493.16	500+	485.29	491.43	
4	-1	1	1	487.54	493.1	500+	493.21	496.16	500+	500+
				483.45	489.48	500+	493.74	495.65	492.99	
5	1	-1	-1	487.89	478.91	490.46	482.31	497.64	483.87	492.14
				490.22	483.14	475.39	488.49	482.59	500+	
6	1	-1	1	500+	500+	494.1	483.43	492.28	480.28	488.82
				488.58	488.99	486.25	500+	494.3	493.38	
7	1	1	-1	497.25	476.03	492.48	495.82	500+	493.55	489.71
				500+	493.49	500+	494.64	500+	494.26	
8	1	1	1	487.3	500+	500+	479.5	487.46	500+	488.1
				481.7	487.01	478.47	500+	500+	486.42	
9	-1.6818	0	0	490.77	491.81	497.81	500+	493.45	494.39	495.14
				484.87	491.27	491.4	484.78	486.2	488.27	
10	1.6818	0	0	498.66	500+	492.2	500+	500+	500+	492.7
				479.7	500+	489.38	482.63	483.34	487.33	
11	0	-1.6818	0	494.54	493.15	484.01	487.66	481.6	474.94	495.89
				486	491.56	490.95	500+	500+	500+	
12	0	1.6818	0	489.11	481.27	481.71	500+	479.88	480.88	500+
				479.64	488.63	480.32	478.63	489.37	486.08	
13	0	0	-1.6818	489.23	493.16	490.49	492.2	486.29	488.58	489.22
				486.15	485.8	493.09	490.11	487.99	492.68	
14	0	0	1.6818	491.17	485.89	495.96	491.42	496.56	495.09	487.21
				484.58	489.85	488.74	486.52	500+	500+	
15	0	0	0	491.77	475.6	477.4	484.54	489.3	492.6	479.5
				487.2	490.6	487.77	492.66	497.1	486.16	
16	0	0	0	500+	482.63	488.45	500+	500+	485.89	488.29
				486.76	500+	500+	488.74	491.71	484.5	
17	0	0	0	488.67	497.35	485.83	481.85	491.62	484.24	489.29
				485.72	491.98	492.07	488	478.05	489.91	
18	0	0	0	497.66	494.57	480.59	488.67	485.43	483.58	482.46
				490.65	486.79	489.4	487.12	486.59	491.09	
19	0	0	0	487.5	481.79	478.6	500+	487	481.67	490.79
				489.6	494.68	500+	500+	493.01	483.6	

Table 4.4 The survival distribution of observations taken at the first and second design points

The first design point (i=1) *where* [x_1, x_2, x_3] = [-1,-1,-1]					*The second design point* (i=2) *where* [x_1, x_2, x_3] = [-1,-1, +1]				
j	t_{1j}	n_{1j}	d_{1j}	$S(t_{1j})$	j	t_{2j}	n_{2j}	d_{2j}	$S(t_{2j})$
1	480.21	13	1	0.9231	1	486.72	13	1	0.9231
2	484.37	12	1	0.8462	2	490.04	12	1	0.8462
3	486.22	11	1	0.7692	3	490.24	11	1	0.7692
4	488.77	10	1	0.6923	4	490.98	10	1	0.6923
5	489.65	9	1	0.6154	5	491.24	9	1	0.6154
6	490.18	8	1	0.5385	6	495.78	8	1	0.5385
7	490.76	7	1	0.4615	7	497.1	7	1	0.4615
8	490.94	6	1	0.3846	8	499.2	6	1	0.3846
9	491.34	5	1	0.3077	9	500	5	0	0.3846
10	491.48	4	1	0.2308	10	500	5	0	0.3846
11	492.9	3	1	0.1538	11	500	5	0	0.3846
12	494.56	2	1	0.0769	12	500	5	0	0.3846
13	494.56	1	1	0.0000	13	500	5	0	0.3846

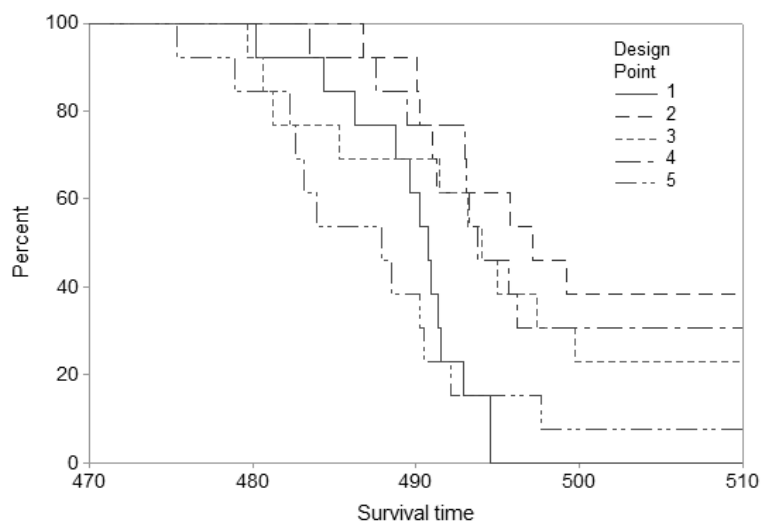


Figure 4.2 Survival curve of design points 1-5

Table 4.5 Median survival time, standard error of median survival time, and variance of median survival time at each design point

Design point i^{th}	x_1	x_2	x_3	M	$Var(M)$	$SE(M)$
1	-1	-1	-1	490.76	0.5974	0.7729
2	-1	-1	1	497.1	22.7452	4.7692
3	-1	1	-1	494.05	4.4226	2.1029
4	-1	1	1	493.74	2.3342	1.5278
5	1	-1	-1	487.89	10.2747	3.2054
6	1	-1	1	492.28	7.4644	2.7321
7	1	1	-1	494.64	1.8498	1.3601
8	1	1	1	487.46	0.4265	0.6531
9	-1.6818	0	0	491.4	0.3883	0.6231
10	1.6818	0	0	492.7	30.9143	5.5601
11	0	-1.6818	0	491.56	10.8195	3.2893
12	0	1.6818	0	481.71	9.7067	3.1155
13	0	0	-1.6818	489.23	0.8403	0.9167
14	0	0	1.6818	491.17	2.5783	1.6057
15	0	0	0	487.77	3.5393	1.8813
16	0	0	0	488.74	4.1987	2.0491
17	0	0	0	488.67	4.2975	2.0731
18	0	0	0	487.12	1.5531	1.2462
19	0	0	0	489.6	5.1563	2.2708

Step 3: Testing the PH assumption. The constant proportional hazards assumption is tested using the scaled Schoenfeld residuals (Grambsch and Therneau, 1994) using the *stphtest* command. The result shows that the PH assumption is invalid for the initial PH model since p-value < 0.05.

Step 4: Adjusting the fitted model. The inactive variables are removed one at a time, and the adjusted model is retested until the PH assumption is satisfied. After removing the four input variables $\left(x_1, x_2^2, x_3^2, x_1x_2\right)$ as shown on the right-hand side of Table 4.6, the fitted function of proportional hazards rate is finally obtained as

$$\hat{H}(\mathbf{x}) = \exp\begin{pmatrix} -0.0996x_2 - 0.2683x_3 - 0.1067x_1^2 \\ -0.1837x_1x_2 + 0.0716x_1x_3 + 0.1813x_2x_3 \end{pmatrix}$$

Using the *stphtest* command the PH assumption is checked once again. The result shows that the the PH assumption of this model is now satisfied with *p*-value = 0.3709. Figure 4.3 presents the scaled Schoenfeld plots where the horizontal trend lines are pproximately constant. This confirms that the proportional hazards rate is approximately constant over time.

Table 4.6 Statistics associated with the fitted proportional hazard model

	x	Coefficient	SE	Z	P-value	95% CI	
Initial fitted PH model	x_1	-0.039	0.089	-0.45	0.656	-0.213	0.134
	x_2	-0.102	0.090	-1.13	0.258	-0.279	0.072
	x_3	-0.262	0.087	-3.03	0.002	-0.431	-0.092
	x_1^2	-0.099	0.092	-1.08	0.282	-0.280	0.081
	x_2^2	0.028	0.092	0.30	0.762	-0.153	0.209
	x_3^2	0.052	0.088	0.58	0.560	-0.122	0.225
	x_1x_2	-0.178	0.109	-1.63	0.103	-0.392	0.036
	x_1x_3	0.067	0.109	0.61	0.540	-0.147	0.282
	x_2x_3	0.179	0.110	1.63	0.103	-0.036	0.393
	Test of the regression model : P-value = 0.0362; Test of the proportional hazards assumption : P-value = 0.0001						
Adjusted fitted PH model	**x**	Coefficient	SE	Z	P-value	95% Confidence interval	
	x_2	-0.100	0.089	-1.12	0.262	-0.274	0.074
	x_3	-0.268	0.088	-3.07	0.002	-0.440	-0.097
	x_1^2	-0.107	0.090	-1.19	0.235	-0.283	0.070
	x_1x_2	-0.184	0.110	-1.67	0.095	-0.399	0.032
	x_1x_3	-0.072	0.110	0.65	0.516	-0.145	0.288
	x_2x_3	0.181	0.111	1.63	0.102	-0.036	0.399
	Test of the regression model : P-value = 0.0082; Test of the proportional hazards assumption : P-value = 0.3595						

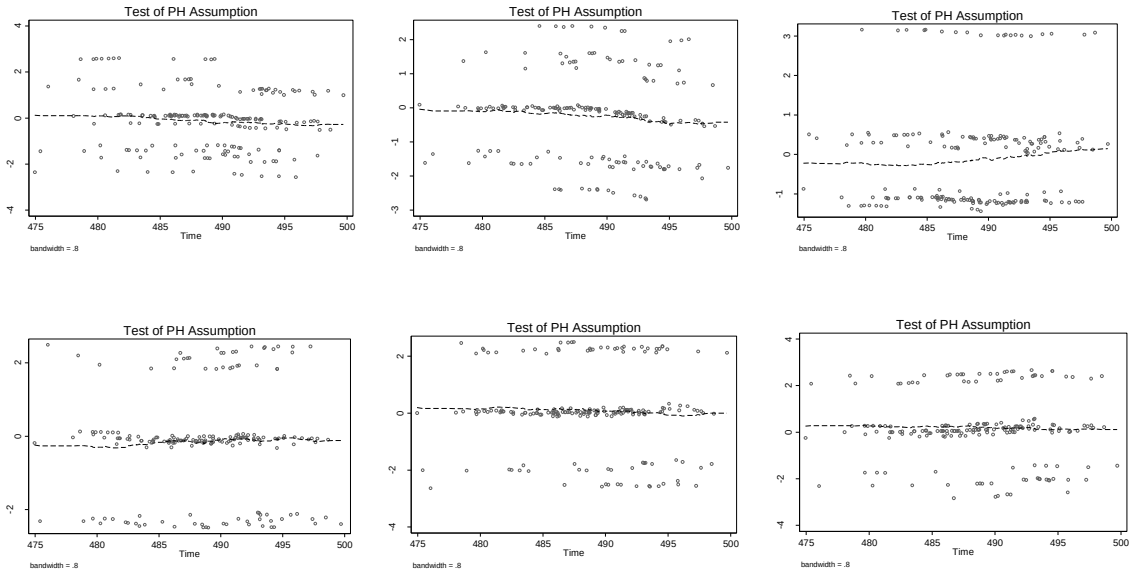


Figure 4.3 The scaled Schoenfeld plots (the first row from left to right: x_2, x_3, x_1^2; the second row from left to right: x_1x_2, x_1x_3, x_2x_3)

4.4.3 Optimization results and findings

Using the fitted functions obtained in Sections 4.4.1 and 4.4.2, the optimization results based on the twelve models proposed in Section 4.3, are compared. Suppose that the minimum survival time allowed is 495, the maximum standard error of the median survival time allowed is 5, and the maximum proportional hazards allowed is 0.75. That is, $\hat{M}(\mathbf{x}) \geq 495$, $\hat{S}E\left[M(\mathbf{x})\right] \leq 5$, and $\hat{H}(\mathbf{x}) \leq 0.75$. Maple® (2016) programming codes for the twelve nonlinear programming models were developed, and the optimization results are summarized in Table 4.8, where italicized numbers represent the predicted values. As expected, all the optimum operating conditions satisfy the requirement for the operability region associated with the proposed type I-right censored central composite design. Also, the addition of constraints decreases the survival time and failure rate and increases the variability of the median survival time.

Table 4.7 Optimization results

Optimization models	x_1^*	x_2^*	x_3^*	$\hat{M}(\mathbf{x}^*)$	$\hat{SE}[M(\mathbf{x}^*)]$	$\hat{H}(\mathbf{x}^*)$
Tier 1 models with one constraint						
Model I: Maximize $\hat{M}(\mathbf{x})$ $s.t.\ \lvert\mathbf{x}\rvert \le \omega$	1.6818	-1.6818	-1.6818	510.60	{*4.6705*}	{*3.1479*}
Model V: Minimize $\hat{SE}(\mathbf{x})$ $s.t.\ \lvert\mathbf{x}\rvert \le \omega$	-1.2079	-0.3095	-1.6818	{*492.45*}	0.8961	{*1.6446*}
Model IX: Minimize $\hat{H}(\mathbf{x})$ $s.t.\ \lvert\mathbf{x}\rvert \le \omega$	1.6818	1.6818	-1.6818	{*492.67*}	{*3.9707*}	0.2857
Tier 2 models with two constraints						
Model II: Maximize $\hat{M}(\mathbf{x})$ $s.t.\ \hat{SE}[M(\mathbf{x})] \le \varepsilon;\ \lvert\mathbf{x}\rvert \le \omega$	1.6818	-1.1996	1.6818	498.42	5.0000	{*0.6531*}
Model III: Maximize $\hat{M}(\mathbf{x})$ $s.t.\ \hat{H}(\mathbf{x}) \le \lambda;\ \lvert\mathbf{x}\rvert \le \omega$	1.6818	-1.6818	1.6818	500.48	{*5.5999*}	0.6866
Model VI: Minimize $\hat{SE}(\mathbf{x})$ $s.t.\ M(\mathbf{x}) \ge \tau;\ \lvert\mathbf{x}\rvert \le \omega$	1.3239	-0.9208	1.6818	495.00	4.3292	{*0.6416*}
Model VII: Minimize $\hat{SE}(\mathbf{x})$ $s.t.\ \hat{H}(\mathbf{x}) \le \lambda;\lvert\mathbf{x}\rvert \le \omega$	-0.6000	-0.5288	1.6818	{*491.52*}	3.1468	0.4825
Model X: Minimize $\hat{H}(\mathbf{x})$ $s.t.\ \hat{SE}[M(\mathbf{x})] \le \varepsilon;\ \lvert\mathbf{x}\rvert \le \omega$	1.6818	0.9824	1.6818	{*488.65*}	5.0000	0.5207
Model XI: Minimize $\hat{H}(\mathbf{x})$ $s.t.\ M(\mathbf{x}) \ge \tau;\ \lvert\mathbf{x}\rvert \le \omega$	1.6818	-0.4176	1.6818	495.00	{*4.4887*}	0.6022
Tier 3 models with three constraints						
Model IX: Maximize $\hat{M}(\mathbf{x})$ $s.t.\ \hat{SE}[M(\mathbf{x})] \le \varepsilon, \hat{H}(\mathbf{x}) \le \lambda, \lvert\mathbf{x}\rvert \le \omega$	1.6818	-1.1996	1.6818	498.42	5.0000	0.6531
Model VIII: Minimize $\hat{SE}(\mathbf{x})$ $s.t.\ M(\mathbf{x}) \ge \tau, \hat{H}(\mathbf{x}) \le \lambda, \lvert\mathbf{x}\rvert \le \omega$	1.3239	-0.9208	1.6818	495.00	4.3292	0.6416
Model XII: Minimize $\hat{H}(\mathbf{x})$ $s.t.\ M(\mathbf{x}) \ge \tau, \hat{SE}[M(\mathbf{x})] \le \varepsilon, \lvert\mathbf{x}\rvert \le \omega$	1.6818	-0.4176	1.6818	495.00	4.4887	0.6022

Note that predicted values are shown in *italics* in brackets

Finally, the optimum operating conditions from Models I, III, and IX provide the longer survival time in tiers 1, 2, and 3, respectively. Also, Models V, VII, and VIII provide a smaller variability associated with the median survival time in each tier. Similarly, the optimum operating conditions from Models IX, VII, and XII result in the lowest proportional hazards rate in each tier.

4.5 Additional remarks on the parametric approach and recommendations

Under the type I-right censoring scheme, the distribution of censored data is very likely to have a right-skewed shape, caused by the observations that are censored at the censoring time. However, the actual distribution of the censored data is unknown and may possibly follow a skewed distribution such as the exponential distribution, the Weibull distribution, or the lognormal distribution. It is important to note that selecting an inappropriate distribution may lead to inaccurate information that affects the ability to make proper decisions on designing production processes and engineering systems, resulting in product recalls and possibly the safety of end users. Therefore, the critical part of analyzing the censored survival data is choosing the distribution that fits the data. One of the methods used for fitting an appropriate distribution is the Anderson-Darling (AD) goodness-of-fit test (see Anderson and Darling (1954)). In the case of censored data, the AD test is computed based on the squared distance between a fitted distribution line and the estimated cumulative probability based on the KM estimate of survival data. As a result, selecting an appropriate distribution of the survival time using the AD test statistic values can be done by seeking the smallest test statistic value among the set of presumed distributions. To illustrate the parametric distribution fitting, the data used in

the numerical example is fitted to eleven parametric distributions. The AD goodness-of-fit tests for the survival data are shown in Table 4.9 and Figure 4.5. For the particular data set given in the numerical example, the most appropriate fitted distribution turns out to be a 3-parameter log-logistic distribution.

As shown in Table 4.8 and Figure 4.4, we compare the mean and variability obtained from the parametric survival analysis using the maximum likelihood estimator (MLE) for the normal distribution, which is the default underlying distribution used in RPD, with the median and standard error from the nonparametric survival analysis using the KM estimator. The percentage of the difference between the MLE estimates x^{MLE} and the KM estimates x^{KM} is given by $x_i^{diff} = \left((x_i^{MLE} - x_i^{KM}) / x_i^{MLE} \right) \times 100\% \, \forall \, i$. Our results show that the mean survival times using the MLE and the median survival times using the KM estimator are very close to each other. However, the standard deviation of the survival time using the MLE is slightly larger than the standard error of the median survival time using the KM estimator. This result agrees well with the findings reported by Breslow and Crowley (1974) that the survival distribution is known to converge weakly to a zero-mean Gaussian process. Another explanation is that the standard deviation is calculated using the entire set of observations, while the standard error is obtained by accounting only the observations around the median survival time under a specific level of the type I error, as shown in Equation (4.7). Finally, the AD test statistics in Table 4.9 indicate that the data lacks a strong evidence for assuming the normal distribution. Hence, for this particular right-censoring data, the use of the nonparametric KM estimator to obtain the optimum operating conditions appears to be an effective method.

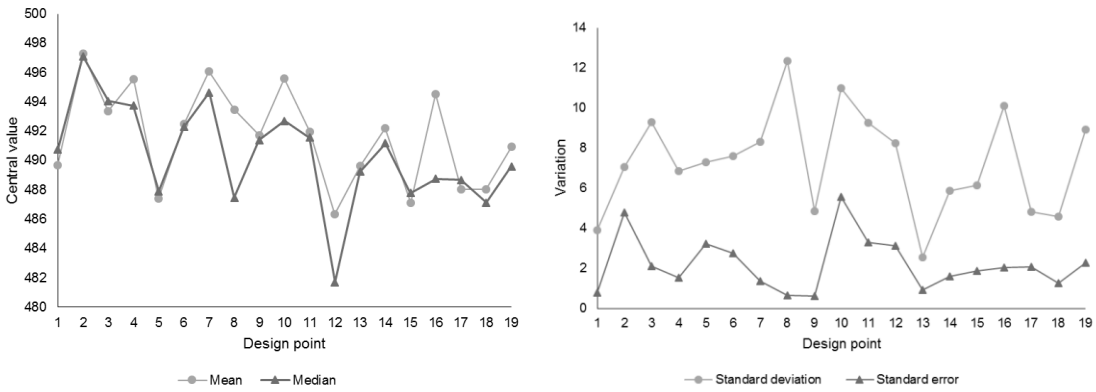


Figure 4.4 Comparison between the MLE estimator (left) and KM estimator (right)

Table 4.8 Comparing statistical estimators

Design point	Number of failure time	Number of censored time	MLE		KM		Percent of difference	
			Mean	Standard deviation	Median	Standard error	Central value	Variation
1	13	0	489.69	3.8994	490.76	2.1494	-0.0022	44.88
2	8	5	497.31	7.0543	497.1	3.4100	0.0004	51.66
3	10	3	493.38	9.3079	494.05	4.0622	-0.0014	56.36
4	9	4	495.52	6.8398	493.74	3.7184	0.0036	45.64
5	12	1	487.39	7.2921	487.89	4.0214	-0.0010	44.85
6	10	3	492.50	7.5987	492.28	3.9384	0.0004	48.17
7	9	4	496.06	8.3056	494.64	5.3856	0.0029	35.16
8	8	5	493.47	12.3592	487.46	5.5284	0.0122	55.27
9	12	1	491.70	4.8523	491.4	2.4870	0.0006	48.75
10	8	5	495.59	10.9868	492.7	5.2126	0.0058	52.56
11	10	3	491.97	9.2540	491.56	5.0049	0.0008	45.92
12	11	2	486.34	8.2484	481.71	3.8411	0.0095	53.43
13	13	0	489.62	2.5323	489.23	1.1024	0.0008	56.47
14	11	2	492.19	5.8776	491.17	2.7716	0.0021	52.84
15	13	0	487.09	6.1523	487.77	3.2204	-0.0014	47.66
16	8	5	494.53	10.1144	488.74	4.4602	0.0117	55.90
17	13	0	488.05	4.8001	488.67	2.8909	-0.0013	39.77
18	13	0	488.05	4.5691	487.12	2.5568	0.0019	44.04
19	10	3	490.94	8.9225	489.6	4.2739	0.0027	52.10

Table 4.9 Anderson-Darling test statistics for the experimental data

Distribution	AD test statistics	Correlation Coefficient
3-Parameter Loglogistic	287.380	0.998
3-Parameter Lognormal	287.415	0.998
3-Parameter Weibull	287.558	0.997
Lognormal	287.655	0.994
Normal	287.692	0.993
Loglogistic	288.120	0.983
Loglogistic	288.186	0.982
Weibull	292.046	0.965
Smallest Extreme Value	292.332	0.963
2-Parameter Exponential	299.491	*
Exponential	335.570	*

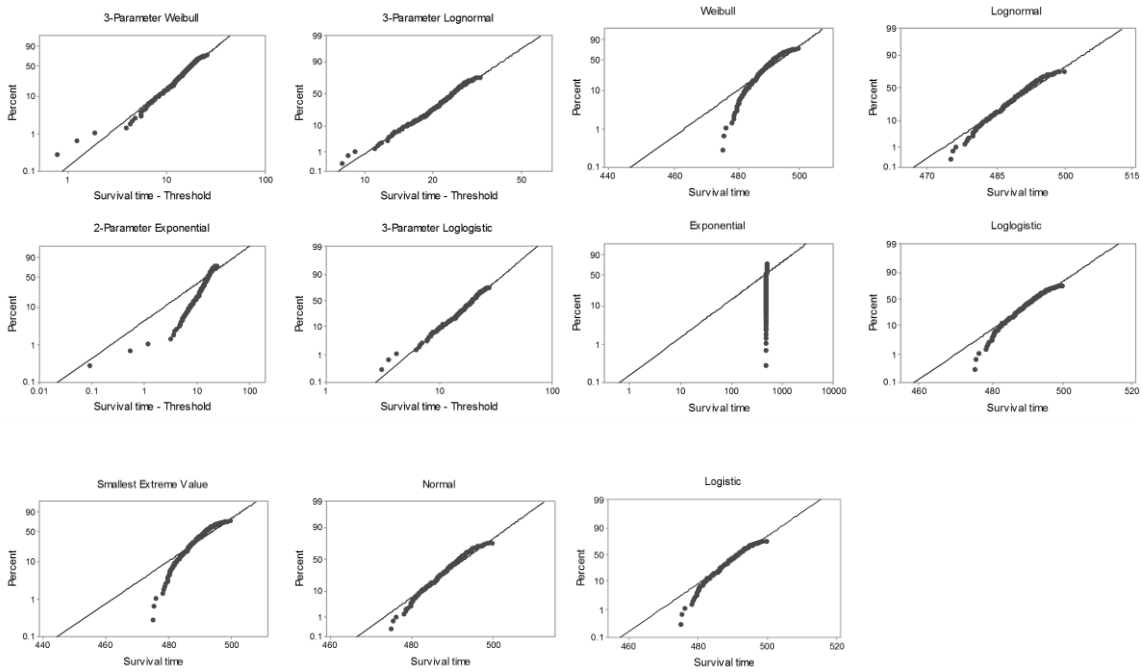


Figure 4.5 Probability plot of the experimental data

4.6 Process capability index for type I-right censored data

The process capability index (PCI) is a measure for evaluating and comparing process performance in manufacturing and production processes. The vast selection of PCI makes them becomes popular in practice, influencing researchers to continually

propose a customized PCI to better measure the capability in process environments. Under censoring schemes, several PCIs are developed for assessing the performance of a product based on its lifetime, known as the lifetime performance index. In particular, Tong et al. (2002) proposed the lifetime index for exponentially distributed data. Along the same lines, extensive studies were conducted by Wu et al. (2008) and Lee et al. (2009), to incorporate the interferences of censoring into the exponential lifetime index. On the other hand, lifetime indices based on other skewed distributions have been proposed. Some noteworthy lifetime indices include the index for a Weibull distribution (Ahmadi et al., 2013), a Weibull distribution with censoring (Lee, 2011; Hong et al., 2012; Dey et al., 2016; and Wu and Lin, 2017), and a Rayleigh distribution with censoring (Lee et al., 2011). Note that the PCI for assessing lifetime are modified based on the same structure, $C_L = (\mu - LSL)/\sigma$, where the μ and σ are estimated based on a specific probability distribution. Although there remain challenging research gaps based on parametric approaches with respect to censored data, this study focuses on the development of PCI using nonparametric approach. It is because, as mentioned in Section 4.5, the distribution fitting of censored data may not provide sufficient evidence to properly selecting the underlying distribution, which may lead to erroneous results of subsequent calculations. In this regard, within the nonparametric scheme, the process capability index for type I-right censored data and its confidence interval estimators are developed in Section 4.6.1. Finally, the numerical example to illustrating the proposed models is presented in Section 4.6.2.

4.6.1 Developments of PCI for type I-right censored data

Prior to the PCI development, the probability distribution of product's lifetime requires identification. Assume that observed survival time of a product is obtained under type I-right censoring scheme and the probability distribution of observed survival time, hereafter referred to as survival distribution, is estimated nonparametrically using KM estimator (see Section 4.2.2). From this point forward, let **T** reduces to a $1 \times n$ matrix of recorded survival time where $t_j \in \mathbf{T}$, $j = 1,...,n$ and $t_j \le t_{j+1} \forall j$; and $S(t_j)$ denotes the survival function of t_j estimated from $S(t_j) = \prod_{\kappa=1}^{j} \left(1 - \left(d_\kappa / n_\kappa\right)\right)$ where d_j ={0, for censored observations; 1, for uncensored observations}, and n_j is the number of units at risk at time t_j. As such, the survival distribution curve of the observed survival time is then obtained by plotting time $t_j \forall j$ versus its survival function $S(t_j) \forall j$.

The definition of the PCI for type I-right censored data based on nonparamatric survival distribution is given as the proportion of the width of actual conforming rate and the width of the expected conforming rate. Since the survival distribution is constructed nonparametrically, the statistical estimates are obtained using percentiles. Given *LSL* is the lower specification limit (i.e., the minimum requirement of product's lifetime), the width of actual conforming rate is estimated from the survival probability correspoding to *LSL*, i.e., the probability that products last longer than the *LSL*, which is defined as $S(LSL) = \min\left(S(t_j) | t_j \le LSL\right)$. Similarly, the expected conforming rate is given as γ,

where $0 \le \gamma \le 1$. Therefore, the process capability index for type I-right censored data based on nonparametric survival distribution is proposed as

$$C_T = \frac{\max\left(t_j \,|\, S(t_j) \ge 0\right) - LSL}{\max\left(t_j \,|\, S(t_j) \ge 0\right) - \min\left(t_j \,|\, S(t_j) \le \gamma\right)} \tag{4.10}$$

where $\max\left(t_j \,|\, S(t_j) \ge 0\right)$ is the maximum recorded survival time of the entire observations, $\min\left(t_j \,|\, S(t_j) \le \gamma\right)$ is the survival time corresponding to the probability of producing conforming units. Additionally, the numerator of C_T, $\max\left(t_j \,|\, S(t_j) \ge 0\right) - LSL$, denotes the width of specifications, and the denominator, $\max\left(t_j \,|\, S(t_j) \ge 0\right) - \min\left(t_j \,|\, S(t_j) \le \gamma\right)$, represents the width of survival distribution, i.e., the variation of data, under the level of conforming rate γ. Therefore, the value of C_T consists of three types. Firstly, if $\min\left(t_j \,|\, S(t_j) \le \gamma\right) = LSL$, we have $C_T = 1$, implying that the conforming rate of a process equals to the expected level γ. Secondly, if $\min\left(t_j \,|\, S(t_j) \le \gamma\right) \le LSL$, then $C_T \le 1$, which indicates that the conforming rate of a process is lesser than γ. Thirdly, if $\min\left(t_j \,|\, S(t_j) \le \gamma\right) \ge LSL$, we get $C_T \ge 1$, meaning that the conforming rate of a process is higher than γ. Thus, the larger C_T value is preferred.

Furthermore, the nonconforming rate (p_{nc}), i.e., the probability that a unit fails to meet a specification limit, is estimated from the survival function at the lower specification limit, $S(LSL)$; see Figure 4.6 in Section 4.6.2. Note that the survival

function is a conditional probability function where $0 \le S(t_j) \le 1$, and is a decreased probability function where $\max S(t_j) = S\left(\min\left(t_j \,|\, S(t_j) \le 1\right)\right) = 1$. Thus, the nonconforming rate is obtained from

$p_{nc} = S\left(\min\left(t_j \,|\, S(t_j) \le 1\right) \le t_j \le LSL\right) = S\left[\min\left(t_j \,|\, S(t_j) \le 1\right)\right] - S(LSL)$; that is,

$p_{nc} = 1 - S(LSL)$. Since $S(LSL) = \min(S(t_i) \,|\, t_i \le LSL)$. Therefore, the nonconforming rate is expressed as

$$p_{nc} = 1 - \min(S(t_i) \,|\, t_i \le LSL) \tag{4.11}$$

The confidence interval approximation of C_T can be manually calculated as follows. For simplicity of the following derivations, let $t_a = \max\left(t_j \,|\, S(t_j) \ge 0\right)$ and

$t_\gamma = \min\left(t_j \,|\, S(t_j) \le \gamma\right)$. Thus, Equation (4.10) becomes $C_T = \dfrac{t_a - LSL}{t_a - t_\gamma}$. Since LSL is given, the fluctuation of C_T can be estimated from the confidence intervals of survival functions associated with t_a and t_γ. Suppose that the confidence intervals of t_a and t_γ are $\left\{t_a^L, t_a^U\right\}$ and $\left\{t_\gamma^L, t_\gamma^U\right\}$, respectively. Based on the structure of C_T, the lower confidence bound of C_T, C_T^L, can be calculated from a proportion of the smallest possible value of the numerator to the largest possible value of the denominator. On the contrary, the upper confidence bound of C_T, C_T^U, is calculated from the ratio of the largest value of numerator and the smallest value of denominator. As a result, the confidence interval approximations of C_T is estimated from

$$\left\{C_T^L,\, C_T^U\right\} = \left\{\frac{t_a^L - LSL}{t_a^U - t_\gamma^L}, \frac{t_a^U - LSL}{t_a^L - t_\gamma^U}\right\} \tag{4.12}$$

Consequently, the confidence intervals of t_j under α level of type I error are estimated based on the corresponding confidence interval of its survival function. Given that $\left\{S^L(t_j), S^U(t_j)\right\}$ are the lower bound and the upper bound of survival function of time t_j. Using Greenwood's formula, $Var\left[S(t_j)\right] = \left(S(t_j)\right)^2 \sum_{\kappa=1}^{i} \frac{d_\kappa}{n_\kappa\left(n_\kappa - d_\kappa\right)}$ (see Section 4.2.2 and Appendix A), the confidence intervals for $S(t_j)$ is written as

$$\left\{S^L(t_j), S^U(t_j)\right\} = \left\{S(t_j) - z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_j)\right]}, S(t_j) + z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_j)\right]}\right\} \tag{4.13}$$

where z_α is the $(1-\alpha)^{th}$ percentile of the standard normal distribution, $N(0,1)$.

Therefore, the confidence intervals of survival time t_j is expressed as

$$\left\{t_j^L, t_j^U\right\} = \left\{\max\left(t_j \mid S(t_j) \geq S^U(t_j)\right), \min\left(t_j \mid S(t_j) \leq S^L(t_j)\right)\right\} \tag{4.14}$$

At this point, we have $\left\{t_a^L, t_a^U\right\} = \left\{\begin{array}{l} \max\left(t_j \mid S(t_j) \geq S(t_a) + z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_a)\right]}\right), \\ \min\left(t_j \mid S(t_j) \leq S(t_a) - z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_a)\right]}\right) \end{array}\right\}$ and

$\left\{t_\gamma^L, t_\gamma^U\right\} = \left\{\begin{array}{l} \max\left(t_j \mid S(t_j) \geq S(t_\gamma) + z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_\gamma)\right]}\right), \\ \min\left(t_j \mid S(t_j) \leq S(t_\gamma) - z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_\gamma)\right]}\right) \end{array}\right\}$. Thus,

$$\left\{C_T^L, C_T^U\right\} = \left\{ \frac{\max\left(t_j \mid S(t_j) \geq S(t_a) + z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_a)\right]}\right) - LSL}{\min\left(t_j \mid S(t_j) \leq S(t_a) - z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_a)\right]}\right) - \max\left(t_j \mid S(t_j) \geq S(t_\gamma) + z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_\gamma)\right]}\right)}, \frac{\min\left(t_j \mid S(t_j) \leq S(t_a) - z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_a)\right]}\right) - LSL}{\max\left(t_j \mid S(t_j) \geq S(t_a) + z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_a)\right]}\right) - \min\left(t_j \mid S(t_j) \leq S(t_\gamma) - z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_\gamma)\right]}\right)} \right\}$$

However, since $t_a = \max\left(t_j | S(t_j) \geq 0\right)$ is the maximum observed survival time available from observations, the upper bound of t_a $\left(t_a^U\right)$ and its corresponding survival function $S^L\left(t_a\right)$ is unobtainable. Therefore, $t_a^U = \min\left(t_j \mid S(t_j) \leq S(t_a) - z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_a)\right]}\right) \approx t_a$ and the previous equation of $\left\{C_T^L, C_T^U\right\}$ becomes

$$\left\{C_T^L, C_T^U\right\} = \left\{ \frac{\max\left(t_j \mid S(t_j) \geq S(t_a) + z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_a)\right]}\right) - LSL}{t_a - \max\left(t_j \mid S(t_j) \geq S(t_\gamma) + z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_\gamma)\right]}\right)}, \frac{t_a - LSL}{\max\left(t_j \mid S(t_j) \geq S(t_a) + z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_a)\right]}\right) - \min\left(t_j \mid S(t_j) \leq S(t_\gamma) - z_{\frac{\alpha}{2}}\sqrt{Var\left[S(t_\gamma)\right]}\right)} \right\}.$$

Finally, the confidence interval estimators of C_T is rewritten in a simpler form as

$$\left\{C_T^L, C_T^U\right\} = \left\{ \frac{t_a^L - LSL}{t_a - t_\gamma^L}, \frac{t_a - LSL}{t_a^L - t_\gamma^U} \right\} \tag{4.15}$$

4.6.2 Numerical example of the PCI for type I-right censored data

The calculation procedures of C_T are illustrated using the data set of observed failure time of 60 electrical appliances in a life test, as an adaptation of Lawless (2003). Assume that the termination time of the test is 5,000 hours, the lower specification limit is 500 hours, and 90% of the product is expected to last longer than 500 hours, i.e., the conforming rate with respect to the lower specification limit be 0.90. Table 4.10 presents

the ranked failure time, survival functions, and the variance of the survival function. The survival distribution based on KM estimator is shown in Figure 4.6. The C_T is calculated as follows. From Table 4.10, we get $\max\left(t_j \,|\, S(t_j) \geq 0\right) = 4{,}584$, $\min\left(t_j \,|\, S(t_j) \leq 0.90\right) =$ 80. Since LSL = 500, using Equation (4.10) we have $C_T = 0.9067$. The nonconforming rate of the electrical appliances is computed based on Equation (4.11), which is $p_{nc} = 1 - \min(S(t_i) \,|\, t_i \leq LSL) = 1 - 0.7833 = 0.2167$. Furthermore, the confidence intervals of C_T are obtained from Equation (4.15). Given $\alpha = 0.05$, $z_{0.025} = 1.96$, $S(t_a) = 0.0833$, and $Var[S(t_a)] = 0.0357$; we have $\left\{S^L(t_a), S^U(t_a)\right\} = \{0.0134, 0.1533\}$. As a result, $t_a^L = 4{,}100$ and $t_a^U = t_a = 4{,}584$. Similarly, since $\left\{S^L(t_\gamma), S^U(t_\gamma)\right\} = \{0.8241, 0.9759\}$, we get $t_\gamma^L = 34$ and $t_\gamma^U = 381$. Finally, we obtain $\left\{C_T^L, C_T^U\right\} = \{0.7912, 1.0981\}$.

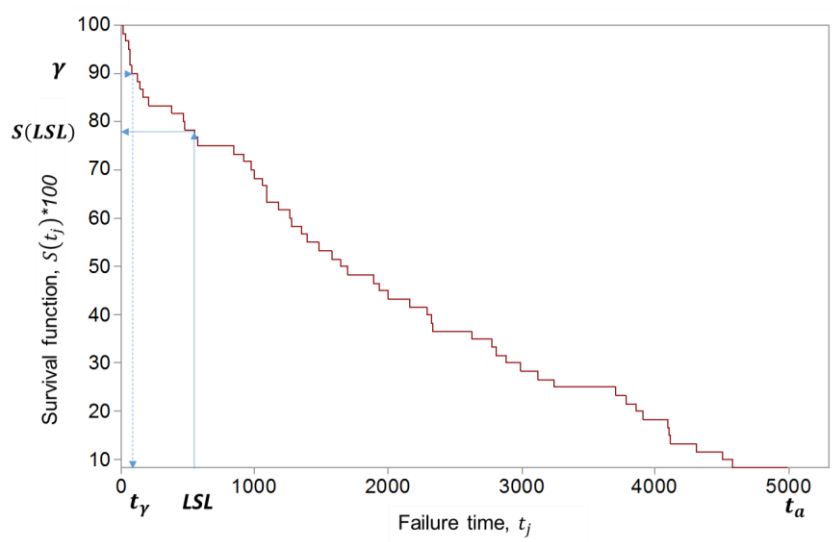


Figure 4.6 Survival distribution of the electrical appliances

Table 4.10 Observed failure time of the electrical appliances and its survival distribution

j	t_j	n_j	d_j	$S(t_j)$	$Var[S(t_j)]$	j	t_j	n_j	d_j	$S(t_j)$	$Var[S(t_j)]$
1	14	60	1	0.9833	0.0165	31	1702	30	1	0.4833	0.0645
2	34	59	1	0.9667	0.0232	32	1893	29	1	0.4667	0.0644
3	59	58	1	0.9500	0.0281	33	1932	28	1	0.4500	0.0642
4	61	57	1	0.9333	0.0322	34	2001	27	1	0.4333	0.0640
5	69	56	1	0.9167	0.0357	35	2161	26	1	0.4167	0.0636
6	80	55	1	0.9000	0.0387	36	2292	25	1	0.4000	0.0632
7	123	54	1	0.8833	0.0414	37	2326	24	1	0.3833	0.0628
8	142	53	1	0.8667	0.0439	38	2337	23	1	0.3667	0.0622
9	165	52	1	0.8500	0.0461	39	2628	22	1	0.3500	0.0616
10	210	51	1	0.8333	0.0481	40	2785	21	1	0.3333	0.0609
11	381	50	1	0.8167	0.0500	41	2811	20	1	0.3167	0.0601
12	464	49	1	0.8000	0.0516	42	2886	19	1	0.3000	0.0592
13	479	48	1	0.7833	0.0532	43	2993	18	1	0.2833	0.0582
14	556	47	1	0.7667	0.0546	44	3122	17	1	0.2667	0.0571
15	574	46	1	0.7500	0.0559	45	3248	16	1	0.2500	0.0559
16	839	45	1	0.7333	0.0571	46	3715	15	1	0.2333	0.0546
17	917	44	1	0.7167	0.0582	47	3790	14	1	0.2167	0.0532
18	969	43	1	0.7000	0.0592	48	3857	13	1	0.2000	0.0516
19	991	42	1	0.6833	0.0601	49	3912	12	1	0.1833	0.0500
20	1064	41	1	0.6667	0.0609	50	4100	11	1	0.1667	0.0481
21	1088	40	1	0.6500	0.0616	51	4106	10	1	0.1500	0.0461
22	1091	39	1	0.6333	0.0622	52	4116	9	1	0.1333	0.0439
23	1174	38	1	0.6167	0.0628	53	4315	8	1	0.1167	0.0414
24	1270	37	1	0.6000	0.0632	54	4510	7	1	0.1000	0.0387
25	1275	36	1	0.5833	0.0636	55	4584	6	1	0.0833	0.0357
26	1355	35	1	0.5667	0.0640	56	5000+	5	0	0.0833	0.0357
27	1397	34	1	0.5500	0.0642	57	5000+	4	0	0.0833	0.0357
28	1477	33	1	0.5333	0.0644	58	5000+	3	0	0.0833	0.0357
29	1578	32	1	0.5167	0.0645	59	5000+	2	0	0.0833	0.0357
30	1649	31	1	0.5000	0.0645	60	5000+	1	0	0.0833	0.0357

4.7 Concluding remarks

The proposed nonparametric RPD provides an effective approach for process parameter optimization associated with time-oriented quality characteristics. Under a censoring scheme, the traditional RPD method based on the normal distribution may not

be a suitable approach. Upon serving the purpose stated above, the guidelines for estimating the response functions based on nonparametric approaches, including the KM estimator, Greenwood's formula, and the Cox PH model, are provided in this study. Further, the applications of the proposed RPD method is not limited to the process optimization problems under type I-right censoring with fixed starting time. The set of methodologies developed in this chapter can also adapt to right censoring with random starting time. As a result, the advantages of the nonparametric approaches employed in RPD can enable applications of RPD in several potential research areas where time is the quality characteristic of interest, such as reliability engineering, medical research, and agriculture.

This chapter also develops optimization models that demonstrate the application of nonparametric-based response functions. The case study results show that, although the optimization models concerning all the three optimization criteria return similar optimal values, the obtained optimum operating condition satisfy the minimum requirement of survival time within the acceptable level of variation and proportional hazards. Therefore, unlike the traditional RPD for time-oriented data, which only focus on the average survival time and the variance of survival time, the proposed optimization models with the consideration of the proportional hazard rate provide additional guidance and confirmative information for making decisions under the quality improvement scheme. In the particular numerical example used in this chapter, Models II and IX provide the same expected median survival time, which implies that the upper bound of the proportional hazards rate λ has been set relatively high. If the lower λ value were

specified, Model II would give longer expected median survival time. Also, it is observed that the standard deviation of the survival time using the MLE is slightly larger than the standard error of the median survival time using the KM estimator, mainly because the survival distribution is known to converge weakly to a zero-mean Gaussian process.

CHAPTER 5

CONCLUSIONS AND FUTURE STUDIES

The original motivation for the dissertation is to develop a set of quality engineering approaches for improving the accuracy of mathematical calculations when solving engineering problems under the complications of incomplete data, i.e., the data with partially known information. Failure to use appropriate statistical foundations for analyzing the incomplete data could lead to aggregate errors of subsequent calculations, resulting in the lack of accuracy when making decisions. Therefore, strengthening the quality engineering approaches, such as the process capability index and the robust parameter design, with consideration of the proper statistical foundations for incomplete data bridges a noteworthy research gap. Despite a significant practical need, this research gap has not been previously explored. Therefore, this chapter develops several quality engineering approaches based on the appropriate statistical foundations for accurately quantifying the incomplete data. These developments are summarized as follows.

Chapter 2: a set of the PCIs focused on the customer perception is developed for accurately measuring the process capability of a product whose probability distribution follows the truncated normal distribution. The truncated normal distribution parameters, i.e., the mean and variance of truncated normal distribution, for two-sided truncated distribution, left-sided truncated distribution, and right-sided truncated distribution are derived in this chapter. Also, the data transformation-based PCI, which is the currently recommended method for obtaining PCI concerning the truncated normal distribution, is

studied and is then compared to the proposed PCIs and the traditional PCIs. The comparative study reveals that the proposed PCIs provide the largest index values and the smallest defective rate, which seems more reasonable for evaluating the process capability of the post-inspection products, which assume that non-conforming products are removed prior to shipping to customers.

Chapter 3: two traditional target-based PCIs are modified to measure the process capability with the consideration of the customer perception. Moreover, to complete the development of the proposed PCIs regarding the truncated normal distribution, the confidence interval approximations for $C_{TN\text{-}p}$, $C_{TN\text{-}pk}$, $C_{TN\text{-}pm}$, and $C_{TN\text{-}pmk}$ are developed for indicating the reliable of PCI values within a specific sample size. Furthermore, a simulation technique is employed to study the features of the proposed PCIs compared to its traditional counterparts across multiple scenarios. As a result, we found that the proposed PCIs provide higher PCI values for a narrow range of truncation points, i.e., when a probability distribution is obviously truncated.

Chapter 4: a series of methods based on RPD is developed to obtain the optimum operating conditions where experimental observations are type I-right censored. The modification is applied to the three phases of RPD. Firstly, for the experimentation phase, the central composite design is modified for dealing with censored data. The nonparametric survival analysis methods, i.e., the KM estimator and Greenwood's formula, are utilized to estimating the median value and the variation of observations at each design point. Secondly, in the response function development phase, the fitted functions for the median survival time and the variance of the median survival time are

obtained based on multiple linear regression, and the fitted function for hazard rate is estimated using the Cox PH model. Thirdly, the optimization models for obtaining the optimum operating conditions based on the three fitted functions are proposed in the optimization phase. From the numerical example, the optimization results indicate that, under nonparametric assumptions, the proposed optimization models return the optimum operating conditions that meet the requirements of the design, with guidance information of the product's reliability based on its survival time. Also, insights regarding the parametric based survival analysis methods for RPD are provided. Furthermore, we expand the application of the nonparametric statistical estimators for type I-right censored data to redesign the process capability index for assessing the performance of a product based on its observed survival times.

The future studies are summarized in Table 5.1. Although the collection of the customer-perceived PCIs developed in this chapter could handle several cases in practices, there are still opportunities for further enhancements. Since the proposed PCIs are a univariate analysis, which are designed to measure the process capability based on a single quality characteristics, a future study regarding how the truncated distribution affects an intersected tolerance region in multivariate PCIs and the development of multivariate PCIs concerning the effects of a truncated distribution would be another interesting topic. Moreover, since the data involving the capability measurement in services industries is generally a time-oriented data, the further extensions may investigate the effects of censoring in a non-normal distribution, such as the exponential

distribution or the Weibull distributions. With extensions of the censoring-based PCI, the service industries may benefit from future process capability analyses.

Moreover, under the RPD scheme, despite the fact that the type I-right censoring is a preferred censoring mechanism for a critical reliability study and medical research, it is considered as an expensive reliability experiment and may be unsuitable for some situations, for instance, to study the failure of an electronic component that is designed to last for years. A counterpart censoring mechanism such as the type I-right censoring which may reduce the duration of an experiment, e.g., an experiment is terminate after the third failure regardless of time, seems to be a considerable alternative. It is important to note that since the data from type II-right censoring is partially observed, the survival distribution of type II-right censoring is then estimated based on parametric approaches. To strengthen the connection between the time-oriented quality characteristics and RPD, it is of interest to develop RPD for time-oriented data based on type II-right censoring. Furthermore, an adaptation of RPD method using another survival analysis approach such as the accelerated failure time, which allows experimenters to speed up failure time by testing a unit under an extreme environmental setting, is also an important future study.

The optimization phase in Chapter 4 could be extended as follows. Since the optimization criteria have different units of measurement, which are also conflicted, the optimization models are developed based on the single objective optimization. Under those circumstances, the most important criterion is prioritized by being set as an objective function while the other criteria are forced to be the optimization constraints. Hence, to optimize the criteria as objective functions simultaneously, the multi-objective

optimization could be incorporated as a potential extension of this study. Ultimately, the future developments of quality improvement methods concerning the effects of incomplete data is not only a remarkable research opportunity for academia, but its contributions could also provide more accurate information for improving the level of quality in both manufacturing and services industries.

Table 5.1 Future studies

Chapters	Limitations	Future studies
Chapters 2 and 3	The proposed PCIs are derived by setting the truncation points at the specification limits. Therefore, these PCIs are invalid if the specification limits and the truncation points are different.	*Extensions of the proposed customer-perceived PCIs* (the truncation points are set at the specification limits) - Sampling plan for the resubmitted lot - Criterion for supplier selection *Manufacturing-based PCIs for the truncated normal distribution* (the specifications and the truncation points are set differently) - Process capability analysis for gap tolerance - Process capability analysis for data with limitation of devices

Chapter 4	Nonparametric method requires a full set of observations, which may be unobtainable in some situations.	*Extensions of the proposed RPD for censored data*
		- RPD for type II-right censored data
		- RPD for time-dependent variables
		- Modifications of the optimization phase by incorporating these following concepts
		  - Multi-objective optimization model
		  - Desirability function
		  - Multivariate optimization
		Extensions of the proposed PCI for censored data
		- Distribution-free PCIs for lifetime performance assessment
		- PCIs for duration data

APPENDICES

Appendix A

Derivation of Greenwood's Formula

To start with, we simplify the product term of Equation (1) by taking the log of both sides of the equation, as shown in Equation (A.1).

$$\log S(t_{ij}) = \sum_{\kappa=1}^{j} \log\left(1 - \frac{d_{i\kappa}}{n_{i\kappa}}\right). \tag{A.1}$$

Thus, the variance of $\log S(t_{ij})$ is given by

$$Var\left[\log S(t_{ij})\right] = Var\left[\sum_{\kappa=1}^{j} \log\left(1 - \frac{d_{i\kappa}}{n_{i\kappa}}\right)\right] \tag{A.2}$$

Note that, for the further derivation, these following assumptions should hold. First, assume that Y_{ij} is the number of units that survive through the interval $\left[t_{ij}, t_{ij+1}\right]$, $Y_{ij} = n_{ij} - d_{ij}$, which is independent and is binomially distributed, $Y_{ij} \sim B\left(n_{ij}, p_{ij}\right)$, where n_{ij} is a number of units at risk at time t_{ij} and the p_{ij} is the success rate of survival, $p_{ij} = \frac{n_{ij} - d_{ij}}{n_{ij}} = \frac{Y_{ij}}{n_{ij}}$. Therefore, from Equation (A.2), $Var\left[\log S(t_{ij})\right]$ can be estimated by

$$Var\left[\log S(t_{ij})\right] = Var\left[\sum_{\kappa=1}^{j} \log\left(p_{i\kappa}\right)\right] = Var\left[\sum_{\kappa=1}^{j} \log\left(Y_{i\kappa}/n_{i\kappa}\right)\right] \tag{A.3}$$

Second, from the first assumption, it yields that the random variable Y_{ij} is uncorrelated. Therefore, the covariance terms are zero, so that $Var\left[\sum_{\kappa=1}^{j} \log\left(Y_{i\kappa}/n_{i\kappa}\right)\right]$ is equal to $\sum_{\kappa=1}^{j} Var\left[\log\left(Y_{i\kappa}/n_{i\kappa}\right)\right]$. Thus, Equation (A.3) becomes

$$Var\left[\log S(t_{ij})\right] = \sum_{\kappa=1}^{j} Var\left[\log\left(Y_{i\kappa}/n_{i\kappa}\right)\right] \tag{A.4}$$

Third, for large N, the binomial random variable $Y_{ij} \sim B\left(n_{ij}, p_{ij}\right)$ can be approximated as $Y_{ij} \sim N\left(n_{ij} p_{ij},\, n_{ij} p_{ij}\left(1-p_{ij}\right)\right)$. Consequently, we obtain $\hat{\mu}_{Y_{ij}} = n_{ij} p_{ij}$ and $\hat{\sigma}^2_{Y_{ij}} = n_{ij} p_{ij}\left(1-p_{ij}\right)$. Thus, using the delta method based on the Taylor series expansions; that is, if $X \sim N\left(\mu, \sigma^2\right)$ and $g(X)$ is a functional form of X, then $g\left(X\right) \sim N\left(g\left(\mu\right), \left[g'\left(\mu\right)\right]^2 \sigma^2\right)$, we obtain $E\left[g(X)\right] = g(\mu)$ and $Var\left[g(X)\right] = \left[g'\left(\mu\right)\right]^2 \sigma^2$. From $Y_{ij} \sim N\left(\hat{\mu}_{Y_{ij}}, \hat{\sigma}^2_{Y_{ij}}\right)$, using the delta method for estimating function $\log(Y_{ij})$, we have

$$\mathrm{Var}\left[\log(Y_{ij})\right] \approx \left[\frac{d}{dx}\log\left(\mu_{Y_{ij}}\right)\right]^2 \sigma^2_{Y_{ij}} \tag{A.5}$$

From $E(Y_{ij}/n_{ij}) = E(Y_{ij})/n_{ij} = (n_{ij} p_{ij})/n_{ij} = p_{ij}$ and $Var(Y_{ij}/n_{ij}) = Var(Y_{ij})/n_{ij}^2 = (n_{ij} p_{ij}(1-p_{ij}))/n_{ij}^2 = (p_{ij}(1-p_{ij}))/n_{ij}$, $Var\left[\log\left(Y_{ij}/n_{ij}\right)\right]$ is calculated as

$Var\left[\log\left(Y_{ij}/n_{ij}\right)\right] = \left[\frac{d}{dx}\log\left(E\left(Y_{ij}/n_{ij}\right)\right)\right]^2 Var\left(Y_{ij}/n_{ij}\right) = \frac{1}{p_{ij}^2} \cdot \frac{p_{ij}\left(1-p_{ij}\right)}{n_{ij}} = \frac{\left(1-p_{ij}\right)}{p_{ij} n_{ij}}$. Since $p_{ij} = \frac{n_{ij} - d_{ij}}{n_{ij}}$, we have $Var\left[\log\left(Y_{ij}/n_{ij}\right)\right] = \frac{d_{ij}}{n_{ij}\left(n_{ij} - d_{ij}\right)}$. Thus, Equation (A.4) becomes

$$Var\left(\log\left[S(t_{ij})\right]\right) \approx \sum_{\kappa=1}^{j} \frac{d_{i\kappa}}{n_{i\kappa}\left(n_{i\kappa} - d_{i\kappa}\right)} \tag{A.6}$$

Similarly, based on the delta method, we have the variance of the exponential function of Y_{ij} as $Var\left(e^{Y_{ij}}\right) \approx \left[\frac{d}{dx} e^{\mu_{Y_{ij}}}\right]^2 \sigma^2_{Y_{ij}}$. From $S(t_{ij}) = e^{\log S(t_{ij})}$ and $Var\left[S(t_{ij})\right] = Var\left[e^{\log\left[S(t_{ij})\right]}\right]$, we obtain

$$Var\left[S(t_{ij})\right] = \left(e^{E\left(\log\left[S(t_{ij})\right]\right)}\right)^2 Var\left(\log\left[S(t_{ij})\right]\right) \tag{A.7}$$

From $E\left(\log\left[S(t_{ij})\right]\right) = E\left(\sum_{\kappa=1}^{j}\left[\log\left(Y_{i\kappa}/n_{i\kappa}\right)\right]\right)$,

$E\left(\sum_{\kappa=1}^{j}\left[\log\left(Y_{i\kappa}/n_{i\kappa}\right)\right]\right) = E\left[\log\left(Y_{ij}/n_{ij}\right)\right] + E\left[\log\left(Y_{ij-1}/n_{ij-1}\right)\right] + \cdots + E\left[\log\left(Y_{i1}/n_{i1}\right)\right]$, and

$E\left[\log\left(Y_{ij}/n_{ij}\right)\right] = \log\left[E\left(Y_{ij}/n_{ij}\right)\right] = \log\left(p_{ij}\right) = \log\left(1-\frac{d_{ij}}{n_{ij}}\right)$, we have

$E\left(\log\left[S(t_{ij})\right]\right) = \sum_{\kappa=1}^{j}\log\left(1-\frac{d_{ij}}{n_{ij}}\right) = \log\left[S(t_{ij})\right]$. Thus,

$Var\left[S(t_{ij})\right] = \left(e^{\log\left[S(t_{ij})\right]}\right)^2 Var\left(\log\left[S(t_{ij})\right]\right)$. Finally, the estimated variance of the survival function, also known as Greenwood's formula (see Greenwood, 1926), is written as

$Var\left[S(t_{ij})\right] \approx \left(S(t_{ij})\right)^2 \sum_{\kappa=1}^{j}\frac{d_{i\kappa}}{n_{i\kappa}\left(n_{i\kappa}-d_{i\kappa}\right)}$.

Although Greenwood's formula is said to be a consistent estimate of the variance of the survival function based on the KM estimator (Brookmeyer and Crowley, 1982) and has been used widely among researchers, there are discussions regarding the assumptions of Greenwood's formula. Additionally, the nonparametric likelihood ratio method is proposed in Thomas and Grunkemeier (1975) as an alternative confidence intervals estimation for the nonparametric survival function without using the normal approximation to the binomial. Li (1995) provides discussions regarding the performance of Greenwood's formula and the nonparametric likelihood ratio method. Moreover, the modification of Greenwood's formula based on the correlated random variables is studied in Kang and Koehler (1997).

Appendix B

Derivation of the Variance of Survival Time

The delta method for estimating the variance of a density function is given as $Var\left[g(X)\right]=\left[\frac{d}{dx}g\left(X\right)\right]^2 Var\left(X\right)$. Since $S(t_{ij})$ is a cumulative probability function of t_{ij}, we have

$$Var\left[S(t(p))\right]=\left[\frac{dS(t(p))}{dt(p)}\right]^2 Var\left(t(p)\right) \tag{B.1}$$

where $t(p)$ is the time at (100p)th percentile of a distribution. Let $f(t(p))=S'(t(p))$ be the estimated probability density function at survival time $t(p)$. Using the mean value theory to estimate the $S'(t(p))$ where $S(t(p))$ is a decreased function, we have $f(t(p))=S'(t(p))=-\frac{\Delta S(t(p))}{\Delta t(p)}$. Given that $p\pm\alpha$ is the confidence intervals of interest where α is the probability of type I error, we obtain $f(t(p))=\frac{S(t(p+\alpha))-S(t(p-\alpha))}{t(p+\alpha)-t(p-\alpha)}$.

Thus, Equation (B.1) becomes $Var\left[S(t(p))\right]=\left[f(t(p))\right]^2 Var\left(t(p)\right)$. Finally, the variance of $t(p)$ is expressed as $Var\left(t(p)\right)=\frac{Var\left[S(t(p))\right]}{\left[f(t(p))\right]^2}$.

Appendix C

Minitab Output for Chapter 4

Response Surface Regression of $\hat{M}(\mathbf{x})$

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Model	9	138.038	15.3375	1.79	0.199
Linear	3	25.486	8.4954	0.99	0.440
X1	1	2.156	2.1561	0.25	0.628
X2	1	15.835	15.8350	1.85	0.207
X3	1	7.495	7.4952	0.88	0.374
Square	3	60.831	20.2771	2.37	0.139
X1*X1	1	46.127	46.1270	5.38	0.045
X2*X2	1	0.079	0.0793	0.01	0.925
X3*X3	1	19.142	19.1418	2.23	0.169
2-Way Interaction	3	51.720	17.2400	2.01	0.183
X1*X2	1	41.496	41.4961	4.84	0.055
X1*X3	1	9.724	9.7241	1.14	0.314
X2*X3	1	0.500	0.5000	0.06	0.815
Error	9	77.093	8.5659		
Lack-of-Fit	5	73.431	14.6862	16.04	0.009
Pure Error	4	3.662	0.9154		
Total	18	215.131			

Model Summary

S	R-sq	R-sq(adj)	R-sq(pred)
2.92675	64.16%	28.33%	0.00%

Coded Coefficients

Term	Effect	Coef	SE Coef	T-Value	P-Value	VIF
Constant		488.28	1.31	373.49	0.000	
X1	0.795	0.397	0.792	0.50	0.628	1.00
X2	-2.154	-1.077	0.792	-1.36	0.207	1.00
X3	-1.482	-0.741	0.792	-0.94	0.374	1.00
X1*X1	3.677	1.838	0.792	2.32	0.045	1.04
X2*X2	-0.152	-0.076	0.792	-0.10	0.925	1.04
X3*X3	2.368	1.184	0.792	1.49	0.169	1.04
X1*X2	-4.56	-2.28	1.03	-2.20	0.055	1.00
X1*X3	-2.21	-1.10	1.03	-1.07	0.314	1.00
X2*X3	0.50	0.25	1.03	0.24	0.815	1.00

Regression Equation in Uncoded Units

Median = 488.28 + 0.397 X1- 1.077 X2 - 0.741 X3 + 1.838 X1*X1 - 0.076 X2*X2
+ 1.184 X3*X3 - 2.28 X1*X2 - 1.10 X1*X3 + 0.25 X2*X3

Fits and Diagnostics for Unusual Observations

Obs	Median	Fit	Resid	Std Resid	
9	481.71	486.26	-4.55	-2.48	R
12	493.74	489.86	3.88	2.30	R

Response Surface Regression for $\hat{S}E\left(M(\mathbf{x})\right)$ versus A, B, X3

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Model	9	15.6511	1.73902	1.65	0.234
Linear	3	8.1166	2.70553	2.56	0.120
X1	1	2.2640	2.26400	2.15	0.177
X2	1	0.7584	0.75840	0.72	0.419
X3	1	5.0942	5.09419	4.83	0.056
Square	3	7.1381	2.37936	2.25	0.151
X1*X1	1	0.9551	0.95505	0.90	0.366
X2*X2	1	2.9791	2.97915	2.82	0.127
X3*X3	1	2.3142	2.31418	2.19	0.173
2-Way Interaction	3	0.3965	0.13216	0.13	0.943
X1*X2	1	0.2375	0.23754	0.23	0.647
X1*X3	1	0.0918	0.09179	0.09	0.775
X2*X3	1	0.0672	0.06716	0.06	0.807
Error	9	9.4991	1.05545		
Lack-of-Fit	5	6.6413	1.32827	1.86	0.284
Pure Error	4	2.8577	0.71444		
Total	18	25.1502			

Model Summary

S	R-sq	R-sq(adj)	R-sq(pred)
1.02735	62.23%	24.46%	0.00%

Coded Coefficients

Term	Effect	Coef	SE Coef	T-Value	P-Value	VIF
Constant		3.456	0.459	7.53	0.000	
X1	0.814	0.407	0.278	1.46	0.177	1.00
X2	0.471	0.236	0.278	0.85	0.419	1.00
X3	1.221	0.611	0.278	2.20	0.056	1.00
X1*X1	0.529	0.265	0.278	0.95	0.366	1.04
X2*X2	0.934	0.467	0.278	1.68	0.127	1.04
X3*X3	-0.823	-0.412	0.278	-1.48	0.173	1.04
X1*X2	-0.345	-0.172	0.363	-0.47	0.647	1.00
X1*X3	-0.214	-0.107	0.363	-0.29	0.775	1.00
X2*X3	0.183	0.092	0.363	0.25	0.807	1.00

Regression Equation in Uncoded Units

SE = 3.456 + 0.407X1 + 0.236X2 + 0.611 X3 + 0.265 X1*X1 + 0.467 X2*X2 - 0.412 X3*X3 - 0.172 X1*X2 - 0.107 X1*X3 + 0.092 X2*X3

Fits and Diagnostics for Unusual Observations

Obs	SE	Fit	Resid	Std Resid	
9	3.841	5.174	-1.333	-2.07	R

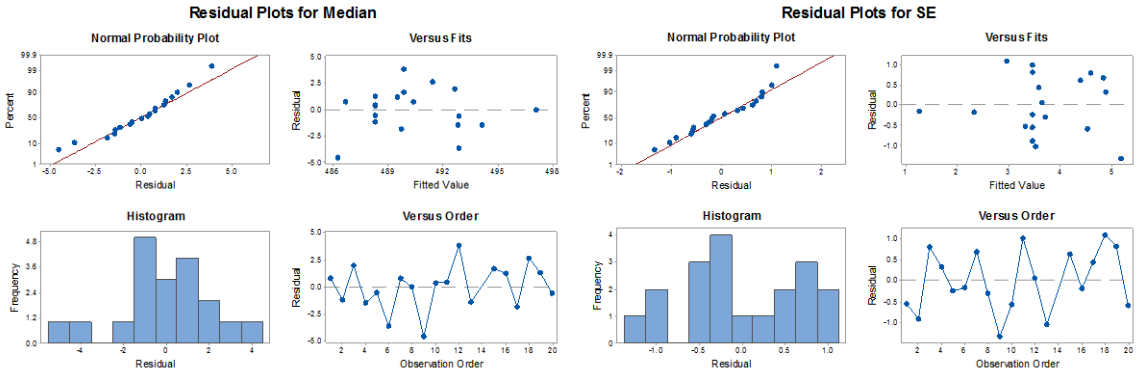


Figure C.1 Residual Plots for (left) $\hat{M}(\mathbf{x})$ (right) $\hat{SE}\left(M(\mathbf{x})\right)$

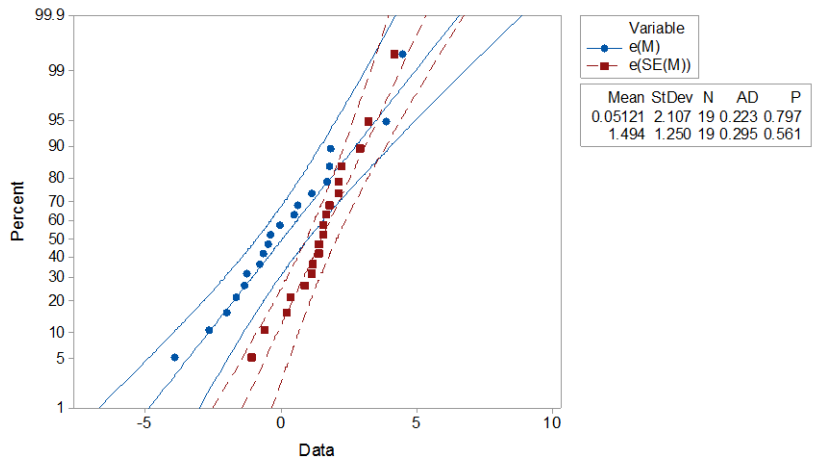


Figure C.2 Normality plot for error terms of $\hat{M}(\mathbf{x})$ and $\hat{SE}\left(M(\mathbf{x})\right)$ at 95% confidence interval

Appendix D

Numerical Programming Code

Simulation code for Chapter 3 (Matlab®)

```
%Scenario1 LSL=-3, USL=3, target=0, mean=0, sigma=1, truncation
effect=Low, mean_location=Centered

clear all; clc;

filename = 'scenario1-n500.xlsx';

rep = 10000;N = 5000; n = 500;
USL = 3; LSL = -3; target = (USL + LSL)/2; mu = 0; sigma = 1;

pop_table = zeros(rep,N);
stat_pop = zeros(rep,10);
PCIs_pop = zeros(rep,12);

samp_table = zeros(rep,N);
stat_samp = zeros(rep,10);
PCIs_samp = zeros(rep,12);

diff_PCIs = zeros(rep,8);

i=1; j=1;

while i <= rep ;

%generate population
pop = mu + (((USL-LSL)/6)*randn(1,N));

mean = sum(pop)/N;
sigma = std(pop);

%%PCIs calculation
pdfLSL = normpdf(LSL,mean,sigma);
cdfLSL = normcdf(LSL,mean,sigma);

pdfUSL = normpdf(USL,mean,sigma);
cdfUSL = normcdf(USL,mean,sigma);

zLSL = (LSL-mean)/sigma ;
zUSL = (USL-mean)/sigma ;

meanTN = mean + sigma*((pdfLSL-pdfUSL)/(cdfUSL-cdfLSL));

sigmaTN= (sqrt( sigma^2 * (1+((zLSL*pdfLSL - zUSL*pdfUSL)/(cdfUSL -
cdfLSL))- ((pdfLSL - pdfUSL)/( cdfUSL - cdfLSL))^2 )));
```

```
Cp =(USL-LSL)/(6*sigma);
Ctnp = (USL-LSL)/(6*sigmaTN);

Cpl =(mean-LSL)/(3*sigma);
Ctnpl = (meanTN-LSL)/(3*sigmaTN);

Cpu =(USL-mean)/(3*sigma);
Ctnpu = (USL-meanTN)/(3*sigmaTN);

Cpk = min(Cpl, Cpu);
Ctnpk = min(Ctnpl, Ctnpu);

Cpm=(USL-LSL)/(6*sqrt(sigma^2+(mean-target)^2));
Ctnm=(USL-LSL)/(6*sqrt(sigmaTN^2+(meanTN-target)^2));

Cpmk = Cpk / sqrt(1+((mean-target)/sigma)^2) ;
Ctnpmk = Ctnpk / sqrt(1+((mean-target)/sigma)^2) ;

%storing values
pop_table(i,:) = pop;
stat_pop(i,:) = [mean; sigma; pdfLSL; cdfLSL; pdfUSL; cdfUSL; zLSL;
zUSL; meanTN; sigmaTN];
PCIs_pop(i,:) = [Cp; Ctnp; Cpl; Ctnpl; Cpu; Ctnpu; Cpk; Ctnpk; Cpm;
Ctnm; Cpmk; Ctnpmk];
%%end PCIs for pop

samp = datasample(pop,n);
mean = sum(samp)/n;
sigma = std(samp);

%%PCIs calculation
pdfLSL = normpdf(LSL,mean,sigma);
cdfLSL = normcdf(LSL,mean,sigma);

pdfUSL = normpdf(USL,mean,sigma);
cdfUSL = normcdf(USL,mean,sigma);

zLSL = (LSL-mean)/sigma ;
zUSL = (USL-mean)/sigma ;

meanTN = mean + sigma*((pdfLSL-pdfUSL)/(cdfUSL-cdfLSL));

sigmaTN= (sqrt( sigma^2 * (1+((zLSL*pdfLSL - zUSL*pdfUSL)/(cdfUSL -
cdfLSL))- ((pdfLSL - pdfUSL)/( cdfUSL - cdfLSL))^2 )));

Cp =(USL-LSL)/(6*sigma);
Ctnp = (USL-LSL)/(6*sigmaTN);

Cpl =(mean-LSL)/(3*sigma);
Ctnpl = (meanTN - LSL)/(3*sigmaTN);
```

```
Cpu =(USL-mean)/(3*sigma);
Ctnpu = (USL-meanTN)/(3*sigmaTN);

Cpk = min(Cpl, Cpu);
Ctnpk = min(Ctnpl, Ctnpu);

Cpm = (USL-LSL)/(6*sqrt(sigma^2+(mean-target)^2));
Ctnm = (USL-LSL)/(6*sqrt(sigmaTN^2+(meanTN-target)^2));

Cpmk = Cpk / sqrt(1+((mean-target)/sigma)^2) ;
Ctnpmk = Ctnpk / sqrt(1+((mean-target)/sigma)^2) ;

%storing values
samp_table(i,:) = pop;
stat_samp(i,:) = [mean; sigma; pdfLSL; cdfLSL; pdfUSL; cdfUSL; zLSL;
zUSL; meanTN; sigmaTN];
PCIs_samp(i,:) = [Cp; Ctnp; Cpl; Ctnpl; Cpu; Ctnpu; Cpk; Ctnpk; Cpm;
Ctnm; Cpmk; Ctnpmk];
%%end PCIs for samples

%diff_PCIs(i,:) =  PCIs_samp(i,:)- PCIs_pop(i,:);

i=i+1;
end;

avg_pop = sum(PCIs_pop)/rep
avg_samp = sum(PCIs_samp)/rep

%write to Excel file

tab_head_stat = {'mean', 'sigma', 'pdf LSL', 'cdf LSL', 'pdf USL', 'cdf
USL', 'meanTN', 'sigmaTN'};
tab_head_PCIs = {'C_p', 'CTN-p','Cpl', 'Ctnpl', 'Cpu', 'Ctnpu', 'Cpk',
'Ctnpk', 'Cpm', 'Ctnm', 'Cpmk', 'Ctnmk'};
tab_head_blank = {' ', ' ',' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ',
' '};

xlswrite(filename, tab_head_stat, 'stat_pop');
xlswrite(filename, stat_pop, 'stat_pop', 'A2');

xlswrite(filename, tab_head_PCIs, 'PCIs_pop', 'A1');
xlswrite(filename, PCIs_pop, 'PCIs_pop', 'A2');
xlswrite(filename, tab_head_PCIs, 'PCIs_samp', 'R1');
xlswrite(filename, avg_pop, 'PCIs_pop', 'R2');

xlswrite(filename, tab_head_stat, 'stat_samp', 'A1');
xlswrite(filename, stat_samp, 'stat_samp', 'A2');

xlswrite(filename, tab_head_PCIs, 'PCIs_samp', 'A1');
xlswrite(filename, PCIs_samp, 'PCIs_samp', 'A2');
```

```
xlswrite(filename, tab_head_PCIs, 'PCIs_samp', 'R1');
xlswrite(filename, avg_samp, 'PCIs_samp', 'R2');

x0=10;y0=10;width=600;height=300;
set(gcf,'units','points','position',[x0,y0,width,height])

%drawing Boxplot

boxplot(PCIs_samp, 'Labels', tab_head_blank);
ylim([0 2])
set(gca,'XTick',[]);
[hx,hy] = format_ticks(gca,{'\itC_p', '\itC_{T-p}','\itC_{pl}',
'\itC_{T-pl}', '\itC_{pu}', '\itC_{T-pu}', '\itC_{pk}', '\itC_{T-pk}',
'\itC_{pm}', '\itC_{T-pm}', '\itC_{pmk}', '\itC_{T-pmk}'},...
                  [],[1:1:12]);
```

Optimization code for numerical example in Chapter 4 (Maple®)

> *restart*;

$$M := 488.28 + 0.397\cdot x1 - 1.077\cdot x2 - 0.741\cdot x3 + 1.838\cdot x1^2 - 0.076\cdot x2^2 + 1.184\cdot x3^2 - 2.28\cdot x1\cdot x2 - 1.10\cdot x1\cdot x3 + 0.25\cdot x2\cdot x3;$$

$$S := 3.456 + 0.407\cdot x1 + 0.236\cdot x2 + 0.611\cdot x3 + 0.265\cdot x1^2 + 0.467\cdot x2^2 - 0.412\cdot x3^2 - 0.172\cdot x1\cdot x2 - 0.107\cdot x1\cdot x3 + 0.092\cdot x2\cdot x3;$$

$$V := 12.55 + 2.97\cdot x1 + 1.81\cdot x2 + 4.07\cdot x3 + 2.16\cdot x1^2 + 3.30\cdot x2^2 - 2.17\cdot x3^2 - 0.93\cdot x1\cdot x2 - 0.43\cdot x1\cdot x3 + 1.73\cdot x2\cdot x3;$$

$$H := e^{-0.0996309\cdot x2 - 0.2683154\cdot x3 - 0.106717\cdot x1^2 - 0.183715\cdot x1\cdot x2 + 0.0716074\cdot x1\cdot x3 + 0.1812551\cdot x2\cdot x3};$$

$t := 495;\ h := 0.75;\ e := 5$

> *with*(*Optimization*);

$$Maximize\left(M, \left\{ x1 \geq -\sqrt[4]{8}, x2 \geq -\sqrt[4]{8}, x3 \geq -\sqrt[4]{8}, x1 \leq \sqrt[4]{8}, x2 \leq \sqrt[4]{8}, x3 \leq \sqrt[4]{8} \right\}\right);$$

$$[510.604907891905, [x1 = 1.68179283050743, x2 = -1.68179283050743, x3 = -1.68179283050743]]$$

$$Maximize\left(M, \left\{ S \leq e, x1 \geq -\sqrt[4]{8}, x2 \geq -\sqrt[4]{8}, x3 \geq -\sqrt[4]{8}, x1 \leq \sqrt[4]{8}, x2 \leq \sqrt[4]{8}, x3 \leq \sqrt[4]{8} \right\}\right);$$

$[498.415914891139153, [x1 = 1.68179283050743, x2 = -1.19963036274432, x3 = 1.68179283050743]]$

$Maximize(M, \{ H \le h, x1 \ge -\sqrt[4]{8}, x2 \ge -\sqrt[4]{8}, x3 \ge -\sqrt[4]{8}, x1 \le \sqrt[4]{8}, x2 \le \sqrt[4]{8}, x3 \le \sqrt[4]{8} \});$

$[500.475737680278087, [x1 = 1.68179283050743, x2 = -1.68179283050743, x3 = 1.68179283050743]]$

$Maximize(M, \{ H \le h, S \le e, x1 \ge -\sqrt[4]{8}, x2 \ge -\sqrt[4]{8}, x3 \ge -\sqrt[4]{8}, x1 \le \sqrt[4]{8}, x2 \le \sqrt[4]{8}, x3 \le \sqrt[4]{8} \});$

$[498.415914891139153, [x1 = 1.68179283050743, x2 = -1.19963036274432, x3 = 1.68179283050743]]$

$Minimize(S, \{x1 \ge -\sqrt[4]{8}, x2 \ge -\sqrt[4]{8}, x3 \ge -\sqrt[4]{8}, x1 \le \sqrt[4]{8}, x2 \le \sqrt[4]{8}, x3 \le \sqrt[4]{8}\});$

$[0.89605249121313, [x1 = -1.20788319370505, x2 = -0.309454998833602, x3 = -1.68179283050743]]$

$Minimize(S, \{M \ge t, x1 \ge -\sqrt[4]{8}, x2 \ge -\sqrt[4]{8}, x3 \ge -\sqrt[4]{8}, x1 \le \sqrt[4]{8}, x2 \le \sqrt[4]{8}, x3 \le \sqrt[4]{8}\});$

$[4.32915739231218755, [x1 = 1.32386204771139, x2 = -0.920835241348141, x3 = 1.68179283050743]]$

$Minimize(S, \{H \le h, x1 \ge -\sqrt[4]{8}, x2 \ge -\sqrt[4]{8}, x3 \ge -\sqrt[4]{8}, x1 \le \sqrt[4]{8}, x2 \le \sqrt[4]{8}, x3 \le \sqrt[4]{8}\});$

$[3.14683398335894982, [x1 = -0.600013057006024, x2 = -0.528829962817893, x3 = 1.68179283050743]]$

$Minimize(S, \{M \ge t, H \le h, x1 \ge -\sqrt[4]{8}, x2 \ge -\sqrt[4]{8}, x3 \ge -\sqrt[4]{8}, x1 \le \sqrt[4]{8}, x2 \le \sqrt[4]{8}, x3 \le \sqrt[4]{8}\});$

$[4.32915739231216978, [x1 = 1.32386204688429, x2 = -0.920835242638038, x3 = 1.68179283050743]]$

$Minimize(H, \{ x1 \ge -\sqrt[4]{8}, x2 \ge -\sqrt[4]{8}, x3 \ge -\sqrt[4]{8}, x1 \le \sqrt[4]{8}, x2 \le \sqrt[4]{8}, x3 \le \sqrt[4]{8}\});$

$[0.285653278139345546, [x1 = 1.68179283050743, x2 = 1.68179283050743, x3 = -1.68179283050743]]$

$$Minimize\left(H, \left\{ S \le e, x1 \ge -\sqrt[4]{8}, x2 \ge -\sqrt[4]{8}, x3 \ge -\sqrt[4]{8}, x1 \le \sqrt[4]{8}, x2 \le \sqrt[4]{8}, x3 \le \sqrt[4]{8} \right\}\right);$$

$$[0.520744757188021601, [x1 = 1.68179283050743, x2 = 0.982378599233812, x3 = 1.68179283050743]]$$

$$Minimize\left(H, \left\{ M \ge t, x1 \ge -\sqrt[4]{8}, x2 \ge -\sqrt[4]{8}, x3 \ge -\sqrt[4]{8}, x1 \le \sqrt[4]{8}, x2 \le \sqrt[4]{8}, x3 \le \sqrt[4]{8} \right\}\right);$$

$$[0.602167198610582766, [x1 = 1.68179283050743, x2 = -0.417621535341181, x3 = 1.68179283050743]]$$

$$Minimize\left(H, \left\{ S \le e, M \ge t, x1 \ge -\sqrt[4]{8}, x2 \ge -\sqrt[4]{8}, x3 \ge -\sqrt[4]{8}, x1 \le \sqrt[4]{8}, x2 \le \sqrt[4]{8}, x3 \le \sqrt[4]{8} \right\}\right);$$

$$[0.602167198610590981, [x1 = 1.68179283050743, x2 = -0.417621535341312, x3 = 1.68179283050743]]$$

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