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Re-Establishing the Theoretical Foundations of a Truncated Normal Distribution: Standardization Statistical Inference, and Convolution

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RE-ESTABLISHING THE THEORETICAL FOUNDATIONS OF A TRUNCATED NORMAL DISTRIBUTION: STANDARDIZATION, STATISTICAL INFERENCE, AND CONVOLUTION

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ABSTRACT

There are special situations where specification limits on a process are implemented externally, and the product is typically reworked or scrapped if its performance does not fall in the range. As such, the actual distribution after inspection is truncated. Despite the practical importance of the role of a truncated distribution, there has been little work on the theoretical foundation of standardization, inference theory, and convolution. The objective of this research is three-fold. First, we derive a standard truncated normal distribution and develop its cumulative probability table by standardizing a truncated normal distribution as a set of guidelines for engineers and scientists. We believe that the proposed standard truncated normal distribution by standardizing a truncated normal distribution makes more sense than the traditionally-known truncated standard normal distribution by truncating a standard normal distribution. Second, we develop the new one-sided and two-sided *z*-test and *t*-test procedures under such special situations, including their associated test statistics, confidence intervals, and *P*-values, using appropriate truncated statistics. We then provide the mathematical justifications that the Central Limit Theorem works quite well for a large sample size, given samples taken from a truncated normal distribution. The proposed hypothesis testing procedures have a wide range of application areas such as statistical process control, process capability analysis, design of experiments, life testing, and reliability engineering. Finally, the convolutions of the combinations of truncated normal and truncated skew normal random variables on double and triple truncations are developed. The proposed convolution framework has not been fully explored in the

literature despite practical importance in engineering areas. It is believed that the particular research task on convolution will help obtain a better understanding of integrated effects of multistage production processes, statistical tolerance analysis and gap analysis in engineering design, ultimately leading to process and quality improvement. We also believe that overall the results from this entire research work may have the potential to impact a wide range of many other engineering and science problems.

DEDICATION

This dissertation is dedicated to my wife, Misun Roh. We have been together for over 17 years. You are the love of my life, my strength and support. I also want to dedicate this to my three children, Eunchan Daniel, Yechan Joshua and Yoochan David Cha. You have brought the most joy to my life and have been a source great learning and healing. I am so proud of each one of you and have a great love for you all.

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TABLE OF CONTENTS

Page

Table of Contents (Continued)

Page

Table of Contents (Continued)

Page

Table of Contents (Continued)

Page

LIST OF TABLES

List of Tables (Continued)

LIST OF FIGURES

List of Figures (Continued)

Figure Page

LIST OF SYMBOLS

Asym $TN_{N-\text{type}}$	An asymmetric doubly truncated normal distribution
f	Probability Density Function of a Normal Distribution
f_T	Probability Density Function of a Truncated Normal Distribution
F	Cumulative Distribution Function of a Normal Distribution
F_T	Cumulative Distribution Function of a Truncated Normal Distribution
I	Indicator Function
Sym $TN_{N-\text{type}}$	A symmetric doubly truncated normal distribution
x_l	Lower Truncation Point
x_u	Upper Truncation Point
X	Random Variable
X_T	Truncated Random Variable
T	Truncated Standard Random Variable
$TN_{L-\text{type}}$	A left truncated normal distribution
$TN_{S-\text{type}}$	A right truncated normal distribution
$TSN^+_{N-\text{type}}$	A doubly truncated positive skew normal distribution
$TSN^-_{N-\text{type}}$	A doubly truncated negative skew normal distribution
$TSN^+_{L-\text{type}}$	A left truncated positive skew normal distribution
$TSN^-_{L-\text{type}}$	A left truncated negative skew normal distribution
$TSN^+_{S-\text{type}}$	A right truncated positive skew normal distribution
$TSN^-_{S-\text{type}}$	A right truncated negative skew normal distribution
z_l	Lower Truncation Point from the Truncated Standard Normal Distribution
z_u	Upper Truncation Point from the Truncated Standard Normal Distribution
Z	Random Variable of the Standard Normal Distribution

z_{T_l} Lower Truncation Point from the Standard Truncated Normal Distribution

z_{T_u} Upper Truncation Point from the Standard Truncated Normal Distribution

Z_{X_T} Random Variable of the Standard Truncated Normal Distribution

μ Mean of a Normal Distribution

μ_T Mean of a Truncated Normal Distribution

σ^2 Variance of a Normal Distribution

σ_T^2 Variance of a Truncated Normal Distribution

ϕ Probability Density Function of the Standard Normal Distribution

Φ Cumulative Distribution Function of the Standard Normal Distribution

ABBREVIATIONS

CI	Confidence Interval
CLT	Central Limit Theorem
DTND	Doubly Truncated Normal Distribution
DTNRV	Doubly Truncated Normal Random Variable
LCI	Lower Confidence Interval
LTND	Left Truncated Normal Distribution
LTNRV	Left Truncated Normal Random Variable
LTP	Lower Truncation Point
NRV	Normal Random Variable
RTND	Right Truncated Normal Distribution
RTNRV	Right Truncated Normal Random Variable
RV	Random Variable
SDTND	Standard Doubly Truncated Normal Distribution
SLTND	Standard Left Truncated Normal Distribution
SRTND	Standard Right Truncated Normal Distribution
STD	Standard Truncated Distribution
STND	Standard Truncated Normal Distribution
TD	Truncated Distribution
TND	Truncated Normal Distribution
TNRV	Truncated Normal Random Variable
TRV	Truncated Random Variable
TSND	Truncated Standard Normal Distribution
UCI	Upper Confidence Interval
UTP	Upper Truncation Point

CHAPTER ONE

INTRODUCTION

The purposes of this research are to reanalyze the theoretical foundations of a truncated normal distribution and to extend new findings to the body of knowledge. More specifically, we develop a new set of hypothesis testing procedures under a truncated normal distribution and derive the sum of a number of types of truncated normal random variables including truncated skew normal random variables based on convolution. To the best of our knowledge, these important questions have remained unanswered in the research community. In Section 1.1, different types of a truncated distribution are introduced with some examples. Based on the concepts of the truncated distribution, the sum of the truncated random variables is then discussed in Section 1.2. In Section 1.3, research significance and questions are posed and the dissertation structure follows in Section 1.4.

1.1 A Truncated Distribution

When a distribution is truncated, the domain of the truncated random variable is restricted based on the truncation points of interest and thus the shape of the distribution changes. A truncated distribution was first introduced by Galton (1898) to analyze speeds of trotting horses for eliminating records which was less than a specific known time. Applications of a truncated distribution can be found in many settings. Khasawneh *et al.* (2004) illustrated examples in quality control. Final products are often subject to screening before being sent to the customer. The usual practice is that if a product's performance falls within certain tolerance limits, it is judged to be conforming and sent to the customer. If the product fails, it is rejected and thus scrapped or reworked. In this case, the distribution of the performance to the customer is truncated. Another example can be found in a multistage production process in which inspection is performed at each production stage. If only conforming items are passed on to the next stage, the distribution of performance of the conforming items is truncated. Accelerated life testing with samples censored is another example of applying a truncated distribution. In fact, the concept of a truncated distribution plays a significant role in analyzing a variety of production processes.

In addition, Lai and Chew (2000) explained the role of a truncated distribution in the gauge repeatability and reproducibility to quantify measurement errors, and illustrated that the distributions of errors associated with measurement data collected from instruments are typically truncated. Field *et al.* (2004) studied truncated distributions associated with measured traffic from different locations in relation to high-performance Ethernet. They experimented with various truncated distributions which were divided into three types: left, right, and doubly truncated distributions. Parsa *et al.* (2009) studied a truncated distribution as the distribution of a noise factor which masks data in data security.

Three types of a truncated distribution were studied in Parsa *et al.* (2009); however, this dissertation categorizes the truncated normal distribution into four different types, such as symmetric double, asymmetric double, left and right truncated distributions. Each type of a truncated normal distribution, where $f_X(x)$ and $f_{X_T}(x)$

represent a normal distribution and its truncated normal distribution, respectively, is shown in Figure 1.1, where plots (a) and (b) show symmetric and asymmetric double truncations, respectively. Left and right truncated normal distributions are shown in plots (c) and (d), respectively. The shapes of a truncated distribution vary based on its truncation point(s) (x_l or x_u), mean (μ), and variance (σ^2). It is noted that a truncated variance after implementing a truncation will be no longer be the same as the original variance associated with the untruncated normal distribution $f_X(x)$. Similarly, unless symmetric double truncations are used, a truncated mean is not the same as the original mean of an untruncated normal distribution.

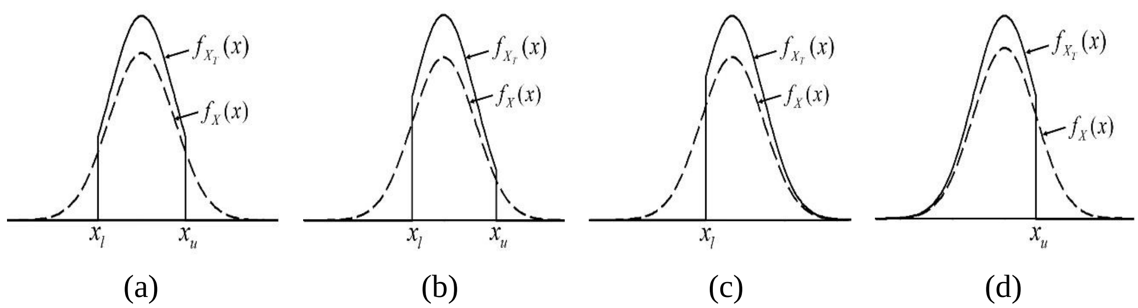


Figure 1.1. Plots of four different types of a truncated normal distribution

As discussed, the application of the truncated distribution can also be found in a multistage production process in which an inspection is performed at each production stage, as shown in Figure 1. Notice that the actual distribution, which moves on to each of the next stage, is a truncated distribution.

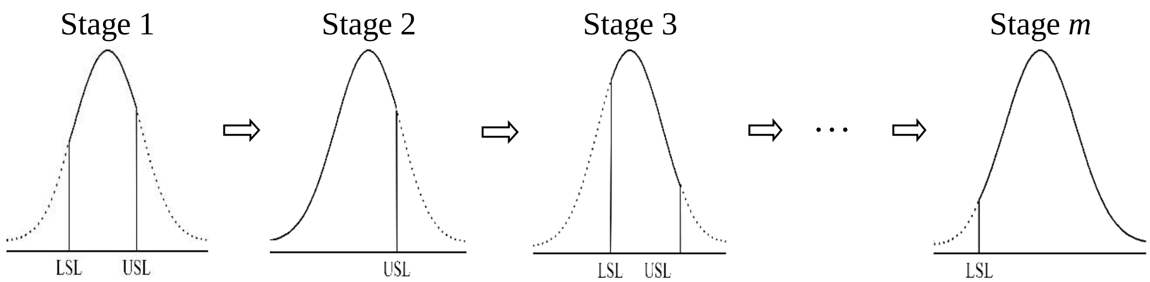


Figure 1.2. Inspections in multistage production process

1.2 Sum of Truncated Random Variables

In this section, the distribution of a sum of the truncated random variables associated with convolution is briefly discussed. Convolution is a mathematical way combining two distributions to form a new distribution. Dominguez-Torres (2010) mentioned that the earliest convolution theorem, $\int_a^b f(u)g(x \pm u)du,$ was introduced by Euler in the middle of the 18th century based on the theories of Taylor series and Beta function. Note that f and g are two real or complex valued functions of real variable μ and x. In the truncated environment, Francis (1946) first used convolution to obtain a density function of a sum of the truncated random variables as follows:

$h_Z(z) = \int_{-\infty}^{\infty} g_{Y_T}(y) f_{X_T}(x)dx = \int_{-\infty}^{\infty} g_{Y_T}(z-x)\, f_{X_T}(x)dx$ where $Z = X_T + Y_T$ and X_T and Y_T are truncated random variables.

It is our observation that convolution may give the closed form of a probability density function of the sum of truncated random variables, when the number of truncated random variables are up to three. Figure 1.2 illustrates the plots of the distribution of the sum of two truncated normal random variables. Plots (a) and (b) show the distributions of two independently, identically distributed symmetric doubly truncated normal random variables, respectively. The distribution of the sum of the truncated normal random variables which is obtained by convolution is shown in plot (c). Note that its probability density function $f_Z(z)$ is different from the density of a traditional normal distribution. d

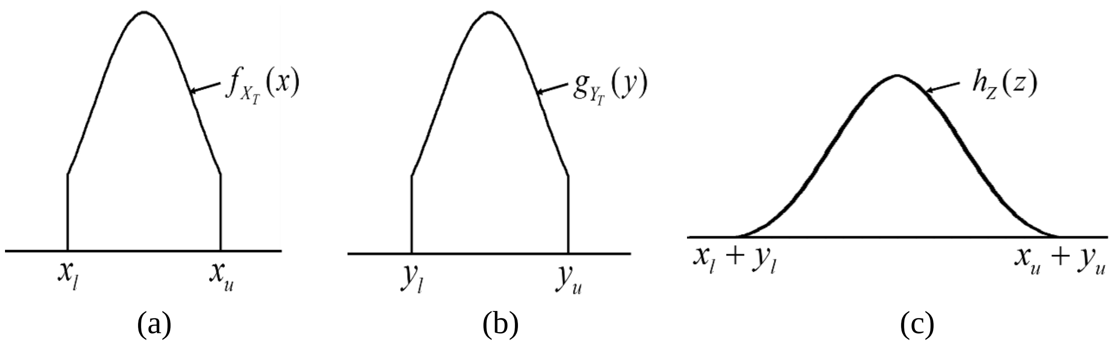


Figure 1.2. Plots of the sum of two truncated normal random variables

Unfortunately, when the number of truncated random variables are four or larger, the closed form of density of the sum of the truncated random variables may not be acquired. However, we have proved that the sum of truncated random variables converges to a normal distribution, when the number of the truncated random variables are large enough. The accuracy of this approximation depends on the number of truncated random variables, truncation point(s), and mean and variance of an untruncated original distribution.

1.3 Research Significance and Questions

As mentioned in Section 1.1, truncated distributions have been used in many areas. In addition to the examples in manufacturing, reliability, quality and data security illustrated in Section 1.1, the application areas of the truncated distribution are also found in economics (Xu *et al.*, 1994), electronics (Dixit and Phal, 2005), biology (Schork *et al.*, 1990), social and behavior science (Cao *et al.*, 2014), physics (Baker, 2008) and education (Hartley, 2010). Although truncated distributions were introduced more than one hundred years ago, there is still ample room for theoretical enhancement.

In my dissertation, there are three research goals: (1) standardization of truncated normal random variables, (2) statistical inference on the mean for truncated samples, and (3) densities of the sum of truncated normal and truncated skew normal random variables.

First, only a few papers have studied the underlying theory associated with the standardization of a truncated distribution. The currently-used traditional truncated standard normal distribution (TSND), derived from truncation of the standard normal distribution, has varying mean and variance, depending on the location of truncation points. As a result, its statistical analysis may not be done on a consistent basis. In order to lay out the theoretical foundation in a more consistent way, we develop the standard truncated normal distribution (STND) which has zero mean and unit variance, regardless of the location of the truncation points. We also develop its properties in this dissertation. In the first part of the dissertation, we answer the following two research questions:

Research question 1: Can we further develop the properties of the proposed standard truncated normal distribution?

Research question 2: Can we develop the cumulative probability table of the truncated normal distribution which might be useful for practitioners?

Second, statistical hypothesis testing is helpful for controlling and improving processes, products, and services. This most fundamental, yet powerful, continuous improvement tool has a wide range of applications in quality and reliability engineering.

Some application areas include statistical process control, process capability analysis, design of experiments, life testing, and reliability analysis. It is well known that most parametric hypothesis tests on a population mean, such as the *z*-test and *t*-test, require a random sample from the population under study. There are special situations in engineering, where the specification limits, such as the lower and upper specification limits, on the process are implemented externally, and the product is typically reworked or scrapped if the performance of a product does not fall in the range. As such, a random sample needs to be taken from a truncated distribution. However, there has been little work on the theoretical foundation of statistical hypothesis procedures under these special situations. In the second part of this research, we pose the following primary research questions:

Research question 3: Can we develop the new statistical inference theory within the truncated normal environment when the sample size is large?

- *Research question 3.1: Can we obtain the confidence intervals?*
- *Research question 3.2: Can we obtain the hypothesis testing?*

Finally, this research lays out the theoretical foundation of sum of truncated normal and skew normal random variables. Specifically, exploring two and three stage screening procedures can substantially reduce errors by understanding the mean and variance of process output. This can be better conceptualized with truncated normal random variables. This paper presents a mathematical framework that exemplifies modeling complex systems. Closed-form expressions of probability density functions are developed for the sums of truncated normal random variables when the number of truncated random variables are two. This is unique in the fact that many types of convolutions of truncated normal random variables were explored. To the authors' knowledge there is no known literature that explores anything other than the convolutions of the same types of singly and doubly truncated normal random variables. This paper adds convolutions of different types of singly and doubly truncated normal random variables, which include S-type, N-type and L-type quality characteristics that include both the symmetric and asymmetric types of normal distributions. A successful completion of the research work will result in a better understanding in gap analysis and tolerance design. Specially, these closed form probability density functions can readily be applied to manufacturing design on the assembly line for rectangular types of sums of truncated normal random variables. Other possible applications include applications in aerospace assembly and watch making for circle types of sums of truncated normal random variables. Consequently, we pose the following primary research questions:

Research question 4: Can we develop the properties of the sums of two truncated skew normal random variables by the convolution?

Research question 5: Can we develop the properties of the sums of three truncated skew normal random variables by the convolution?

The goal of the literature review was to support this thesis' effort to enhance the understanding of the cross-ambiguity function by integrating a wide range of mathematical concepts into an engineering framework.

1.4 Overview and Strategy for the Dissertation

Figures 1.3 and 1.4 show the overall strategy and roadmap of the dissertation. Chapter 2 reviews the literature and support the validity of the research questions. In Chapter 3, we extend our research effort to achieve associated the properties of the standard truncated normal distribution which is different from the truncated standard normal distribution we normally see in the literature. We then develop the cumulative probability tables based on the proposed standard truncated normal distribution. Chaper 4 develops statistical inference for hypothesis testing and confidence intervals in the trucated normal enviroment, when the sample size is large. In Chapers 5, twenty-one cases of convolutions of truncated normal and truncated skew normal random variables are highlighted. The cases presented here represent all the possible types of convolutions of double truncations (i.e., the sum of all the possible combinations, containing two truncated random variables, of normal and skew normal probability distributions). Fifty-six cases of the convolutions of triple truncations (i.e., the sums of all the possible combinations, containing three truncated random variables, of normal and skew normal probability distributions) are then illustrated. Numerical examples illustrate the application of convolutions of truncated normal random variables and truncated skew normal random variables to highlight the improved accuracy of tolerance analysis and gap analysis techniques.

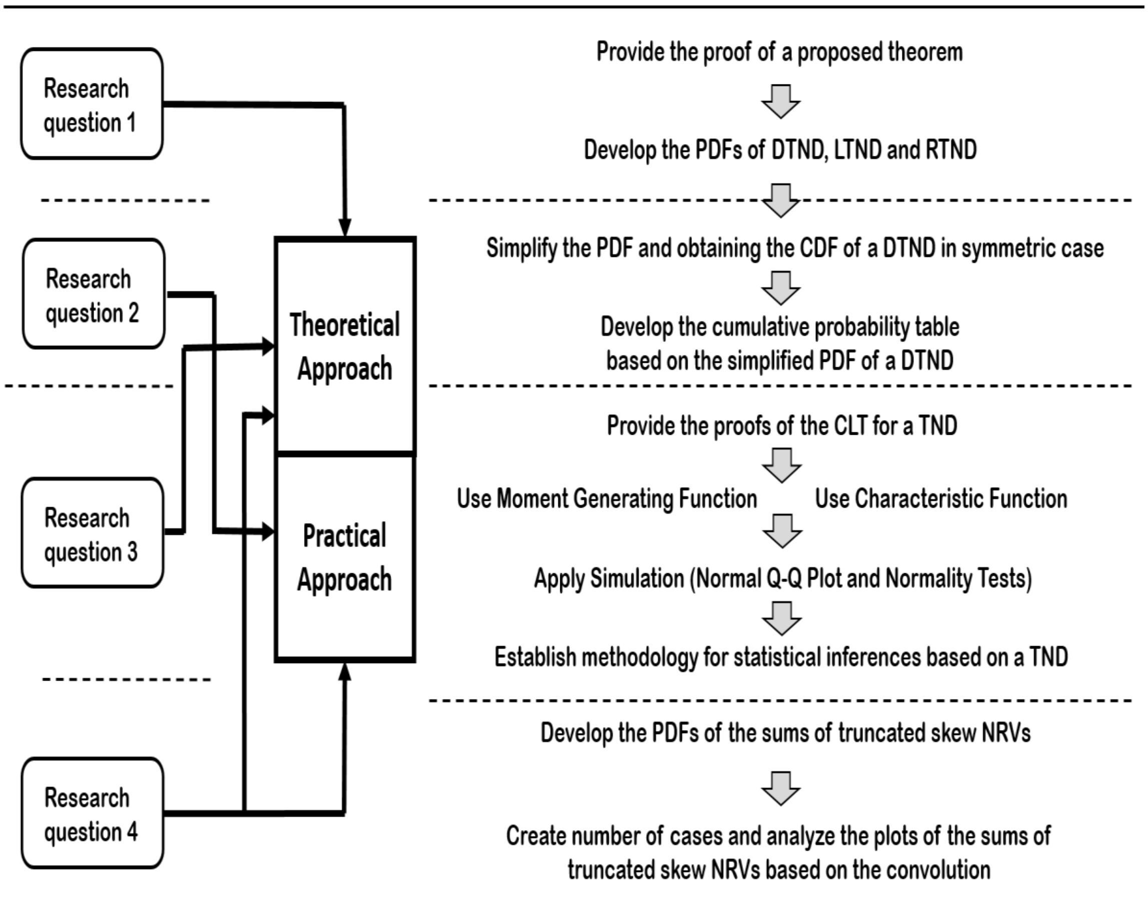


Figure 1.3. Strategy of the dissertation

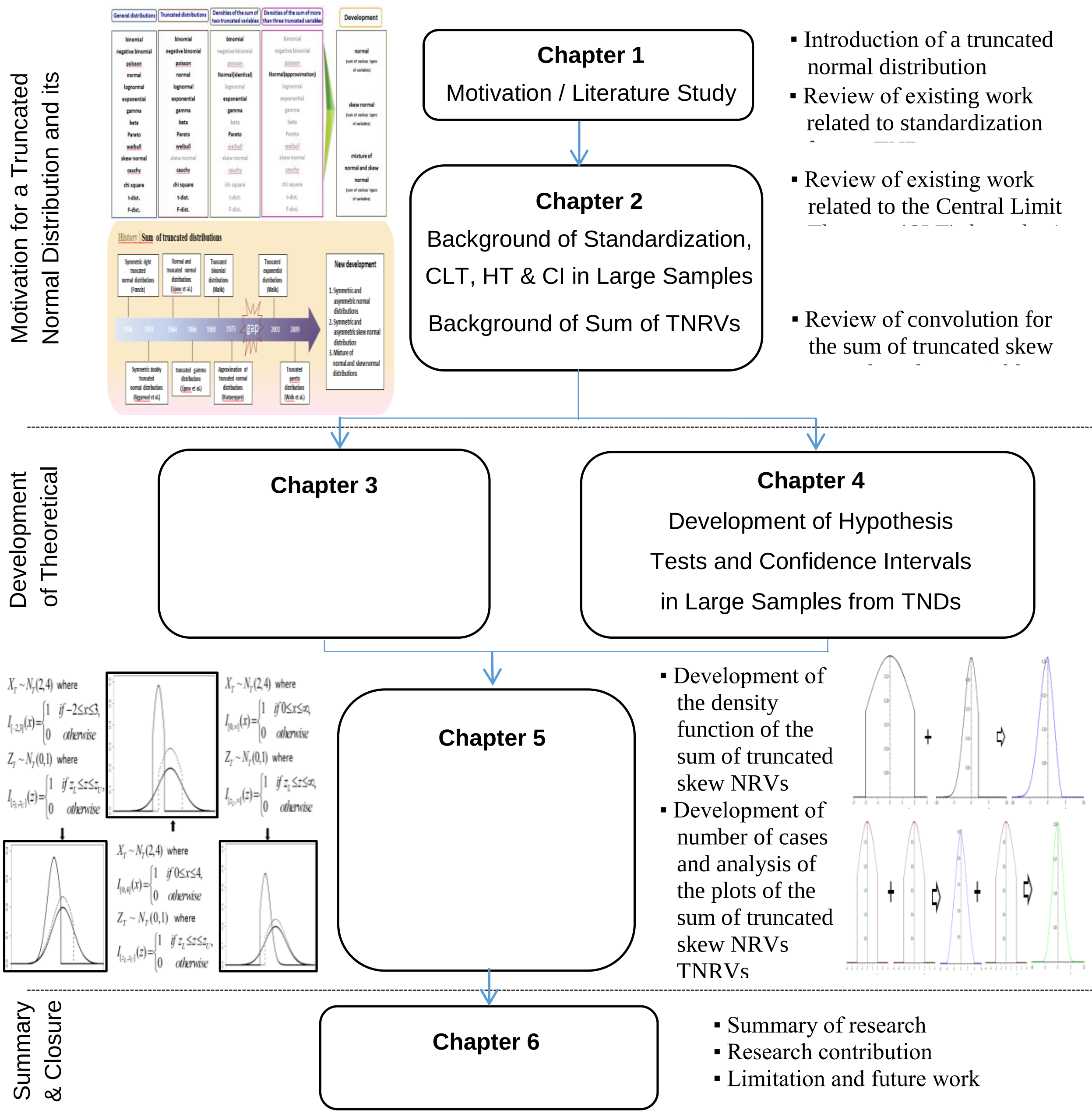


Figure 1.4. Dissertation overview and roadmap

CHAPTER TWO

LITERATURE REVIEW

This chapter comprises three sections. Section 2.1 reviews discrete and continuous truncated distributions and several estimation methods such as maximum likelihood estimation and goodness-fit-tests in the truncated environment. Section 2.2 discusses well-known properties of a truncated normal distribution and the standardization of a truncated normal random variable. Section 2.3 examines the Central Limit Theorem and the sum of random variables incorporating the convolution concept.

2.1 Truncated Distributions, Samples and Estimations

In this section, we review fifteen truncated distributions, examine truncated and censored samples, and investigate five estimation methods. In particular, twelve continuous and three discrete truncated distributions are studied in Section 2.1.1. We then discuss truncated and censored samples in Section 2.1.2. Five different estimation methods based on these samples are investigated in Section 2.1.3.

2.1.1 Truncated Distributions

Since Galton (1898) and Pearson and Lee (1908) introduced the basic concepts of left and right truncated distributions, several types of truncated distributions have been developed. For discrete distributions, David and Johnson (1952), and Moore (1954) implemented a truncated Poisson distribution to examine the number of accidents per worker. Finney (1949) and Sampford (1955) discussed the doubly truncated binomial and negative-binomial distributions with examples in biology with respect to the number of abnormals in sibships of specified size.

Truncated gamma, Pareto, exponential, Cauchy, t, F, normal, Weibull, skew and Beta distributions have also been studied by researchers. Chapman (1956) discussed a truncated gamma distribution with right truncation to analyze an animal migration pattern. A truncated Pareto distribution was considered to find the appropriate distribution due to the lack of the Pareto distribution, in which the whole range of income and tax is not rarely fitted over, in income-tax statistics by Bhattacharya (1963). Cosentino *et al.* (1977) investigated the frequency magnitude relationship to solve a problem concerning the statistical analysis of earthquakes with a truncated exponential distribution. A truncated Cauchy distribution was introduced to overcome the weakness of the Cauchy distribution by Nadarajah and Kotz (2006). Kotz and Nadarajah (2004) also introduced the truncated t and F distributions to inspect the moments and estimation procedures by the method of moments and the method of maximum likelihood. A truncated Weibull distribution was studied to solve the problem of nonexistence of the maximum likelihood estimators by Mittal and Dahiya (1989). Jamalizadeh *et al.* (2009) examined the cumulative density function and the moment generating function of a truncated skew normal distribution. Zaninetti (2013), recently, found that a left truncated beta distribution fits to the initial mass function for stars better than the lognormal distribution which has been commonly used in astrophysics.

2.1.2 Truncated and Censured Samples

Before Hald (1949) had the meaning of 'censored' in writing, truncated and censored samples had not been used without any separation. Hald (1949) used two papers (Fisher; 1931, Stevens; 1937) to explain truncated and censored samples. According to the examples of the paper of Hald (1949), samples in the case in which all record is eliminated of observations below a given value are truncated samples. In this case, the observations make a random sample taken from a truncated distribution. Instead, samples in the case in which the frequency of observations below a given value is recorded but the individual values of these observations are not specified, are censored samples. The samples, in this case, are drawn from an untruncated distribution in which the obtainable information in a sense has been censored.

For lifetime testing, most researchers have examined truncated distributions based on censored samples which are classified into types I and II. In type I samples, censoring points are known, whereas the number of censored samples is unknown. Thus, the size of the censored samples is the observed value of a random variable. In contrast, in type II samples, the size of the censored samples is known, whereas a censoring point is an unknown random variable.

2.1.3 Estimations of Truncated and Censored Means

We review the maximum likelihood estimation and moment generating estimation for truncated and censored samples in Section 2.1.3.1 and 2.1.3.2, respectively. Then, we discuss the goodness fit test followed by the inferences, including hypothesis testing for censored samples and their confidence intervals.

2.1.3.1 Methods of Maximum Likelihood and Moments

For the estimation of the parameters of a truncated normal distribution, Cohen (1941, 1955, 1961), Cochran (1946), Gupta (1952), and Saw (1961) studied the method of moments with singly or doubly truncated normal distributions. Stevens (1937), Hald

(1949), and Halperin (1952) examined the method of maximum likelihood with singly or doubly truncated normal distributions. Accordingly, Shah and Jaiswal (1966) showed that the results from the likelihood estimators were similar to the results from the first four moments for a doubly truncated case. Later, Schneider (1986) and Cohen (1991) investigated the methods of maximum likelihood and moments for left and right truncated cases. However, Schneider (1986) and Cohen (1991) found that there were sampling errors for the odd number of moment estimators. They calculated that the sampling errors of the odd number of moment estimators were greater than those of relevant maximum likelihood estimators. Along the same line, Jawitz (2004) revealed the way to reduce the errors by using the order statistics.

2.1.3.2 Goodness Fit Test

In terms of goodness of fit tests for censored samples, Barr and Davidson (1973) developed the modified Kolmogorov–Smirnov test statistic, which is invariant under the probability integral transformation of the underlying data for types I and II censored samples. Pettitt and Stephens (1976) modified the Cramer–von Mises test statistics for singly censored samples, which may not depend on the specific form of the distribution, and developed tables of asymptotic percentage points. Mihalko and Moore (1980) showed that the vector of standardized cell probabilities is asymptotically normally distributed for type II singly or doubly censored samples based on the Chi-square test of fit. The Shapiro–Wilk test was applied to the normality test for censored samples by Verril and Johnson (1987). Monte Carlo simulation was then used to find the critical values such as the total number of samples, the number of censored samples, and the significant level. Chernobai *et al.* (2006) compared the results of the goodness of fit test using the modified Anderson–Darling test statistic they developed from six different censored data sets.

2.1.3.3 Confidence Interval

Halperin (1952), Nadarajah (1978), Schneider (1986), and Schneider and Weissfeld (1986) studied the confidence intervals for the mean μ of random variable X, which is normally distributed with mean μ and variance σ^2, both unknown, in type II censoring. Especially, Schneider (1986) studied the effect of symmetric and asymmetrical censoring on the probability of type I error for a t-test and on the confidence level of a confidence interval and concluded that the t-statistic is only reliable for symmetrical censoring. In addition, Schneider and Weissfeld (1986) analyzed that the confidence intervals are unreliable even for the sample size as large as 100 and then obtained more accurate confidence intervals by using bias correction methods for the computation of $\hat{\mu}$ and $\hat{\sigma}$ in small samples.

For the confidence limits of μ and σ^2 from types I and II censored samples, Dumonceaux (1969) developed the tables based on the maximum likelihood estimators by Monte Carlo simulation. Later, Schmee *et al.* (1985) found that the confidence limits are valid only for type II censored samples where the sample size is less than 20. Clarke (1998) investigated the confidence limits for type II censored samples under less than 10 sample sizes among 500 samples using simulation.

2.1.3.4 Hypothesis Testing

Aggarwal and Guttman (1959) examined a one-sided hypothesis testing for the truncated mean of a symmetric doubly truncated normal distribution (DTND) based on the small sample size, which is less than 4. They investigated the loss of power, which is the difference of power functions between a normal distribution and its truncated normal distribution and found that the loss of power decreases very rapidly with the distance of the alternative value of the mean from the test and also with the distance of the truncation from the mean.

Later, Williams (1965) extended a one-sided hypothesis testing to asymmetric single or double truncations and arbitrary sample size. The author then discovered that the loss of power is very little when the sample size is greater than 10 and the true value of the mean is more than 0.5 standard deviations away from the hypothesized value specified in the null hypothesis. Tiku *et al.* (2000) derived the modified maximum likelihood estimators, which showed that they are highly efficient, and then developed hypothesis testing procedures for censored samples with the estimators. However, the testing procedures developed by Aggarwal and Guttman (1959), Williams (1965), and Tiku *et al.* (2000) focused on a hypothesis testing for censored samples from a normal distribution, rendering a limited applicability.

2.2 A Truncated Normal Distribution

Section 2.2.1 discusses the properties of a truncated normal distribution such as the probability density function, cumulative distribution function, mean and variance. The truncated standard normal distribution is then reviewed in Section 2.2.2.

2.2.1 Properties of a TND

If a random variable X is normally distributed with mean μ and variance σ^2, its well-known probability density function is defined as $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ where $-\infty \leq x \leq \infty$. When the random variable $X \sim N\left(\mu, \sigma^2\right)$ is transformed by $Z = \left(X - \mu\right)/\sigma$, the random variable Z follows a $N\left(0,1\right)$ distribution, known as the standard normal distribution. The probability density function of Z is written as $f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{1}{2}z^2}$ where $-\infty \leq x \leq \infty$.

When the distribution of X is truncated at the lower and/or upper truncation point(s), its truncated distribution is called a truncated normal distribution. There are four types of truncated normal distributions such as symmetric doubly truncated normal distribution (symmetric DTND), asymmetric doubly truncated normal distribution (asymmetric DTND), left truncated normal distribution (LTND), and right truncated normal distribution (RTND). LTND or RTND is often called a singly truncated normal distribution. Furthermore, a DTND can be symmetric or asymmetric, depending on the location of the lower and upper truncation points.

When the distribution of X is doubly truncated at the lower and upper truncation points, x_l and x_u, the probability density function of the DTND is expressed as $f_{X_T}(x) = \frac{f_X(x)}{\int_{x_l}^{x_u} f_X(y)dy}$ where $x_l \leq x \leq x_u$ and its cumulative distribution function is written as $F_{X_T}(x)=\int_{-\infty}^{x}\frac{f_X(h)}{\int_{x_l}^{x_u} f_X(y)dy}dh$ where $x_l \le h \le x_u$. Based on the probability density function of the DTND, the probability density functions of the LTND and RTND are then obtained as $f_{X_T}(x)=\frac{f_X(x)}{\int_{x_l}^{\infty} f_X(y)dy}$ where $x_l \le x \le \infty$ and $f_{X_T}(x)=\frac{f_X(x)}{\int_{-\infty}^{x_u} f_X(y)dy}$ where $-\infty \le x \le x_u$, respectively, because the left (right) truncated distribution has only a lower (upper) truncation point, x_l $\left(x_u\right)$.

The mean and variance of the truncated normal random variable X_T are derived from the formulas $\mu_T = \int_{-\infty}^{\infty} x \cdot f_{X_T}(x)dx$ and $\sigma_T^2 = \int_{-\infty}^{\infty} x^2 \cdot f_{X_T}(x)dx - \left(\int_{-\infty}^{\infty} x \cdot f_{X_T}(x)dx\right)^2$. Table 2.1 shows the formulas of means and variances of the DTND, LTND, and RTND (see Johnson *et al*., 1998), where $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function and the cumulative distribution function, respectively, of a standard normal random variable Z, respectively. Detailed proofs for the mean and variance of the DTND can be found in Cha *et al.* (2014). Table 2.1 shows that both $\phi\left(\frac{x_l-\mu}{\sigma}\right)$ and $\Phi\left(\frac{x_l-\mu}{\sigma}\right)$ converge to zero in the mean and variance of the DTND as the lower truncation point, x_l, goes negative infinity. On the contrary, $\phi\left(\frac{x_u-\mu}{\sigma}\right)$ and $\Phi\left(\frac{x_u-\mu}{\sigma}\right)$ converge to zero and one, respectively, as the upper truncation point, x_u, goes positive infinity.

Table 2.1. Mean μ_T and variance σ_T^2 of doubly, left and right truncated normal distributions (Johnson *et al.*, 1998)

DTND	$\mu_T = \mu + \frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\sigma$ $\sigma_T^2 = \sigma^2\left[1 + \frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)-\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]$
LTND	$\mu_T = \mu + \frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\sigma$ $\sigma_T^2 = \sigma^2\left[1 + \frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]$
RTND	$\mu_T = \mu + \frac{-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\sigma$ $\sigma_T^2 = \sigma^2\left[1 - \frac{\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right)^2\right]$

2.2.2 Standardization of a TNRVs

In previous studies, a random variable $(X_T-\mu)/\sigma$ was used to estimate the mean and variance of a truncated normal random variable X_T. For example, Cohen (1991), Barr and Sherrill (1999), and Khasawneh *et al.* (2004, 2005) defined $T=(X_T-\mu)/\sigma$ as a truncated standard normal random variable. Even though various truncated distributions have been introduced, only a few papers investigated the standardization of a truncated normal random variable. Cohen (1991) denoted the random variable, $T=(X_T-\mu)/\sigma$ as the standardized truncated normal random variable for the method of moment estimation. Barr and Sherrill (1999) also defined the random variable, $T=(X_T-\mu)/\sigma$ as the truncated standard normal random variable for maximum likelihood estimators. Khasawneh *et al.* (2004, 2005) used the same truncated standard normal random variable, developed by Cohen (1991) and Barr and Sherrill (1999), to build tables of the distribution's cumulative probability, mean, and variance.

2.2.3 A truncated skew NRV

A skew normal distribution represents a parametric class of probability distributions, reflecting varying degrees of skewness, which includes the standard normal distribution as a special case. The skewness parameter makes it possible for probabilistic modeling of the data obtained from skewed population. The skew normal distributions are also useful in the study of the robustness and as priors in Bayesian analysis of the data. Birnbaum (1950) first explored skew normal distributions while investigating educational testing using truncated normal random variables. Roberts (1966) was another early pioneer in skew normal distributions by studying correlation models of twins. The term, the skew normal distribution, was formally introduced by Azzalini (1985, 1986), who explored the distribution in depth. Gupta *et al.* (2004) classified several multivariate skew-normal models. Nadarajah and Kotz (2006) showed skewed distributions from different families of distributions, whereas Azzalini (2005, 2006) discussed the skew normal distribution and related multivariate families. Jamalizadeh, *et al.* (2008) and Kazemi *et al.* (2011) discussed generalizations of the skew normal distribution based on various families. Multivariate versions of the skew normal distribution have also been proposed. Among them Azzalini and Valle (1996), Azzalini and Capitanio (1999), Arellano-Valle *et al.* (2002), Gupta and Chen (2004), and Vernic (2006) are notable. In many applications, the probability distribution function of some observed variables can be skewed and their values restricted to a fixed interval, as shown in Fletcher *et al.* (2010)

where the skew normal distribution was used to represent daily relative humidity measurements. As mentioned earlier, convolutions play an important role in statistical tolerance analysis. Most of the research work, however, considered untruncated normal distributions. See, for example, Gilson (1951), Mansoor (1963), Fortini (1967), Wade (1967), Evans (1975), Cox (1986), Greenwood and Chase (1987), Kirschling (1988), Bjorke (1989), Henzold (1995), and Nigam and Turner (1995), and Scholz (1995).

If a random variable Y is distributed with its location parameter μ, scale parameter σ, and shape parameter α, its probability density function is defined as

$$f_Y(y)=\frac{2}{\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\int_{-\infty}^{\alpha\frac{y-\mu}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt \text{, where } -\infty < y < \infty.$$

It is noted that the probability density function of Y becomes a normal distribution when the shape parameter α is zero. When the skew normal distribution of Y is truncated with the lower and upper truncation points, y_l and y_u, the probability density function of the truncated skew normal distribution is then expressed as

$f_{Y_{TS}}(y)=\dfrac{f_Y(y)}{\int_{y_l}^{y_u} f_Y(y)dy}$ where $y_l < y < y_u$. Similarly, $f_{Y_{TS}}(y)=\dfrac{f_Y(y)}{\int_{y_l}^{y_u} f_Y(y)dy} I_{[y_l, y_u]}(y)$ where the indicator function $I_{[y_l, y_u]}(y)$ is then defined as $I_{[y_l, y_u]}(y)=\begin{cases}1 & \text{if } y\in[y_l, y_u] \\ 0 & \text{otherwise}\end{cases}$. The truncated mean μ_{TS} and truncated variance σ_{TS}^2 of Y_{TS} are given by $\int_{y_l}^{y_u} y\cdot f_{Y_{TS}}(y)dy$ and $\int_{y_l}^{y_u} y^2\cdot f_{Y_{TS}}(y)dy-\left(\int_{y_l}^{y_u} y\cdot f_{Y_{TS}}(y)dy\right)^2$, respectively.

2.3 Central Limit Theorem and Sums of Random Variables

In this dissertation, the Central Limit Theorem is the key developing statistical inference in Chapter 3, when the sample size is large. In addition, the Central Limit Theorem might also pave the way to support that the distribution of the sum of independent random variables converges a normal distribution as the number of random variables increase. Thus, we first review the Central Limit Theorem in Section 2.3.1 and then discuss the ways to obtain the sums of truncated random variables in Section 2.3.2.

2.3.1 Central Limit Theorem

According to Fischer (2010), the fundamental foundation of the Central Limit Theorem was built in the middle of 1950s. De Morvre (1733) examined the sums of the independent binomial random variables, and Bernoulli (1778) showed that the distribution of the sum of the binomial random variables converge as the number of trials are getting large. Later, many researchers including Laplace (1810), Poission (1829), Dirichlet (1846), Cauchy (1853) and Lyapunov (1901) attempted to prove the Central Limit Theorem. Von Mises (1919) contributed to developing the local limit theorems for sums of continuous random variables based on the characteristic function. Meanwhile, Polya (1919, 1920) devoted to developing the theory of numbers associated with the Law of Large Number depending on the moment generating function, and first coined the term, Central Limit Theorem.

Lindeberg (1922) fundamentally generalized the proof of the Central Limit Theorem under the “Lindeberg condition” which is called a very weak condition. Levy

(1922, 1935, 1937) proved the Central Limit Theorem with the characteristic function by considering limit distributions for sums of independent, but not identically distributed random variables and developed the generalization of Fourier's integral formula to the case of Fourier transforms expressed by Stieltjes integrals. Furthermore, Donsker (1949) examined the Central Limit Theorem for sums of independent random elements in a Hilbert space. In terms of stochastic point of view, Gnedenko and Kolmogorov (1954) inspected limit distributions of sums of independent random variables with regard to the Central Limit Theorem. Fortet and Mourier (1955) developed the limit theorem associating the Central Limit Theorem in Banach spaces.

In Chapter 4, we provide two proposed theorems to prove the Central Limit Theorem with the moment generating and characteristic functions for a truncated normal distribution. In the future research, the proposed theorems are utilized to assume that the distribution of the sum of the truncated normal random variables has an approximate normal distribution, when the number of random variables are sufficiently large.

2.3.2 Sums of Truncated Random Variables

As discussed in Section 1.2, convolution is the composition of two distributions for deriving the combined distribution. In this section, we first review the sums of truncated normal random variables based on convolution where the number of truncated random variables are generally less than four. When the number of random variables are larger than four, approximation methods might need to be applied. Two of the most popular methods are the Laplace and Fourier transforms.

To be more specific, Francis (1946) and Aggarwal and Guttman (1959) examined the probability density functions of the sums of singly and doubly truncated normal random variables and developed their cumulative probability tables under the assumption that the random variables are independently and identically distributed. Lipow *et al.* (1964) then investigated the density functions of the sums of a standard normal random variable and a left truncated normal random variable. Francis (1946), Aggarwal and Guttman (1959), and Lipow *et al.* (1964) have not been able to obtain the closed density functions of the sums, when the number of truncated normal random variables are equal and greater than five due to the computational complexity.

For the sum of more than four truncated normal random variables, Kratuengarn (1973) compared the means and variances of the sums of left truncated normal random variables numerically through Laplace and Fourier transforms. Although the Laplace and Fourier transforms allowed the consideration of the sum of the large number of variables, the results of the transformations included some errors. Recently, Fletcher *et al.* (2010) examined an expression of the moments an expression of the moments based on a truncated skew normal distribution. Tsai and Kuo (2012) applied the Monte Carlo method to obtain the densities of the sums of truncated normal random variables with 1,000,000 samples.

However, most studies focused on which are identically truncated normal distributions. In this research, we consider both identical and non-identical truncated normal distributions. Furthermore, we extend our research to a truncated skew normal distribution which has not been studied in the research community.

2.3.3 Multistage convolutions

Multistage convolutions may also be common in linear systems used in the electronics industry. Note that a system's impulse response specifies a linear system's characteristics, which are governed by the mathematics of convolution. This is the key support in many signal processing methods. For example, echo suppression in long distance phone calls is achieved by utilizing an impulse response that counteracts the impulse response of reverberation. Aircraft are detected by radar through analyzing a measured impulse response and digital filters are created by designing an appropriate impulse response (Smith, 1997). In Digital Signal Processing (DSP), the convolution the input signal function with the impulse response function yields a linear time-invariant system (LTI) as an output. The LTI output is an accumulated effect of all the prior values of the input function, with the most recent values typically having the most influence on the output. Using exact two and three stage truncated normal random variables in this model can result in heightened accuracy of DSP algorithms. This may result in faster processing times for common DSP algorithms. Note that multistage signal processing convolution methods are common when they are used in two dimensional Gaussian functions for Gaussian blurs of images (Hummel *et al*. 1987). Gaussian blur can be used in order to create a smoother digital image of halftone prints. Convolutions of functions and similar functional operators in general have several important applications in engineering, science and mathematics. Several important applications of convolutions are prominent in digital signal processing. For example, in digital image processing, convolutional filtering plays an important role in many important algorithms in edge detection and related processes. See Ieng, *et al*. (2014), Fournier (2011), and Reddy and Reddy (1979) for more examples.

2.3.4 Simulation Algorithms

Another research approach to the truncated normal distribution comes from the development of algorithms in computer software. Chou (1981) introduced the Markov Chain Monte Carlo algorithm using Gibbs-sampler from singly truncated bivariate normal distributions. Breslaw (1994), Robert (1995), Foulley (2000), Fernandez *et al.* (2007) and Yu *et al.* (2011) developed algorithms using Gibbs-sampler for singly and doubly truncated multivariate normal distributions.

2.4 Justification of Research Questions

First, based on the previous literature reviews, this research provides additional proposed theorems, in which variance of a normal distribution is compared with and variances of four different types of its truncated normal distribution, to solve Research questions 1 and 2. Second, for illustration of the Central Limit Theorem for a truncated normal distribution with respect to Research questions 3, this dissertation examines how the normal quantile–quantile (Q–Q) plots change according to four different sample sizes based on the four types of a truncated normal distribution and diagnoses the normality by applying the Shapiro–Wilk test (Shapiro and Wilk; 1968, Shapiro; 1990). Third, sums of truncated normal and truncated skew normal random variables are extended by double and triple truncations for examples of two application areas. To solve Research question

4, three different generalized probability density functions under double truncation and four different generalized probability density functions under triple truncation are developed on convolution that have not been explored previously. By using those seven probability density functions, sixty five cases are investigated based on double truncations while two hundred twenty cases are examined triple truncations. Density, mean and variance of the sum in each case are obtained and those results are analyzed to draw the critical concepts in multistage production process, statistical tolerance analysis, and gap analysis.

CHAPTER THREE

DEVELOPMENT OF STANDARDIZATION OF A TND

As indicated in Chapters 1 and 2, the traditional truncated standard normal distribution, derived from the truncation of a standard normal distribution (TSND), has varying mean and variance, depending on the location of truncation points. In contrast, we develop a standard truncated normal distribution (STND) by standardizing a truncated normal distribution in this chapter. In Section 3.1, to ensure the validity of the development of the STND, we compare the variance of a normal distribution and its truncated normal distribution by proposing three theorems. Within the properties of the STND which are developed in Sections 3.2, we develop the cumulative probability table of the STND as a set of guidelines for engineers and scientists in Section 3.3. A numerical example and conclusions are followed by Sections 3.4 and 3.5, respectively.

3.1 Comparison of Variances between an NRV and its TNRV

In Section 3.1.1, the variance of a doubly truncated normal distribution is examined to compare the one of its original normal distribution. Then, the variance of normal distribution is compared to ones of its left and right truncated normal distributions in Sections 3.1.2 and 3.1.3, respectively.

3.1.1 Case of a DTNRV

Once a normal distribution is truncated, its variance changes. Intuitively, the variance of the truncated normal random variable is smaller than the variance of the original normal random variable. In this section, we provide a proposed theorem to compare the variances between a normal random variable and its doubly truncated normal random variable.

Proposed Theorem 1 Let $X \sim N(\mu, \sigma^2)$ where $\sigma > 0$ and let X_T be its doubly truncated normal random variable where $E(X_T) = \mu_T$, $V(X_T) = \sigma_T^2$, and the lower and upper truncation points are denoted by x_l and x_u, respectively. Then, σ_T^2 is always less than σ^2. That is, $\sigma_T^2 < \sigma^2$.

Proof

We will show $\sigma^2 - \sigma_T^2 > 0$. From Table 2.1, the difference of variances, $\sigma^2 - \sigma_T^2$, is written as

$$\sigma^2 \left[\frac{\left(-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right) + \frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)\right) \cdot \left(\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)\right)}{\left(\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)\right)^2} + \frac{\left(\phi\left(\frac{x_l-\mu}{\sigma}\right) - \phi\left(\frac{x_u-\mu}{\sigma}\right)\right)^2}{\left(\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)\right)^2} \right]. \tag{1}$$

By the properties of the standard normal distribution, $\phi\left(\frac{x_l-\mu}{\sigma}\right) > 0$, $\phi\left(\frac{x_u-\mu}{\sigma}\right) > 0$, and $\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right) > 0$.

Since the second term inside the brackets in Eq. (1) is always greater than or equal to zero, the first term inside the brackets should be investigated.

There are three cases associated with the first term we need to consider. Fig. 3.1 shows the plots of the three cases which can occur from the double truncations. To prove $\sigma^2 - \sigma_T^2 > 0$, we need to check whether $\left(-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right) + \frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)\right) > 0$ since $\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right) > 0$.

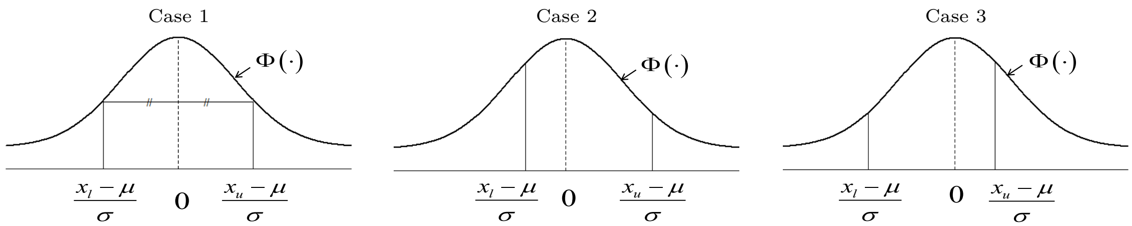


Figure 3.1. Plots of three cases under double truncations

Case 1: Consider a symmetric case. Note that $\frac{x_l-\mu}{\sigma} < 0$, $\frac{x_u-\mu}{\sigma} > 0$, and $\frac{x_l-\mu}{\sigma} = -\frac{x_u-\mu}{\sigma}$. Since $\frac{x_l-\mu}{\sigma} = -\frac{x_u-\mu}{\sigma}$, $\phi\left(\frac{x_l-\mu}{\sigma}\right)$ is equal to $\phi\left(\frac{x_u-\mu}{\sigma}\right)$. Thus, the first term indicates that $-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right) + \frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right) = 2\,\frac{x_u-\mu}{\sigma}\,\phi\left(\frac{x_u-\mu}{\sigma}\right) > 0$. Therefore, $\sigma^2 - \sigma_T^2 > 0$.

Case 2: Consider an asymmetric case in which $-\frac{x_l-\mu}{\sigma} \leq 0$, $\frac{x_u-\mu}{\sigma} > 0$, and $\left|\frac{x_l-\mu}{\sigma}\right| < \left|\frac{x_u-\mu}{\sigma}\right|$. $\phi\left(\frac{x_l-\mu}{\sigma}\right)$ is greater than $\phi\left(\frac{x_u-\mu}{\sigma}\right)$ since $\left|\frac{x_l-\mu}{\sigma}\right| < \left|\frac{x_u-\mu}{\sigma}\right|$. Hence, $-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right) + \frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right) > 0$. Therefore, $\sigma^2 - \sigma_T^2 > 0$.

Case 3: Now, consider an aymmetric case in which $\frac{x_l-\mu}{\sigma} < 0, \frac{x_u-\mu}{\sigma} \geq 0$, and $\left|\frac{x_l-\mu}{\sigma}\right| > \left|\frac{x_u-\mu}{\sigma}\right|$. Since $\left|\frac{x_l-\mu}{\sigma}\right| > \left|\frac{x_u-\mu}{\sigma}\right|$, $\phi\left(\frac{x_l-\mu}{\sigma}\right)$ is less than $\phi\left(\frac{x_u-\mu}{\sigma}\right)$ and $\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right) > \frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)$. Therefore, $\sigma^2 - \sigma_T^2 > 0$,

Q. E. D.

We have demonstrated that the variance of a normal random variable is always greater than the variance of its doubly truncated normal random variable. This indicates that the variance of its doubly truncated standard normal distribution is always less than the one of the standard normal distribution.

3.1.2 Cases of an LTNRV

The variances of a normal distribution and its left truncated normal distribution are compared by a proposed theorem in this section.

Proposed Theorem 2 Let $X \sim N(\mu, \sigma^2)$ where $\sigma > 0$ and let X_T be its left truncated normal random variable (mean μ_T, variance σ_T^2, the lower truncation point x_l). Then, σ_T^2 is always less than σ^2. That is, $\sigma_T^2 < \sigma^2$.

Proof

Based on Table 2.1, the difference of variances, $\sigma^2 - \sigma_T^2$, is expressed as

$$\sigma^2 \left[-\frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)} + \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)} \right)^2 \right]. \tag{2}$$

Since $\sigma > 0$, we will show $-\frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)} + \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)} \right)^2 > 0$. A plot of the case under left truncation is shown in Fig. 3.2.

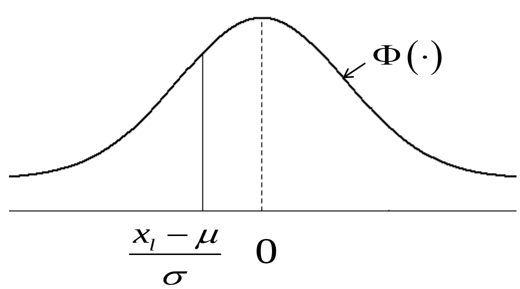


Figure 3.2. A plot of the case under left truncation

Let $t = \frac{x_l-\mu}{\sigma}$ and $g(t) = -\frac{t\phi(t)}{1-\Phi(t)} + \left(\frac{\phi(t)}{1-\Phi(t)}\right)^2 = \frac{\phi(t)}{1-\Phi(t)}\left(\frac{\phi(t)}{1-\Phi(t)} - t\right)$ where $-\infty \le t \le 0$. Since $\sigma_T^2 = \sigma^2\left(1 - g(t)\right)$, $\sigma^2 - \sigma_T^2$ is written as $\sigma^2 - \sigma_T^2 = \sigma^2 \cdot g(t)$.

Again, let $h(t) = \frac{\phi(t)}{1-\Phi(t)}$. Then, $g(t)$ is obtained as $g(t) = h(t) \cdot (h(t) - t)$ where $-\infty \le t \le 0$ since the value of $h(t)$ is greater than zero. It is noted that the derivative of $h(t)$ is given by $h^{'}(t) = \frac{d}{dt}h(t) = \frac{d}{dt}\left(\frac{\phi(t)}{1-\Phi(t)}\right)$. Since $\frac{d}{dt}\phi(t) = -t\phi(t)$ and $\frac{d}{dt}\left(\frac{1}{1-\Phi(t)}\right) = \frac{\phi(t)}{(1-\Phi(t))^2}$, we have $h^{'}(t) = \frac{-t\phi(t)}{1-\Phi(t)} + \left(\frac{\phi(t)}{1-\Phi(t)}\right)^2 = \frac{\phi(t)}{1-\Phi(t)}\left(\frac{\phi(t)}{1-\Phi(t)} - t\right)$. Thus, $h^{'}(t)$ is expressed as $h^{'}(t) = g(t) = h(t)\,(h(t) - t)$. Based on $h^{'}(t)$, $g^{'}(t)$ is obtained as $g^{'}(t) = h^{'}(t)\,(h(t) - t) + h(t)\left(h^{'}(t) - 1\right)h(t)\left[(h(t) - t)^2 + h(t)\,(h(t) - t) - 1\right]$.

Let $t^* \in (-\infty, 0]$ which makes $g^{'}(t^*) = 0$. Then, $(h(t^*) - t^*)^2 + h(t^*)\,(h(t^*) - t^*) - 1 = 0$ since $h(t^*) > 0$ for $\forall\, t^* \in (-\infty, 0]$. Hence, $g(t^*)$ is written as $g(t^*) = h(t^*)\,(h(t^*) - t^*) = 1 - \left(h^{'}(t^*) - t^*\right)^2$. Since $(h(t^*) - t^*)^2 > 0$, we find $g(t^*) < 1$. In addition, since $\lim_{t \to -\infty} h(t) = \frac{\phi(t)}{1-\Phi(t)} = 0$ and $\lim_{t \to -\infty} t\,h(t) = t\,\frac{\phi(t)}{1-\Phi(t)} = 0$, we have $\lim_{t \to -\infty} g(t) = 0$. Note that $1 - \Phi(t)$ and $t\phi(t)$ converge to one and zero, respectively, as t goes negative infinity. Thus, $0 < g(t) < 1$. Therefore, $\sigma^2 - \sigma_T^2 = \sigma^2 \cdot g(t)$ is always greater than zero and $0 < \sigma^2 - \sigma_T^2 < \sigma^2$,

Q. E. D.

3.1.3 Case of an RTNRV

In this section, we provide a proposed theorem to compare the variances of a normal distribution and its right truncated normal distribution.

Proposed Theorem 3 Let $X \sim N(\mu, \sigma^2)$ where $\sigma > 0$ and let X_T be its right truncated normal random variable where $E(X_T) = \mu_T$, $V(X_T) = \sigma_T^2$, and the upper truncation point is denoted by and x_u, respectively. Then, σ_T^2 is always less than σ^2. That is, $\sigma_T^2 < \sigma^2$.

Proof

According to Table 2.1,$\sigma^2 - \sigma_T^2$ is written as

$$\sigma^2 \left[\frac{\frac{x_U-\mu}{\sigma}\phi\left(\frac{x_U-\mu}{\sigma}\right)}{\Phi\left(\frac{x_U-\mu}{\sigma}\right)} + \left(\frac{\phi\left(\frac{x_U-\mu}{\sigma}\right)}{\Phi\left(\frac{x_U-\mu}{\sigma}\right)} \right)^2 \right] \tag{3}$$

A plot of the case under right truncation is illustrated in Fig. 3.3.

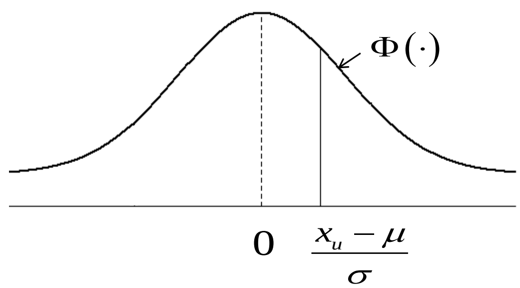


Figure 3.3. A plot of the case under right truncation

Let $g(t) = \frac{t\phi(t)}{\Phi(t)} + \left(\frac{\phi(t)}{\Phi(t)}\right)^2 = \frac{\phi(t)}{\Phi(t)}\left(\frac{\phi(t)}{\Phi(t)} + t\right)$ where $0 \leq t \leq \infty$. Eq. (3) is expressed as $\sigma^2 \cdot g(t)$. Since $\sigma > 0$, we will show $g(t) > 0$. Let $h(t) = \frac{\phi(t)}{\Phi(t)}$. It is noted that $h(t) > 0$. Based on we obtain $h(t)$, $g(t)$ is obtained as

$g(t) = h(t) \cdot (h(t) + t)$ where $0 \leq t \leq \infty$. Notice that $h^{'}(t) = \frac{d}{dt}h(t) = \frac{d}{dt}\left(\frac{\phi(t)}{\Phi(t)}\right)$.

Since $\frac{d}{dt}\phi(t) = -t\phi(t)$ and $\frac{d}{dt}\left(\frac{1}{1-\Phi(t)}\right) = \frac{\phi(t)}{(1-\Phi(t))^2}$, $h^{'}(t)$ is given by

$h^{'}(t) = \frac{-t\phi(t)}{\Phi(t)} - \left(\frac{\phi(t)}{\Phi(t)}\right)^2 = \frac{\phi(t)}{\Phi(t)}\left(\frac{\phi(t)}{\Phi(t)} + t\right) = -g(t) = -h(t)\left(h(t) + t\right)$. Thus, $g^{'}(t)$ is written as $h^{'}(t)\left(h(t) + t\right) + h(t)\left(h^{'}(t) + 1\right) = h(t)\left[-\left(h(t) + t\right)^2 + h(t)\left(-h(t) - t\right)\right.$ $\left.+1\right]$.

Let $t^* \in (-\infty, 0]$ which leads to $g^{'}(t^*) = 0$. Then,

$-\left(h(t^*) - t^*\right)^2 + h(t^*)\left(-h(t^*) - t\right) + 1 = 0$ for $\forall\, t \in [0, \infty)$. Thus, we have

$.h(t^*)\left(h(t^*)+t^*\right)=1-\left(h^{'}(t^*)+t^*\right)^2$. Therefore, $g(t^*)$ is expressed as $g(t^*)=h(t^*)\left(h(t^*)+t^*\right)=1-\left(h^{'}(t^*)+t^*\right)^2$. Since $\left(h(t^*)+t^*\right)^2>0$, $g(t^*)$ is less than one. It is noted that $\lim_{t\to\infty}h(t)=\frac{\phi(t)}{\Phi(t)}=0$. As t converges to ∞, $g(t)$ becomes zero since $\lim_{t\to\infty}h(t)=\frac{\phi(t)}{\Phi(t)}=0$ and $\lim_{t\to\infty}t\,h(t)=t\,\frac{\phi(t)}{\Phi(t)}=0$. Note that $\Phi(t)$ and $t\phi(t)$ converge to 1 and zero, respectively, as t goes infinity. Thus, we find $0<g(t)<1$. Therefore, $0<\sigma^2-\sigma_T^2<\sigma^2$,

Q. E. D.

3.2 Rethinking Standardization of a TND

The development of the properties of the STNRV is discussed with respect to Research Question 1. In Section 3.2.1, the terms associated with the STNRV is explained by comparing the terms of the traditional truncated standard normal random variable. In Section 3.2.2, we develop the probability density functions of the standard singly and doubly truncated normal distributions. Within those distributions, we concentrate on the standard doubly truncated normal distribution, which is symmetric, in order to obtain the simplified forms of its probability density function and cumulative distribution function in Section 3.3.3. Based on the results, we develop the cumulative probability table in Section 3.3.4.

3.2.1 Standardized TNRVs

In this research, we propose a standard truncated normal random variable as $Z_T=\frac{X_T-\mu_T}{\sigma_T}$, whose mean and variance are zero and one, respectively. Table 3.1 shows the terms for the standardization of the truncated normal distribution where its random variable is X_T, and x_l and x_u are its lower and upper truncation points, respectively. Furthermore, T denotes a truncated standard normal random variable, and $z_l = (x_l - \mu)/\sigma$ and $z_u = (x_u - \mu)/\sigma$ denote the lower and upper truncation points of T, respectively. In contrast, we define $z_{T_l} = (x_l - \mu_T)/\sigma_T$ and $z_{T_u} = (x_u - \mu_T)/\sigma_T$.

Table 3.1. The terms for the standardization of a truncated normal random variable

	Truncated standard normal	Standard truncated normal
Random variable	$T = \frac{X_T - \mu}{\sigma}$	$Z_{X_T} = \frac{X_T - \mu_T}{\sigma_T}$
Lower truncation point	$z_l = \frac{x_l - \mu}{\sigma}$	$z_{T_l} = \frac{x_l - \mu_T}{\sigma_T}$
Upper truncation point	$z = \frac{x_l - \mu}{\sigma}$	$z_{T_u} = \frac{x_u - \mu_T}{\sigma_T}$

Khasawneh *et al.* (2005) introduced tables of cumulative probability, mean, and variance of the doubly truncated standard normal distribution. The plot of variance of the symmetric doubly truncated standard normal distribution is shown in Fig. 3.4. It is noted that the values of variance are less than one and that the mean of a doubly truncated standard normal distribution is zero in a symmetric case. If the distribution is asymmetric, its mean values are not constant. That is, the values of mean and variance vary, depending on z_l and z_u.

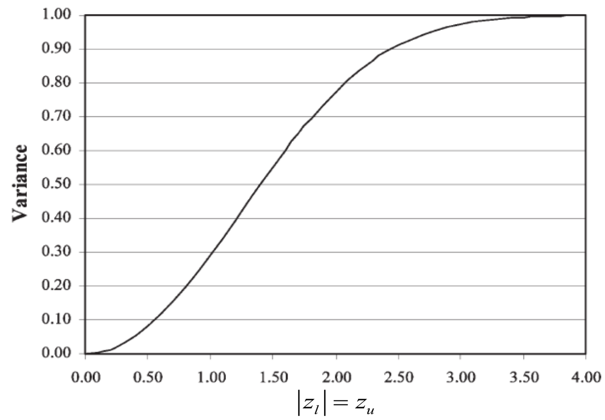


Figure 3.4. A plot of variance for doubly truncated standard normal distribution in a symmetric case by Khasawneh *et al.*(2005)

Fig. 3.5 shows a portion of the mean and variance tables from Khasewneh *et al.* (2005). It is noted that the truncated mean and variance are changed by the lower and upper truncation points. When $|z_l| \neq z_u$, the values of mean and variance are not zero and one, respectively.

Mean

z_l/z_u	0.00	0.10	0.20
0.00	0.00000	0.04996	0.09967
−0.10	−0.04996	0.00000	0.04963
−0.20	−0.09967	−0.04963	0.00000
−0.30	−0.14888	−0.09867	−0.04897
−0.40	−0.19735	−0.14690	−0.09704
−0.50	−0.24484	−0.19407	−0.14398
−0.60	−0.29111	−0.23996	−0.18956
−0.70	−0.33595	−0.28435	−0.23359
−0.80	−0.37915	−0.32704	−0.27586
−0.90	−0.42051	−0.36785	−0.31619
−1.00	−0.45986	−0.40659	−0.35442

Variance

z_l/z_u	0.00	0.10	0.20
0.0	0.00000	0.00083	0.00333
−0.1	0.00083	0.00333	0.00748
−0.2	0.00333	0.00748	0.01326
−0.3	0.00748	0.01326	0.02066
−0.4	0.01326	0.02065	0.02964
−0.5	0.02064	0.02962	0.04015
−0.6	0.02959	0.04011	0.05214
−0.7	0.04005	0.05206	0.06554
−0.8	0.05195	0.06539	0.08025
−0.9	0.06519	0.07999	0.09616
−1.0	0.07965	0.09574	0.11315

Figure 3.5. A portion of the tables of mean and variance in an asymmetric case for the truncated standard normal distributions by Khasawneh *et al.* (2005)

3.2.2 Development of the Properties of Standardization of a TND

In Section 3.2.2.1, we provide a proposed theorem to develop the probability density function of the standard doubly truncated normal distribution (SDTND). Based on the proposed theorem, the probability density functions of standard left and right truncated normal distributions are developed in Section 3.3.2.2.

3.2.2.1 Standardization of a DTND

In this section, we propose the probability density function of a random variable $Z_T = \frac{X_T - \mu_T}{\sigma_T}$ with mean zero and variance one.

Proposed Theorem 4 Let X_T be a random variable with mean μ_T and variance σ_T^2 which has a doubly truncated normal distribution with the probability density function

$$f_{X_T}(x) = \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\int_{x_l}^{x_u} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy},\ x_l \leq x \leq x_u \ .$$

A random variable $Z_T = \frac{X_T - \mu_T}{\sigma_T}$ has a standard doubly truncated normal distribution with the probability density function

$$f_{Z_T}(z) = \frac{\frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-\left(\frac{\mu-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2}}{\int_{z_{T_l}}^{z_{T_u}} \frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{p-\left(\frac{\mu-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2} dp}$$

where $z_{T_l} \leq z \leq z_{T_u},\ z_{T_l} = \frac{x_l - u_T}{\sigma_T},$ and $z_{T_u} = \frac{x_u - u_T}{\sigma_T}$. We then have $E(Z_T) = 0$ and $Var(Z_T) = 1$.

Proof

We first obtain the probability density function of Z_T and then show $E(Z_T) = 0$ and $Var(Z_T) = 1$. Let $Z_T = g(X_T) = \frac{X_T - \mu_T}{\sigma_T}$. For the sample space of X_T and Z_T, let $\mathcal{X} = \{x\colon f_{X_T}(x) > 0\}$ and $\mathcal{Z} = \{z\colon z = g(x) \text{ for some } x \in \mathcal{X}\}$. Since $\frac{d}{dx}g(x) = \frac{d}{dt}\left(\frac{x-\mu_T}{\sigma_T}\right) = \frac{1}{\sigma_T} > 0$ for $-\infty < x_l < x < x_u < \infty$, $g(x)$ is an increasing function. Note that $X_T \in [x_l, x_u]$ and $\frac{X_T-\mu_T}{\sigma_T} \in \left[\frac{x_l-\mu_T}{\sigma_T}, \frac{x_u-\mu_T}{\sigma_T}\right]$. Also note that $f_{X_T}(x)$ is continuous on $\mathcal{X}$ and $g^{-1}(z)$ has a continuous derivative on $\mathcal{Z}$. If we let $z = g(x)$, then $g^{-1}(z) = z\sigma_T + \mu_T$ and $\frac{d}{dz}g^{-1}(z) = \sigma_T$ since $z = \frac{x-\mu_T}{\sigma_T}$ implies $x = z\sigma_T + \mu_T$. By the chain rule, we have $f_{Z_T}(z) = f_{X_T}(g^{-1}(z))\frac{d}{dz}g^{-1}(z)$. Thus, the probability density function of Z_T is written as

$$
\begin{aligned}
f_{Z_T}(z) &= f_{X_T}(g^{-1}(z))\frac{d}{dz}g^{-1}(z) = f_{X_T}\left(z\sigma_T + \mu_T\right)\sigma_T \\
&= \frac{\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z\sigma_T+\mu_T-\mu}{\sigma}\right)^2}}{\int_{x_l}^{x_u}\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}dy}\,\sigma_T,\; x_l \le z\sigma_T + \mu_T \le x_u \\
&= \frac{\frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z-\left(\frac{\mu-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2}}{\int_{x_l}^{x_u}\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}dy},\; \frac{x_l-u_T}{\sigma_T} \le z \le \frac{x_u-u_T}{\sigma_T}. \qquad (4)
\end{aligned}
$$

It is observed that the numerator of $f_{Z_T}(z)$ has a normal distribution whose mean and variance are $\frac{\mu-\mu_T}{\sigma_T}$ and $\frac{\sigma}{\sigma_T}$, respectively, and that the denominator of $f_{Z_T}(z)$ is constant since x_l and x_u are given. Let $z_{T_l} = \frac{x_l-u_T}{\sigma_T}$, $z_{T_u} = \frac{x_u-u_T}{\sigma_T}$ and $f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$. Then, the denominator of $f_{Z_T}(z)$ is obtained as $\int_{x_l}^{x_u} f_Y(y)dy$. If we let $P = q(Y) = \frac{Y-\mu_T}{\sigma_T}$, then $\mathcal{Y} = \{y\colon x_l < y < x_u\}$ and $\mathcal{P} = \{p\colon p = q(y) \text{ for some } y \in \mathcal{Y}\}$. Since $\frac{d}{dy}q(y) = \frac{d}{dy}\left(\frac{y-\mu_T}{\sigma_T}\right) = \frac{1}{\sigma_T} > 0$ for

$x_l < y < x_u$, $q(y)$ is an increasing function. Consequently, $Y \in [x_l,\, x_u]$ and $\frac{Y-\mu_T}{\sigma_T} \in \left[\frac{x_l-\mu_T}{\sigma_T},\, \frac{x_u-\mu_T}{\sigma_T}\right]$. Similarly, $f_Y(y)$ is continuous on $\mathcal{Y}$ and $q^{-1}(p)$ has a continuous derivative on $\mathcal{P}$. By letting $p = q(y)$, we have $q^{-1}(p) = p\sigma_T + \mu_T$ and $\frac{d}{dp}q^{-1}(p) = \sigma_T$ since $p = \frac{y-\mu_T}{\sigma_T}$ implies $y = p\sigma_T + \mu_T$. Using the chain rule, we have $f_P(p) = f_Y(q^{-1}(p))\frac{d}{dp}q^{-1}(p)$. Then, the probability density function of P is expressed as

$$\begin{aligned} f_P(p) &= f_Y(q^{-1}(p))\frac{d}{dp}q^{-1}(p) = f_Y\left(p\sigma_T + \mu_T\right)\,\sigma_T \\ &= \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{p\sigma_T+\mu_T-\mu}{\sigma}\right)^2} = \frac{1}{\left(\sigma/\sigma_T\right)\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{p-\left(\frac{\mu-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2}. \end{aligned} \tag{5}$$

Since $q(y)$ is an increasing function, the denominator of Z_T is expressed as

$$\begin{aligned} \int_{x_l}^{x_u} f_Y(y)dy &= \int_{q(x_l)}^{q(x_u)} f_P(p)dp \\ &= \int_{z_{T_l}}^{z_{T_u}} \frac{1}{\left(\sigma/\sigma_T\right)\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{p-\left(\frac{\mu-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2} dp. \end{aligned} \tag{6}$$

Therefore, based on Eqs. (5) and (6), the probability density function of Z_T is obtained as

$$f_{Z_T}(z) = \frac{\frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{z-\left(\frac{u-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2}}{\int_{z_{T_l}}^{z_{T_u}}\frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{p-\left(\frac{u-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2}dp},\; z_{T_l} \le z \le z_{T_u}. \tag{7}$$

Finally, $E(Z_T) = 0$ and $V(Z_T) = 1$ as follows: $E(Z_T) = E\left(\frac{X_T - \mu_T}{\sigma_T}\right) = \frac{1}{\sigma_T}\left(E(X_T) - \mu_T\right) = \frac{1}{\sigma_T}(\mu_T - \mu_T) = 0$ and

$Var(Z_T) = Var\left(\frac{X_T - \mu_T}{\sigma_T}\right) = \frac{1}{\sigma_T^2} Var\left(X_T - \mu_T\right) = \frac{1}{\sigma_T^2}\left(\sigma_T^2 + 0\right) = 1,$

Q. E. D.

The results shown in this section are now consistent with the ones of the well-known standard normal distribution, and support the theoretical foundations of the standard truncated normal random variable which we propose in this dissertation.

3.2.2.2 Standardization of Left and Right TNDs

The probability density functions of standard left and right truncated normal distributions are shown in Table 3.2. It is noted that means and variances of the SLTND and SRTND are also zero and one, respectively.

Table 3.2. Probability density functions of standard left and right truncated normal distributions

	Probability Density Function
LTND	$f_{Z_T}(z) = \dfrac{\frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z - \left(\frac{u-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2}}{\int_{z_{T_l}}^{\infty} \frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{p - \left(\frac{u-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2} dp}$ where $z_{T_l} \leq z \leq \infty$
RTND	$f_{Z_T}(z) = \dfrac{\frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z - \left(\frac{u-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2}}{\int_{-\infty}^{z_{T_u}} \frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{p - \left(\frac{u-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2} dp}$ where $-\infty \leq z \leq z_{T_u}$

3.2.3 Simplifying PDF of the SDTND

In this section, a table of cumulative probabilities of the standard symmetric doubly truncated normal distribution is developed. When a random variable X_T with mean μ_T and variance σ_T^2 is doubly truncated and symmetric, its probability density function of X_T is expressed as $f_{X_T}(x) = \frac{\frac{1}{\sqrt{2\pi}\cdot\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\int_{x_l}^{x_u}\frac{1}{\sqrt{2\pi}\cdot\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx}$ where $x_l \le x \le x_u$. Since the distribution of X_T is symmetric,
$\mu_T = \mu$, $x_u - \mu = \mu - x_l$, $\phi\left(\frac{x_l-\mu}{\sigma}\right) = \phi\left(\frac{x_u-\mu}{\sigma}\right)$ and $\Phi\left(\frac{x_l-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{x_u-\mu}{\sigma}\right)$.

Fig. 3.6 shows a symmetric doubly truncated normal distribution where $x_u - \mu = \Delta$.

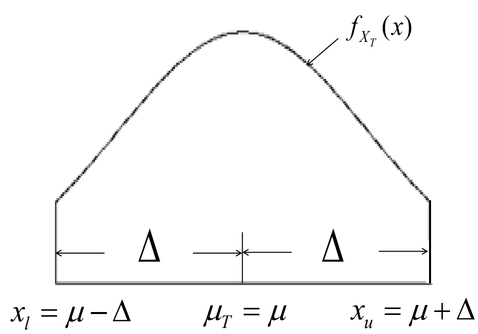


Figure 3.6. A plot of the symmetric doubly truncated normal distribution

Based on σ_T^2 in Table 2.1, the variance of X_T is expressed as

$$\begin{aligned}
\sigma_T^2 &= \sigma^2\left[1 + \frac{\frac{x_l-\mu}{\sigma}\cdot\phi\left(\frac{x_l-\mu}{\sigma}\right) - \frac{x_u-\mu}{\sigma}\cdot\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right) - \phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right] \\
&= \sigma^2\left[1 + \frac{-\frac{\Delta}{\sigma}\cdot\phi\left(-\frac{\Delta}{\sigma}\right) - \frac{\Delta}{\sigma}\cdot\phi\left(\frac{\Delta}{\sigma}\right)}{\Phi\left(\frac{\Delta}{\sigma}\right) - \left[1 - \Phi\left(\frac{\Delta}{\sigma}\right)\right]} - \left(\frac{\phi\left(-\frac{\Delta}{\sigma}\right) - \phi\left(\frac{\Delta}{\sigma}\right)}{\Phi\left(\frac{\Delta}{\sigma}\right) - \left[1 - \Phi\left(\frac{\Delta}{\sigma}\right)\right]}\right)^2\right] \\
&\quad \sigma^2\left[1 - \frac{2\frac{\Delta}{\sigma}\phi\left(\frac{\Delta}{\sigma}\right)}{2\Phi\left(\frac{\Delta}{\sigma}\right) - 1}\right]. \qquad (8)
\end{aligned}$$

The upper truncation point, z_{T_u}, of $Z_T = \frac{X_T - \mu_T}{\sigma_T}$ is written as

$$z_{T_u} = \frac{x_u - u_T}{\sigma_T} = \frac{\Delta}{\sigma\sqrt{1 - \frac{2\frac{\Delta}{\sigma}\phi\left(\frac{\Delta}{\sigma}\right)}{2\Phi\left(\frac{\Delta}{\sigma}\right)-1}}} \tag{9}$$

and $z_{T_u} = -z_{T_l}$. Therefore, the probability density function of Z_T is represented as

$$\begin{aligned} f_{Z_T}(z) &= \frac{\frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z-\left(\frac{u-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2}}{\int_{-z_{T_u}}^{z_{T_u}} \frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z-\left(\frac{u-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2} dz} \\ &\qquad \textit{where } \text{-}z_{T_u} \le z \le z_{T_u},\ z_{T_u} = \frac{x_u - u_T}{\sigma_T} \\ &= \frac{\frac{A}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(A\cdot z)^2}}{\int_{-\frac{\Delta}{\sigma\cdot A}}^{\frac{\Delta}{\sigma\cdot A}} \frac{A}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(A\cdot z)^2} dz} \\ &\qquad \textit{where } \text{-}\frac{\Delta}{\sigma\cdot A} \le z \le \frac{\Delta}{\sigma\cdot A},\ A = \sqrt{1 - \frac{2\frac{\Delta}{\sigma}\phi\left(\frac{\Delta}{\sigma}\right)}{2\Phi\left(\frac{\Delta}{\sigma}\right)-1}}. \end{aligned} \tag{10}$$

By denoting $k = \frac{\Delta}{\sigma}$, the probability density function of Z_T is expressed as

$$\begin{aligned} f_{Z_T}(z) &= \frac{\frac{B}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(B\cdot z)^2}}{\int_{-\frac{k}{B}}^{\frac{k}{B}} \frac{B}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(B\cdot z)^2} dz} \\ &\qquad \text{where } -\frac{k}{B} \le z \le \frac{k}{B},\ B = \sqrt{1 - \frac{2k\phi(k)}{2\Phi(k) - 1}} \end{aligned} \tag{11}$$

and the variance of X_T is given by

$$\sigma_T^2 = \sigma^2 \left[1 - \frac{2k\phi(k)}{2\Phi(k) - 1}\right]. \tag{12}$$

Hence, the cumulative distribution function of Z_T is written as

$$\begin{aligned} F_{Z_T}(z) &= \int_{-\infty}^{z} f_{Z_T}(y)dy \\ &\quad \textit{where } z_l \leq y \leq z_u,\ z_l = \frac{x_l - u_T}{\sigma_T},\ z_u = \frac{x_u - u_T}{\sigma_T} \\ &= \int_{-\infty}^{z} \frac{\frac{B}{\sqrt{2\pi}} e^{-\frac{1}{2}(B\cdot y)^2}}{\int_{-\frac{k}{B}}^{\frac{k}{B}} \frac{B}{\sqrt{2\pi}} e^{-\frac{1}{2}(B\cdot z)^2} dz} dy \\ &\quad \textit{where } -\frac{k}{B} \leq y \leq \frac{k}{B},\ B = \sqrt{1 - \frac{2k\phi(k)}{2\Phi(k) - 1}}. \end{aligned} \tag{13}$$

3.3 Development of a Cumulative Probability Table of the SDTND in a Symmetric Case

We are now ready to develop a table of cumulative probabilities of the standard doubly truncated normal distribution in a symmetric case. Once the values of Δ and σ are chosen, k can be determined since $k = \frac{\Delta}{\sigma}$; this relationship implies that we only need to consider k to decide its probability density function of Z_T. For example, consider a symmetric doubly truncated random normal variable X_{T_1} with $\Delta_1 = 1$ and $\sigma_1 = 5$. Also, consider a symmetric doubly truncated random normal variable X_{T_2} with $\Delta_2 = 1/2$ and $\sigma_2 = 2/5$. Then, both X_{T_1} and X_{T_2} have the same probability density function with $k = 1.5$.

The table of cumulative probabilities of Z_T based on Eq. (13) is shown in Table 3.7, where k values range between 0 and 6. When the value of k is greater than 6, the cumulative probability of Z_T is close to 1. It is noted that z_{T_u} increases as k increases, and $z_{T_u} = 6$ when $k = 6$. When the k values are 3 and 6, the z_{T_u} values become 3.041 and 6, respectively. The cumulative probabilities of the standard symmetric doubly truncated normal distribution shown in Table 3 are worked out numerically by the *Maple* software.

The cumulative probabilities for the doubly symmetric standard normal distribution are shown in Fig. 3.7.

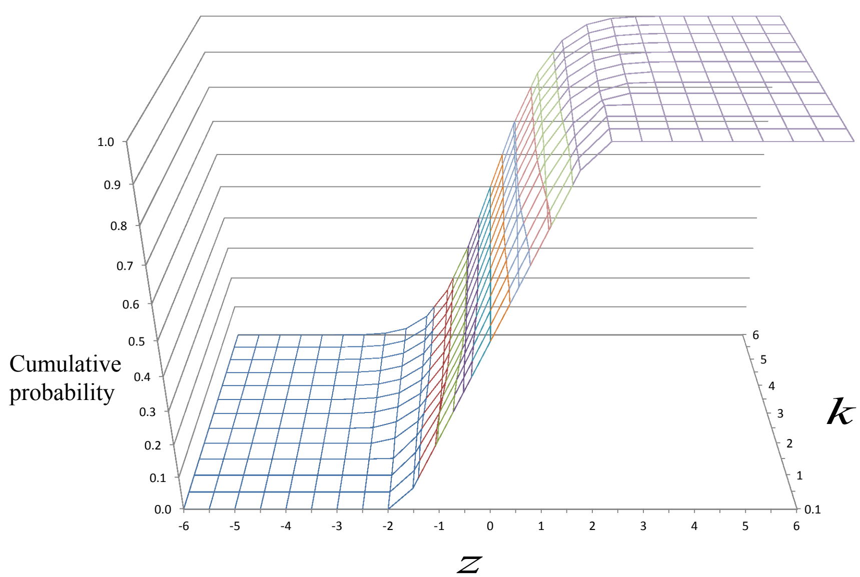


Figure 3.7. Cumulative area of the truncated standard normal distribution in a symmetric doubly truncated case

Table 3.3. Cumulative area of truncated standard normal distribution in a symmetric doubly truncated case

$k = \frac{\Delta}{\sigma}$	$z_{T_u} = \frac{k}{\sigma_T}$	z: -6.0	-5.9	-5.8	-5.7	-5.6	-5.5	-5.4	-5.3	-5.2	-5.1	-5.0	-4.9	-4.8	-4.7	-4.6	-4.5	-4.4	-4.3	-4.2	-4.1	-4.0
0.1	1.73320	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	1.73668	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.3	1.74249	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.4	1.75068	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.5	1.76129	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.6	1.77439	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.7	1.79006	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.8	1.80838	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.9	1.82944	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.0	1.85336	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.1	1.88025	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.2	1.91022	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.3	1.94339	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.4	1.97998	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.5	2.01980	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.6	2.06325	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.7	2.11031	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.8	2.16104	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.9	2.21550	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2.0	2.27369	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2.1	2.33561	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2.2	2.40119	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2.3	2.47035	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2.4	2.54297	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2.5	2.61890	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2.6	2.69796	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2.7	2.77991	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2.8	2.86454	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2.9	2.95159	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.0	3.04081	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.1	3.13194	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.2	3.22473	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.3	3.31894	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.4	3.41434	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.5	3.51074	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.6	3.60796	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.7	3.70583	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.8	3.80422	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.9	3.90303	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
4.0	4.00214	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
4.1	4.10150	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
4.2	4.20104	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00002
4.3	4.30071	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00002
4.4	4.40048	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00002	0.00003
4.5	4.50032	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00002	0.00003
4.6	4.60021	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00002	0.00003
4.7	4.70014	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00002	0.00003
4.8	4.80009	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00002	0.00003
4.9	4.90006	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00002	0.00003
5.0	5.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003
5.1	5.10002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003
5.2	5.20001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003
5.3	5.30001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003
5.4	5.40001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003
5.5	5.50000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003
5.6	5.60000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003
5.7	5.70000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003
5.8	5.80000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003
5.9	5.90000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003
6.0	6.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00002	0.00003

Table 3.3.

Continued

$k = \frac{\Delta}{\sigma}$	$z_{T_u} = \frac{k}{\sigma_T}$	z: -4.0	-3.9	-3.8	-3.7	-3.6	-3.5	-3.4	-3.3	-3.2	-3.1	-3.0	-2.9	-2.8	-2.7	-2.6	-2.5	-2.4	-2.3	-2.2	-2.1	-2.0
0.1	1.73320	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	1.73668	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.3	1.74249	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.4	1.75068	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.5	1.76129	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.6	1.77439	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.7	1.79006	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.8	1.80838	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.9	1.82944	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.0	1.85336	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.1	1.88025	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.2	1.91022	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.3	1.94339	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.4	1.97998	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.5	2.01980	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00222
1.6	2.06325	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00635
1.7	2.11031	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00086	0.00989
1.8	2.16104	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00453	0.01286
1.9	2.21550	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00094	0.00757	0.01532
2.0	2.27369	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00391	0.01007	0.01730
2.1	2.33561	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00151	0.00632	0.01207	0.01888
2.2	2.40119	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00004	0.00375	0.00824	0.01365	0.02010
2.3	2.47035	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00204	0.00552	0.00975	0.01486	0.02102
2.4	2.54297	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00097	0.00362	0.00689	0.01091	0.01581	0.02170
2.5	2.61890	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00033	0.00232	0.00483	0.00795	0.01180	0.01650	0.02218
2.6	2.69796	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00146	0.00336	0.00576	0.00875	0.01245	0.01699	0.02251
2.7	2.77991	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00091	0.00233	0.00415	0.00645	0.00934	0.01293	0.01735	0.02273
2.8	2.86454	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00055	0.00161	0.00298	0.00474	0.00697	0.00978	0.01327	0.01759	0.02286
2.9	2.95159	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00033	0.00111	0.00213	0.00346	0.00517	0.00735	0.01009	0.01351	0.01774	0.02292
3.0	3.04081	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00019	0.00076	0.00152	0.00252	0.00382	0.00549	0.00762	0.01031	0.01367	0.01784	0.02295
3.1	3.13194	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00011	0.00053	0.00108	0.00182	0.00280	0.00407	0.00571	0.00781	0.01046	0.01378	0.01789	0.02295
3.2	3.22473	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00006	0.00036	0.00077	0.00132	0.00205	0.00301	0.00426	0.00587	0.00794	0.01056	0.01385	0.01792	0.02294
3.3	3.31894	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00003	0.00025	0.00054	0.00095	0.00148	0.00220	0.00315	0.00439	0.00599	0.00803	0.01063	0.01388	0.01793	0.02291
3.4	3.41434	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00002	0.00017	0.00038	0.00067	0.00107	0.00160	0.00231	0.00325	0.00448	0.00606	0.00809	0.01067	0.01391	0.01793	0.02289
3.5	3.51074	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00012	0.00027	0.00048	0.00077	0.00116	0.00169	0.00239	0.00332	0.00454	0.00612	0.00813	0.01070	0.01392	0.01792	0.02286
3.6	3.60796	0.00000	0.00000	0.00000	0.00000	0.00000	0.00008	0.00019	0.00034	0.00055	0.00083	0.00122	0.00175	0.00245	0.00337	0.00458	0.00615	0.00816	0.01071	0.01392	0.01792	0.02284
3.7	3.70583	0.00000	0.00000	0.00000	0.00000	0.00005	0.00013	0.00024	0.00038	0.00059	0.00088	0.00126	0.00179	0.00248	0.00340	0.00461	0.00617	0.00818	0.01072	0.01392	0.01791	0.02282
3.8	3.80422	0.00000	0.00000	0.00000	0.00004	0.00009	0.00016	0.00027	0.00042	0.00062	0.00091	0.00129	0.00181	0.00251	0.00343	0.00463	0.00619	0.00819	0.01073	0.01392	0.01790	0.02280
3.9	3.90303	0.00000	0.00000	0.00003	0.00006	0.00011	0.00019	0.00029	0.00044	0.00065	0.00093	0.00131	0.00183	0.00252	0.00344	0.00464	0.00620	0.00819	0.01073	0.01392	0.01789	0.02279
4.0	4.00214	0.00000	0.00002	0.00004	0.00008	0.00013	0.00020	0.00031	0.00045	0.00066	0.00094	0.00133	0.00184	0.00254	0.00345	0.00465	0.00620	0.00820	0.01073	0.01391	0.01788	0.02278
4.1	4.10150	0.00001	0.00003	0.00005	0.00009	0.00014	0.00021	0.00032	0.00046	0.00067	0.00095	0.00133	0.00185	0.00254	0.00346	0.00465	0.00621	0.00820	0.01073	0.01391	0.01788	0.02277
4.2	4.20104	0.00002	0.00003	0.00005	0.00009	0.00015	0.00022	0.00032	0.00047	0.00068	0.00096	0.00134	0.00186	0.00255	0.00346	0.00466	0.00621	0.00820	0.01073	0.01391	0.01787	0.02276
4.3	4.30071	0.00002	0.00004	0.00006	0.00010	0.00015	0.00022	0.00033	0.00048	0.00068	0.00096	0.00134	0.00186	0.00255	0.00346	0.00466	0.00621	0.00820	0.01073	0.01391	0.01787	0.02276
4.4	4.40048	0.00003	0.00004	0.00007	0.00010	0.00016	0.00023	0.00033	0.00048	0.00068	0.00097	0.00135	0.00187	0.00255	0.00346	0.00466	0.00621	0.00820	0.01073	0.01391	0.01787	0.02276
4.5	4.50032	0.00003	0.00004	0.00007	0.00010	0.00016	0.00023	0.00033	0.00048	0.00068	0.00097	0.00135	0.00187	0.00255	0.00347	0.00466	0.00621	0.00820	0.01073	0.01391	0.01787	0.02275
4.6	4.60021	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00255	0.00347	0.00466	0.00621	0.00820	0.01073	0.01391	0.01787	0.02275
4.7	4.70014	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00255	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01787	0.02275
4.8	4.80009	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00255	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01787	0.02275
4.9	4.90006	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00255	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01787	0.02275
5.0	5.00004	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00255	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01787	0.02275
5.1	5.10002	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00256	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01786	0.02275
5.2	5.20001	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00256	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01786	0.02275
5.3	5.30001	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00256	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01786	0.02275
5.4	5.40001	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00256	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01786	0.02275
5.5	5.50000	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00256	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01786	0.02275
5.6	5.60000	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00256	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01786	0.02275
5.7	5.70000	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00256	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01786	0.02275
5.8	5.80000	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00256	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01786	0.02275
5.9	5.90000	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00256	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01786	0.02275
6.0	6.00000	0.00003	0.00005	0.00007	0.00011	0.00016	0.00023	0.00034	0.00048	0.00069	0.00097	0.00135	0.00187	0.00256	0.00347	0.00466	0.00621	0.00820	0.01072	0.01390	0.01786	0.02275

Table 3.3.

Continued

$k = \frac{\Delta}{\sigma}$	$z_{T_u} = \frac{k}{\sigma_T}$	z: -2.0	-1.9	-1.8	-1.7	-1.6	-1.5	-1.4	-1.3	-1.2	-1.1	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0
0.1	1.73320	0.00000	0.00000	0.00000	0.00955	0.03831	0.06709	0.09589	0.12470	0.15352	0.18235	0.21120	0.24005	0.26891	0.29778	0.32666	0.35554	0.38442	0.41332	0.44221	0.47110	0.50000
0.2	1.73668	0.00000	0.00000	0.00000	0.01042	0.03889	0.06741	0.09599	0.12463	0.15331	0.18204	0.21081	0.23962	0.26847	0.29734	0.32624	0.35517	0.38411	0.41307	0.44204	0.47102	0.50000
0.3	1.74249	0.00000	0.00000	0.00000	0.01184	0.03982	0.06792	0.09616	0.12451	0.15296	0.18152	0.21018	0.23891	0.26773	0.29661	0.32556	0.35455	0.38359	0.41266	0.44176	0.47088	0.50000
0.4	1.75068	0.00000	0.00000	0.00000	0.01375	0.04106	0.06860	0.09636	0.12432	0.15247	0.18080	0.20929	0.23793	0.26671	0.29560	0.32461	0.35370	0.38287	0.41210	0.44137	0.47068	0.50000
0.5	1.76129	0.00000	0.00000	0.00000	0.01607	0.04257	0.06941	0.09659	0.12407	0.15184	0.17988	0.20817	0.23669	0.26541	0.29432	0.32340	0.35262	0.38195	0.41138	0.44088	0.47043	0.50000
0.6	1.77439	0.00000	0.00000	0.00000	0.01871	0.04428	0.07032	0.09681	0.12373	0.15106	0.17876	0.20681	0.23519	0.26386	0.29279	0.32195	0.35132	0.38085	0.41052	0.44029	0.47013	0.50000
0.7	1.79006	0.00000	0.00000	0.00000	0.02157	0.04612	0.07127	0.09701	0.12331	0.15013	0.17745	0.20524	0.23345	0.26205	0.29101	0.32027	0.34981	0.37957	0.40952	0.43960	0.46978	0.50000
0.8	1.80838	0.00000	0.00000	0.00187	0.02456	0.04802	0.07223	0.09716	0.12278	0.14906	0.17597	0.20346	0.23149	0.26002	0.28901	0.31838	0.34812	0.37814	0.40840	0.43883	0.46939	0.50000
0.9	1.82944	0.00000	0.00000	0.00614	0.02758	0.04992	0.07314	0.09723	0.12214	0.14784	0.17431	0.20149	0.22933	0.25779	0.28680	0.31631	0.34625	0.37656	0.40716	0.43798	0.46895	0.50000
1.0	1.85336	0.00000	0.00000	0.01035	0.03054	0.05175	0.07398	0.09720	0.12138	0.14649	0.17250	0.19935	0.22700	0.25538	0.28443	0.31407	0.34424	0.37485	0.40582	0.43706	0.46849	0.50000
1.1	1.88025	0.00000	0.00000	0.01440	0.03336	0.05347	0.07470	0.09705	0.12049	0.14501	0.17055	0.19707	0.22451	0.25281	0.28190	0.31169	0.34210	0.37304	0.40440	0.43609	0.46799	0.50000
1.2	1.91022	0.00000	0.00163	0.01820	0.03599	0.05501	0.07527	0.09677	0.11949	0.14341	0.16848	0.19467	0.22191	0.25013	0.27926	0.30921	0.33987	0.37114	0.40291	0.43506	0.46747	0.50000
1.3	1.94339	0.00000	0.00629	0.02168	0.03836	0.05635	0.07569	0.09636	0.11837	0.14170	0.16632	0.19217	0.21921	0.24736	0.27654	0.30664	0.33757	0.36919	0.40138	0.43401	0.46693	0.50000
1.4	1.97998	0.00000	0.01049	0.02479	0.04044	0.05747	0.07593	0.09582	0.11715	0.13991	0.16408	0.18962	0.21646	0.24454	0.27377	0.30403	0.33522	0.36720	0.39982	0.43294	0.46638	0.50000
1.5	2.01980	0.00222	0.01421	0.02752	0.04222	0.05836	0.07599	0.09514	0.11583	0.13806	0.16180	0.18703	0.21369	0.24170	0.27098	0.30141	0.33287	0.36520	0.39825	0.43186	0.46583	0.50000
1.6	2.06325	0.00635	0.01743	0.02985	0.04369	0.05901	0.07589	0.09436	0.11444	0.13616	0.15951	0.18444	0.21093	0.23889	0.26822	0.29881	0.33053	0.36322	0.39670	0.43079	0.46529	0.50000
1.7	2.11031	0.00989	0.02017	0.03179	0.04486	0.05945	0.07563	0.09347	0.11300	0.13425	0.15722	0.18189	0.20821	0.23612	0.26551	0.29627	0.32824	0.36128	0.39517	0.42974	0.46476	0.50000
1.8	2.16104	0.01286	0.02244	0.03337	0.04575	0.05967	0.07524	0.09250	0.11153	0.13235	0.15498	0.17940	0.20558	0.23344	0.26289	0.29381	0.32603	0.35940	0.39370	0.42873	0.46424	0.50000
1.9	2.21550	0.01531	0.02428	0.03461	0.04638	0.05972	0.07473	0.09148	0.11006	0.13049	0.15281	0.17701	0.20305	0.23088	0.26039	0.29145	0.32392	0.35761	0.39230	0.42776	0.46375	0.50000
2.0	2.27369	0.01730	0.02575	0.03554	0.04679	0.05962	0.07413	0.09044	0.10860	0.12869	0.15073	0.17473	0.20066	0.22845	0.25802	0.28924	0.32193	0.35592	0.39097	0.42685	0.46328	0.50000
2.1	2.33561	0.01888	0.02688	0.03622	0.04701	0.05939	0.07348	0.08939	0.10720	0.12698	0.14877	0.17260	0.19842	0.22620	0.25582	0.28718	0.32008	0.35435	0.38974	0.42600	0.46285	0.50000
2.2	2.40119	0.02010	0.02773	0.03667	0.04707	0.05907	0.07279	0.08835	0.10585	0.12537	0.14695	0.17062	0.19636	0.22411	0.25380	0.28528	0.31839	0.35291	0.38861	0.42522	0.46246	0.50000
2.3	2.47035	0.02102	0.02833	0.03695	0.04702	0.05869	0.07209	0.08736	0.10459	0.12388	0.14527	0.16881	0.19448	0.22223	0.25197	0.28357	0.31685	0.35161	0.38759	0.42452	0.46210	0.50000
2.4	2.54297	0.02170	0.02875	0.03709	0.04688	0.05827	0.07141	0.08642	0.10343	0.12252	0.14376	0.16719	0.19279	0.22054	0.25033	0.28203	0.31548	0.35045	0.38668	0.42389	0.46178	0.50000
2.5	2.61890	0.02218	0.02901	0.03712	0.04668	0.05784	0.07076	0.08556	0.10237	0.12129	0.14240	0.16574	0.19130	0.21904	0.24888	0.28068	0.31427	0.34942	0.38588	0.42334	0.46150	0.50000
2.6	2.69796	0.02251	0.02916	0.03709	0.04645	0.05742	0.07015	0.08477	0.10142	0.12021	0.14121	0.16447	0.18999	0.21774	0.24762	0.27950	0.31322	0.34853	0.38518	0.42286	0.46125	0.50000
2.7	2.77991	0.02273	0.02923	0.03700	0.04621	0.05702	0.06959	0.08407	0.10059	0.11927	0.14018	0.16338	0.18887	0.21661	0.24653	0.27849	0.31231	0.34777	0.38458	0.42245	0.46104	0.50000
2.8	2.86454	0.02286	0.02924	0.03688	0.04597	0.05665	0.06909	0.08346	0.09987	0.11846	0.13930	0.16244	0.18791	0.21566	0.24562	0.27764	0.31155	0.34712	0.38408	0.42211	0.46087	0.50000
2.9	2.95159	0.02292	0.02921	0.03676	0.04574	0.05632	0.06866	0.08293	0.09926	0.11777	0.13855	0.16166	0.18711	0.21486	0.24485	0.27693	0.31092	0.34658	0.38366	0.42182	0.46072	0.50000
3.0	3.04081	0.02295	0.02916	0.03663	0.04553	0.05602	0.06829	0.08248	0.09874	0.11719	0.13793	0.16101	0.18644	0.21421	0.24422	0.27634	0.31039	0.34614	0.38331	0.42158	0.46060	0.50000
3.1	3.13194	0.02295	0.02910	0.03651	0.04534	0.05577	0.06798	0.08211	0.09831	0.11672	0.13742	0.16048	0.18590	0.21367	0.24370	0.27586	0.30997	0.34578	0.38303	0.42139	0.46050	0.50000
3.2	3.22473	0.02294	0.02904	0.03640	0.04518	0.05556	0.06772	0.08180	0.09797	0.11634	0.13701	0.16005	0.18547	0.21324	0.24329	0.27548	0.30963	0.34550	0.38281	0.42123	0.46042	0.50000
3.3	3.31894	0.02291	0.02898	0.03630	0.04505	0.05539	0.06750	0.08155	0.09769	0.11603	0.13669	0.15971	0.18512	0.21290	0.24296	0.27518	0.30936	0.34527	0.38263	0.42111	0.46036	0.50000
3.4	3.41434	0.02289	0.02893	0.03622	0.04493	0.05525	0.06734	0.08136	0.09747	0.11580	0.13643	0.15944	0.18486	0.21264	0.24271	0.27494	0.30915	0.34509	0.38249	0.42102	0.46031	0.50000
3.5	3.51074	0.02286	0.02888	0.03615	0.04485	0.05514	0.06720	0.08121	0.09730	0.11561	0.13623	0.15924	0.18465	0.21243	0.24251	0.27476	0.30899	0.34496	0.38238	0.42094	0.46028	0.50000
3.6	3.60796	0.02284	0.02884	0.03610	0.04478	0.05505	0.06710	0.08109	0.09716	0.11546	0.13608	0.15908	0.18449	0.21228	0.24236	0.27462	0.30887	0.34485	0.38230	0.42089	0.46025	0.50000
3.7	3.70583	0.02282	0.02881	0.03605	0.04472	0.05498	0.06702	0.08100	0.09706	0.11535	0.13596	0.15896	0.18437	0.21216	0.24225	0.27452	0.30877	0.34478	0.38224	0.42085	0.46023	0.50000
3.8	3.80422	0.02280	0.02879	0.03602	0.04468	0.05493	0.06696	0.08093	0.09699	0.11527	0.13588	0.15887	0.18428	0.21207	0.24217	0.27444	0.30871	0.34472	0.38220	0.42082	0.46021	0.50000
3.9	3.90303	0.02279	0.02877	0.03600	0.04465	0.05489	0.06692	0.08088	0.09693	0.11521	0.13582	0.15881	0.18422	0.21201	0.24211	0.27439	0.30866	0.34468	0.38217	0.42079	0.46020	0.50000
4.0	4.00214	0.02278	0.02875	0.03598	0.04462	0.05487	0.06688	0.08084	0.09689	0.11517	0.13577	0.15876	0.18417	0.21196	0.24206	0.27435	0.30862	0.34465	0.38214	0.42078	0.46019	0.50000
4.1	4.10150	0.02277	0.02874	0.03596	0.04461	0.05485	0.06686	0.08082	0.09686	0.11514	0.13574	0.15873	0.18413	0.21193	0.24203	0.27432	0.30859	0.34463	0.38213	0.42077	0.46019	0.50000
4.2	4.20104	0.02276	0.02873	0.03595	0.04459	0.05483	0.06684	0.08080	0.09684	0.11512	0.13572	0.15871	0.18411	0.21191	0.24201	0.27430	0.30858	0.34461	0.38211	0.42076	0.46018	0.50000
4.3	4.30071	0.02276	0.02873	0.03595	0.04458	0.05482	0.06683	0.08078	0.09683	0.11510	0.13700	0.15869	0.18409	0.21189	0.24200	0.27428	0.30856	0.34460	0.38211	0.42075	0.46018	0.50000
4.4	4.40048	0.02276	0.02873	0.03594	0.04458	0.05481	0.06682	0.08078	0.09682	0.11509	0.13569	0.15868	0.18408	0.21188	0.24198	0.27427	0.30855	0.34459	0.38210	0.42075	0.46018	0.50000
4.5	4.50032	0.02275	0.02872	0.03594	0.04457	0.05481	0.06682	0.08077	0.09681	0.11508	0.13568	0.15867	0.18408	0.21187	0.24198	0.27427	0.30855	0.34459	0.38210	0.42075	0.46018	0.50000
4.6	4.60021	0.02275	0.02872	0.03593	0.04457	0.05481	0.06681	0.08076	0.09681	0.11508	0.13568	0.15867	0.18407	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
4.7	4.70014	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11508	0.13567	0.15866	0.18407	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
4.8	4.80009	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
4.9	4.90006	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
5.0	5.00004	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
5.1	5.10002	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
5.2	5.20001	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
5.3	5.30001	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
5.4	5.40001	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
5.5	5.50000	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
5.6	5.60000	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
5.7	5.70000	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
5.8	5.80000	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
5.9	5.90000	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000
6.0	6.00000	0.02275	0.02872	0.03593	0.04457	0.05480	0.06681	0.08076	0.09681	0.11507	0.13567	0.15866	0.18406	0.21186	0.24197	0.27426	0.30854	0.34458	0.38209	0.42074	0.46017	0.50000

Table 3.3.

Continued

$k = \frac{\Delta}{\sigma}$	$z_{T_u} = \frac{k}{\sigma_T}$	z 0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0.1	1.73320	0.50000	0.52890	0.55779	0.58668	0.61558	0.64446	0.67334	0.70222	0.73109	0.75995	0.78880	0.81765	0.84648	0.87530	0.90411	0.93291	0.96169	0.99045	1.00000	1.00000	1.00000
0.2	1.73668	0.50000	0.52898	0.55796	0.58693	0.61589	0.64483	0.67376	0.70266	0.73153	0.76038	0.78919	0.81796	0.84669	0.87537	0.90401	0.93259	0.96111	0.98958	1.00000	1.00000	1.00000
0.3	1.74249	0.50000	0.52912	0.55824	0.58734	0.61641	0.64545	0.67444	0.70339	0.73227	0.76109	0.78982	0.81848	0.84704	0.87549	0.90384	0.93208	0.96018	0.98816	1.00000	1.00000	1.00000
0.4	1.75068	0.50000	0.52932	0.55863	0.58790	0.61713	0.64630	0.67539	0.70440	0.73329	0.76207	0.79071	0.81920	0.84753	0.87568	0.90364	0.93140	0.95894	0.98625	1.00000	1.00000	1.00000
0.5	1.76129	0.50000	0.52957	0.55912	0.58862	0.61805	0.64738	0.67660	0.70568	0.73459	0.76331	0.79183	0.82012	0.84816	0.87593	0.90341	0.93059	0.95743	0.98393	1.00000	1.00000	1.00000
0.6	1.77439	0.50000	0.52987	0.55971	0.58948	0.61915	0.64868	0.67805	0.70721	0.73614	0.76481	0.79319	0.82124	0.84894	0.87627	0.90319	0.92968	0.95572	0.98129	1.00000	1.00000	1.00000
0.7	1.79006	0.50000	0.53022	0.56040	0.59048	0.62043	0.65019	0.67973	0.70899	0.73795	0.76655	0.79476	0.82255	0.84987	0.87669	0.90299	0.92873	0.95388	0.97843	1.00000	1.00000	1.00000
0.8	1.80838	0.50000	0.53061	0.56117	0.59160	0.62186	0.65188	0.68161	0.71099	0.73998	0.76851	0.79654	0.82403	0.85094	0.87722	0.90284	0.92777	0.95198	0.97544	0.99813	1.00000	1.00000
0.9	1.82944	0.50000	0.53105	0.56202	0.59284	0.62344	0.65375	0.68369	0.71320	0.74221	0.77067	0.79851	0.82569	0.85216	0.87786	0.90277	0.92686	0.95008	0.97242	0.99386	1.00000	1.00000
1.0	1.85336	0.50000	0.53151	0.56294	0.59418	0.62515	0.65580	0.68593	0.71557	0.74462	0.77300	0.80065	0.82750	0.85351	0.87862	0.90280	0.92602	0.94825	0.96946	0.98965	1.00000	1.00000
1.1	1.88025	0.50000	0.53201	0.56391	0.59560	0.62696	0.65790	0.68831	0.71810	0.74719	0.77549	0.80293	0.82945	0.85499	0.87951	0.90295	0.92530	0.94653	0.96664	0.98560	1.00000	1.00000
1.2	1.91022	0.50000	0.53253	0.56494	0.59709	0.62886	0.66013	0.69079	0.72074	0.74987	0.77809	0.80533	0.83152	0.85659	0.88051	0.90323	0.92473	0.94499	0.96401	0.98180	0.99837	1.00000
1.3	1.94339	0.50000	0.53307	0.56599	0.59862	0.63081	0.66243	0.69336	0.72346	0.75264	0.78079	0.80783	0.83368	0.85830	0.88163	0.90364	0.92431	0.94365	0.96164	0.97832	0.99371	1.00000
1.4	1.97998	0.50000	0.53362	0.56706	0.60018	0.63280	0.66478	0.69597	0.72623	0.75546	0.78354	0.81038	0.83592	0.86009	0.88285	0.90418	0.92407	0.94253	0.95956	0.97521	0.98951	1.00000
1.5	2.01980	0.50000	0.53417	0.56814	0.60175	0.63480	0.66713	0.69859	0.72902	0.75830	0.78631	0.81297	0.83820	0.86194	0.88417	0.90486	0.92401	0.94164	0.95778	0.97248	0.98579	0.99778
1.6	2.06325	0.50000	0.53471	0.56921	0.60330	0.63678	0.66947	0.70119	0.73178	0.76111	0.78907	0.81556	0.84049	0.86384	0.88556	0.90564	0.92411	0.94098	0.95631	0.97015	0.98257	0.99365
1.7	2.11031	0.50000	0.53524	0.57026	0.60483	0.63872	0.67176	0.70373	0.73449	0.76388	0.79179	0.81811	0.84278	0.86575	0.88700	0.90653	0.92437	0.94055	0.95514	0.96821	0.97983	0.99011
1.8	2.16104	0.50000	0.53576	0.57127	0.60630	0.64060	0.67397	0.70619	0.73711	0.76656	0.79442	0.82060	0.84501	0.86765	0.88847	0.90750	0.92476	0.94033	0.95425	0.96663	0.97756	0.98714
1.9	2.21550	0.50000	0.53625	0.57224	0.60770	0.64239	0.67608	0.70855	0.73961	0.76912	0.79695	0.82299	0.84719	0.86951	0.88994	0.90852	0.92527	0.94028	0.95362	0.96539	0.97571	0.98468
2.0	2.27369	0.50000	0.53672	0.57315	0.60903	0.64408	0.67807	0.71076	0.74198	0.77155	0.79934	0.82527	0.84927	0.87131	0.89140	0.90956	0.92587	0.94038	0.95321	0.96446	0.97425	0.98270
2.1	2.33561	0.50000	0.53715	0.57400	0.61026	0.64565	0.67992	0.71282	0.74418	0.77380	0.80158	0.82740	0.85123	0.87302	0.89280	0.91061	0.92652	0.94061	0.95299	0.96378	0.97312	0.98112
2.2	2.40119	0.50000	0.53754	0.57478	0.61139	0.64709	0.68161	0.71472	0.74620	0.77588	0.80364	0.82938	0.85305	0.87463	0.89415	0.91165	0.92721	0.94093	0.95293	0.96333	0.97227	0.97990
2.3	2.47035	0.50000	0.53790	0.57548	0.61241	0.64839	0.68315	0.71643	0.74803	0.77777	0.80552	0.83118	0.85473	0.87612	0.89541	0.91264	0.92791	0.94131	0.95298	0.96305	0.97167	0.97898
2.4	2.54297	0.50000	0.53822	0.57611	0.61332	0.64955	0.68452	0.71797	0.74967	0.77946	0.80721	0.83281	0.85624	0.87748	0.89657	0.91358	0.92859	0.94173	0.95312	0.96291	0.97125	0.97830
2.5	2.61890	0.50000	0.53850	0.57666	0.61412	0.65058	0.68573	0.71932	0.75112	0.78096	0.80870	0.83426	0.85760	0.87871	0.89763	0.91444	0.92924	0.94216	0.95332	0.96288	0.97099	0.97782
2.6	2.69796	0.50000	0.53875	0.57714	0.61482	0.65147	0.68678	0.72050	0.75238	0.78226	0.81001	0.83553	0.85879	0.87979	0.89858	0.91523	0.92985	0.94258	0.95355	0.96291	0.97084	0.97749
2.7	2.77991	0.50000	0.53896	0.57755	0.61542	0.65223	0.68769	0.72151	0.75347	0.78339	0.81113	0.83662	0.85982	0.88073	0.89941	0.91593	0.93041	0.94298	0.95379	0.96300	0.97077	0.97727
2.8	2.86454	0.50000	0.53913	0.57789	0.61592	0.65288	0.68845	0.72236	0.75438	0.78434	0.81209	0.83756	0.86070	0.88154	0.90013	0.91654	0.93091	0.94335	0.95403	0.96312	0.97076	0.97714
2.9	2.95159	0.50000	0.53928	0.57818	0.61634	0.65342	0.68908	0.72307	0.75515	0.78514	0.81289	0.83834	0.86145	0.88223	0.90074	0.91707	0.93134	0.94368	0.95426	0.96324	0.97079	0.97708
3.0	3.04081	0.50000	0.53940	0.57842	0.61669	0.65386	0.68961	0.72366	0.75578	0.78579	0.81355	0.83899	0.86207	0.88281	0.90126	0.91752	0.93171	0.94398	0.95447	0.96337	0.97084	0.97705
3.1	3.13194	0.50000	0.53950	0.57861	0.61697	0.65422	0.69003	0.72414	0.75630	0.78633	0.81410	0.83952	0.86258	0.88328	0.90169	0.91789	0.93202	0.94423	0.95466	0.96349	0.97090	0.97705
3.2	3.22473	0.50000	0.53958	0.57877	0.61719	0.65450	0.69037	0.72452	0.75671	0.78676	0.81453	0.83995	0.86299	0.88366	0.90203	0.91820	0.93228	0.94444	0.95482	0.96360	0.97096	0.97706
3.3	3.31894	0.50000	0.53964	0.57889	0.61737	0.65473	0.69064	0.72482	0.75704	0.78710	0.81487	0.84029	0.86331	0.88397	0.90231	0.91845	0.93250	0.94461	0.95495	0.96370	0.97102	0.97709
3.4	3.41434	0.50000	0.53968	0.57898	0.61751	0.65491	0.69085	0.72506	0.75729	0.78736	0.81514	0.84056	0.86357	0.88421	0.90253	0.91864	0.93266	0.94475	0.95507	0.96378	0.97107	0.97711
3.5	3.51074	0.50000	0.53972	0.57906	0.61762	0.65504	0.69101	0.72524	0.75749	0.78757	0.81535	0.84076	0.86377	0.88439	0.90270	0.91879	0.93280	0.94486	0.95515	0.96385	0.97112	0.97714
3.6	3.60796	0.50000	0.53975	0.57911	0.61770	0.65515	0.69113	0.72538	0.75764	0.78772	0.81551	0.84092	0.86392	0.88454	0.90284	0.91891	0.93290	0.94495	0.95523	0.96390	0.97116	0.97716
3.7	3.70583	0.50000	0.53977	0.57915	0.61776	0.65522	0.69123	0.72548	0.75775	0.78784	0.81563	0.84104	0.86404	0.88465	0.90294	0.91900	0.93298	0.94502	0.95528	0.96395	0.97119	0.97718
3.8	3.80422	0.50000	0.53979	0.57918	0.61780	0.65528	0.69129	0.72556	0.75783	0.78793	0.81572	0.84113	0.86412	0.88473	0.90301	0.91907	0.93304	0.94507	0.95532	0.96398	0.97121	0.97720
3.9	3.90303	0.50000	0.53980	0.57921	0.61783	0.65532	0.69134	0.72561	0.75789	0.78799	0.81578	0.84119	0.86418	0.88479	0.90307	0.91912	0.93308	0.94511	0.95535	0.96400	0.97123	0.97721
4.0	4.00214	0.50000	0.53981	0.57922	0.61786	0.65535	0.69138	0.72565	0.75794	0.78804	0.81583	0.84124	0.86423	0.88483	0.90311	0.91916	0.93312	0.94513	0.95538	0.96402	0.97125	0.97722
4.1	4.10150	0.50000	0.53981	0.57923	0.61787	0.65537	0.69141	0.72568	0.75797	0.78807	0.81587	0.84127	0.86426	0.88486	0.90313	0.91918	0.93314	0.94515	0.95539	0.96404	0.97126	0.97723
4.2	4.20104	0.50000	0.53982	0.57924	0.61789	0.65539	0.69142	0.72570	0.75799	0.78809	0.81589	0.84129	0.86428	0.88488	0.90316	0.91920	0.93316	0.94517	0.95541	0.96405	0.97127	0.97724
4.3	4.30071	0.50000	0.53982	0.57925	0.61789	0.65540	0.69144	0.72572	0.75800	0.78811	0.81591	0.84131	0.86430	0.88490	0.90317	0.91922	0.93317	0.94518	0.95542	0.96405	0.97127	0.97724
4.4	4.40048	0.50000	0.53982	0.57925	0.61790	0.65541	0.69145	0.72573	0.75802	0.78812	0.81592	0.84132	0.86431	0.88491	0.90318	0.91922	0.93318	0.94519	0.95542	0.96406	0.97127	0.97724
4.5	4.50032	0.50000	0.53983	0.57925	0.61790	0.65541	0.69145	0.72573	0.75802	0.78813	0.81592	0.84133	0.86432	0.88492	0.90319	0.91923	0.93318	0.94519	0.95543	0.96406	0.97128	0.97725
4.6	4.60021	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72574	0.75803	0.78814	0.81593	0.84133	0.86432	0.88492	0.90319	0.91924	0.93318	0.94519	0.95543	0.96407	0.97128	0.97725
4.7	4.70014	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72574	0.75803	0.78814	0.81593	0.84133	0.86433	0.88492	0.90319	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
4.8	4.80009	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72574	0.75803	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
4.9	4.90006	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72574	0.75803	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
5.0	5.00004	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
5.1	5.10002	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
5.2	5.20001	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
5.3	5.30001	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
5.4	5.40001	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
5.5	5.50000	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
5.6	5.60000	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
5.7	5.70000	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
5.8	5.80000	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
5.9	5.90000	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725
6.0	6.00000	0.50000	0.53983	0.57926	0.61791	0.65542	0.69146	0.72575	0.75804	0.78814	0.81594	0.84134	0.86433	0.88493	0.90320	0.91924	0.93319	0.94520	0.95543	0.96407	0.97128	0.97725

Table 3.3.

Continued

$k = \frac{\Delta}{\sigma}$	$z_{T_u} = \frac{k}{\sigma_T}$	z 2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0
0.1	1.73320	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.2	1.73668	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.3	1.74249	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.4	1.75068	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.5	1.76129	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.6	1.77439	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.7	1.79006	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.8	1.80838	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.9	1.82944	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.0	1.85336	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.1	1.88025	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.2	1.91022	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.3	1.94339	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.4	1.97998	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.5	2.01980	0.99778	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.6	2.06325	0.99365	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.7	2.11031	0.99011	0.99914	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.8	2.16104	0.98714	0.99547	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.9	2.21550	0.98468	0.99243	0.99906	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.0	2.27369	0.98270	0.98993	0.99609	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.1	2.33561	0.98112	0.98793	0.99368	0.99849	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.2	2.40119	0.97990	0.98635	0.99176	0.99625	0.99996	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.3	2.47035	0.97898	0.98512	0.99025	0.99448	0.99796	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99981	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.4	2.54297	0.97830	0.98419	0.98909	0.99311	0.99638	0.99903	1.00000	1.00000	1.00000	1.00000	1.00000	0.99981	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.5	2.61890	0.97782	0.98350	0.98820	0.99205	0.99517	0.99768	0.99967	1.00000	1.00000	1.00000	1.00000	0.99981	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.6	2.69796	0.97749	0.98301	0.98755	0.99125	0.99424	0.99554	0.99854	1.00000	1.00000	1.00000	1.00000	0.99981	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.7	2.77991	0.97727	0.98265	0.98673	0.99022	0.99303	0.99526	0.99702	0.99839	0.99945	1.00000	1.00000	0.99981	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.8	2.86454	0.97714	0.98241	0.98707	0.99066	0.99355	0.99585	0.99767	0.99909	1.00000	1.00000	1.00000	0.99981	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.9	2.95159	0.97708	0.98226	0.98649	0.98991	0.99265	0.99483	0.99654	0.99787	0.99889	0.99967	1.00000	0.99981	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.0	3.04081	0.97705	0.98216	0.98633	0.98969	0.99238	0.99451	0.99618	0.99748	0.99848	0.99924	0.99981	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.1	3.13194	0.97705	0.98211	0.98622	0.98951	0.99219	0.99429	0.99593	0.99720	0.99817	0.99891	0.99947	0.99989	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.2	3.22473	0.97706	0.98208	0.98615	0.98944	0.99206	0.99413	0.99574	0.99699	0.99795	0.99868	0.99923	0.99964	0.99994	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.3	3.31894	0.97709	0.98207	0.98612	0.98937	0.99197	0.99401	0.99561	0.99685	0.99780	0.99852	0.99905	0.99946	0.99975	0.99997	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.4	3.41434	0.97711	0.98207	0.98609	0.98933	0.99191	0.99394	0.99552	0.99675	0.99769	0.99840	0.99893	0.99933	0.99962	0.99983	0.99998	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.5	3.51074	0.97714	0.98208	0.98608	0.98930	0.99187	0.99388	0.99546	0.99668	0.99761	0.99831	0.99884	0.99923	0.99952	0.99973	0.99988	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000
3.6	3.60796	0.97716	0.98208	0.98608	0.98929	0.99184	0.99385	0.99542	0.99663	0.99755	0.99825	0.99878	0.99917	0.99945	0.99966	0.99981	0.99991	1.00000	1.00000	1.00000	1.00000	1.00000
3.7	3.70583	0.97718	0.98209	0.98608	0.98928	0.99182	0.99383	0.99539	0.99660	0.99752	0.99821	0.99874	0.99912	0.99941	0.99962	0.99976	0.99987	0.99995	1.00000	1.00000	1.00000	1.00000
3.8	3.80422	0.97720	0.98210	0.98608	0.98927	0.99181	0.99381	0.99537	0.99657	0.99749	0.99819	0.99871	0.99909	0.99938	0.99958	0.99973	0.99984	0.99991	0.99996	1.00000	1.00000	1.00000
3.9	3.90303	0.97721	0.98211	0.98608	0.98927	0.99181	0.99380	0.99536	0.99656	0.99748	0.99817	0.99869	0.99907	0.99935	0.99956	0.99971	0.99981	0.99989	0.99994	0.99997	1.00000	1.00000
4.0	4.00214	0.97722	0.98212	0.98609	0.98927	0.99180	0.99380	0.99535	0.99655	0.99746	0.99816	0.99867	0.99906	0.99934	0.99955	0.99969	0.99980	0.99987	0.99992	0.99996	0.99998	1.00000
4.1	4.10150	0.97723	0.98212	0.98609	0.98927	0.99180	0.99379	0.99535	0.99654	0.99746	0.99815	0.99867	0.99905	0.99933	0.99954	0.99968	0.99979	0.99986	0.99991	0.99995	0.99997	0.99999
4.2	4.20104	0.97724	0.98213	0.98609	0.98927	0.99180	0.99379	0.99534	0.99654	0.99745	0.99814	0.99866	0.99904	0.99932	0.99953	0.99968	0.99978	0.99985	0.99991	0.99994	0.99997	0.99998
4.3	4.30071	0.97724	0.98213	0.98609	0.98927	0.99180	0.99379	0.99534	0.99654	0.99745	0.99814	0.99865	0.99904	0.99932	0.99952	0.99967	0.99978	0.99985	0.99990	0.99994	0.99996	0.99998
4.4	4.40048	0.97724	0.98213	0.98609	0.98927	0.99180	0.99379	0.99534	0.99654	0.99745	0.99814	0.99865	0.99904	0.99932	0.99952	0.99967	0.99977	0.99985	0.99990	0.99993	0.99996	0.99997
4.5	4.50032	0.97725	0.98213	0.98609	0.98927	0.99180	0.99379	0.99534	0.99653	0.99745	0.99814	0.99865	0.99904	0.99932	0.99952	0.99967	0.99977	0.99984	0.99990	0.99993	0.99996	0.99997
4.6	4.60021	0.97725	0.98213	0.98609	0.98927	0.99180	0.99379	0.99534	0.99653	0.99746	0.99814	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
4.7	4.70014	0.97725	0.98213	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
4.8	4.80009	0.97725	0.98213	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
4.9	4.90006	0.97725	0.98213	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
5.0	5.00004	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
5.1	5.10002	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
5.2	5.20001	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
5.3	5.30001	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
5.4	5.40001	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
5.5	5.50000	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
5.6	5.60000	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
5.7	5.70000	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
5.8	5.80000	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
5.9	5.90000	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997
6.0	6.00000	0.97725	0.98214	0.98610	0.98928	0.99180	0.99379	0.99534	0.99653	0.99745	0.99813	0.99865	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.99993	0.99995	0.99997

Table 3.3.

Continued

$k = \frac{\Delta}{\sigma}$	$z_{T_u} = \frac{k}{\sigma_T}$	z: 4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0
0.1	1.73320	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.2	1.73668	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.3	1.74249	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.4	1.75068	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.5	1.76129	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.6	1.77439	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.7	1.79006	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.8	1.80838	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.9	1.82944	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.0	1.85336	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.1	1.88025	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.2	1.91022	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.3	1.94339	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.4	1.97998	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.5	2.01980	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.6	2.06325	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.7	2.11031	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.8	2.16104	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.9	2.21550	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00001	1.00001	1.00001	1.00002	1.00000
2.0	2.27369	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.1	2.33561	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.2	2.40119	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.3	2.47035	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.4	2.54297	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00001	1.00001	1.00001	1.00002	1.00000
2.5	2.61890	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.6	2.69796	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.7	2.77991	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.8	2.86454	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.9	2.95159	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.0	3.04081	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00001	1.00001	1.00001	1.00002	1.00000
3.1	3.13194	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.2	3.22473	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.3	3.31894	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.4	3.41434	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.5	3.51074	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.6	3.60796	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.7	3.70583	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.8	3.80422	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.9	3.90303	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.0	4.00214	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.1	4.10150	0.99998	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.2	4.20104	0.99997	0.99998	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.3	4.30071	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.4	4.40048	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.5	4.50032	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.6	4.60021	0.99997	0.99998	0.99999	99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.7	4.70014	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00001	1.00001	1.00001	1.00002	1.00000
4.8	4.80009	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.9	4.90006	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5.0	5.00004	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5.1	5.10002	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5.2	5.20001	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5.3	5.30001	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5.4	5.40001	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5.5	5.50000	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5.6	5.60000	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5.7	5.70000	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5.8	5.80000	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5.9	5.90000	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
6.0	6.00000	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

3.4 Numerical Example

As an example, Table 3.4 shows the procedure to develop the standard doubly truncated normal distribution. If $\mu = 2$, $\sigma = 2$, $x_l = 2$ and $x_u = 4$, the probability function of X_T is obtained as

$$f_{X_T}(x) = \frac{\frac{1}{2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-2}{2}\right)^2}}{\int_2^4 \frac{1}{2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y-2}{2}\right)^2} dy},\ 2 \le x \le 4.$$

From Table 1, $\mu_T = 2$ and $\sigma_T = 1.079$, and consequently, $\frac{\mu-\mu_T}{\sigma_T} = 0$ and $\frac{\sigma}{\sigma_T} = 1.853$. Moreover, the lower and upper truncation points of Z_T are calculated and obtained as $z_{T_l} = \frac{x_l-\mu_T}{\sigma_T} = -1.853$ and $z_{T_u} = \frac{x_u-\mu_T}{\sigma_T} = 1.853$. Then, we obtain the probability density function of Z_T $f_{Z_T}(z) = \frac{\frac{1}{1.853\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z}{1.853}\right)^2}}{\int_{-1.853}^{1.853} \frac{1}{1.853\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{p}{1.853}\right)^2} dp}$ where $-1.853 \le z \le 1.853$. $E(Z_T)$ and $Var(Z_T)$ are then obtained as $E(Z_T) = \int_{-\infty}^{\infty} z\, f_{Z_T}(z)dz = 0$ and $Var(Z_T) = \int_{-\infty}^{\infty} z^2\, f_{Z_T}(z)dz - \left(\int_{-\infty}^{\infty} z\, f_{Z_T}(z)dz\right)^2 = 1$.

Fig. 3.8. shows the density plots of the random variables X_T and Z_T defined in Table 3.4 for the numerical example.

X_T (before standardization) $\rightarrow$ Z_T (after standardization)

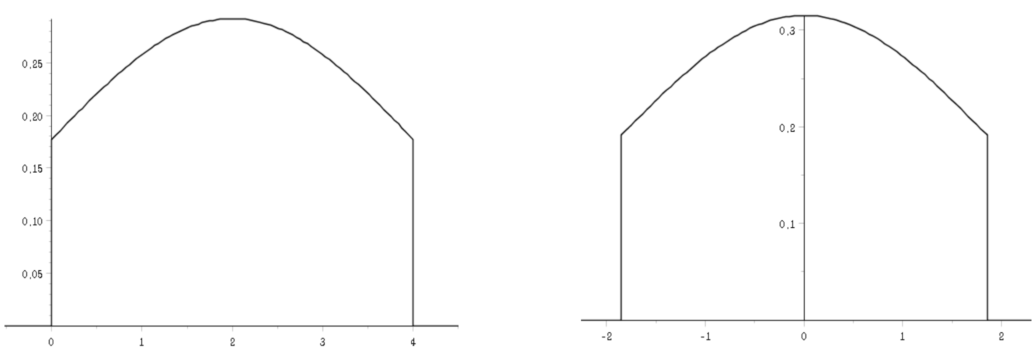


Figure 3.8. Density plots of X_T and Z_T

Table 3.4. The procedure to develop the standard doubly truncated normal distribution and its mean and variance

Given	$\mu = 2,\ \sigma = 2,\ x_l = 0,\ x_u = 4$
PDF of X_T	$f_{X_T}(x) = \dfrac{\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\int_{x_l}^{x_u} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{p-\mu}{\sigma}\right)^2} dp},\ x_l \le x \le x_u$
	$= \dfrac{\frac{1}{2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-2}{2}\right)^2}}{\int_{0}^{4} \frac{1}{2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{p-2}{2}\right)^2} dp},\ 0 \le x \le 4.$
Find	$\mu_T = \mu + \dfrac{\phi\left(\frac{x_l-\mu}{\sigma}\right) - \phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)}\sigma = 2,$
	$\sigma_T = \sigma \cdot \sqrt{\left[1 + \dfrac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right) - \frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)} - \left(\dfrac{\phi\left(\frac{x_l-\mu}{\sigma}\right) - \phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]} = 1.079,$
	$\frac{u-\mu_T}{\sigma_T} = 0,\ \frac{\sigma}{\sigma_T} = 1.853,\ z_{T_l} = \frac{x_l-u_T}{\sigma_T} = -1.853,\ and\ z_{T_u} = \frac{x_u-u_T}{\sigma_T} = 1.853.$
PDF of Z_T	$f_{Z_T}(z) = \dfrac{\frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z-\left(\frac{\mu-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2}}{\int_{z_{T_l}}^{z_{T_u}} \frac{1}{(\sigma/\sigma_T)\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z-\left(\frac{\mu-\mu_T}{\sigma_T}\right)}{\sigma/\sigma_T}\right)^2} dz}$
	$where\ \ z_{T_l} \le z \le z_{T_u},\ z_{T_l} = \frac{x_l-u_T}{\sigma_T},\ and\, z_{T_u} = \frac{x_u-u_T}{\sigma_T}$
	$= \dfrac{\frac{1}{1.853\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z}{1.853}\right)^2}}{\int_{-1.853}^{1.853} \frac{1}{1.853\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z}{1.853}\right)^2} dz},\ -1.853 \le z \le 1.853.$
$E(Z_T)$	$E(Z_T) = \int_{-\infty}^{\infty} z\, f_{Z_T}(z) dz$
	$= \int_{-1.853}^{1.853} z\, \dfrac{\frac{1}{1.853\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z}{1.853}\right)^2}}{\int_{-1.853}^{1.853} \frac{1}{1.853\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{p}{1.853}\right)^2} dp} dz$
	$= 0.$
$Var(Z_T)$	$Var(Z_T) = \int_{-\infty}^{\infty} z^2\, f_{Z_T}(z) dz - \left(\int_{-\infty}^{\infty} z\, f_{Z_T}(z) dz\right)^2$
	$= \int_{-1.853}^{1.853} z^2\, \dfrac{\frac{1}{1.853\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z}{1.853}\right)^2}}{\int_{-1.853}^{1.853} \frac{1}{1.853\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{p}{1.853}\right)^2} dp} dz$
	$- \left(\int_{-3.358}^{3.358} z\, \dfrac{\frac{1}{1.853\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z}{1.853}\right)^2}}{\int_{-1.853}^{1.853} \frac{1}{1.853\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{p}{1.853}\right)^2} dp} dz\right)^2$
	$= 1.$

3.5 Conclusions and Future Work

There are practical necessities in which a truncated normal distribution is required to be considered. This dissertation developed the probability density function of a standard doubly truncated normal distribution, and showed that the mean and variance of the standard truncated normal distribution are always zero and one regardless of its truncation points. Based on the cumulative distribution function of a standard truncated random variable, we also developed the cumulative probability table of the standard truncated normal distribution in a symmetric case, which might be useful for practitioners.

One interesting fact we observed is that the standard truncated normal distribution is the same probability density function once two different truncated normal distributions have the same k values where $k = \frac{\Delta}{\sigma}$. Mathematical proofs were performed in order to compare the variances between the normal distribution and its truncated normal distributions. We then verified that the variance of the truncated normal distribution is always smaller than the one of its original normal distribution. As a future study, the cumulative probability tables of standard left and right truncated normal distributions need to be developed. Due to the fact that both left and right truncated normal distributions are not symmetric, it is believed that one mathematical hurdle we need to overcome would be the curse of dimensionality associated with the conditions of k. Note that the function of k is the expression of the ratio of the difference between the truncation point of interest in the asymmetric case, such as lower and upper truncation points, and its untruncated standard deviation. In other words, the simple condition associated with k we derived in Section 3.2.3, cannot be applied to the cases of the standard asymmetric doubly truncated normal distributions. We encourage researchers to develop the simplified conditions associated with k in the asymmetric cases which can map into one set of the cumulative probability tables.

CHAPTER FOUR

DEVELOPMENT OF STATISTICAL INFERENCE FROM A TND

In this chapter, statistical inference for a truncated normal distribution associated with Research Question 3 is developed. Note that we consider large truncated samples to assure the appropriate use of the Central Limit Theorem throughout this chapter. In Section 4.1, two proposed theorems are provided to prove the Central Limit Theorem within the truncated normal environment. Section 4.2 examines how the Central Limit Theorem works based on different sample sizes from four types of a truncated normal distribution by performing simulations. We then identify the methodologies for the new statistical inference theory in Section 4.3. The confidence intervals and hypothesis tests which are of critical importance in order to give the direct answers to Research Question 3, are developed in Sections 4.4, 4.5 and 4.6, respectively. A numerical example follows in Section 4.7. Finally, we discuss the conclusions and future work in Section 4.8.

4.1 Mathematical Proofs of the Central Limit Theorem for a TND

It is well known that the limiting form of the distribution of a sample mean, $\overline{X}$, is the standard normal distribution as the sample size goes infinity, if $X_1, X_2, \ldots, X_n$ is an independently, identically distributed random sample from a normal population with a finite variance. The Central Limit Theorem says that the distribution of the mean of a random sample taken from any population with a finite variance converges to the standard normal distribution as the sample size becomes large. As discussed in Cha *et al.* (2014), the variance of its truncated normal distribution, σ_T, becomes finite if the variance of the normal distribution is finite. Sections 4.1.1 and 4.1.2 provide proposed theorems which prove the Central Limit Theorem for a truncated normal distribution by using the moment generating function and characteristic function, respectively.

4.1.1 Moment Generating Function

For the mathematical proof, we assume the finiteness of the moment generating function of X_T which implies the finiteness of all the moments.

Proposed Theorem 5 Let $X_{T_1}, X_{T_2}, \ldots, X_{T_n}$ be independent and identically distributed truncated normal random variables with mean, μ_T, variance, σ_T^2, where $\sigma_T^2 < \infty$, and the probability density function $f_{X_{T_i}}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} / \int_{x_l}^{x_u} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$ where $x_l \leq x \leq x_u$ for $i = 1, 2, \ldots, n$. Suppose that all of the moments are finite. That is, $M_{X_T}(t)$ converges for $|t| < \delta$ for some positive δ. Then, the random variable $\sqrt{n}\left(\overline{X}_T - \mu_T\right)/\sigma_T$ where $\overline{X}_T = \left(X_{T_1} + \cdots + X_{T_n}\right)/n$ is approximately normally distributed when n is large. That is, $\sqrt{n}\left(\overline{X}_T - \mu_T\right)/\sigma_T \rightarrow N(0, 1)$.

Proof

We define the k^{th} moment of X_T as μ'_{T_k}. By the definition of moment, the k^{th} moment is written as $\mu'_{T_k} = E[X_T^k] = \int_{-\infty}^{\infty} x^k f_{X_T}(x)dx$. It is noted that $\mu_T = \mu'_{T_1}$ since $\mu'_{T_1} = \int_{-\infty}^{\infty} x^k f_{X_T}(x)dx = \mu_T$. By definition, the moment generating function of X_T is written as $M_{X_T}(t) = E\left[e^{tX_T}\right]$ for $t \in \mathbb{R}$.The random variable, $\sqrt{n}\left(\overline{X}_T - \mu_T\right)/\sigma_T$, is expressed as

$$\sqrt{n}\left(\overline{X}_T-\mu_T\right)/\sigma_T=\frac{\sqrt{n}\left[\frac{X_{T_1}+\cdots+X_{T_n}}{n}-\mu_T\right]}{\sigma_T}=\frac{X_{T_1}+\cdots+X_{T_n}-n\mu_T}{\sqrt{n}\sigma_T}=\sum_{i=1}^{n}\left[\left(X_{T_i}-\mu_T\right)/\sigma_T\sqrt{n}\right].$$

Thus, the moment generating function of $\sqrt{n}\left(\overline{X}_{T_n}-\mu_T\right)/\sigma_T$ is obtained as

$$\begin{aligned}
M_{\frac{\overline{X}_T-\mu_T}{\sigma_T/\sqrt{n}}}(t) &= M_{\sum_{i=1}^{n}\left(\frac{X_{T_i}-\mu_T}{\sigma_T\sqrt{n}}\right)}(t)=M_{\frac{X_{T_1}-\mu_T}{\sigma_T\sqrt{n}}+\frac{X_{T_2}-\mu_T}{\sigma_T\sqrt{n}}+\cdots+\frac{X_{T_n}-\mu_T}{\sigma_T\sqrt{n}}}(t)\\
&= E\left[e^{t\cdot\left(\frac{X_{T_1}-\mu_T}{\sigma_T\sqrt{n}}+\frac{X_{T_2}-\mu_T}{\sigma_T\sqrt{n}}+\cdots+\frac{X_{T_n}-\mu_T}{\sigma_T\sqrt{n}}\right)}\right]=E\left[e^{t\cdot\frac{X_{T_1}-\mu_T}{\sigma_T\sqrt{n}}}e^{t\cdot\frac{X_{T_2}-\mu_T}{\sigma_T\sqrt{n}}}\cdots e^{t\cdot\frac{X_{T_n}-\mu_T}{\sigma_T\sqrt{n}}}\right]\\
&= E\left[e^{t\cdot\frac{X_{T_1}-\mu_T}{\sigma_T\sqrt{n}}}\right]E\left[e^{t\cdot\frac{X_{T_2}-\mu_T}{\sigma_T\sqrt{n}}}\right]\cdots E\left[e^{t\cdot\frac{X_{T_n}-\mu_T}{\sigma_T\sqrt{n}}}\right]\\
&= \prod_{i=1}^{n}E\left[e^{t\cdot\frac{X_{T_i}-\mu_T}{\sigma_T\sqrt{n}}}\right]=\prod_{i=1}^{n}e^{\frac{-\mu_T t}{\sigma_T\sqrt{n}}}E\left[e^{t\cdot\frac{X_{T_i}}{\sigma_T\sqrt{n}}}\right]=\prod_{i=1}^{n}e^{\frac{-\mu_T t}{\sigma_T\sqrt{n}}}M_{\frac{X_{T_i}}{\sigma_T\sqrt{n}}}(t)\\
&= \prod_{i=1}^{n}e^{\frac{-\mu_T t}{\sigma_T\sqrt{n}}}M_{X_{T_i}}\left(\frac{t}{\sigma_T\sqrt{n}}\right)=e^{\frac{-\mu_T t\sqrt{n}}{\sigma_T}}M_{X_T}\left(\frac{t}{\sigma_T\sqrt{n}}\right)^{n}. \qquad (14)
\end{aligned}$$

Note that $M_{X_{T_i}}\left(t/\sigma_T\sqrt{n}\right)$ is written as $M_{X_T}\left(t/\sigma_T\sqrt{n}\right)^n$ since each $M_{X_{T_i}}\left(t/\sigma_T\sqrt{n}\right)$ is identically distributed for $i=1,2,\cdots,n$. Additionally, since the logarithm of a product is the sum of the logarithms, Eq. (14) is expressed as

$$\log M_{\frac{\overline{X}_T-\mu_T}{\sigma_T/\sqrt{n}}}(t)=-\frac{\mu_T t\sqrt{n}}{\sigma_T}+n\log M_{X_T}\left(\frac{t}{\sigma_T\sqrt{n}}\right). \qquad (15)$$

By using the Talyor series expansion using the exponential function $e^{tx}=\sum_{j=0}^{\infty}(tx)^j/j!$ and the convergence of the moments where $M_{X_T}(t)$ converges for $|t|<\delta$ for some positive δ, we have

$$M_{X_T}(t) = E\left[e^{tX_T}\right]=\int_{-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{x^j t^j}{j!}f_{X_T}(x)dx=\sum_{j=0}^{\infty}\frac{t^j}{j!}\int_{-\infty}^{\infty}x^j f_{X_T}(x)dx. \qquad (16)$$

Since $\int_{-\infty}^{\infty} x^j f_{X_T}(x)dx$ is μ'_{T_j}, $M_{X_T}(t)$ is obtained as

$$\begin{aligned}
M_{X_T}(t) &= \sum_{j=0}^{\infty}\frac{t^j}{j!}\int_{-\infty}^{\infty} x^j f_{X_T}(x)dx = \sum_{j=0}^{\infty}\frac{t^j}{j!}\mu'_{T_j} \\
&= 1+\mu'_{T_1}t+\frac{\mu'_{T_2}t^2}{2!}+\frac{\mu'_{T_3}t^3}{3!}+\cdots \\
&= 1+\mu_T t+\frac{\mu'_{T_2}t^2}{2!}+\frac{\mu'_{T_3}t^3}{3!}+\cdots \\
&= 1+t\left(\mu_T+\frac{\mu'_{T_2}t}{2!}+\frac{\mu'_{T_3}t^2}{3!}+\cdots\right). \qquad (17)
\end{aligned}$$

Expanding $\log(1+a)$ into a Taylor series where

$\log(1+a) = a - a^2/2! + a^3/3! - a^4/4! + \cdots$, we have

$$\begin{aligned}
\log M_{X_T}(t) &= \log\left[1+t\left(\mu_T+\frac{\mu'_{T_2}t}{2!}+\frac{\mu'_{T_3}t^2}{3!}+\cdots\right)\right] \\
&= t\left(\mu_T+\frac{\mu'_{T_2}t}{2!}+\frac{\mu'_{T_3}t^2}{3!}+\cdots\right) - \frac{t^2\left(\mu_T+\frac{\mu'_{T_2}t}{2!}+\frac{\mu'_{T_3}t^2}{3!}+\cdots\right)^2}{2!}+\cdots \\
&= \mu_T t + \frac{\mu'_{T_2}-\mu_T^2}{2}t^2 + O\left(t^3\right) \qquad (18)
\end{aligned}$$

where $O(t^3)$ represents higher-order terms in t. Thus, $\log M_{X_T}(t/\sigma_T\sqrt{n})$ is expressed as

$$\log M_{X_T}\left(\frac{t}{\sigma_T\sqrt{n}}\right) = \frac{\mu_T t}{\sigma_T\sqrt{n}} + \frac{\mu'_{T_2}-\mu_T^2}{2}\frac{t^2}{\sigma_T^2 n} + O\left(1/n^{3/2}\right) \qquad (19)$$

where $O\left(1/n^{3/2}\right)$ represents lower-order terms in n. Eq. (15) is then written as

$$
\begin{aligned}
\log M_{\frac{\overline{X}_T-\mu_T}{\sigma_T/\sqrt{n}}}(t) &= -\frac{\mu_T t\sqrt{n}}{\sigma_T} + n\log M_{X_T}\left(\frac{t}{\sigma_T\sqrt{n}}\right) \\
&= -\frac{\mu_T t\sqrt{n}}{\sigma_T} + n\left[\frac{\mu_T t}{\sigma_T\sqrt{n}} + \frac{\mu'_{T_2}-\mu_T^2}{2}\frac{t^2}{\sigma_T^2 n} + O\left(n^{-3/2}\right)\right] \\
&= -\frac{\mu_T t\sqrt{n}}{\sigma_T} + \frac{\mu_T t\sqrt{n}}{\sigma_T} + \frac{\mu'_{T_2}-\mu_T^2}{2}\frac{t^2}{\sigma_T^2} + O\left(n^{-1/2}\right) \\
&= \frac{\mu'_{T_2}-\mu_T^2}{2}\frac{t^2}{\sigma_T^2} + O\left(n^{-1/2}\right) = \frac{\sigma_T^2}{2}\frac{t^2}{\sigma_T^2} + O\left(n^{-1/2}\right) \\
&= \frac{t^2}{2} + O\left(n^{-1/2}\right). \qquad (20)
\end{aligned}
$$

Note that $\mu'_{T_2} - \mu_T^2 = E[X_T^2] - E[X_T]^2 = \sigma_T^2$. Thus, we have

$$
\begin{aligned}
\log M_{\frac{\overline{X}_T-\mu_T}{\sigma_T/\sqrt{n}}}(t) &= \frac{\mu'_{T_2}-\mu_T^2}{2}\frac{t^2}{\sigma_T^2} + O\left(n^{-1/2}\right) \\
&= \frac{\sigma_T^2}{2}\frac{t^2}{\sigma_T^2} + O\left(n^{-1/2}\right) \\
&= \frac{t^2}{2} + O\left(n^{-1/2}\right). \qquad (21)
\end{aligned}
$$

Therefore, Eq. (21) can be expressed as

$$
M_{\frac{\overline{X}_T-\mu_T}{\sigma_T/\sqrt{n}}}(t) = e^{\frac{t^2}{2}+O\left(n^{-1/2}\right)}. \qquad (22)
$$

Meanwhile, according to the definition of $M_{X_T}(t)$, the moment generating function of the standard normal random variable, Z, whose probability density function $f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\, z^2}$ where $-\infty \le z \le \infty$, is expressed as

$$
\begin{aligned}
M_Z(t) &= E\left[e^{tZ}\right] = \int_{-\infty}^{\infty} e^{tz} f_Z(z) dz = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{tz-\frac{1}{2}z^2} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{\frac{2tz-z^2}{2}} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{\frac{t^2-t^2+2tz-z^2}{2}} dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{\frac{t^2}{2}-\frac{(z-t)^2}{2}} dz = e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(z-t)^2}{2}} dz = e^{\frac{t^2}{2}}. \qquad (23)
\end{aligned}
$$

Finally, based on Eqs. (22) and (23), we conclude $\sqrt{n}\left(\overline{X}_T - \mu_T\right)/\sigma_T \to N\left(0, 1\right),$

Q. E. D.

Notice that the limiting form of the distribution of $\overline{X_T}$ as $n \to \infty$ is the normal distribution with mean, μ_T, and variance, σ_T^2/n. That is, $\overline{X}_T \sim N(\mu_T, \sigma_T^2/n)$.

4.1.2 Characteristic Function

The characteristic function, which is always in existence for any real-valued random variable, is considered in this section.

Proposed Theorem 6 Let $X_{T_1}, X_{T_2}, \ldots, X_{T_n}$ be independent and identically distributed truncated normal random variables with mean μ_T where $\mu_T < \infty$, variance σ_T^2 where $\sigma_T^2 < \infty$, and probability density function $f_{X_T}(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} / \int_{x_l}^{x_u} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$ where $x_l \le x \le x_u$. Then, the random variable $\sqrt{n}\left(\overline{X_{T_n}} - \mu_T\right)/\sigma_T$ where $\overline{X}_{T_n} = \left(X_{T_1} + X_{T_2} + \cdots + X_{T_n}\right)/n$ is approximately normally distributed when n is large. That is, $\sqrt{n}\left(\overline{X}_{T_n} - \mu_T\right)/\sigma_T \to N\left(0, 1\right)$.

Proof

Let Z_{T_i} and be $\left(X_{T_i} - \mu_T\right)/\sigma_T$ and let $\overline{Z}_{T_n}$= $\left(Z_{T_1} + Z_{T_2} \cdots + Z_{T_n}\right)/n$. It is noted that $\sqrt{n}\overline{Z}_{T_n} = \sqrt{n}\left(Z_{T_1} + Z_{T_2} \cdots + Z_{T_n}\right)/n =$
$\sqrt{n}\left(X_{T_1} + X_{T_2} + \cdots + X_{T_n} - n\mu_T\right)/n\sigma_T = \sqrt{n}\left\{\left(X_{T_1} + X_{T_2} + \cdots + X_{T_n}\right)/n\right.$
$\left.-\mu_T/\sigma_T\right\} = \sqrt{n}\left(\overline{X}_{T_n} - \mu_T\right)/\sigma_T$.

We first show $E\left[\sqrt{n}\left(\overline{X}_{T_n} - \mu_T\right)/\sigma_T\right] = 0$ and $Var\left[\sqrt{n}\left(\overline{X}_{T_n} - \mu_T\right)/\sigma_T\right]) = 1$. Since $E(Z_{T_i}) = E\left[\left(X_{T_i} - \mu_T\right)/\sigma_T\right] = 0$ and $Var(Z_{T_i}) = Var\left[\left(X_{T_i} - \mu_T\right)/\sigma_T\right] = 1$, the mean and variance of $\sqrt{n}\left(\overline{X}_{T_n} - \mu_T\right)/\sigma_T = \sqrt{n}\overline{Z}_{T_n}$ are given by
$E(\sqrt{n}\overline{Z}_{T_n})$= $E\left[\sqrt{n}\left(Z_{T_1} + Z_{T_2} \cdots + Z_{T_n}\right)/n\right]$= $E\left[\left(Z_{T_1} + Z_{T_2} \cdots + Z_{T_n}\right)\right]/\sqrt{n} = 0$ and $Var(\sqrt{n}\overline{Z}_{T_n})$= $Var\left[\sqrt{n}\left(Z_{T_1} + Z_{T_2} \cdots + Z_{T_n}\right)/n\right]$=
$Var\left[\left(Z_{T_1} + Z_{T_2} \cdots + Z_{T_n}\right)\right]/n = 1$.

Now we show that $\sqrt{n}\left(\overline{X}_{T_n} - \mu_T\right)/\sigma_T$ has an approximate normal distribution. By definition, the characteristic function of Z_T is written as $\varphi_{Z_T}(t) = E\left[e^{itZ_T}\right] = \int e^{itZ_T} dF$ for $t \in \mathbb{R}$. So, when $t = 0$, we have $\varphi_{Z_T}(0) = E\left(1\right) = 1$. Meanwhile, the derivative of $\varphi_{Z_T}(t)$ is given by $\varphi'_{Z_T}(t) = \frac{d}{dt}E\left(e^{itZ_T}\right) = E\left(\frac{d}{dt}e^{itZ_T}\right) = E\left(iZ_Te^{itZ_T}\right)$ and thus $\varphi'_{Z_T}(0) = E\left(iZ_T\right) = iE\left(Z_T\right) = 0$. Moreover, the second derivative of $\varphi_{Z_T}(t)$ is obtained as $\varphi''_{Z_T}(t) = \frac{d}{dt}\varphi'_{Z_T}(t) = \frac{d}{dt}E\left(iZ_Te^{itZ_T}\right) = \frac{d}{dt}E\left(i^2Z_T^2e^{itZ_T}\right)$ and hence $\varphi''_{Z_T}(0) = E\left(i^2Z_T^2\right) = i^2E\left(Z_T^2\right) = i^2\left[Var(Z_T) + E\left(Z_T\right)^2\right] = i^2\left(1 + 0\right) = -1$.

Let $g(t)$= $\log\varphi_{Z_T}(t)$. Then, we have $\varphi_{Z_T}(t) = e^{g(t)}$. Based on $\varphi_{Z_T}(t) = e^{g(t)}$, the first and second derivatives are given by $g'(t) = \frac{d}{dt}\log\varphi_{Z_T}(t) = \frac{\varphi'_{Z_T}(t)}{\varphi_{Z_T}(t)}$ and $g''(t) = \frac{d}{dt}g'(t) = \frac{d}{dt}\frac{\varphi'_{Z_T}(t)}{\varphi_{Z_T}(t)} = \frac{\varphi''_{Z_T}(t)}{\varphi_{Z_T}(t)} - \left[\frac{\varphi'_{Z_T}(t)}{\varphi_{Z_T}(t)}\right]^2$, respectively. Therefore, when the value of t is zero, $g(0)=\log\varphi_{Z_T}(0)=0$, $g'(0)=\frac{d}{dt}\log\varphi_{Z_T}(0)=\frac{\varphi'_{Z_T}(0)}{\varphi_{Z_T}(0)}=0$ and $g''(0)=\frac{-1}{1}-\left(\frac{0}{1}\right)^2=-1$.

By using the Maclaurin expansion of $g(t)$, $g(t)$ is obtained as

$g(t)$= $g(0)+tg'(0)+\frac{t^2}{2!}g''(0)+O(t^2)0+0-\frac{1}{2}t^2+O(t^2)=-\frac{1}{2}t^2+O(t^2)$ for t near zero. Hence, the characteristic function of $\sqrt{n}\left(\overline{X}_{T_n}-\mu_T\right)=\sqrt{n}\overline{Z}_{T_n}$ is written as

$$
\begin{aligned}
\varphi_{\sqrt{n}\overline{Z}_{T_n}}(t) &= \varphi_{\frac{Z_{T_1}+Z_{T_2}+\cdots+Z_{T_n}}{\sqrt{n}}}(t)=\varphi_{\frac{Z_{T_1}}{\sqrt{n}}}(t)\cdot\varphi_{\frac{Z_{T_2}}{\sqrt{n}}}(t)\cdot\cdots\cdot\varphi_{\frac{Z_{T_n}}{\sqrt{n}}}(t)\\
&= \varphi_{\frac{Z_T}{\sqrt{n}}}(t)\cdot\varphi_{\frac{Z_T}{\sqrt{n}}}(t)\cdot\cdots\cdot\varphi_{\frac{Z_T}{\sqrt{n}}}(t)\\
&= E\left(e^{it\frac{Z_T}{\sqrt{n}}}\right)\cdot E\left(e^{it\frac{Z_T}{\sqrt{n}}}\right)\cdot\cdots\cdot E\left(e^{it\frac{Z_T}{\sqrt{n}}}\right)\\
&= E\left(e^{i\frac{t}{\sqrt{n}}Z_T}\right)\cdot E\left(e^{i\frac{t}{\sqrt{n}}Z_T}\right)\cdot\cdots\cdot E\left(e^{i\frac{t}{\sqrt{n}}Z_T}\right)\\
&= \varphi_{Z_T}(\frac{t}{\sqrt{n}})\cdot\varphi_{Z_T}(\frac{t}{\sqrt{n}})\cdot\cdots\cdot\varphi_{Z_T}(\frac{t}{\sqrt{n}})\\
&= \left[\varphi_{Z_T}(\frac{t}{\sqrt{n}})\right]^n=\left[e^{g\left(\frac{t}{\sqrt{n}}\right)}\right]^n=e^{ng\left(\frac{t}{\sqrt{n}}\right)}=e^{n\left\{-\frac{1}{2}\left(\frac{t}{\sqrt{n}}\right)^2+O\left[\left(\frac{t}{\sqrt{n}}\right)^2\right]\right\}}\\
&= e^{-\frac{1}{2}t^2+nO\left(\frac{t^2}{n}\right)}=e^{-\frac{1}{2}t^2+O\left(t^2\right)}\approx e^{-\frac{1}{2}t^2}. \qquad (24)
\end{aligned}
$$

We proved that the random variable $\sqrt{n}\left(\overline{X}_{T_n}-\mu_T\right)/\sigma_T$ has an approximate normal distribution when n is large. Therefore, we conclude

$\sqrt{n}\left(\overline{X}_{T_n}-\mu_T\right)/\sigma_T\rightarrow N\left(0,1\right)$,

Q. E. D.

4.2 Simulation

In Section 4.1, we examined the Central Limit Theorem within the truncated normal environment. In this section, the results of simulation are presented for a verification purpose.

4.2.1 Sampling Distribution

The probability distribution of $\overline{X}_T = (X_{T_1} + X_{T_2} + \cdots + X_{T_n})/n$, which is the sampling distribution of the mean from a truncated normal population, is depicted in Fig. 4.1. It is noted that $\overline{x}_T$ and s_T are the truncated sample mean and truncated sample standard deviation from the truncated normal population, respectively. Based on the Central Limit Theorem (CLT) discussed in Section 4.1, the sampling distribution of $\overline{X_T}$ is approximately normal with mean μ_T and variance σ_T^2/n when the sample size is large.

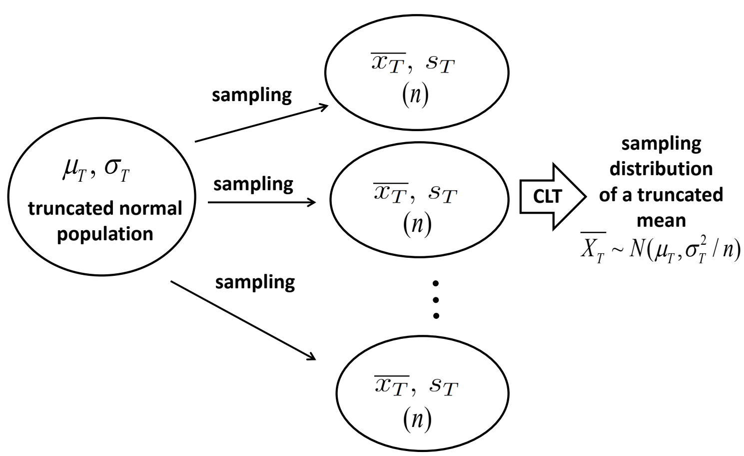


Figure 4.1. Samples from normal and truncated normal distributions

Plots of samples from the normal and truncated normal populations are shown in Fig. 4.2. Plot (a) shows samples, which are denoted by $\times$, from the normal population, while in plot (b), the samples denoted by ● are truncated samples from the truncated normal population.

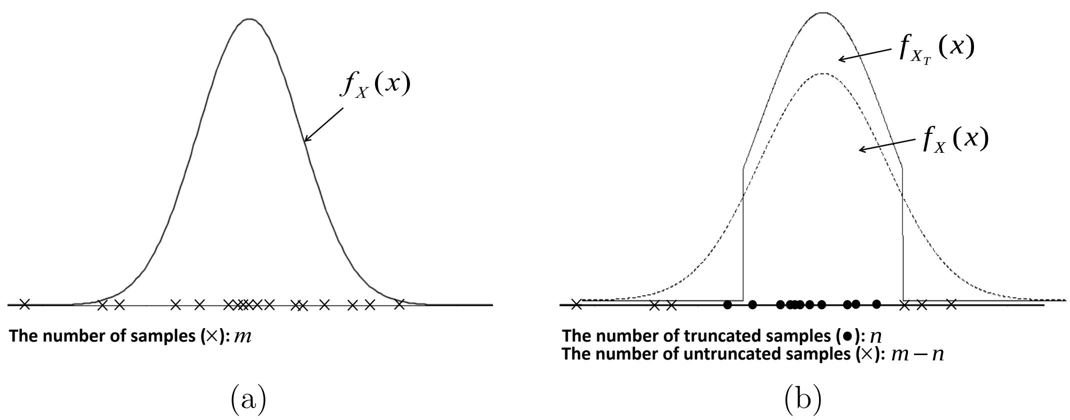


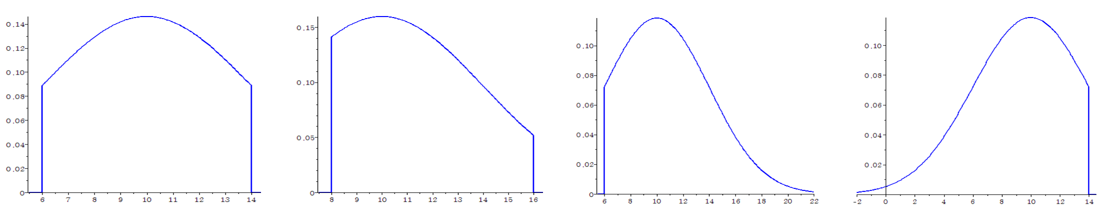
Figure 4.2. Samples from normal and truncated normal distributions

4.2.2 Four Types of TDs

To verify numerically that the distribution for the truncated sample mean follows the Central Limit Theorem, simulation is performed using *R* software. We consider four different truncated normal distributions as shown in Table 4.1 and Fig. 4.3 where plots (a) and (b) represent symmetric and asymmetric doubly truncated normal distributions (symmetric DTND and asymmetric DTND), respectively, while plots (c) and (d) represent left and right truncated normal distributions (LTND and RTND), respectively. The truncated mean, μ_T, and truncated variance, σ_T^2, are calculated by using the formulas shown in Table 4.1.

Table 4.1. Truncated normal population distributions for simulation

	Probability density function	Mean μ_T	SD σ_T	Var σ_T^2
(a)	$f_{X_T}(x) = \dfrac{\frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-10}{4}\right)^2}}{\int_6^{14} \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-10}{4}\right)^2} dy}, 6 \leq x \leq 14$	10	2.158	4.658
(b)	$f_{X_T}(x) = \dfrac{\frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-10}{4}\right)^2}}{\int_8^{16} \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-10}{4}\right)^2} dy}, 8 \leq x \leq 16$	11.425	2.118	4.484
(c)	$f_{X_T}(x) = \dfrac{\frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-10}{4}\right)^2}}{\int_6^{\infty} \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-10}{4}\right)^2} dy}, 6 \leq x \leq \infty$	11.150	3.174	10.075
(d)	$f_{X_T}(x) = \dfrac{\frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-10}{4}\right)^2}}{\int_{-\infty}^{14} \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-10}{4}\right)^2} dy}, -\infty \leq x \leq 14$	8.847	3.174	10.075



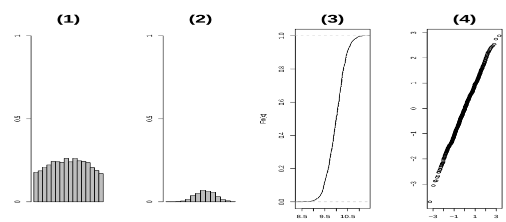
(a) (b) (c) (d)

Figure 4.3. Plots of the truncated population distributions illustrated in Table 4.1

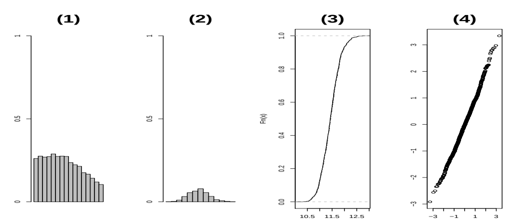
4.2.3 Normality Tests

Based on the truncated normal population distributions in Table 4.1, we generated 1,000 random samples of sample size 30, with truncated sample means denoted by $\overline{X_T}_{30,1}, \overline{X_T}_{30,2}, \ldots, \overline{X_T}_{30,1000}$. We show the simulation results for the CLT depicted in Fig. 5. In each truncated normal distribution, plot (1) represents a histogram for truncated samples from the truncated normal distribution. It is noted that the histogram in each plot (1) is similar to the population distribution in Fig.

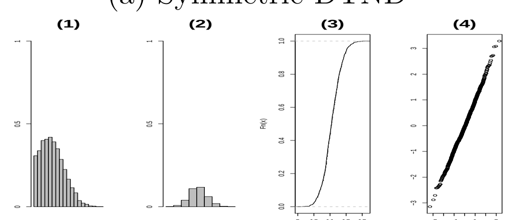
4.4. In each truncated normal distribution, plots (2), (3), and (4) represent histogram, cumulative density curve and normal quantile-quantile (Q-Q) plot for the sampling distribution of the truncated mean under the CLT, respectively. Based on plots (2), (3), and (4), we see that the sampling distribution for the mean from four different types of a truncated normal distribution is normally distributed when the sample size is large.



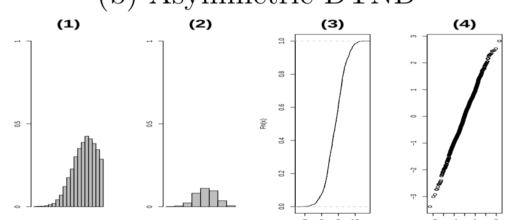
(a) Symmetric DTND



(b) Asymmetric DTND



(c) LTND

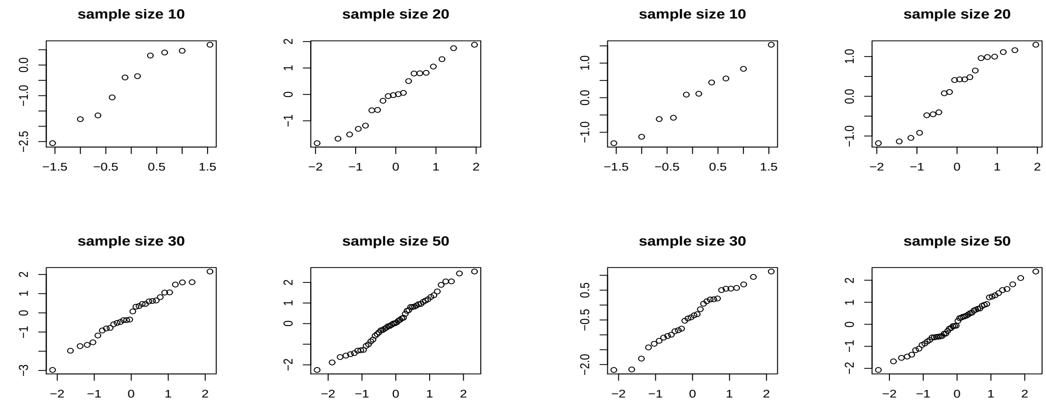


(d) RTND

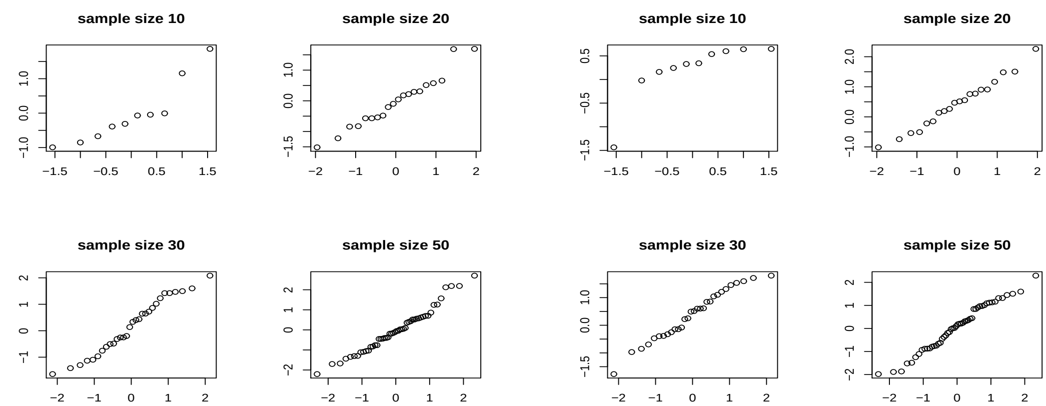
Figure 4.4. Simulation for the Central Limit Theorem by samples from the truncated normal distributions with n=30

Fig. 4.5 shows different normal Q-Q plots for the sampling distributions with four different sample sizes where n = 10, 20, 30, 50. As the sample size increases, it is observed that the curves come closer to a straight line in each truncated distribution.

To support the normality of the sampling distribution of the mean more analytically, the Shapiro-Wilk normality test will be used.



(a) Symmetric DTND (b) Asymmetric DTND



(c) LTND (d) RTND

Figure 4.5. Simulation for the CLT from the truncated normal distributions (four different sample sizes: 10, 20, 30, 50)

Shapiro and Wilk (1968) noted that the Shapiro-Wilk test is comparatively sensitive to a wide range of non-normality, even for small samples ($n < 20$) or with outliers. Pearson *et al.* (1977) explained that the Shapiro-Wilk test is a very sensitive omnibus test against skewed alternatives, and that it is the most powerful for many skewed alternatives. Royston (1982) also noted that the Shapiro-Wilk's W test statistic provides the best omnibus test of normality when the sample sizes are less than 50. The Shapiro-Wilk W test statistic is defined as

$$W = \left\{ \sum_{i=1}^{h} a_{in} (x_{(n-i+1)} - x_{(i)}) \right\}^2 \bigg/ \sum_{i=1}^{n} (x_i - \overline{x})^2, \ x_{(1)} \leq \cdots \leq x_{(n)} \tag{25}$$

where $h = n/2$ when n is even or $h = (n-1)/2$ when n is odd, and a_{in} is a constant which is obtained by the expected values of the order statistics of independent and identically distributed random variables and the covariance matrix of those order statistics. When the P-value of the test statistic, W, is greater than 0.05, it is assumed that the sampling distribution is normally distributed. Five iterations are performed to acquire the average of the P-values, as shown in Table 4.2. Based on the Central Limit Theorem, we expect that P-value increases as the sample size increases. As shown in Table 4.2 and Fig. 4.6, the average of the P-values shows that the Central Limit Theorem works fairly well, regardless of a truncation type.

Table 4.2. P-values of the Shapiro-Wilk test for the sampling distribution of the sample means from truncated normal distributions

	Sample size	Iteration 1	Iteration 2	Iteration 3	Iteration 4	Iteration 5	Average
Symmetric	$n = 10$	0.1676	0.1030	0.2028	0.1112	0.1112	**0.1837**
DTND	$n = 20$	0.5978	0.1639	0.2109	0.4641	0.4101	**0.3694**
	$n = 30$	0.6069	0.2058	0.2176	0.7890	0.4837	**0.4606**
	$n = 50$	0.9223	0.4705	0.3683	0.8440	0.8397	**0.6890**
Asymmetric	$n = 10$	0.0887	0.0824	0.0590	0.0074	0.0157	**0.0506**
DTND	$n = 20$	0.3398	0.1651	0.1444	0.1121	0.2087	**0.1940**
	$n = 30$	0.4283	0.4848	0.1149	0.1193	0.5812	**0.3457**
	$n = 50$	0.7782	0.6213	0.9985	0.1586	0.9154	**0.6944**
LTND	$n = 10$	0.0002	0.0001	0.0007	0.0006	0.0002	**0.0004**
	$n = 20$	0.0110	0.0627	0.0040	0.0024	0.0021	**0.0164**
	$n = 30$	0.0291	0.1664	0.1062	0.1301	0.0435	**0.0951**
	$n = 50$	0.3233	0.2453	0.1069	0.5223	0.1180	**0.2632**
RTND	$n = 10$	0.0001	0.0005	0.0002	0.0001	0.0008	**0.0004**
	$n = 20$	0.0013	0.0107	0.0016	0.0085	0.0873	**0.0219**
	$n = 30$	0.0646	0.0284	0.1386	0.0214	0.0232	**0.0632**
	$n = 50$	0.3983	0.3070	0.1465	0.4048	0.1230	**0.2759**

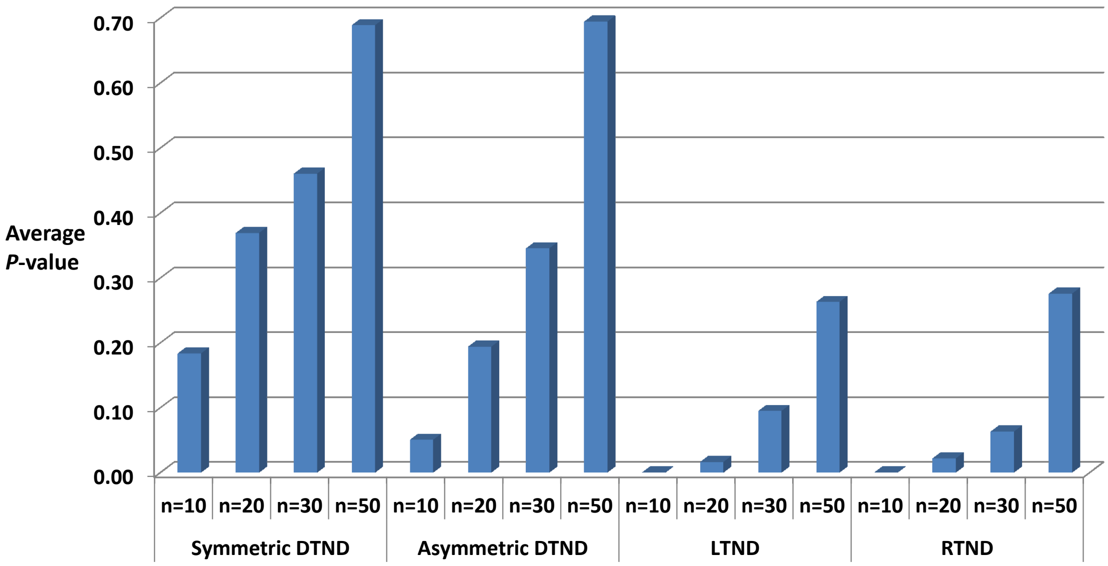


Figure 4.6. Average P-values of the Shapiro-Wilk test for the sampling distribution of the sample mean from a truncated normal distribution

4.3 Methodology Development for Statistical Inferences on the Mean of a TND

In Section 4.2.2, we learned that the CLT for the truncated sample mean works properly regardless of a shape of the population distribution and its truncation type. That is, the distributions of sample means with known and unknown variance are assumed to be normally distributed when those sampling sizes are large. Shown in Fig. 4.7 is the methodology, which shows the way to choose appropriate test statistics from a truncated normal population, to develop the statistical inferences on the mean for truncated samples. Two test statistics, $\sqrt{n}\left(\overline{X_T} - \mu_T\right)/\sigma_T$ and $\sqrt{n}\left(\overline{X_T} - \mu_T\right)/s_T$ where s_T represents the truncated sample standard deviation, are applied.

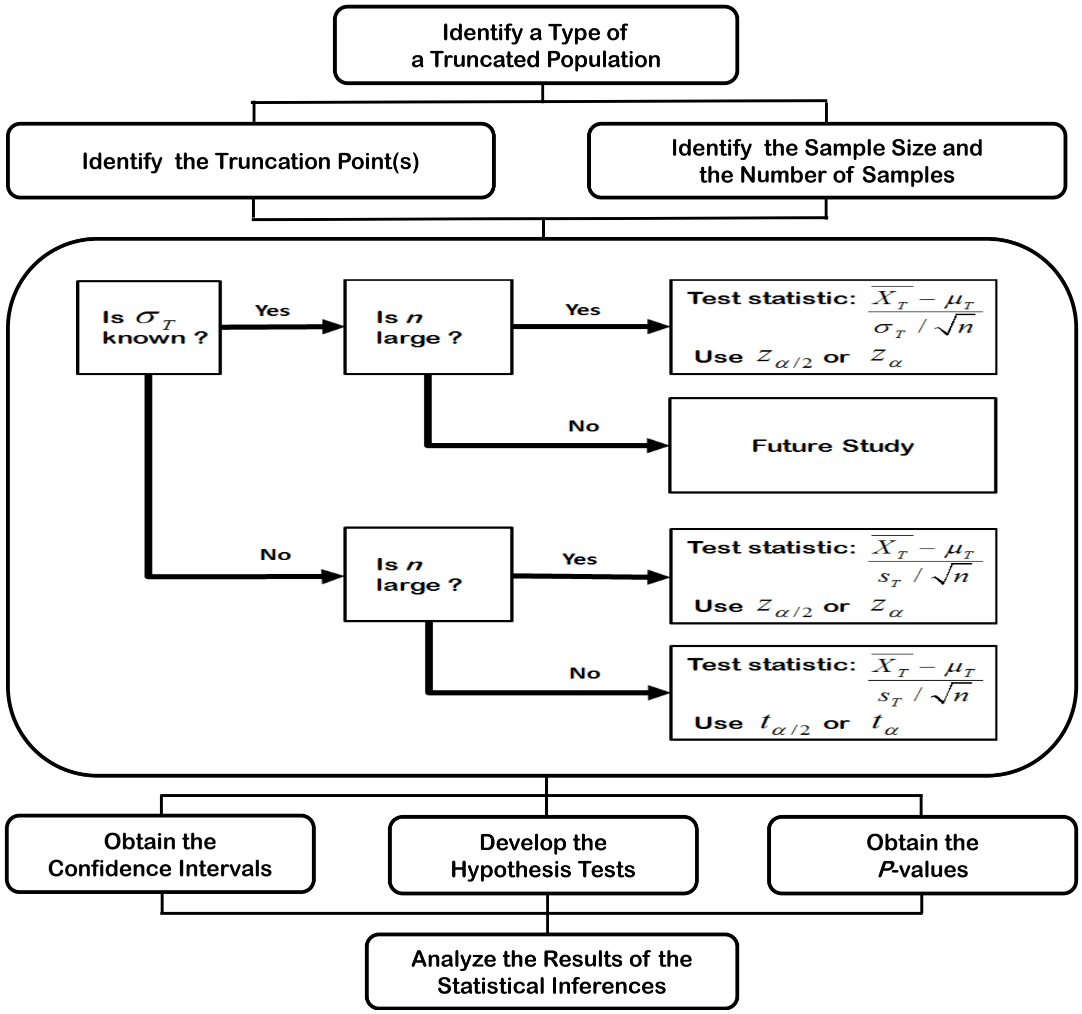


Figure 4.7. Decision diagram for statistical inferences based on a truncated normal population

4.4 Development of Confidence Intervals for the Mean of a TND

In this section, confidence intervals for the truncated mean are developed. In Sections 4.4.1 and 4.4.2, the z and t confidence intervals with known variances are developed. The z and t confidence intervals with unknown variances are then developed in Sections 4.4.3.

4.4.1 Variance Known under a DTND

In Section 4.4.1.1, a 100(1-α)% two-sided confidence interval for μ_T is discussed, and in Sections 4.4.1.2 and 4.4.1.3, the 100(1-α)% one-sided confidence intervals with lower and upper bounds for μ_T are examined, respectively. It should be noted that the truncated variance can be easily obtained when the variance of the original untruncated normal distribution with truncation point(s) are known.

4.4.1.1 Two-Sided Confidence Intervals

The distribution of $\overline{X_T}$, a sampling distribution of the truncated mean, is getting close to a normal distribution based on the Central Limit Theorem as the sample size n increases. Hence, the random variable $\sqrt{n}\left(\overline{X_T}-\mu_T\right)/\sigma_T$ approximately becomes a standard normal distribution for large n. The probability 1-α, called a confidence coefficient, is then expressed as $P\left(-z_{\alpha/2} \leq \sqrt{n}\left(\overline{X_T}-\mu_T\right)/\sigma_T \leq z_{\alpha/2}\right)$ which is written as

$$\begin{aligned} 1-\alpha &= P\left(-z_{\alpha/2} \leq \frac{\overline{x_T}-\mu_T}{\sigma_T/\sqrt{n}} \leq z_{\alpha/2}\right) \\ &= P\left(-z_{\alpha/2}\frac{\sigma_T}{\sqrt{n}} \leq \overline{x_T}-\mu_T \leq z_{\alpha/2}\frac{\sigma_T}{\sqrt{n}}\right) \\ &= P\left(-z_{\alpha/2}\frac{\sigma_T}{\sqrt{n}} \leq \mu_T-\overline{x_T} \leq z_{\alpha/2}\frac{\sigma_T}{\sqrt{n}}\right) \\ &= P\left(\overline{x_T}-z_{\alpha/2}\frac{\sigma_T}{\sqrt{n}} \leq \mu_T \leq \overline{x_T}+z_{\alpha/2}\frac{\sigma_T}{\sqrt{n}}\right). \end{aligned} \tag{26}$$

Based on the doubly truncated normal distribution shown in Table 2.1, the confidence coefficient is obtained as

$$1-\alpha = P\left(\overline{x_T} - z_{\alpha/2}\sigma\sqrt{\frac{1+\frac{-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)+\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2}{n}} \leq \mu_T \right.$$

$$\left. \leq \overline{x_T} + z_{\alpha/2}\sigma\sqrt{\frac{1+\frac{-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)+\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2}{n}}\right). \quad (27)$$

Therefore, the 100(1-α)% confidence interval for μ_T is written as

$$\left[\overline{x_T} - z_{\alpha/2}\sigma\sqrt{\frac{1+\frac{\frac{x_l-\mu}{\sigma}\cdot\phi(\frac{x_l-\mu}{\sigma})-\frac{x_u-\mu}{\sigma}\cdot\phi(\frac{x_u-\mu}{\sigma})}{\Phi(\frac{x_u-\mu}{\sigma})-\Phi(\frac{x_l-\mu}{\sigma})} - \left[\frac{\phi(\frac{x_l-\mu}{\sigma})-\phi(\frac{x_u-\mu}{\sigma})}{\Phi(\frac{x_u-\mu}{\sigma})-\Phi(\frac{x_l-\mu}{\sigma})}\right]^2}{n}}\right.,$$

$$\left.\overline{x_T} + z_{\alpha/2}\sigma\sqrt{\frac{1+\frac{\frac{x_l-\mu}{\sigma}\cdot\phi(\frac{x_l-\mu}{\sigma})-\frac{x_u-\mu}{\sigma}\cdot\phi(\frac{x_u-\mu}{\sigma})}{\Phi(\frac{x_u-\mu}{\sigma})-\Phi(\frac{x_l-\mu}{\sigma})} - \left[\frac{\phi(\frac{x_l-\mu}{\sigma})-\phi(\frac{x_u-\mu}{\sigma})}{\Phi(\frac{x_u-\mu}{\sigma})-\Phi(\frac{x_l-\mu}{\sigma})}\right]^2}{n}}\right]. \quad (28)$$

4.4.1.2 One-Sided Confidence Intervals for Lower Bound

Under the scheme of lower confidence bound for μ_T, we have that $1-\alpha = P\left(\sqrt{n}\left(\overline{X_T}-\mu_T\right)/\sigma_T \leq z_\alpha\right)$ when n is large. By noting $x_l \leq \mu_T \leq x_u$, the confidence coefficient 1-α is obtained as

$$\begin{aligned} 1-\alpha &= P\left(\frac{\overline{x_T}-\mu_T}{\sigma_T/\sqrt{n}} \leq z_\alpha\right) = P\left(\overline{x_T}-\mu_T \leq z_\alpha\frac{\sigma_T}{\sqrt{n}}\right) = P\left(-z_\alpha\frac{\sigma_T}{\sqrt{n}} \leq \mu_T - \overline{x_T}\right) \\ &= P\left(\overline{x_T} - z_\alpha\frac{\sigma_T}{\sqrt{n}} \leq \mu_T\right) = P\left(\overline{x_T} - z_\alpha\frac{\sigma_T}{\sqrt{n}} \leq \mu_T \leq x_u\right). \end{aligned}$$

Since the truncated mean should be less than the upper truncation point x_u, the 100(1-α)% confidence interval with the lower bound for μ_T is then written as

$$\left[\overline{x_T} - z_\alpha \sigma \sqrt{\frac{1 + \frac{\frac{x_l-\mu}{\sigma}\cdot\phi(\frac{x_l-\mu}{\sigma}) - \frac{x_u-\mu}{\sigma}\cdot\phi(\frac{x_u-\mu}{\sigma})}{\Phi(\frac{x_u-\mu}{\sigma}) - \Phi(\frac{x_l-\mu}{\sigma})} - \left[\frac{\phi(\frac{x_l-\mu}{\sigma}) - \phi(\frac{x_u-\mu}{\sigma})}{\Phi(\frac{x_u-\mu}{\sigma}) - \Phi(\frac{x_l-\mu}{\sigma})}\right]^2}{n}}, x_u\right]. \tag{29}$$

4.4.1.3 One-Sided Confidence Intervals for Upper Bound

For a 100(1-α)% upper confidence bound for μ_T, the confidence coefficient is expressed as $1 - \alpha = P\left(\sqrt{n}\left(\overline{X_T} - \mu_T\right)/\sigma_T \geq -z_\alpha\right)$ when the sample size is large. The probability of 1-α is then defined as

$$\begin{aligned} 1-\alpha &= P\left(-z_\alpha \leq \frac{\overline{x_T}-\mu_T}{\sigma_T/\sqrt{n}}\right) = P\left(-z_\alpha \frac{\sigma_T}{\sqrt{n}} \leq \overline{x_T}-\mu_T\right) \\ &= P\left(\mu_T - \overline{x_T} \leq z_\alpha \frac{\sigma_T}{\sqrt{n}}\right) = P\left(x_l \leq \mu_T \leq \overline{x_T} + z_\alpha \frac{\sigma_T}{\sqrt{n}}\right). \end{aligned}$$

Thus, the 100(1-α)% confidence interval with the upper bound for μ_T is given by

$$\left[x_l, \overline{x_T} + z_\alpha \sigma \sqrt{\frac{1 + \frac{\frac{x_l-\mu}{\sigma}\cdot\phi(\frac{x_l-\mu}{\sigma}) - \frac{x_u-\mu}{\sigma}\cdot\phi(\frac{x_u-\mu}{\sigma})}{\Phi(\frac{x_u-\mu}{\sigma}) - \Phi(\frac{x_l-\mu}{\sigma})} - \left[\frac{\phi(\frac{x_l-\mu}{\sigma}) - \phi(\frac{x_u-\mu}{\sigma})}{\Phi(\frac{x_u-\mu}{\sigma}) - \Phi(\frac{x_l-\mu}{\sigma})}\right]^2}{n}}\right]. \tag{30}$$

4.4.2 Variance Known under Singly TNDs

Based on Table 2.1 and the results of Section 4.4.1, we develop the confidence intervals for mean μ_T from left and right truncated normal distributions as shown in Table 4.3, where CI, LCI and UCI stand for the confidence intervals for lower and upper bounds, the confidence interval for a lower bound, and the confidence interval for an upper bound, respectively.

Table 4.3. CIs for mean of left and right truncated normal distributions

LTND	a two-sided CI	$\left[\overline{x_T} - z_{\alpha/2}\sigma\sqrt{\frac{1+\frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2}{n}},\ \overline{x_T} + z_{\alpha/2}\sigma\sqrt{\frac{1+\frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2}{n}}\right]$
	a one-sided LCI	$\left[\overline{x_T} - z_{\alpha}\sigma\sqrt{\frac{1+\frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2}{n}},\ x_u\right]$
	a one-sided UCI	$\left[x_l, \overline{x_T} + z_{\alpha}\sigma\sqrt{\frac{1+\frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2}{n}}\right]$
RTND	a two-sided CI	$\left[\overline{x_T} - z_{\alpha/2}\sigma\sqrt{\frac{1-\frac{\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right)^2}{n}},\ \overline{x_T} + z_{\alpha/2}\sigma\sqrt{\frac{1-\frac{\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right)^2}{n}}\right]$
	a one-sided LCI	$\left[\overline{x_T} - z_{\alpha}\sigma\sqrt{\frac{1-\frac{\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right)^2}{n}},\ x_u\right]$
	a one-sided UCI	$\left[x_l, \overline{x_T} + z_{\alpha}\sigma\sqrt{\frac{1-\frac{\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right)^2}{n}}\right]$

4.4.3 Variance Unknown

When the variance σ_T is unknown and the sample size is large, σ_T is replaced with the truncated sample standard deviation, $S_T = \sqrt{[1/(n-1)] \sum_{i=1}^{n} \left(X_{T_i} - \overline{X_T}\right)^2}$. Accordingly, the random variable $\sqrt{n}\left(\overline{X_T} - \mu_T\right)/S_T$ has an approximately standard normal distribution which leads to the confidence intervals shown in Table 4.4. It is suggested that the sample size required is at least 40 (see Montgomery and Runger, 2011) as shown in Fig. 4.7.

Table 4.4. z CIs for mean of a truncated normal distribution when n is large

a two-sided CI	$\left[\overline{x_T} - z_{\alpha/2}\sqrt{\frac{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}{n}},\ \overline{x_T} + z_{\alpha/2}\sqrt{\frac{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}{n}}\right]$
a one-sided LCI	$\left[\overline{x_T} - z_{\alpha}\sqrt{\frac{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}{n}},\ x_u\right]$
a one-sided UCI	$\left[x_l,\ \overline{x_T} + z_{\alpha}\sqrt{\frac{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}{n}}\right]$

Similarly, we can develop the t confidence intervals with the random variables by incorporating $\sqrt{n}\left(\overline{X_T} - \mu_T\right)/S_T$ which follows a t distribution with $n-1$ degrees of freedom, as shown in Table 4.5.

Table 4.5. t CIs for mean of a truncated normal distribution when n is small

a two-sided CI	$\left[\overline{x_T} - t_{\alpha/2,n-1}\sqrt{\frac{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}{n}}, \overline{x_T} + t_{\alpha/2,n-1}\sqrt{\frac{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}{n}}\right]$
a one-sided LCI	$\left[\overline{x_T} - t_{\alpha,n-1}\sqrt{\frac{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}{n}}, x_u\right]$
a one-sided UCI	$\left[x_l, \overline{x_T} + t_{\alpha,n-1}\sqrt{\frac{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}{n}}\right]$

4.5 Development of Hypothesis Tests on the Mean of a TND

The hypothesis tests on a truncated mean are developed with known and unknown variances based on the CLT in this section. For the hypothesis tests, the random variables $\sqrt{n}\left(\overline{X_T} - \mu_T\right)/\sigma_T$ and $\sqrt{n}\left(\overline{X_T} - \mu_T\right)/S_T$ are used as a test statistics developed in Sections 4.5.1 and 4.5.2, respectively.

4.5.1 Variance Known

The sample mean $\overline{X_T}$ is an unbiased point estimator of μ_T with variance σ_T^2/n. When the sampling distribution of the truncated mean is approximately normally distributed, the test statistic, $Z_{T_0} = \sqrt{n}\left(\overline{X_T} - \theta\right)/\sigma_T$, has a standard normal distribution with mean 0 and variance 1, when n is large. Three types of test statistics are developed and shown in Table 4.6. When the alternative hypothesis is H_1: $\mu_T \neq \theta$, H_0 will be rejected if the observed value of the test statistic $z_{T_0} = \sqrt{n}\left(\overline{x_T} - \theta\right)/\sigma_T$ is either $z_{T_0}{>}{-z_{\alpha/2}}$ or $z_{T_0}{<}z_{\alpha/2}$. If the value of $z_{T_0}{>}z_\alpha$, H_0 will be rejected under H_1: $\mu_T > \theta$. In contrast, the value of z_{T_0}<-z_α, H_0 will be rejected under H_1: $\mu_T < \theta$.

Table 4.6. Hypothesis tests with known variance

Null hypothesis	H_0: $\mu_T = \theta$
Test statistics	
DTND	$Z_{T_0} = \frac{\overline{X_T}-\theta}{\sigma_T/\sqrt{n}}$ $= \frac{\overline{X_T}-\theta}{\sigma\sqrt{\left[1+\frac{-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)+\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]/n}}$
LTND	$Z_{T_0} = \frac{\overline{X_T}-\theta}{\sigma_T/\sqrt{n}}$ $= \frac{\overline{X_T}-\theta}{\sigma\sqrt{\left[1+\frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]/n}}$
RTND	$Z_{T_0} = \frac{\overline{X_T}-\theta}{\sigma_T/\sqrt{n}}$ $= \frac{\overline{X_T}-\theta}{\sigma\sqrt{\left[1-\frac{\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right)^2\right]/n}}$
Alternative hypotheses	Rejection criteria
H_1: $\mu_T \neq \theta$	$z_{T_0}>-z_{\alpha/2}$ or $z_{T_0}<z_{\alpha/2}$
H_1: $\mu_T > \theta$	$z_{T_0}>z_{\alpha}$
H_1: $\mu_T < \theta$	$z_{T_0}<-z_{\alpha}$

4.5.2 Variance Unknown

As shown in Fig. 4.7, the random variable $\sqrt{n}\left(\overline{X_T}-\mu_T\right)/S_T$ has an approximate normal distribution or an approximate t distribution, depending on a sample size. By referring to Sections 4.4.3, we develop hypothesis tests on the truncated mean with unknown variance as shown in Tables 4.7 and 4.8, respectively.

Table 4.7. Hypothesis tests with unknown variance when n is large

Null hypothesis	H_0: $\mu_T = \theta$
Test statistic	$Z_{T_0} = \frac{\sqrt{n}\left(\overline{X_T}-\theta\right)}{s_T} = \frac{\sqrt{n}\left(\overline{X_T}-\theta\right)}{\sqrt{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}}$
Alternative hypothesis	Rejection criteria
H_1: $\mu_T \neq \theta$	$z_{T_0}>-z_{\alpha/2}$ or $z<z_{\alpha/2}$
H_1: $\mu_T > \theta$	$z_{T_0}>z_{\alpha}$
H_1: $\mu_T < \theta$	$z_{T_0}<-z_{\alpha}$

Table 4.8. Hypothesis tests with unknown variance when n is small

Null hypothesis	H_0: $\mu_T = \theta$
Test statistic	$T_{T_0} = \frac{\sqrt{n}\left(\overline{X_T}-\theta\right)}{s_T} = \frac{\sqrt{n}\left(\overline{X_T}-\theta\right)}{\sqrt{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}}$
Alternative hypothesis	Rejection criteria
H_1: $\mu_T \neq \theta$	$t_{T_0}>-t_{\alpha/2}$ or $t_{T_0}<t_{\alpha/2}$
H_1: $\mu_T > \theta$	$t_{T_0}>t_{\alpha}$
H_1: $\mu_T < \theta$	$t_{T_0}<-t_{\alpha}$

4.6 Development of P-values for the Mean of a TND

In Sections 4.6.1 and 4.6.2, we develop the P-values for the truncated mean when variance of a population distribution is known and unknown.

4.6.1 Variance Known

4.6.1.1 P-values for the Mean of a Doubly TND

For the foregoing test from a doubly truncated normal distribution, it is relatively easy to interpret the P-values. If $z_{T_0} = \sqrt{n}\left(\overline{x_T} - \theta\right)/\sigma_T$ is the computed value of the test statistic when the sample size is large, the P-values are obtained as

$$
P\text{-value} = \begin{cases}
2\left[1 - \Phi\left(\left|z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma_T}\right|\right)\right] = \\
\qquad 2\left[1 - \Phi\left(\left|z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma\sqrt{\left[1+\frac{-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)+\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]}}\right|\right)\right] \\
\qquad\qquad \text{for a two-tailed test under } H_1\text{: } \mu_T \neq \theta, \\
1 - \Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma_T}\right) = \\
\qquad 1 - \Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma\sqrt{\left[1+\frac{-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)+\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]}}\right) \\
\qquad\qquad \text{for an upper-tailed test under } H_1\text{: } \mu_T > \theta, \\
\Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma_T}\right) = \\
\qquad \Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma\sqrt{\left[1+\frac{-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)+\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]}}\right) \\
\qquad\qquad \text{for a lower-tailed test under } H_1\text{: } \mu_T < \theta.
\end{cases}
$$

4.6.1.2 P-values for the Mean of Singly TNDs

The P-values for the means of left and right truncated normal distributions (LTND and RTND) are shown in Table 4.9.

Table 4.9. P-values under the left and right truncated normal distributions

LTDN	$P\text{-value} = \begin{cases} 2\left[1-\Phi\left(\left\lvert z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma_T}\right\rvert\right)\right] = 2\left[1-\Phi\left(\left\lvert z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma\sqrt{\left[1+\frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]}}\right\rvert\right)\right] & \text{for a two-tailed test under } H_1\text{: } \mu_T \neq \theta, \\ 1-\Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma_T}\right) = 1-\Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma\sqrt{\left[1+\frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]}}\right) & \text{for an upper-tailed test under } H_1\text{: } \mu_T > \theta, \\ \Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma_T}\right) = \Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma\sqrt{\left[1+\frac{\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)}{1-\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right]}}\right) & \text{for a lower-tailed test under } H_1\text{: } \mu_T < \theta. \end{cases}$
RTDN	$P\text{-value} = \begin{cases} 2\left[1-\Phi\left(\left\lvert z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma_T}\right\rvert\right)\right] = 2\left[1-\Phi\left(\left\lvert z = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma\sqrt{\left[1-\frac{\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right)^2\right]}}\right\rvert\right)\right] & \text{for a two-tailed test under } H_1\text{: } \mu_T \neq \theta, \\ 1-\Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma_T}\right) = 1-\Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma\sqrt{\left[1-\frac{\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right)^2\right]}}\right) & \text{for an upper-tailed test under } H_1\text{: } \mu_T > \theta, \\ \Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma_T}\right) = \Phi\left(z_{T_0} = \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sigma\sqrt{\left[1-\frac{\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)}\right)^2\right]}}\right) & \text{for a lower-tailed test under } H_1\text{: } \mu_T < \theta. \end{cases}$

4.6.2 Variance Unknown

Tables 4.10 and 4.11 show the associated P-values when variance is unknown.

Table 4.10. P-values with unknown variance when n is large

$$P\text{-value} = \begin{cases} 2\left[1-\Phi\left(\left|z=\frac{\sqrt{n}(\overline{x_T}-\theta)}{S_T}\right|\right)\right] = 2\left[1-\Phi\left(\left|z=\frac{\sqrt{n}(\overline{x_T}-\theta)}{\sqrt{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}}\right|\right)\right] \\ \qquad \text{for a two-tailed test under } H_1\text{: } \mu_T \neq \theta, \\ 1-\Phi\left(\left|z=\frac{\sqrt{n}(\overline{x_T}-\theta)}{S_T}\right|\right) = 1-\Phi\left(\left|z=\frac{\sqrt{n}(\overline{x_T}-\theta)}{\sqrt{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}}\right|\right) \\ \qquad \text{for an upper-tailed test under } H_1\text{: } \mu_T > \theta, \\ \Phi\left(\left|z=\frac{\sqrt{n}(\overline{x_T}-\theta)}{S_T}\right|\right) = \Phi\left(\left|z=\frac{\sqrt{n}(\overline{x_T}-\theta)}{\sqrt{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}}\right|\right) \\ \qquad \text{for a lower-tailed test under } H_1\text{: } \mu_T < \theta. \end{cases}$$

Table 4.11. P-values with unknown variance when n is small

$$P-value = \begin{cases} 2\left[1-P\left(|t_{T_0}| \leq \frac{\sqrt{n}(\overline{x_T}-\theta)}{S_T}\right)\right] = 2\left[1-P\left(|t_{T_0}| \leq \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sqrt{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}}\right)\right] \\ \qquad \text{for a two-tailed test under } H_1\text{: } \mu_T \neq \theta, \\ 1-P\left(t_{T_0} \leq \frac{\sqrt{n}(\overline{x_T}-\theta)}{S_T}\right) = 1-\Phi\left(t \leq \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sqrt{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}}\right) \\ \qquad \text{for an upper-tailed test under } H_1\text{: } \mu_T > \theta, \\ P\left(t_{T_0} \leq \frac{\sqrt{n}(\overline{x_T}-\theta)}{S_T}\right) = \Phi\left(t \leq \frac{\sqrt{n}(\overline{x_T}-\theta)}{\sqrt{[1/(n-1)]\sum_{i=1}^{n}\left(x_{T_i}-\overline{x_T}\right)^2}}\right) \\ \qquad \text{for a lower-tailed test under } H_1\text{: } \mu_T < \theta. \end{cases}$$

4.7 Numerical Example

In this section, we provide a numerical example to illustrate the proposed confidence intervals, hypothesis tests, and P-values. Let $X_{T_1}, X_{T_2}, \ldots, X_{T_n}$ be independent, identically distributed, and assume the truncated normal random sample with $x_l = 6$, $x_u = 14$, σ_T=2.158, $n = 35$, $\overline{x_T} = 10.3$ and $\alpha = 0.05$. Using Eqs. (27), (28) and (29), the results based on the symmetric doubly truncated normal distribution are shown in Table 4.1. First, the $100(1-\alpha)\%$ two-sided confidence interval for μ_T is obtained as $[10.3 - 1.96 \times 2.158/\sqrt{35},\ 10.3 + 1.96 \times 2.158/\sqrt{35}] = [9.585,\ 11.015]$. Second, the $100(1-\alpha)\%$ one-sided confidence interval with the lower bound for μ_T is given by $[10.3 - 1.65 \times 2.158/\sqrt{35},\ 14] = [9.698,\ 14]$. Finally, the $100(1-\alpha)\%$ one-sided confidence interval with the upper bound for μ_T is expressed as $[6,\ 10.3 + 1.65 \times 2.158/\sqrt{35}] = [6,\ 10.902]$. Table 4.12 shows the confidence intervals for μ_T under the four different truncated normal distributions. Fig. 4.8 shows the corresponding the confidence intervals for μ where its probability density function is $f_X(t) = \left(1/4\sqrt{2\pi}\right)\, e^{-\frac{1}{2}\left(\frac{t-10}{4}\right)^2}$, $-\infty \le t \le \infty$. It is our finding that the confidence intervals for a truncated normal population are always smaller than the ones for a untruncated normal population.

Table 4.12. Confidence intervals (α=0.05)

	Mean	SD	100(1-α)% CI	100(1-α)% LCI	100(1-α)% UCI
ND	10	4	[8.975, 11.625]	[9.184, ∞]	[-∞, 11.415]
Symmetric DTND	10	2.158	[9.585, 11.015]	[9.698, 14]	[6, 10.902]
Asymmetric DTND	11.425	2.118	[9.585, 11.015]	[9.709, 14]	[6, 10.891]
LTND	11.150	3.174	[9.248, 11.351]	[9.414, 14]	[6, 11.185]
RTND	8.847	2.975	[9.248, 11.351]	[9.414, 14]	[6, 11.185]

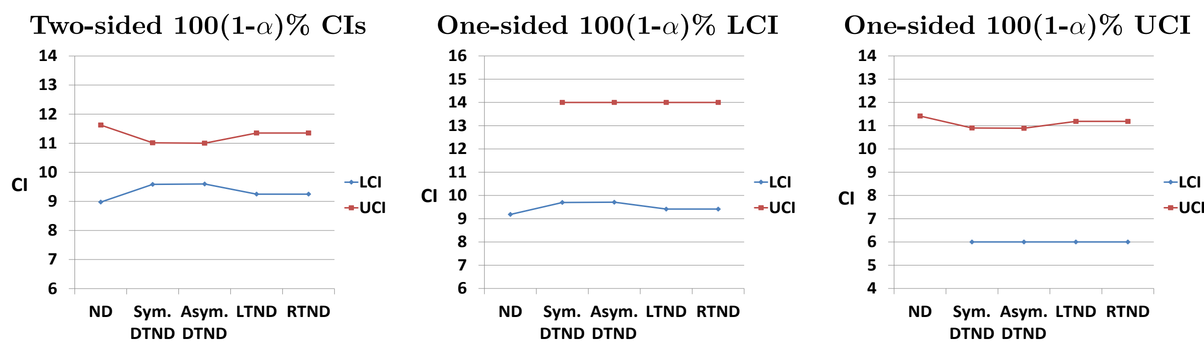


Figure 4.8. Comparisons of the confidence intervals

For the doubly truncated normal distribution, consider the null hypothesis, $H_0 : \mu_T = 10$, and the significant level 0.05. Then, the statistic $Z_{T_0} = \sqrt{n}\left(\overline{X_T} - \theta\right)/\sigma_T$ shown in Table 4.13 will be applied as since the sample size is large and the variance is known. Consequently, under the alternative hypothesis $H_1 : \mu_T \neq 10$, there is no strong evidence that μ_T is different from 10.3 since the value of $z_{T_0}(= 0.822)$ does not fall in the rejection region $[-1.96, 1.96]$. When the alternative hypothesis is $H_1 :\ \mu_T < 10$, there is also no strong evidence that μ_T is less than 10.3 because $z_{T_0} > -1.64$.

Table 4.13. Hypothesis tests with variance known under the doubly truncated normal distribution

Null hypothesis	$H_0 : \mu_T = 10$
Test statistic	$Z_{T_0} = \frac{\sqrt{n}\left(\overline{X_T}-\theta\right)}{\sigma_T} = \frac{\sqrt{n}\left(\overline{X_T}-\theta\right)}{\sigma\sqrt{1+\frac{-\frac{x_l-\mu}{\sigma}\phi\left(\frac{x_l-\mu}{\sigma}\right)+\frac{x_u-\mu}{\sigma}\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}-\left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2}}$ $z_{T_0} = \frac{\sqrt{35}(10.3-10)}{2.158}$ =0.822
Alternative hypothesis	Rejection criteria
$H_1 : \mu_T \neq 10$	z_{T_0}>-1.96 or z_{T_0}<1.96
$H_1 : \mu_T > 10$	z_{T_0}>1.64
$H_1 : \mu_T < 10$	z_{T_0}<-1.64

The P-values are then obtained as

$$
P\text{-value} = \begin{cases} 2\left[1-\Phi\left(\left|z_{T_0}=\frac{\sqrt{n}\left(\overline{X_T}-\theta\right)}{\sigma_T}\right|\right)\right]=2\left[1-\Phi\left(\left|z_{T_0}=\frac{\sqrt{35}(10.3-10)}{2.158}\right|\right)\right]=2\left[1-\Phi\left(0.822\right)\right]=0.412 \\ \qquad \text{for a two-tailed test under } H_1\text{: } \mu_T\neq 10, \\ 1-\Phi\left(z_{T_0}=\frac{\sqrt{n}\left(\overline{X_T}-\theta\right)}{\sigma_T}\right)=1-\Phi\left(z_{T_0}=\frac{\sqrt{35}(10.3-10)}{2.158}\right)=1-\Phi\left(0.822\right)=0.206 \\ \qquad \text{for an upper-tailed test under } H_1\text{: } \mu_T>10, \\ \Phi\left(z_{T_0}=\frac{\sqrt{n}\left(\overline{X_T}-\theta\right)}{\sigma_T}\right)=\Phi\left(z_{T_0}=\frac{\sqrt{35}(10.3-10)}{2.158}\right)=\Phi\left(0.822\right)=0.794 \\ \qquad \text{for a lower-tailed test under } H_1\text{: } \mu_T<10. \end{cases}
$$

4.8 Conclusions and Future Work

In many quality and reliability engineering problems, specifications are implemented on products, and hence the resulting distributions of conforming products are truncated. However, the current statistical inference typically does not incorporate a random sample from a truncated distribution into hypothesis testing. This research has provided the mathematical proofs of the Central Limit Theorem within a truncated environment and also verified the theorem through simulation. Based on the Central Limit Theorem, we have then developed the new one-sided and two-sided z-test and t-test procedures, including their test statistics, confidence intervals, and P-values, using appropriate truncated test statistics. As a future study, the work done in this dissertation can be extended to several different areas. Statistical inference on a population proportion is one example. Inference on population means for two samples with variances known and unknown can also be developed by extending the truncated statistics. The sample size determination associated with the probability of type II error is another fruitful future research area.

CHAPTER FIVE

DEVELOPMENT OF STATISTICAL CONVOLUTIONS OF TRUNCATED NORMAL AND TRUNCATED SKEW NORMAL RANDOM VARIABLES WITH APPLICATIONS

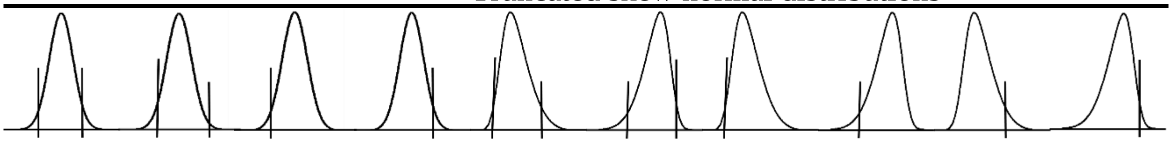
As discussed in Chapter 2, several crucial contributions to the literature on convolutions that have not been explored previously is offered in Chapter 5. Convolutions are analogous to the sum of random variables and are critical concepts in multistage production processes, statistical tolerance analysis, and gap analysis. More specifically, the focus is on the convolutions resulting from double and triple truncations associated with symmetric and asymmetric normal and skew normal distributions under three types of quality characteristics, such as nominal-the-best type (*N*-type), smaller-the-better type (*S*-type), and larger-the-better type (*L*-type). The convolutions of the combinations of truncated normal and truncated skew normal random variables have never been fully explored in the literature. This is a critical issue because specification limits on a process are implemented externally in most manufacturing and service processes, which implies that the product is typically reworked or scrapped if its performance does not fall in the range of the specifications. As such, the actual distribution after inspection becomes truncated. In Section 5.1, we first provide notations of four cases of truncated normal and six cases of truncated skew normal random variables. Then, the convolutions of truncated normal and truncated skew normal random variables on doubly truncations is investigated. We extend the convolution on triple truncations in Section 5.2. Finally, numerical examples for statistical tolerance analysis and gap analysis follow in Section 5.3.

5.1 Development of the convolutions of truncated normal and truncated skew normal random variables on double truncations

In the convolution theorem, the order of truncated random variables does not affect the probability density function of the sum of those random variables. In this paper, truncated normal and truncated skew normal random variables are considered independent but are not necessarily identically distributed. By using truncated normal and skew normal distributions, we can design various cases of the sums on double truncations. As shown in Figure 3, four types of a truncated normal distribution and six types of a truncated skew normal distribution are categorized. In the notation of the truncated normal distribution, '*Sym*' and '*Asym*' denote symmetric and asymmetric, respectively, and *TN* stands for 'truncated normal.' Similarly, for the truncated skew normal distribution, '+' indicates a positive α value which means the untruncated original distribution is positively skewed. In contrast, '−' means that α is negative and the untruncated original distribution is negatively skewed, and *TSN* denotes 'truncated skew normal.'

This section has three subsections. First, the sums of two truncated normal random variables are derived in Section 5.1. Second, the sums of two truncated skew normal random variables are examined in Section 5.2. Finally, in Section 5.3, we investigate the sums of truncated normal and truncated skew normal random variables.

Truncated normal distributions	Truncated skew normal distributions



(a) (b) (c) (d) (e) (f) (g) (h) (i) (j)

Notation

(a)	Sym $TN_{N-\text{type}}$	A symmetric doubly truncated normal distribution	(e)	$TSN^{+}_{N-\text{type}}$	A doubly truncated positive skew normal distribution
(b)	Asym $TN_{N-\text{type}}$	An asymmetric doubly truncated normal distribution	(f)	$TSN^{-}_{N-\text{type}}$	A doubly truncated negative skew normal distribution
(c)	$TN_{L-\text{type}}$	A left truncated normal distribution	(g)	$TSN^{+}_{L-\text{type}}$	A left truncated positive skew normal distribution
(d)	$TN_{S-\text{type}}$	A right truncated normal distribution	(h)	$TSN^{-}_{L-\text{type}}$	A left truncated negative skew normal distribution
			(i)	$TSN^{+}_{S-\text{type}}$	A right truncated positive skew normal distribution
			(j)	$TSN^{-}_{S-\text{type}}$	A right truncated negative skew normal distribution

Figure 5.1. Ten cases of truncated normal and truncated skew normal random variables and notation

5.1.1 The convolutions of truncated normal random variables on double truncations

To develop the sums of two independent truncated normal random variables, we consider the following two truncated normal random variables, X_{T_1} and X_{T_2}, where those probability density functions are

$$f_{X_{T_1}}(x)=\frac{\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\int_{x_{l_1}}^{x_{u_1}}\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2}dh}I_{[x_{l_1},x_{u_1}]}(x)$$ and

$$f_{X_{T_2}}(y)=\frac{\frac{1}{\sigma_2\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{\int_{x_{l_2}}^{x_{u_2}}\frac{1}{\sigma_2\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}dp}I_{[x_{l_2},x_{u_2}]}(y),$$ respectively.

Let Z_2 be $X_{T_1} + X_{T_2}$. Based on the convolution theorem, the probability density function of the sum of the above two truncated normal random variables is obtained as:

$$f_{Z_2}(z) = \int_{-\infty}^{\infty} f_{X_{T_2}}(z-x) f_{X_{T_1}}(x)dx$$

$$= \int_{-\infty}^{\infty} \frac{\frac{1}{\sigma_2\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}}{\int_{x_{l_2}}^{x_{u_2}} \frac{1}{\sigma_2\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2} dp} \frac{\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\int_{x_{l_1}}^{x_{u_1}} \frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2} dh} dx$$

where $x_{l_2} \le z - x \le x_{u_2}$ and $x_{l_1} \le x \le x_{u_1}$

$$= \int_{-\infty}^{\infty} \frac{\frac{1}{\sigma_2\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}}{\int_{x_{l_2}}^{x_{u_2}} \frac{1}{\sigma_2\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2} dp} \frac{\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\int_{x_{l_1}}^{x_{u_1}} \frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2} dh} I_{[z-x_{u_2}, z-x_{l_2}]}(x) I_{[x_{l_1}, x_{u_1}]}(x)dx.$$

Note that $I_{[x_{l_2}, x_{u_2}]}(z-x)$ can be expressed as $I_{[z-x_{u_2}, z-x_{l_2}]}(x)$ since $z = x + y$. Ten cases of the sums of two truncated normal random variables are illustrated in Figure 4. The distributions, means and variances of the sums of truncated normal random variables are also shown in Table 2, where $E(Z_2)$ is equal to the sum of $E(X_{T_1}) = \mu_{T_1}$ and $E(X_{T_2}) = \mu_{T_2}$, and $Var(Z_2)$ is equal to the sum of $Var(X_{T_1}) = \sigma_{T_1}^2$ and $Var(X_{T_2}) = \sigma_{T_2}^2$.

In Figure 5.2, we assume that $\mu_1 = \mu_2 = 8$ and $\sigma_1 = \sigma_2 = 2$. In addition, the lower and upper truncation points are considered according to different types of truncation as shown in Table 5.1.

Table 5.1. Lower and upper truncation points based on a TNRV

Type	LTP	UTP	Type	LTP	UTP
Sym $TN_{N-\text{type}}$	6.5	9.5	Asym $TN_{N-\text{type}}$	7.5	10
$TN_{L-\text{type}}$	7	∞	$TN_{S-\text{type}}$	-∞	9

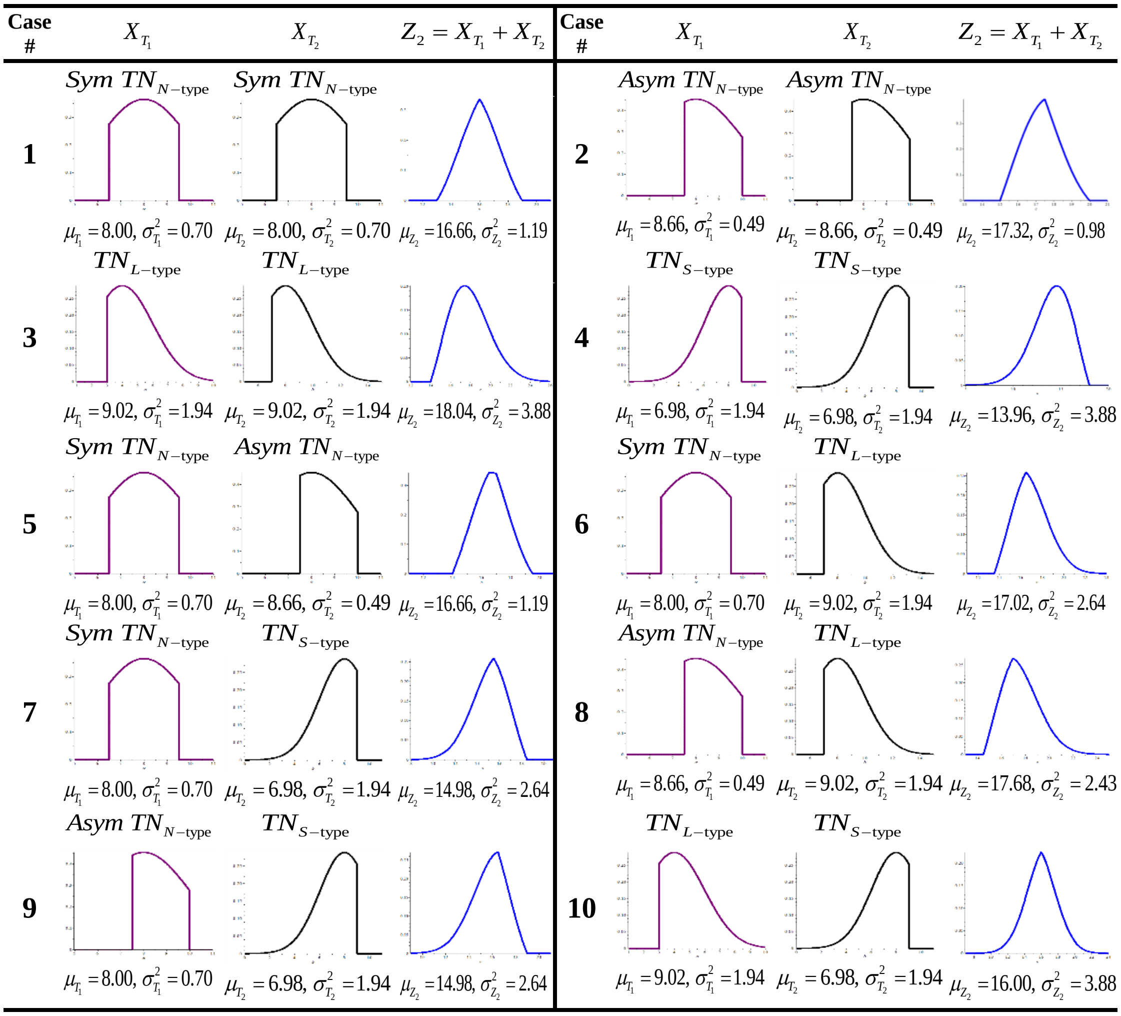


Figure 5.2. Ten different cases of the sums of two TNRVs

5.1.2 The convolutions of truncated skew normal random variables on double truncations

The convolutions of the sums of two independent truncated skew normal random variables, Y_{TS_1} and Y_{TS_2}, are developed in this section as follows

$$f_{Y_{TS_1}}(x)=\frac{\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y-\mu_1}{\sigma_1}\right)^2}\int_{-\infty}^{\alpha_1\frac{y-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_1}}^{y_{u_1}}\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2}\left(\int_{-\infty}^{\alpha_1\frac{h-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dh}I_{[y_l,y_{u_1}]}(x) \text{ and}$$

$$f_{Y_{TS_2}}(y)=\frac{\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}\int_{-\infty}^{\alpha_2\frac{y-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_2}}^{y_{u_2}}\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}\left(\int_{-\infty}^{\alpha_2\frac{p-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dp}I_{[y_{l_2},y_{u_2}]}(y) \text{, respectively.}$$

Letting $Z_2 = Y_{TS_1} + Y_{TS_2}$, the probability density function of the sum of the two truncated skew normal random variables is obtained as

$$f_{Z_2}(z)=\int_{-\infty}^{\infty}f_{Y_{TS_2}}(z-x)f_{Y_{TS_1}}(x)dx$$

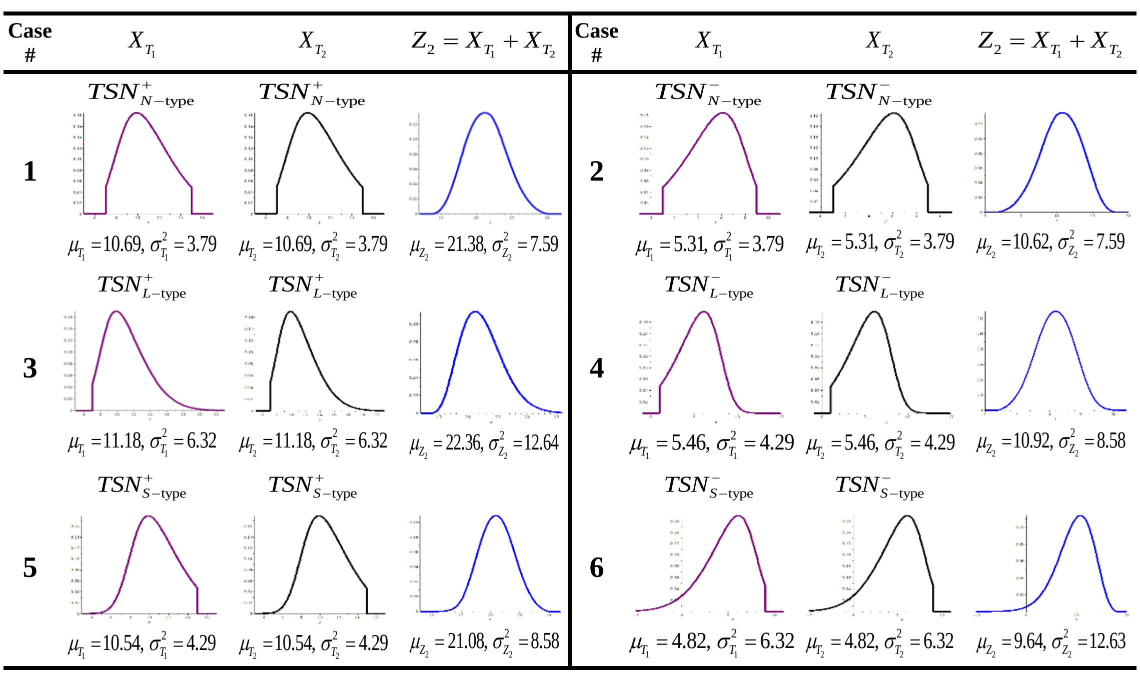
$$=\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}\int_{-\infty}^{\alpha_2\frac{y-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_2}}^{y_{u_2}}\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}\left(\int_{-\infty}^{\alpha_2\frac{p-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dp}\cdot$$

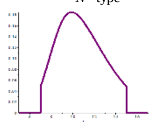
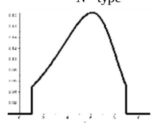
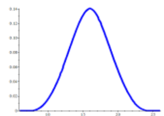
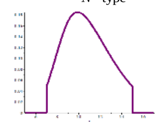
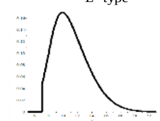
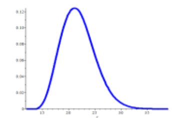
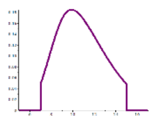
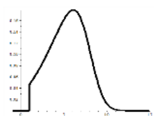
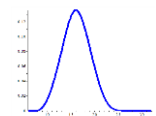
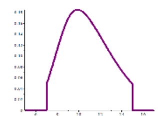
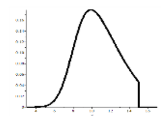
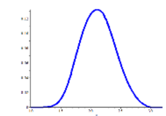
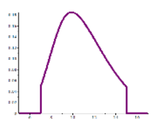
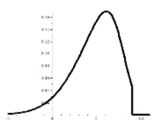
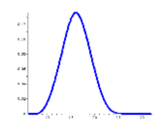
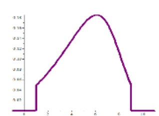
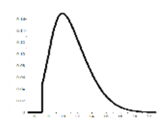
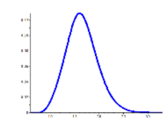
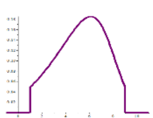
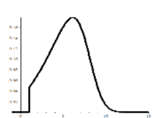
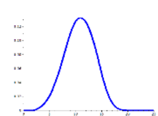
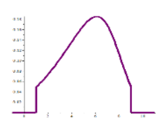
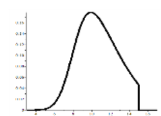
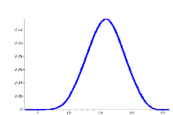
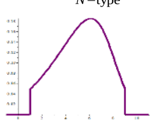
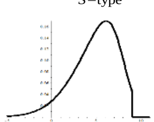
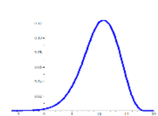
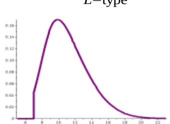
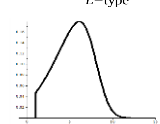
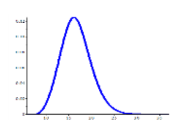
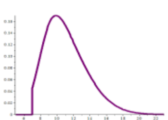
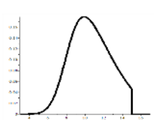
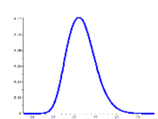
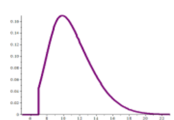
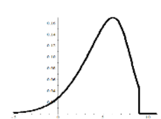
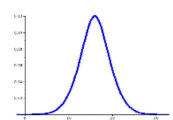
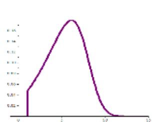
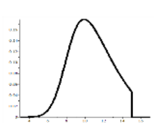
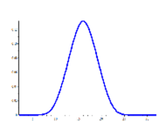
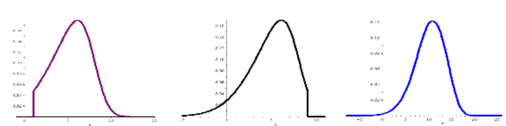
$$\frac{\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}\int_{-\infty}^{\alpha_1\frac{x-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_1}}^{y_{u_1}}\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2}\left(\int_{-\infty}^{\alpha_1\frac{h-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dh}dx$$

where $y_{l_2} \le z-x \le y_{u_2}$ and $y_{l_1} \le x \le y_{u_1}$

$$= \int_{-\infty}^{\infty} \frac{\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2} \int_{-\infty}^{\alpha_2 \frac{z-x-\mu_2}{\sigma_2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt}{\int_{y_{l_2}}^{y_{u_2}} \frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2} \left(\int_{-\infty}^{\alpha_2 \frac{p-\mu_2}{\sigma_2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \right) dp} \cdot \frac{\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \int_{-\infty}^{\alpha_1 \frac{x-\mu_1}{\sigma_1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt}{\int_{y_{l_1}}^{y_{u_1}} \frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2} \left(\int_{-\infty}^{\alpha_1 \frac{h-\mu_1}{\sigma_1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \right) dh} I_{[z-y_{u_2}, z-y_{l_2}]}(x) I_{[y_{l_1}, y_{u_1}]}(x) dx.$$

$I_{[y_{l_2}, y_{u_2}]}(z-x)$ can be given by $I_{[z-y_{u_2}, z-y_{l_2}]}(x)$. Twenty-one cases of the sums of two truncated skew normal random variables are listed in Figure 5.3. It is assumed that the parameters, μ_1 and μ_2 are 8, and the parameters, σ_1 and σ_2 are 4. In addition, the shape parameter α discussed in Section 2.2.3, and the lower and upper truncation points are utilized according to six different types of truncation as shown in Table 5.2.



Case #	X_{T_1}	X_{T_2}	$Z_2 = X_{T_1} + X_{T_2}$	Case #	X_{T_1}	X_{T_2}	$Z_2 = X_{T_1} + X_{T_2}$
7	$TSN^{+}_{N\text{-type}}$  $\mu_{T_1}=10.69,\ \sigma^2_{T_1}=3.79$	$TSN^{-}_{N\text{-type}}$  $\mu_{T_2}=5.31,\ \sigma^2_{T_2}=3.79$	 $\mu_{Z_2}=16.00,\ \sigma^2_{Z_2}=7.59$	**8**	$TSN^{+}_{N\text{-type}}$  $\mu_{T_1}=10.69,\ \sigma^2_{T_1}=3.79$	$TSN^{+}_{L\text{-type}}$  $\mu_{T_2}=11.18,\ \sigma^2_{T_2}=6.32$	 $\mu_{Z_2}=21.87,\ \sigma^2_{Z_2}=10.11$
9	$TSN^{+}_{N\text{-type}}$  $\mu_{T_1}=10.69,\ \sigma^2_{T_1}=3.79$	$TSN^{-}_{L\text{-type}}$  $\mu_{T_2}=5.46,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=16.15,\ \sigma^2_{Z_2}=8.08$	**10**	$TSN^{+}_{N\text{-type}}$  $\mu_{T_1}=10.69,\ \sigma^2_{T_1}=3.79$	$TSN^{+}_{S\text{-type}}$  $\mu_{T_2}=10.54,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=21.23,\ \sigma^2_{Z_2}=8.08$
11	$TSN^{+}_{N\text{-type}}$  $\mu_{T_1}=10.69,\ \sigma^2_{T_1}=3.79$	$TSN^{-}_{S\text{-type}}$  $\mu_{T_2}=4.82,\ \sigma^2_{T_2}=6.32$	 $\mu_{Z_2}=15.51,\ \sigma^2_{Z_2}=10.11$	**12**	$TSN^{-}_{N\text{-type}}$  $\mu_{T_1}=5.31,\ \sigma^2_{T_1}=3.79$	$TSN^{+}_{L\text{-type}}$  $\mu_{T_2}=11.18,\ \sigma^2_{T_2}=6.32$	 $\mu_{Z_2}=16.49,\ \sigma^2_{Z_2}=10.11$
13	$TSN^{-}_{N\text{-type}}$  $\mu_{T_1}=5.31,\ \sigma^2_{T_1}=3.79$	$TSN^{-}_{L\text{-type}}$  $\mu_{T_2}=5.46,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=10.77,\ \sigma^2_{Z_2}=8.08$	**14**	$TSN^{-}_{N\text{-type}}$  $\mu_{T_1}=5.31,\ \sigma^2_{T_1}=3.79$	$TSN^{+}_{S\text{-type}}$  $\mu_{T_2}=10.54,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=15.85,\ \sigma^2_{Z_2}=8.08$
15	$TSN^{-}_{N\text{-type}}$  $\mu_{T_1}=5.31,\ \sigma^2_{T_1}=3.79$	$TSN^{-}_{S\text{-type}}$  $\mu_{T_2}=4.82,\ \sigma^2_{T_2}=6.32$	 $\mu_{Z_2}=10.13,\ \sigma^2_{Z_2}=10.11$	**16**	$TSN^{+}_{L\text{-type}}$  $\mu_{T_1}=11.18,\ \sigma^2_{T_1}=6.32$	$TSN^{-}_{L\text{-type}}$  $\mu_{T_2}=5.46,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=16.64,\ \sigma^2_{Z_2}=10.61$
17	$TSN^{+}_{L\text{-type}}$  $\mu_{T_1}=11.18,\ \sigma^2_{T_1}=6.32$	$TSN^{+}_{S\text{-type}}$  $\mu_{T_2}=10.54,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=21.72,\ \sigma^2_{Z_2}=10.61$	**18**	$TSN^{+}_{L\text{-type}}$  $\mu_{T_1}=11.18,\ \sigma^2_{T_1}=6.32$	$TSN^{-}_{S\text{-type}}$  $\mu_{T_2}=4.82,\ \sigma^2_{T_2}=6.32$	 $\mu_{Z_2}=16.00,\ \sigma^2_{Z_2}=12.64$
19	$TSN^{-}_{L\text{-type}}$  $\mu_{T_1}=5.46,\ \sigma^2_{T_1}=4.29$	$TSN^{+}_{S\text{-type}}$  $\mu_{T_2}=10.54,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=16.00,\ \sigma^2_{Z_2}=8.58$	**20**	$TSN^{-}_{L\text{-type}}$  $\mu_{T_1}=5.46,\ \sigma^2_{T_1}=4.29$	$TSN^{-}_{S\text{-type}}$ $\mu_{T_2}=4.82,\ \sigma^2_{T_2}=6.32$	$\mu_{Z_2}=10.28,\ \sigma^2_{Z_2}=10.61$

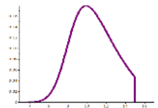
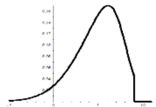
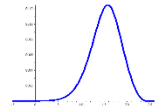
Case #	X_{T_1}	X_{T_2}	$Z_2 = X_{T_1} + X_{T_2}$
21	$TSN^{+}_{S-\text{type}}$  $\mu_{T_1} = 10.54, \sigma^2_{T_1} = 4.29$	$TSN^{-}_{S-\text{type}}$  $\mu_{T_2} = 4.82, \sigma^2_{T_2} = 6.32$	 $\mu_{Z_2} = 15.36, \sigma^2_{Z_2} = 10.61$

Figure 5.3. Twenty-one different cases of the sums of truncated skew NRVs

Table 5.2. Shape parameter α and lower and upper truncation points

Type	α	LTP	UTP	Type	α	LTP	UTP
$TSN^{+}_{N-\text{type}}$	3	7	15	$TSN^{-}_{N-\text{type}}$	-3	1	9
$TSN^{+}_{L-\text{type}}$	3	7	∞	$TSN^{-}_{L-\text{type}}$	-3	1	∞
$TSN^{+}_{S-\text{type}}$	3	$-\infty$	15	$TSN^{-}_{S-\text{type}}$	-3	$-\infty$	9

5.1.3 The convolutions of the sum of truncated normal and truncated skew normal random variables on double truncations

An example of the sum of independent truncated normal and a truncated skew normal random variables is shown in Figure 5.4, where the sum of a doubly truncated skew normal random variable X_{T_1} and a doubly truncated normal random variable X_{T_2} is illustrated.

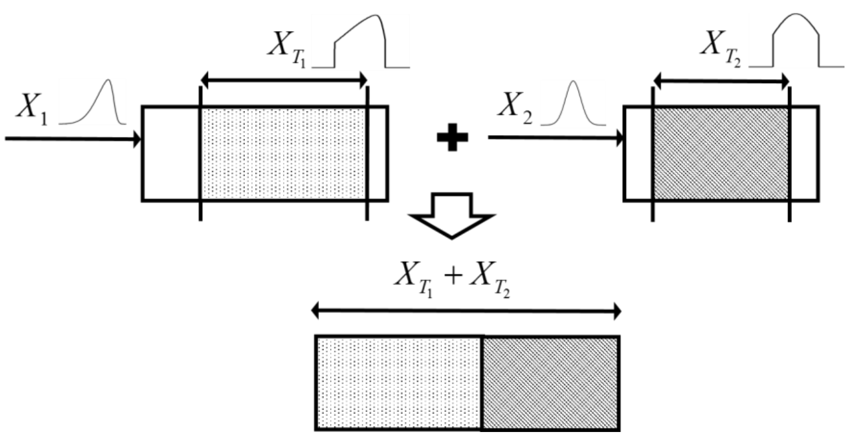


Figure 5.4. Illustration of a sum of truncated normal and truncated skew normal random variables on double truncations

The probability density function of the sum of truncated normal and truncated skew normal random variables is derived as follows

$$f_{X_{T_1}}(x)=\frac{\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\int_{x_{l_1}}^{x_{u_1}}\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{p-\mu_1}{\sigma_1}\right)^2}dp}I_{[x_{l_1},x_{u_1}]}(x) \text{ and}$$

$$f_{Y_{TS_2}}(y)=\frac{\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}\int_{-\infty}^{\alpha_2\frac{y-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_2}}^{y_{u_2}}\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}\left(\int_{-\infty}^{\alpha_2\frac{p-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dp}I_{[y_{l_2},y_{u_2}]}(y), \text{ respectively.}$$

Equating $Z_2 = X_{T_1}+Y_{TS_2}$,

$$f_{Z_2}(z)=\int_{-\infty}^{\infty}f_{Y_{TS_2}}(z-x)f_{X_{T_1}}(x)dx$$

$$=\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}\int_{-\infty}^{\alpha_2\frac{z-x-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_2}}^{y_{u_2}}\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}\left(\int_{-\infty}^{\alpha_2\frac{p-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)ds}\frac{\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\int_{x_{l_1}}^{x_{u_1}}\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{h-u_1}{\sigma_1}\right)^2}dp}dx$$

where $y_{l_2}\le z-x\le y_{u_2}$ and $x_{l_1}\le x\le x_{u_1}$

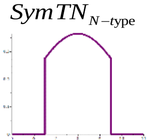
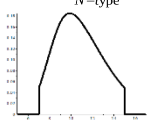
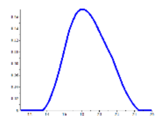
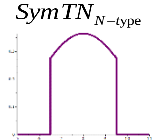
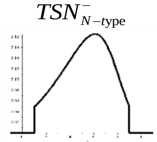
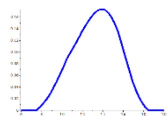
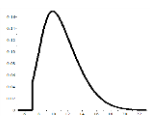
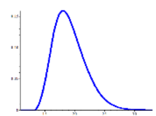
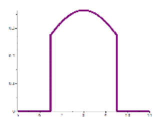
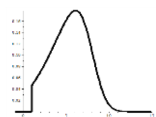
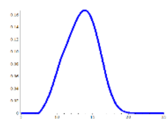
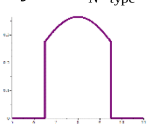
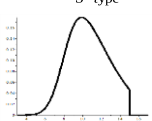
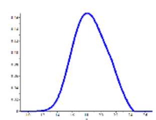
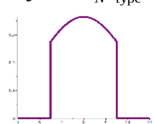
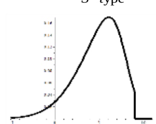
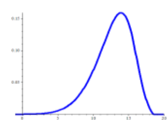
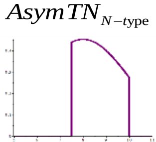
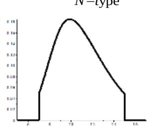
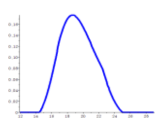
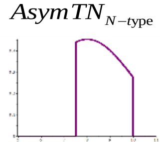
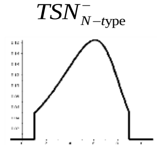
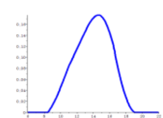
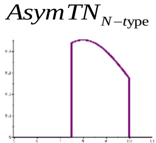
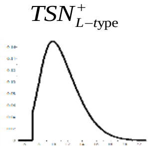
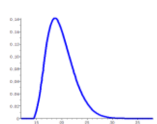
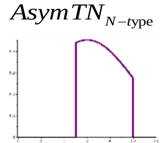
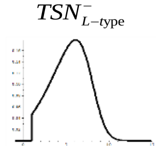
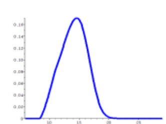
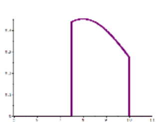
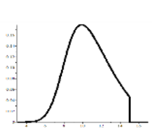
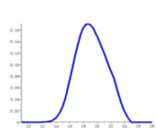
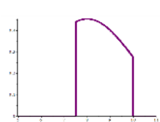
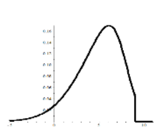
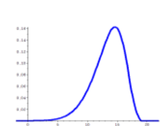
$$=\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}\int_{-\infty}^{\alpha_2\frac{z-x-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_2}}^{y_{u_2}}\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}\left(\int_{-\infty}^{\alpha_2\frac{p-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dp}\frac{\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\int_{x_{l_1}}^{x_{u_1}}\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{h-u_1}{\sigma_1}\right)^2}dh}.$$

$$I_{[z-y_{u_2},z-y_{l_2}]}(x)I_{[x_{l_1},x_{u_1}]}(x)dx.$$

$I_{[y_{l_2},y_{u_2}]}(y)$ can be written as $I_{[z-y_{u_2},z-y_{l_2}]}(x)$ since $z=x+y$.

Twenty-four cases of the sums of truncated normal and truncated skew normal random variables are listed in Figure 5.5. We assume that $\mu_1=\mu_2=8$, $\sigma_1=2$ and $\sigma_2=4$.

As shown in Table 5.3, the shape parameters and the lower and upper truncation points are utilized. It is noted that the shape parameters are zero when truncated normal distributions are considered.

Case #	X_{T_1}	X_{T_2}	$Z_2 = X_{T_1} + X_{T_2}$	Case #	X_{T_1}	X_{T_2}	$Z_2 = X_{T_1} + X_{T_2}$
1	$SymTN_{N-type}$  $\mu_{T_1}=8.00,\ \sigma^2_{T_1}=0.70$	TSN^{+}_{N-type}  $\mu_{T_2}=10.69,\ \sigma^2_{T_2}=3.79$	 $\mu_{Z_2}=18.69,\ \sigma^2_{Z_2}=4.49$	**2**	$SymTN_{N-type}$  $\mu_{T_1}=8.00,\ \sigma^2_{T_1}=0.70$	TSN^{-}_{N-type}  $\mu_{T_2}=5.31,\ \sigma^2_{T_2}=3.79$	 $\mu_{Z_2}=13.31,\ \sigma^2_{Z_2}=4.49$
3	$SymTN_{N-type}$  $\mu_{T_1}=8.00,\ \sigma^2_{T_1}=0.70$	TSN^{+}_{L-type}  $\mu_{T_2}=11.18,\ \sigma^2_{T_2}=6.32$	 $\mu_{Z_2}=19.18,\ \sigma^2_{Z_2}=7.02$	**4**	$SymTN_{N-type}$  $\mu_{T_1}=8.00,\ \sigma^2_{T_1}=0.70$	TSN^{-}_{L-type}  $\mu_{T_2}=5.46,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=13.46,\ \sigma^2_{Z_2}=4.99$
5	$SymTN_{N-type}$  $\mu_{T_1}=8.00,\ \sigma^2_{T_1}=0.70$	TSN^{+}_{S-type}  $\mu_{T_2}=10.54,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=18.54,\ \sigma^2_{Z_2}=4.99$	**6**	$SymTN_{N-type}$  $\mu_{T_1}=8.00,\ \sigma^2_{T_1}=0.70$	TSN^{-}_{S-type}  $\mu_{T_2}=4.82,\ \sigma^2_{T_2}=6.32$	 $\mu_{Z_2}=12.82,\ \sigma^2_{Z_2}=7.02$
7	$AsymTN_{N-type}$  $\mu_{T_1}=8.66,\ \sigma^2_{T_1}=0.49$	TSN^{+}_{N-type}  $\mu_{T_2}=10.69,\ \sigma^2_{T_2}=3.79$	 $\mu_{Z_2}=19.35,\ \sigma^2_{Z_2}=4.28$	**8**	$AsymTN_{N-type}$  $\mu_{T_1}=8.66,\ \sigma^2_{T_1}=0.49$	TSN^{-}_{N-type}  $\mu_{T_2}=5.31,\ \sigma^2_{T_2}=3.79$	 $\mu_{Z_2}=13.97,\ \sigma^2_{Z_2}=4.28$
9	$AsymTN_{N-type}$  $\mu_{T_1}=8.66,\ \sigma^2_{T_1}=0.49$	TSN^{+}_{L-type}  $\mu_{T_2}=11.18,\ \sigma^2_{T_2}=6.32$	 $\mu_{Z_2}=19.84,\ \sigma^2_{Z_2}=6.81$	**10**	$AsymTN_{N-type}$  $\mu_{T_1}=8.66,\ \sigma^2_{T_1}=0.49$	TSN^{-}_{L-type}  $\mu_{T_2}=5.46,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=14.12,\ \sigma^2_{Z_2}=4.78$
11	$AsymTN_{N-type}$  $\mu_{T_1}=8.66,\ \sigma^2_{T_1}=0.49$	TSN^{+}_{S-type}  $\mu_{T_2}=10.54,\ \sigma^2_{T_2}=4.29$	 $\mu_{Z_2}=19.20,\ \sigma^2_{Z_2}=4.78$	**12**	$AsymTN_{N-type}$  $\mu_{T_1}=8.66,\ \sigma^2_{T_1}=0.49$	TSN^{-}_{S-type}  $\mu_{T_2}=4.82,\ \sigma^2_{T_2}=6.32$	 $\mu_{Z_2}=13.48,\ \sigma^2_{Z_2}=6.81$

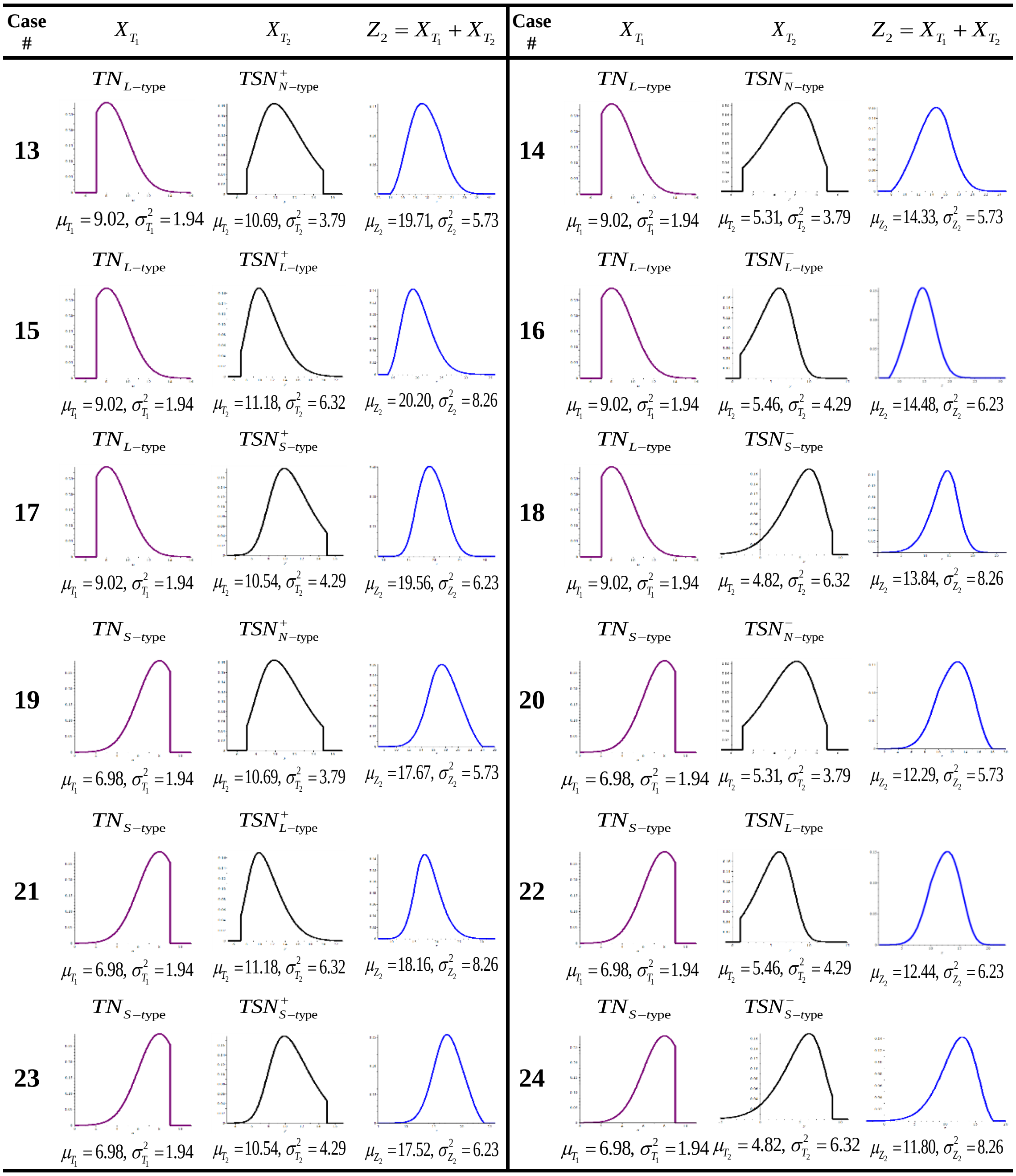


Figure 5.5. Twenty four different cases of sums of TN and truncated skew NRV

Table 5.3. Shape parameter α and lower and upper truncation points based on a truncated skew normal random variable

Type	α	LTP	UTP	Type	α	LTP	UTP
$SymTN_{N-\text{type}}$	0	6.5	9.5	$AsymTN_{N-\text{type}}$	0	7.5	10
$TN_{L-\text{type}}$	0	7	∞	$TN_{S-\text{type}}$	0	$-\infty$	9
$TSN^{+}_{N-\text{type}}$	3	7	15	$TSN^{-}_{N-\text{type}}$	-3	1	9
$TSN^{+}_{L-\text{type}}$	3	7	∞	$TSN^{-}_{L-\text{type}}$	-3	1	∞

5.2 Development of the convolutions of the combinations of truncated normal and truncated skew normal random variables on triple truncations

In this section, we develop the convolutions of the sums of independent truncated normal and truncated skew normal random variables on triple truncations. First, the sums of three truncated normal random variables are discussed in Section 5.2.1. Second, the sums of three truncated skew normal random variables are then examined in Section 5.2.2. Finally, in Section 5.2.3, the sums of the combinations of truncated normal and truncated skew normal random variables on triple truncations are studied.

5.2.1 The convolutions of truncated normal random variables on triple truncations

The probability density function of X_{T_3} is defined as

$$f_{X_{T_3}}(k)=\frac{\dfrac{1}{\sigma_3\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{k-\mu_3}{\sigma_3}\right)^2}}{\displaystyle\int_{x_{l_3}}^{x_{u_3}}\frac{1}{\sigma_3\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2}dv}\,I_{[x_{l_3},x_{u_3}]}(k)\,.$$

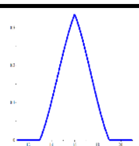
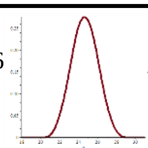
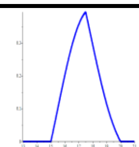
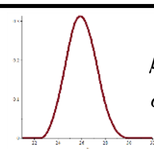
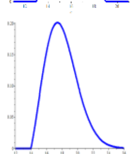
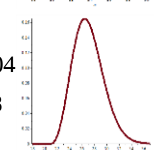
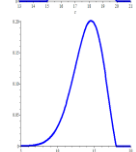
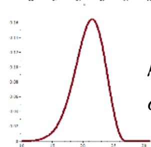
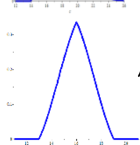
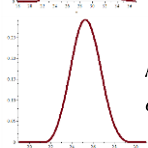
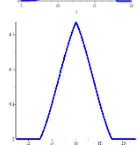
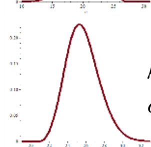
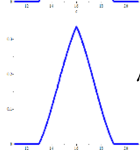
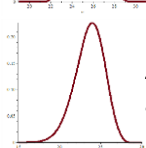
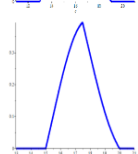
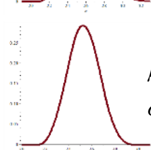
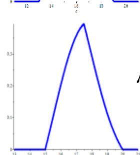
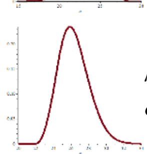
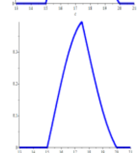
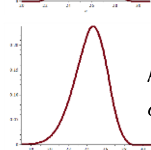
Denoting $Z_3 = Z_2 + X_{T_3}$ where $Z_2 = X_{T_1} + X_{T_2}$, the probability density function of Z_3 is then given by

$$
\begin{aligned}
f_{Z_3}(s) &= \int_{-\infty}^{\infty} f_{X_{T_3}}(s-z) f_{Z_2}(z)dz \\
&= \int_{-\infty}^{\infty} \frac{\frac{1}{\sigma_3\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}}{\int_{x_{l_3}}^{x_{u_3}} \frac{1}{\sigma_3\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2} dv} I_{[x_{l_3}, x_{u_3}]}(s-z) f_{Z_2}(z)dz \\
&= \int_{-\infty}^{\infty} \frac{\frac{1}{\sigma_3\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}}{\int_{x_{l_3}}^{x_{u_3}} \frac{1}{\sigma_3\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2} dv} I_{[x_{l_3}, x_{u_3}]}(s-z) \left(\int_{-\infty}^{\infty} \frac{\frac{1}{\sigma_2\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}}{\int_{x_{l_2}}^{x_{u_2}} \frac{1}{\sigma_2\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2} dp} \cdot \right. \\
&\qquad \left. \frac{\frac{1}{\sigma_1\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\int_{x_{l_1}}^{x_{u_1}} \frac{1}{\sigma_1\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2} dh} I_{[z-x_{u_2}, z-x_{l_2}]}(x) I_{[x_{l_1}, x_{u_1}]}(x)dx \right) dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\frac{1}{\sigma_3\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}}{\int_{x_{l_3}}^{x_{u_3}} \frac{1}{\sigma_3\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2} dv} \frac{\frac{1}{\sigma_2\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}}{\int_{x_{l_2}}^{x_{u_2}} \frac{1}{\sigma_2\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2} dp} \cdot \\
&\qquad \frac{\frac{1}{\sigma_1\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\int_{x_{l_1}}^{x_{u_1}} \frac{1}{\sigma_1\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2} dh} I_{[x_{l_1}, x_{u_1}]}(x) I_{[z-x_{u_2}, z-x_{l_2}]}(x) I_{[s-x_{u_3}, s-x_{l_3}]}(z)dxdz.
\end{aligned}
$$

It is noted that $I_{[x_{l_3}, x_{u_3}]}(s-z)$ can be written as $I_{[s-x_{u_3}, s-x_{l_3}]}(z)$. Twenty cases for triple convolutions of the combinations of truncate normal and truncated skew normal random variables are listed in Table 5.4. The values of parameters and lower and upper truncation points in Section 5.1.1 are utilized. Also, illustrations of the probability densities of Z_3 are shown in Figure 5.6.

Table 5.4. Twenty different cases based on a TNRV

Case #	X_{T_1}	X_{T_2}	X_{T_3}	Case #	X_{T_1}	X_{T_2}	X_{T_3}
1	$Sym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**2**	$Asym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$
3	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	**4**	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$
5	$Sym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**6**	$Sym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$
7	$Sym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	**8**	$Asym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$
9	$Asym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	**10**	$Asym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$
11	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**12**	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	$Asym\ TN_{N-\text{type}}$
13	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	$TN_{S-\text{type}}$	**14**	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$
15	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**16**	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$	$TN_{L-\text{type}}$
17	$Sym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	**18**	$Sym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$
19	$Sym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TN_{S-\text{type}}$	**20**	$Asym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TN_{S-\text{type}}$

Case #	$Z_2 = X_{T_1} + X_{T_2}$	$Z_3 = X_{T_1} + X_{T_2} + X_{T_3}$	Case #	$Z_2 = X_{T_1} + X_{T_2}$	$Z_3 = X_{T_1} + X_{T_2} + X_{T_3}$
1	 $\mu_{Z_2} = 16.66$, $\sigma^2_{Z_2} = 1.19$	 $\mu_{Z_3} = 24.66$, $\sigma^2_{Z_3} = 1.89$	**2**	 $\mu_{Z_2} = 17.32$, $\sigma^2_{Z_2} = 0.98$	 $\mu_{Z_3} = 25.98$, $\sigma^2_{Z_3} = 1.47$
3	 $\mu_{Z_2} = 18.04$, $\sigma^2_{Z_2} = 3.88$	 $\mu_{Z_3} = 27.06$, $\sigma^2_{Z_3} = 5.82$	**4**	 $\mu_{Z_2} = 13.96$, $\sigma^2_{Z_2} = 3.88$	 $\mu_{Z_3} = 20.94$, $\sigma^2_{Z_3} = 5.82$
5	 $\mu_{Z_2} = 16.00$, $\sigma^2_{Z_2} = 1.40$	 $\mu_{Z_3} = 24.66$, $\sigma^2_{Z_3} = 1.89$	**6**	 $\mu_{Z_2} = 16.00$, $\sigma^2_{Z_2} = 1.40$	 $\mu_{Z_3} = 27.02$, $\sigma^2_{Z_3} = 3.34$
7	 $\mu_{Z_2} = 16.00$, $\sigma^2_{Z_2} = 1.40$	 $\mu_{Z_3} = 22.98$, $\sigma^2_{Z_3} = 3.34$	**8**	 $\mu_{Z_2} = 17.32$, $\sigma^2_{Z_2} = 0.98$	 $\mu_{Z_3} = 25.32$, $\sigma^2_{Z_3} = 1.68$
9	 $\mu_{Z_2} = 17.32$, $\sigma^2_{Z_2} = 0.98$	 $\mu_{Z_3} = 26.34$, $\sigma^2_{Z_3} = 2.92$	**10**	 $\mu_{Z_2} = 17.32$, $\sigma^2_{Z_2} = 0.98$	 $\mu_{Z_3} = 24.30$, $\sigma^2_{Z_3} = 2.92$

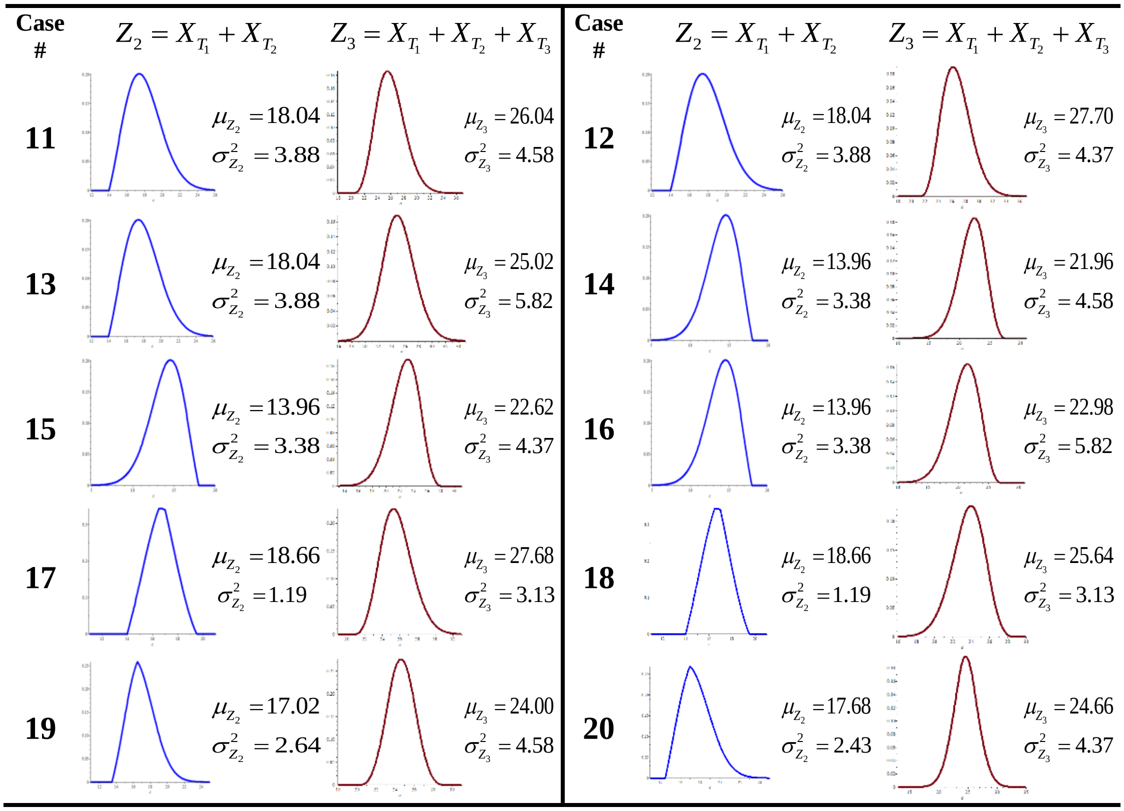


Figure 5.6. Twenty different cases of the sums as listed in Table 5.4

5.2.2 The convolutions of truncated skew normal random variables on triple truncations

The probability density function of Y_{TS_3} is defined as

$$f_{Y_{TS_3}}(k) = \frac{\dfrac{2}{\sigma_3}\dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{k-\mu_3}{\sigma_3}\right)^2} \displaystyle\int_{-\infty}^{\alpha_3 \frac{k-\mu_3}{\sigma_3}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t)^2} dt}{\displaystyle\int_{y_{l_3}}^{y_{u_3}} \frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2} \left(\int_{-\infty}^{\alpha_3 \frac{v-\mu_3}{\sigma_3}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t)^2} dt \right) dv} I_{[y_{l_3}, y_{u_3}]}(k).$$

By denoting $Z_{TS_3} = Z_{TS_2} + Y_{TS_3}$ where $Z_{TS_2} = Y_{TS_1} + Y_{TS_2}$, the probability density function of Z_{TS_3} is obtained as

$$f_{Z_{TS_3}}(s)=\int_{-\infty}^{\infty} f_{Y_{TS_3}}(s-z)f_{Z_{TS_2}}(z)dz$$

$$=\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}\int_{-\infty}^{\alpha_3\frac{s-z-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_3}}^{y_{u_3}}\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2}\left(\int_{-\infty}^{\alpha_3\frac{v-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dv}I_{[y_{l_3},y_{u_3}]}(s-z)\,f_{Z_2}(z)dz$$

$$=\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}\int_{-\infty}^{\alpha_3\frac{s-z-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_3}}^{y_{u_3}}\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2}\left(\int_{-\infty}^{\alpha_3\frac{v-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dv}I_{[y_{l_3},y_{u_3}]}(s-z)\cdot$$

$$\left(\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}\int_{-\infty}^{\alpha_2\frac{z-x-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_2}}^{y_{u_2}}\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}\left(\int_{-\infty}^{\alpha_2\frac{p-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dp}\cdot\right.$$

$$\left.\frac{\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}\int_{-\infty}^{\alpha_1\frac{x-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_1}}^{y_{u_1}}\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2}\int_{-\infty}^{\alpha_1\frac{h-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dtdh}I_{[z-y_{u_2},z-y_{l_2}]}(x)\,I_{[y_{l_1},y_{u_1}]}(x)dx\right)dz$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}\int_{-\infty}^{\alpha_3\frac{s-z-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_3}}^{y_{u_3}}\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2}\left(\int_{-\infty}^{\alpha_3\frac{v-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dv}\cdot$$

$$\frac{\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}\int_{-\infty}^{\alpha_2\frac{z-x-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_2}}^{y_{u_2}}\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}\left(\int_{-\infty}^{\alpha_2\frac{p-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dp}\cdot$$

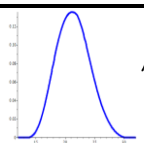
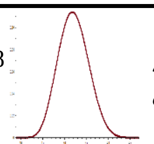
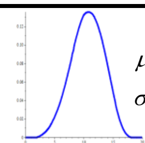
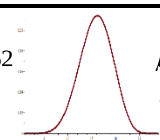
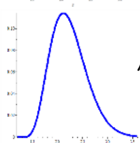
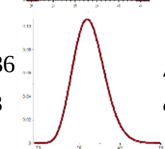
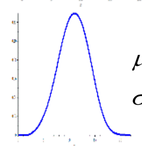
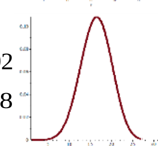
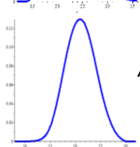
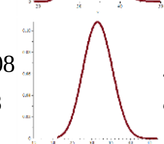
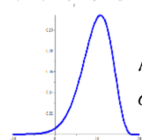
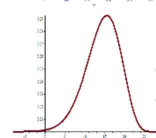
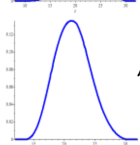
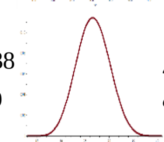
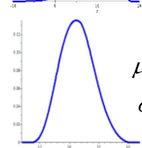
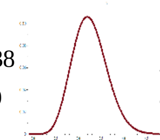
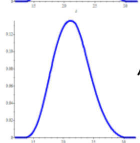
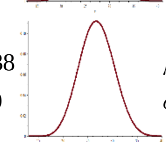
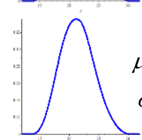
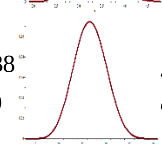
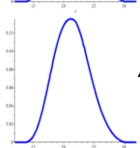
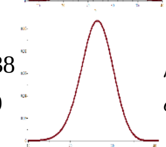
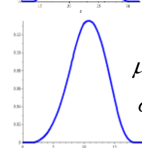
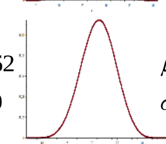
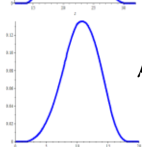
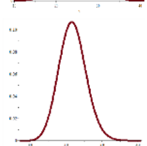
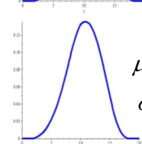
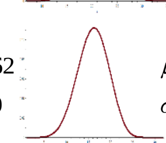
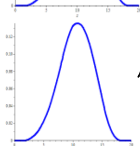
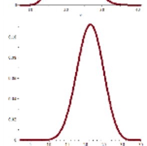
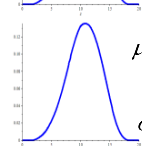
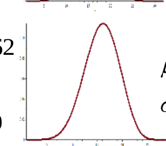
$$\frac{\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}\int_{-\infty}^{\alpha_1\frac{x-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_1}}^{y_{u_1}}\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2}\left(\int_{-\infty}^{\alpha_1\frac{h-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dh} I_{[x_{l_1},x_{u_1}]}(x) I_{[z-y_{u_2},z-y_{l_2}]}(x) I_{[s-y_{u_3},s-y_{l_3}]}(z)dxdz.$$

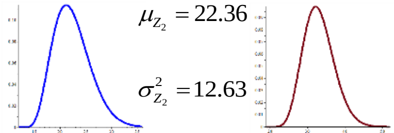
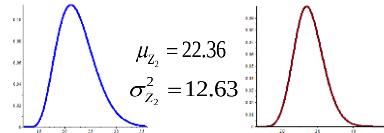
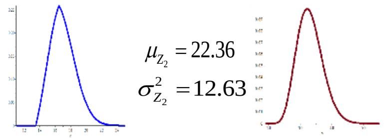
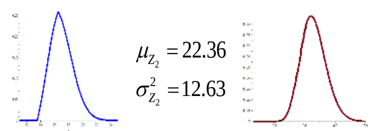
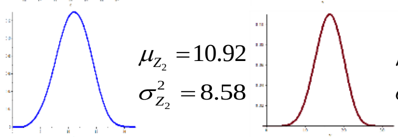
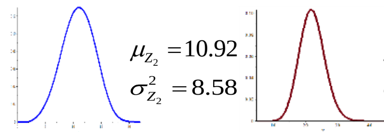
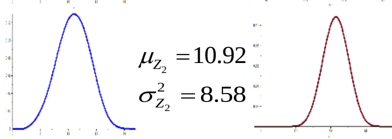
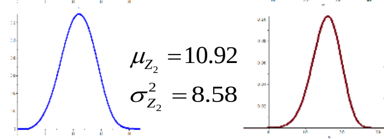
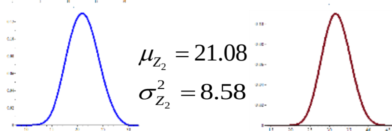
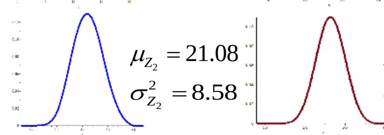
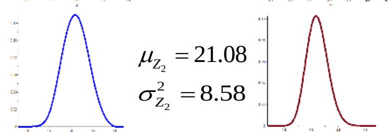
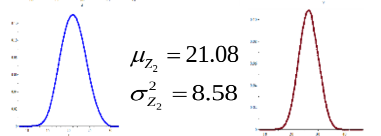
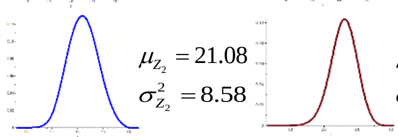
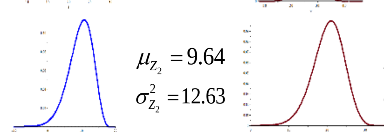
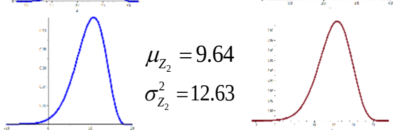
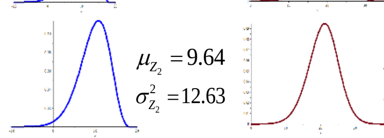
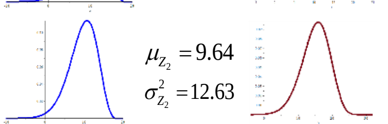
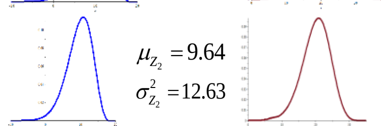
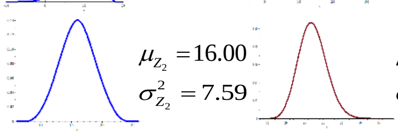
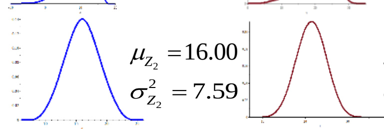
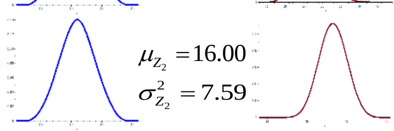
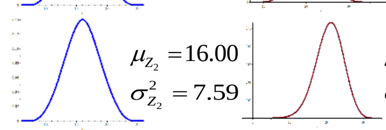
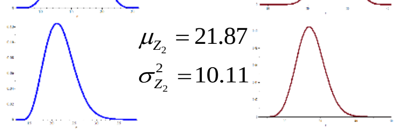
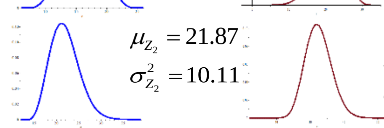
Since $s = z + k$, $I_{[y_{l_3},y_{u_3}]}(s-z)$ can be written as $I_{[s-y_{u_3},s-y_{l_3}]}(z)$. Fifty-six cases are presented in Table 5.5 and Figure 5.7. The values of parameters and lower and upper truncation points utilized in Section 5.1.2 are applied

Table 5.5. Fifty six different cases based on a TN and truncated skew NRV

Case #	X_{T_1}	X_{T_2}	X_{T_3}	Case #	X_{T_1}	X_{T_2}	X_{T_3}
1	$TSN^+_{N-\text{type}}$	$TSN^+_{N-\text{type}}$	$TSN^+_{N-\text{type}}$	**2**	$TSN^-_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	$TSN^-_{N-\text{type}}$
3	$TSN^+_{L-\text{type}}$	$TSN^+_{L-\text{type}}$	$TSN^+_{L-\text{type}}$	**4**	$TSN^-_{L-\text{type}}$	$TSN^-_{L-\text{type}}$	$TSN^-_{L-\text{type}}$
5	$TSN^+_{S-\text{type}}$	$TSN^+_{S-\text{type}}$	$TSN^+_{S-\text{type}}$	**6**	$TSN^-_{S-\text{type}}$	$TSN^-_{S-\text{type}}$	$TSN^-_{S-\text{type}}$
7	$TSN^+_{N-\text{type}}$	$TSN^+_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	**8**	$TSN^+_{N-\text{type}}$	$TSN^+_{N-\text{type}}$	$TSN^+_{L-\text{type}}$
9	$TSN^+_{N-\text{type}}$	$TSN^+_{N-\text{type}}$	$TSN^-_{L-\text{type}}$	**10**	$TSN^+_{N-\text{type}}$	$TSN^+_{N-\text{type}}$	$TSN^+_{S-\text{type}}$
11	$TSN^+_{N-\text{type}}$	$TSN^+_{N-\text{type}}$	$TSN^-_{S-\text{type}}$	**12**	$TSN^-_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	$TSN^+_{N-\text{type}}$
13	$TSN^-_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	$TSN^+_{L-\text{type}}$	**14**	$TSN^-_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	$TSN^-_{L-\text{type}}$
15	$TSN^-_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	$TSN^+_{S-\text{type}}$	**16**	$TSN^-_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	$TSN^-_{S-\text{type}}$
17	$TSN^+_{L-\text{type}}$	$TSN^+_{L-\text{type}}$	$TSN^+_{N-\text{type}}$	**18**	$TSN^+_{L-\text{type}}$	$TSN^+_{L-\text{type}}$	$TSN^-_{N-\text{type}}$
19	$TSN^+_{L-\text{type}}$	$TSN^+_{L-\text{type}}$	$TSN^-_{L-\text{type}}$	**20**	$TSN^+_{L-\text{type}}$	$TSN^+_{L-\text{type}}$	$TSN^+_{S-\text{type}}$
21	$TSN^+_{L-\text{type}}$	$TSN^+_{L-\text{type}}$	$TSN^-_{S-\text{type}}$	**22**	$TSN^-_{L-\text{type}}$	$TSN^-_{L-\text{type}}$	$TSN^+_{N-\text{type}}$
23	$TSN^-_{L-\text{type}}$	$TSN^-_{L-\text{type}}$	$TSN^-_{N-\text{type}}$	**24**	$TSN^-_{L-\text{type}}$	$TSN^-_{L-\text{type}}$	$TSN^-_{L-\text{type}}$
25	$TSN^-_{L-\text{type}}$	$TSN^-_{L-\text{type}}$	$TSN^+_{S-\text{type}}$	**26**	$TSN^-_{L-\text{type}}$	$TSN^-_{L-\text{type}}$	$TSN^-_{S-\text{type}}$
27	$TSN^+_{S-\text{type}}$	$TSN^+_{S-\text{type}}$	$TSN^+_{N-\text{type}}$	**28**	$TSN^+_{S-\text{type}}$	$TSN^+_{S-\text{type}}$	$TSN^-_{N-\text{type}}$
29	$TSN^+_{S-\text{type}}$	$TSN^+_{S-\text{type}}$	$TSN^+_{L-\text{type}}$	**30**	$TSN^+_{S-\text{type}}$	$TSN^+_{S-\text{type}}$	$TSN^-_{L-\text{type}}$
31	$TSN^+_{S-\text{type}}$	$TSN^+_{S-\text{type}}$	$TSN^-_{S-\text{type}}$	**32**	$TSN^-_{S-\text{type}}$	$TSN^-_{S-\text{type}}$	$TSN^+_{N-\text{type}}$
33	$TSN^-_{S-\text{type}}$	$TSN^-_{S-\text{type}}$	$TSN^-_{N-\text{type}}$	**34**	$TSN^-_{S-\text{type}}$	$TSN^-_{S-\text{type}}$	$TSN^+_{L-\text{type}}$
35	$TSN^-_{S-\text{type}}$	$TSN^-_{S-\text{type}}$	$TSN^-_{L-\text{type}}$	**36**	$TSN^-_{S-\text{type}}$	$TSN^-_{S-\text{type}}$	$TSN^+_{S-\text{type}}$
37	$TSN^+_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	$TSN^+_{L-\text{type}}$	**38**	$TSN^+_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	$TSN^-_{L-\text{type}}$
39	$TSN^+_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	$TSN^+_{S-\text{type}}$	**40**	$TSN^+_{N-\text{type}}$	$TSN^-_{N-\text{type}}$	$TSN^-_{S-\text{type}}$
41	$TSN^+_{N-\text{type}}$	$TSN^+_{L-\text{type}}$	$TSN^-_{L-\text{type}}$	**42**	$TSN^+_{N-\text{type}}$	$TSN^+_{L-\text{type}}$	$TSN^+_{S-\text{type}}$
43	$TSN^+_{N-\text{type}}$	$TSN^+_{L-\text{type}}$	$TSN^-_{S-\text{type}}$	**44**	$TSN^+_{N-\text{type}}$	$TSN^-_{L-\text{type}}$	$TSN^+_{S-\text{type}}$
45	$TSN^+_{N-\text{type}}$	$TSN^-_{L-\text{type}}$	$TSN^-_{S-\text{type}}$	**46**	$TSN^+_{N-\text{type}}$	$TSN^+_{S-\text{type}}$	$TSN^-_{S-\text{type}}$

Case #	X_{T_1}	X_{T_2}	X_{T_3}	Case #	X_{T_1}	X_{T_2}	X_{T_3}
47	$TSN^-_{N\text{-type}}$	$TSN^+_{L\text{-type}}$	$TSN^-_{L\text{-type}}$	**48**	$TSN^-_{N\text{-type}}$	$TSN^+_{L\text{-type}}$	$TSN^+_{S\text{-type}}$
49	$TSN^-_{N\text{-type}}$	$TSN^+_{L\text{-type}}$	$TSN^-_{S\text{-type}}$	**50**	$TSN^-_{N\text{-type}}$	$TSN^-_{L\text{-type}}$	$TSN^+_{S\text{-type}}$
51	$TSN^-_{N\text{-type}}$	$TSN^-_{L\text{-type}}$	$TSN^-_{S\text{-type}}$	**52**	$TSN^-_{N\text{-type}}$	$TSN^+_{S\text{-type}}$	$TSN^-_{S\text{-type}}$
53	$TSN^+_{L\text{-type}}$	$TSN^-_{L\text{-type}}$	$TSN^+_{S\text{-type}}$	**54**	$TSN^+_{L\text{-type}}$	$TSN^-_{L\text{-type}}$	$TSN^-_{S\text{-type}}$
55	$TSN^+_{L\text{-type}}$	$TSN^+_{S\text{-type}}$	$TSN^-_{S\text{-type}}$	**56**	$TSN^-_{L\text{-type}}$	$TSN^+_{S\text{-type}}$	$TSN^-_{S\text{-type}}$

Case #	$Z_2 = Y_{T_1} + Y_{T_2}$		$Z_3 = Y_{T_1} + Y_{T_2} + Y_{T_3}$		Case #	$Z_2 = Y_{T_1} + Y_{T_2}$		$Z_3 = Y_{T_1} + Y_{T_2} + Y_{T_3}$	
1		$\mu_{Z_2} = 21.38$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 32.07$ $\sigma^2_{Z_3} = 11.38$	**2**		$\mu_{Z_2} = 10.62$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 15.93$ $\sigma^2_{Z_3} = 11.38$
3		$\mu_{Z_2} = 22.36$ $\sigma^2_{Z_2} = 12.63$		$\mu_{Z_3} = 33.54$ $\sigma^2_{Z_3} = 18.95$	**4**		$\mu_{Z_2} = 10.92$ $\sigma^2_{Z_2} = 8.58$		$\mu_{Z_3} = 16.38$ $\sigma^2_{Z_3} = 12.87$
5		$\mu_{Z_2} = 21.08$ $\sigma^2_{Z_2} = 8.58$		$\mu_{Z_3} = 31.62$ $\sigma^2_{Z_3} = 12.87$	**6**		$\mu_{Z_2} = 9.64$ $\sigma^2_{Z_2} = 12.63$		$\mu_{Z_3} = 14.46$ $\sigma^2_{Z_3} = 18.95$
7		$\mu_{Z_2} = 21.38$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 26.69$ $\sigma^2_{Z_3} = 11.38$	**8**		$\mu_{Z_2} = 21.38$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 32.56$ $\sigma^2_{Z_3} = 13.90$
9		$\mu_{Z_2} = 21.38$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 26.84$ $\sigma^2_{Z_3} = 11.88$	**10**		$\mu_{Z_2} = 21.38$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 31.92$ $\sigma^2_{Z_3} = 11.88$
11		$\mu_{Z_2} = 21.38$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 26.20$ $\sigma^2_{Z_3} = 13.90$	**12**		$\mu_{Z_2} = 10.62$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 21.31$ $\sigma^2_{Z_3} = 11.38$
13		$\mu_{Z_2} = 10.62$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 21.80$ $\sigma^2_{Z_3} = 13.90$	**14**		$\mu_{Z_2} = 10.62$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 16.08$ $\sigma^2_{Z_3} = 11.88$
15		$\mu_{Z_2} = 10.62$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 21.16$ $\sigma^2_{Z_3} = 11.88$	**16**		$\mu_{Z_2} = 10.62$ $\sigma^2_{Z_2} = 7.59$		$\mu_{Z_3} = 15.44$ $\sigma^2_{Z_3} = 13.90$

Case #	$Z_2 = Y_{T_1} + Y_{T_2}$	$Z_3 = Y_{T_1} + Y_{T_2} + Y_{T_3}$	Case #	$Z_2 = Y_{T_1} + Y_{T_2}$	$Z_3 = Y_{T_1} + Y_{T_2} + Y_{T_3}$
17	 $\mu_{Z_2} = 22.36$, $\sigma^2_{Z_2} = 12.63$	$\mu_{Z_3} = 33.05$, $\sigma^2_{Z_3} = 16.43$	**18**	 $\mu_{Z_2} = 22.36$, $\sigma^2_{Z_2} = 12.63$	$\mu_{Z_3} = 27.67$, $\sigma^2_{Z_3} = 16.43$
19	 $\mu_{Z_2} = 22.36$, $\sigma^2_{Z_2} = 12.63$	$\mu_{Z_3} = 27.82$, $\sigma^2_{Z_3} = 16.92$	**20**	 $\mu_{Z_2} = 22.36$, $\sigma^2_{Z_2} = 12.63$	$\mu_{Z_3} = 32.90$, $\sigma^2_{Z_3} = 16.92$
23	 $\mu_{Z_2} = 10.92$, $\sigma^2_{Z_2} = 8.58$	$\mu_{Z_3} = 16.23$, $\sigma^2_{Z_3} = 12.37$	**24**	 $\mu_{Z_2} = 10.92$, $\sigma^2_{Z_2} = 8.58$	$\mu_{Z_3} = 22.10$, $\sigma^2_{Z_3} = 14.90$
25	 $\mu_{Z_2} = 10.92$, $\sigma^2_{Z_2} = 8.58$	$\mu_{Z_3} = 21.46$, $\sigma^2_{Z_3} = 12.87$	**26**	 $\mu_{Z_2} = 10.92$, $\sigma^2_{Z_2} = 8.58$	$\mu_{Z_3} = 15.74$, $\sigma^2_{Z_3} = 14.90$
27	 $\mu_{Z_2} = 21.08$, $\sigma^2_{Z_2} = 8.58$	$\mu_{Z_3} = 31.77$, $\sigma^2_{Z_3} = 12.37$	**28**	 $\mu_{Z_2} = 21.08$, $\sigma^2_{Z_2} = 8.58$	$\mu_{Z_3} = 26.39$, $\sigma^2_{Z_3} = 12.37$
29	 $\mu_{Z_2} = 21.08$, $\sigma^2_{Z_2} = 8.58$	$\mu_{Z_3} = 32.26$, $\sigma^2_{Z_3} = 14.90$	**30**	 $\mu_{Z_2} = 21.08$, $\sigma^2_{Z_2} = 8.58$	$\mu_{Z_3} = 26.54$, $\sigma^2_{Z_3} = 12.87$
31	 $\mu_{Z_2} = 21.08$, $\sigma^2_{Z_2} = 8.58$	$\mu_{Z_3} = 25.90$, $\sigma^2_{Z_3} = 14.90$	**32**	 $\mu_{Z_2} = 9.64$, $\sigma^2_{Z_2} = 12.63$	$\mu_{Z_3} = 20.33$, $\sigma^2_{Z_3} = 16.43$
33	 $\mu_{Z_2} = 9.64$, $\sigma^2_{Z_2} = 12.63$	$\mu_{Z_3} = 14.95$, $\sigma^2_{Z_3} = 16.43$	**34**	 $\mu_{Z_2} = 9.64$, $\sigma^2_{Z_2} = 12.63$	$\mu_{Z_3} = 20.82$, $\sigma^2_{Z_3} = 18.95$
35	 $\mu_{Z_2} = 9.64$, $\sigma^2_{Z_2} = 12.63$	$\mu_{Z_3} = 15.10$, $\sigma^2_{Z_3} = 16.92$	**36**	 $\mu_{Z_2} = 9.64$, $\sigma^2_{Z_2} = 12.63$	$\mu_{Z_3} = 20.18$, $\sigma^2_{Z_3} = 16.92$
37	 $\mu_{Z_2} = 16.00$, $\sigma^2_{Z_2} = 7.59$	$\mu_{Z_3} = 27.18$, $\sigma^2_{Z_3} = 13.90$	**38**	 $\mu_{Z_2} = 16.00$, $\sigma^2_{Z_2} = 7.59$	$\mu_{Z_3} = 21.46$, $\sigma^2_{Z_3} = 11.88$
39	 $\mu_{Z_2} = 16.00$, $\sigma^2_{Z_2} = 7.59$	$\mu_{Z_3} = 26.54$, $\sigma^2_{Z_3} = 11.88$	**40**	 $\mu_{Z_2} = 16.00$, $\sigma^2_{Z_2} = 7.59$	$\mu_{Z_3} = 20.82$, $\sigma^2_{Z_3} = 13.90$
41	 $\mu_{Z_2} = 21.87$, $\sigma^2_{Z_2} = 10.11$	$\mu_{Z_3} = 27.33$, $\sigma^2_{Z_3} = 14.40$	**42**	 $\mu_{Z_2} = 21.87$, $\sigma^2_{Z_2} = 10.11$	$\mu_{Z_3} = 32.41$, $\sigma^2_{Z_3} = 14.40$

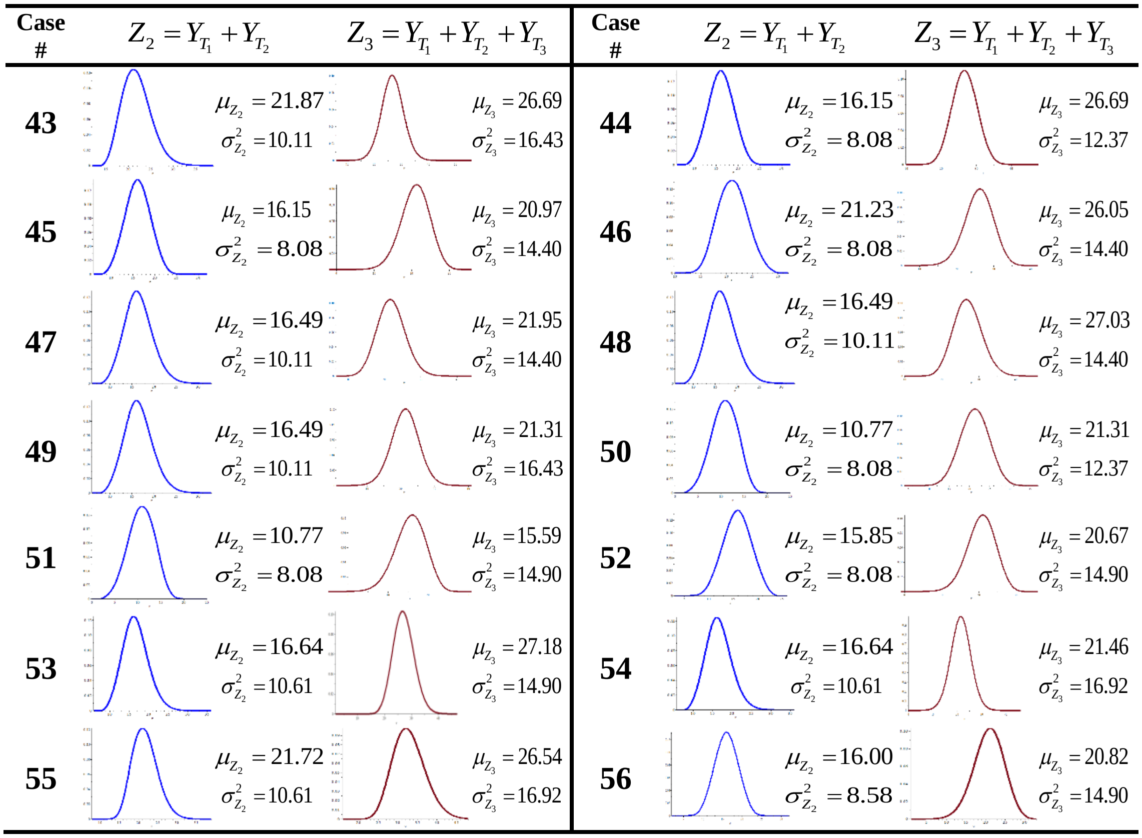


Figure 5.7. Fifty-six cases of the sums as listed in Table 5.5

5.2.3 The convolutions of the combinations of truncated normal and truncated skew normal random variables on triple truncations

Figure 5.8 illustrates an example of the sum of truncated normal and truncated skew normal random variables on triple truncations. The mean and variance of $X_{T_1} + X_{T_2} + X_{T_3}$ are the sums of means and variances of X_{T_1}, X_{T_2} and since X_{T_1}, X_{T_2} and X_{T_3} are independent of each other.

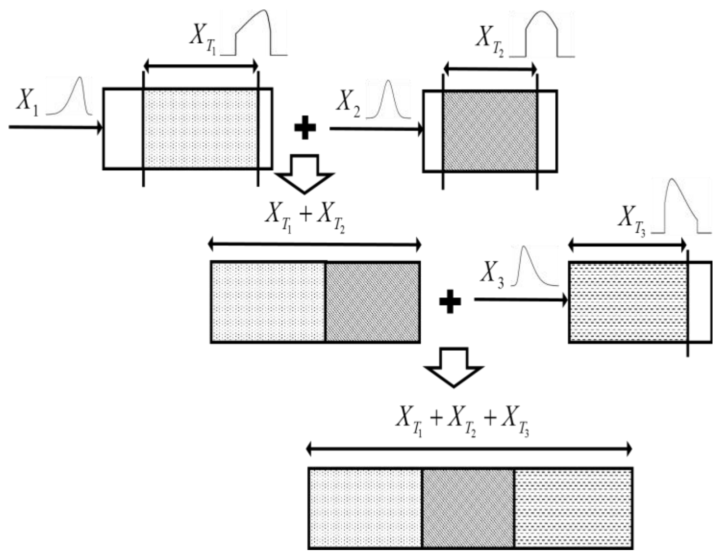


Figure 5.8. Illustration of a sum of truncated normal and truncated skew normal random variables on triple convolutions

In this section, we have two subsections. First, the sums of two truncated normal random variables and one truncated skew normal random variable are examined in Section 5.3.1. Second, the sums of one truncated normal random variable and two truncated skew normal random variables are investigated in Section 5.3.2. We provide only cases of the sums without the properties such as distributions, means and variances of the sums because cases are too many to discuss. In Section 6.1, however, we will discuss a numerical example.

5.2.3.1 Sums of two truncated NRVs and one truncated skew NRV

Let Z_2 and Z_3 be $X_{T_1}+X_{T_2}$ and $Z_2+Y_{TS_3}$, respectively. Therefore, the probability density function of Z_3 is obtained as

$$f_{Z_3}(s)=\int_{-\infty}^{\infty} f_{Y_{TS_3}}(s-z)f_{Z_2}(z)dz$$

$$=\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}\int_{-\infty}^{\alpha_3\frac{s-z-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_3}}^{y_{u_3}}\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2}\left(\int_{-\infty}^{\alpha_3\frac{v-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dv}I_{[y_{l_3},y_{u_3}]}(s-z)f_{Z_2}(z)dz$$

$$=\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}\int_{-\infty}^{\alpha_3\frac{s-z-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_3}}^{y_{u_3}}\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2}\left(\int_{-\infty}^{\alpha_3\frac{v-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dv}I_{[y_{l_3},y_{u_3}]}(s-z)\cdot$$

$$\left(\int_{-\infty}^{\infty}\frac{\frac{1}{\sigma_2\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}}{\int_{x_{l_2}}^{x_{u_2}}\frac{1}{\sigma_2\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}dp}\frac{\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\int_{x_{l_1}}^{x_{u_1}}\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2}dh}I_{[z-x_{u_2},z-x_{l_2}]}(x)I_{[x_{l_1},x_{u_1}]}(x)dx\right)dz$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}\int_{-\infty}^{\alpha_3\frac{s-z-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_3}}^{y_{u_3}}\frac{2}{\sigma_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2}\left(\int_{-\infty}^{\alpha_3\frac{v-\mu_3}{\sigma_3}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dv}\frac{\frac{1}{\sigma_2\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}}{\int_{x_{l_2}}^{x_{u_2}}\frac{1}{\sigma_2\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{s-\mu_2}{\sigma_2}\right)^2}ds}\cdot$$

$$\frac{\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\int_{x_{l_1}}^{x_{u_1}}\frac{1}{\sigma_1\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2}dh}I_{[x_{l_1},x_{u_1}]}(x)I_{[z-x_{u_2},z-x_{l_2}]}(x)I_{[s-y_{u_3},s-y_{l_3}]}(z)dxdz.$$

Sixty cases are summarized in Table 5.6.

Table 5.6. Sixty different cases based on two TNRVs and one truncated skew NRV

Case #	X_{T_1}	X_{T_2}	Y_{TS_3}	**Case #**	X_{T_1}	X_{T_2}	Y_{TS_3}
1	*Sym* $TN_{N-\text{type}}$	*Sym* $TN_{N-\text{type}}$	$TSN^+_{N-\text{type}}$	**2**	*Asym* $TN_{N-\text{type}}$	*Asym* $TN_{N-\text{type}}$	$TSN^+_{N-\text{type}}$
3	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	$TSN^+_{N-\text{type}}$	**4**	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$	$TSN^+_{N-\text{type}}$
5	*Sym* $TN_{N-\text{type}}$	*Asym* $TN_{N-\text{type}}$	$TSN^+_{N-\text{type}}$	**6**	*Sym* $TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^+_{N-\text{type}}$
7	*Sym* $TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^+_{N-\text{type}}$	**8**	*Asym* $TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^+_{N-\text{type}}$

Case #	X_{T_1}	X_{T_2}	Y_{TS_3}	Case #	X_{T_1}	X_{T_2}	Y_{TS_3}
9	$Asym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{+}_{N-\text{type}}$	**10**	$TN_{L-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{+}_{N-\text{type}}$
11	$Sym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$	**12**	$Asym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$
13	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{-}_{N-\text{type}}$	**14**	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{N-\text{type}}$
15	$Sym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$	**16**	$Sym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{-}_{N-\text{type}}$
17	$Sym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{N-\text{type}}$	**18**	$Asym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{-}_{N-\text{type}}$
19	$Asym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{N-\text{type}}$	**20**	$TN_{L-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{N-\text{type}}$
21	$Sym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$	**22**	$Asym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$
23	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{+}_{L-\text{type}}$	**24**	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{+}_{L-\text{type}}$
25	$Sym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$	**26**	$Sym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{+}_{L-\text{type}}$
27	$Sym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{+}_{L-\text{type}}$	**28**	$Asym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{+}_{L-\text{type}}$
29	$Asym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{+}_{L-\text{type}}$	**30**	$TN_{L-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{+}_{L-\text{type}}$
31	$Sym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$	**32**	$Asym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$
33	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$	**34**	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{L-\text{type}}$
35	$Sym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$	**36**	$Sym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$
37	$Sym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{L-\text{type}}$	**38**	$Asym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$
39	$Asym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{L-\text{type}}$	**40**	$TN_{L-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{L-\text{type}}$
41	$Sym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$	**42**	$Asym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$
43	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$	**44**	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{+}_{S-\text{type}}$
45	$Sym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$	**46**	$Sym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$
47	$Sym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{+}_{S-\text{type}}$	**48**	$Asym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$
49	$Asym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{+}_{S-\text{type}}$	**50**	$TN_{L-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{+}_{S-\text{type}}$
51	$Sym\ TN_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$	**52**	$Asym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$
53	$TN_{L-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$	**54**	$TN_{S-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$
55	$Sym\ TN_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$	**56**	$Sym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$
57	$Sym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$	**58**	$Asym\ TN_{N-\text{type}}$	$TN_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$
59	$Asym\ TN_{N-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$	**60**	$TN_{L-\text{type}}$	$TN_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$

5.2.3.2 Sums of one truncated NRVs and two truncated skew NRVs

Denoting Z_2 be $Y_{TS_1}+Y_{TS_2}$ and Z_3 be $Y_{TS_1}+Y_{TS_2}+X_{T_3}$, Z_3 can be expressed as $Z_2+X_{T_3}$. Therefore, the probability density function of Z_3 is expressed as

$$f_{Z_3}(s)=\int_{-\infty}^{\infty} f_{X_{TS_3}}(s-z)f_{Z_2}(z)dz$$

$$=\int_{-\infty}^{\infty}\frac{\frac{1}{\sigma_3\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}}{\int_{x_{l_3}}^{x_{u_3}}\frac{1}{\sigma_3\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2}dv} I_{[x_{l_3},x_{u_3}]}(s-z)f_{Z_2}(z)dz$$

$$=\int_{-\infty}^{\infty}\frac{\frac{1}{\sigma_3\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}}{\int_{x_{l_3}}^{x_{u_3}}\frac{1}{\sigma_3\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2}dv} I_{[x_{l_3},x_{u_3}]}(s-z)\cdot$$

$$\left(\int_{-\infty}^{\infty}\frac{\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}\int_{-\infty}^{\alpha_2\frac{z-x-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_2}}^{y_{u_2}}\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}\left(\int_{-\infty}^{\alpha_2\frac{p-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dp}\cdot\right.$$

$$\left.\frac{\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}\int_{-\infty}^{\alpha_1\frac{x-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(t)^2}dt}{\int_{y_{l_1}}^{y_{u_1}}\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2}\left(\int_{-\infty}^{\alpha_1\frac{h-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(t)^2}dt\right)dh} I_{[z-y_{u_2},z-y_{l_2}]}(x)I_{[y_{l_1},y_{u_1}]}(x)dx\right)dz$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\frac{1}{\sigma_3\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{s-z-\mu_3}{\sigma_3}\right)^2}}{\int_{x_{l_3}}^{x_{u_3}}\frac{1}{\sigma_3\sqrt{2\pi}}\exp^{-\frac{1}{2}\left(\frac{v-\mu_3}{\sigma_3}\right)^2}dv}\frac{\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}\int_{-\infty}^{\alpha_2\frac{z-x-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_2}}^{y_{u_2}}\frac{2}{\sigma_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{p-\mu_2}{\sigma_2}\right)^2}\left(\int_{-\infty}^{\alpha_2\frac{p-\mu_2}{\sigma_2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dp}\cdot$$

$$\frac{\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}\int_{-\infty}^{\alpha_1\frac{x-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt}{\int_{y_{l_1}}^{y_{u_1}}\frac{2}{\sigma_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{h-\mu_1}{\sigma_1}\right)^2}\left(\int_{-\infty}^{\alpha_1\frac{h-\mu_1}{\sigma_1}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\right)dh}\cdot$$

$$I_{[y_{l_1},y_{u_1}]}(x)I_{[z-y_{u_2},z-y_{l_2}]}(x)I_{[s-z-x_{u_3},s-z-x_{l_3}]}(z)dxdz.$$

There are eighty-four cases for the combinations of one truncated normal and two truncated skew normal random variables, which is listed in Table 5.7.

Table 5.7. Eight four different cases based on one TNRVs and one truncated skew NRVs

Case #	Y_{TS_1}	Y_{TS_2}	X_{T_3}	**Case #**	Y_{TS_1}	Y_{TS_2}	X_{T_3}
1	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**2**	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$
3	$TSN^{+}_{L-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**4**	$TSN^{-}_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$Sym\ TN_{N-\text{type}}$
5	$TSN^{+}_{S-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**6**	$TSN^{-}_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$
7	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**8**	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$Sym\ TN_{N-\text{type}}$
9	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**10**	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$
11	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**12**	$TSN^{-}_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$Sym\ TN_{N-\text{type}}$
13	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**14**	$TSN^{-}_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$
15	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**16**	$TSN^{+}_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$Sym\ TN_{N-\text{type}}$
17	$TSN^{+}_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**18**	$TSN^{+}_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$
19	$TSN^{-}_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**20**	$TSN^{-}_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$
21	$TSN^{+}_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Sym\ TN_{N-\text{type}}$	**22**	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$
23	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**24**	$TSN^{+}_{L-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$Asym\ TN_{N-\text{type}}$
25	$TSN^{-}_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**26**	$TSN^{+}_{S-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$
27	$TSN^{-}_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**28**	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$	$Asym\ TN_{N-\text{type}}$
29	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**30**	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$Asym\ TN_{N-\text{type}}$
31	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**32**	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$
33	$TSN^{-}_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**34**	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$Asym\ TN_{N-\text{type}}$
35	$TSN^{-}_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**36**	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$
37	$TSN^{+}_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**38**	$TSN^{+}_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$
39	$TSN^{+}_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**40**	$TSN^{-}_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$
41	$TSN^{-}_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$	**42**	$TSN^{+}_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$Asym\ TN_{N-\text{type}}$
43	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{N-\text{type}}$	$TN_{L-\text{type}}$	**44**	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$	$TN_{L-\text{type}}$
45	$TSN^{+}_{L-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$TN_{L-\text{type}}$	**46**	$TSN^{-}_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$TN_{L-\text{type}}$
47	$TSN^{+}_{S-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$TN_{L-\text{type}}$	**48**	$TSN^{-}_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{L-\text{type}}$
49	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$	$TN_{L-\text{type}}$	**50**	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$TN_{L-\text{type}}$
51	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$TN_{L-\text{type}}$	**52**	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$TN_{L-\text{type}}$
53	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{L-\text{type}}$	**54**	$TSN^{-}_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$TN_{L-\text{type}}$
55	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$TN_{L-\text{type}}$	**56**	$TSN^{-}_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$TN_{L-\text{type}}$
57	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{L-\text{type}}$	**58**	$TSN^{+}_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$TN_{L-\text{type}}$
59	$TSN^{+}_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$TN_{L-\text{type}}$	**60**	$TSN^{+}_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{L-\text{type}}$
61	$TSN^{-}_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$TN_{L-\text{type}}$	**62**	$TSN^{-}_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{L-\text{type}}$
63	$TSN^{+}_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{L-\text{type}}$	**64**	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{N-\text{type}}$	$TN_{L-\text{type}}$
65	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$	$TN_{S-\text{type}}$	**66**	$TSN^{+}_{L-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$TN_{S-\text{type}}$
67	$TSN^{-}_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$TN_{S-\text{type}}$	**68**	$TSN^{+}_{S-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$TN_{S-\text{type}}$
69	$TSN^{-}_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{S-\text{type}}$	**70**	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{N-\text{type}}$	$TN_{S-\text{type}}$

Case #	Y_{TS_1}	Y_{TS_2}	X_{T_3}	Case #	Y_{TS_1}	Y_{TS_2}	X_{T_3}
71	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$TN_{S-\text{type}}$	**72**	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$TN_{S-\text{type}}$
73	$TSN^{+}_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$TN_{S-\text{type}}$	**74**	$TSN^{+}_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{S-\text{type}}$
75	$TSN^{-}_{N-\text{type}}$	$TSN^{+}_{L-\text{type}}$	$TN_{S-\text{type}}$	**76**	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$TN_{S-\text{type}}$
77	$TSN^{-}_{N-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$TN_{S-\text{type}}$	**78**	$TSN^{-}_{N-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{S-\text{type}}$
79	$TSN^{+}_{L-\text{type}}$	$TSN^{-}_{L-\text{type}}$	$TN_{S-\text{type}}$	**80**	$TSN^{+}_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$TN_{S-\text{type}}$
81	$TSN^{+}_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{S-\text{type}}$	**82**	$TSN^{-}_{L-\text{type}}$	$TSN^{+}_{S-\text{type}}$	$TN_{S-\text{type}}$
83	$TSN^{-}_{L-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{S-\text{type}}$	**84**	$TSN^{+}_{S-\text{type}}$	$TSN^{-}_{S-\text{type}}$	$TN_{S-\text{type}}$

5.3 Numerical Examples

Results of the convolutions developed in this paper are applied to two key application areas: statistical tolerance analysis and gap analysis. In Section 5.3.2, we provide an example, of the sum of one truncated normal and two truncated skew normal random variables being related to Section 5.2.3.2.

5.3.1 Application to statistical tolerance analysis

In assembly design, as shown in Figure 5.9, the width of component 1 is a normal random variable X_1 and the width of component 2 is a positively skew normal random variable Y_2. Similarly, the width of component 3 is a negatively skew normal random variable Y_3. Suppose that the parameters, μ_1, μ_2, and μ_3, of X_1, Y_2 and Y_3 are 10, 8 and 16, and the parameters, σ_1, σ_2, and σ_3, of X_1, Y_2 and X_3 are 3, 4 and 4, respectively. We also assume that the random variable X_1 is doubly truncated at the lower and upper truncation points, 7 and 13, respectively, the random variable Y_2 is left truncated at 7, and the random variable Y_3 is right truncated at 17. Since Y_2 and Y_3 are negatively and positively skew, respectively, we consider the shape parameters of Y_2 and Y_3 as 3 and -3, respectively.

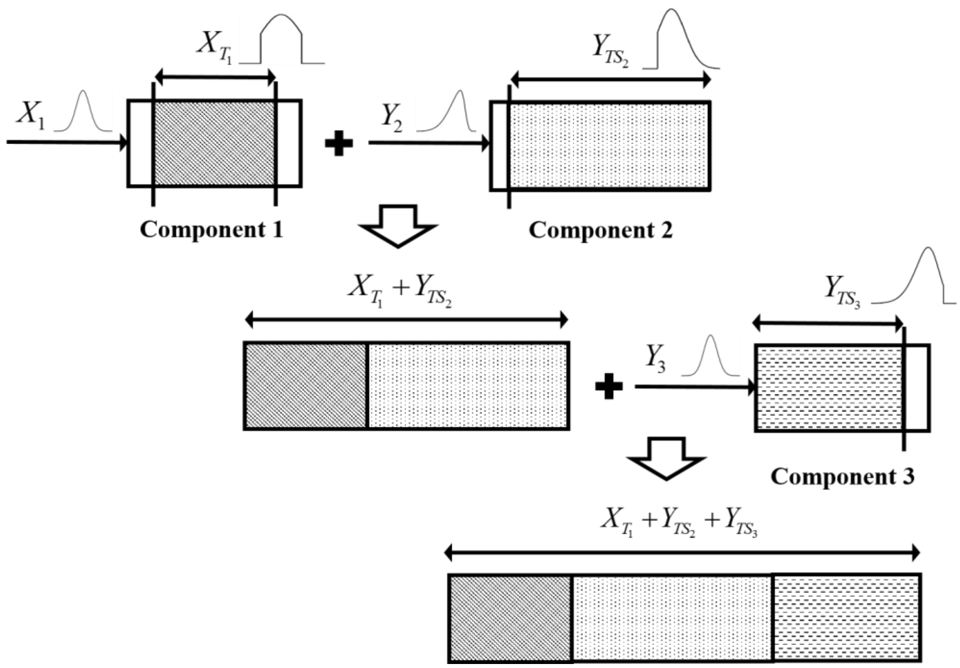


Figure 5. 9. Assembly design of statistical tolerance design for three truncated components

Let $Z_2 = X_{T_1} + Y_{TS_2}$. By referring to equations in Section 5.1.3, the probability density function of the sum of the above two truncated normal random variables is expressed as

$$f_{Z_2}(z) = \int_{-\infty}^{\infty} f_{Y_{TS_2}}(y) f_{X_{T_1}}(x) dx$$

$$= \int_{-\infty}^{\infty} \frac{\frac{2}{4}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-x-8}{4}\right)^2} \int_{-\infty}^{3\frac{z-x-8}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt}{\int_7^{\infty} \frac{2}{4}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{p-8}{4}\right)^2} \left(\int_{-\infty}^{3\frac{p-8}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt\right) dp} \frac{\frac{1}{2\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-10}{3}\right)^2}}{\int_7^{13} \frac{1}{2\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{h-10}{3}\right)^2} dh} I_{[-\infty, z-7]}(x) I_{[7,13]}(x) dx.$$

Furthermore, the mean and variance of Z_2 are obtained as 21.18 and 7.63, respectively. In a similar fashion, let Z_3 be $X_{T_1} + Y_{TS_2} + X_{TS_3}$. Based on equations in Section 6.3.2, the probability density function of Z_3 is then obtained as

$$
\begin{aligned}
f_{Z_3}(s) &= \int_{-\infty}^{\infty} f_{X_{T_3}}(s-z) f_{Z_2}(z)dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\frac{2}{4}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-x-16}{4}\right)^2} \int_{-\infty}^{-3\frac{z-x-16}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt}{\int_{-\infty}^{17} \frac{2}{4}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{p-16}{4}\right)^2} \left(\int_{-\infty}^{-3\frac{p-16}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt\right) dp} \cdot \\
&\qquad \frac{\frac{2}{4}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-x-8}{4}\right)^2} \int_{-\infty}^{3\frac{z-x-8}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt}{\int_{7}^{\infty} \frac{2}{4}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{p-8}{4}\right)^2} \left(\int_{-\infty}^{3\frac{p-8}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt\right) dp} \frac{\frac{1}{2\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-10}{3}\right)^2}}{\int_{7}^{13} \frac{1}{2\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{h-10}{3}\right)^2} dh} \cdot \\
&\qquad I_{[s-z-17,\infty]}(z) I_{[-\infty,z-7]}(x) I_{[7,13]}(x)\, dxdz.
\end{aligned}
$$

Finally, the mean and variance of Z_3 are obtained as 34.00 and 15.25, respectively. Figure 5.10 shows the properties of X_{T_1}, Y_{TS_2}, Z_2, Y_{ST_3}, and Z_3.

X_{T_1}	Y_{TS_2}	$Z_2 = X_{T_1} + Y_{TS_2}$	X_{TS_3}	$Z_3 = X_{T_1} + Y_{TS_2} + X_{TS_3}$
Sym $TN_{N-\text{type}}$	$TN_{S-\text{type}}$			
$\mu_{T_1} = 10.00$, $\sigma^2_{T_1} = 2.62$	$\mu_{T_2} = 11.18$, $\sigma^2_{T_2} = 6.32$	$\mu_{Z_2} = 21.18$, $\sigma^2_{Z_2} = 8.94$	$\mu_{T_3} = 12.82$, $\sigma^2_{T_3} = 6.32$	$\mu_{Z_3} = 34.00$, $\sigma^2_{Z_3} = 15.25$

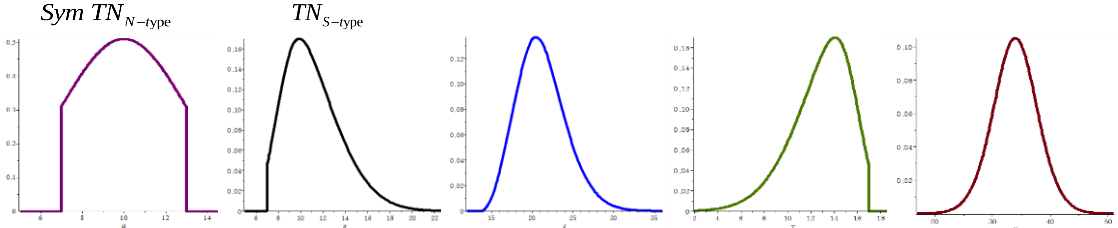


Figure 5.10. The statistical tolerance analysis example

5.3.2 Application to gap analysis

Gap is defined as $G = X_A - X_{C1i} - X_{C2j} - X_{C3k}$ for i = 1, 2, 3, j = 1, 2, and k = 1, 2, 3, where X_A, X_{C1i}, X_{C2j} and X_{C3k} are the dimension of an assembly and a respective dimension of components. Suppose that the truncated mean of X_A is 41. Nine different distributions of assembly components are illustrated in Table 8, and the means and variances of G are shown in Table 9 and Figure 13.

Table 5.8. Gap analysis data set 1

	Type	α	μ	σ	LTP	UTP	Truncated mean	Truncated variance
X_{C11}	$Sym\ TN_{N-\text{type}}$	0	15	2	13.5	16.5	15.0000	0.6953
X_{C12}	$TN_{L-\text{type}}$	0	15	2	13.5	∞	15.7788	2.2254
X_{C13}	$TN_{S-\text{type}}$	0	15	2	$-\infty$	16.5	14.2212	2.2254
X_{C21}	$TSN^{+}_{L-\text{type}}$	5	10	1.5	10.2	∞	11.3533	0.7478
X_{C22}	$TSN^{+}_{S-\text{type}}$	5	10	1.5	$-\infty$	12.0	10.8336	0.3514
X_{C31}	$Sym\ TN_{N-\text{type}}$	0	12	3	11.0	13.0	12.0000	0.3284
X_{C32}	$TN_{L-\text{type}}$	0	12	3	11.0	∞	13.7955	3.9808
X_{C33}	$TN_{S-\text{type}}$	0	12	3	$-\infty$	13.0	10.2045	3.9808
X_A	$Sym\ TN_{N-\text{type}}$	0	41	1	40.5	41.5	41.0000	0.0806

Table 5.9. Mean and variance of gap for data set 1

	X_{C1}	X_{C2}	X_{C3}	X_A	μ_G	σ_G^2
1	X_{C11}	X_{C21}	X_{C31}	X_A	2.6467	1.8522
2	X_{C11}	X_{C21}	X_{C32}	X_A	0.8512	5.5046
3	X_{C11}	X_{C21}	X_{C33}	X_A	4.4422	5.5046
4	X_{C11}	X_{C22}	X_{C31}	X_A	3.1664	1.4558
5	X_{C11}	X_{C22}	X_{C32}	X_A	1.3709	5.1082
6	X_{C11}	X_{C22}	X_{C33}	X_A	4.9618	5.1082
7	X_{C12}	X_{C21}	X_{C31}	X_A	1.8679	3.3822
8	X_{C12}	X_{C21}	X_{C32}	X_A	0.0725	7.0346

	X_{C1}	X_{C2}	X_{C3}	X_A	μ_G	σ_G^2
9	X_{C12}	X_{C21}	X_{C33}	X_A	3.6634	7.0346
10	X_{C12}	X_{C22}	X_{C31}	X_A	2.3876	2.9858
11	X_{C12}	X_{C22}	X_{C32}	X_A	0.5921	6.6382
12	X_{C12}	X_{C22}	X_{C33}	X_A	4.1831	6.6382
13	X_{C13}	X_{C21}	X_{C31}	X_A	3.4255	3.3822
14	X_{C13}	X_{C21}	X_{C32}	X_A	1.6300	7.0346
15	X_{C13}	X_{C21}	X_{C33}	X_A	5.2209	7.0346
16	X_{C13}	X_{C22}	X_{C31}	X_A	3.9451	2.9858
17	X_{C13}	X_{C22}	X_{C32}	X_A	2.1497	6.6382
18	X_{C13}	X_{C22}	X_{C33}	X_A	5.7406	6.6382

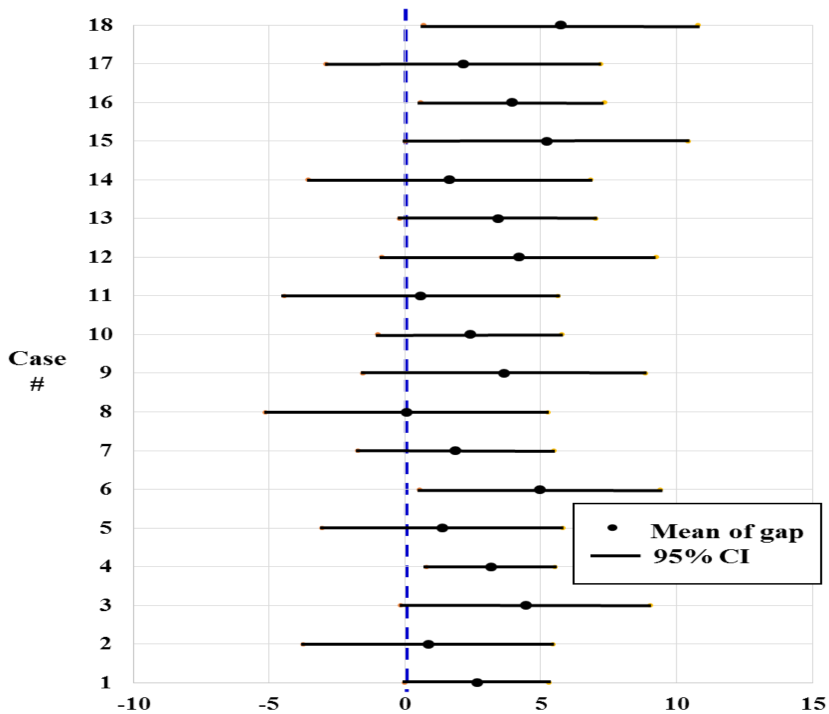


Figure 5.11. 95% CI of means of gap using data set 1 when the number of sample size for assembly product is large

Note that dimensional interference occurs when the gap becomes negative (i.e., $X_A < X_{C1} + X_{C2} + X_{C3}$) which often results in assembled products being scrapped or reworked. The convolutions developed in this paper could be an effective tool to help predict the dimensional interference. Now assuming that the truncated mean of X_A is 39, nine different distributions of assembly components are illustrated in Table 5.10, and the means and variances of *G* are shown in Table 5.11. In this particular example, there are six cases where the mean of gap is negative, creating the extreme dimensional interference. This highlights the importance of using truncated normal and skew normal distributions in gap analysis.

Table 5.10. Gap analysis data set 2

	Type	α	μ	σ	LTP	UTP	Truncated mean	Truncated variance
X_{C11}	$Sym\ TN_{N-\text{type}}$	0	15	2	13.5	16.5	15.0000	0.6953
X_{C12}	$TN_{L-\text{type}}$	0	15	2	13.5	∞	15.7788	2.2254
X_{C13}	$TN_{S-\text{type}}$	0	15	2	$-\infty$	16.5	14.2212	2.2254
X_{C21}	$TSN^{+}_{L-\text{type}}$	5	10	1.5	10.2	∞	11.3533	0.7478
X_{C22}	$TSN^{+}_{S-\text{type}}$	5	10	1.5	$-\infty$	12.0	10.8336	0.3514
X_{C31}	$Sym\ TN_{N-\text{type}}$	0	12	3	11.0	13.0	12.0000	0.3284
X_{C32}	$TN_{L-\text{type}}$	0	12	3	11.0	∞	13.7955	3.9808
X_{C33}	$TN_{S-\text{type}}$	0	12	3	$-\infty$	13.0	10.2045	3.9808
X_A	$Sym\ TN_{N-\text{type}}$	0	39	1	38.5	39.5	39.0000	0.0806

Table 5.11. Mean and variance of gap for data set 2

	X_{C1}	X_{C2}	X_{C3}	X_A	μ_G	σ_G^2
1	X_{C11}	X_{C21}	X_{C31}	X_A	0.6467	1.8522
2	X_{C11}	X_{C21}	X_{C32}	X_A	**-1.1488**	5.5046
3	X_{C11}	X_{C21}	X_{C33}	X_A	2.4422	5.5046
4	X_{C11}	X_{C22}	X_{C31}	X_A	1.1664	1.4558
5	X_{C11}	X_{C22}	X_{C32}	X_A	**-0.6291**	5.1082
6	X_{C11}	X_{C22}	X_{C33}	X_A	2.9618	5.1082

	X_{C1}	X_{C2}	X_{C3}	X_A	μ_G	σ_G^2
7	X_{C12}	X_{C21}	X_{C31}	X_A	**-0.1321**	3.3822
8	X_{C12}	X_{C21}	X_{C32}	X_A	**-1.9275**	7.0346
9	X_{C12}	X_{C21}	X_{C33}	X_A	1.6634	7.0346
10	X_{C12}	X_{C22}	X_{C31}	X_A	0.3876	2.9858
11	X_{C12}	X_{C22}	X_{C32}	X_A	**-1.4079**	6.6382
12	X_{C12}	X_{C22}	X_{C33}	X_A	2.1831	6.6382
13	X_{C13}	X_{C21}	X_{C31}	X_A	1.4255	3.3822
14	X_{C13}	X_{C21}	X_{C32}	X_A	**-0.3700**	7.0346
15	X_{C13}	X_{C21}	X_{C33}	X_A	3.2209	7.0346
16	X_{C13}	X_{C22}	X_{C31}	X_A	1.9451	2.9858
17	X_{C13}	X_{C22}	X_{C32}	X_A	0.1497	6.6382
18	X_{C13}	X_{C22}	X_{C33}	X_A	3.7406	6.6382

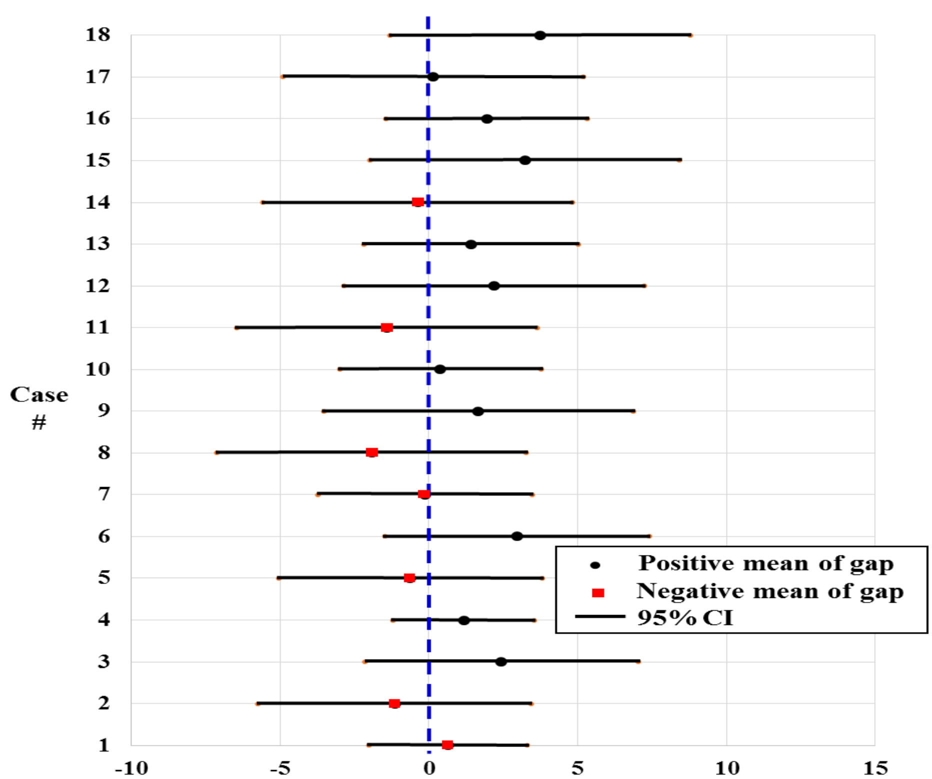


Figure 5.12. 95% CI of means of gap using data set 2 when the number of sample size for assembly product is large

5.5 Concluding Remarks

Chapter 5 laid out the theoretical foundations of convolutions of truncated normal and skew normal distributions based on double and triple truncations. Convolutions of truncated normal and truncated skew normal random variables were highlighted. The cases presented in this chapter illustrate the possible types of convolutions of double truncations. This includes the sum of all the possible combinations containing two truncated random variables with normal and skew normal probability distributions. Numerical examples illustrate the application of convolutions of truncated normal random variables and truncated skew normal random variables to highlight the improved accuracy of tolerance analysis and gap analysis techniques. New findings have the potential to impact a wide range of many other engineering and science problems such as those found in statistical tolerance analysis, more specifically, tolerance stack analysis methods. By utilizing skew normal distributions in tolerance stack analysis methods this allows the tolerance interval to be covered more precisely, allowing for a more accurate understanding of the variation in the gap.

CHAPTER SIX

CONCLUSION AND FUTURE STUDY

For solving engineering problems including truncation concepts, many quality practitioners have used untruncated original distributions to analyze testing and inspection procedures in production or process according to the computational complexity and the pursuit of easy usefulness. There are researchers who have made an effort to improve the accuracy of methods of maximum likelihood and moments, to examine methods of analyzing order statistics and regression, and to develop statistical inferences based on truncated data and distributions. However, much room for research in order to enhance by using truncated normal and truncated skew normal distributions still exists. The objective of this research was to pioneer in a particular area of research and contribute to the research community. In Chapter 3, the standardization of a truncated normal distribution which is different from a traditional truncated standard normal distribution was established theoretically by proposing theorems. Its cumulative table will be very useful for practitioners. Then, as an extension of the standardization, the new one-sided and two-sided z-test and t-test procedures including their associated test statistics, confidence intervals and *P*-values were developed in Chapter 4. Since the specific formulas or equations based on four different types of a truncated normal distribution were suggested to apply by quality practitioners.

Mathematical convolution was another important concept within the truncated normal environment. In Chapter 5, a mathematical framework for the convolutions of truncated normal random variables under three different types of quality characteristics was developed. One of the critical contribution is to provide closed forms of density for the sums of two truncated normal random variables regardless of four different types of a truncated normal distribution. Extension to truncated skew normal random variables was performed with proposed general forms of probability density function for the sums of two and three truncated normal and truncated skew normal random variables. The successful completion of this research will help obtain a better understanding of the integrated effects of statistical tolerance analysis and gap analysis, ultimately leading to process and quality improvement. This research also advances the state of knowledge of the inherent complexities arising from issues related to prediction of system performance. Although this research will primarily focus on statistical tolerance analysis and gap analysis, the results have the potential to impact a wide range of tasks in many engineering problems, including process control monitoring.

APPENDICES

A: Derivation of Mean and Variance of a TNRV for Chapter 3

A.1 Mean of a DTRV, X_T in Figure 2.1

Each Sections 3.1.1, 3.1.2, and 3.1.3 provides a proposed theorem to prove the fact that the variance of the truncated normal random variable is smaller than the variance of the original normal random variable. Double, left and right truncations of a normal distribution are applied in Sections 3.1.1, 3.1.2, and 3.1.3, respectively.
By definition, the mean of X_T is written as

$$\begin{aligned} E(X_T) = \mu_T &= \int_{-\infty}^{\infty} x\, f_{X_T}(x)dx \\ &= \int_{-\infty}^{\infty} x \frac{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy} dx \quad \text{where}\, x_l \le x \le x_u \\ &= \frac{\int_{x_l}^{x_u} x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx}{\int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy}. \end{aligned}$$

Let $A = \int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$. Then we have

$$\begin{aligned} \mu_T &= \frac{1}{A} \cdot \int_{x_l}^{x_u} x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= \frac{1}{A} \cdot \left[\int_{x_l}^{x_u} \frac{\mu}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{x_l}^{x_u} \left(\frac{x-\mu}{\sigma}\right) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \right]. \end{aligned}$$

By letting $z = \frac{x-\mu}{\sigma}$, $\sigma dz = dx$. Thus,

$$\begin{aligned} \mu_T &= \frac{1}{A} \cdot \left[\mu \int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma dz \right] \\ &= \frac{1}{A} \cdot \left[\mu A + \sigma \int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} z\, e^{-\frac{1}{2}z^2} dz \right] \\ &= \mu - \frac{\sigma}{A} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} \Big|_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}}. \end{aligned}$$

Notice that A can be expressed as $\int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy = \int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds = \Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)$. Therefore, the mean of X_T, μ_T, is obtained as

$$\mu + \sigma \cdot \frac{\phi\left(\frac{x_l-\mu}{\sigma}\right) - \phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right) - \Phi\left(\frac{x_l-\mu}{\sigma}\right)}.$$

A.2 Variance of a DTRV, X_T in Figure 2.1

By definition,

$$\begin{aligned}
E(X_T^2) &= \int_{-\infty}^{\infty} x^2 f_{X_T}(x)dx \\
&= \int_{-\infty}^{\infty} \frac{x^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy} dx \text{ where } x_l \leq x \leq x_u \\
&= \frac{\int_{x_l}^{x_u} x^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx}{\int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy} \\
&= \frac{1}{A}\left[\int_{x_l}^{x_u} \sigma \left(\frac{x^2 - 2\mu x + 2\mu x - \mu^2 + \mu^2}{\sigma^2}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx\right] \\
&= \frac{1}{A}\left[\int_{x_l}^{x_u} \sigma \left(\frac{x^2 - 2\mu x + \mu^2}{\sigma^2}\right) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{x_l}^{x_u} \sigma \left(\frac{2\mu x - \mu^2}{\sigma^2}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx\right] \\
&= \frac{1}{A}\left[\sigma \int_{x_l}^{x_u} \left(\frac{x-\mu}{\sigma}\right)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + 2\mu \int_{x_l}^{x_u} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \right. \\
&\quad \left. -\mu^2 \int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx\right].
\end{aligned}$$

Since $z = \frac{x-\mu}{\sigma}$ and $\sigma dz = dx$,

$$\begin{aligned}
E(X_T^2) &= \frac{1}{A}\left[\sigma \int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma dz + 2\mu \frac{\int_{x_l}^{x_u} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx}{A} A - \mu^2 \int_{x_l}^{x_u} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx\right] \\
&= \frac{1}{A}\left[\int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \frac{\sigma^2}{\sqrt{2\pi}} z^2 e^{-\frac{1}{2}z^2} dz + 2\mu\mu_T A - \mu^2 A\right]
\end{aligned}$$

In the meantime, $\frac{d}{dz}\left(-\frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\right) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} + \frac{z^2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. Thus, $\frac{z^2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} = \frac{d}{dz}\left(-\frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. After taking the integral in the above equation, we obtain $\int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = -\frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\Big|_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} + \int_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. Therefore,

$$\begin{aligned}
E(X_T^2) &= \frac{1}{A}\left[\sigma^2 \left(-\frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\Big|_{\frac{x_l-\mu}{\sigma}}^{\frac{x_u-\mu}{\sigma}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\right) + 2\mu\mu_T A - \mu^2 A\right] \\
&= \frac{1}{A}\left[-\sigma^2 \left(\frac{x_u-\mu}{\sigma}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_u-\mu}{\sigma}\right)^2} + \sigma^2 \left(\frac{x_l-\mu}{\sigma}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_l-\mu}{\sigma}\right)^2}\right.
\end{aligned}$$

$$+\sigma^2 A + 2\mu\mu_T A - \mu^2 A\Big].$$

The variance of X_T, σ_T^2, is represented as

$$\begin{aligned}
Var(X_T) &= E(X_T^2) - E(X_T)^2 \\
&= \frac{1}{A}\left[-\sigma^2\left(\frac{x_u-\mu}{\sigma}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x_u-\mu}{\sigma}\right)^2} + \sigma^2\left(\frac{x_l-\mu}{\sigma}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x_l-\mu}{\sigma}\right)^2}\right. \\
&\quad \left. +\sigma^2 A + 2\mu\mu_T A - \mu^2 A\right] - \mu_T^2 \\
&= \frac{1}{A}\left[-\sigma^2\left(\frac{x_u-\mu}{\sigma}\right)\cdot\phi\left(\frac{x_u-\mu}{\sigma}\right) + \sigma^2\left(\frac{x_l-\mu}{\sigma}\right)\cdot\phi\left(\frac{x_l-\mu}{\sigma}\right)\right. \\
&\quad \left. +\sigma^2 A + 2\mu\mu_T A - \mu^2 A\right] - \mu_T^2
\end{aligned}$$

Since $\mu_T = \mu + \frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\sigma = \mu + \frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{A}\sigma$,

$$\begin{aligned}
Var(X_T) &= -\frac{\sigma^2}{A}\left(\frac{x_u-\mu}{\sigma}\right)\cdot\phi\left(\frac{x_u-\mu}{\sigma}\right) + \frac{\sigma^2}{A}\left(\frac{x_l-\mu}{\sigma}\right)\cdot\phi\left(\frac{x_l-\mu}{\sigma}\right) + \sigma^2 + \\
&\quad 2\mu\cdot\left(\mu + \frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{A}\right) - \mu^2 - \left(\mu + \sigma\cdot\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{A}\right) \\
&= \sigma^2\left[1 + \frac{\frac{x_l-\mu}{\sigma}\cdot\phi\left(\frac{x_l-\mu}{\sigma}\right) - \frac{x_u-\mu}{\sigma}\cdot\phi\left(\frac{x_u-\mu}{\sigma}\right)}{A} - \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{A}\right)^2\right].
\end{aligned}$$

As a result, the variance of X_T, σ_T^2, is obtained as

$$\sigma^2\left[1 + \frac{\frac{x_l-\mu}{\sigma}\cdot\phi\left(\frac{x_l-\mu}{\sigma}\right) - \frac{x_u-\mu}{\sigma}\cdot\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{x_l-\mu}{\sigma}\right)-\phi\left(\frac{x_u-\mu}{\sigma}\right)}{\Phi\left(\frac{x_u-\mu}{\sigma}\right)-\Phi\left(\frac{x_l-\mu}{\sigma}\right)}\right)^2\right].$$

B: Supporting for R Programing code for Chapter 4

B.1 R simulation code for the Central Limit Theorem by samples from the truncated normal distribution with sample size, 30 in Figure 4.4

```
# Call up required packages or libraries in R
require(truncnorm)

# (a) Symmetric DTND
x_double <- rtruncnorm(10000,a=6,b=14,mean=10,sd=4)
par(mfrow=c(1,4))
hist(x_double,xlab="",ylab="",ylim=c(0,2800),yaxt="n",xaxt="n",main="(1)",col="gray"
,cex.main=2.5)
axis(2,at=c(0,1400,2800),labels=c(0,0.5,1))
sampmeans <- matrix(NA,nrow=1000,ncol=1)
for (i in 1:1000){
  samp <- sample(x_double,30,replace=T)
  sampmeans[i,] <- mean(samp)
  }
hist(sampmeans,xlab="",ylab="",ylim=c(0,2800),yaxt="n",xaxt="n",main="(2)",col="gra
y",cex.main=2.5)
  axis(2,at=c(0,1400,2800),labels=c(0,0.5,1))
 plot(ecdf(sampmeans),xlab="",main="(3)",cex.main=2.5)
 z<-(sampmeans-mean(x_double))/sd(x_double)*sqrt(30)
 qqnorm(z, lty=1,xlab="",ylab="",main="(4)",cex.main=2.5)

# (b) Asymmetric DTND
x_asym_double <- rtruncnorm(10000,a=8,b=16,mean=10,sd=4)
par(mfrow=c(1,4))
hist(x_asym_double,xlab="",ylab="",ylim=c(0,2800),yaxt="n",xaxt="n",main="(1)",col=
"gray",cex.main=2.5)
axis(2,at=c(0,1400,2800),labels=c(0,0.5,1))
sampmeans <- matrix(NA,nrow=1000,ncol=1)
for (i in 1:1000){
  samp <- sample(x_asym_double,30,replace=T)
  sampmeans[i,] <- mean(samp)
  }
hist(sampmeans,xlab="",ylab="",ylim=c(0,2800),yaxt="n",xaxt="n",main="(2)",col="gra
y",cex.main=2.5)
  axis(2,at=c(0,1400,2800),labels=c(0,0.5,1))
 plot(ecdf(sampmeans),xlab="",main="(3)",cex.main=2.5)
 z<-(sampmeans-mean(x_asym_double))/sd(x_asym_double)*sqrt(30)
 qqnorm(z, lty=1,xlab="",ylab="",main="(4)",cex.main=2.5)
```

```r
# (c) LTND
x_left <- rtruncnorm(10000,a=6,mean=10,sd=4)
par(mfrow=c(1,4))
hist(x_left,xlab="",ylab="",ylim=c(0,2800),yaxt="n",xaxt="n",main="(1)",col="gray",cex.main=2.5)
axis(2,at=c(0,1400,2800),labels=c(0,0.5,1))
sampmeans <- matrix(NA,nrow=1000,ncol=1)
for (i in 1:1000){
  samp <- sample(x_left,30,replace=T)
  sampmeans[i,] <- mean(samp)
  }
hist(sampmeans,xlab="",ylab="",ylim=c(0,2800),yaxt="n",xaxt="n",main="(2)",col="gray",cex.main=2.5)
  axis(2,at=c(0,1400,2800),labels=c(0,0.5,1))
 plot(ecdf(sampmeans),xlab="",main="(3)",cex.main=2.5)
 z<-(sampmeans-mean(x_left))/sd(x_left)*sqrt(30)
 qqnorm(z, lty=1,xlab="",ylab="",main="(4)",cex.main=2.5)
```

```r
# (d) RTND
x_right <- rtruncnorm(10000,b=14,mean=10,sd=4)
par(mfrow=c(1,4))
hist(x_right,xlab="",ylab="",ylim=c(0,2800),yaxt="n",xaxt="n",main="(1)",col="gray",cex.main=2.5)
axis(2,at=c(0,1400,2800),labels=c(0,0.5,1))
sampmeans <- matrix(NA,nrow=1000,ncol=1)
for (i in 1:1000){
  samp <- sample(x_right,30,replace=T)
  sampmeans[i,] <- mean(samp)
  }
hist(sampmeans,xlab="",ylab="",ylim=c(0,2800),yaxt="n",xaxt="n",main="(2)",col="gray",cex.main=2.5)
  axis(2,at=c(0,1400,2800),labels=c(0,0.5,1))
 plot(ecdf(sampmeans),xlab="",main="(3)",cex.main=2.5)
 z<-(sampmeans-mean(x_right))/sd(x_right)*sqrt(30)
 qqnorm(z, lty=1,xlab="",ylab="",main="(4)",cex.main=2.5)
```

C: Supporting for Maple code for Chapter 5

C.1 Maple code for the statistical analysis example in Figure 5.10

C.1.1 Maple code captured for a DTNRV

 # Probability density function of X_{T_1} :

$$f_{f_{XT1}(x)}(x) = \frac{\frac{1}{2\sqrt{2} * \sqrt{Pi}} e^{-\frac{1}{2} * \left(\frac{x-10}{3}\right)^2} \cdot \begin{cases} 0 & x < 7 \\ 1 & x \geq 7 \end{cases} \cdot \begin{cases} 1 & x < 13 \\ 0 & x \geq 13 \end{cases}}{\int_7^{13} \frac{1}{2\sqrt{2} * \sqrt{Pi}} e^{-\frac{1}{2} * \left(\frac{x-10}{3}\right)^2} dx}$$

Result: $\frac{1}{6} \frac{e^{-\frac{1}{2}\left(\frac{1}{3}x - \frac{10}{3}\right)^2} \left(\begin{cases} 0 & x < 7 \\ 1 & 7 \leq x \end{cases}\right) \left(\begin{cases} 1 & x < 13 \\ 0 & 13 \leq x \end{cases}\right) \sqrt{2}}{\sqrt{\pi}\, \mathrm{erf}\left(\frac{1}{2}\sqrt{2}\right)}$

simplify($f_{XT1}(x)$)

$$\begin{cases} 0 & x < 7 \\ \frac{1}{6} \frac{e^{-\frac{1}{18}(x-10)^2}\sqrt{2}}{\sqrt{\pi}\, \mathrm{erf}\left(\frac{1}{2}\sqrt{2}\right)} & x < 13 \\ 0 & 13 \leq x \end{cases}$$

 # Mean of X_{T_1} :

$$E\left(X_{T_1}\right) = \int_7^{13} x \cdot \frac{1}{6} \frac{e^{-\frac{1}{2}\left(\frac{1}{3}x - \frac{10}{3}\right)^2} \left(\begin{cases} 0 & x < 7 \\ 1 & 7 \leq x \end{cases}\right) \left(\begin{cases} 1 & x < 13 \\ 0 & 13 \leq x \end{cases}\right) \sqrt{2}}{\sqrt{\pi}\, \mathrm{erf}\left(\frac{1}{2}\sqrt{2}\right)} dx$$

Result: 10

 # Variacne of X_{T_1} :

$$Var\left(X_{T_1}\right) = \int_7^{13} (x-10)^2 \cdot \frac{1}{6} \frac{e^{-\frac{1}{2}\left(\frac{1}{3}x - \frac{10}{3}\right)^2} \left(\begin{cases} 0 & x < 7 \\ 1 & 7 \leq x \end{cases}\right) \left(\begin{cases} 1 & x < 13 \\ 0 & 13 \leq x \end{cases}\right) \sqrt{2}}{\sqrt{\pi}\, \mathrm{erf}\left(\frac{1}{2}\sqrt{2}\right)} dx$$

Result: 2.6207

C.1.2 Maple code for a left truncated positive skew NRV

 # Probability density function of Y_{TS_2} :

$f_{f_{YST2}}(y) =$ (2*(1/4))*exp(-(1/2)*((y-8)*(1/4))^2)*(int(exp(-(1/2)*t^2)/sqrt(2*Pi), t = -infinity .. 3*((y-8)*(1/4))))*piecewise(y < 7, 0, 7 <= y, 1)/(sqrt(2*Pi)*(int((2*(1/4))*exp(-(1/2)*((h-8)*(1/4))^2)*(int(exp(-(1/2)*t^2)/sqrt(2*Pi), t = -infinity .. 3*((h-8)*(1/4))))/sqrt(2*Pi), h = 7 .. infinity)))

Result:
$$\frac{1}{4}\,\frac{e^{-\frac{1}{2}\left(\frac{1}{4}y-2\right)^2}\left(\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{3}{8}\sqrt{2}\,(y-8)\right)\right)\left(\begin{cases}0 & y<7\\ 1 & 7\le y\end{cases}\right)\sqrt{2}}{\sqrt{\pi}\left(\int_7^{\infty}\frac{1}{4}\,\frac{e^{-\frac{1}{2}\left(\frac{1}{4}h-2\right)^2}\left(\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{3}{8}\sqrt{2}\,(h-8)\right)\right)\sqrt{2}}{\sqrt{\pi}}\,dh\right)}$$

 # Mean of Y_{TS_2} :

$E\left(Y_{TS_2}\right) =$ int((1/4)*y*exp(-(1/2)*((1/4)*y-2)^2)*(1/2+(1/2)*erf((3/8)*sqrt(2)*(y-8)))*piecewise(y < 7, 0, 7 <= y, 1)*sqrt(2)/(sqrt(Pi)*(int((1/4)*exp(-(1/2)*((1/4)*h-2)^2)*(1/2+(1/2)*erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity))), y = 7 .. infinity)

Result: 11.18015321

 # Variacne of Y_{TS_2} :

$Var\left(Y_{TS_2}\right) =$ int((1/4)*(y-11.18015321)^2*exp(-(1/2)*((1/4)*y-2)^2)*(1/2+(1/2)*erf((3/8)*sqrt(2)*(y-8)))*piecewise(y < 7, 0, 7 <= y, 1)*sqrt(2)/(sqrt(Pi)*(int((1/4)*exp(-(1/2)*((1/4)*h-2)^2)*(1/2+(1/2)*erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity))), y = 7 .. infinity)

Result: 6.317101607

C.1.3 Maple code for a right truncated negative skew NRV

 # Probability density function of Y_{TS_3} :

$f_{f_{YST3}}(y) =$ (2*(1/4))*exp(-(1/2)*((k-16)*(1/4))^2)*(int(exp(-(1/2)*t^2)/sqrt(2*Pi), t = -infinity .. -3*((k-16)*(1/4))))*piecewise(k <= 17, 1, 17 > k, 0)/(sqrt(2*Pi)*(int((2*(1/4))*exp(-(1/2)*((h-16)*(1/4))^2)*(int(exp(-(1/2)*t^2)/sqrt(2*Pi), t = -infinity .. -3*((h-16)*(1/4))))/sqrt(2*Pi), h = -infinity .. 17)))

Result: $$\frac{1}{4}\frac{e^{-\frac{1}{2}\left(\frac{1}{4}k-4\right)^2}\left(\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{3}{8}\sqrt{2}\,(k-16)\right)\right)\left(\begin{cases}1 & k\le 17\\ 0 & k<17\end{cases}\right)\sqrt{2}}{\sqrt{\pi}\left(\int_{-\infty}^{17}\frac{1}{4}\frac{e^{-\frac{1}{2}\left(\frac{1}{4}h-4\right)^2}\left(\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{3}{8}\sqrt{2}\,(h-16)\right)\right)\sqrt{2}}{\sqrt{\pi}}\,\mathrm{d}h\right)}$$

 # Mean of Y_{TS_3} :

$E\left(Y_{TS_3}\right)=$ int((1/4)*y*exp(-(1/2)*((1/4)*y-2)^2)*(1/2+(1/2)*erf((3/8)*sqrt(2)*(y-8)))*piecewise(y < 7, 0, 7 <= y, 1)*sqrt(2)/(sqrt(Pi)*(int((1/4)*exp(-(1/2)*((1/4)*h-2)^2)*(1/2+(1/2)*erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity))), y = 7 .. infinity)

Result: 12.81984679

 # Variacne of Y_{TS_3} :

$Var\left(Y_{TS_3}\right)=$ int((1/4)*(k-12.81984679)^2*exp(-(1/2)*((1/4)*k-4)^2)*(1/2-(1/2)*erf((3/8)*sqrt(2)*(k-16)))*piecewise(k <= 17, 1, k < 17, 0)*sqrt(2)/(sqrt(Pi)*(int((1/4)*exp(-(1/2)*((1/4)*h-4)^2)*(1/2-(1/2)*erf((3/8)*sqrt(2)*(h-16)))*sqrt(2)/sqrt(Pi), h = -infinity .. 17))), k = -infinity .. 17)

Result: 6.31710160

C.1.4 Maple code for $Z_2 = X_{T_1} + Y_{TS_2}$

 # Probability density function of Z_2 :

$f_{Z_2}(z)=$ int(piecewise(z-y < 7, 0, z-y < 13, (1/6)*exp(-(1/18)*(z-y-10)^2)*sqrt(2)/(sqrt(Pi)*erf((1/2)*sqrt(2))), 13 <= z-y, 0)*piecewise(y < 7, 0, 7 <= y, (1/8)*exp(-(1/32)*(y-8)^2)*(1+erf((3/8)*sqrt(2)*(y-8)))*sqrt(2)/(sqrt(Pi)*(int((1/8)*exp(-(1/32)*(h-8)^2)*(1+erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity)))), y = -infinity .. infinity)

Result: piecewise(z < 14, 0, z < 20, int((1/24)*exp(-(1/18)*(z-y-10)^2)*exp(-(1/32)*(y-8)^2)*(1+erf((3/8)*sqrt(2)*(y-8)))/(Pi*erf((1/2)*sqrt(2))*(int((1/8)*exp(-(1/32)*(h-8)^2)*(1+erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity))), y = 7 .. z-7), 20 <= z, int((1/24)*exp(-(1/18)*(z-y-10)^2)*exp(-(1/32)*(y-8)^2)*(1+erf((3/8)*sqrt(2)*(y-8)))/(Pi*erf((1/2)*sqrt(2))*(int((1/8)*exp(-(1/32)*(h-8)^2)*(1+erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity))), y = z-13 .. z-7))

plot(*f*(Z_2), z=12 .. 26, color = blue, thickness = 5)

C.1.5 Maple code for $Z_3 = X_{T_1} + Y_{TS_2} + X_{TS_3}$

 # Probability density function of Z_3:

$f_{Z_3}(s) =$ int(piecewise(s-z < 7, 0, v-z < 13, (1/6)*exp(-(1/18)*(s-z-10)^2)*sqrt(2)/(sqrt(Pi)*erf((1/2)*sqrt(2))), 13 <= s-z, 0)*piecewise(z < 24, (1/32)*(int(exp(-(1/16)*x^2+(1/16)*z*x-(1/32)*z^2-(1/2)*x+z-10)*(1+erf((3/8)*sqrt(2)*(-z+x+16)))*(1+erf((3/8)*sqrt(2)*(x-8))), x = 7 .. infinity))/(Pi*(int(-(1/8)*exp(-(1/32)*(h-16)^2)*(-1+erf((3/8)*sqrt(2)*(h-16)))*sqrt(2)/sqrt(Pi), h = -infinity .. 17))*(int((1/8)*exp(-(1/32)*(h-8)^2)*(1+erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity))), 24 <= z, (1/32)*(int(exp(-(1/16)*x^2+(1/16)*z*x-(1/32)*z^2-(1/2)*x+z-10)*(1+erf((3/8)*sqrt(2)*(-z+x+16)))*(1+erf((3/8)*sqrt(2)*(x-8))), x = z-17 .. infinity))/(Pi*(int(-(1/8)*exp(-(1/32)*(h-16)^2)*(-1+erf((3/8)*sqrt(2)*(h-16)))*sqrt(2)/sqrt(Pi), h = -infinity .. 17))*(int((1/8)*exp(-(1/32)*(h-8)^2)*(1+erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity)))), z = -infinity .. infinity)

Result: piecewise(s < 31, (1/192)*sqrt(2)*(int(exp(-(1/18)*(s-z-10)^2)*(int(exp(-(1/16)*h^2+(1/16)*z*h-(1/32)*z^2-(1/2)*h+z-10)*(1+erf((3/8)*sqrt(2)*(-z+h+16)))*(1+erf((3/8)*sqrt(2)*(h-8))), h = 7 .. infinity)), z = -13+s .. -7+s))/(Pi^(3/2)*erf((1/2)*sqrt(2))*(int(-(1/8)*exp(-(1/32)*(h-16)^2)*(-1+erf((3/8)*sqrt(2)*(h-16)))*sqrt(2)/sqrt(Pi), h = -infinity .. 17))*(int((1/8)*exp(-(1/32)*(h-8)^2)*(1+erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity))), v < 37, (1/192)*sqrt(2)*(int(exp(-(1/18)*(v-z-10)^2)*(int(exp(-(1/16)*h^2+(1/16)*z*h-(1/32)*z^2-(1/2)*h+z-10)*(1+erf((3/8)*sqrt(2)*(-z+h+16)))*(1+erf((3/8)*sqrt(2)*(h-8))), h = 7 .. infinity)), z = -13+v .. 24)+int(exp(-(1/18)*(v-z-10)^2)*(int(exp(-(1/16)*h^2+(1/16)*z*h-(1/32)*z^2-(1/2)*h+z-10)*(1+erf((3/8)*sqrt(2)*(-z+h+16)))*(1+erf((3/8)*sqrt(2)*(h-8))), h = z-17 .. infinity)), z = 24 .. -7+s))/(Pi^(3/2)*erf((1/2)*sqrt(2))*(int(-(1/8)*exp(-(1/32)*(h-16)^2)*(-1+erf((3/8)*sqrt(2)*(h-16)))*sqrt(2)/sqrt(Pi), h = -infinity .. 17))*(int((1/8)*exp(-(1/32)*(h-8)^2)*(1+erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity))), 37 <= v, (1/192)*sqrt(2)*(int(exp(-(1/18)*(v-z-10)^2)*(int(exp(-(1/16)*h^2+(1/16)*z*h-(1/32)*z^2-(1/2)*h+z-10)*(1+erf((3/8)*sqrt(2)*(-z+h+16)))*(1+erf((3/8)*sqrt(2)*(h-8))), h = z-17 .. infinity)), z = -13+s .. -7+s))/(Pi^(3/2)*erf((1/2)*sqrt(2))*(int(-(1/8)*exp(-(1/32)*(h-16)^2)*(-1+erf((3/8)*sqrt(2)*(h-16)))*sqrt(2)/sqrt(Pi), h = -infinity .. 17))*(int((1/8)*exp(-(1/32)*(h-8)^2)*(1+erf((3/8)*sqrt(2)*(h-8)))*sqrt(2)/sqrt(Pi), h = 7 .. infinity))))

plot($f(Z_3)$, s=17 .. 51, color = red, thickness = 5)

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