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Problems in Domination and Graph Products

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Problems in domination and graph products

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Abstract

The "domination chain," first proved by Cockayne, Hedetniemi, and Miller in 1978, has been the focus of much research. In this work, we continue this study by considering unique realizations of its parameters. We first consider unique minimum dominating sets in Cartesian product graphs. Our attention then turns to unique minimum independent dominating sets in trees, and in some direct product graphs. Next, we consider an extremal graph theory problem and determine the maximum number of edges in a graph having a unique minimum independent dominating set or a unique minimum maximal irredundant set of cardinality two. Finally, we consider a variation of domination, called identifying codes, in the Cartesian product of a complete graph and a path.

Dedication

I dedicate this dissertation to my parents, Sandra and Stephen Hedetniemi. For your encouragement and support, this is for you.

Acknowledgments

My thanks go out to several people for helping me create this thesis. I would like to thank my two research advisors Dr. Kevin James and Dr. Doug Rall. Dr. James' keen insight was invaluable in helping me prove several of the theorems you will find in this text. Dr. Rall's impressive graph theory knowledge not only helped me research these subjects, but his tireless editing also helped make this dissertation what it is. I also thank Dr. Hui Xue and Dr. Wayne Goddard for agreeing to be valued members of my PhD committee.

I thank my parents, Sandra and Stephen, for their encouragement and support. I also thank my brother, Kevin, and my two sisters, Traci and Laura, for their support and friendship. Many thanks go to my girlfriend Michelle for always being there for me throughout my entire graduate career.

There is no way I could have completed this project by myself. So, to all the people big and small who in one way or another helped me in this endeavor, you have my thanks.

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Chapter 1

Definitions

Graph theory is a field of many definitions. Thus, in this chapter, we devote our attention to defining the major terms and key concepts that we will be using throughout the remainder of this work. A reader well-versed in the terminology of graph theory can safely skip this chapter and continue with Chapter 2.

1.1 Basics

We begin with the following definition.

Definition 1. A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$, called the *vertex set of* G, is a non-empty set, and $E(G)$, called the *edge set of* G, is a, possibly empty, set of distinct unordered pairs of elements from $V(G)$. For notational convenience, we denote the edge $\{u, v\}$ by either uv or vu.

The elements in $V(G)$ are called vertices, while the elements in $E(G)$ are called edges. We note that if the graph G is understood from context, then $V(G)$ and $E(G)$ are often denoted by V and E respectively for simplicity. The cardinality of $V(G)$ is the *order* of G, while the cardinality of $E(G)$ is the *size* of G. If $V(G)$ is a finite set, then G is a *finite graph.* We will exclusively consider finite graphs in our work. Additionally, if $|V(G)| = 1$, then G is trivial. All other graphs are nontrivial. If u and v are vertices of G, we say that u and v are *adjacent* if uv is an edge in $E(G)$. Moreover, we say that u and v are *incident* with the edge uv and vice versa. In a similar manner, two edges are adjacent if they share a vertex in common. Note that by Definition 1, two vertices may be joined by at most one edge. Thus, we are only considering *simple* graphs in our work.

As an example, the following defines a graph.

$$G = (\{a, b, c, d\}, \{ab, ac, ad, bd, bc\})$$

Here, G has four vertices and five edges. We see that vertex a, for example, is adjacent to vertices b, c, and d. Additionally, the edges ab and bd are adjacent.

Graphs are typically visualized by placing a dot for each vertex, and connecting two dots if their corresponding vertices form an edge. For example, the graph G above could be depicted as follows.

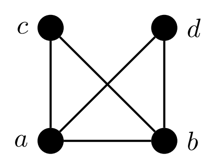


Figure 1.1: An example of a graph

Notice, however, that G could also be depicted in the following manner.

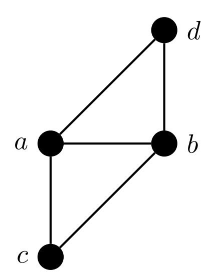


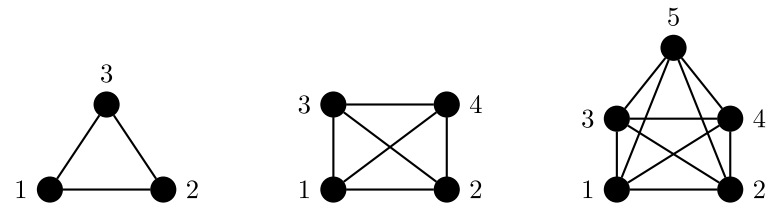
Figure 1.2: The same graph depicted differently

Thus, we see that the relative positions of the vertices and edges is irrelevant. Importance lies only in what vertices are present, and which are adjacent.

To better acquaint ourselves with graph notation, we now consider a few special graph families. These families will appear and be used frequently throughout our work.

First, we consider the complete graph on $n \geq 1$ vertices, denoted K_n. It has vertex set

$V(K_n) = \{1, 2, \ldots, n\}$ and edge set $E(K_n) = \{ij : i, j \in V(K_n), i \neq j\}$. As illustrations, the graphs K_3, K_4, and K_5 are depicted below.



K_3 K_4 K_5

Figure 1.3: Complete graphs

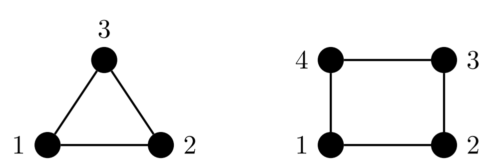
The completely disconnected graph on $n \geq 1$ vertices, denoted $\overline{K_n}$, has $V(\overline{K_n}) = \{1, 2, \ldots, n\}$ and $E(\overline{K_n}) = \emptyset$.

The path on $n \geq 1$ vertices, denoted P_n, has vertex set $V(P_n) = \{1, 2, \ldots, n\}$ and edge set $E(P_n) = \{i(i+1) : 1 \leq i \leq n-1\}$. For example, P_3 and P_4 are depicted below.

P_3 P_4

Figure 1.4: Paths

The cycle on $n \geq 3$ vertices, denoted C_n, has vertex set $V(C_n) = \{1, 2, \ldots, n\}$ and edge set defined by $E(C_n) = E(P_n) \cup \{1n\}$. C_3 and C_4 are depicted below.



C_3 C_4

Figure 1.5: Cycles

For $n \geq 1$, the n-dimensional hypercube, denoted Q_n, has as its vertex set the set of all n-digit binary strings. That is, $V(Q_n) = \{i_1 i_2 \cdots i_n : i_j \in \{0,1\}, 1 \leq j \leq n\}$. Two vertices in Q_n are adjacent if their corresponding strings differ in exactly one position. For illustration, Q_1, Q_2, and Q_3 are shown below.

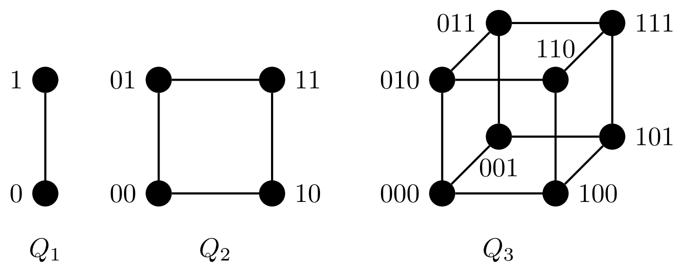


Figure 1.6: Hypercubes

Finally, a graph G is *bipartite* if its vertex set can be partitioned into two sets V_1 and V_2 (called *partite sets*) such that each edge in $E(G)$ is incident with one vertex in V_1 and one vertex in V_2. The *complete bipartite graph* $K_{m,n}$ has $V(K_{m,n}) = V_1 \cup V_2$ with $V_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, $V_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ and $E(K_{m,n}) = \{\alpha_i \beta_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. See Figure 1.7 for an example. In a similar manner, one can define the *complete multipartite graph* $K_{m_1,m_2,\ldots,m_k}$. The vertex set for this graph is the disjoint union of k sets of cardinalities $m_1, m_2, \ldots, m_k$ respectively. Each vertex in $K_{m_1,m_2,\ldots,m_k}$ is adjacent to every vertex outside of its own partite set.

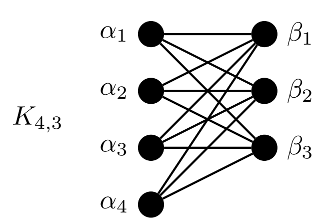


Figure 1.7: Complete bipartite graph

Two graphs G and H are *equal* if $V(G) = V(H)$ and $E(G) = E(H)$. Thus, when we consider the two graphs A and B in Figure 1.8, we see that they are not equal since their respective vertex sets are distinct.

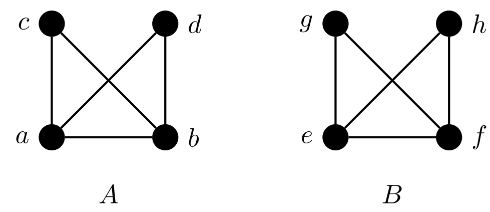


Figure 1.8: Two isomorphic graphs

However, A and B essentially depict the same relationships. We say that A and B are *isomorphic*. A formal definition follows.

Definition 2. Two graphs G and H are *isomorphic* if there exists a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$.

If G and H are isomorphic, we denote this by writing $G \cong H$. The bijection guaranteed by the definition is called an *isomorphism*. In our example above, an isomorphism ϕ is defined by $\phi : \{a, b, c, d\} \to \{e, f, g, h\}$ such that $\phi(a) = e, \phi(b) = f, \phi(c) = g$, and $\phi(d) = h$.

It is often the case that when studying graphs, the particular names of the vertices are not important. In such situations, isomorphic graphs, for all intents and purposes, are identical. Hence, we may at times write $G = H$ to denote the fact that G and H are two representatives from the same isomorphism class, even if their sets of vertices and edges are distinct. This convention, for example, allows us to refer to *the* complete graph on three vertices, regardless of the particular vertex labels used. It also allows us to depict graphs without explicitly labeling each vertex.

If v is a vertex of G, the *neighbors of* v are those vertices adjacent to v. The *degree* of v is equal to the number of vertices adjacent to v. We denote this by $\deg_G(v)$ or $\deg(v)$ if G is understood from context. A vertex of degree one is called a *leaf*, while a vertex adjacent to a leaf is called a *support vertex*. We let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum vertex degrees in G respectively. If $\delta(G) = \Delta(G)$, then G is a *regular* graph. For example, K_n and C_n are regular graphs.

Given a graph G, a *subgraph* of G is any graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, we denote this by writing $H \subseteq G$. If $S \subseteq V(G)$, the subgraph of G *induced* by S, denoted $G\langle S\rangle$ is the graph H such that $V(H) = S$ and $E(H) = \{ij : i, j \in S, ij \in E(G)\}$. For example, in Figure 1.9, S is a subgraph of G and T is the subgraph of G induced by

$\{a, b, c, e\}$.

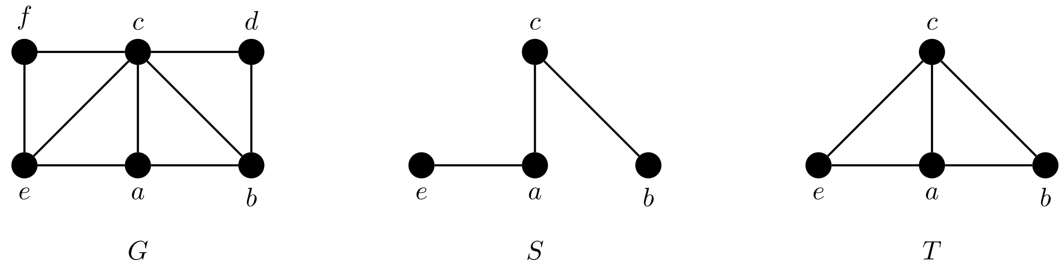


Figure 1.9: A subgraph and an induced subgraph

Subgraphs obtained by deleting either a subset of vertices or a subset of edges will appear frequently throughout our work. In particular, if $S \subseteq V(G)$, the graph $G - S$ is defined to be the graph $G\langle V(G) - S\rangle$. If S is a singleton set, say $\{v\}$, we simplify the notation $G - \{v\}$ to $G - v$ for convenience. If $B \subseteq E(G)$, the graph $G - B$ is defined to be the graph H with $V(H) = V(G)$, and $E(H) = E(G) - B$. Once again, we simplify the notation from $G - \{e\}$ to $G - e$ in the case of $B = \{e\}$, a singleton edge. As an illustration, the graphs G, $G - a$ and $G - \{ab, ae\}$ are shown in Figure 1.10 below.

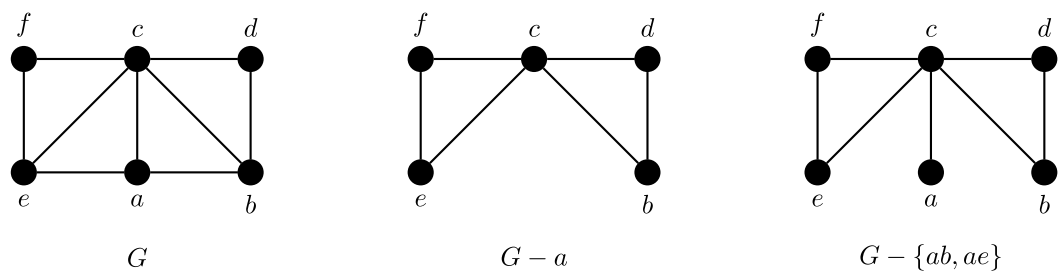


Figure 1.10: Deletion subgraphs

Given a graph G, a *supergraph* of G is any graph H such that $G \subseteq H$. For example, if G and H are distinct graphs, then their *union*, denoted $G \cup H$, is the graph defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. As another example, the *join* of disjoint graphs G and H is the graph $G + H$ defined by $V(G + H) = V(G) \cup V(H)$ with $E(G + H) = E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$. Similar to deletion subgraphs, we can also consider *addition subgraphs*. For example, if G is a graph and S is a set of vertices not in $V(G)$, then $G + S$ is the graph defined by $V(G + S) = V(G) \cup S$ and $E(G + S) = E(G)$. In a similar manner, edges can be added to G. For example, if u and v are two nonadjacent vertices in G, then the graph $G + uv$ is defined by $V(G + uv) = V(G)$ and $E(G + uv) = E(G) \cup \{uv\}$. Vertices and edges can also simultaneously be added to a graph. For example, given a graph G, the graph $G \circ K_1$ is the graph formed from G by adding a vertex u' and the edge uu' for each $u \in V(G)$. This is a specific example of a more general *corona of two graphs.* An illustrative example is shown below.

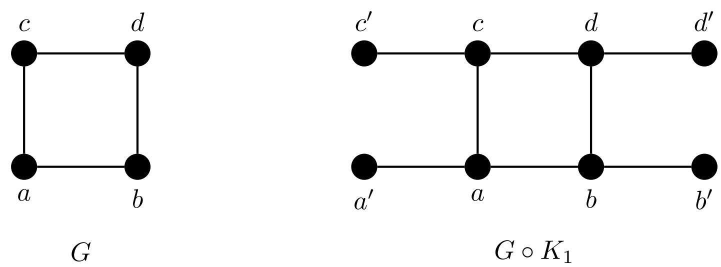


Figure 1.11: A corona

A *walk* W in a graph is an alternating sequence of vertices and edges

$$W = v_0, e_1, v_1, e_2, \ldots, e_k, v_k$$

such that $e_i = v_{i-1}v_i$ is an edge of the graph. The *initial vertex* of the walk is v_0, while v_k is called the *terminal vertex* of the walk. This makes W a $v_0 - v_k$ walk. The *length* of W is the number of edges it contains. If $v_0 = v_k$, then the walk is said to be *closed.* If a walk contains no repeated vertex, then the walk is a *path*, while a closed walk with no repeated vertices (other than the initial and terminal vertex) is a *cycle.* If for each pair of vertices u and v in a graph G there is a path from u to v, then G is said to be *connected.* The maximally connected induced subgraphs of G are called the *components* of G. A graph containing no cycles is called an *acyclic graph*, or a *forest*, while a connected, acyclic graph is called a *tree.*

The distance between two vertices u and v, denoted $d_G(u, v)$ (or $d(u, v)$ if G is known from context), is the length of a shortest path in G from u to v. If there are no paths from u to v, then $d(u, v) = \infty$. The diameter of a connected graph G, denoted $\text{diam}(G)$, is defined by

$\mathrm{diam}(G) = \max_{u,v \in V(G)} d(u,v)$. A *diametral path* in G is a path between two vertices u and v such that $d(u,v) = \mathrm{diam}(G)$.

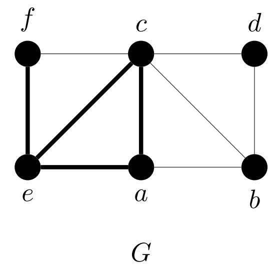


Figure 1.12: A walk

As an illustration of a few of these definitions, consider Figure 1.12. In the figure, the walk $e, ea, a, ac, c, ce, e, ef, f$ is highlighted by darkened edges. Observe that a, b, and c form a cycle, and that e, ea, a, ac, c and e, ec, c are two paths from e to c. We see that $d(e,b) = 2$, $d(e,c) = 1$, and that $\mathrm{diam}(G) = 2$. Thus, e, ea, a, ab, b is a diametral path.

1.2 The Domination Chain

Most of our work in the following chapters concern a family of graph invariants that are linked together by a well-known and often-studied inequality chain, called the domination chain, presented as Theorem 5 below. The three big concepts in the chain: domination, irredundance, and independence, are discussed in this section. After presenting the requisite definitions, we illustrate the chain itself by showing how the three concepts are related.

1.2.1 Domination

Let v be a vertex of G. The *open neighborhood of* v, which we denote by $N(v)$, is the set of vertices sharing an edge with v. That is, $N(v) = \{u\,;\, uv \in E(G)\}$. The *closed neighborhood of* v, denoted $N[v]$, is the set $N(v) \cup \{v\}$; v is said to *dominate* every vertex in its closed neighborhood. Given a set $S \subseteq V(G)$, the open neighborhood of S, denoted $N(S)$, is defined by $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood of S, denoted $N[S]$, is given by $N[S] = N(S) \cup S$. Similar to the above, S is said to dominate every vertex in its closed neighborhood.

Definition 3. A *dominating set* in G is any set $S \subseteq V(G)$ such that $N[S] = V(G)$. Equivalently, S is a dominating set if every vertex in G is either in S, or is adjacent to a vertex in S.

Note that every graph G contains a dominating set since $V(G)$ itself satisfies $N[V(G)] = V(G)$.

Definition 4. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G.

The following distinction between *minimum* and *minimal* is somewhat unique to graph theory, thus we carefully make the following definitions.

Definition 5. A *minimum dominating set* of G, also called a γ-set of G, is a dominating set S of G for which $|S| = \gamma(G)$.

Definition 6. A *minimal dominating set* of G is any dominating set S such that for each $v \in S$, $S - \{v\}$ is no longer a dominating set.

Thus, we see that a *minimum* dominating set is, necessarily, a *minimal* dominating set, but a *minimal* dominating set need not be a *minimum* dominating set. With this distinction now made, we can also define the following parameter.

Definition 7. The *upper domination number* of G, denoted $\Gamma(G)$, is the maximum cardinality of a minimal dominating set. A minimal dominating set of cardinality $\Gamma(G)$ is called a Γ-set.

Given our observations above, note that the domination number of G can also be defined as the minimum cardinality of a minimal dominating set. Thus, in particular, we see that $\gamma(G) \leq \Gamma(G)$ for any graph G.

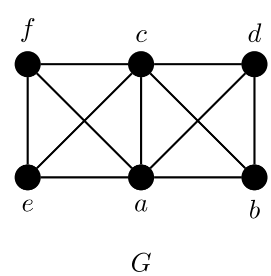


Figure 1.13: Domination example

As an illustration, consider the graph G in Figure 1.13. $S = \{f, c, d\}$ is a dominating set since every vertex in G is either in S (in the case of f, c, and d) or is adjacent to a vertex in S (in the case of e, a, and b). Notice, however, that S is not minimal since $S - \{f\}$ is still a dominating set. For this graph, $\gamma(G) = 1$ since $\{c\}$ is a dominating set. Hence, $\{c\}$ is a minimum dominating set or a γ-set. Notice additionally that $\{a\}$ is a γ-set. Thus, we see that graphs, in general, can have many γ-sets. Now, consider the set $\{e, b\}$. This set is clearly a dominating set. Moreover, it is a minimal dominating set since neither of the sets $\{b\}$ or $\{e\}$ is dominating. Thus, $\{e, b\}$ is a minimal dominating set that is *not* a minimum dominating set. In fact, $\{e, b\}$ is a Γ-set.

Thus far, we have presented vertex domination. This is only one of many different types of domination. For example, one can consider selecting a set of edges of a graph so that every edge is either in the set, or is adjacent to an edge in the set. Such a set is then called an *edge dominating set*. In vertex domination, vertices in the dominating set need not be adjacent to another vertex in the set. If we add this requirement, the result is *total domination*. *Distance-k domination* is a weakening of vertex domination in that every vertex not in the dominating set now only needs to be within distance k of a vertex in the set. Other examples of domination include *Roman domination*, *independence domination* (not to be confused with *independent domination* considered below), *rainbow domination*, *k-domination*, *signed domination*, and *broadcast domination*, just to name a few.

1.2.2 Irredundance

To understand irredundance, we first need the following definition.

Definition 8. Let G be a graph and let $S \subseteq V(G)$ with $x \in S$. A *private neighbor of x with respect to S* is any vertex v for which $N[v] \cap S = \{x\}$. If v is a private neighbor of x and $v \neq x$, then v is also called an *external private neighbor of x with respect to S*. The *private neighborhood of x with respect to S*, denoted $pn(x, S)$, is the set of private neighbors of x with respect to S.

Notice that x can be a private neighbor of itself with respect to S. An example will help illustrate this definition. Let G be the graph below.

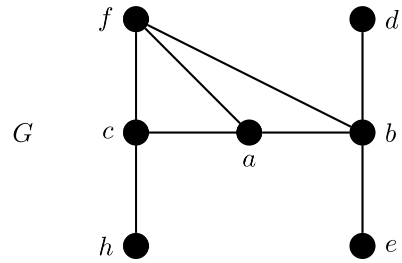


Figure 1.14: Irredundance example

Consider the set S defined by $S = \{a, b\}$. Notice that S is not a dominating set since h is not adjacent to any vertex in S. Observe that a is adjacent to c, but b is not adjacent to c. Thus, c is a private neighbor of a with respect to S. Since f is adjacent to both a and b, f is not a private neighbor of either a or b. In fact, we see that $pn(a, S) = \{c\}$ and $pn(b, S) = \{d, e\}$.

Definition 9. A set $S \subseteq V(G)$ is *irredundant* (or is an irredundant set), if for each $x \in S$, $|pn(x, S)| > 0$. Equivalently, S is irredundant if for each $x \in S$, $N[x] - N[S - \{x\}] \neq \emptyset$. If S is not irredundant, it is *redundant*.

Definition 10. A set $S \subseteq V(G)$ is *maximal irredundant* if S is irredundant and for each $x \in V - S$, $S \cup \{x\}$ is redundant.

Definition 11. The *irredundance number* of G, denoted $ir(G)$, is the minimum cardinality of a maximal irredundant set. A maximal irredundant set of cardinality $ir(G)$ is called an ir-set. The *upper irredundance number* of G, denoted $IR(G)$, is the maximum cardinality of an irredundant set. A maximal irredundant set of cardinality $IR(G)$ is called an IR-set.

Consider once again the graph G from Figure 1.14 and the set $S = \{a, b\}$. First, note that S is irredundant since $pn(a, S) \neq \emptyset$ and $pn(b, S) \neq \emptyset$. Is S *maximal* irredundant? Consider creating a new set from S, call it S', by adding the vertex c. We see that $pn(c, S') = \{h\}$. Thus, c has a private neighbor with respect to S'. However, by adding c, a no longer has a private neighbor. Thus, $S' = \{a, b, c\}$ is redundant. In fact, S is maximal irredundant. To see this, observe the following.

- $S' = \{a, b, c\}$ is redundant since $pn(a, S') = \emptyset$.
- $S' = \{a, b, d\}$ is redundant since $pn(d, S') = \emptyset$.

- $S' = \{a, b, e\}$ is redundant since $pn(e, S') = \emptyset$.

- $S' = \{a, b, f\}$ is redundant since $pn(a, S') = pn(f, S') = \emptyset$.

- $S' = \{a, b, h\}$ is redundant since $pn(a, S') = \emptyset$.

In fact, $ir(G) = 2$. Thus, $S = \{a, b\}$ is an ir-set. Also observe that $IR(G) = 4$ and that $S = \{d, e, f, h\}$ is an IR-set with each element of S a self-private neighbor.

As in domination, there are also different types of irredundance. As we have defined irredundance, a set $S \subseteq V(G)$ is irredundant if for all $x \in S$, $N[x] - N[S - \{x\}] \neq \emptyset$. In *open irredundance*, we say that a set $S \subseteq V(G)$ is open irredundant if for all $x \in S$, $N(x) - N[S - \{x\}] \neq \emptyset$. Hence, every vertex in the set must have an external private neighbor. By altering the open and closed neighborhoods, we also have *open-open irredundance* in which each vertex $x \in S$ satisfies $N(x) - N(S - \{x\}) \neq \emptyset$, and *closed-open irredundance* in which each vertex $x \in S$ satisfies $N[x] - N(S - \{x\}) \neq \emptyset$.

1.2.3 Independence

The last of the three domination chain concepts is independence.

Definition 12. Let G be a graph and $S \subseteq V(G)$. The set S is *independent* (or is an independent set) if no two vertices in S share an edge.

Equivalently, $S \subseteq V(G)$ is independent if $G\langle S\rangle$ has no edges.

Definition 13. A set of vertices $S \subseteq V(G)$ is maximal independent if S is independent and for each $x \in V - S$, $S \cup \{x\}$ is not independent.

Definition 14. The *independence number* of G, which we denote by $\beta_0(G)$, is the maximum cardinality of an independent set. An independent set of cardinality $\beta_0(G)$ is called a β_0-set. The minimum cardinality of a maximal independent set is denoted $i(G)$, and a maximal independent set of cardinality $i(G)$ is called an i-set.

The parameter $i(G)$ is also called the *independent domination number* for reasons that will be made clear in the following subsection. To illustrate these definitions, consider the graph in Figure 1.15.

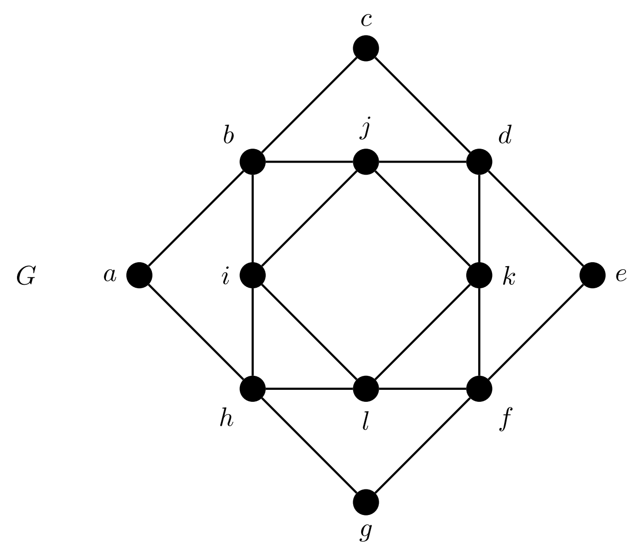


Figure 1.15: Independence example

The set $S = \{b, d, h, f\}$ is independent, as no two vertices in S share an edge. Moreover, S is maximal independent, since every vertex in $V - S$ shares an edge with a vertex in S. In fact, $i(G) = 4$, in which case S is an i-set. For this graph, $\beta_0(G) = 6$ with $S' = \{a, c, e, g, i, k\}$ a β_0-set.

1.2.4 Relationships

We conclude this section by showing how domination, irredundance, and independence are related to one another. We begin with the following.

Proposition 1. *Every maximal independent set of vertices in a graph is a minimal dominating set.*

Proof. Let G be a graph and let S be a maximal independent set. Let $x \in V(G) - S$. If x is not adjacent to any vertex in S, then $S \cup \{x\}$ is independent. Thus, since S is maximal independent, there exists $v \in S$ such that $x \in N[v]$. Hence, S is a dominating set. To see that S is minimal dominating, let $v \in S$ and consider $S - \{v\}$. Since S is independent, $S - \{v\}$ fails to dominate v. Thus, S is minimal dominating and our result follows. $\square$

This result gives us the following corollary, which illustrates how domination and independence are related.

Corollary 2. *For any graph G, $\gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G)$.*

Next, we observe the following.

Proposition 3. *Every minimal dominating set in a graph is a maximal irredundant set.*

Proof. Let G be a graph and let S be a minimal dominating set. Let $x \in S$ and consider $pn(x, S)$. Suppose that $pn(x, S) = \emptyset$. This implies that there exists $v \in S$ for which $vx \in E(G)$ (since otherwise $x \in pn(x, S)$) and that there exists no $u \in V(G) - S$ for which $N[u] \cap S = \{x\}$ (since otherwise $u \in pn(x, S)$). Hence, $S - \{x\}$ is a dominating set, a contradiction. Thus, since $pn(x, S) \neq \emptyset$ for all $x \in S$, we see that S is irredundant. To see that S is maximal irredundant, suppose there exists $z \in V - S$ for which $pn(z, S \cup \{z\}) \neq \emptyset$. This implies there exists $y \in V(G)$ for which $N[y] \cap (S \cup \{z\}) = \{z\}$. This, however, implies that S does not dominate y, a contradiction. Thus, S is maximal irredundant. $\square$

Corollary 4. *For any graph G, $ir(G) \leq \gamma(G) \leq \Gamma(G) \leq IR(G)$.*

By combining Corollary 2 and Corollary 4, we arrive at the domination chain first proved by Cockayne, Hedetniemi, and Miller in 1978 [2].

Theorem 5. *For any graph G, $ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$.*

For more on domination, see [19].

1.3 Graph Products

We have already seen a few methods for constructing a new graph from a given graph. For example, the deletion subgraphs considered in Section 1 produce a smaller graph from a given graph. In this section, we consider methods, called graph products, for constructing a new larger graph given two starting graphs. There are many different types of graph products. For our purposes, we will only need to focus on two: the Cartesian product, and the direct product.

1.3.1 The Cartesian product

We begin with a definition.

Definition 15. Let G and H be graphs. The *Cartesian product* of G and H, denoted $G \square H$, is the graph defined by $V(G \square H) = V(G) \times V(H)$ with $(g_1, h_1)(g_2, h_2) \in E(G \square H)$ if and only if either $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $h_1 = h_2$ and $g_1g_2 \in E(G)$.

Recall that $V(G) \times V(H)$ is the Cartesian product of sets. Thus, each vertex of $G \,\square\, H$ is an ordered pair (g, h) where $g \in V(G)$ and $h \in V(H)$. Let's consider an example.

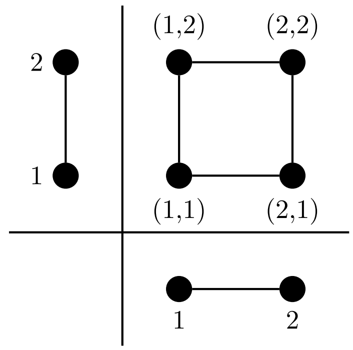


Figure 1.16: $K_2 \,\square\, K_2$

We see that $K_2 \,\square\, K_2$ produces a "box". This, in fact, is the motivation for the $\square$ notation for the product itself.

The Cartesian product produces a graph which has similarities to both of its factors. It is this fact that often makes the Cartesian product useful and easy to work with. For example, properties of $G \,\square\, H$ often depend on the corresponding properties holding in one of G or H. We will see an example of this in Chapters 2 and 4 to come. To better illustrate this discussion, we consider another example.

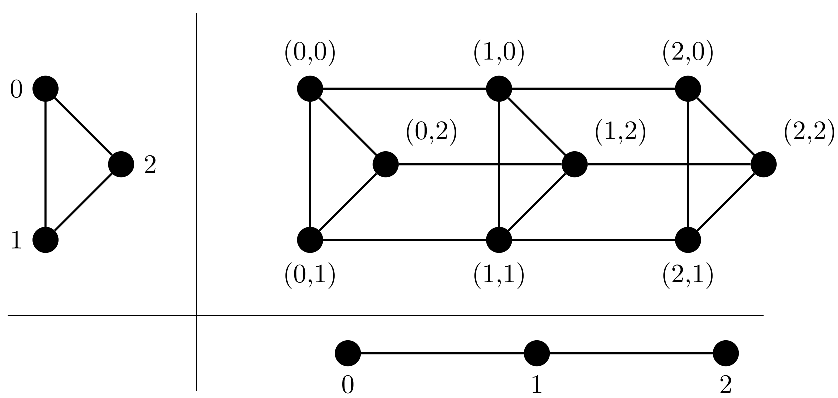


Figure 1.17: $P_3 \,\square\, C_3$

In Figure 1.17, we see that there are three copies of C_3 ($(0, 0), (0, 2)$, and $(0, 1)$ induce one), and three copies of P_3 ($(0,0),(1,0)$, and $(2,0)$ induce one). We refer to these "copies" as either *layers* or *fibers*.

1.3.2 The Direct Product

The direct product of two graphs is defined as follows.

Definition 16. Let G and H be graphs. The *direct product* of G and H, denoted $G \times H$, is the graph defined by $V(G \times H) = V(G) \times V(H)$ with $(g_1,h_1)(g_2,h_2) \in E(G \times H)$ if and only if $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$.

As in the Cartesian product, the notation $\times$ used for the direct product is motivated by considering the product of two K_2s.

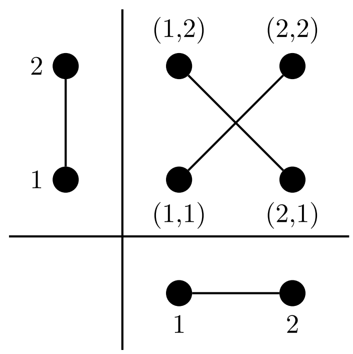


Figure 1.18: $K_2 \times K_2$

The direct product behaves quite differently to the Cartesian product. For example, the Cartesian product of two connected graphs is a connected graph. The case of $K_2 \times K_2$ above illustrates that the direct product of two connected graphs may not be a connected graph. Additionally, the "copies" of the factors found in the Cartesian product are not as obvious in the direct product. That is, if we fix the first component of a vertex in $G \,\square\, H$ and let the second component vary, we induce a copy of H. If we fix the first component of a vertex in $G \times H$ and let the second component vary, we induce an independent set. These differences, among others, make the direct product interesting to work with. For illustrative purposes, the graph $K_3 \times K_2$ is given below.

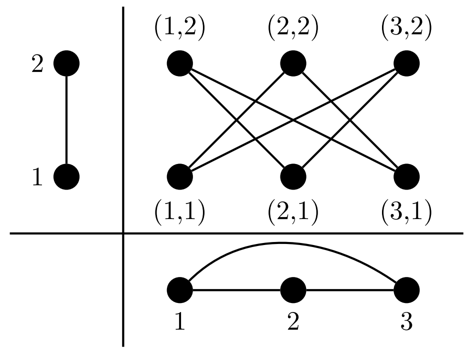


Figure 1.19: $K_3 \times K_2$

Before concluding, we note that in the literature, the Cartesian product is sometimes denoted $G \times H$. In our work, the Cartesian product will always be denoted $G \,\Box\, H$ and the direct product will always be denoted by $G \times H$.

For more on the Cartesian and direct product, and for more on product graphs in general, see [18].

Chapter 2

Unique minimum dominating sets in some Cartesian product graphs

In our discussion of domination in Chapter 1, we saw an example of a graph that had two distinct minimum dominating sets. In this chapter, we consider graphs having a unique minimum dominating set. More precisely, we consider unique minimum dominating sets in graphs $G \,\Box\, K_n$ where G is a finite, simple, connected, nontrivial graph and K_n is the complete graph on n vertices. In our discussion, we first characterize the unique γ-sets in such graphs in Section 2.3 by illustrating the special form they take. Using this characterization, we then generalize a main result of [17] in Section 2.4, thereby giving a method for recognizing a γ-set in $G \,\Box\, K_n$ as unique when the first factor G is a tree. In Section 2.5, we consider the ways two such graphs, each having a unique minimum dominating set, can be combined while preserving a unique γ-set. We use these operations in Section 2.6 to present the proof of our first main result, that if T is a nontrivial tree with minimum dominating set D, then $T \,\Box\, K_n$ has a unique minimum dominating set if and only if every vertex in D has at least $n+1$ external private neighbors. We thus characterize those trees whose Cartesian product with a complete graph has a unique γ-set. In Section 2.7, we apply our results thus far to graphs $G \,\Box\, K_{n_1} \,\Box\, K_{n_2} \,\Box\, \cdots \,\Box\, K_{n_r}$. These so-called "repeated products" share properties similar to both the graphs $G \,\Box\, K_n$ considered in the section, and the graphs $G \,\Box\, H$ considered in Chapter 4. Note that the work in this chapter is a more thorough discussion of the two publications [23] and [24]. As such, the results in this chapter will not be specifically referenced to those papers.

2.1 Introduction and background

The study of unique minimum vertex dominating sets began with Gunther, Hartnell, Markus and Rall in [17] where the authors established a method for recognizing unique γ-sets in trees, and provided a characterization of those trees which have a unique γ-set. Their work was later expanded upon by Fischermann in [5] where block graphs were considered, and by Fischermann and Volkmann in [10] where cactus graphs were considered. The maximum number of edges contained in graphs with unique γ-sets was studied in [7] and [13]. We will continue this study in Chapter 4. For complexity results concerning unique γ-sets, see [8]. Uniqueness of other types of dominating sets has also been studied. For example, edge domination was studied in [36] and [9]. Distance k domination was analyzed in [9]. Total domination was first studied in [20] and later in [6]. Mixed domination was considered in [10], and paired domination was studied in [1]. Connections between unique minimum dominating sets and unique irredundant and independent dominating sets was studied in [12], while connections between maximum independent sets and unique upper dominating sets can be found in [11]. Finally, properties of unique domination were used in [27] and [26] to study properties of Roman dominating sets.

2.2 Notation

In Chapter 1, we defined $pn(x, D)$ to be the set of private neighbors of $x \in D$ with respect to D, where D is an arbitrary subset of vertices. In this chapter, we will be particularly interested in *external* private neighbors. Thus, for notational convenience, we let $epn(x, D)$ denote the set of external private neighbors of $x \in D$ with respect to D. Recall that since every minimal dominating set is a maximal irredundant set, if D is a minimum dominating set of some graph G, then $pn(x, D) \neq \emptyset$ for all $x \in D$. However, $epn(x, D)$ may or may not be empty.

As we are considering Cartesian product graphs in this chapter, we will make extensive use of the following. Given the graph $G_1 \square G_2$, for $i = 1, 2$, let the projection $\pi_{G_i} : G_1 \square G_2 \to G_i$ be defined by $\pi_{G_i}((u_1, u_2)) = u_i$. As an example, consider the graph $P_3 \square K_3$ in Figure 2.1 below. Let D be the set of white vertices, that is $D = \{(1, 0), (0, 2)\}$. We see that $\pi_{P_3}(D) \subseteq V(P_3)$ is the set $\{0, 1\}$, while $\pi_{K_3}(D) \subseteq V(K_3)$ is the set $\{0, 2\}$.

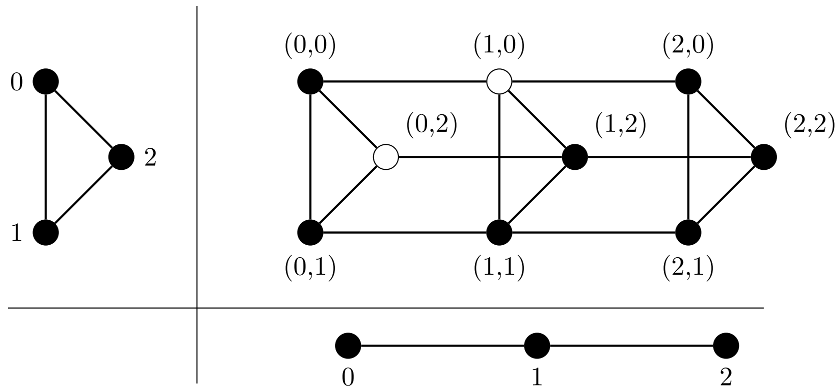


Figure 2.1: Projections

We note that if A is a dominating set of $G_1 \square G_2$, then $\pi_{G_i}(A)$ dominates G_i for $i = 1$ and $i = 2$.

As further notational convenience, for $(u_1, u_2) \in V(G_1 \square G_2)$, we let $G_i^{(u_1,u_2)}$ denote the induced subgraph

$$G_i^{(u_1,u_2)} = \langle \{(v_1, v_2) : \pi_{G_{3-i}}((v_1, v_2)) = \pi_{G_{3-i}}((u_1, u_2))\} \rangle$$

and refer to it as the G_i-layer through (u_1, u_2). For example, in Figure 2.2 below, the G_1-layer through $(v, 2)$ is the subgraph induced by the white vertices.

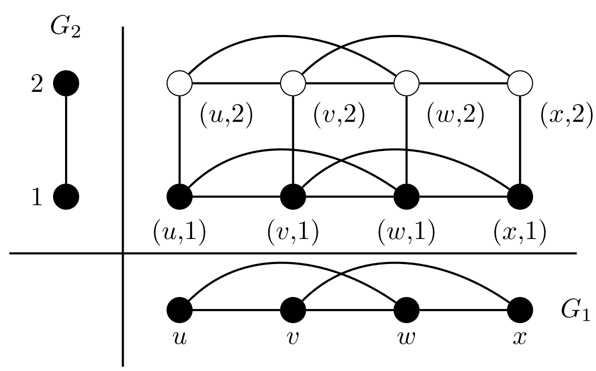


Figure 2.2: T-layer illustration

For other graph product terminology not explicitly mentioned, we follow [18].

We first consider graphs $G \square K_n$ where G is a connected, finite, simple, nontrivial graph.

As in Chapter 1, we assume that the vertex set of K_n is $\{1, 2, \ldots, n\}$ which we will denote by $[n]$ for brevity. For convenience, for $u \in V(G)$ and for $k \in [n]$, we denote the G-layer of $G \square K_n$ through (u, k) as G^k. We let $\mathcal{U}$ denote the class of all finite simple graphs that have a unique minimum dominating set. If $G \in \mathcal{U}$, then we let $UD(G)$ denote the unique γ-set for G.

With our notation now defined, our main result is the following.

Theorem 6. *Let n be a positive integer, let T be a nontrivial tree, and let D be a γ-set of T. The graph $T \square K_n$ is in $\mathcal{U}$ if and only if for all $v \in D$, $|epn(v, D)| \geq n + 1$.*

2.3 Basic Structure

Suppose that $G \square K_n \in \mathcal{U}$. What, if anything, can we say about $UD(G \square K_n)$? We begin with the following observation.

Lemma 7. *If $G \square K_n \in \mathcal{U}$, then there exists $S \subseteq V(G)$ such that $UD(G \square K_n) = S \times [n]$.*

Proof. Denote $UD(G \square K_n)$ by D. Without loss of generality, suppose that $(v, 1) \in D$ but $(v, 2) \notin D$. Let

$$D' = \{(x, 1) : (x, 2) \in D\} \cup \{(y, 2) : (y, 1) \in D\} \cup \{(w, j) : (w, j) \in D, 3 \leq j \leq n\}.$$

We claim that D' is also a γ-set for $G \square K_n$.

- If $x \in \pi_G(D)$, then by the definition of D', it follows that the K_n-layer through $(x, 1)$ is contained in $N[D']$.

- If $x \notin \pi_G(D)$, then for $1 \leq j \leq n$, each (x, j) is dominated by some (v_j, j) in D. Thus, $(x, 1)$ is dominated by $(v_2, 1)$ in D', $(x, 2)$ is dominated by $(v_1, 2)$ in D', and (x, j) is dominated by (v_j, j) in D' for $3 \leq j \leq n$. Hence, every vertex in the K_n-layer through $(x, 1)$ is contained in $N[D']$.

Thus, we see that D' is a γ-set of $G \square K_n$ distinct from D, proving our result. $\square$

This result can be used to quickly determine that a graph $G \square K_n$ is *not* an element of $\mathcal{U}$. For example, the graph $P_3 \square K_2$ is shown below in Figure 2.3, with a γ-set indicated in white. Since this γ-set is not of the form $S \times [2]$ for any $S \subseteq V(P_3)$, we immediately know that $P_3 \square K_2 \notin \mathcal{U}$.

Notice, however, that $\{(2,1),(2,2)\}$ is also a γ-set of $P_3 \,\square\, K_2$ and it *is* of the form $S \times [2]$ with $S = \{2\}$. Thus, having a γ-set of the form $S \times [n]$ for $S \subseteq V(G)$ does not imply that $G \,\square\, K_n \in \mathcal{U}$.

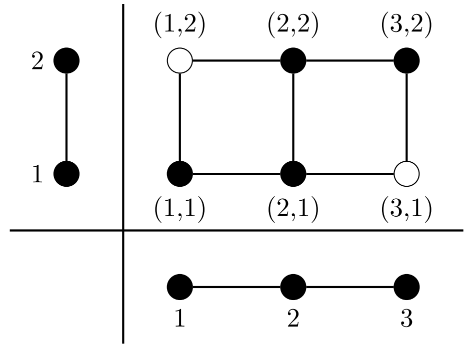


Figure 2.3: $P_3 \,\square\, K_2$

Our first result also has the following corollary.

Corollary 8. *If $G \,\square\, K_n \in \mathcal{U}$, then $\gamma(G \,\square\, K_n)$ is a multiple of n.*

As illustrated in $P_3 \,\square\, K_2$ above, $\gamma(P_3 \,\square\, K_2) = 2$ is a multiple of 2. Thus, our corollary demonstrates that knowledge of $\gamma(G \square K_n)$ may be sufficient to deduce that $G \square K_n \notin \mathcal{U}$, however, more knowledge may be required.

Given Lemma 7, we say that any subset A of $V(G \,\square\, K_n)$ such that $A = S \times [n]$ for some subset S of $V(G)$ has the *stacked property*. Before proceeding to our next result, we recall the following lemma from [17].

Lemma 9 ([17])**.** *Let G be a graph with a unique γ-set D. Let uv be any edge in G other than an edge connecting a vertex in D to one of its private neighbors. If G^- is the graph obtained from G by deleting the edge uv, then G^- has D as the unique γ-set.*

We now consider the following consequence of Lemma 7.

Proposition 10. *If $G \,\square\, K_n \in \mathcal{U}$, then $G \in \mathcal{U}$. Moreover, $G \,\square\, K_m \in \mathcal{U}$ for $1 \leq m \leq n$.*

Proof. Denote $UD(G \,\square\, K_n)$ by D. By Lemma 7, there exists $S \subseteq V(G)$ such that $D = S \times [n]$. Thus, for any $(x,i) \in D$, the external private neighbors of (x,i) with respect to D all belong to G^i. Define H to be the graph

$$G \,\square\, K_n - \{(v,n)(v,j) \,:\, v \in V(G), 1 \leq j \leq n-1\}.$$

We see that H is isomorphic to $(G \,\Box\, K_{n-1}) \cup G$. By Lemma 9, D is still the unique γ-set for H. The proposition follows by induction. $\square$

Thus, we see that if $G \Box K_n \in \mathcal{U}$, then $UD(G \Box K_n) = UD(G) \times [n]$ and thus, $\pi_G(UD(G \Box K_n)) = UD(G)$. Once again, $P_3 \Box K_2$ illustrates that the converse of Proposition 10 does not hold since $P_3 \in \mathcal{U}$, however $P_3 \Box K_2 \notin \mathcal{U}$.

In [17], the authors prove the following lemma for general graphs.

Lemma 11 ([17])**.** *If G has a unique γ-set D, then every vertex in D that is not an isolated vertex has at least two private neighbors other than itself.*

In our specialized setting, we can improve upon this bound as follows.

Suppose that $A \subseteq V(G \,\Box\, K_n)$ has the stacked property and that $\{v\} \times [n] \subseteq A$. If $(u, j) \in epn((v, j), A)$ for some j, then $(u, i) \in epn((v, i), A)$ for $1 \leq i \leq n$. Bearing this in mind, suppose that D is a γ-set of $G \,\Box\, K_n$ with the stacked property. Additionally, suppose that $(v, 1) \in D$ has $epn((v, 1), D) = \{(u_1, 1), (u_2, 1), \ldots, (u_j, 1)\}$ for some $j \leq n$. This implies that $epn((v, i), D) = \{(u_1, i), (u_2, i), \ldots, (u_j, i)\}$ for $2 \leq i \leq n$. The set D' defined by

$$D' = \Big(D - \{(v,1), (v,2), \ldots, (v,j)\}\Big) \cup \{(u_1,1), (u_2,2), \ldots, (u_j,j)\}$$

is a γ-set of $G \,\Box\, K_n$ distinct from D. Thus, we have the following.

Lemma 12. *If $G \,\Box\, K_n \in \mathcal{U}$, then for each element $v \in UD(G \,\Box\, K_n)$, $|epn(v, UD(G \,\Box\, K_n))| \geq n + 1$.*

The graph $K_{1,n+1} \Box K_n$ demonstrates that this "bound" is sharp. That is, $K_{1,n+1} \Box K_n \in \mathcal{U}$ and each vertex $v \in UD(K_{1,n+1} \Box K_n)$ has exactly $n + 1$ external private neighbors. The family of graphs $K_m \,\Box\, K_n$, $m \geq n$, demonstrates that no condition on the number of external private neighbors for vertices in a minimum dominating set is, by itself, sufficient to force the product with K_n to have a unique γ-set. For use in the proof of Theorem 21 to follow, we note here the following.

Observation 13. *If $v \in V(G)$ has at least $n + 1$ leaf neighbors, then $\{v\} \times [n]$ is contained in every γ-set of $G \,\Box\, K_n$.*

Thus far, we have considered the structural properties of $UD(G \Box K_n)$ when $G \Box K_n \in \mathcal{U}$. We now consider the structural properties of $G \Box K_n$ when $G \Box K_n \in \mathcal{U}$. To that end, in [17], the authors prove the following lemma.

Lemma 14 ([17]). *Let D be a γ-set of a graph G. If for every $x \in D$, $\gamma(G - x) > \gamma(G)$, then D is the unique γ-set of G.*

The following statement is a generalization of this result to our setting.

> If $G \,\square\, K_n$ has a γ-set D satisfying the stacked property such that for every $v \in \pi_G(D)$, $\gamma(G \,\square\, K_n - (\{v\} \times [n])) > \gamma(G \,\square\, K_n)$, then D is the unique γ-set of $G \,\square\, K_n$.

This statement, however, does not hold for a general product $G \,\square\, K_n$. The graph G illustrated in Figure 2.4 provides a counterexample. Define H to be the graph $G \,\square\, K_2$. The set D defined by

$$D = \{1, 2, 3, 4, 5, 6\} \times \{1, 2\}$$

is a γ-set satisfying the stacked property such that for every $v \in \pi_G(D)$, $\gamma(H - \{(v, 1), (v, 2)\}) > \gamma(H)$. However, D is not a unique γ-set since the set

$$\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (5, 1), (6, 1), (10, 2), (14, 2), (18, 2)\}$$

is also a γ-set of H.

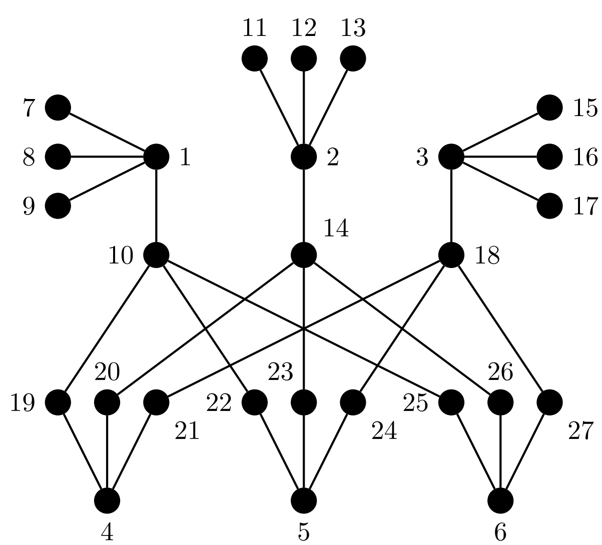


Figure 2.4: Counterexample

In the next section, we will show that if G is a tree, then the conditions above *do* imply that $G \,\square\, K_n \in \mathcal{U}$. The following lemma will be used in the proof.

Lemma 15. *If $G \,\square\, K_n$ has a γ-set D satisfying the stacked property such that for every $v \in \pi_G(D)$, $\gamma(G \,\square\, K_n - (\{v\} \times [n])) > \gamma(G \,\square\, K_n)$, then for all $y \in D$, $|epn(y, D)| \geq n+1$.*

Proof. Let $v \in \pi_G(D)$. Suppose for some $j \leq n$ that

$$epn((v,1), D) = \{(u_1, 1), (u_2, 1), \ldots, (u_j, 1)\}.$$

Since D satisfies the stacked property,

$$epn((v,i), D) = \{(u_1, i), (u_2, i), \ldots, (u_j, i)\}$$

for $1 \leq i \leq n$. The set

$$\Big(D - (\{v\} \times [n])\Big) \cup \{(u_1, 1), (u_2, 2), \ldots (u_j, j), (u_j, j+1), \ldots, (u_j, n)\}$$

is a dominating set of $G \,\square\, K_n - (\{v\} \times [n])$ of cardinality equal to $|D|$, a contradiction. Thus, our result follows. $\square$

Before we proceed to our first theorem, we recall the following two lemmas from [17].

Lemma 16 ([17])**.** *If G is a graph that has a unique γ-set D, then for any $x \in G - D$, $\gamma(G - x) = \gamma(G)$.*

Lemma 17 ([17])**.** *If G is a graph that has a unique γ-set D, then $\gamma(G - x) \geq \gamma(G)$ for all $x \in D$.*

We can now generalize these lemmas to our setting.

Lemma 18. *Let $G \,\square\, K_n \in \mathcal{U}$ and let $v \notin \pi_G(UD(G \,\square\, K_n))$. For any subset B of $\{v\} \times [n]$, $\gamma(G \,\square\, K_n - B) = \gamma(G \,\square\, K_n)$.*

Proof. Suppose that $\gamma(G \,\square\, K_n - B) < \gamma(G \,\square\, K_n)$. This implies that $G \,\square\, K_n - B$ is dominated by a set D' with $|D'| < |UD(G \,\square\, K_n)|$. However, for any $(v, i) \in B$, $D' \cup \{(v, i)\}$ is a dominating set of $G \,\square\, K_n$ distinct from $UD(G \,\square\, K_n)$ of cardinality less than or equal to $|UD(G \,\square\, K_n)|$, a contradiction. Thus, $\gamma(G \,\square\, K_n - B) \geq \gamma(G \,\square\, K_n)$. Since $UD(G \,\square\, K_n)$ dominates $G \,\square\, K_n - B$, we see that $\gamma(G \,\square\, K_n - B) = \gamma(G \,\square\, K_n)$. $\square$

Lemma 19. *Let $G \,\square\, K_n \in \mathcal{U}$ and let $v \in \pi_G(UD(G \,\square\, K_n))$. For any subset B of $\{v\} \times [n]$, $\gamma(G \,\square\, K_n - B) \geq \gamma(G \,\square\, K_n)$.*

Proof. For the sake of contradiction, suppose that $\gamma(G \,\square\, K_n - B) < \gamma(G \,\square\, K_n)$ for some $B \subseteq \{v\} \times [n]$. If D' is a γ-set of $G \,\square\, K_n - B$, then $|D'| < |UD(G \,\square\, K_n)|$ and D' dominates all of the external private neighbors of the vertices in B with respect to $UD(G \,\square\, K_n)$. However, for any $(v, i) \in B$, $D' \cup \{(v, i)\}$ is a γ-set of $G \,\square\, K_n$ and $UD(G \,\square\, K_n) \neq D' \cup \{(v, i)\}$, a contradiction. $\square$

2.4 Trees

In this section, we restrict our attention to graphs $T \,\square\, K_n$ where T is a nontrivial tree. We prove a set of equivalences which can be used to determine whether a γ-set in $T \,\square\, K_n$ is unique. This result, formulated as Theorem 21 below, is a generalization of the following theorem from [17], and as such, the notation and proof structure are similar.

Theorem 20 ([17])**.** *If T is a tree of order at least 3, then the following conditions are equivalent:*

1. *T has a unique γ-set D.*

2. *T has a γ-set D for which every vertex $x \in D$ has at least two private neighbors other than itself.*

3. *T has a γ-set D for which every vertex $x \in D$ has the property that $\gamma(T - x) > \gamma(T)$.*

Theorem 21. *Let T be a nontrivial tree. The following conditions are equivalent.*

1. *$T \,\square\, K_n \in \mathcal{U}$.*

2. *$T \,\square\, K_n$ has a stacked γ-set D such that for all $v \in D$, $|epn(v, D)| \geq n + 1$.*

3. *$T \,\square\, K_n$ has a stacked γ-set A such that for every $v \in \pi_T(A)$, $\gamma(T \,\square\, K_n - (\{v\} \times [n])) > \gamma(T \,\square\, K_n)$.*

Proof. By Lemma 7 and Lemma 12, we see that statement (1) implies statement (2). We first show that statement (2) implies statement (1). We proceed by induction on $|V(T)|$.

The base case is given by $T = K_{1,n+1}$ where the result holds. We note that for any other tree T on $n + 2$ vertices, statement (2) does not hold for $T \,\square\, K_n$. Suppose then that the result has been shown whenever $|V(T)| < r$. Let T be a tree on r vertices for which there exists a subset $S \subseteq V(T)$ such that $S \times [n]$ is a γ-set for $T \,\Box\, K_n$ and such that every element $v \in S \times [n]$ satisfies $|epn(v, S \times [n])| \geq n+1$. To simplify notation, we let $D = S \times [n]$ and $H = T \,\Box\, K_n$. Suppose that $H - D$ contains two vertices $(u,1)$, $(v,1)$ which are connected by the edge $(u,1)(v,1)$. Let $H(u)$ be the component of $(T - uv) \,\Box\, K_n$ containing $(u,1)$, and let $H(v)$ be the component containing $(v,1)$. Let $D(u) = D \cap V(H(u))$ and $D(v) = D \cap V(H(v))$. We first claim that $D(u)$ and $D(v)$ are γ-sets for $H(u)$ and $H(v)$ respectively. To see this, note that $D(u)$ and $D(v)$ dominate $H(u)$ and $H(v)$. Additionally, if $H(u)$, for example, has a γ-set A of cardinality smaller than $|D(u)|$, then $A \cup D(v)$ is a dominating set of $T \,\Box\, K_n$ smaller than D, a contradiction. Since all private neighbors with respect to D are preserved in the individual components, our induction hypothesis implies that $D(u)$ and $D(v)$ are the unique γ-sets for $H(u)$ and $H(v)$ respectively.

Assume now that D' is a γ-set of H distinct from D. If $D' \cap (\{u,v\} \times [n]) = \emptyset$ then $D' \cap V(H(u)) = D(u)$ and $D' \cap V(H(v)) = D(v)$, a contradiction. Thus, $D' \cap (\{u,v\} \times [n]) \neq \emptyset$.

- If $D' \cap (\{u\} \times [n]) \neq \emptyset$, then $D' \cap V(H(u))$ dominates $H(u)$ in which case $|D' \cap V(H(u))| > |D(u)|$. Similarly, if $D' \cap (\{v\} \times [n]) \neq \emptyset$, then $|D' \cap V(H(v))| > |D(v)|$.

- If $D' \cap (\{u\} \times [n]) = \emptyset$ but $D' \cap (\{v\} \times [n]) \neq \emptyset$, then certainly $D' \cap V(H(u))$ dominates $H(u) - (\{u\} \times [n])$ in which case by Lemma 18, $|D' \cap V(H(u))| \geq |D(u)|$. Similarly, if $D' \cap (\{v\} \times [n]) = \emptyset$ but $D' \cap (\{u\} \times [n]) \neq \emptyset$, then $|D' \cap V(H(v))| \geq |D(v)|$.

Thus, since $D' \cap (\{u,v\} \times [n]) \neq \emptyset$, we see that $|D'| = |D' \cap V(H(u))| + |D' \cap V(H(v))| > |D(u)| + |D(v)| = |D|$, a contradiction. Hence, in this case, D is the unique γ-set for H.

Our last case assumes there are no edges in H of the form $(u,1)(v,1)$ with $(u,1),(v,1) \in V(H) - D$. In this case, let $(x,i) \in D$. If (y,i) is an external private neighbor of (x,i) with respect to D, then y is a leaf of T. Hence, $x \in V(T)$ has at least $n+1$ leaf neighbors. As observed above, this implies that $\{x\} \times [n]$ is contained in every γ-set of H. Since $(x,i) \in D$ was arbitrary, we see that D is the unique γ-set of H. Hence, we have now shown that (1) and (2) are equivalent.

Assume now that statement (3) holds. By Lemma 15, statement (2) holds. Our work above then implies that statement (1) also holds. Thus, we next prove that statement (1) implies statement (3).

Let $T \,\Box\, K_n \in \mathcal{U}$. Let $D = UD(T \,\Box\, K_n)$ and let $H = T \,\Box\, K_n$. By Lemma 7, there exists $S \subseteq V(T)$ such that $D = S \times [n]$. Suppose that $\{v\} \times [n] \subseteq D$. Partition $N((v,1)) \cap V(G^1)$ as

$epn((v,1),D) \cup Q((v,1))$. Let

$$epn((v,1),D) = \{(p_1,1),(p_2,1),\ldots,(p_m,1)\}$$

and

$$Q((v,1)) = \{(q_1,1),(q_2,1),\ldots,(q_k,1)\}.$$

We know that $m \geq n+1$ and that $k \geq 0$. Let $H(p_i)$, respectively $H(q_j)$, be the component of $H - (\{v\} \times [n])$ containing $(p_i,1)$, respectively $(q_j,1)$. For $1 \leq i \leq m$, let $D(p_i) = D \cap V(H(p_i))$ and define $D(q_j)$ similarly. Since T is a tree, we see that

$$\gamma(H) = |D| = n + \sum_{i=1}^{m} |D(p_i)| + \sum_{j=1}^{k} |D(q_j)|.$$

Since $H - (\{v\} \times [n])$ is the disjoint union

$$\left[\bigcup_{i=1}^{m} H(p_i)\right] \bigcup \left[\bigcup_{j=1}^{k} H(q_j)\right],$$

we can calculate $\gamma(H - (\{v\} \times [n]))$ by calculating $\gamma(H(p_i))$ and $\gamma(H(q_j))$ for each i and j and summing the results.

First, we consider $H(p_i)$. If $V(H(p_i)) = \{p_i\} \times [n]$, then $D(p_i) = \emptyset$. In this case, it is easy to see that $\gamma(H(p_i)) = 1 = |D(p_i)| + 1$.

If $V(H(p_i)) \neq \{p_i\} \times [n]$, then $D(p_i) \neq \emptyset$. Moreover, for each j such that $1 \leq j \leq n$, no neighbor of (p_i, j) in the graph $H(p_i)$ is in $D(p_i)$, since $(p_i, j) \in epn((v,j),D)$. Thus, $D(p_i)$ is not a γ-set for $H(p_i)$ since it does not dominate $(p_i,1)$. Nevertheless, suppose that $\gamma(H(p_i)) = |D(p_i)|$, and let B be a γ-set of $H(p_i)$. It follows that $(D - D(p_i)) \cup B$ is a dominating set of H of cardinality equal to $|D|$, contradicting the uniqueness of D. Hence, $\gamma(H(p_i)) > |D(p_i)|$. Since $D(p_i) \cup \{(p_i,1)\}$ dominates $H(p_i)$, we see, once again, that $\gamma(H(p_i)) = |D(p_i)| + 1$.

Next, we consider $H(q_j)$. Since $(q_j, i) \notin epn((v,i),D)$ for $1 \leq i \leq n$, we see that $D(q_j)$ is a γ-set of $H(q_j)$. Moreover, for each $v \in D(q_j)$, $|epn(v, D(q_j))| \geq n+1$. Thus, $D(q_j)$ is the unique γ-set of $H(q_j)$, giving us that $\gamma(H(q_j)) = |D(q_j)|$.

Thus, we can now compute $\gamma(H - (\{v\} \times [n]))$:

$$\begin{aligned}\gamma(H - (\{v\} \times [n])) &= \sum_{i=1}^{m} \gamma(H(p_i)) + \sum_{j=1}^{k} \gamma(H(q_j)) \\ &= \sum_{i=1}^{m} (|D(p_i)| + 1) + \sum_{j=1}^{k} |D(q_j)| \\ &= \gamma(H) + m - n \\ &\geq \gamma(H) + (n+1) - n \\ &= \gamma(H) + 1 \\ &> \gamma(H).\end{aligned}$$

Thus, we see that statement (1) implies statement (3), and our proof is complete. □

This result is useful in that if we are given a γ-set in a graph $T \,\square\, K_n$, we can immediately determine whether $T \,\square\, K_n \in \mathcal{U}$ simply by analyzing the γ-set. That is, all we must do is determine if the γ-set is stacked, and if so, count the number of external private neighbors for each vertex. However, there is a problem. If we are simply given a graph $T \,\square\, K_n$, to use this theorem, we must first find a γ-set for the graph. This may be difficult to do. Thus, this theorem is of limited use on its own. In Section 2.6 to follow, we will use this result to show that finding a γ-set in $T \,\square\, K_n$ is not required to determine whether $T \,\square\, K_n \in \mathcal{U}$. We will show that analysis of a γ-set of T will suffice.

2.5 Combining Graphs With Unique γ-sets

Suppose that $G_1 \,\square\, K_n$ and $G_2 \,\square\, K_n$ have unique minimum dominating sets. In this section, we consider the ways in which these two graphs can be combined to produce a new graph having a unique minimum dominating set. We discuss four operations. Throughout this section, $G_1 \,\square\, K_n$ and $G_2 \,\square\, K_n$, denoted H_1 and H_2 respectively, are graphs in $\mathcal{U}$ with G_1 and G_2 nontrivial. Let D_1 and D_2 denote the sets $UD(G_1 \,\square\, K_n)$ and $UD(G_2 \,\square\, K_n)$ respectively.

Operation 1. *If $x \notin \pi_{G_1}(D_1)$ and $y \notin \pi_{G_2}(D_2)$, then $((G_1 \cup G_2) + xy) \,\square\, K_n \in \mathcal{U}$ and $UD(((G_1 \cup G_2) + xy) \,\square\, K_n) = D_1 \cup D_2$.*

Proof. Let H denote the graph $((G_1 \cup G_2) + xy) \,\square\, K_n$. First, we see that $D_1 \cup D_2$ dominates all of H. Let D be a γ-set for H. It follows that

$$|D| \leq |D_1 \cup D_2| = |D_1| + |D_2|.$$

Without loss of generality, suppose that $|D \cap V(H_1)| \leq |D_1|$. Since the only vertices of H_1 that could be dominated from outside of H_1 are elements of $\{x\} \times [n]$, we see that either $D \cap V(H_1)$ dominates all of H_1, or $D \cap V(H_1)$ fails to dominate a subset B of $\{x\} \times [n]$.

First, suppose that $D \cap V(H_1)$ dominates all of H_1. Since H_1 has a unique γ-set, and since we're assuming $|D \cap V(H_1)| \leq |D_1|$, we have that $D \cap V(H_1) = D_1$. However, if $D \cap V(H_1) = D_1$, then we also have $D \cap V(H_2) = D_2$ since $x \notin \pi_{G_1}(D_1)$. Thus, in this case, we have that $D = D_1 \cup D_2$.

Now suppose that $D \cap V(H_1)$ fails to dominate a subset B of $\{x\} \times [n]$. By Lemma 18, we have that $|D \cap V(H_1)| \geq |D_1|$. Since $|D| \leq |D_1| + |D_2|$, we have that $|D \cap V(H_2)| \leq |D_2|$. Note, however, that $D \cap V(H_2)$ intersects $\{y\} \times [n]$, in which case we have a set of cardinality at most $|D_2|$ that is distinct from D_2 and dominates H_2. This contradicts the uniqueness of D_2. Our result now follows. $\square$

We have seen that $K_{1,3} \,\square\, K_2 \in \mathcal{U}$. Operation 1 implies that we can connect two copies of $K_{1,3} \,\square\, K_2$ as in the Figure 2.5 below, and the result is still in $\mathcal{U}$.

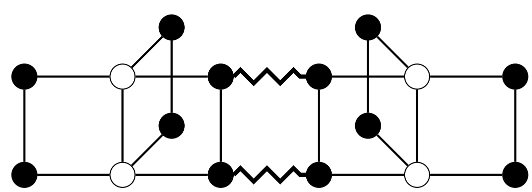


Figure 2.5: Operation 1 example

Operation 2. *Let $x \in \pi_{G_1}(D_1)$ and $y \in \pi_{G_2}(D_2)$. If u is a new vertex in neither G_1 nor G_2, then $((G_1 \cup G_2 \cup \{u\}) + \{ux, uy\}) \,\square\, K_n \in \mathcal{U}$ and $UD(((G_1 \cup G_2) + \{ux, uy\}) \,\square\, K_n) = D_1 \cup D_2$.*

Proof. Let H denote the graph $((G_1 \cup G_2 \cup \{u\}) + \{ux, uy\}) \,\square\, K_n$. First, note that $D_1 \cup D_2$ dominates H. If D is a γ-set of H with $|D| < |D_1| + |D_2|$, then $D \cap (\{u\} \times [n]) \neq \emptyset$. Suppose that $\{(u, i_1), (u, i_2), \ldots, (u, i_k)\} \subseteq D$. In this case, the vertices in $\{(x, i_1), (x, i_2), \ldots, (x, i_k)\}$ and $\{(y, i_1), (y, i_2), \ldots, (y, i_k)\}$ need not be dominated from H_1 and H_2 respectively. However, by Lemma 19, we know that $|D \cap V(H_1)| \geq |D_1|$ and that $|D \cap V(H_2)| \geq |D_2|$. Thus, $|D| \geq$

$|D_1| + |D_2| + k > |D_1 \cup D_2|$. Thus, no γ-set of H intersects $\{u\} \times [n]$. Hence, any γ-set of H intersects each of $V(H_1)$ and $V(H_2)$ in a γ-set, in which case $D = D_1 \cup D_2$. $\square$

A visual example of Operation 2 is given in Figure 2.6

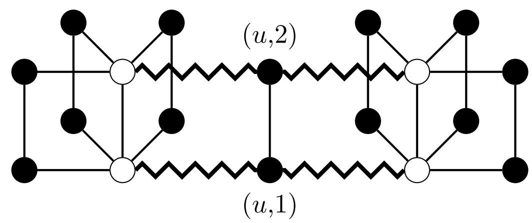


Figure 2.6: Operation 2 example

Before we discuss the next operation, we need the following lemma.

Lemma 22. *Let T be a tree, and let $T \,\square\, K_n \in \mathcal{U}$. If $(v, i) \notin UD(T \,\square\, K_n)$ is adjacent to at least two elements of $UD(T \,\square\, K_n)$, then $(T - v) \,\square\, K_n \in \mathcal{U}$ and $UD((T - v) \,\square\, K_n) = UD(T \,\square\, K_n)$.*

Proof. Let H' denote the graph $(T-v) \,\square\, K_n$ and let D denote the set $UD(T \,\square\, K_n)$. By Lemma 18, we know that $\gamma(H') = \gamma(T \,\square\, K_n)$. Thus, D is a γ-set for H'. We must show that D is the only γ-set for H'. Note that the removal of $(v, 1), (v, 2), \ldots, (v, n)$ from $T \,\square\, K_n$ breaks $T \,\square\, K_n$ into $k \geq 2$ components; call them $H_1, H_2, \ldots, H_k$.

We claim that for $i = 1, 2, \ldots k$, $D_i = D \cap V(H_i)$ is the unique γ-set for H_i. Without loss of generality, consider D_1. Clearly D_1 is a dominating set for H_1. If D_1' were a smaller dominating set of H_1, then $D_1' \cup D_2 \cup \cdots \cup D_k$ would be a smaller γ-set for $T \,\square\, K_n$. Thus, D_1 is a γ-set. By the same logic, D_1 is the unique γ-set for H_1.

Thus, each H_i has D_i as its unique minimum dominating set, in which case H' has $D = D_1 \cup D_2 \cup \cdots \cup D_k$ as its unique γ-set. $\square$

Operation 3. *Let G_2 be a tree. If $x \in \pi_{G_1}(D_1)$, $y \notin \pi_{G_2}(D_2)$, and y is a neighbor of at least two vertices in $\pi_{G_2}(D_2)$, then $((G_1 \cup G_2) + xy) \,\square\, K_n \in \mathcal{U}$ and $UD(((G_1 \cup G_2) + xy) \,\square\, K_n) = D_1 \cup D_2$.*

Proof. Let H denote the graph $((G_1 \cup G_2) + xy) \,\square\, K_n$. Note that $D_1 \cup D_2$ dominates H. Let D be a γ-set of H. Suppose that $(\{y\} \times [n]) \cap D \neq \emptyset$. This implies that some subset of $\{x\} \times [n]$ will be dominated from outside of H_1. By Lemma 19, we still have that $|D \cap V(H_1)| \geq |D_1|$. Additionally, $D \cap V(H_2)$ dominates H_2, in which case $|D \cap V(H_2)| > |D_2|$ since D_2 is the unique γ-set for H_2 and

$y \notin \pi_{G_2}(D_2)$. Thus, we have $|D| > |D_1 \cup D_2|$. This implies that no γ-set of H intersects $\{y\} \times [n]$. Hence, if D is a γ-set for H, then $D \cap V(H_1) = D_1$. Lemma 22 then implies that $D \cap V(H_2) = D_2$. Thus, $D_1 \cup D_2$ is the unique γ-set for H. $\square$

An example of Operation 3 is shown below.

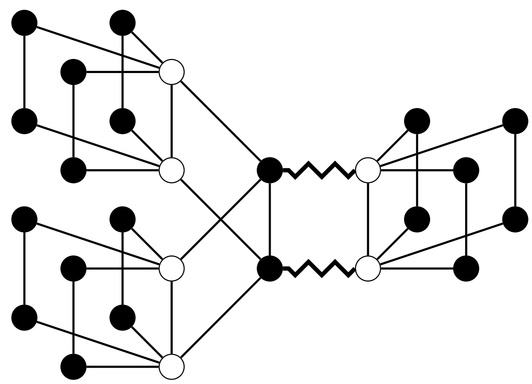


Figure 2.7: Operation 3 example

Operation 4. *Let G_1 and G_2 be trees. If $x \in \pi_{G_1}(D_1)$ and $y \in \pi_{G_2}(D_2)$, then $((G_1 \cup G_2) + \{xy\}) \,\square\, K_n \in \mathcal{U}$ and $UD(((G_1 \cup G_2) + \{xy\}) \,\square\, K_n) = D_1 \cup D_2$.*

Proof. Once again, let H denote the graph $((G_1 \cup G_2) + \{xy\}) \,\square\, K_n$. Since $D_1 \cup D_2$ dominates H, we have that $\gamma(H) \leq |D_1 \cup D_2| = |D_1| + |D_2|$. Let A denote the set $\{x\} \times [n]$, let B denote the set $\{y\} \times [n]$, and suppose that D is a γ-set for H.

- If $A \subseteq D$ and $B \subseteq D$, then $D \cap V(H_1)$ and $D \cap V(H_2)$ are γ-sets for H_1 and H_2, respectively, in which case $D = D_1 \cup D_2$.

- Suppose that $D \cap A = \emptyset$. This implies that $D \cap V(H_1)$ dominates $H_1 - A$. However, by Theorem 21, we know that $\gamma(H_1 - A) > \gamma(H_1) = |D_1|$. Additionally, in this case $D \cap V(H_2)$ is a γ-set of H_2 implying that $D \cap H_2 = D_2$. Thus, we have $|D| > |D_1| + |D_2| = |D_1 \cup D_2|$. The same contradiction arises if $D \cap B = \emptyset$.

- This leaves us with one case to consider. Without loss of generality, suppose that $0 < |D \cap A| < |A|$ and that $D \cap B \neq \emptyset$. In this case, $D \cap V(H_1)$ dominates H_1 and $D \cap V(H_2)$ dominates H_2. However, since D_1 and D_2 are the unique γ-sets for H_1 and H_2 respectively, and since

$A \subseteq D_1$, we have that $|D \cap V(H_1)| > |D_1|$ and that $|D \cap V(H_2)| \geq |D_2|$. Thus, we have $|D| > |D_1 \cup D_2|$, a contradiction.

Thus, we have $D = D_1 \cup D_2$, which implies $D_1 \cup D_2$ is the unique γ-set for H. □

Our final visual example, of Operation 4, is below.

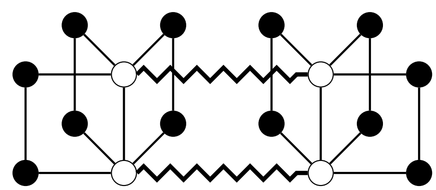


Figure 2.8: Operation 4 example

2.6 First Main Result

We are now able to prove our first main result, which we restate for your convenience.

Theorem 6. *Let n be a positive integer, let T be a nontrivial tree, and let D be a γ-set of T. The graph $T \square K_n$ is in $\mathcal{U}$ if and only if for all $v \in D$, $|epn(v, D)| \geq n + 1$.*

Proof. First, suppose that $T \square K_n$ has a unique γ-set, denoted UD. Since $T \square K_n \in \mathcal{U}$, Proposition 10 implies that T also has a unique γ-set. Hence, our γ-set D of T is the unique γ-set of T. Moreover, by our observations following Proposition 10, we see that $UD = D \times V(K_n)$. By Lemma 12, for every element $w \in UD$, $|epn(w, UD)| \geq n + 1$. Since UD satisfies the stacked property, this implies that every element $v \in D$ also satisfies $|epn(v, D)| \geq n + 1$. Thus, we see that if $T \square K_n \in \mathcal{U}$, then for all $v \in D$, $|epn(v, D)| \geq n + 1$.

Now suppose that for all $v \in D$, $|epn(v, D)| \geq n + 1$. First note that since n is a positive integer, every vertex $v \in D$ has at least two private neighbors other than themselves. Thus, by Theorem 20, D is the unique γ-set of T. Let H be the graph $T \square K_n$, and let $D' = D \times [n]$. Since D is a dominating set of T, D' is clearly a dominating set of H. Futhermore, since for all $v \in D$, $|epn(v, D)| \geq n + 1$, we see that for all $w \in D'$, $|epn(w, D')| \geq n + 1$. Thus, if we can prove that D' is a γ-set for H, then Theorem 21 will imply that D' is the unique γ-set for H. We show this by induction on $\gamma(T) = |D|$.

The base case is given by $\gamma(T) = 1$. If $\gamma(T) = 1$, then $T = K_{1,m}$ with $m \geq n + 1$ where the result holds. Thus, assume the result holds whenever $\gamma(T) < r$. Suppose that $\gamma(T) = r$. Consider a diametral path in T, call it $v_1 v_2 \cdots v_k$. Note that v_k is a leaf and is not an element of D. This implies that $v_{k-1} \in D$. Since v_{k-1} has at least $n+1$ external private neighbors with respect to D, we see that v_{k-1} is adjacent to at least $n-1$ leaves besides v_k. Let A be the set $\{v_{k-1}\} \cup epn(v_{k-1}, D)$ and let $B = \{v \in N(v_{k-1}) : v \notin D, |N(v) \cap D| \geq 2\}$. We note that B equals either the empty set or $\{v_{k-2}\}$. By Theorem 20, $\{v_{k-1}\}$ and $D - \{v_{k-1}\}$ are the unique minimum dominating sets for $T\langle A \rangle$ and $T - (A \cup B)$ respectively. By our induction hypothesis, $\{v_{k-1}\} \times [n]$ and $D' - (\{v_{k-1}\} \times [n])$ are the unique minimum dominating sets for $T\langle A \rangle \,\square\, K_n$ and $(T - (A \cup B)) \,\square\, K_n$ respectively. Our graph H can be reconstructed from $T\langle A \rangle \,\square\, K_n$ and $(T - (A \cup B)) \,\square\, K_n$ by performing at least one of the operations discussed in Section 5 above. Hence, not only is D' a γ-set for H, but it is also the unique γ-set for H. $\square$

Theorem 6 implies that in order to determine whether $T \,\square\, K_n \in \mathcal{U}$ it is sufficient to consider T alone through the following procedure. First, find a γ-set in T, call it D. Next, for each vertex $v \in D$, determine the number of external private neighbors v has with respect to D. Let $m = \min_{v \in D} |epn(v, D)|$. If $m = 1$, then Theorem 20 implies that $T \notin \mathcal{U}$. If $T \notin \mathcal{U}$, then $T \,\square\, K_n \notin \mathcal{U}$ by Proposition 10. If $m \geq 2$, then Theorem 20 implies that D is the unique γ-set of T, and Theorem 6 implies that $T \,\square\, K_n \in \mathcal{U}$ if and only if $m \geq n + 1$. Notice that this is an improvement upon Theorem 21 since finding a γ-set in T can be done in linear time (see [15]). Thus, we ultimately see that the problem of determining for which K_n, $T \,\square\, K_n$ has a unique minimum dominating set can be solved in polynomial time.

We now consider an example of the procedure outlined above. Consider the tree T in Figure 2.9 below. The set $D = \{u, v, w\}$ is a γ-set for T. We see that $|epn(u, D)| = 3$, $|epn(v, D)| = 5$, and $|epn(w, D)| = 6$. Thus, according to Theorem 6, $T \,\square\, K_1$, and $T \,\square\, K_2$ each have a unique γ-set. However, $T \,\square\, K_n \notin \mathcal{U}$ for $n \geq 3$.

$d = \lvert V(T) \rvert$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
2	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0
4	1	1	0	0	0	0	0	0	0
5	1	1	1	0	0	0	0	0	0
6	3	1	1	1	0	0	0	0	0
7	4	1	1	1	1	0	0	0	0
8	6	3	1	1	1	1	0	0	0
9	11	4	1	1	1	1	1	0	0
10	22	6	3	1	1	1	1	1	0
11	38	7	4	1	1	1	1	1	1
12	75	13	6	3	1	1	1	1	1
13	153	23	7	4	1	1	1	1	1
14	308	40	9	6	3	1	1	1	1
15	616	61	14	7	4	1	1	1	1
16	1310	106	25	9	6	3	1	1	1
17	2776	188	41	10	7	4	1	1	1
18	5884	351	63	16	9	6	3	1	1
19	12697	609	91	26	10	7	4	1	1
20	27810	1073	140	43	12	9	6	3	1
21	60771	1936	229	64	17	10	7	4	1

Table 2.1: The number of trees T on d vertices for which $T \square K_n \in \mathcal{U}$

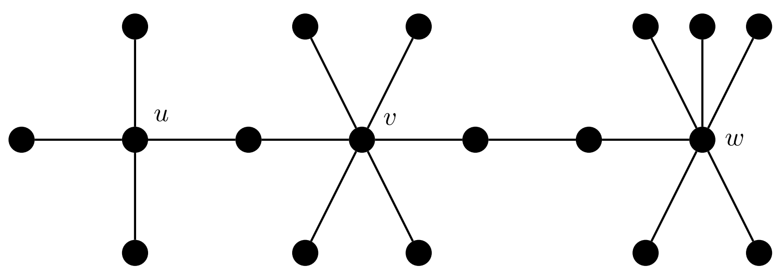


Figure 2.9: Tree Example

By applying this procedure to every tree on d vertices, we can enumerate the number of trees on d vertices for which $T \square K_n \in \mathcal{U}$. Table 2.1 presents the results of this work. We invite the interested reader to determine a closed form expression for the number of trees on d vertices for which $T \square K_n \in \mathcal{U}$.

2.7 Repeated Products

Up to this point, we have only considered unique γ-sets in $G \,\square\, K_n$. In this section, we consider graphs $G \,\square\, K_{n_1} \,\square\, K_{n_2} \,\square \cdots \square\, K_{n_r}$. A Cartesian product of complete graphs is called a *Hamming graph*. Thus, we can alternatively say that in this section we are considering unique minimum dominating sets in the Cartesian product of a nontrivial graph and a Hamming graph. As notational convenience, if the Cartesian product of m K_n factors is performed, we simplify the notation $K_n \,\square\, K_n \,\square \cdots \square\, K_n$ to K_n^m. According to this convention, the n-dimensional hypercube Q_n is denoted K_2^n. Our first result is the following.

Lemma 23. *If $G \,\square\, K_n^m \in \mathcal{U}$, then $UD(G \,\square\, K_n^m) = UD(G) \times V(K_n^m)$.*

Proof. As noted just after the proof of Proposition 10, if $G \,\square\, K_n \in \mathcal{U}$, then $UD(G \,\square\, K_n) = UD(G) \times [n]$. Thus, we see that

$$UD(G \,\square\, K_n^m) = UD(G \,\square\, K_n^{m-1} \,\square\, K_n) = UD(G \,\square\, K_n^{m-1}) \times [n].$$

By induction, we see that $UD(G \,\square\, K_n^m) = UD(G) \times V(K_n^m)$. $\square$

Additionally, since the Cartesian product is both commutative and associative, Proposition 10 gives us the following result.

Proposition 24. *If $G \,\square\, K_n^m \in \mathcal{U}$, then $G \,\square\, K_{n_1} \,\square\, K_{n_2} \,\square \cdots \square\, K_{n_r} \in \mathcal{U}$ for $1 \leq n_i \leq n$ and $1 \leq i \leq r \leq m$.*

Proof. Suppose that $G \,\square\, K_n^m \in \mathcal{U}$. By associativity, $(G \,\square\, K_n^{m-1}) \,\square\, K_n \in \mathcal{U}$. By Proposition 10, we then have that $(G \,\square\, K_n^{m-1}) \,\square\, K_{n_1} \in \mathcal{U}$ so long as $1 \leq n_1 \leq n$. By commutativity, we have that $(G \,\square\, K_{n_1}) \,\square K_n^{m-1} \in \mathcal{U}$. By induction, our result follows. $\square$

As a result of Proposition 24, in order to determine whether $G \,\square\, K_{n_1} \,\square\, K_{n_2} \,\square \cdots \square\, K_{n_r} \in \mathcal{U}$, it *may* suffice to consider whether $G \,\square\, K_n^r \in \mathcal{U}$ where $n = \max\{n_1, n_2, \ldots, n_r\}$. Thus, we are motivated to define the following parameter.

Definition 17. Let $G \in \mathcal{U}$ and let $U_n^{\square}(G)$ denote the integer m such that $G \,\square\, K_n^m \in \mathcal{U}$, but $G \,\square\, K_n^{m+1} \notin \mathcal{U}$. If $G \,\square\, K_n^m \notin \mathcal{U}$ for any $m \geq 1$, define $U_n^{\square}(G) = 0$, while if $G \,\square\, K_n^m \in \mathcal{U}$ for all $m \geq 1$, define $U_n^{\square}(G) = \infty$.

As an illustration of this definition, consider the following examples. The graph $K_{1,2} \in \mathcal{U}$ but $K_{1,2} \,\Box\, K_2 \notin \mathcal{U}$ (see Figure 2.10). Thus, $U_2^{\Box}(K_{1,2}) = 0$. When we consider the graph $K_{1,3}$, we see that $K_{1,3} \,\Box\, K_2 \in \mathcal{U}$ but $K_{1,3} \,\Box\, K_2^2 \notin \mathcal{U}$. Hence, $U_2^{\Box}(K_{1,3}) = 1$. Finally, when considering the graph $K_{1,4}$, we see that $K_{1,4} \,\Box\, K_2^2 \in \mathcal{U}$, but $K_{1,4} \,\Box\, K_2^3 \notin \mathcal{U}$. Thus, $U_2^{\Box}(K_{1,4}) = 2$.

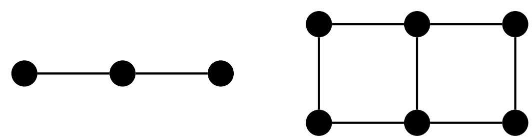


Figure 2.10: $K_{1,2} \in \mathcal{U}$ but $K_{1,2} \,\Box\, K_2 \notin \mathcal{U}$

We now determine $U_n^{\Box}(K_{1,p})$ for $n \geq 2$. For notational purposes, let $V(K_{1,p}) = \{0, 1, \ldots, p\}$ with 0 denoting the support vertex (non-leaf). Additionally, denote the vertices of K_n^m as strings of length m over the alphabet $[n]$. By the *jth cube* of $K_{1,p} \,\Box\, K_n^m$, we mean the subgraph of $K_{1,p} \,\Box\, K_n^m$ induced by $\{j\} \times V(K_n^m)$. The zeroth cube will be referred to as the *central cube*, while all other cubes will be referred to as the *outer cubes*.

Proposition 25. *If $2 \leq p \leq n$, then $U_n^{\Box}(K_{1,p}) = 0$. If $p > n \geq 2$, then $U_n^{\Box}(K_{1,p}) = \left\lfloor \frac{p-2}{n-1} \right\rfloor$.*

Proof. By Theorem 6, we see that if $2 \leq p \leq n$, then $K_{1,p} \,\Box\, K_n \notin \mathcal{U}$. Hence, in this instance, $U_n^{\Box}(K_{1,p}) = 0$ as claimed.

Suppose then that $p > n$. Let $m = \lfloor \frac{p-2}{n-1} \rfloor$, and consider $K_{1,p} \,\Box\, K_n^m$. Let D be the set $\{0\} \times V(K_n^m)$, and note that D is certainly a dominating set for $K_{1,p} \,\Box\, K_n^m$. Suppose that D' is a γ-set for $K_{1,p} \,\Box\, K_n^m$ and that for some $k > 0$, $|D - D'| = k$. In K_n^m, every vertex is of degree $(n-1)m$. Thus, D' contains at least $\lceil \frac{k}{(n-1)m+1} \rceil$ vertices from each of the p outer cubes of $K_{1,p} \,\Box\, K_n^m$. Hence, we see that

$$|D'| \quad \geq \quad n^m - k + (p) \left\lceil \frac{k}{(n-1)m+1} \right\rceil .$$

Since $m < \frac{p-1}{n-1}$, we see that $(n-1)m + 1 < p$ in which case $(p) \left\lceil \frac{k}{(n-1)m+1} \right\rceil > k$. Hence $|D'| > n^m$, a contradiction. Thus, D is the unique γ-set for $K_{1,p} \,\Box\, K_n^m$.

Now consider $K_{1,p} \,\Box\, K_n^{m+1}$. Once again, $D = \{0\} \times V(K_n^{m+1})$ is a dominating set for

$K_{1,p} \square K_n^{m+1}$. Construct a new set D' from D by deleting $(0, 11\cdots 1)$ and all of its neighbors in the central cube from D. Since $(m+1) \geq 2$, $|D'| > 0$. Thus, the only vertex of the central cube not dominated by D' is $(0, 11\cdots 1)$. Let $D'' = D' \cup \{(i, 11\cdots 1) \mid 1 \leq i \leq p\}$. D'' is a dominating set for $K_{1,p} \square K_n^{m+1}$. Additionally, we see that

$$\begin{aligned}
|D''| &= |D| - [1 + (n-1)(m+1)] + p \\
&\leq |D| - \left[1 + (n-1)\frac{p-1}{n-1}\right] + p \\
&= |D| - p + p \\
&= |D|.
\end{aligned}$$

Hence, we have constructed a dominating set D'' distinct from D of cardinality at most $|D|$. Thus, $K_{1,p} \square K_n^{m+1}$ cannot have a unique γ-set by Lemma 23. Our result now follows. $\square$

Proposition 25 provides us with the following result.

Lemma 26. *If $G \square K_n^m \in \mathcal{U}$, then for all $v \in UD(G \square K_n^m)$, $|epn(v, UD(G \square K_n^m))| \geq m(n-1)+2$.*

Proof. For notational convenience, let D denote the set $UD(G \square K_n^m)$ and let D' denote the set $UD(G)$. Recall that by Lemma 23, $D = D' \times V(K_n^m)$. This implies that if $v \in D'$ with $epn(v, D') = \{p_1, p_2, \ldots, p_k\}$, then for all $x \in V(K_n^m)$, $(v, x) \in D$ with $epn((v, x), D) = \{(p_1, x), (p_2, x), \ldots, (p_k, x)\}$. For the sake of contradiction, suppose that $(u, w) \in D$ has $epn((u, w), D) = \{(p_1, w), (p_2, w), \ldots, (p_j, w)\}$ for some $j < m(n-1)+2$. Since $U_n^{\square}(K_{1,j}) < m$, this implies that the subgraph of $G \square K_n^m$ induced by $\{u, p_1, p_2, \ldots, p_j\} \times V(K_n^m)$ has a γ-set, call it B, distinct from $\{u\} \times V(K_n^m)$. In that case, $(D - (\{u\} \times V(K_n^m))) \cup B$ is a dominating set for $G \square K_n^m$ distinct from D of cardinality at most $|D|$, a contradiction. $\square$

This lemma, along with Theorem 6 allows us to classify the trees T for which $T \square K_n^m$ has a unique γ-set. For notational purposes, if $v \in V(T)$, then we let the *vth cube* of $T \square K_n^m$ denote the subgraph of $T \square K_n^m$ induced by $\{v\} \times V(K_n^m)$.

Theorem 27. *Let $n \geq 2$, $m \geq 1$, and let T be a tree. The Cartesian product $T \square K_n^m$ has a unique γ-set if and only if $T \square K_{m(n-1)+1}$ has a unique γ-set.*

Proof. First, suppose that $T \square K_n^m \in \mathcal{U}$. By Lemma 23, $UD(T \square K_n^m) = UD(T) \times V(K_n^m)$. By Lemma 26, we know that for each $v \in UD(T \square K_n^m)$,

$|epn(v, UD(T \,\square\, K_n^m))| \geq m(n-1)+2$. This implies that for each $w \in UD(T)$, $|epn(w, UD(T))| \geq m(n-1)+2$. By Theorem 6, it follows that $T \,\square\, K_{m(n-1)+1}$ has a unique γ-set.

Now suppose that $T \,\square\, K_{m(n-1)+1} \in \mathcal{U}$. By Proposition 10 and Theorem 6, we see that T has a unique γ-set S so that every element in S has at least $m(n-1)+2$ external private neighbors with respect to S. Consider then $T \,\square\, K_n^m$. Note that the set $S \times V(K_n^m)$ is a dominating set for $T \,\square\, K_n^m$. We must show that it is a γ-set for $T \,\square\, K_n^m$, and that it is the unique γ-set for $T \,\square\, K_n^m$.

We proceed by induction on $\gamma(T)$. If $\gamma(T) = 1$, then T is a star $K_{1,p}$ with $p \geq m(n-1)+2$. By Proposition 25, we see that $T \,\square\, K_n^m$ has $UD(T) \times V(K_n^m)$ as its unique γ-set. Thus, suppose the result has been proven whenever $\gamma(T) < q$. Let T be a tree such that $\gamma(T) = q$ and such that $T \,\square\, K_{m(n-1)+1}$ has a unique γ-set. Let S be the unique γ-set for T. We know that for all $x \in S$, $|epn(x, S)| \geq m(n-1)+2$. Consider a diametral path $x_1 x_2 \ldots x_{t-1} x_t x_{t+1}$ in T. Note that $x_t \in S$ and that $t \geq 3$.

Case One

First, suppose that $x_{t-1} \notin epn(x_t, S)$. In this case, since $|epn(x_t, S)| \geq m(n-1)+2$, we see that x_t is adjacent to at least $m(n-1)+2$ leaves. Thus, by the proof of Proposition 25, every vertex of the x_tth cube in $T \,\square\, K_n^m$ will be selected for inclusion in every γ-set of $T \,\square\, K_n^m$. Let T' denote the tree obtained by removing x_t and all of its private neighbors with respect to S from T. Note that by Lemma 9, $T' \in \mathcal{U}$ with $UD(T') = S - \{x_t\}$. Additionally, observe that if $x \in S - \{x_t\}$, then $epn(x, S - \{x_t\}) \supseteq epn(x, S)$. Thus, by Theorem 6, we also see that $T' \,\square\, K_{m(n-1)+1} \in \mathcal{U}$. Since $\gamma(T') < \gamma(T)$, our induction hypothesis implies that $T' \,\square\, K_n^m \in \mathcal{U}$ and that $UD(T' \,\square\, K_n^m) = (S - \{x_t\}) \times V(K_n^m)$.

Suppose then that D is a γ-set for $T \,\square\, K_n^m$ and that $D \neq S \times V(K_n^m)$. By our observations above, we know that $\{x_t\} \times V(K_n^m) \subseteq D$. Let $B = D - (\{x_t\} \times V(K_n^m))$ and note that $B \subseteq V(T' \,\square\, K_n^m)$. If B dominates $T' \,\square\, K_n^m$, then since $UD(T' \,\square\, K_n^m) = (S - \{x_t\}) \times V(K_n^m)$ and since $B \neq (S - \{x_t\}) \times V(K_n^m)$, this implies that $|B| > |(S - \{x_t\}) \times V(K_n^m)|$. This, however, implies that $S \times V(K_n^m)$ is a smaller cardinality dominating set for $T \,\square\, K_n^m$, a contradiction.

Thus, assume that B does not dominate $T' \,\square\, K_n^m$. Since D is a dominating set of $T \,\square\, K_n^m$, this implies that B fails to dominate some subset of the x_{t-1}-cube in $T' \,\square\, K_n^m$. In particular, this implies that some subset of the x_{t-1}-cube is not contained in B. We consider two sub cases.

Subcase One

Suppose that $x_{t-1} \notin S$.

- First, suppose that $N(x_{t-1}) = \{x_{t-2}, x_t\}$. Since $x_{t-1} \notin epn(x_t, S)$, this implies that $x_{t-2} \in S$. Apply Lemma 9 to T, and remove the edge $x_{t-2}x_{t-1}$. It follows that $T' - x_{t-1} \in \mathcal{U}$ and that $UD(T' - x_{t-1}) = S - \{x_t\}$. This further implies, by the same logic as above, that $(T' - x_{t-1}) \square K_n^m \in \mathcal{U}$ with unique γ-set given by $(S - \{x_t\}) \times V(K_n^m)$. Note that since B does not dominate all of the x_{t-1}-cube in $T' \square K_n^m$, this implies that B does not contain all of the x_{t-2}-cube.

  If B contains no vertices from the x_{t-1}-cube, then B is a dominating set for $(T' - x_{t-1}) \square K_n^m$ distinct from $(S - \{x_t\}) \times V(K_n^m)$. This contradicts our assumption that D was a γ-set for $T \square K_n^m$.

  Hence, we see that B contains some subset of the x_{t-1}-cube. Let $\{(x_{t-1}, p_1), (x_{t-1}, p_2), \ldots, (x_{t-1}, p_j)\} \subseteq B$. This implies that

$$B \cap \{(x_{t-2}, p_1), (x_{t-2}, p_2), \ldots, (x_{t-2}, p_j)\} = \emptyset$$

  since otherwise D would not be a $\gamma-$set for $T \square K_n^m$. Thus, consider the set

$$(B - \{(x_{t-1}, p_1), \ldots, (x_{t-1}, p_j)\}) \cup \{(x_{t-2}, p_1), \ldots, (x_{t-2}, p_j)\}.$$

  This is a dominating set for $(T' - x_{t-1}) \square K_n^m$ distinct from $(S - \{x_t\}) \times V(K_n^m)$, a contradiction.

- Now suppose that x_{t-1} is adjacent to a vertex, call it y, not on the diametral path. First, note that $y \in S$. If $y \notin S$, then since $x_{t-1} \notin S$, y would have a neighbor in S which, with its external private neighbors, could be used to create a longer path in T. In particular, any neighbors of x_{t-1} in T not on the diametral path are in S and have only leaf neighbors. Since our initial assumption was that each element of S has at least $m(n-1)+2$ external private neighbors, this implies that y has $m(n-1)+2$ leaf-neighbors in T. Hence, by the same logic as applied to x_t above, every vertex of the y-cube will be contained in every γ-set for $T \square K_n^m$. However, this implies that $\{y\} \times V(K_n^m) \subseteq D$ which further implies that B dominates $T' \square K_n^m$, a contradiction.

Thus, in both cases, $x_{t-1} \notin S$ leads to a contradiction.

Subcase Two

Suppose now that $x_{t-1} \in S$. This implies that $|epn(x_{t-1}, S)| \geq m(n-1)+2$ by our earlier assumption. If x_{t-1} has an external private neighbor other than x_{t-2} that is not a leaf, then a longer path in T can be found. Hence, we see that x_{t-1} has at least $m(n-1)+1$ leaf-neighbors in T, call them $l_1, l_2, \ldots, l_r$. Note that if $r \geq m(n-1)+2$, then every vertex of the x_{t-1}-cube will be contained in every γ-set of $T \,\Box\, K_n^m$ implying that B is a dominating set for $T' \,\Box\, K_n^m$, a contradiction.

Thus, we see that x_{t-1} has exactly $m(n-1)+1$ leaf-neighbors and $x_{t-2} \in epn(x_{t-1}, S)$. Recall that some subset of the x_{t-1}-cube in $T \,\Box\, K_n^m$ is not contained in B. To be specific, assume k vertices of the x_{t-1}-cube are not contained in B. This implies that at least $\lceil \frac{k}{m(n-1)+1} \rceil$ vertices from each of the $l_1, l_2, \ldots, l_r$-cubes are contained in B. Additionally, the vertices in the x_{t-2}-cube that are adjacent to vertices in $(\{x_{t-1}\} \times V(K_n^m)) - B$ will be dominated by vertices outside of the x_{t-1}-cube. Since

$$[m(n-1)+1] \cdot \lceil \frac{k}{m(n-1)+1} \rceil \geq k$$

we see that B contains exactly k vertices from the $l_1, l_2, \ldots, l_r$-cubes in total, since otherwise a smaller dominating set for $T \,\Box\, K_n^m$ could be constructed. Consider the set obtained from B by removing the k vertices from the $l_1, l_2, \ldots, l_r$-cubes and including the k missing vertices from the x_{t-1}-cube. This set will be a dominating set for $T' \,\Box\, K_n^m$ distinct from $(S - \{x_t\}) \times V(K_n^m)$, a contradiction.

Case Two

Finally, suppose that $x_{t-1} \in epn(x_t, S)$. In this case, x_t is adjacent to at least $m(n-1)+1$ leaves, call them $l_1, l_2, \ldots, l_p$. Note that the only neighbors of x_{t-1} are x_t and x_{t-2}. If x_{t-1} had any other neighbors, either a longer path in T could be found, or x_{t-1} would not be an external private neighbor of x_t with respect to S.

Suppose that D is a γ-set of $T \,\Box\, K_n^m$ which does not contain k vertices of the x_tth cube. This implies that D contains at least $\lceil \frac{k}{(n-1)m+1} \rceil$ vertices from each of the $l_1, l_2, \ldots, l_p$-cubes. In fact, if $(m(n-1)+1)\lceil \frac{k}{(n-1)m+1} \rceil > k$, then we have reached a contradiction since a smaller dominating set for $T \,\Box\, K_n^m$ could be found simply by including every vertex of the x_tth cube. In particular, this implies that $(m(n-1)+1)\lceil \frac{k}{m(n-1)+1} \rceil = k$.

We now claim that D contains at least one vertex from the x_{t-1}-cube. To see this, first note that the tree T'' defined by $T'' = T - \{x_t, x_{t-1}, l_1, \ldots, l_p\}$ belongs to $\mathcal{U}$ with $UD(T'') = S - \{x_t\}$. Additionally, since $epn(x, S - \{x_t\}) = epn(x, S)$ for all $x \in S - \{x_t\}$, Theorem 6 implies that

$T'' \square K_{m(n-1)+1} \in \mathcal{U}$. Thus, our induction hypothesis implies that $T'' \square K_n^m$ has a unique γ-set given by $(S - \{x_t\}) \times V(K_n^m)$. If no vertices from the x_{t-1}-cube are included in D, then

$$D \cap V(T'' \square K_n^m) = (S - \{x_t\}) \times V(K_n^m).$$

This, however, results in at least k vertices of the x_{t-1}-cube being undominated by D since $x_{t-2} \notin S - \{x_t\}$. This is a contradiction.

Thus, D contains at least one vertex from the x_{t-1}-cube. If we "shift" these vertices to their corresponding positions in the x_{t-2}-cube, remove the vertices from D in the $l_1, l_2, \ldots, l_p$-cubes, and add in the missing vertices from the x_t-cube, we will create a γ-set D' distinct from D which induces a γ-set distinct from $(S - \{x_t\}) \times V(K_n^m)$ on the subgraph $T'' \square K_n^m$, a contradiction.

Hence, if D is a γ-set for $T \square K_n^m$, then every vertex of the x_t-cube is included in D. By the logic applied above, this implies that $S \times V(K_n^m)$ is the unique γ-set for $T \square K_n^m$.

Thus, we see that if $T \square K_{m(n-1)+1} \in \mathcal{U}$, then $T \square K_n^m \in \mathcal{U}$. $\square$

Before we conclude, we note that Theorem 27 and Theorem 6 together imply the following corollary concerning hypercubes.

Corollary 28. *Let T be a nontrivial tree, and let $m \geq 1$. The following conditions are equivalent.*

- $T \square Q_m \in \mathcal{U}$.
- $T \square K_{m+1} \in \mathcal{U}$.
- *T has a γ-set D such that for all $v \in D$, $|epn(v, D)| \geq m + 2$.*

As we observed above, since a γ-set in a tree can be found in linear time (see [15]), the problem of determining for which m, $T \square Q_m \in \mathcal{U}$ can be solved in polynomial time.

2.8 Moving Forward

In this chapter, we considered unique minimum dominating sets. In the next chapter we turn our attention to unique minimum *independent* dominating sets. We will see that there are many interesting correspondences between the results in this chapter and the results in the next. In Chapter 4, we will then expand upon the results of Chapters 2 and 3 to discuss unique γ-sets and i-sets in graphs $G \square H$ and $G \times H$ where H is a vertex-transitive graph.

Chapter 3

Unique minimum independent dominating sets

3.1 Introduction

In this chapter we consider graphs having a unique minimum independent dominating set. As we saw in the previous chapter, graphs having a unique minimum dominating set have been much studied. Graphs containing a unique minimum *independent* dominating set have received less attention. In [12], the authors discussed a hereditary class of graphs containing all graphs G for which every induced subgraph of G has a unique i-set if and only if it has a unique γ-set. Unique i-sets in trees T satisfying $\gamma(T) = i(T)$ were also considered. In Chapter 5, we consider the maximum number of edges in a graph having a unique i-set of cardinality 2. We note that minimum independent dominating sets can also be viewed as maximal independent sets of minimum cardinality. Quite a bit of work has been done on graphs having a unique maximum independent set, and, in general, the total number of maximal independent sets in a given graph. We direct the reader towards [28], [36], [35], [11], and [37] for just a few examples of such work.

In this chapter, we begin in Section 3.2 by discussing the effects of deleting a vertex, or the closed neighborhood of a vertex, from a graph having a unique minimum independent dominating set. We then turn our attention to trees in Section 3.3, where we strengthen some of our earlier results. In Section 3.4, we consider a collection of operations which can be used to combine two graphs having a unique i-set to produce a new graph also having a unique i-set. Finally, in Section 3.5, we use these operations to characterize those trees having a unique minimum independent dominating set. As in Chapter 2, the work in this chapter is a more thorough discussion of the work in [21]. Any results here not already referenced can be assumed to be a result from this paper.

As notational conventions, we let $\mathcal{UI}$ represent the class of graphs having a unique minimum independent dominating set. If $G \in \mathcal{UI}$, we let $I(G)$ denote the unique i-set of G.

3.2 Deleting vertices and closed neighborhoods

In [12], the authors prove the following.

Lemma 29 ([12])**.** *If any graph G has a unique i-set $I(G)$, then every vertex in $I(G)$ fulfills either $|epn(x, I(G))| = 0$ or $|epn(x, I(G))| \geq 2$.*

We are thus motivated to make the following definitions.

Definition 18. Given a graph $G \in \mathcal{UI}$ and its unique i-set $I(G)$, we define the following sets.

$$\begin{aligned} \mathcal{A}(I(G)) &= \{v \in I(G) : |epn(v, I(G))| \geq 2\} \\ \mathcal{B}(I(G)) &= \{v \in I(G) : |epn(v, I(G))| = 0\} \end{aligned}$$

We see that if $G \in \mathcal{UI}$, then we can partition $V(G)$ into $V(G) = \mathcal{A}(I(G)) \cup \mathcal{B}(I(G)) \cup (V(G) - I(G))$. Bearing this is mind, we now consider the implications of deleting a vertex, or the closed neighborhood of a vertex, chosen from each of these classes.

We begin with the following.

Lemma 30. *Let $G \in \mathcal{UI}$. For any $v \in V(G) - I(G)$, $i(G - v) = i(G)$.*

Proof. Since $v \notin I(G)$, we see that $I(G)$ dominates $G - v$. Hence, $i(G - v) \leq i(G)$. Suppose that $i(G - v) < i(G)$, and let D be an i-set for $G - v$. Consider then D in G. If D dominates G, then we arrive at a contradiction since this implies that $I(G)$ is not a *minimum* independent dominating set. Thus, D fails to dominate v. In this case, $D \cup \{v\}$ is an independent dominating set of cardinality at most $|I(G)|$. This contradicts the *uniqueness* of $I(G)$. Our result is shown. $\square$

We briefly note that if $G \in \mathcal{UI}$ and we delete a vertex $v \in V(G) - I(G)$, it is not guaranteed that

$G - v \in \mathcal{UI}$. For example, $P_3 \in \mathcal{UI}$, but if we delete a leaf from P_3, the resulting graph, P_2, is not in $\mathcal{UI}$.

The conditions in Lemma 29, while necessary, are not sufficient for general graphs (take $K_{3,3}$ for example). They are, however, sufficient for trees T satisfying $\gamma(T) = i(T)$ as illustrated in [12]. The following set of conditions provide a necessary and sufficient condition for $G \in \mathcal{UI}$ for an arbitrary graph G.

Lemma 31. *A graph G has a unique minimum independent dominating set if and only if there exists an i-set D of G such that for all $v \in V(G) - D$, $i(G - N[v]) \geq i(G)$.*

Proof. First, suppose that $G \in \mathcal{UI}$. In this case, let $D = I(G)$, and consider $v \in V(G) - D$. For the sake of contradiction, suppose that $i(G - N[v]) < i(G)$. Let D' be an i-set for $G - N[v]$. We see that $D' \cup \{v\}$ is then an independent dominating set for G of cardinality at most $|I(G)|$, a contradiction. Thus, we see that $i(G - N[v]) \geq i(G)$ as claimed.

Now suppose that G has an i-set D such that for all $v \in V(G) - D$, $i(G - N[v]) \geq i(G)$. For the sake of contradiction, suppose that $G \notin \mathcal{UI}$. Let D' be an i-set distinct from D, and let $v \in D' - D$. We see that $D' - \{v\}$ is an i-set for $G - N[v]$. Thus, $i(G - N[v]) = |D' - \{v\}| = |D'| - 1 = |D| - 1 < i(G)$. This, however, is a contradiction since D does not satisfy the assumed property. □

We now consider deleting a vertex from $I(G)$.

Lemma 32. *Let $G \in \mathcal{UI}$. For any $v \in \mathcal{A}(I(G))$, $i(G - v) \geq i(G)$.*

Proof. Let $v \in \mathcal{A}(I(G))$ and suppose that $i(G - v) < i(G)$. If D is an i-set for $G - v$, then D dominates every vertex in $epn(v, I(G))$. Consider then D in G. If D dominates G, then $I(G)$ is not a *minimum* independent dominating set, a contradiction. Hence, D fails to dominate v. In this case, $D \cup \{v\}$ is an independent dominating set of cardinality at most $|I(G)|$ distinct from $I(G)$, contradicting the *uniqueness* of $I(G)$. □

We briefly note that it is possible for $i(G - v) = i(G)$ for some $v \in \mathcal{A}(I(G))$ as the following example illustrates.

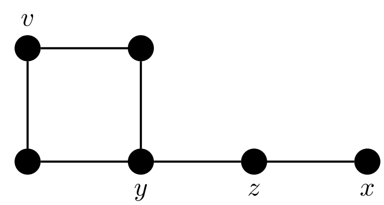


Figure 3.1: $i(G - v) = i(G)$

We see that $i(G) = 2$, $I(G) = \{v, z\}$, and that $i(G - v) = 2$ with an i-set given by $\{x, y\}$. We also note that if $u \in \mathcal{A}(I(G))$, then $G - u$ is not guaranteed to be in $\mathcal{UI}$. This is in contrast to the following result.

Lemma 33. *Let $G \in \mathcal{UI}$. For any $v \in \mathcal{B}(I(G))$, $G - v \in \mathcal{UI}$, and $I(G - v) = I(G) - \{v\}$.*

Proof. Since $v \in \mathcal{B}(I(G))$, v has no external private neighbors. Thus, $I(G) - \{v\}$ dominates $G - v$. Hence, $i(G - v) \leq i(G) - 1$. Suppose that $i(G - v) < i(G) - 1$ and let D be an i-set for $G - v$. If D dominates G, then $I(G)$ is not a minimum independent dominating set for G. Thus, D does not dominate v. This implies that $D \cup \{v\}$ is an independent dominating set of G of cardinality at most $|I(G)| - 1$, a contradiction. Thus, we see that $i(G - v) = i(G) - 1$, with $I(G) - \{v\}$ an i-set for $G - v$. Suppose $G - v$ has another i-set, call it D'. Note that D' dominates $G - v$ but does not dominate G, else we would have $i(G) = i(G) - 1$. In that case, D' fails to dominate v in G. This implies that $D' \cup \{v\}$ is an independent dominating set of G of cardinality at most $|I(G)|$. Since $D' \neq I(G) - \{v\}$ we see that $D' \cup \{v\} \neq I(G)$, a contradiction. Thus, $G - v \in \mathcal{UI}$ with $I(G - v) = I(G) - \{v\}$. $\square$

$\mathcal{A}(I(G))$ and $\mathcal{B}(I(G))$ are similar in the following respect.

Lemma 34. *Let $G \in \mathcal{UI}$. For any $v \in I(G)$, $i(G - N[v]) = i(G) - 1$, $G - N[v] \in \mathcal{UI}$, and $I(G - N[v]) = I(G) - \{v\}$.*

Proof. First note that $I(G) - \{v\}$ is an independent dominating set for $G - N[v]$. Thus, $i(G - N[v]) \leq i(G) - 1$. Suppose $i(G - N[v]) < i(G) - 1$, and let D be an i-set for $G - N[v]$. In this case, $D \cup \{v\}$ is an independent dominating set for G of cardinality at most $i(G) - 1$, a contradiction. Thus, we see that $i(G - N[v]) = i(G) - 1$. If $G - N[v]$ has an i-set distinct from $I(G) - \{v\}$, call it D', then $D' \cup \{v\}$ is an i-set of G distinct from $I(G)$, a contradiction. Thus, we see that $G - N[v] \in \mathcal{UI}$ with $I(G - N[v]) = I(G) - \{v\}$. $\square$

Our last result does not concern deleting a vertex or a private neighbor. Nevertheless, these ideas are used in the proof, in which case we present the result here.

Lemma 35. *If G is a tree in $\mathcal{UI}$ with $v \in V(G) - I(G)$, then $N(v) \cap \mathcal{A}(I(G)) \neq \emptyset$.*

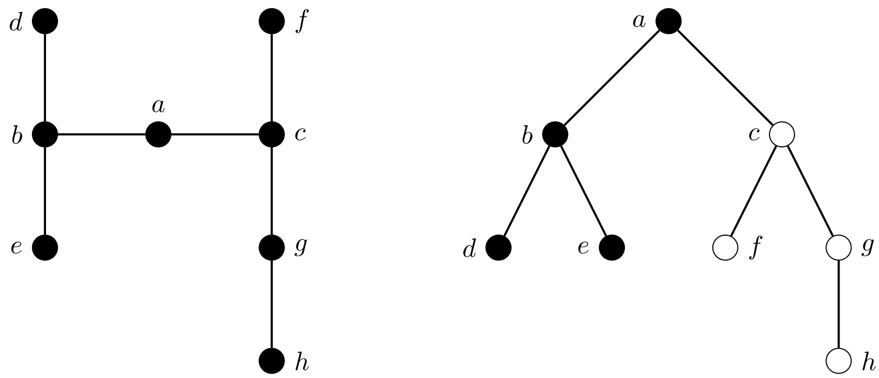
Proof. Note that since $I(G)$ is a dominating set, $|N(v) \cap I(G)| \geq 1$. For the sake of contradiction, suppose that $(N(v) \cap I(G)) \subseteq \mathcal{B}(I(G))$ with $N(v) \cap I(G) = \{b_1, b_2, \ldots, b_k\}$. Consider then $G - N[v]$. Since G is a tree, it is acyclic. This implies that $I(G) - \{b_1, b_2, \ldots, b_k\}$ is an independent dominating set for $G - N[v]$. Thus, $i(G - N[v]) \leq i(G) - k$ for some $k \geq 1$. This, however, contradicts Lemma 31. Thus, v has a neighbor in $\mathcal{A}(I(G))$. □

We will make use of this result in our characterization to come.

3.3 Trees

In this section, we seek to improve upon Lemma 32 in the case when G is a tree. Throughout this section we will be discussing *rooted* trees. Thus, a preliminary discussion of rooted trees is required.

A rooted tree is simply a tree T in which a specific vertex, called the *root*, has been identified. When a rooted tree is depicted, the root typically appears at the top. The neighbors of the root then appear below the root and are called the *children* of the root. The root, in a similar manner, is then called their *parent*. The vertices at distance two from the root are then drawn below the children of the root, and so on and so forth. By rooting the tree, a parent/child relationship is established for each edge $uv \in E(T)$. In particular, if $uv \in E(T)$, then u is the parent of v if the distance from u to the root is less than the distance from v to the root; otherwise v is the parent of u. By establishing this parent/child relationship, we can then set the *descendants* of a vertex v to be the set of children of v together with their children and so on and so forth. For notational convenience, if T is a rooted tree, then for $v \in V(T)$, we let T_v denote the subgraph of T induced by v and all of its descendants. A graphical example will help to clarify these definitions.



T $\qquad\qquad$ T rooted at a

Figure 3.2: A rooted tree

In Figure 3.2, a tree T appears on the left. On the right, the same tree T is depicted rooted at a. We see that b and c are the children of a, while d and e are the children of b. Additionally, g is the parent of h while the descendants of c are f, g, and h. Finally, the subtree T_c is the graph $T\langle\{c, f, g, h\}\rangle$ (the vertices of which are shown in white). Observe that our decision to root T at a was arbitrary. In fact, a tree can be rooted at any vertex. Notice, however, that if we change the root, the parent/child relationships will change.

We begin with the following.

Lemma 36. *Let $T \in \mathcal{UI}$, $v \in \mathcal{A}(I(T))$, and $epn(v, I(T)) = \{p_1, p_2, \ldots, p_k\}$. For $1 \leq j \leq k$, $i(T_{p_j}) = |I(T) \cap V(T_{p_j})| + 1$.*

Proof. Root T at v. Consider T_{p_1}, the subtree of T induced by p_1 and all of its descendants. By Lemma 34, $T - N[v] \in \mathcal{UI}$ with $I(T - N[v]) = I(T) - \{v\}$. This implies that $T_{p_1} - p_1 \in \mathcal{UI}$ with $I(T_{p_1} - p_1) = V(T_{p_i}) \cap I(T)$. Notice, however, that $V(T_{p_1}) \cap I(T)$ does not dominate p_1 since p_1 is an external private neighbor of v with respect to $I(T)$. In particular, this implies that none of the descendants of p_1 are contained in $I(T)$. Thus, let D be an i-set of T_{p_1}. There are two cases to consider.

- First, suppose that $p_1 \notin D$. In this case, some descendant of p_1 is contained in D, and D is an independent dominating set for $T_{p_1} - p_1$. Since $T_{p_1} - p_1 \in \mathcal{UI}$ and no descendant of p_1 is contained in $I(T_{p_1} - p_1)$, we see that $|D| > |I(T_{p_1} - p_1)| = |I(T) \cap V(T_{p_1})|$.

- Now suppose that $p_1 \in D$. In this case, no descendant of p_1 is contained in D. Let $d_1, d_2, \ldots, d_n$ denote the descendants of p_1. Now, note that if we delete p_1 from T_{p_1}, we are left with a forest whose components, namely $T_{d_1}, T_{d_2}, \ldots, T_{d_n}$, are also found in $T - N[v]$. Hence, by Lemma 34 the components of $T_{p_1} - p_1$ are each graphs in $\mathcal{UI}$. Hence, we see that

$$\begin{aligned}
|D| &= 1 + |D \cap V(T_{p_1} - p_1)| \\
&= 1 + \sum_{j=1}^{n} |D \cap V(T_{d_j})| \\
&= 1 + \sum_{j=1}^{n} |I(T) \cap V(T_{d_j})| \quad \text{by Lemma 30} \\
&= 1 + |I(T) \cap V(T_{p_1} - p_1)| \\
&= 1 + |I(T) \cap V(T_{p_1})|.
\end{aligned}$$

Thus, we see that $i(T_{p_j}) > |I(T) \cap V(T_{p_j})|$ for $1 \le j \le k$. Moreover, we also see that $p_j \cup (I(T) \cap V(T_{p_j}))$ is an independent dominating set for T_{p_j}. Thus, our result is proven. $\square$

This lemma is particularly nice since it implies the following.

Proposition 37. *Let $T \in \mathcal{UI}$. For all $v \in \mathcal{A}(I(T))$, $i(T - v) > i(T)$.*

Proof. Root T at v. Let $epn(v, I(T)) = \{p_1, p_2, \ldots, p_k\}$ and let $N(v) - epn(v, I(T)) = \{n_1, n_2, \ldots, n_m\}$. If we delete v from T, are left with $k + m$ components, namely

$$T_{p_1}, T_{p_2}, \ldots, T_{p_k}, T_{n_1}, T_{n_2}, \ldots, T_{n_m}.$$

Recall from the proof of Lemma 36 that $p_j \cup (I(T) \cap V(T_{p_j}))$ is an i-set for T_{p_j}, $1 \le j \le k$. Let F denote the subforest of $T - v$ given by $T_{n_1} \cup T_{n_2} \cup \cdots \cup T_{n_m}$, and let $\alpha = |I(T) \cap V(F)|$. Note then that

$$|I(T)| = 1 + \sum_{s=1}^{k} |I(T) \cap V(T_{p_s})| + \alpha.$$

Moreover, we see that

$$\begin{aligned} i(T-v) &= \sum_{s=1}^{k} i(T_{p_s}) + \sum_{t=1}^{m} i(T_{n_t}) \\ &= \sum_{s=1}^{k} (1 + |I(T) \cap V(T_{p_s})|) + i(F) \\ &= k + \sum_{s=1}^{k} |I(T) \cap V(T_{p_s})| + i(F) \\ &= k + (|I(T)| - 1 - \alpha) + i(F) \end{aligned}$$

Consider $i(F)$. We see that if $i(F) > \alpha - k + 1$, then our result is shown.

Suppose, then, that $i(F) \leq \alpha - k + 1$. Let D be an i-set for F. In this case, we see that

$$D \cup \bigcup_{s=1}^{k} (p_j \cup (I(T) \cap V(T_{p_j})))$$

is an independent dominating set of T, distinct from $I(T)$, of cardinality at most $|I(T)|$. This contradicts the uniqueness of $I(T)$.

Thus, we see that $i(F) > \alpha - k + 1$, in which case $i(T-v) > i(T)$. □

Thus, we see that when we consider trees in $\mathcal{UI}$, the result of Lemma 32 can be improved upon.

Continuing on, our next result will be used in Section 3.4.

Lemma 38. *If $T \in \mathcal{UI}$ is a tree with $v \in V(T) - I(T)$ a shared neighbor of at least two vertices in $I(T)$, then $T - v \in \mathcal{UI}$ with $I(T-v) = I(T)$.*

Proof. Let $T_1, T_2, \ldots, T_k$ be the components of $T - v$, and let $I_j = I(T) \cap V(T_j)$ for $1 \leq j \leq k$. Note that I_j is an independent dominating set for T_j. If D is an i-set for T_j, then

$$D \cup \bigcup_{s \neq j} I_s$$

is an independent dominating set for T. This observation implies that I_j is, in fact, an i-set for T_j, and that each $T_j \in \mathcal{UI}$. Since each $T_j \in \mathcal{UI}$, $T - v \in \mathcal{UI}$ as well. Our result is shown. □

3.4 Operations

Using our observations above, we now illustrate a collection of operations which allow us to construct a new graph in $\mathcal{UI}$ by combining two graphs in $\mathcal{UI}$. In particular, throughout this section, G_1 and G_2 are assumed to both be graphs in $\mathcal{UI}$. We let I_1 denote the unique i-set of G_1 and I_2 denote the unique i-set of G_2.

Operation 5. *For $j = 1, 2$, choose $u_j \in V(G_j) - I_j$. If G is the graph defined by $G = (G_1 \cup G_2) + u_1u_2$, then G has the unique i-set $I_1 \cup I_2$.*

Proof. First, observe that $I_1 \cup I_2$ is an independent dominating set for G. Thus, $i(G) \le |I_1 \cup I_2| = |I_1| + |I_2|$. Suppose that $i(G) < |I_1| + |I_2|$, and let D be an i-set of G. In particular, this implies that $D \neq I_1 \cup I_2$. Let $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$. Without loss of generality, suppose that $|D_1| \le |I_1|$. If D_1 dominates G_1, then we have $D_1 = I_1$ and thus $D_2 = I_2$ as well, a contradiction. Thus, D_1 does not dominate G_1. Since the only vertex of $V(G_1)$ that can be dominated from outside of $V(G_1)$ by D is u_1, we see that D_1 fails to dominate u_1, in which case $u_2 \in D_2$. This implies each of the following.

- D_2 independently dominates $V(G_2)$. Since I_2 is the unique i-set of G_2, and since $u_2 \notin I_2$, we see that $|D_2| > |I_2|$.
- D_1 independently dominates $G - u_1$. Hence, by Lemma 30, $|D_1| \ge |I_1|$.

Hence, we see that $|D| = |D_1| + |D_2| > |I_1| + |I_2|$, a contradiction.

Thus, we see that $i(G) = |I_1 \cup I_2|$. By the logic applied above, if D is any i-set of G containing one of u_1 or u_2, then $|D| > |I_1 \cup I_2|$. Thus, we see that $I_1 \cup I_2$ is the unique i-set of G. $\square$

Operation 6. *For $j = 1, 2$, choose $v_j \in \mathcal{A}(I_j)$. Let u be a new vertex, not in G_1 nor G_2. If G is the graph defined by $V(G) = V(G_1) \cup V(G_2) \cup \{u\}$ and $E(G) = E(G_1) \cup E(G_2) \cup \{v_1u, uv_2\}$, then G has the unique i-set $I_1 \cup I_2$.*

Proof. First, observe that $I_1 \cup I_2$ is an independent dominating set for G. Thus, $i(G) \le |I_1 \cup I_2| = |I_1| + |I_2|$. Let D be an i-set for G. Once again, let $D_1 = D \cap V(G_1)$ and let $D_2 = D \cap V(G_2)$. There are two cases to consider.

- First, suppose that $u \in D$. Since D is independent, this implies that $v_1 \notin D$ and that $v_2 \notin D$. Hence, D_1 is an independent dominating set for $G_1 - v_1$ and D_2 is an independent dominating set for $G_2 - v_2$. By Lemma 32, this implies that $|D_1| \geq |I_1|$ and $|D_2| \geq |I_2|$. Hence, we see that $|D| = |D_1 \cup D_2 \cup \{u\}| = |D_1| + |D_2| + 1 \geq |I_1| + |I_2| + 1 > |I_1| + |I_2|$, a contradiction.

- Now suppose that $u \notin D$. In this case, D_1 is an independent dominating set for G_1 and D_2 is an independent dominating set for G_2. This implies that $D_1 = I_1$ and $D_2 = I_2$. Thus, $D = I_1 \cup I_2$.

Hence, we see that G has a unique i-set given by $I_1 \cup I_2$. $\square$

Operation 7. *Let $G_1 \in \mathcal{UI}$ be a tree and let $G_2 \in \mathcal{UI}$. Let $v_1 \in \mathcal{A}(I_1)$ and $v_2 \in \mathcal{B}(I_2)$. Let u be a new vertex, not in G_1 nor G_2. If G is the graph defined as in Operation 6, then G has the unique i-set $I_1 \cup I_2$.*

Proof. First, observe that $I_1 \cup I_2$ is an independent dominating set for G. Thus, $i(G) \leq |I_1 \cup I_2| = |I_1| + |I_2|$. Let D be an i-set for G. Let $D_1 = D \cap V(G_1)$ and let $D_2 = D \cap V(G_2)$. Once again, we consider two cases.

- First, suppose that $u \in D$. Since D is independent, this implies that $v_1 \notin D$ and that $v_2 \notin D$. Hence, D_1 is an independent dominating set for $G_1 - v_1$ and D_2 is an independent dominating set for $G_2 - v_2$. By Proposition 37 and Lemma 33, we see that

$$\begin{aligned} |D| &= 1 + |D_1| + |D_2| \\ &\geq 1 + |D_1| + |I_2| - 1 \\ &= |D_1| + |I_2| \\ &> |I_1| + |I_2| \\ &= |I_1 \cup I_2|. \end{aligned}$$

  Thus, we have arrived at a contradiction. Hence, u is not a member of any i-set of G.

- Now suppose that $u \notin D$. In this case, D_1 is an independent dominating set for G_1 and D_2 is an independent dominating set for G_2. This implies that $D_1 = I_1$ and $D_2 = I_2$. Thus, $D = I_1 \cup I_2$.

Thus, we see that G has a unique i-set given by $I_1 \cup I_2$. $\square$

We note that if G_1 is not a tree, then Operation 7 is not guaranteed to produce a graph in $\mathcal{UI}$. For example, if we let G_1 be the graph from Figure 3.1 with $v_1 = v$, and let $G_2 = K_1$, then Operation 7 will produce the graph below, which does not have a unique i-set.

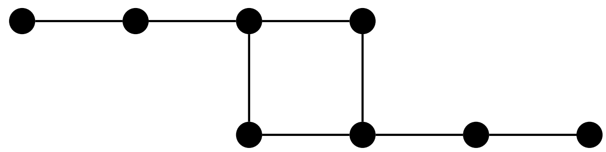


Figure 3.3: Operation 7 requires G_1 a tree

Operation 8. *Let $G_1 \in \mathcal{UI}$ be a tree and let $G_2 \in \mathcal{UI}$. Let $v_1 \in V(G_1) - I_1$ be a common neighbor of at least two vertices in I_1, and let $v_2 \in A(I_2)$. If G is the graph formed by joining G_1 and G_2 with the new edge v_1v_2, then G has the unique i-set $I_1 \cup I_2$.*

Proof. Once again, we see that $I_1 \cup I_2$ is an independent dominating set for G. Thus, $i(G) \leq |I_1| + |I_2|$. Let D be an i-set for G. Let $D_1 = D \cap V(G_1)$ and let $D_2 = D \cap V(G_2)$. We consider two cases.

- First, suppose that $v_1 \in D$. In this case, D_1 is an independent dominating set for G_1. Since $v_1 \notin I_1$, this implies that $|D_1| > |I_1|$. Additionally, if $v_1 \in D$ then $v_2 \notin D$. Hence, D_2 is a dominating set for $G_2 - v_2$. By Lemma 32, we see that $|D_2| \geq |I_2|$. Hence, we see that $|D| = |D_1| + |D_2| > |I_1| + |I_2|$, a contradiction.

- Now suppose that $v_1 \notin D$. This implies that D_2 is a minimum independent dominating set for G_2. Thus, $D_2 = I_2$. This implies that D_1 is a minimum independent dominating set for $G_1 - v_1$. Thus, by Lemma 38, we see that $D_1 = I_1$. Thus, $D = I_1 \cup I_2$.

Hence, we see that $G \in \mathcal{UI}$ and that $I(G) = I_1 \cup I_2$. $\square$

In the operation above, if $v_2 \in \mathcal{B}(I_2)$, then the resulting graph G is not guaranteed to have a unique i-set. For example, in the figure below, if we add in the dashed edge v_1v_2, the resulting graph no longer has a unique i-set.

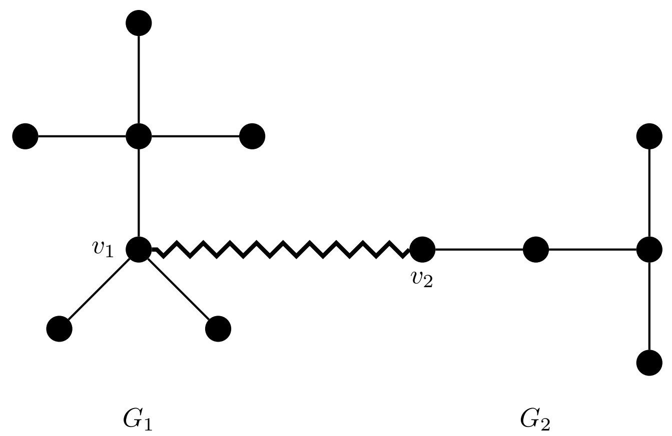


Figure 3.4: Operation 8 requires $v_2 \in \mathcal{A}(I_2)$

The ultimate problem in this example is that $i(G_1 - N[v_1]) = i(G_1)$. Thus, given a graph $G \in \mathcal{UI}$, let

$$\mathcal{C}(G) = \{v \in V(G) - I(G) : |N(v) \cap I(G)| \geq 2 \text{ and } i(G - N[v]) > i(G)\}.$$

With this notation established, we have the following operation.

Operation 9. *Let $G_1 \in \mathcal{UI}$ be a tree and let $G_2 \in \mathcal{UI}$. Let $v_1 \in \mathcal{C}(G_1)$ and let $v_2 \in \mathcal{B}(I_2)$. If G is formed by joining G_1 and G_2 with the new edge v_1v_2, then G has the unique i-set $I_1 \cup I_2$.*

Proof. As $I_1 \cup I_2$ is an independent dominating set for G, we once again have that $i(G) \leq |I_1| + |I_2|$. Thus, let D be an i-set for G, and let $D_1 = D \cap V(G_1)$ and let $D_2 = D \cap V(G_2)$. We consider two cases.

- First, suppose that $v_1 \in D$. In this case, D_1 is an independent dominating set for G_1. Since $v_1 \in \mathcal{C}(G_1)$, this implies that $|D_1| \geq |I_1| + 2$. Additionally, if $v_1 \in D$ then $v_2 \notin D$. Hence, D_2 is an independent dominating set for $G_2 - v_2$. By Lemma 33, we see that $|D_2| \geq |I_2| - 1$. Hence, we see that $|D| = |D_1| + |D_2| \geq |I_1| + 2 + |I_2| - 1 > |I_1| + |I_2|$, a contradiction.

- Now suppose that $v_1 \notin D$. This implies that D_2 is a minimum independent dominating set for G_2. Thus, $D_2 = I_2$. This implies that D_1 is a minimum independent dominating set for $G_1 - v_1$. Thus, by Lemma 38, we see that $D_1 = I_1$. Thus, $D = I_1 \cup I_2$.

Hence, we see that $G \in \mathcal{UI}$ and that $I(G) = I_1 \cup I_2$. □

It is important to notice that after performing each of these five operations, $\mathcal{A}(I(G)) = \mathcal{A}(I_1) \cup \mathcal{A}(I_2)$ and that $\mathcal{B}(I(G)) = \mathcal{B}(I_1) \cup \mathcal{B}(I_2)$.

3.5 Characterizing Trees

In this section, we characterize the trees $T \in \mathcal{UI}$.

Theorem 39. *If T is a tree in $\mathcal{UI}$, then T can be constructed from a disjoint union of isolated vertices and stars, each with at least two leaves, by a finite sequence of Operations 5 through 9.*

Proof. We proceed by induction on $i(T)$. If $i(T) = 1$, then T is either K_1 or a star with at least 2 leaves. In either case, the result holds.

Assume the result holds for all trees T in $\mathcal{UI}$ satisfying $i(T) < k$, $k \geq 2$. Let $T \in \mathcal{UI}$ be a tree satisfying $i(T) = k$. We consider two cases, and many subcases.

Case One: T has a leaf in $I(T)$.

Suppose that T has a leaf, call it l, in $I(T)$. Let v denote the support vertex of l. First, notice that $v \notin I(T)$, since $I(T)$ is independent. Additionally, by Lemma 35, some neighbor of v, distinct from l, is in $\mathcal{A}(I(T))$. Let $a_1 \in N(v) \cap \mathcal{A}(I(T))$. We consider the following two subcases.

Subcase One: $|N(v) \cap I(T)| = 2$.

First suppose that $|N(v) \cap I(T)| = 2$. Let $N(v) = \{l, a_1, o_1, o_2, \ldots, o_k\}$. Observe that $o_1, o_2, \ldots, o_k$ are not in $I(T)$. Root T at v. By Lemma 38, each of T_{a_1}, $T_{o_1}, T_{o_2}, \ldots, T_{o_k}$ has a unique i-set. Thus, by our induction hypothesis, each of these subtrees can be constructed from a disjoint union of isolates and stars by a finite sequence of Operations 5 through 9. To construct T, first note that since $a_1 \in \mathcal{A}(I(T))$, we also have $a_1 \in \mathcal{A}(I(T_{a_1}))$. Thus, we can connect v, l, and T_{a_1} by applying Operation 7. Call this resulting graph F. From there, we can construct T by connecting $T_{o_1}, T_{o_2}, \ldots, T_{o_k}$ to F by performing Operation 5 k-times.

Subcase Two: $|N(v) \cap I(T)| > 2$.

Now suppose that $|N(v) \cap I(T)| > 2$. Once again, root T at v. Let

$$N(v) = \{l, a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_k, o_1, o_2, \ldots, o_m\}$$

where $a_1, a_2, \ldots, a_j \in \mathcal{A}(I(T))$, $b_1, b_2, \ldots, b_k \in \mathcal{B}(I(T))$ and $o_1, o_2, \ldots, o_m \in V(T) - I(T)$. Let $T' = T - l$. Recall that since $T \in \mathcal{UI}$, Lemma 31 implies that $i(T - N[v]) \geq i(T)$. Thus, in particular, we have that

$$\begin{aligned} i(T' - N_{T'}[v]) &= i(T - N[v]) \\ &\geq i(T) \\ &> i(T) - 1 \\ &= i(T - l) \\ &= i(T'). \end{aligned}$$

Thus, we see that $i(T' - N_{T'}[v]) > i(T')$. Thus, $v \in \mathcal{C}(T')$. Recall that $T' \in \mathcal{UI}$ by Lemma 33. Thus, by our induction hypothesis, T' can be constructed from a disjoint union of isolated vertices and stars by a finite sequence of Operations 5 through 9. We can then reconstruct T from T' and l by applying Operation 9.

Case Two: No leaf of T is in $I(T)$.

Suppose now that no leaf of T is in $I(T)$. Consider a diametral path $v_1 v_2 \cdots v_{k-2} v_{k-1} v_k v_{k+1}$ in T. Since $i(T) \geq 2$, we see that $k \geq 4$. Observe that $v_{k+1} \notin I(T)$ in which case $v_k \in I(T)$. This further implies that $v_k \in \mathcal{A}(I(T))$. We once again consider two subcases.

Subcase One: $v_{k-1} \in epn(v_k, I(T))$.

First suppose that $v_{k-1} \in epn(v_k, I(T))$. In this case, observe that $N(v_{k-1}) = \{v_{k-2}, v_k\}$ since otherwise either $I(T)$ contains a leaf or $v_1 v_2 \cdots v_{k+1}$ is not a diametral path. Moreover, since $v_{k-1} \in epn(v_k, I(T))$, we see that $v_{k-2} \notin I(T)$. Thus, consider $T - N[v_k]$. By Lemma 34, $T - N[v_k] \in \mathcal{UI}$ and $i(T - N[v_k]) = i(T) - 1$. Thus, we can apply our induction hypothesis to $T - N[v_k]$. We can then reconstruct T from $T - N[v_k]$ and $N[v_k]$ by applying Operation 5.

Subcase Two: $v_{k-1} \notin epn(v_k, I(T))$.

Finally, suppose that $v_{k_1} \notin epn(v_k, I(T))$. Since $v_k \in \mathcal{A}(I(T))$, this implies that v_k has at least two leaf neighbors. Consider $N(v_{k-1})$. We see that $|N(v_{k-1}) \cap I(T)| \geq 2$, and that v_{k-1} has no leaf neighbors.

First suppose $N(v_{k-1}) = \{v_{k-2}, v_k\}$. In this case, $v_{k-2} \in I(T)$. Since $T - N[v_k] \in \mathcal{UI}$ by Lemma 34, we can apply our induction hypothesis to $T - N[v_k]$. We can then reconstruct T from $T - N[v_k]$, v_{k-1}, and $\{v_k\} \cup epn(v_k, I(T))$ by applying either Operation 6 or Operation 7.

Suppose now that $N(v_{k-1}) = \{v_{k-2}, v_k, o_1, o_2, \ldots, o_r\}$. Since $I(T)$ contains no leaves, we see that $o_1, o_2, \ldots, o_r$ are each in $\mathcal{A}(I(T))$. In particular, this implies that each has at least two leaf neighbors. Root T at v_{k-1}. By Lemma 38, $T_{v_{k-2}} \in \mathcal{UI}$ in which case our induction hypothesis implies it can be reconstructed using a finite sequence of our operations. We can then reconstruct T as follows. First, combine $\{v_k\} \cup epn(v_k, I(T))$, $\{o_1\} \cup epn(o_1, I(T)), \ldots, \{o_r\} \cup epn(o_r, I(T))$ through one Operation 6 followed by Operation 8 $(r-1)$-times. From there, we can reconstruct T by performing Operation 9 if $v_{k-2} \in \mathcal{B}(I(T))$, Operation 8 if $v_{k-2} \in \mathcal{A}(I(T))$, or Operation 5 if $v_{k-2} \notin I(T)$. $\square$

We conclude this chapter by giving a concrete example of the constructions discussed above. Consider the tree T in Figure 3.5. This tree has a unique i-set given by $\{a, e, g, j\}$. The construction of T using our five operations above is depicted in Figure 3.6. We begin with a disjoint union of stars and isolates in T_1. We then apply Operation 7 letting our new vertex be d. We then connect the vertices d and f through an application of Operation 5. Our initial tree T is then constructed through one more application of Operation 7 by letting our new vertex be i.

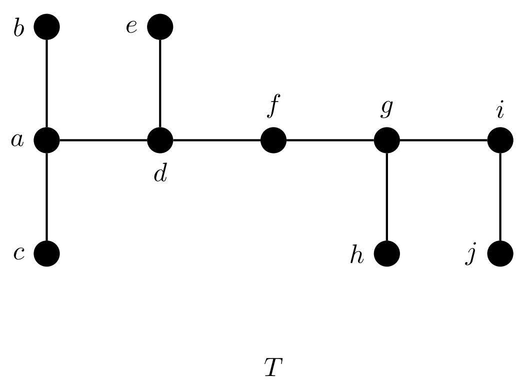


Figure 3.5: Construction example

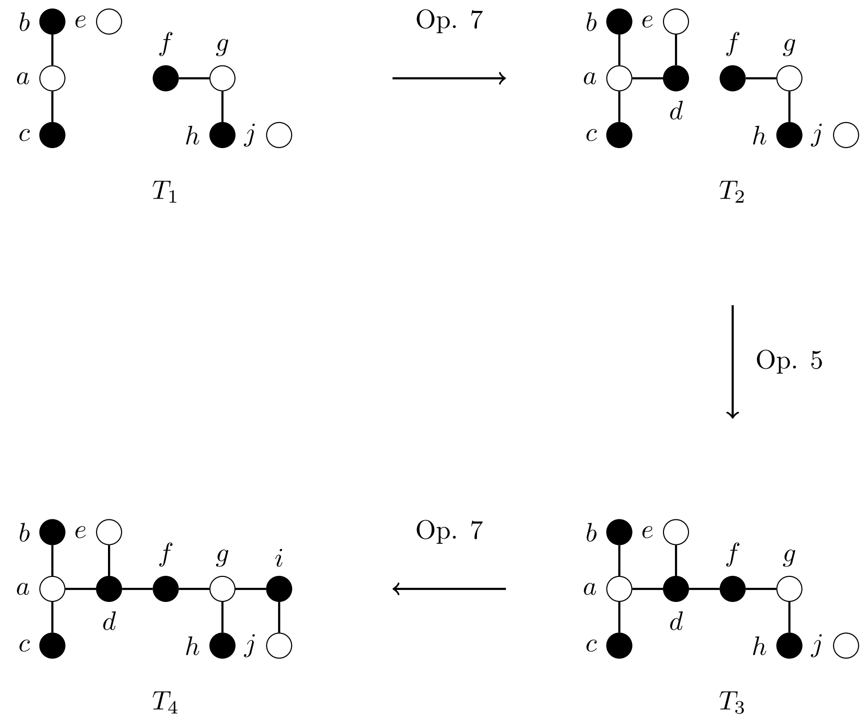


Figure 3.6: Construction steps

Chapter 4

Graph products of a vertex-transitive graph

At the conclusion of Chapter 2, we considered the existence of unique minimum dominating sets in $T \,\square\, K_n^m$. The Hamming graph K_n^m is a specific example of a vertex-transitive graph. In this chapter, we continue our work from Chapter 2 and consider unique minimum dominating sets and unique minimum independent dominating sets in graphs $G \,\square\, H$ and $G \times H$ where G is a nontrivial graph containing at least one edge and H is a connected, nontrivial, vertex-transitive graph. After briefly discussing and defining vertex-transitive graphs in Section 4.1, we then focus on unique minimum dominating sets in the Cartesian product in Section 4.2. We then turn our attention to the direct product in Section 4.3, where unique minimum independent dominating sets will be our main subject.

We continue with the same notation used in the previous two chapters. That is, $\mathcal{U}$ represents the class of graphs G having a unique minimum dominating set, denoted $UD(G)$, and $\mathcal{UI}$ represents the class of graphs having a unique minimum independent dominating set, denoted $I(G)$.

4.1 Vertex-transitive graphs

Recall from Chapter 1 that an isomorphism between two graphs G_1 and G_2 is a bijection $\phi : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. As we have seen, isomorphisms are primarily used to show that two graphs are essentially the same. An *automorphism* is a special kind of isomorphism.

Definition 19. An automorphism on a graph G is a bijection $\phi : V(G) \to V(G)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(G)$.

Thus, we see that an automorphism is an isomorphism of a graph with itself. As a trivial example, the identity mapping is an automorphism. A nontrivial example follows.

Consider the graph G below.

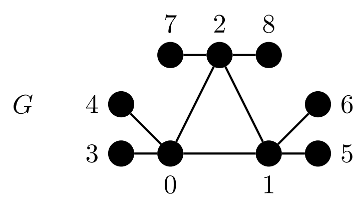


Figure 4.1: Automorphism

We claim that the following mapping defines an automorphism.

v	0	1	2	3	4	5	6	7	8
$\phi(v)$	1	2	0	5	6	7	8	3	4

Clearly ϕ is a bijection. Additionally, ϕ preserves all adjacencies and non-adjacencies. For example, we see that the edge 04 is mapped to the edge 16, while the edge 03 is mapped to the edge 15. We leave it to the reader to verify that all other adjacencies and non-adjacencies are preserved.

Since automorphisms preserve vertex adjacencies and non-adjacencies, we see that an automorphism ϕ, among other things, also preserves each of the following.

- Vertex Degrees: For all vertices $v \in V(G)$, $\deg(v) = \deg(\phi(v))$.
- Dominating Sets: If D is a dominating set of G, then $\phi(D) = \{\phi(v) : v \in D\}$ is a dominating set of G.
- Independent Sets: If I is an independent set of G, then $\phi(I) = \{\phi(v) : v \in I\}$ is an independent set of G.

Consider once again the graph G in Figure 4.1. Our automorphism ϕ mapped 0 to 1. Another automorphism of G exists which maps 0 to 2. However, given our statements above, no automorphism will map 0 to any other vertex, since $\deg(0) = 4$ and each of the remaining vertices are of degree 1. This consideration provides us with the motivation for vertex-transitive graphs.

Definition 20. A graph G is *vertex-transitive* if for any distinct pair of vertices u and v in $V(G)$, there exists an automorphism ϕ of G such that $\phi(u) = v$.

The cycle C_n, the complete graph K_n, the hypercube Q_n, and the complete bipartite graph $K_{n,n}$ are all examples of vertex-transitive graphs. The following famous graph, known as the Petersen Graph, is also vertex-transitive. Note that since automorphisms preserve vertex degrees, and since for any two distinct vertices u and v there is an automorphism which maps u to v, we see that all vertex-transitive graphs are regular graphs.

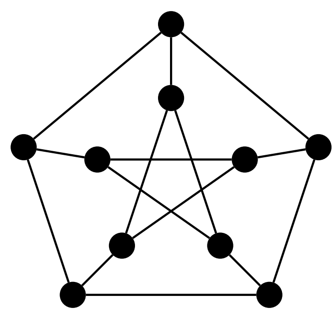


Figure 4.2: Petersen Graph

In our graph G from Figure 4.1, if we imagine being dropped onto a vertex and looking out at the rest of the graph, the "view" from each of 0, 1, and 2 is the same (provided we ignore the vertex labels). However, the view from, say, 4 looks quite different. In a vertex-transitive graph, the view from each and every vertex is the same.

4.2 Unique γ-sets and i-sets in $G \,\square\, H$

In this section, we wish to consider unique γ-sets and i-sets in graphs $G \,\square\, H$. Throughout this section, we assume that G is nontrivial and connected. We also assume that H is connected and vertex-transitive. Since K_1 is vertex-transitive and $G \,\square\, K_1 \cong G$, in order to avoid this trivial case, we additionally assume that H is nontrivial. We begin with the following well-known result.

Lemma 40. *Let σ be an automorphism of G, and let ϕ be an automorphism of H. The mapping $f : V(G \,\square\, H) \to V(G \,\square\, H)$ defined by $f(g, h) = (\sigma(g), \phi(h))$ is an automorphism of $G \,\square\, H$.*

Proof. First, observe that f is a bijection on $V(G \,\square\, H)$ since both σ and ϕ are bijections on $V(G)$ and $V(H)$ respectively. Next, suppose that $(g_1, h_1)(g_2, h_2) \in E(G \,\square\, H)$. There are two cases.

- Suppose $g_1 = g_2$ and $h_1h_2 \in E(H)$. In this case, we see that $\sigma(g_1) = \sigma(g_2)$ since σ is a well-defined function, and that $\phi(h_1)\phi(h_2) \in E(H)$ since ϕ is an automorphism on H. Thus, $f(g_1, h_1)f(g_2, h_2) \in E(G \,\square\, H)$.

- Suppose now that $h_1 = h_2$ and that $g_1g_2 \in E(G)$. In this case, we see that $\phi(h_1) = \phi(h_2)$ since ϕ is a well-defined function, and that $\sigma(g_1)\sigma(g_2) \in E(G)$ since σ is an automorphism on G. Hence, once again $f(g_1, h_1)f(g_2, h_2) \in E(G \,\square\, H)$.

Next, suppose that $f(g_1, h_1)f(g_2, h_2) \in E(G \,\square\, H)$. Once again, there are two cases.

- Suppose $\sigma(g_1) = \sigma(g_2)$ and that $\phi(h_1)\phi(h_2) \in E(H)$. In this case, since σ is one-to-one, $g_1 = g_2$. Additionally, since ϕ is an automorphism, $h_1h_2 \in E(H)$. Hence, $(g_1, h_1)(g_2, h_2) \in E(G \,\square\, H)$.

- Finally, suppose that $\sigma(g_1)\sigma(g_2) \in E(G)$ and that $\phi(h_1) = \phi(h_2)$. Since ϕ is injective we see that $h_1 = h_2$. Additionally, since σ is an automorphism, $g_1g_2 \in E(G)$. Thus, we see that $(g_1, h_1)(g_2, h_2) \in E(G \,\square\, H)$.

Thus, we see that $(g_1, h_1)(g_2, h_2) \in E(G \,\square\, H)$ if and only if $f(g_1, h_1)f(g_2, h_2) \in E(G \,\square\, H)$. Thus, f is an automorphism of $G \,\square\, H$. $\square$

This result allows us to generalize Lemma 7 as follows.

Lemma 41. *Let H be a vertex-transitive graph. If $G \,\square\, H \in \mathcal{U}$, then there exists $S \subseteq V(G)$ such that $UD(G \,\square\, H) = S \times V(H)$. Similarly, if $G \,\square\, H \in \mathcal{UI}$, then there exists $S \subseteq V(G)$ such that $I(G \,\square\, H) = S \times V(H)$.*

Proof. Let D be a γ-set (or i-set) of $G \,\square\, H$. Let $v \in V(G)$ and suppose that $(v, h_1) \in D$ but $(v, h_2) \notin D$ for some h_1 and h_2 in $V(H)$. Let ϕ be an automorphism of H for which $\phi(h_1) = h_2$. Construct the automorphism f as in Lemma 40 by taking the identity mapping as the automorphism on G and ϕ as the automorphism on H. Since automorphisms preserve dominating sets and independence, we see that the image of D under f is another γ-set (or i-set) of $G \,\square\, H$. $\square$

Once again, for convenience, we say that a set $A \subseteq V(G \,\Box\, H)$ is *stacked* if $A = S \times V(H)$ for some $S \subseteq V(G)$. The keen observer now sees that this causes trouble in the case of *independent* domination.

Proposition 42. *If H is a connected, nontrivial, vertex-transitive graph, then $G \,\Box\, H \notin \mathcal{UI}$ for any graph G.*

Proof. Suppose $G \,\Box\, H \in \mathcal{UI}$. By Lemma 41, $I(G \,\Box\, H) = S \times V(H)$ for some $S \subseteq V(G)$. Since H is nontrivial and connected, this implies there exists two vertices (v, h_1) and (v, h_2) in $I(G \,\Box\, H)$ for which $h_1h_2 \in E(H)$. However, since $(v, h_1)(v, h_2) \in E(G \,\Box\, H)$, we see that $I(G \,\Box\, H)$ is not independent, a contradiction. $\square$

Upon further reflection, this result is not at all surprising given how the Cartesian product is defined.

Unique minimum dominating sets, however, *do* exist in $G \,\Box\, H$, and they share many of the same characteristics as the unique γ-sets discussed in Chapter 2. For example, our observations following the proof of Proposition 10 still hold.

Proposition 43. *Let H be a vertex-transitive graph. If $G \,\Box\, H \in \mathcal{U}$, then $G \in \mathcal{U}$ and $UD(G \,\Box\, H) = UD(G) \times V(H)$.*

Proof. Suppose that $G \,\Box\, H \in \mathcal{U}$. By Lemma 41, this implies there exists $S \subseteq V(G)$ such that $UD(G \,\Box\, H) = S \times V(H)$. If we consider the set S, we see that not only is S a dominating set of G, but it is also the unique γ-set of G. This follows since if S is not a dominating set of G, then $S \times V(H)$ is not a dominating set of $G \,\Box\, H$. Moreover, if S' is another dominating set of G with $|S'| \leq |S|$, then $S' \times V(H)$ is a dominating set of $G \,\Box\, H$ distinct from $S \times V(H)$ and $|S' \times V(H)| \leq |S \times V(H)|$. Thus, our result is shown. $\square$

It comes as no surprise to note that the converse of Proposition 43 once again does not hold. That is, if $G \in \mathcal{U}$, it does not follow that $G \,\Box\, H \in \mathcal{U}$. Our example from Chapter 2, in fact, still holds here. The graph $P_3 \in \mathcal{U}$, but $P_3 \,\Box\, K_2 \notin \mathcal{U}$.

Following the lead of Chapter 2, we now consider the number of external private neighbors a vertex in $UD(G \,\Box\, H)$ is guaranteed to have. Recall that if $G \in \mathcal{U}$, then every vertex in $UD(G)$ is guaranteed to have at least two external private neighbors, while if $G \,\Box\, K_n \in \mathcal{U}$, then every vertex in $UD(G \,\Box\, K_n)$ is guaranteed to have at least $n + 1$ external private neighbors. To make progress in our new general setting, we must first restrict our vertex-transitive graph H to be *twin-free*. Two vertices u and v in a graph G are called *twins* if $N[u] = N[v]$. Thus, we assume that if u and v are two distinct vertices in H, then $N[u] \neq N[v]$. The reason for this restriction will be made evident in the proof of Proposition 44. We note that by assuming H to be twin-free, we exclude the cases of $H \cong K_n$.

Taking a cue from Section 2.7, we first consider stars $K_{1,n}$. Note that if $n = 0$, then we have $K_{1,0} \square H \cong H$, which never has a unique γ-set so long as H is non-trivial. If $n = 1$, then $K_{1,1} \cong P_2$ and $P_2 \notin \mathcal{U}$. Thus, $K_{1,1} \square H \notin \mathcal{U}$ for any (vertex-transitive) H by Proposition 43. Thus, we restrict our attention to $n \geq 2$. We once again let $V(K_{1,n}) = \{0, 1, \ldots, n\}$ with 0 denoting the support vertex. Additionally, for notational convenience, we let $H(j)$ denote the H-layer of $K_{1,n} \square H$ through (j, h).

Proposition 44. *Let H be a twin-free, vertex-transitive graph of vertex degree regularity d. If $K_{1,n} \square H \in \mathcal{U}$, then $n \geq d + 2$.*

Proof. Suppose that $2 \leq n \leq d + 1$. If $K_{1,n} \square H \in \mathcal{U}$, then by Proposition 43, $UD(K_{1,n} \square H) = \{0\} \times V(H)$. For convenience, let D denote the set $\{0\} \times V(H)$. Pick an arbitrary vertex $h \in V(H)$, and construct the set

$$D' = (D - (\{0\} \times N[h])) \cup (\{1, 2, \ldots, n\} \times \{h\}).$$

Since h is twin-free, the only vertex in $H(0)$ not dominated by $D - (\{0\} \times N[h])$ is $(0, h)$. Thus, by including $\{1, 2, \ldots, n\} \times \{h\}$ in D', we see that D' is a dominating set of $K_{1,n} \square H$. Moreover, since $|N[h]| = d + 1$, and since $n \leq d + 1$, we see that the cardinality of D' is less than or equal to $|D|$. Thus, $K_{1,n} \square H \notin \mathcal{U}$. $\square$

Before proceeding, we note that in the proof above, had we not restricted H to be twin-free, then the set D' constructed would not necessarily be a dominating set. For example, if $H = K_m$, then $\{0\} \times N[h] = \{0\} \times V(K_m)$ for all $h \in V(K_m)$. Thus, in this case, the set D' would dominate each of $H(1), H(2), \ldots, H(n)$, but it would not dominate $H(0)$.

Proposition 45. *Let H be a graph having maximum vertex degree d. If $n \geq d+2$, then $K_{1,n} \square H \in \mathcal{U}$.*

Proof. Let D' be a dominating set of $K_{1,n} \square H$, and let k be the number of vertices in $H(0)$ but not D'. Since H satisfies $\Delta(H) = d$, we see that D' contains at least $\lceil \frac{k}{d+1} \rceil$ vertices from each of

$H(1), H(2), \ldots, H(n)$. Hence, we see that

$$|D'| \quad \geq \quad |V(H)| - k + (n)\left\lceil \frac{k}{d+1} \right\rceil.$$

However, we see that

$$(n)\left\lceil \frac{k}{d+1} \right\rceil \geq n \cdot \frac{k}{d+1} \geq (d+2) \cdot \frac{k}{d+1} > k.$$

Thus, D' contains more vertices than the dominating set $\{0\} \times V(H)$. Thus, $\{0\} \times V(H)$ is the unique γ-set for $K_{1,n} \,\Box\, H$. $\square$

Corollary 46. *If H is a twin-free, vertex-transitive graph of vertex degree regularity d, then $K_{1,n} \,\Box\, H \in \mathcal{U}$ if and only if $n \geq d+2$.*

Having established this result, we can now give our general characterization for the number of external private neighbors of a vertex in $UD(G \,\Box\, H)$.

Proposition 47. *Let H be a nontrivial, twin-free, vertex-transitive graph of vertex degree regularity d. If $G \,\Box\, H \in \mathcal{U}$, then for all $v \in UD(G \,\Box\, H)$, $|epn(v, UD(G \,\Box\, H))| \geq d+2$.*

Proof. Let D denote the set $UD(G \,\Box\, H)$ and let D' denote the set $UD(G)$. By Proposition 43, $D = D' \times V(H)$. Observe that if $v \in D'$ with $epn(v, D') = \{p_1, p_2, \ldots, p_k\}$, then for all $x \in V(H)$, $(v, x) \in D$ with $epn((v, x), D) = \{(p_1, x), (p_2, x), \ldots, (p_k, x)\}$. Bearing this in mind, suppose, for the sake of contradiction, that $(u, w) \in D$ has

$$epn((u, w), D) = \{(p_1, w), (p_2, w), \ldots, (p_j, w)\}$$

for some $j < d+2$. Since $K_{1,j} \,\Box\, H \notin \mathcal{U}$, this implies that the subgraph of $G \,\Box\, H$ induced by $\{u, p_1, p_2, \ldots, p_j\} \times V(H)$ has a γ-set, call it B, distinct from $\{u\} \times V(H)$. In that case, $(D - (\{u\} \times V(H))) \cup B$ is a dominating set of $G \,\Box\, H$ of cardinality at most $|D|$ distinct from D, a contradiction. $\square$

As in Chapter 2, we can now use this result to classify the trees T for which $T \,\Box\, H$ has a unique γ-set. We once again let $H(v)$ denote the H-layer of $T \,\Box\, H$ through the vertex (v, h). We note here that the proof of Theorem 48 is almost identical to that of Theorem 27. We have included it here for completeness.

Theorem 48. *Let H be a nontrivial, twin-free, vertex-transitive graph of vertex degree regularity d, let T be a tree, and let S be a γ-set of T. The Cartesian product $T \Box H$ has a unique minimum dominating set if and only if every vertex $v \in S$ has at least $d+2$ external private neighbors.*

Proof. First, suppose that $T \Box H \in \mathcal{U}$. By Proposition 43, we see that $T \in \mathcal{U}$ and that $UD(T \Box H) = UD(T) \times V(H)$. Thus, in this case, $S = UD(T)$. By Proposition 47, we know that for each $v \in UD(T \Box H)$, $|epn(v, UD(T \Box H))| \geq d+2$. This, however, implies that for each $w \in S$, $|epn(w,S)| \geq d+2$. This completes the forward direction of our proof.

We now suppose that every element in S has at least $d+2$ external private neighbors. We first note that since $d+2 \geq 2$, Theorem 20 implies that $T \in \mathcal{U}$. That is, S is the unique minimum dominating set of T. We wish to show that $T \Box H \in \mathcal{U}$. The set $S \times V(H)$ is clearly a dominating set for $T \Box H$. We must show that it is a γ-set for $T \Box H$, and that it is the unique γ-set for $T \Box H$.

We proceed by induction on $\gamma(T) = |S|$. If $\gamma(T) = 1$, then T is a star $K_{1,p}$ with $p \geq d+2$. By Corollary 46, we see that $T \Box H$ has $S \times V(H)$ as its unique γ-set. Thus, suppose the result has been proven whenever $\gamma(T) < q$. Suppose now that $\gamma(T) = q$. Consider a diametral path $x_1 x_2 \ldots x_{t-1} x_t x_{t+1}$ in T. Note that $x_t \in S$ and that $t \geq 3$.

Case One

First, suppose that $x_{t-1} \notin epn(x_t, S)$. In this case, since $|epn(x_t, S)| \geq d+2$, we see that x_t is adjacent to at least $d+2$ leaves. Thus, by the proof of Proposition 45, every vertex of $H(x_t)$ is included in every γ-set of $T \Box H$. Let T' denote the tree obtained by removing x_t and all of its private neighbors with respect to S from T. Note that by Lemma 9, $T' \in \mathcal{U}$ with $UD(T') = S - \{x_t\}$. Additionally, observe that if $x \in S - \{x_t\}$, then $epn(x, S - \{x_t\}) \supseteq epn(x, S)$. Since $\gamma(T') < \gamma(T)$, our induction hypothesis implies that $T' \Box H \in \mathcal{U}$ and that $UD(T' \Box H) = (S - \{x_t\}) \times V(H)$.

Suppose then that D is a γ-set for $T \Box H$ and that $D \neq S \times V(H)$. By our observations above, we know that $\{x_t\} \times V(H) \subseteq D$. Let $B = D - (\{x_t\} \times V(H))$ and note that $B \subseteq V(T' \Box H)$. If B dominates $T' \Box H$, then since $UD(T' \Box H) = (S - \{x_t\}) \times V(H)$ and since $B \neq (S - \{x_t\}) \times V(H)$, this implies that $|B| > |(S - \{x_t\}) \times V(H)|$. This, however, implies that $S \times V(H)$ is a smaller cardinality dominating set for $T \Box H$ than D, a contradiction.

Thus, assume that B does not dominate $T' \Box H$. Since D is a dominating set of $T \Box H$, this implies that B fails to dominate some subset of $H(x_{t-1})$ in $T' \Box H$. In particular, this implies that some subset of $H(x_{t-1})$ is not contained in B. We consider two subcases.

Subcase One

Suppose that $x_{t-1} \notin S$.

- First, suppose that $N(x_{t-1}) = \{x_{t-2}, x_t\}$. Since $x_{t-1} \notin epn(x_t, S)$, this implies that $x_{t-2} \in S$. Apply Lemma 9 to T, and remove the edge $x_{t-2}x_{t-1}$. It follows that $T' - x_{t-1} \in \mathcal{U}$ and that $UD(T' - x_{t-1}) = S - \{x_t\}$. This further implies, by the same logic as above, that $(T' - x_{t-1}) \,\Box\, H \in \mathcal{U}$ with unique γ-set given by $(S - \{x_t\}) \times V(H)$. Note that since B does not dominate all of $H(x_{t-1})$ in $T' \,\Box\, H$, this implies that B does not contain all of $H(x_{t-2})$.

  If B contains no vertices from $H(x_{t-1})$, then B is a dominating set for $(T' - x_{t-1}) \Box H$ distinct from $(S - \{x_t\}) \times V(H)$. Thus, $|B| > |(S - \{x_t\}) \times V(H)|$ implying that $|D| > |S \times V(H)|$. This contradicts our assumption that D was a γ-set for $T \,\Box\, H$.

  Hence, we see that B contains some subset of $H(x_{t-1})$. Let this subset be

  $$\{(x_{t-1}, p_1), (x_{t-1}, p_2), \ldots, (x_{t-1}, p_j)\} \subseteq B.$$

  This implies that

  $$B \cap \{(x_{t-2}, p_1), (x_{t-2}, p_2), \ldots, (x_{t-2}, p_j)\} = \emptyset$$

  since otherwise D is not a minimum cardinality dominating set for $T \,\Box\, H$. Thus, consider the set

  $$(B - \{(x_{t-1}, p_1), \ldots, (x_{t-1}, p_j)\}) \cup \{(x_{t-2}, p_1), \ldots, (x_{t-2}, p_j)\}.$$

  This is a dominating set for $(T' - x_{t-1}) \Box H$ distinct from $(S - \{x_t\}) \times V(H)$, a contradiction.

- Now suppose that x_{t-1} is adjacent to a vertex, call it y, not on the diametral path. First, note that $y \in S$. If $y \notin S$, then since $x_{t-1} \notin S$, y would have a neighbor in S which, with its external private neighbors, could be used to create a longer path in T. In particular, any neighbors of x_{t-1} in T not on the diametral path are in S and have only leaf neighbors. Since our initial assumption was that each element of S has at least $d+2$ external private neighbors, this implies that y has $d+2$ leaf-neighbors in T. Hence, by the same logic as applied to x_t above, every vertex of $H(y)$ is contained in every γ-set for $T \,\Box\, H$. However, this implies that $\{y\} \times V(H) \subseteq D$ which further implies that B dominates $T' \,\Box\, H$, a contradiction.

Thus, in both cases, $x_{t-1} \notin S$ leads to a contradiction.

Subcase Two

Suppose now that $x_{t-1} \in S$. This implies that $|epn(x_{t-1}, S)| \geq d+2$ by our earlier assumption. If x_{t-1} has an external private neighbor other than x_{t-2} that is not a leaf, then a longer path in T can be found. Hence, we see that x_{t-1} has at least $d+1$ leaf-neighbors in T, call them $l_1, l_2, \ldots, l_r$. Note that if $r \geq d+2$, then every vertex of $H(x_{t-1})$ is contained in every γ-set of $T \,\Box\, H$ implying that B is a dominating set for $T' \,\Box\, H$, a contradiction.

Thus, we see that x_{t-1} has exactly $d+1$ leaf-neighbors and $x_{t-2} \in epn(x_{t-1}, S)$. Recall that some subset of $H(x_{t-1})$ in $T \,\Box\, H$ is not contained in B. To be specific, assume k vertices of $H(x_{t-1})$ are not contained in B. This implies that at least $\lceil \frac{k}{d+1} \rceil$ vertices from each of $H(l_1), H(l_2), \ldots, H(l_r)$ are contained in B. Additionally, the vertices in $H(x_{t-2})$ that are adjacent to vertices in $(\{x_{t-1}\} \times V(H)) - B$ are dominated by vertices outside of $H(x_{t-1})$. Since

$$[d+1] \cdot \lceil \frac{k}{d+1} \rceil \geq k$$

we see that B contains exactly k vertices from $H(l_1), H(l_2), \ldots, H(l_r)$ in total, since otherwise a smaller dominating set for $T \,\Box\, H$ could be constructed. Consider the set obtained from B by removing the k vertices from $H(l_1), H(l_2), \ldots, H(l_r)$ and including the k missing vertices from $H(x_{t-1})$. This set is a dominating set for $T' \,\Box\, H$ distinct from $(S - \{x_t\}) \times V(H)$, a contradiction.

Case Two

Finally, suppose that $x_{t-1} \in epn(x_t, S)$. In this case, x_t is adjacent to at least $d+1$ leaves, call them $l_1, l_2, \ldots, l_p$. Note that the only neighbors of x_{t-1} are x_t and x_{t-2}. If x_{t-1} had any other neighbors, either a longer path in T could be found, or x_{t-1} would not be an external private neighbor of x_t with respect to S.

Suppose that D is a γ-set of $T \,\Box\, H$ which does not contain k vertices of $H(x_t)$. This implies that D contains at least $\lceil \frac{k}{d+1} \rceil$ vertices from each of $H(l_1), H(l_2), \ldots, H(l_p)$. In fact, if $p\lceil \frac{k}{d+1} \rceil > k$, then we have reached a contradiction since a smaller dominating set for $T \,\Box\, H$ could be found simply by including every vertex of $H(x_t)$. In particular, this implies that $p\lceil \frac{k}{d+1} \rceil = k$.

We now claim that D contains at least one vertex from $H(x_{t-1})$. To see this, first note that the tree T'' defined by $T'' = T - \{x_t, x_{t-1}, l_1, \ldots, l_p\}$ belongs to $\mathcal{U}$ with $UD(T'') = S - \{x_t\}$. Additionally, since $epn(x, S - \{x_t\}) = epn(x, S)$ for all $x \in S - \{x_t\}$, our induction hypothesis implies that $T'' \,\Box\, H$ has a unique γ-set given by $(S - \{x_t\}) \times V(H)$. If no vertices from $H(x_{t-1})$

are included in D, then

$$D \cap V(T'' \,\Box\, H) = (S - \{x_t\}) \times V(H).$$

This, however, results in at least k vertices of $H(x_{t-1})$ being undominated by D since $x_{t-2} \notin S - \{x_t\}$. This is a contradiction.

Thus, D contains at least one vertex from $H(x_{t-1})$. If we "shift" these vertices to their corresponding positions in $H(x_{t-2})$, remove the vertices from D in $H(l_1), H(l_2), \ldots, H(l_p)$, and add in the missing vertices from $H(x_t)$, we create a γ-set D' distinct from D which induces a dominating set distinct from $(S - \{x_t\}) \times V(H)$ on the subgraph $T'' \,\Box\, H$. Hence, D is not a minimum dominating set, a contradiction.

We thus see that if D is a γ-set for $T \,\Box\, H$, then every vertex of $H(x_t)$ is included in D. By the logic applied in the above paragraph, this implies that $S \times V(H)$ is the unique γ-set for $T \,\Box\, H$.

Thus, we see that if T has a minimum dominating set S for which every element in S has at least $d + 2$ external private neighbors, then $T \,\Box\, H \in \mathcal{U}$. $\Box$

By combining this result with Theorem 6, we have the following corollary.

Corollary 49. *Let T be a nontrivial tree and let D be a γ-set of T.*

1. $T \,\Box\, K_n \in \mathcal{U}$ *if and only if for all* $v \in D$, $|epn(v, D)| \geq n + 1$.

2. $T \,\Box\, Q_n \in \mathcal{U}$ *if and only if for all* $v \in D$, $|epn(v, D)| \geq n + 2$.

3. $T \,\Box\, C_n \in \mathcal{U}$ *if and only if for all* $v \in D$, $|epn(v, D)| \geq 4$.

4. *Let P denote the Petersen graph.* $T \,\Box\, P \in \mathcal{U}$ *if and only if for all* $v \in D$, $|epn(v, D)| \geq 5$.

5. *Let $H(n_1, n_2, \ldots, n_k)$ denote the Hamming graph $K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_k}$.* $T \,\Box\, H(n_1, n_2, \ldots, n_k) \in \mathcal{U}$ *if and only if for all* $v \in D$, $|epn(v, D)| \geq (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) + 2$.

4.3 Unique γ-sets and i-sets in $G \times H$

In this section, we turn our attention to the direct product. We once again assume that G is nontrivial and connected. Additionally, we assume that H is connected, nontrivial, and vertex-transitive. As in the previous section, we begin by discussing an automorphism of $G \times H$. Like Lemma 40, the following result is well known.

Lemma 50. *Let σ be an automorphism of G and let ϕ be an automorphism of H. The mapping $f : V(G \times H) \to V(G \times H)$ defined by $f(g, h) = (\sigma(g), \phi(h))$ is an automorphism of $G \times H$.*

Proof. Since σ and ϕ are bijections on $V(G)$ and $V(H)$ respectively, f is a bijection on $V(G \times H)$. Observe that $(g_1, h_1)(g_2, h_2) \in E(G \times H)$ if and only if $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. This holds if and only if $\sigma(g_1)\sigma(g_2) \in E(G)$ and $\phi(h_1)\phi(h_2) \in E(H)$ since both of σ and ϕ are automorphisms. Thus, we see that $(g_1, h_1)(g_2, h_2) \in E(G \times H)$ if and only if $f(g_1, h_1)f(g_2, h_2) \in E(G \times H)$. Thus, f is an automorphism on $G \times H$. $\square$

This result again implies that unique γ-sets and unique i-sets in $G \times H$ satisfy the stacked property, as we now prove.

Lemma 51. *Let H be a vertex-transitive graph. If $G \times H \in \mathcal{U}$, then $UD(G \times H) = UD(G) \times V(H)$. If $G \times H \in \mathcal{UI}$, then $I(G \times H) = I(G) \times V(H)$.*

Proof. Let D be the unique γ-set of $G \times H$, and let $(v, h_1) \in D$. If $h_2 \in V(H)$, then, since H is vertex-transitive, there exists an automorphism ϕ of H for which $\phi(h_1) = h_2$. Construct the automorphism f as in Lemma 50 with the identity mapping as the automorphism on G and ϕ as the automorphism on H. Since $f(D) = D$ and since $f(v, h_1) = (v, \phi(h_1)) = (v, h_2)$, we see that $(v, h_2) \in D$. Thus, there exists $S \subseteq V(G)$ such that $D = S \times V(H)$. Similarly, if J is the unique i-set for $G \times H$, then there exists a set $T \subseteq V(G)$ such that $J = T \times V(H)$.

By our observations above, if $G \times H \in \mathcal{U}$, then $UD(G \times H) = S \times V(H)$ for some $S \subseteq V(G)$. S is certainly a dominating set for G, since if S is not a dominating set for G, then $S \times V(H)$ is not a dominating set for $G \times H$. If S is not a γ-set of G, then a smaller dominating set for $G \times H$ can be found. In fact, if S is not the unique γ-set for G, then $G \times H$ has two distinct γ-sets. Thus, if $G \times H \in \mathcal{U}$, then $UD(G \times H) = UD(G) \times V(H)$. The case of $G \times H \in \mathcal{UI}$ follows similarly. $\square$

Recall from Proposition 42 that if H is connected and nontrivial, then $G \,\square\, H \notin \mathcal{UI}$ for any G. The following is an interesting "dual" to this result.

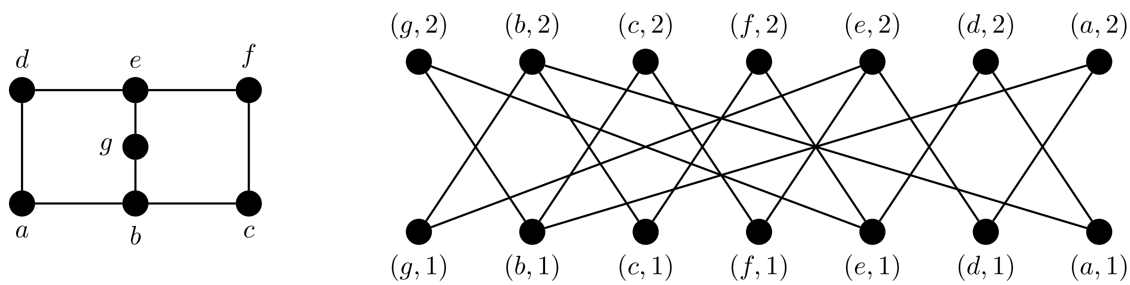
Proposition 52. *If G is a connected graph on at least two vertices, and H is a connected, vertex-transitive graph on at least three vertices, then $G \times H \notin \mathcal{U}$.*

Proof. Suppose $G \times H \in \mathcal{U}$. By Lemma 51, $UD(G \times H) = S \times V(H)$ for some $S \subseteq V(G)$. Let $x \in S$, $h \in V(H)$, and consider (x, h). We claim that $epn(x, h) = \emptyset$. To see this, suppose that

$(y, h_1) \in N((x, h))$. This implies that $xy \in E(G)$ and $hh_1 \in E(H)$. Since $|V(H)| \geq 3$, and since H is connected, there exists $h_2 \in V(H)$ such that $h_1h_2 \in E(H)$. Thus, $(x, h_2) \in UD(G \times H)$ implying that (y, h_1) is not an external private neighbor of (x, h). Thus, since (y, h_1) was an arbitrary element in $N((x, h))$, we see that (x, h) does not have at least two private neighbors other than itself. Thus, by Lemma 11, $G \times H \notin \mathcal{U}$. □

Given this, we see that if H is vertex-transitive, then $G \times H \in \mathcal{U}$ only if $H = K_2$. For the moment, we consider this case. Recall that when we considered $G \,\square\, K_n \in \mathcal{U}$ in Chapter 2, we were able to improve upon the general bound for external private neighbors given by Lemma 11 in Lemma 12. Unfortunately, Lemma 11 cannot be improved upon in the case of $G \times K_2 \in \mathcal{U}$ as the case of $P_3 \times K_2$ attests. That is, if $G \times K_2 \in \mathcal{U}$, then every vertex $v \in UD(G \times K_2)$ has at least two external private neighbors and this bound is sharp.

We have seen that if $G \times K_2 \in \mathcal{U}$, then $G \in \mathcal{U}$. However, as in the Cartesian product cases above, if $G \in \mathcal{U}$, it does not imply that $G \times K_2 \in \mathcal{U}$ as the graph G in Figure 4.3 illustrates. We see that $UD(G) = \{b, e\}$ but $\{(b, 1), (c, 1), (d, 2), (e, 2)\}$ is a γ-set for $G \times K_2$ that is not stacked.



G $\qquad$ $G \times K_2$

Figure 4.3: $G \in \mathcal{U}$ but $G \times K_2 \notin \mathcal{U}$

However, we do have the following.

Proposition 53. *Let B be a bipartite graph. If $B \in \mathcal{U}$, then $B \times K_2 \in \mathcal{U}$.*

Proof. If B is a bipartite graph, then $B \times K_2 \cong B \cup B$. Hence, if $B \in \mathcal{U}$, then $B \cup B \in \mathcal{U}$. □

Since every tree T is bipartite, Proposition 53 has the following corollary.

Corollary 54. *Let T be a nontrivial tree. The following are equivalent.*

1. *The tree T has a unique minimum dominating set.*

2. *The tree T has a minimum dominating set D such that for all $v \in D$, $|epn(v, D)| \geq 2$.*

3. *The graph $T \times K_2$ has a unique minimum dominating set.*

Thus, we see that this result is a natural parallel to Theorem 6.

Unique i-sets in $G \times H$ prove to be both more interesting, and more difficult to work with. In order to mirror our earlier work, we restrict our attention to $H = K_n$ for $n \geq 2$ ($G \times K_1 \in \mathcal{UI}$ for all G). We first make the following definition.

Definition 21. Let D be an i-set for $G \times K_n$. For $v \in V(G)$, let $f_D(v) = |\{(v, h) : h \in V(H), (v, h) \in D\}|$. If D is understood from context, we simplify the notation to $f(v)$.

With this notation established, we have the following result.

Lemma 55. *If D is an i-set in $G \times K_n$, then $f_D(v) \in \{0, 1, n\}$ for all $v \in V(G)$.*

Proof. Suppose $2 \leq f_D(v) \leq n - 1$ for some $v \in V(G)$. If $uv \in E(G)$, then $f_D(u) = 0$, since otherwise D is not independent. This, however, implies that D is not a dominating set since there is at least one vertex in $\{v\} \times V(H)$ that is not contained in D, and is thus not dominated. $\square$

By considering this result with Lemma 51, we see that if $G \times K_n \in \mathcal{UI}$, then $f_{I(G)}(v) \in \{0, n\}$ for all $v \in V(G)$. This observation, while simple, actually highlights a very interesting point. Observe that for any $j \in V(K_n)$, the set $V(G) \times \{j\}$ is an independent dominating set for $G \times K_n$ since G is assumed to be connected. Thus, if $G \times K_n \in \mathcal{UI}$, then it is the case that $i(G \times K_n) < |V(G)|$. Moreover, if $G \times K_n \in \mathcal{UI}$, then $i(G \times K_n) = n \cdot i(G)$ by Lemma 51. Thus, we actually have a stronger result, namely that $i(G) < \frac{|V(G)|}{n}$. Hence, if $i(G) \geq \frac{|V(G)|}{n}$, then we immediately know that $G \times K_n \notin \mathcal{UI}$.

Proposition 56. *If $G \times K_n \in \mathcal{UI}$, then $i(G) < \frac{|V(G)|}{n}$.*

What else can be said concerning $I(G \times K_n)$? If $n \geq 3$, then Lemma 51 implies that if $G \times K_n \in \mathcal{UI}$, then $\mathcal{B}(I(G \times K_n)) = I(G \times K_n)$. That is, unlike in the general graph case considered in Chapter 3, no vertex in the unique i-set in $G \times K_n$ has any external private neighbors. Thus, thanks to Lemma 33, we have the following result.

Proposition 57. *Let $n \geq 3$. If $G \times K_n \in \mathcal{UI}$, then for all $v \in I(G \times K_n)$, $i(G \times K_n - v) = i(G \times K_n) - 1$, $G \times K_n - v \in \mathcal{UI}$ and $I(G \times K_n - v) = I(G \times K_n) - \{v\}$.*

While no vertex in $I(G \times K_n)$ has any external private neighbors, what can be said concerning the external private neighbors of $I(G)$ when $G \times K_n \in \mathcal{UI}$?

Proposition 58. *If $G \times K_n \in \mathcal{UI}$, then for all $v \in I(G)$, $|epn(v, I(G))| = 0$ or $|epn(v, I(G))| \geq n$.*

Proof. Suppose that $G \times K_n \in \mathcal{UI}$ with $I(G \times K_n) = I(G) \times V(K_n)$. Let D denote $I(G \times K_n)$. Consider $v \in I(G)$. Suppose for some j, $1 \leq j \leq n-1$, that $epn(v, I(G)) = \{p_1, p_2, \ldots, p_j\}$. This implies that $(\{p_1, p_2, \ldots, p_j\} \times V(K_n)) \cap D = \emptyset$. In this case, $(D - \{(v, 2), (v, 3), \ldots, (v, n)\}) \cup \{(p_1, 1), (p_2, 1), \ldots, (p_j, 1)\}$ is an independent dominating set of $G \times K_n$ of cardinality at most $|D|$ that is distinct from D, a contradiction. $\square$

Recall that if T is a nontrivial tree, then $T \,\square\, K_n \in \mathcal{U}$ if and only if T has a γ-set D such that $|epn(v, D)| \geq n+1$ for all $v \in D$. One might hazard a guess at this point, and predict that $T \times K_n \in \mathcal{UI}$ if and only if $T \in \mathcal{UI}$ with all $v \in I(T)$ either satisfying $|epn(v, I(T))| = 0$ or $|epn(v, I(T))| \geq n$, then $T \times K_n \in \mathcal{UI}$. This, however, does not turn out to be the case, as the graph T in Figure 4.4 illustrates.

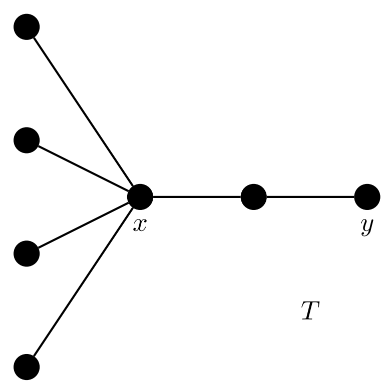


Figure 4.4: $T \in \mathcal{UI}$ but $T \times K_4 \notin \mathcal{UI}$

We see that $T \in \mathcal{UI}$ with $I(T) = \{x, y\}$. Additionally, $|epn(x, I(T))| = 4$ while $|epn(y, I(T))| = 0$. However, $T \times K_4 \notin \mathcal{UI}$ by Proposition 57, since $i(T) = 2 > \frac{7}{4} = \frac{|V(T)|}{n}$. In fact, $\{x, y\} \times V(K_4)$ is not an i-set for $T \times K_4$. We see that $V(T) \times \{j\}$ is a smaller independent dominating set for $T \times K_4$ where j is an arbitrary vertex in $V(K_4)$.

4.4 Looking ahead

We have now considered unique γ-sets, and unique i-sets. Thinking of the domination chain presented as Theorem 5, we have yet to consider unique ir-sets, unique β_0-sets, unique Γ-sets, or unique IR-sets. We do so in the following chapter in an extremal graph setting.

Chapter 5

Maximum Graphs

5.1 Background

In our introductory chapter, we were introduced to the domination chain

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$$

which holds for every graph G. Many research papers have been devoted to its study. In this chapter, we consider a specialization of this chain as it pertains to *unique realizations* of these parameters in an extremal graph setting.

In [7] and [13] the problem of determining the maximum number of edges in a graph having a unique γ-set was considered. In this chapter, we build on this work and consider the maximum number of edges in a graph having a unique ir-set, i-set, β_0-set, Γ-set, or IR-set of cardinality 2. In so doing, we show that the domination chain above still holds in this different setting.

The work that follows appears in [25] which has been accepted for publication and is to appear. For ease, we do not specifically reference this paper when stating and proving theorems in this chapter.

5.2 Definitions and Notation

Let $\mathcal{P}$ be one of ir, γ, i, β_0, Γ, or IR. In this chapter we are not only interested in graphs having a unique $\mathcal{P}$-set, but also the number of edges they contain. We make the following definitions.

Definition 22. Let $m^*_{\mathcal{P}}(n,k)$ denote the maximum number of edges in a graph on n vertices having a unique $\mathcal{P}$-set of cardinality k.

Definition 23. Let $m_{\mathcal{P}}(n,k)$ denote the maximum number of edges in an isolate free graph on n vertices having a unique $\mathcal{P}$-set of cardinality k.

Observe that $m_{\mathcal{P}}(n,k) \leq m^*_{\mathcal{P}}(n,k)$. Furthermore, if one can find a graph G of order n and $\delta(G) \geq 1$ having a unique $\mathcal{P}$-set such that $|E(G)| = m^*_{\mathcal{P}}(n,k)$, then $m_{\mathcal{P}}(n,k) = m^*_{\mathcal{P}}(n,k)$.

With this notation now defined, our main result in this chapter is as follows.

Theorem 59. *For $n \geq 6$*

$$m_{ir}(n,2) = m_\gamma(n,2) \leq m_i(n,2) \leq m_{\beta_0}(n,2) = m_\Gamma(n,2) = m_{IR}(n,2)$$

and

$$m^*_{ir}(n,2) = m^*_\gamma(n,2) \leq m^*_i(n,2) \leq m^*_{\beta_0}(n,2) = m^*_\Gamma(n,2) = m^*_{IR}(n,2)$$

We prove this theorem by computing, or recalling in the case of $m_\gamma(n,2)$, the exact values for each of these parameters.

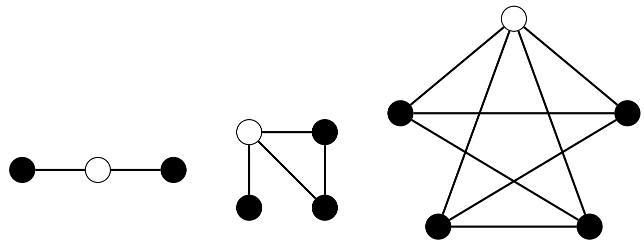
In Section 5.3, we collect the results from [7] and [13] which we need for our discussion of unique irredundant sets in Section 5.4. In Section 5.5, we return our attention to unique minimum independent dominating sets of cardinality 2, while in Section 5.6 we consider unique β_0-sets, Γ-sets, and IR-sets of cardinality k for $k \geq 2$. Finally, in Section 5.7, as we bring our discussion of unique domination chain parameters to a close, we pose several open problems.

5.3 Unique minimum dominating sets

In this section, we briefly collect the results from [7] and [13] concerning unique minimum dominating sets that we will need in our work to come. We first note the following result from [7].

Proposition 60 ([7])**.** *For $n \geq 3$, $m_\gamma(n,1) = \binom{n}{2} - \lceil \frac{n-1}{2} \rceil$.*

This bound is achieved by constructing a graph G on n vertices having a single vertex of degree $n-1$, with all other vertices having degree at most $n-2$. For example, the graphs in Figure 5.1 below all have a unique γ-set of cardinality one (given by the white vertex in each case) and have a maximum number of edges.



$n = 3$ $n = 4$ $n = 5$

Figure 5.1: $m_\gamma(n, 1) = \binom{n}{2} - \lceil \frac{n-1}{2} \rceil$

For purely comparative purposes in Section 5.5 to follow, we also note the following result from [7].

Theorem 61 ([7])**.** *For $k \geq 2$, $m_\gamma(3k, k) = 2k + 2\binom{k}{2}$.*

The following result, from [13], will be used in Section 4 below.

Theorem 62 ([13])**.** *For $n \geq 6$, $m_\gamma(n, 2) = \binom{n-2}{2}$.*

There are precisely two graphs on six vertices having a unique minimum dominating set of cardinality 2 that achieve the bound from Theorem 62 and they are pictured in Figure 5.2 below.

G_1 G_2

Figure 5.2: $m_\gamma(n, 2) = \binom{n-2}{2}$

By combining Proposition 60 and Theorem 62, we see the following.

Proposition 63. *For $n \geq 6$, $m^*_\gamma(n, 2) = m_\gamma(n-1, 1) = \binom{n-1}{2} - \lceil \frac{n-2}{2} \rceil$.*

Proof. Suppose G is a graph on $n \geq 6$ vertices having a unique γ-set of cardinality 2. Note that G has at most one isolated vertex. If G is isolate free, then $|E(G)| \leq \binom{n-2}{2}$ by Theorem 62. However, if G has one isolate and a unique γ-set of cardinality 2, then the component of G containing $n-1$ vertices necessarily induces an isolate free graph having a unique γ-set of cardinality 1. Hence, if G has an isolate, then $|E(G)| \leq \binom{n-1}{2} - \lceil \frac{n-2}{2} \rceil$ by Proposition 60. Since this upper bound is strictly greater than $\binom{n-2}{2}$ and can be achieved, our result follows. $\square$

5.4 Unique minimum maximal irredundant sets

We now turn out attention to the maximum number of edges in a graph G on n vertices having a unique ir-set of cardinality 2. The only graph on two vertices having a unique ir-set of cardinality 2 is the graph consisting of two isolated vertices, namely $\overline{K_2}$. Additionally, no graph on three vertices has a unique ir-set of cardinality 2. Thus, we first restrict ourselves to graphs on $n \geq 4$ vertices. Note that if G has two or more isolated vertices, then $ir(G) \geq 3$. Thus, we first suppose that G has one isolated vertex. In this case, G has a component of order $n-1$, call it C, which necessarily satisfies $ir(C) = 1$. Since $ir(C) = 1$ if and only if $\gamma(C) = 1$, we see that, in fact, C has a unique γ-set of cardinality 1. Hence, by Proposition 60, C, and thus G, has at most $\binom{n-1}{2} - \lceil \frac{n-2}{2} \rceil$ edges. As we saw above, this bound can be achieved. Thus, we have that $m^*_{ir}(n,2) \geq \binom{n-1}{2} - \lceil \frac{n-2}{2} \rceil$. Hence, by Proposition 63, if $n \geq 6$, then $m^*_{ir}(n,2) \geq m^*_{\gamma}(n,2)$.

We now restrict ourselves to isolate free graphs. It can be readily checked that no isolate free graph on two, three, or four vertices has a unique ir-set of cardinality 2. Among the isolate free graphs on five vertices, twelve satisfy $ir(G) = 2$, and none has a unique ir-set (see Figure 5.3). Thus, we restrict ourselves to graphs on $n \geq 6$ vertices.

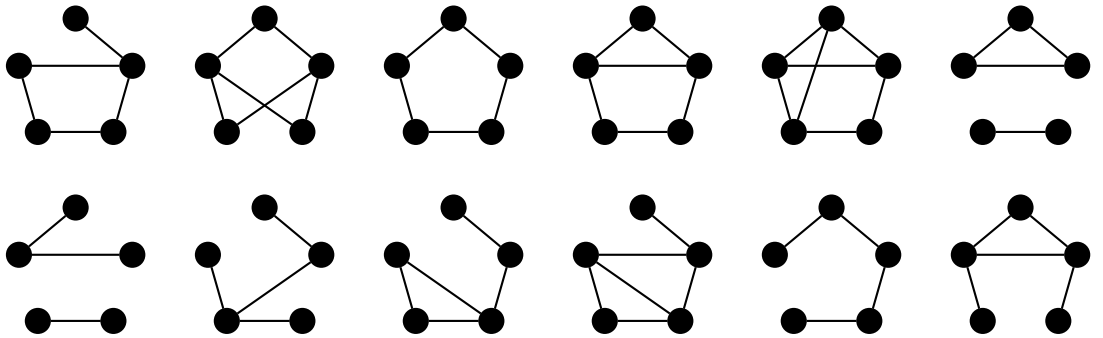


Figure 5.3: Isolate free graphs on five vertices satisfying $ir(G) = 2$

Suppose G is an isolate free graph on $n \geq 6$ vertices having a unique ir-set of cardinality 2, call it D. We see that D is either a dominating set, or it is not a dominating set. In the following two subsections, we consider each case.

5.4.1 D is not a dominating set

We first consider the case when D is not a dominating set. Observe that in this case, $ir(G) < \gamma(G)$, for if $\gamma(G) = 2$ also, then D is not a unique ir-set in G since every minimal dominating set is maximal irredundant.

We begin by considering an example. Let the graph F on $n \geq 7$ vertices be constructed as follows. Let $V(F) = \{x, y, x', y', w, z, v, b_1, b_2, \ldots, b_{n-7}\}$. Let $F\langle\{v, b_1, b_2, \ldots, b_{n-7}\}\rangle$ be complete. Additionally, let

$$\begin{aligned}
N(x) &= \{y, v, x', b_1, b_2, \ldots, b_{n-7}\} \\
N(y) &= \{x, v, y', b_1, b_2, \ldots, b_{n-7}\} \\
N(v) &= \{x, y, b_1, b_2, \ldots, b_{n-7}\} \\
N(x') &= \{x, w, y', b_1, b_2, \ldots, b_{n-7}\} \\
N(y') &= \{y, z, x', b_1, b_2, \ldots, b_{n-7}\} \\
N(w) &= \{x'\} \\
N(z) &= \{y'\}.
\end{aligned}$$

The case of $n = 8$ is illustrated below for convenience.

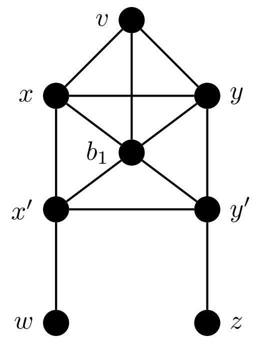


Figure 5.4: $n = 8$ case

We claim that $\{x, y\}$ is the unique ir-set of F.

Proof. First, note that $ir(F) \geq 2$ since $\Delta(F) < n-1$. Consider $\{x, y\}$. This set is irredundant, since x has x' as a private neighbor and y has y' as a private neighbor. Moreover, this set is maximal irredundant since the inclusion of w, z, x', y', or $b_1, b_2, \ldots, b_{n-7}$ eliminates the private neighbor of x or y while the inclusion of v results in v not having a private neighbor. Thus, $ir(F) = 2$ and $\{x, y\}$ is an ir-set. It remains to show that $\{x, y\}$ is the *only* maximal irredundant set of cardinality 2.

To see that no other two element subsets of $V(F)$ containing x are maximal irredundant, observe first that $\{x, v\}$ and $\{x, b_i\}$ for $1 \leq i \leq n-7$ are both redundant sets. Additionally, since the sets $\{x, x', z\}, \{x, y', w\}$, and $\{x, w, z\}$ are irredundant, we see that $\{x, x'\}, \{x, y'\}, \{x, w\}$ and $\{x, z\}$ are not *maximal* irredundant.

We leave it to the interested reader to verify, in a manner similar to the above, that any other two element subset of $V(F)$ distinct from $\{x, y\}$ is either redundant, or is contained in a larger irredundant set. We thus have that $\{x, y\}$ is the the unique ir-set of F. $\square$

Before proceeding, we note that $|E(F)| = \binom{n-2}{2} - 2$.

We now prove the following theorem through a sequence of claims.

Theorem 64. *Let $n \geq 7$. If G is a graph of order n such that $\delta(G) \geq 1$, $ir(G) = 2$, $\gamma(G) \geq 3$, and G has a unique ir-set D, then $|E(G)| \leq \binom{n-2}{2} - 2$. Furthermore, there exists such a G that has exactly $\binom{n-2}{2} - 2$ edges.*

Proof. Among all isolate free graphs on n vertices with domination number at least 3 and having a unique ir-set of cardinality 2, suppose that G has the maximum number of edges. Note that since

$\gamma(G) \geq 3$, $\Delta(G) \leq n-3$. Let $D = \{x, y\}$ denote the unique ir-set of G. Define the following sets.

- $D_x = N(x) - N[y]$
- $D_y = N(y) - N[x]$
- $D_{xy} = N(x) \cap N(y)$
- $R = V(G) - (N[x] \cup N[y])$

We note that R is the set of vertices not dominated by x or y. Since D is not a dominating set, we have that $|R| > 0$. This implies that $xy \in E(G)$, since if $xy \notin E(G)$, then $\{x, y, w\}$, where $w \in R$, is independent and is thus irredundant. The fact that $xy \in E(G)$ further implies that $D_x \neq \emptyset$ and that $D_y \neq \emptyset$.

We first consider the set R.

Claim 1. $|R| \geq 2$.

Proof of Claim: For the sake of contradiction, suppose $|R| = 1$ with $w \in R$. Since D is maximal irredundant, $\{x, y, w\}$ is redundant. Since w is a self-private neighbor with respect to $\{x, y, w\}$, we see that either w dominates D_x or w dominates D_y. Without loss of generality, assume w dominates D_x. In this case, $\{y, w\}$ is a dominating set of G, a contradiction. Hence, $|R| \geq 2$.

Claim 2. *Every vertex in R either dominates D_x or D_y.*

Proof of Claim: Suppose $w \in R$ does not dominate D_x or D_y. Since w is not dominated by D, we see that $\{x, y, w\}$ is irredundant, a contradiction to the maximality of D.

Claim 3. $\gamma(G\langle R \rangle) \geq 2$.

Proof of Claim: Suppose that $\gamma(G\langle R \rangle]) = 1$. Let $w \in R$ dominate $G\langle R \rangle$. By Claim 2, w dominates D_x or D_y. Without loss of generality, assume w dominates D_x. In this case, $\{w, y\}$ is a dominating set of G, a contradiction.

Claim 4. *There exists $w \in R$ that dominates D_x but not D_y and $z \in R$ that dominates D_y but not D_x.*

Proof of Claim: Without loss of generality, suppose every vertex in R dominates D_x. This implies every vertex in D_x dominates R. Consider $\gamma(G\langle D_x \rangle)$. First, assume that $\gamma(G\langle D_x \rangle) = 1$. In this case, if $x' \in D_x$ dominates $G\langle D_x \rangle$, then $\{x', y\}$ is a dominating set of G, a contradiction. Thus, $\gamma(G\langle D_x \rangle) \geq 2$. This, implies that $|D_x| > 1$. Let $x' \in D_x$ and $x'' \in D_x$ be non-adjacent. Consider x'. We have seen that x' dominates R. If x' also dominates D_y, then $\{x, x'\}$ dominates all of G, a contradiction. Thus, there exists $y' \in D_y$ such that $x'y' \notin E(G)$. However, we now see that $\{x, y\}$ is not maximal irredundant, since $\{x, x', y\}$ is irredundant (x has x'' as a private neighbor, y has y' as a private neighbor, and x' has any vertex in R as a private neighbor). Thus, our result follows.

Next, we consider the set D_{xy}.

Claim 5. *No vertex in D_{xy} dominates R.*

Proof of Claim: Suppose $v \in D_{xy}$ dominates R. Recall that $deg(v) \leq n - 3$. Since v is adjacent to x, y, and every vertex in R, this implies that there are at least two vertices in $D_x \cup D_y \cup D_{xy}$ that are not neighbors of v. Suppose the only vertices not adjacent to v are in D_x and D_{xy}. This implies that v dominates D_y in which case $\{x, v\}$ is a dominating set, a contradiction. By similar reasoning, the vertices not adjacent to v cannot lie in only D_y and D_{xy}, only D_x, only D_y, or only D_{xy}. Thus, there exists $x' \in D_x$ and $y' \in D_y$ for which $x'v \notin E(G)$ and $y'v \notin E(G)$. This, however, implies that $D = \{x, y\}$ is not maximal irredundant, since $\{x, y, v\}$ is irredundant, a contradiction.

Claim 6. *$D_{xy} \neq \emptyset$.*

Proof of Claim: Suppose that $D_{xy} = \emptyset$. If $\gamma(G\langle D_x \rangle) = \gamma(G\langle D_y \rangle) = 1$, with $x' \in D_x$ dominating D_x and $y' \in D_y$ dominating D_y, then $\{x', y'\}$ is a dominating set of G, a contradiction. Thus, either $\gamma(G\langle D_x \rangle) \geq 2$ or $\gamma(G\langle D_y \rangle) \geq 2$. Without loss of generality, assume that $\gamma(G\langle D_x \rangle) \geq 2$. Let $x' \in D_x$. If x' does not dominate D_y, then $\{x, y, x'\}$ is irredundant, a contradiction. Hence, we see that every vertex in D_x dominates D_y. This implies that every vertex in D_y dominates D_x as well. Hence, if $x' \in D_x$ and $y' \in D_y$, then $\{x', y'\}$ is a dominating set of G, a contradiction.

Corollary 65. *If G is an isolate free graph on six vertices having a unique ir-set of cardinality 2, then $\gamma(G) = 2$.*

Finally, we consider the sets D_x and D_y.

Claim 7. *If $v \in D_x$ or $v \in D_y$, then $deg(v) \leq n - 4$.*

Proof of Claim: Since $\Delta(G) \leq n-3$, we immediately know that if $x' \in D_x$, then $deg(x') \leq n-3$. For the sake of contradiction, suppose that $deg(x') = n-3$ and that x' is not adjacent to a and y. (We note that $x' \in D_x$ implies x' is not adjacent to y.)

- If $a \in D_{xy}$ or $a \in D_y$, then $\{x', y\}$ dominates G, a contradiction.
- If $a \in D_x$, then $\{x', x\}$ dominates G, a contradiction.
- If $a \in R$, then since every vertex in R either dominates D_x or D_y, we see that a dominates D_y. This however, implies that $\{x', y'\}$ dominates G for any $y' \in D_y$.

Hence, we have arrived at a contradiction. Our claim follows.

Claim 8. *If $x' \in D_x$, then x' either dominates D_x or D_y. If $y' \in D_y$, then y' either dominates D_x or D_y.*

Proof of Claim: If $x' \in D_x$ does not dominate D_x and does not dominate D_y, then $\{x, x', y\}$ is irredundant by Claim 4.

Corollary 66. *If $\gamma(G\langle D_x \rangle) > 1$ or $\gamma(G\langle D_y \rangle) > 1$, then each vertex in D_x dominates D_y and each vertex in D_y dominates D_x.*

Proof. Suppose $\gamma(G\langle D_x \rangle) > 1$. This implies that no vertex in D_x dominates D_x. Hence, by Claim 8, each vertex in D_x dominates D_y. This also implies that each vertex in D_y dominates D_x. The case of $\gamma(G\langle D_y \rangle) > 1$ follows similarly. □

Claim 9. *No vertex in D_x or D_y dominates D_{xy}.*

Proof of Claim: Suppose $x' \in D_x$ dominates D_{xy}. x' itself either dominates D_x or D_y by Claim 8. We consider each case.

First suppose x' dominates D_x. If there exists $y' \in D_y$ that dominates D_y, then $\{x', y'\}$ is a dominating set of G, a contradiction. Thus, no vertex in D_y dominates D_y. Hence, every vertex in D_y dominates D_x. This, however, implies that every vertex in D_x also dominates D_y. Hence, $\{x', y'\}$ is a dominating set of G for any $y' \in D_y$, a contradiction.

Suppose now that x' dominates D_y but not D_x. If there exists $y' \in D_y$ such that y' dominates D_x then $\{x', y'\}$ is a dominating set of G, a contradiction. Thus, no vertex in D_y dominates D_x, in which case every vertex in D_y dominates D_y. Let x'' be a vertex in D_x not dominated by x'.

If x'' does not dominate D_y, then $\{x, x'', y\}$ is irredundant, a contradiction. Thus, x'' dominates D_y. Since this is true for every vertex in D_x not dominated by x', we see that each vertex in D_y dominates the vertices in D_x not dominated by x'. This implies that $\{x', y'\}$ is a dominating set of G for any $y' \in D_y$.

Claim 10. *No vertex in D_x dominates R. No vertex in D_y dominates R.*

Proof of Claim: Suppose $x' \in D_x$ dominates R. If x' dominates D_x, then $\{x', y\}$ dominates G, a contradiction. Thus, x' dominates D_y. This, however, implies that $\{x, x'\}$ dominates G, again a contradiction. Hence, no vertex in D_x dominates R. By the same logic, no vertex in D_y dominates R.

Claim 11. *For each pair of vertices $\{x', y'\}$ such that $x' \in D_x$ and $y' \in D_y$, there exists a vertex $v \in D_{xy}$ not adjacent to either x' or y'.*

Proof of Claim: Suppose $x' \in D_x$, $y' \in D_y$, and that $D_{xy} \subseteq (N[x'] \cup N[y'])$. We consider several cases.

- If x' dominates D_x and y' dominates D_y, then $\{x', y'\}$ dominates G, a contradiction.
- If x' dominates D_y and y' dominates D_x, then $\{x', y'\}$ dominates G, a contradiction.
- Suppose both x' and y' dominate D_y but not D_x. In this case, consider $x'' \in (D_x - N[x'])$. If x'' does not dominate D_y, then $\{x, x'', y\}$ is irredundant, a contradiction. Thus, we see that each vertex in $D_x - N[x']$ dominates D_y, in which case every vertex in D_y dominates $D_x - N[x']$. Thus, once again we see that $\{x', y'\}$ dominates all of G, a contradiction.
- If both x' and y' dominate D_x but not D_y, then $\{x', y'\}$ will dominate all of G in a manner similar to the case above.

Our result follows.

Considering the results above, we see that G has one of the four graphs below as an induced subgraph.

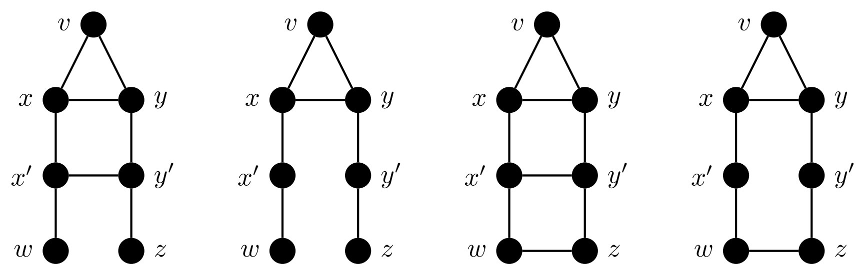


Figure 5.5: Induced Subgraphs

We now show that if $|R| \geq 3$, $|D_x| \geq 2$, or if $|D_y| \geq 2$, then $|E(G)| \leq |E(F)| = \binom{n-2}{2} - 2$, where F is the graph considered at the beginning of this section. We prove this by constructing a graph G' from G satisfying $|E(G)| \leq |E(G')|$ and $G' \subseteq F$.

Suppose that at least one of the following is true concerning G.

- $|R| \geq 3$
- $|D_x| \geq 2$
- $|D_y| \geq 2$

Find, and label, vertices w and z in R such that w dominates D_x but not D_y and such that z dominates D_y but not D_x. Next, find, and label, vertex x' in D_x that is not dominated by z and vertex y' in D_y that is not dominated by w. Finally, for the pair $\{x', y'\}$, find, and label, the vertex v in D_{xy} that is not dominated by x' or y'. Observe that v is not adjacent to w or z, since if v is adjacent to either w or z, then $\{x, y, v\}$ is irredundant. Define the following sets.

- $D_x^* = D_x - \{x'\}$
- $D_y^* = D_y - \{y'\}$
- $R^* = R - \{w, z\}$
- $D_{xy}^* = D_{xy} - \{v\}$

Note that if there are no edges between $\{w, z\}$ and D_x^*, D_y^*, D_{xy}^*, and R^*, then G is isomorphic to a subgraph of F above. Bearing this in mind, consider the following procedure.

Let G' be a distinct copy of G. We alter G' after considering G.

1. If $wz \notin E(G)$, continue to Step 2 below. If $wz \in E(G)$, then there exists $r \in R^*$ for which $wr \notin E(G)$ by Claim 3. In G and G', delete the edge wz and add the edge wr.

2. Consider D_x^* in G. If $|D_x^*| = 0$, continue to Step 3 below. Otherwise, for each $s \in D_x^*$ proceed as follows. Note that s is adjacent to w, but is not adjacent to y (by definition of D_x) and at least one vertex in R (by Claim 10), call it r_s. In G', delete the edge sw and add the edge sy. If s is adjacent to z, then in G' delete the edge sz and add the edge sr_s. Note that after the completion of Step 2, $|E(G)| = |E(G')|$ and $N_{G'}(\{w, z\}) \cap D_x^* = \emptyset$.

3. Consider D_y^* in G. If $|D_y^*| = 0$, continue to Step 4 below. Otherwise, for each $t \in D_y^*$, proceed as follows. Note that t is adjacent to z, but is not adjacent to x (by definition of D_y) and at least one vertex in R (by Claim 10), call it r_t. In G', delete the edge tz and add the edge tx. Additionally, if t is adjacent to w, in G' delete the edge tw and add the edge tr_t. Note that after the completion of Step 3, $|E(G)| = |E(G')|$ and $N_{G'}(\{w, z\}) \cap D_y^* = \emptyset$.

4. Consider R^* in G. If $|R^*| = 0$, continue to Step 5 below. Otherwise, for each $u \in R^*$, proceed as follows. Note that u is not adjacent to x, y, or v. In G', add the edges uv, ux, and uy. If u is adjacent to w in G, delete the edge uw from G'. If u is adjacent to z in G, delete the edge uz from G'. Note that after the completion of Step 4, $|E(G)| \leq |E(G')|$ and $N_{G'}(\{w, z\}) \cap R^* = \emptyset$.

5. Finally, consider D_{xy}^* in G. If $|D_{xy}^*| = 0$, then we are done. Otherwise, partition D_{xy}^* as follows.

   - Let S_{1A} denote the set of vertices p in D_{xy}^* which dominate all but one vertex in R, and which do not dominate $D_x \cup D_y \cup \{v\}$.
   - Let S_{1B} denote the set of vertices p in D_{xy}^* which dominate all but one vertex in R, and which dominate $D_x \cup D_y \cup \{v\}$.
   - Let S_2 denote the set of vertices p in D_{xy}^* which do not dominate two or more vertices in R.

   For each $p \in S_{1A}$, proceed as follows. Let o_p denote the vertex in $D_x \cup D_y \cup \{v\}$ that p is not adjacent to, and let r_p denote the vertex in R that p is not adjacent to. In G', delete the edges pw and pz (if they exist), add the edge po_p, and add the edge pr_p if and only if r_p is distinct from w and z.

For each $p \in S_{1B}$, proceed as follows. Let r_p denote the vertex in R that p is not adjacent to. First, observe that there is at least one vertex in $D_{xy} - N[p]$ since $deg(p) \leq n-3$. Next, note that $(D_{xy} - N[p]) \cap (S_{1A} \cup S_2) \neq \emptyset$, since otherwise $\{p, x'\}$ or $\{p, y'\}$ dominates G (depending upon whether r_p dominates D_x or D_y). Thus, let $o_p \in (D_{xy} - N[p]) \cap (S_{1A} \cup S_2)$. In G', delete the edges pw and pz (if they exist), add the edge po_p, and add the edge pr_p if and only if r_p is distinct from w and z.

For each $p \in S_2$, let r_1 and r_2 denote the vertices in R that p is not adjacent to. In G', delete the edges pw and pz (if they exist), and add the edges pr_1 and pr_2 if and only if r_1 and r_2 are distinct from w and z respectively.

Note that after the completion of Step 5, $|E(G)| = |E(G')|$ and that $N_{G'}(\{w,z\}) \cap D^*_{xy} = \emptyset$.

Upon completion of Step 5, we see that the graph G' is isomorphic to a subgraph of the graph F constructed above and that $|E(G)| \leq |E(G')|$. Hence, we have proven that if $|R| > 2$, $|D_x| > 1$, or $|D_y| > 1$, then $|E(G)| \leq \binom{n-2}{2} - 2$.

Suppose now that G satisfies $|R| = 2$, $|D_x| = 1$, $|D_y| = 1$ and $|D_{xy}| = n-6$. As before, find, and label, the vertices w, z, x', y' and v as before. Note that x' is not adjacent to z and that y' is not adjacent to w by Claim 10. Additionally, v is not adjacent to either w or z, since otherwise $\{x, y, v\}$ is irredundant. If no vertex in D_{xy} shares an edge with w or z, then G is a subgraph of F, in which case $|E(G)| \leq \binom{n-2}{2} - 2$. Thus, suppose there exists a vertex, call it p, in D_{xy} which is adjacent to a vertex in R. Note that $|N(p) \cap R| \leq 1$ by Claim 5. Create a copy G' of G, and proceed as follows.

1. For each vertex $p \in D_{xy}$, if p is adjacent to a vertex in R but is not adjacent to one of x', y', or v, delete the edge from p to R in G' and add the edge px', py', or pv as appropriate.

2. If there are still vertices in D_{xy} that share an edge with a vertex in R, proceed as follows. Suppose that $p \in D_{xy}$ is adjacent to a vertex in R, say z without loss of generality. Consider then x'. Note that x' does not dominate $V(G) - N[p]$ since other $\{x', p\}$ dominates G. Hence, there exists a vertex which neither x' nor p is adjacent to, call it c. Note that $c \neq z$. In G', exchange the pz edge for the pc edge. Since $x'c \notin E(G)$, c does not share an edge with a vertex in R.

Since the total number of edges is preserved in Step 1, and since the only edges added in Step 2 are from a vertex sharing an edge with a vertex in R to a vertex not sharing an edge with a vertex in

R, we see that $|E(G)| = |E(G')|$. Since G' is a subgraph of F, we see that $|E(G)| \leq \binom{n-2}{2} - 2$. We have thus proven our result. $\square$

5.4.2 D is a dominating set

Suppose now that G is an isolate free graph on n-vertices ($n \geq 6$) having a unique ir-set of cardinality 2, call it D, which is a dominating set. Since D is a dominating set, this implies that D is a γ-set, since if $\gamma(G) = 1$, then $ir(G) = 1$ as well, a contradiction. Moreover, D is a unique γ-set in G, since if G has a γ-set distinct from D, call it D', then D' is maximal irredundant, contradicting the uniqueness of D. Hence, by Theorem 62, $|E(G)| \leq \binom{n-2}{2}$. To see that this bound can be achieved, consider the following two constructions.

First, the following graph on six vertices, together with Corollary 65, shows that $m_{ir}(6, 2) = \binom{6-2}{2} = 6$.

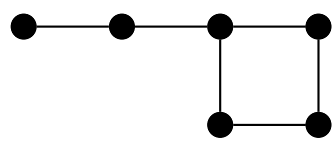


Figure 5.6: $n = 6$ case

For the $n \geq 7$ case, consider the following. Let

$$V(H) = \{x_1, x_2, \ldots, x_{n-5}, y_1, y_2, s, l, p\}.$$

Let $H\langle\{x_1, x_2, \ldots, x_{n-5}\}\rangle$ be complete. Additionally, let

$$\begin{aligned}
N(y_1) &= \{x_1, x_2, \ldots, x_{n-5}, s\} \\
N(y_2) &= \{x_1, x_2, \ldots, x_{n-5}, s, p\} \\
N(s) &= \{y_1, y_2\} \\
N(l) &= \{x_1\} \\
N(p) &= \{x_2, x_3, \ldots, x_{n-5}, y_2\}.
\end{aligned}$$

The case of $n = 7$ is illustrated in the figure below.

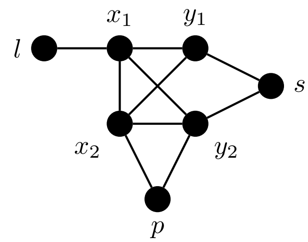


Figure 5.7: $n \geq 7$ case

Similar to the graph F considered in the previous section, the reader can verify that H has a unique ir-set of cardinality 2 given by $\{x_1, y_2\}$. Since $|E(H)| = \binom{n-2}{2}$, we have the following result.

Theorem 67. *For $n \geq 6$, $m_{ir}(n, 2) = \binom{n-2}{2}$.*

Additionally, by considering our work at the beginning of this section, we have the following.

Theorem 68. *For $n \geq 4$, $m^*_{ir}(n, 2) = \binom{n-1}{2} - \lceil \frac{n-2}{2} \rceil$.*

Before concluding this section, we note that not every graph G having a unique γ-set of cardinality 2 has a unique ir-set of cardinality 2, even when $ir(G) = \gamma(G) = 2$. For example, the graph P_6 has a unique γ-set of cardinality 2, but does not have a unique ir-set. Moreover, the set of isolate free graphs on n vertices having a unique ir-set of cardinality 2 and a maximum number of edges is a *proper* subset of the set of all isolate free graphs on n vertices having a unique γ-set of cardinality 2 and a maximum number of edges. For example, the graph below has a unique γ-set of cardinality 2 (and a maximum number of edges), but does not have a unique ir-set, even though $ir(G) = 2$.

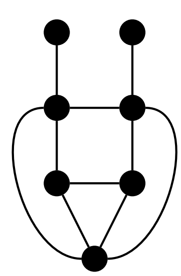


Figure 5.8: $\gamma(G) = ir(G) = 2$, unique γ-set, not a unique ir-set

5.5 Unique minimum independent dominating sets

In this section, we consider the maximum number of edges in a graph on n vertices having a unique i-set of cardinality 2. First, we note that no graph on one or three vertices has a unique i-set of cardinality 2. The only graph on two vertices having a unique i-set of cardinality two is the completely disconnected graph $\overline{K_2}$. Additionally, the only graph on four vertices which has a unique i-set of cardinality two is $P_3 \cup K_1$. Putting these trivial cases aside, we now restrict ourselves to graphs on at least five vertices. Let G be a graph on $n \geq 5$ vertices having a unique minimum independent dominating set of cardinality 2 and having a maximum number of edges. Let D denote the unique i-set of G, and for notational purposes, let $D = \{x, y\}$. Additionally, let $R = V(G) - D$. Partition R as follows.

- $D_x = N(x) - N(y)$
- $D_y = N(y) - N(x)$
- $D_{xy} = N(x) \cap N(y)$.

Consider a vertex $v \in R$. What is the maximum degree of v? We see that if $deg(v) = n-1$, then $\{v\}$ itself is an independent dominating set of cardinality one, a contradiction. If $deg(v) = n-2$, with v not adjacent to u, then $\{u, v\}$ is an independent dominating set of cardinality two distinct from D, a contradiction. Thus, $deg(v) \leq n-3$. Additionally, note that if $deg(v) = n-3$, then the two vertices not adjacent to v are not adjacent.

Using the familiar result that $|E(G)| = \frac{1}{2}\sum_{v \in V(G)} deg(v)$, we see that

$$\begin{aligned}
|E(G)| &\leq \frac{1}{2}\left(|D_x| + 2|D_{xy}| + |D_y| + (n-3)(n-2)\right) \\
&\leq \frac{1}{2}\left(2(n-2) + (n-3)(n-2)\right) \\
&= \frac{1}{2}\left((n-2)(n-1)\right) \\
&= \binom{n-1}{2}.
\end{aligned}$$

When $n = 3k+2$, $k \geq 1$, this upper bound on $|E(G)|$ can be achieved if $R = D_{xy}$ and $G\langle R\rangle$ is a complete graph minus the edges of k disjoint triangles (see Figure 5.9). Thus, we see that $m_i^*(3k+2, 2) = \binom{3k+1}{2}$.

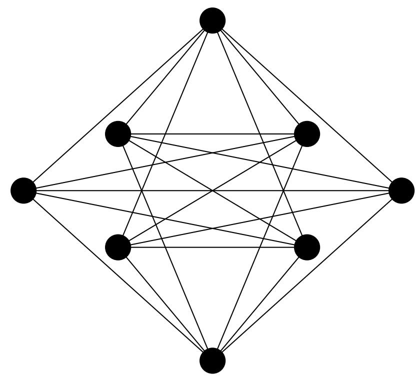


Figure 5.9: $m_i(8,2) = 21$

Suppose now that $n = 3k$ or $n = 3k+1$, for $k \geq 2$. Note that the upper bound on $|E(G)|$ found above is achievable if and only if $R = D_{xy}$ with each vertex $v \in R$ satisfying $deg(v) = n-3$. Thus, suppose that $R = D_{xy}$ and that each vertex in R has degree $n-3$. If $v \in R$ is not adjacent to vertices v_1 and v_2, by our observations above, $v_1v_2 \notin E(G)$. Since we are assuming $R = D_{xy}$, we see that $v_1 \in D_{xy}$ and $v_2 \in D_{xy}$. Hence, $deg(v_1) = n-3$ and $deg(v_2) = n-3$. Moreover, the two vertices not adjacent to v_1 are v and v_2, while the two vertices not adjacent to v_2 are v and v_1. Thus, each of v, v_1 and v_2 dominates $R - \{v, v_1, v_2\}$. Since the above logic can be applied to each vertex in R, R can be partitioned into sets of cardinality 3 such that each set S induces an independent set in G that dominates $R - S$. This, however, is clearly a contradiction since $|R| \not\equiv 0 \pmod 3$. Thus, if $n = 3k$ or $n = 3k+1$, we have $|E(G)| \leq \binom{n-1}{2} - 1$. This upper bound can be achieved in each case as follows.

If $n = 3k$, we let $R = D_{xy}$. Initially, we let $G\langle R\rangle$ be complete. After removing the edges of $k-1$ disjoint triangles from R, find the one remaining vertex in R of degree $n-1$ and make it not adjacent to any two other vertices in R. An example construction is illustrated in Figure 5.5.

If $n = 3k+1$, we once again let $R = D_{xy}$ and once again initially set $G\langle R\rangle$ to be complete. In this case, after removing the edges of $k-1$ disjoint triangles from R, we find the remaining two vertices in R of degree $n-1$, call them v_1 and v_2. We remove the edge v_1v_2 from G, and then pick an arbitrary vertex $v \in R$, ($v \neq v_1$ and $v \neq v_2$) and remove the edges vv_1 and vv_2. An example construction is illustrated in Figure 5.5.

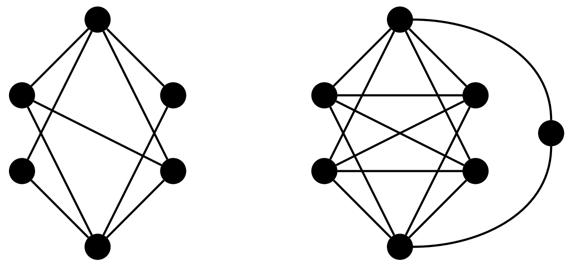


Figure 5.10: $m_i(6,2) = 9$ and $m_i(7) = 14$

Thus, we summarize our results as follows.

Theorem 69. *For $n = 2$, $n = 4$, or $n \geq 5$,*

$$m_i^*(n,2) = \begin{cases} \binom{n-1}{2} & \text{if } n \equiv 2 \pmod 3, \\ \binom{n-1}{2} - 1 & \text{otherwise.} \end{cases}$$

By observing that the graphs constructed in the $n \geq 5$ case are all isolate free, we also have the following.

Corollary 70. *For $n \geq 5$,*

$$m_i(n,2) = \begin{cases} \binom{n-1}{2} & \text{if } n \equiv 2 \pmod 3, \\ \binom{n-1}{2} - 1 & \text{otherwise.} \end{cases}$$

5.5.1 Unique minimum independent dominating sets of cardinalities greater than 2

Determining the exact value for $m_i^*(n,k)$ is more difficult when $k \geq 2$. However, if n satisfies certain modular congruences, then determining exact values is possible.

Let G be a graph having a unique i-set of cardinality $k \geq 2$, call it D. As above, let $R = V(G) - D$. Since D is independent, we see that the maximum degree of a vertex $v \in D$ is $n-k$. That is, v is not adjacent to itself and $k-1$ other vertices. By an argument similar to that above, we see that the maximum degree of a vertex $v \in R$ is $n-k-1$. Thus, if G has a unique i-set of cardinality k, then we have

$$\begin{aligned}|E(G)| &= \frac{1}{2}\sum_{v\in V(G)}\deg(v)\\ &= \frac{1}{2}\left(\sum_{v\in D}\deg(v)+\sum_{u\in R}\deg(u)\right)\\ &\le \frac{1}{2}\left(k\cdot(n-k)+(n-k)\cdot(n-k-1)\right)\\ &= \frac{(n-k)(n-1)}{2}.\end{aligned}$$

Consider this upper bound. In the case of $n = 8$ and $k = 3$, we see that $|E(G)| \le \frac{(8-3)(8-1)}{2} = \frac{35}{2}$, which is not an integer. Thus, unlike the case of $k = 2$ in the previous subsection, the trivially produced upper bound on $|E(G)|$ cannot always be achieved. However, we do have the following result.

Proposition 71. *Let $k \ge 2$ and let $n > k$. If $n \equiv k \pmod{k+1}$, then $m_i^*(n,k) = m_i(n,k) = \frac{(n-k)(n-1)}{2}$.*

Proof. By our observations above, we know that $m_i^*(n,k) \le \frac{(n-k)(n-1)}{2}$. We show that this upper bound can be achieved. Let $\alpha = \frac{n-k}{k+1}$. Since $n \equiv k \pmod{k+1}$, α is an integer. Let G be the complete multipartite graph on α partite sets each of cardinality $k+1$. That is, let

$$G = K_{\underbrace{k+1,k+1,\ldots,k+1}_{\alpha\text{ times}}}.$$

Note that

$$\begin{aligned}|E(G)| &= \binom{\alpha}{2}(k+1)^2\\ &= \frac{\alpha\cdot(\alpha-1)}{2}\cdot(k+1)^2\\ &= \frac{\frac{n-k}{k+1}\cdot\left(\frac{n-k}{k+1}-1\right)}{2}\cdot(k+1)^2\\ &= \frac{(n-k)\cdot(n-k-(k+1))}{2}\\ &= \frac{(n-k)\cdot(n-2k-1)}{2}.\end{aligned}$$

Let H be the *join* of G and $\overline{K_k}$. That is, $H = K_{k+1,k+1,\ldots,k+1} + \overline{K_k}$. Recall that H is formed from

G by adding k new vertices and making each new vertex adjacent to every vertex in $V(G)$. We see that these k new vertices form an independent dominating set of H of cardinality k. This set, in fact, is the unique i-set of H since any independent dominating set containing a vertex from $V(G)$ is necessarily of cardinality $k+1$. We see that

$$\begin{aligned} |E(H)| &= |E(G)| + k(n-k) \\ &= \frac{(n-k)\cdot(n-2k-1)}{2} + k(n-k) \\ &= (n-k)\cdot\left(\frac{n-2k-1}{2} + k\right) \\ &= (n-k)\cdot\left(\frac{n-2k-1}{2} + \frac{2k}{2}\right) \\ &= \frac{(n-k)(n-1)}{2}. \end{aligned}$$

Since H is isolate free when $n > k$, our result is shown. □

5.6 Unique maximum independent, maximum minimal dominating, and maximum irredundant sets

In this section, we consider unique β_0-sets, Γ-sets, and IR-sets of cardinalities at least 2. We begin with unique β_0-sets.

Theorem 72. *Let $k \geq 2$. For $n \geq k$, $m^*_{\beta_0}(n,k) = \binom{n}{2} - \binom{k}{2}$.*

Proof. Let $k \geq 2$, and let G be a graph on $n \geq k$ vertices having a unique β_0-set of cardinality k. First, observe that since G contains k mutually non-adjacent vertices, $|E(G)| \leq \binom{n}{2} - \binom{k}{2}$. Since the graph $K_n - E(K_k)$ has a unique β_0-set of cardinality k and has $\binom{n}{2} - \binom{k}{2}$ edges, our result follows. □

Before proceeding, note that if $n > k$, then $K_n - E(K_k)$ is isolate free in which case $m_{\beta_0}(n,k) = \binom{n}{2} - \binom{k}{2}$ as well.

As it turns out, unique Γ-sets and unique IR-sets of cardinality k can be handled similarly.

Theorem 73. *Let $k \geq 2$. For $n \geq k$, $m^*_{\Gamma}(n,k) = m^*_{IR}(n,k) = \binom{n}{2} - \binom{k}{2}$.*

Proof. Let $k \geq 2$, and let G be a graph on $n \geq k$ vertices having a unique Γ-set or a unique IR-set of cardinality k, call it D. In either case, we have that D is maximal irredundant.

Partition D into two subsets S_1 and S_2 such that every vertex in S_1 has an external private neighbor, while every vertex in S_2 does not have an external private neighbor. If $S_1 = \emptyset$, then D is an independent set in which case $|E(G)| \leq \binom{n}{2} - \binom{k}{2}$ as illustrated in the proof of Theorem 72.

Thus, suppose that $S_1 \neq \emptyset$. Note that for each vertex v in S_1, there is a vertex $u_v \in V(G) - D$ satisfying $N[u_v] \cap D = \{v\}$. We construct a graph G' from G as follows. For each $v \in S_1$, let the vertices in $N(v) \cap D$ be $w_1, w_2, \ldots, w_r$. Delete the edge vw_i and add the edges $w_i u_v$ and vu_{w_i} for each $1 \leq i \leq r$. If v has no neighbors in S_1, no edges need be deleted or added. Once this has been completed, we see that D forms an independent set in G'. Since $|E(G')| > |E(G)|$, we see that $|E(G)| < \binom{n}{2} - \binom{n}{k}$.

The graph $K_n - E(K_k)$ satisfies $\Gamma(K_n - E(K_k)) = IR(K_n - E(K_k)) = k$ and has a unique Γ-set and a unique IR-set. Thus, our result follows. $\square$

Once again, if $n > k$, then we have $m_\Gamma(n,k) = m_{IR}(n,k) = \binom{n}{2} - \binom{k}{2}$.

By combining Proposition 63 and Theorems 62, 67, 68, 69, 72, and 73, we have our main result.

Theorem. *For $n \geq 6$*

$$m_{ir}(n,2) = m_\gamma(n,2) \leq m_i(n,2) \leq m_{\beta_0}(n,2) = m_\Gamma(n,2) = m_{IR}(n,2)$$

and

$$m^*_{ir}(n,2) = m^*_\gamma(n,2) \leq m^*_i(n,2) \leq m^*_{\beta_0}(n,2) = m^*_\Gamma(n,2) = m^*_{IR}(n,2).$$

5.7 Open Problems

Before concluding this chapter, we pose a few open problems.

- What are the values for $m_{ir}(n,k)$ and $m^*_{ir}(n,k)$ for $k \geq 3$?
- What are the values for $m_i(n,k)$ and $m^*_i(n,k)$ for $k \geq 3$ not handled in our result above?

- For $k \geq 3$ and n sufficiently large, does the following inequality hold?

$$m_{ir}(n,k) \leq m_{\gamma}(n,k) \leq m_i^*(n,k) \leq m_{\beta_0}^*(n,k) \leq m_{\Gamma}^*(n,k) \leq m_{IR}^*(n,k)$$

Chapter 6

Identifying codes in some Cartesian product graphs

6.1 Introduction

While discussing domination in Chapter 1, we saw that there are many different types of domination. In this concluding chapter, we consider a special kind of dominating set called an identifying code. The study of identifying codes in graphs began with Chakrabarty, Karpovsky, and Levitin in [29] in 1998. Since then, identifying codes have been studied in many different classes of graphs, including product graphs. For example, identifying codes in the direct product were studied by Rall and Wash in [34]. Identifying codes in lexicographic product graphs were studied by Feng and Wang in [4], and while not strictly a graph product, identifying codes have also been studied in corona graphs by Feng, Xu, and Wang in [3]. Our interest will be in the Cartesian product, where, again, much work has been done (see [29], [31], [33], and [30]). Of particular interest to our work, Gravier, Moncel, and Semri considered identifying codes in the Cartesian product of two complete graphs of the same size in [16]. This work was later expanded upon to consider identifying codes in the Cartesian product of two arbitrary complete graphs by Goddard and Wash in [14].

In this chapter, we study identifying codes in the Cartesian product of a complete graph and a path. We first characterize identifying codes in such graphs, providing a set of conditions which can be used to determine whether a given set in such a graph is an identifying code. Using these conditions, we then determine the minimum cardinality of an identifying code in $K_n \,\Box\, P_m$ for all $m \geq 3$ when $n = 3$ and when $n \geq 5$ by constructing codes whose cardinalities attain a provable lower bound.

For more on identifying codes in graphs, we direct the reader to the excellent bibliography maintained by Lobstein [32]. As in previous chapters, we note here that the work in this chapter is a discussion of the work in [22]. The results and proofs in this chapter are assumed to be from this paper unless otherwise stated.

6.2 Definitions and Notation

If A is a subset of $V(G)$, we call $N[x] \cap A$ the set of *identifiers* of x with respect to A, or simply the set of identifiers of x if A is understood from context. The set A is said to *separate* vertices y and z in $V(G)$ if the sets of identifiers for y and z are distinct. In other words, A separates y and z if there exists some $x \in A$ for which $x \in N[y] \cap A$ but $x \notin N[z] \cap A$ or for which $x \in N[z] \cap A$ but $x \notin N[y] \cap A$.

For example, consider the graph in Figure 6.1. Let A be given by the set $A = \{a, c, f\}$ (the white vertices). The set of identifiers for b is $N[b] \cap A = \{a, c\}$ while the set of identifiers for c is $\{c\}$. We thus see that b and c are separated by A since $N[b] \cap A \neq N[c] \cap A$. A, however, does not separate e and g since $N[e] \cap A = \{f\} = N[g] \cap A$.

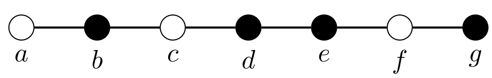


Figure 6.1: Identifiers and separating sets

Any subset C of $V(G)$ which is a dominating set that separates all distinct pairs of vertices is called an *identifying code*. Note that if a distinct pair of vertices, say x and y, satisfy $N[x] = N[y]$, then G contains no identifying codes. However, if G is twin-free, then G contains an identifying code, namely $V(G)$ itself. An identifying code of G of minimum cardinality is called a minimum identifying code, and its cardinality is denoted by $\gamma^{ID}(G)$. A minimum identifying code will henceforth be referred to as a γ^{ID}-set. Vertices in an identifying code C are commonly referred to as *codewords*.

Considering once again the graph from Figure 6.1, we see that $A = \{a, c, f\}$ is not an identifying code. While A is dominating, it does not separate each pair of distinct vertices. The set $C = \{a, c, e, g\}$, however, is an identifying code, since each vertex has a *unique* set of identifiers.

- $N[a] \cap C = \{a\}$
- $N[b] \cap C = \{a, c\}$
- $N[c] \cap C = \{c\}$
- $N[d] \cap C = \{c, e\}$
- $N[e] \cap C = \{c\}$
- $N[f] \cap C = \{e, g\}$
- $N[g] \cap C = \{g\}$.

In our work to follow, we consider minimum identifying codes in $K_n \,\Box\, P_m$ where, once again, K_n denotes the complete graph on n vertices and P_m denotes the path on m vertices. Unless stated otherwise, we assume that $n \geq 3$ and that $m \geq 3$. As in Chapter 1, we define $V(K_n) = \{1, 2, \ldots, n\}$ and $V(P_m) = \{1, 2, \ldots, m\}$ for ease. For convenience, we denote the K_n-layer through (i, y) by K^y. That is, K^y is the subgraph of $K_n \,\Box\, P_m$ induced by $V(K_n) \times \{y\}$. The two K_n-layers K^i and K^j are adjacent if $ij \in E(P_m)$, and are non-adjacent otherwise. Finally, if $C \subseteq V(K_n \,\Box\, P_m)$, we define the function $f_C : V(P_m) \to \{0, 1, \ldots, n\}$ by $f_C(i) = |C \cap V(K^i)|$. For other graph product terminology, we follow [18].

6.3 Properties

Suppose that $C \subseteq V(K_n \,\Box\, P_m)$ is a dominating set. To determine whether C is an identifying code, we need to determine whether $N[u] \cap C \neq N[v] \cap C$ for all distinct vertices u and v in $V(K_n \Box P_m)$. For $1 \leq i \leq m$, define the set $B_i(C)$ as follows.

$$B_i(C) = \begin{cases} \{v \,:\, C \cap \{(v, 2)\} \neq \emptyset\} & \text{if } i = 1, \\ \{v \,:\, C \cap \{(v, i-1), (v, i+1)\} \neq \emptyset\} & \text{if } 2 \leq i \leq m-1, \\ \{v \,:\, C \cap \{(v, m-1)\} \neq \emptyset\} & \text{if } i = m. \end{cases}$$

We have the following result.

Proposition 74. *If C is an identifying code for $K_n \,\square\, P_m$, then $|B_i(C)| \geq n-1$ for $1 \leq i \leq m$.*

Proof. For the sake of contradiction, suppose that $|B_i(C)| \leq n-2$ for some i. Let $\{u, v\} \cap B_i(C) = \emptyset$. In this case, we see that $N[(v, i)] \cap C = C \cap V(K^i) = N[(u, i)] \cap C$, in which case C is not an identifying code. Our result follows. $\square$

Corollary 75. *If C is an identifying code for $K_n \,\square\, P_m$, then $f_C(2) \geq n-1$, $f_C(m-1) \geq n-1$, and $f_C(i) + f_C(i+2) \geq n-1$ for $1 \leq i \leq m-2$.*

After factoring in that C is a dominating set, we also have the following.

Corollary 76. *If C is an identifying code for $K_n \,\square\, P_m$, then $f_C(1) + f_C(2) \geq n$, $f_C(m-1) + f_C(m) \geq n$, and $f_C(i) + f_C(i+1) + f_C(i+2) \geq n$ for $1 \leq i \leq m-2$.*

These two corollaries, will be used extensively in our identifying code constructions in Section 6.4 to come.

Proposition 77. *If $C \subseteq V(K_n \,\square\, P_m)$ is a dominating set satisfying $|B_i(C)| \geq n-1$ for $1 \leq i \leq m$, then every pair of vertices that both belong to the same K_n-layer or which respectively belong to non-adjacent K_n-layers are separated by C.*

Proof. First, note that since C is dominating, for all $v \in V(K_n \,\square\, P_m)$, $N[v] \cap C \neq \emptyset$. Next, note that for all distinct vertices (u, i) and (v, i) in $V(K^i)$, $N[(u, i)] \cap C \neq N[(v, i)] \cap C$ since $|B_i(C)| \geq n-1$. Thus, each pair of distinct vertices within a given K_n-layer is separated by C.

For the sake of contradiction, suppose that $N[(u, i)] \cap C = N[(v, i+j)] \cap C$ for some $j \geq 2$. If $j \geq 3$, then $N[(u, i)] \cap N[(v, i+j)] = \emptyset$, a contradiction. By the same reasoning, we see that if $j = 2$, then $u = v$. However, if $N[(u, i)] \cap C = N[(u, i+2)] \cap C$, then $N[(u, i)] \cap C = \{(u, i+1)\}$ and $f_C(i) = f_C(i+2) = 0$. This, however, implies that $|B_{i+1}(C)| = 0$, contradicting our earlier assumption. $\square$

The following proposition gives a sufficient condition for vertices in *adjacent* K_n-layers to be separated from each other.

Proposition 78. *Let $C \subseteq V(K_n \,\square\, P_m)$. If $f_C(i) \geq 2$, then any vertex in $V(K^i)$ is separated from any vertex in $V(K_n \,\square\, P_m) - V(K^i)$ by C.*

Proof. If $f_C(i) \geq 2$, then each vertex of K^i has at least two vertices from K^i in its set of identifiers. Any vertex outside of K^i contains at most one vertex from K^i in its set of identifiers. Our result follows. $\square$

We have now proven the following proposition, which will be critically important in Section 6.4 to follow.

Proposition 79. *Let C be a dominating set of $K_n \,\square\, P_m$ for which $|B_i(C)| \geq n-1$ for $1 \leq i \leq m$. If each K_n-layer either contains at least two vertices in C or is only adjacent to K_n-layers containing at least two vertices in C, then C is an identifying code.*

In Section 6.5, we consider identifying codes for which adjacent K_n-layers each contain a single vertex in the code. Thus, assume now that $C \subseteq V(K_n \,\square\, P_m)$ is a dominating set, satisfies $|B_i(C)| \geq n-1$ for $1 \leq i \leq m$, and that $f_C(i) = f_C(i+1) = 1$ for some i. Note that if $i \in \{1, 2, m-1, m-2\}$ or if $m \leq 5$, then C is not an identifying code by Corollary 75. Thus, assume that $m \geq 6$ and that $3 \leq i \leq m-3$. Without loss of generality, assume that $(1, i) \in C$. We consider two cases.

1. Suppose that $(1, i+1) \in C$. In this case, $(1, i)$ and $(1, i+1)$ are separated if and only if $\{(1, i-1), (1, i+2)\} \cap C \neq \emptyset$. This case illustrates that if C contains an isolated edge, then C is not an identifying code.

2. Suppose that $(j, i+1) \in C$ for $j \neq 1$. In this case, (j, i) and $(1, i+1)$ are separated if and only if $\{(j, i-1), (1, i+2)\} \cap C \neq \emptyset$.

These two cases will be used frequently in the proof of Theorem 87.

6.4 Code Constructions for $n \geq 5$

In this section, we determine $\gamma^{ID}(K_n \,\square\, P_m)$ when $n \geq 5$ and $m \geq 3$. We first need the following lemma, which actually holds so long as $n \geq 3$.

Lemma 80. *Let $C \subseteq V(K_n \,\square\, P_m)$ where $n \geq 3, m \geq 4$. If $f_C(i) = n-2, f_C(i+1) = 1, f_C(i+2) = 1$, and $f_C(i+3) = n-2$ for some i, then C is not an identifying code.*

Proof. Suppose that $(j, i+1), (k, i+2) \in C$. Since $f_C(i) = n-2$ and $f_C(i+2) = 1$, in order for $|B_{i+1}(C)| \geq n-1$, we see that $(k, i) \notin C$. Similarly, since $f_C(i+1) = 1$ and $f_C(i+3) = n-2$, we see

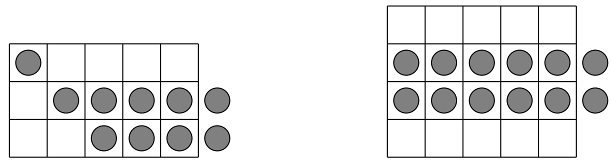


Figure 6.2: γ^{ID}-set for $K_n \square P_3$ and $K_n \square P_4$ respectively

that $(j, i{+}3) \notin C$. If $j = k$, then $N[(j,i{+}1)] \cap C = \{(j,i{+}1),(j,i{+}2)\} = N[(j,i{+}2)] \cap C$, in which case C is not an identifying code. If $j \neq k$, then $N[(k,i{+}1)] \cap C = \{(j,i{+}1),(k,i{+}2)\} = N[(j,i{+}2)] \cap C$, in which case C is not an identifying code. □

We now begin our construction of minimum identifying codes. For ease, we will depict our code constructions using a grid in which the K_n-layers are represented by rows, and the P_m-layers are represented by columns. If vertex (i,j) is included in the constructed code, then a circle will appear in the ith cell of the jth row. For conciseness, we depict our codes in the case of $K_5 \square P_m$. However, in each case, we designate how a code for $K_n \square P_m$ with $n \geq 6$ can be constructed. In particular, when a column is appended on the right of the grid, the circles appearing outside of the grid indicate which vertices in that new column to include.

For each proof in the remainder of this section, we assume that we have been given a γ^{ID}-set and its corresponding function f.

Proposition 81. *For $n \geq 3$, $\gamma^{ID}(K_n \square P_3) = 2n - 2$.*

Proof. By Corollary 75, we see that $f(1) + f(3) \geq n - 1$. Additionally, by Corollary 75, we see that $f(2) \geq n - 1$. Hence, $\gamma^{ID}(K_n \square P_3) \geq 2n - 2$. Since

$$\{(1,1),(2,2),(3,2),\ldots,(n,2),(3,3),(4,3),\ldots,(n,3)\}$$

is an identifying code (see Figure 6.2), the result follows. □

Proposition 82. *For $n \geq 3$, $\gamma^{ID}(K_n \square P_4) = 2n$.*

Proof. By Corollary 76, $f(1) + f(2) \geq n$ and $f(3) + f(4) \geq n$. Hence, $\gamma^{ID}(K_n \square P_4) \geq 2n$. As $\{1,2,\ldots,n\} \times \{2,3\}$ (see Figure 6.2) is an identifying code, the result follows. □

Proposition 83. *For $n \geq 5$ and $k \geq 1$, $\gamma^{ID}(K_n \square P_{4k+1}) = (2k+1)(n-1) + 1$.*

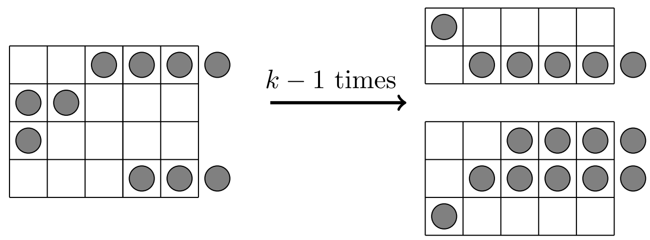


Figure 6.3: γ^{ID}-set for $K_n \,\square\, P_{4k+1}$, $k \geq 1$

Proof. Let $m = 4k+1$. By Corollary 76, we see that $f(1)+f(2) \geq n$. Additionally, by Corollary 75, $f(m-2)+f(m) \geq n-1$ and $f(m-1) \geq n-1$. Each consecutive four K_n-layer collection between the K^2 and K^{m-2}-layers includes at least $2n-2$ vertices from any γ^{ID}-set according to Corollary 75. Hence, we see that

$$\gamma^{ID}(K_n \,\square\, P_{4k+1}) \geq 3n-2+(k-1)\cdot 2(n-1) = (2k+1)(n-1)+1.$$

The construction illustrated in Figure 6.3 gives a set which satisfies this lower bound as well as Proposition 79. Thus, $\gamma^{ID}(K_n \,\square\, P_{4k+1}) = (2k+1)(n-1)+1$. We note that the four layer collection shown on the left can be inserted into the figure to the right and repeated (one on top of the other) $k-1$ times. $\square$

Proposition 84. *For $n \geq 5$ and $k \geq 1$, $\gamma^{ID}(K_n \,\square\, P_{4k+2}) = (2k+2)(n-1)$.*

Proof. Let $m = 4k+2$. Corollary 75 implies that $f(1)+f(3) \geq n-1$, $f(2) \geq n-1$, $f(m-2)+f(m) \geq n-1$, and $f(m-1) \geq n-1$. In addition, every γ^{ID}-set contains at least $2n-2$ vertices from each consecutive four K_n-layer collection between K^3 and K^{m-2}. Hence, $\gamma^{ID}(K_n \,\square\, P_{4k+2}) \geq 4(n-1)+(k-1)\cdot 2(n-1) = (2k+2)(n-1)$. The construction in Figure 6.4 satisfies Proposition 79 and satisfies this lower bound. Hence our result is shown. $\square$

Proposition 85. *For $n \geq 5$ and $k \geq 1$,*

$$\gamma^{ID}(K_n \,\square\, P_{4k+3}) = \begin{cases} (k+1)\cdot 2(n-1) & \text{if } n \geq k+3, \\ (k+1)\cdot 2(n-1)+1 & \text{otherwise.} \end{cases}$$

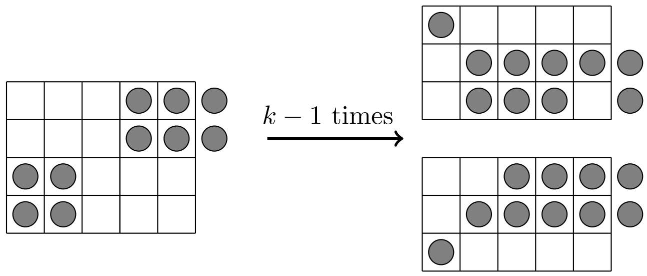


Figure 6.4: γ^{ID}-set for $K_n \,\square\, P_{4k+2}$, $k \geq 1$

Proof. First, suppose that $n \geq k+3$ and let C be any γ^{ID}-set of $K_n \,\square\, P_{4k+3}$. Among any four consecutive K_n-layers, at least $2(n-1)$ vertices are included in C. Additionally, $|C \cap (V(K^{4k+1}) \cup V(K^{4k+2}) \cup V(K^{4k+3}))| \geq 2(n-1)$. Thus, we find that $\gamma^{ID}(K_n \,\square\, P_{4k+3}) \geq (k+1) \cdot 2(n-1)$. This bound can be shown to be exact under the following construction. When considering K^j:

- If j is congruent to 1 modulo 4, add $(1,j),(2,j),\ldots,(p+1,j)$ where $p = \lfloor \frac{j}{4} \rfloor$.
- If j is congruent to 2 modulo 4, add $(1,j),(2,j),\ldots,(n-1,j)$.
- If j is congruent to 3 modulo 4, add $(n,j),(n-1,j),\ldots,(p+3,j)$ where $p = \lfloor \frac{j}{4} \rfloor$.
- If j is congruent to 0 modulo 4, do not add any vertices.

This construction produces a set satisfying the given lower bound as well as Proposition 79. Hence, the first part of our result is shown.

Now suppose that $n < k+3$, and let C be an identifying code of $K_n \,\square\, P_{4k+3}$. Let $f = f_C$ be as defined in Section 6.2. Consider the following system of inequalities:

$$\begin{aligned} f(s) + f(s+2) &\geq n-1 \quad (s \equiv 1 \pmod 4, 1 \leq s \leq 4k+1) \\ f(t) + f(t+2) &\geq n-1 \quad (t \equiv 2 \pmod 4, 1 \leq t \leq 4k-2) \\ f(4k+2) &\geq n-1. \end{aligned}$$

By Corollary 75, each of these inequalities holds. Thus, once again, we see that $\gamma^{ID}(K_n \,\square\, P_{4k+3}) \geq (k+1) \cdot 2(n-1)$. We claim that at least one of these inequalities is a strict inequality.

To see this, suppose not. First, consider $f(2)$. By Corollary 75, $f(2) \geq n-1$. Since $f(2) = n$ would give us $f(2) + f(4) > n-1$, we see that $f(2) = n-1$. This implies that $f(1) \geq 1$ since C is a

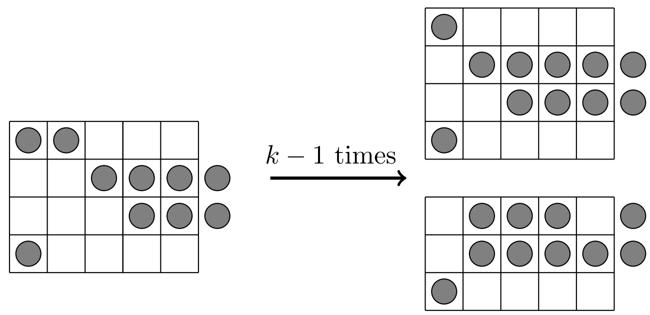


Figure 6.5: γ^{ID}-set for $K_n \,\square\, P_{4k+3}$, $k \geq 1$, $n < k+3$

dominating set, that $f(4) = 0$ by our equalities above, and that the vertices selected in K^3 and K^5 dominate all of the vertices in K^4. Since $f(1)+f(3) = n-1$ with $f(1) \geq 1$, we see that $f(3) \leq n-2$. Hence, we see that $f(5) \geq 2$. Finally, since $f(4) = 0$, we have $f(6) \geq n-1$ by Corollary 75. As above, this implies that both $f(6) = n-1$ and that $f(8) = 0$.

By continuing the logic applied above, we see that, in general, $f(4j+1) \geq j+1$ for $1 \leq j \leq k$. Thus, since $k \geq n-2$, this implies that $f(4k+1) \geq n-1$. However, this implies that $f(4k+3) = 0$. If $f(4k+3) = 0$, then $f(4k+2) = n$ in order for C to be a dominating set. This, however, contradicts our assumption that $f(4k+2) = n-1$. Thus, if $n < k+3$, at least one of the inequalities mentioned above is strict. If we let the $f(2)+f(4) \geq n-1$ inequality be strict, we can achieve the construction in Figure 6.5 letting all of the other inequalities hold at equality. Our result is shown. $\square$

Proposition 86. *For $n \geq 5$ and $k \geq 2$, $\gamma^{ID}(K_n \,\square\, P_{4k}) = 2k(n-1)+3$.*

Proof. Let $m = 4k$, let C be an identifying code for $K_n \,\square\, P_m$, and let $f = f_C$ be defined as in Section 2. Consider the inequalities

$$\begin{aligned} f(1)+f(2) &\geq n \\ f(s)+f(s+2) &\geq n-1 \quad (s \equiv 3 \pmod 4, 3 \leq s \leq 4k-5) \\ f(t)+f(t+2) &\geq n-1 \quad (t \equiv 0 \pmod 4, 4 \leq t \leq 4k-4) \\ f(4k-1)+f(4k) &\geq n. \end{aligned}$$

By Corollaries 75 and 76, each of these inequalities holds, implying that $\gamma^{ID}(K_n \,\square\, P_{4k}) \geq 2k(n-1)+2$. We claim that at least one of these inequalities is a strict inequality.

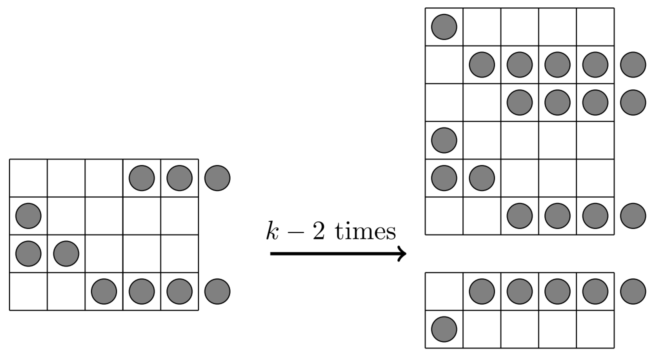


Figure 6.6: γ^{ID}-set for $K_n \,\square\, P_{4k}$, $k \geq 2$

For the sake of contradiction, suppose each inequality above holds at equality. First, note that $f(2) \geq n-1$ by Corollary 75. Thus, by our equations above, we have that $f(1) \leq 1$. By Corollary 75, this implies that $f(3) \geq n-2$. Observe that if $f(3) = n-1$, then by our equations above, $f(5) = 0$. This, and the fact that C is a dominating set, implies that $f(4) + f(6) = n$, which contradicts our assumption that each inequality holds at equality. Thus, we have $f(3) = n-2$, $f(1) = 1$, and $f(2) = n-1$. By the same logic, we have that $f(4k-2) = n-2, f(4k) = 1$, and $f(4k-1) = n-1$.

Once these six values are determined, we see that all of the remaining values for f are also determined. That is,

- If $s \equiv 3 \pmod 4$ with $3 \leq s \leq 4k-5$, then $f(s) = n-2$.
- If $s \equiv 1 \pmod 4$ with $5 \leq s \leq 4k-3$, then $f(s) = 1$.
- If $s \equiv 2 \pmod 4$ with $6 \leq s \leq 4k-2$, then $f(s) = n-2$.
- If $s \equiv 0 \pmod 4$ with $4 \leq s \leq 4k-4$, then $f(s) = 1$.

This, however, implies that $f(3) = n-2, f(4) = 1, f(5) = 1$, and that $f(6) = n-2$. This contradicts Lemma 80. Hence, we see that at least one of our inequalities cannot hold at equality.

A code can be constructed in which only one inequality is strict. This code, illustrated in Figure 6.6, shows that $\gamma^{ID}(K_n \,\square\, P_{4k}) = 2k(n-1) + 3$. $\square$

6.5 $K_3 \square P_m$

Given an identifying code C for $K_3 \square P_m$, it is possible for $f_C(i) = 0$ for some i. We, however, wish to first show that for $m \geq 3$, $K_3 \square P_m$ has a minimum identifying code C for which $f_C(i) > 0$ for $1 \leq i \leq m$. For conciseness, if $f_C(i) = 0$, we say that K^i is *empty* with respect to C.

Theorem 87. *For $m \geq 3$, $K_3 \square P_m$ has a minimum identifying code C for which $f_C(i) > 0$ for $1 \leq i \leq m$.*

Proof. We first consider $3 \leq m \leq 5$. In these cases, the sets

$$\{(1,1),(2,2),(3,2),(3,3)\}$$

$$\{(1,1),(2,2),(3,2),(1,3),(2,3),(3,4)\}$$

$$\{(3,1),(2,2),(3,2),(1,3),(1,4),(2,4),(3,5)\}$$

are minimum identifying codes for $K_3 \square P_3$, $K_3 \square P_4$, and $K_3 \square P_5$ respectively satisfying the condition in the statement of the theorem. Thus, we now restrict ourselves to the family $K_3 \square P_m$ with $m \geq 6$.

Among all minimum identifying codes for $K_3 \square P_m$, let C be such that the cardinality of $\{i : f_C(i) = 0\}$ is a minimum. If $f_C(i) > 0$ for $1 \leq i \leq m$, then we are done. Thus, for the sake of contradiction, suppose that $f_C(i) = 0$ for some i. For notational ease, we denote f_C by f for the remainder of the proof.

First, suppose that for some i, $f(i) = 0$ and $f(i+1) = 0$. By Corollary 75, $f(2) > 0$ and $f(m-1) > 0$. Thus, we have $3 \leq i \leq m-3$. Since C is a dominating set, $f(i-1) = 3$ and $f(i+2) = 3$. Additionally, by Corollary 75, C satisfies $f(i-2) \geq 2$ and $f(i+3) \geq 2$. However, in this case, the set $(C - \{(1,i-1),(1,i+2)\}) \cup \{(2,i),(2,i+1)\}$ is a minimum identifying code with fewer empty K_3 layers, a contradiction. Thus, from this point on, we assume that no adjacent K_3 layers are both empty with respect to C.

Consider the smallest i for which $f(i) = 0$.

- First, suppose $i = 1$. Since C is a dominating set, $f(2) = 3$. Additionally, by Corollary 75, $f(3) \geq 2$. In this case, $(C - \{(1,2)\}) \cup \{(2,1)\}$ is a γ^{ID}-set with fewer empty K_3 layers, a contradiction. By similar logic, if $f(m) = 0$, then an alternative γ^{ID}-set can be found with fewer empty K_3 layers. Thus, we assume that $f(1) \neq 0$ and that $f(m) \neq 0$.

- By Corollary 75, we see that $i = 2$ is not possible. Similarly, $i = m - 1$ is not possible.

- Next, suppose that $i = 3$. By Corollary 75, $f(1) \geq 2$, $f(2) \geq 2$, and $f(5) \geq 2$. Additionally, note that $f(4) > 0$ since we are assuming that no adjacent K_3 layers are both empty with respect to C.

  Suppose that for some k, $(k, 4) \in C$ and $(k, 5) \in C$. In this case, let $C' = C - (V(K^1) \cup V(K^2))$, and let $j \in \{1, 2, 3\}$ such that $j \neq k$. The set C'' defined by $C'' = C' \cup \{(j, 1), (j, 2), (k, 2), (k, 3)\}$ is a γ^{ID}-set with fewer empty K_3 layers, a contradiction.

  Now suppose that no k exists for which $(k, 4) \in C$ and $(k, 5) \in C$. In this case, we see that $f(4) = 1$ and $f(5) = 2$. Without loss of generality, assume that $\{(1, 4), (2, 5), (3, 5)\} \subseteq C$. Let $C' = C - (V(K^1) \cup V(K^2))$. The set C'' defined by $C'' = C' \cup \{(3, 1), (2, 2), (3, 2), (2, 3)\}$ is a γ^{ID}-set with fewer empty K_3-layers, a contradiction.

  Thus, we now assume that $f(3) \neq 0$ and that $f(m - 2) \neq 0$.

- We now assume that $4 \leq i \leq m - 3$. Since $f(i) = 0$, by Corollary 75 we see that $f(i - 2) \geq 2$ and that $f(i+2) \geq 2$. We additionally see that $f(i-3) > 0$ and that $f(i-1) > 0$ by our choice of i. Since we assume there are no adjacent, empty K_3 layers, we also have that $f(i + 1) > 0$. Without loss of generality, assume that $(1, i - 3) \in C$.

  Since C is a dominating set, note that either $f(i - 1) \geq 2$ or $f(i + 1) \geq 2$.

  - First, suppose that $f(i - 1) = 3$.

    Suppose for some k that $(k, i + 1) \in C$ and that $(k, i + 2) \in C$. The set C' defined by $C' = (C - \{(2, i - 1)\}) \cup \{(k, i)\}$ is a γ^{ID}-set with fewer empty K_3 layers, a contradiction.

    Suppose now that no such k exists. Let $j \in \{1, 2, 3\}$ be such that $(j, i + 1) \notin C$. We then see that the set C' defined by $C' = (C - \{(2, i - 1)\}) \cup \{(j, i)\}$ is a γ^{ID}-set with fewer empty K_3 layers.

    By applying similar logic, if $f(i + 1) = 3$, then a γ^{ID}-set with fewer empty layers can be found. Hence, we now assume that $f(i - 1) < 3$ and that $f(i + 1) < 3$.

  - Next, suppose that $f(i - 1) = 2$ and that $f(i + 1) = 2$. In this case, let k be such that $(k, i + 1) \notin C$. Let C' be the set defined by $C' = (C - V(K^{i-1})) \cup \{(k, i)\}$.

    If $f(i - 3) = 1$, then let j be such that $(j, i - 3) \notin C$ and such that $j \neq k$. The set $C' \cup \{(j, i - 1)\}$ is a γ^{ID}-set with fewer empty K_3 layers.

If $f(i-3) \geq 2$, then let $j \in \{1, 2, 3\}$ be such that $j \neq k$. The set $C' \cup \{(j, i-1)\}$ is a γ^{ID}-set with fewer empty K_3 layers.

Hence, we now have just two cases left to consider: $f(i-1) = 2$, $f(i+1) = 1$ and $f(i-1) = 1$ and $f(i+1) = 2$.

– First, suppose that $f(i-1) = 2$ and that $f(i+1) = 1$. We note that in this case, $f(i+2) = 3$ in order to separate the vertices in $V(K^{i+1})$ from those in $V(K^i)$.

  If $(1, i-1), (2, i-1) \in C$, then $(3, i+1) \in C$ since C is dominating. In this case, if $f(i-2) = 2$ with $(1, i-2), (3, i-2) \in C$, then the set $(C - \{(1, i-1)\}) \cup \{(1, i)\}$ gives a contradiction. Otherwise, $(2, i-2) \in C$ in which case the set $(C - \{(1, i-1)\}) \cup \{(2, i)\}$ gives a contradiction.

  If $(1, i-1), (3, i-1) \in C$, then $(2, i+1) \in C$. If $f(i-2) = 2$ with $(1, i-2), (2, i-2) \in C$, then the set $(C - \{(1, i-1)\} \cup \{(1, i)\}$ gives a contradiction. Otherwise, $(3, i-2) \in C$ in which case the set $(C - \{(1, i-1)\}) \cup \{(3, i)\}$ is a γ^{ID}-set with fewer empty K_3 layers.

  If $(2, i-1), (3, i-1) \in C$, then $(1, i+1) \in C$. If $(1, i-2), (2, i-2) \in C$, then the set $(C - \{(2, i-1)\}) \cup \{(2, i)\}$ yields a contradiction. If $(1, i-2), (3, i-2) \in C$, then the set $(C - \{(3, i-1)\}) \cup \{(3, i)\}$ gives a contradiction, while if $(2, i-2), (3, i-2) \in C$, then $(C - \{(3, i-1)\}) \cup \{(2, i)\}$ gives a contradiction.

– Finally, suppose that $f(i-1) = 1$ and $f(i+1) = 2$. Since our initial assumption was that $(1, i-3) \in C$, we see that either $(2, i-1) \in C$ or $(3, i-1) \in C$. Without loss of generality, we assume that $(2, i-1) \in C$. This implies that $(1, i+1), (3, i+1) \in C$ as well.

  First, note that $f(i-2) = 3$. This follows from the fact that the vertices of K^{i-1} are separated from $(2, i)$ by C. If $i = 4$, then $(C - \{(3, i-2)\}) \cup (2, i)$ is a γ^{ID}-set with fewer empty K_3 layers. If $i > 4$, then $f(i-4) > 0$ by our choice of i. If $(3, i-4) \notin C$, then $(C - \{(3, i-2)\}) \cup (2, i)$ is a γ^{ID}-set with fewer empty K_3 layers. If $(3, i-4) \in C$, then $(C - \{(2, i-2)\}) \cup (1, i)$ is a γ^{ID}-set with fewer empty K_3 layers.

Thus, C satisfies the property that $f_C(i) > 0$ for $1 \leq i \leq m$, completing our proof. $\square$

By combining this result with Lemma 80, we have the following.

Proposition 88. *Let C be a γ^{ID}-set for $K_3 \,\Box\, P_m$ $(m \geq 4)$, satisfying $f_C(i) > 0$ for $1 \leq i \leq m$. One of $f(i)$, $f(i+1)$, $f(i+2)$, or $f(i+3)$ is at least 2 for each $1 \leq i \leq m-3$.*

This gives us the following corollary.

Corollary 89. *For $m \geq 4$, $\gamma^{ID}(K_3 \,\Box\, P_m) \geq m + \lfloor \frac{m-4}{4} \rfloor + 2$.*

Proof. Let C be a γ^{ID}-set for $K_3 \,\Box\, P_m$ such that $f_C(i) > 0$ for $1 \leq i \leq m$. This implies that $|C| \geq m$. Since $f(2) \geq 2$ and $f(n-1) \geq 2$, we see that $|C| \geq m+2$. Finally, by Proposition 88 we see that among the K_3 layers $K^3, K^4, \ldots, K^{m-2}$, $\lfloor \frac{m-4}{4} \rfloor$ layers will contain at least two vertices in C, giving us the result. □

We can now determine $\gamma^{ID}(K_3 \,\Box\, P_m)$ for all $m \geq 4$.

Theorem 90. *For $m \geq 4$, $\gamma^{ID}(K_3 \,\Box\, P_m) = m + 2 + \lfloor \frac{m-4}{4} \rfloor$.*

Proof. We construct an identifying code C of cardinality $m + 2 + \lfloor \frac{m-4}{4} \rfloor$. We proceed by adding vertices from K^1, then K^2, and so on, stopping after K^{m-2}. When adding vertices from K^i, we follow the rules below.

- If $i \equiv 1 \pmod 4$, add (k,i) such that $(k,i-1) \notin C$ and such that $(k,i-2) \notin C$. If $i = 1$, add any vertex from $V(K^1)$ to C.
- If $i \equiv 2 \pmod 4$, add (k,i) and (j,i) such that $(k,i-1) \notin C$ and $(j,i-1) \notin C$.
- If $i \equiv 3 \pmod 4$, add (k,i) such that $(k,i-2) \notin C$.
- If $i \equiv 0 \pmod 4$, add (k,i) such that $(k,i-1) \notin C$, $(k,i-2) \in C$.

After this has been done, consider $V(K^{m-1})$. If $m-2 \equiv 2 \pmod 4$, add any two vertices from $V(K^{m-1})$ to C, and then add any vertex from $V(K^m)$ to C. If $m-2 \not\equiv 2 \pmod 4$, add $(k,m-1),(j,m-1)$ to C so that $(k,m-2)$ and $(j,m-2)$ are not in C, and then add (p,m) such that $(p,m-2) \notin C$. This will then be an identifying code of cardinality $m + 2 + \lfloor \frac{m-4}{4} \rfloor$. Since we know that $\gamma^{ID}(K_3 \,\Box\, P_m) \geq m + 2 + \lfloor \frac{m-4}{4} \rfloor$, we see that C is a γ^{ID}-set. □

6.6 Closing Remarks

Computing $\gamma^{ID}(K_4 \,\Box\, P_m)$ proves to be more difficult. Determining exact values is possible for certain values of m. However, in other cases, only upper bounds can be obtained.

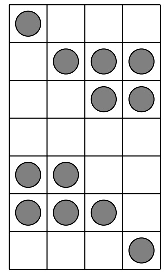


Figure 6.7: $\gamma^{ID}(K_4 \,\square\, P_7) = 12$

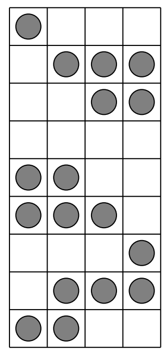


Figure 6.8: $\gamma^{ID}(K_4 \,\square\, P_9) \leq 17$

For example, consider the case of $K_4 \,\square\, P_7$. If C is a minimum identifying code for $K_4 \,\square\, P_7$, then by Corollary 75 we have $f_C(1) + f_C(3) \geq 3$, $f_C(2) \geq 3$, $f_C(5) + f_C(7) \geq 3$ and $f_C(6) \geq 3$. That is, $\gamma^{ID}(K_4 \,\square\, P_7) \geq 12$. The identifying code constructed in Figure 6.7 achieves this lower bound. Thus, $\gamma^{ID}(K_4 \,\square\, P_7) = 12$.

Now consider the case of $K_4 \,\square\, P_9$. Similar to the proof of Proposition 83, if C is a minimum identifying code for $K_4 \,\square\, P_9$, then by Corollary 75 and Corollary 76 we have that $f_C(1) + f_C(3) \geq 3$, $f_C(2) \geq 3$, $f_C(4) + f_C(6) \geq 3$, $f_C(5) + f_C(7) \geq 3$, and $f_C(8) + f_C(9) \geq 4$. Thus, $\gamma^{ID}(K_4 \,\square\, P_9) \geq 16$. The identifying code illustrated in Figure 6.8 illustrates that $\gamma^{ID}(K_4 \,\square\, P_9) \leq 17$. Does $K_4 \square P_9$ have a minimum identifying code of cardinality 16? Answering this question involves considering many cases. For larger values of m, similar questions arise only with more cases to consider.

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