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Multiplier Operator on Framed Hilbert Spaces

Haodong Li
Clemson University, haodonl@g.clemson.edu

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Haodong Li
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Shitao Liu
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Abstract

Very often the operators that we study appear most naturally in highly non-diagonal representation. The main goal of spectral theory is to solve this problem by exhibiting for many operators a natural orthonormal basis with respect to which the operators have diagonal representations. However, this can be done only for certain classes of operators. The most important such class is probably the class of compact operators. The problem is that it is often hard to tell whether an operator is compact looking at its non-diagonal representation. In this thesis, we will study a class of operators for which we can determine all of their basic operator-theoretic properties from their original representation which is not diagonal in the classical sense. There are many important subclasses of operators which belong in our class, including Toeplitz operators on various function spaces, some pseudo-differential operators, some singular integral operators, etc.

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Chapter 1

Introduction

1.1 Operator-theoretic properties

The basic operator-theoretic properties that we will study are boundedness, compactness and Schatten class membership. For completeness, let's briefly recall the definition of these properties.

Definition 1.1.1. *Let X and Y be Hilbert spaces and $T : X \longrightarrow Y$ be a linear operator. T is said to be bounded if there is a real number c such that for all $x \in X$,*

$$\|Tx\| \leq c\|x\|$$

Definition 1.1.2. *Let X and Y be Hilbert spaces and $T : X \longrightarrow Y$ be a linear operator. T is said to be compact if T maps any bounded sets into precompact sets. Equivalently, T maps any weakly convergent sequence in X into a convergent sequence in Y.*

Another equivalent definition of compact operators is given as follows. Let H be an infinite-dimensional Hilbert space and $T : H \to H$ be a linear operator. T is said to be compact if it can be written in the form

$$Tf = \sum_n \lambda_n \langle f|f_n\rangle g_n$$

where $f_1, f_2, \cdots$ and $g_1, g_2, \cdots$ are (not necessarily complete) orthonormal sequences, and $\lambda_1, \lambda_2, \cdots$ is a sequence of positive numbers with limit zero.

Definition 1.1.3. *Let X and Y be Hilbert spaces and $T : X \longrightarrow Y$ be a compact linear operator. The operator T is in Schatten Class S_p $(p > 1)$ if*

$$\sum_{n \geq 1} s_n^p(T) < \infty$$

where $s_1(T) \geq s_2(T) \geq \ldots s_n(T) \geq \ldots \geq 0$ are the singular values of T, $\|T\|_p := \left(\sum_n s_n^p(T) \right)^{\frac{1}{p}}$ is the Schatten p-norm of T. If T is positive symmetric operator, the singular value of T is the same with the eigenvalue of T.

The class of operators that we will study can be defined roughly as those operators $T : H \to H$ which can be represented as : $Tf = \int_X a(x) \langle f|k_x \rangle k_x d\lambda(x)$ where $\{k_x\}_{x \in X}$ is some "basis" index by a measure space (X, λ) which can be continuous. We will be interested in describing the operator-theoretic properties of T in term of the "symbol function" $a : X \to \mathbb{C}$. We will call these operators multiplier operators. We will first give a detailed answer in the simplest case when the "basis" $\{k_x\}_{x \in X}$ is an orthonormal basis indexed as usual by $X = \mathbb{N}$, equipped with the counting measure.

1.2 Toy example

Let $\{e_n\}_{n=1}^{\infty}$ be some orthonormal basis for a separable Hilbert space H and $\{a_n\}_{n=1}^{\infty}$ is some positive sequence of numbers. The multiplier operator $T : H \to H$ is given by $Tf = \sum_n a_n \langle f|e_n \rangle e_n$. It is easy to see that T is densely defined symmetric positive operator.

Theorem 1. *T can be extended to a bounded operator on H iff $\{a_n\}_{n=1}^{\infty}$ is bounded*

Proof. $\Rightarrow$

$$\begin{aligned} \|Te_n\| &= \| \sum_m a_m \langle e_n|e_m \rangle e_m \| \\ &= \|a_n e_n\| \\ &= a_n \leq \|T\| \quad \forall n \in \mathbb{N} \end{aligned}$$

It follows that a_n is bounded.

$\Leftarrow$

For any $f = \sum_n k_n e_n \in H$, we have

$$\begin{aligned}
\|Tf\|^2 &= \langle Tf|Tf\rangle \\
&= \left\langle T\sum_n k_n e_n | T\sum_m k_m e_m \right\rangle \\
&= \left\langle \sum_n k_n a_n e_n | \sum_m k_m a_m e_m \right\rangle \\
&= \sum_n |k_n a_n|^2 \\
&\leq C^2 \sum_n |k_n|^2 \\
&= C^2 \|f\|^2
\end{aligned}$$

It follows that T is bounded. □

Theorem 2. *T is compact iff $\{a_n\}_{n=1}^{\infty} \in c_0$ i.e. $a_n \to 0$ as $n \to \infty$.*

Proof. $\Rightarrow$

Notice $Te_n = a_n e_n$, so $\{e_n\}_{n=1}^{\infty}$ is nothing but orthonormal eigenbasis of T associated with eigenvalue a_n. By spectral theorem, the compactness of the positive symmetric operator T follows that $a_n \to 0$ as $n \to \infty$.

$\Leftarrow$

$Tf = \sum_n a_n \langle f|e_n\rangle e_n$ where $\{e_n\}_{n=1}^{\infty}$ is orthonormal set and a_n has limit 0, the compactness of T follows from the equivalent definition of compact operators. □

Theorem 3. *$T \in S_p$(schatten class $p \geq 1$) iff $\{a_n\}_{n=1}^{\infty} \in l^p(\mathbb{N})$*

Proof. $\{e_n\}_{n=1}^{\infty}$ is the orthonormal eigenbasis of T associated with eigenvalue $\{a_n\}_{n=1}^{\infty}$. Since T is a positive symmetric operator, T is in Schatten class iff $\sum_n a_n^p < \infty$. □

The aim of this paper is to generalize the conclusions above. In the case when the orthonormal basis $\{e_n\}_{n=1}^{\infty}$ is replaced by a much more general "basis" called generalized Parseval frame $\{k_x\}_{x\in X}$. Notice in the example $a_n = \langle Te_n|e_n\rangle$, naturally a_n will be replaced by $\langle Tk_x|k_x\rangle$ which plays the same role as a_n in the above toy example.

We now give the basic assumptions that we impose on $\{k_x\}_{x\in X}$.

1.3 Basic definitions and assumptions

1.3.1 Definition of framed Hilbert space

A framed Hilbert space is a triple (H, X, k) such that

1. H is a Hilbert Space

2. (X, d, λ) is metric measure space, d is the metric on X, λ is the Borel measure with respect to the metric d.

3. k is a generalized Parseval Frame which is a continuous function given by

$$k : X \longrightarrow H$$
$$x \longrightarrow k_x$$

and satisfies

$$\|f\|^2 = \int_X |\langle f|k_x\rangle|^2 d\lambda(x), \ \forall f \in H$$

If $\{k_x\}$ is a generalized Parseval frame, then we have $f = \int_X \langle f|k_x\rangle k_x d\lambda(x)$, for all $f \in H$. In this case, $\langle f|g\rangle = \int_X \langle f|k_x\rangle \langle k_x|g\rangle d\lambda(x)$ for all $f, g \in H$. When $X = \mathbb{Z}$ with the counting measure λ, k is the usual Parseval frame.

1.3.2 Assumptions on the indexing metric measure space (X, d, λ)

From now on, we will assume (X, d, λ) is a metric measure space satisfying the following properties

S1. λ is Borel measure with respect to d.

S2. (X, d, λ) has the following covering property: For any $r > 0$, there exists N and the collection of Borel sets $\{F_n\}_{n=1}^{\infty}$ which satisfies

(i) $X = \bigcup_{n=1}^{\infty} F_n$

(ii) $F_n \bigcap F_m = \emptyset$, $n \neq m$

(iii) Any $x \in X$ is contained in at most N sets of $\{G_n\}_{n=1}^{\infty}$, where $G_n := \{x \in X : d(x, F_n) \leqslant r\}$ is the r-neighborhood of F_n.

(iv) $diam(F_n) \leqslant r$, $\forall n \in \mathbb{N}$

(v) There exist constants A_r and B_r, such that for any $n \in \mathbb{N}$

$$A_r \leqslant \lambda(F_n) \leqslant \lambda(G_n) \leqslant B_r$$

Recall if a metric space (X, d) satisfies (i) (ii) (iii) (iv), we say (X, d) has a finite asymptotic dimension. If (X, d) is a Gromov hyperbolic geodesic metric space with bounded growth, then X has a finite asymptotic dimension. See [1] [4] [8] for more information.

1.3.3 Assumptions on generalized Parseval frame

Furthermore, we will make the following assumptions for the generalized Parseval frame $\{k_x\}_{x\in X}$.

F1. $\|k_x\| = 1$, $\forall x \in X$

F2. (Mean value property) For any $r > 0$, there exists $C_r > 0$, such that for any $x \in X, f \in H$

$$|\langle f|k_x\rangle|^2 \leqslant C_r \int_{B(x,r)} |\langle f|k_y\rangle|^2 d\lambda(y)$$

F3. For any $r > 0$, there exists $D_r > 0$, such that when $d(x, y) < r$ for any $x, y \in X$ we have

$$|\langle k_x|k_y\rangle| \geqslant D_r$$

F4. When $d(x, a) \longrightarrow \infty$ for some/any fixed point $a \in X$

$$k_x \xrightarrow{w} 0 \quad \text{(weakly convergent)}$$

Recall, $k_x \xrightarrow{w} 0$ iff $\langle f|k_x\rangle \to 0$, $\forall f \in H$.

1.4 Examples of framed Hilbert space

1.4.1 Bergman space

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$

For $-1 < \alpha < \infty$, define $dA_\alpha(z) := \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha dxdy$,

The Bergman space $\mathcal{A}^2_\alpha$ is given by:

$\mathcal{A}^2_\alpha := \{f : \mathbb{D} \to \mathbb{C} | f$ is holomorphic on $\mathbb{D}$ and $\int\limits_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < \infty\}$

$\mathcal{A}^2_\alpha$ is a Hilbert space equipped with the inner product

$$\langle f|g\rangle := \int\limits_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z)$$

The Bergman metric on $\mathbb{D}$ is given by

$$d(z,w) := inf\{\ell(\gamma)| \text{ all piecewise } \mathcal{C}^1 \text{ curves } \gamma \text{ such that } \gamma(0) = z \text{ and } \gamma(1) = w\}$$

The hyperbolic measure on $\mathbb{D}$ is given by

$$d\lambda(z) := \frac{\alpha+1}{\pi}(1-|z|^2)^{-2} dxdy$$

which is invariant under Möbius transformations.

$(\mathbb{D}, d, \lambda(z))$ is the metric measure space which satisfies S1 and S2.

The Bergman space $\mathcal{A}^2_\alpha$ is a reproducing kernel Hilbert space with reproducing kernel $K_z(w) = (1-\bar{z}w)^{-(2+\alpha)}$, such that $\langle f|K_z\rangle = f(z)$. Let $k_z = \frac{K_z}{\|K_z\|}$ be the normalized reproducing kernel. We have

$$\begin{aligned}\int_{\mathbb{D}} |\langle f|k_z\rangle|^2 d\lambda(z) &= \int_{\mathbb{D}} \left|\left\langle f|\frac{K_z}{\|K_z\|}\right\rangle\right|^2 d\lambda(z)\\ &= \int_{\mathbb{D}} \frac{|\langle f|K_z\rangle|^2}{\|K_z\|^2} d\lambda(z)\\ &= \int_{\mathbb{D}} \frac{|f(z)|^2}{\|K_z\|^2} d\lambda(z)\\ &= \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^{(2+\alpha)}\frac{\alpha+1}{\pi}(1-|z|^2)^{-2}dxdy\\ &= \int_{\mathbb{D}} |f(z)|^2\frac{\alpha+1}{\pi}(1-|z|^2)^{\alpha}dxdy\\ &= \int_{\mathbb{D}} |f(z)|^2 dA^2_\alpha(z) = \|f\|^2\end{aligned}$$

So the normalized reproducing kernel $\{k_z\}_{z\in\mathbb{D}}$ is nothing but the generalized Parseval frame given by

$$\begin{aligned} k :&\mathbb{D} \longrightarrow \mathcal{A}^2_\alpha\\ &z \longrightarrow k_z(w) := \frac{(1-|z|^2)^{\frac{2+\alpha}{2}}}{(1-\bar{z}w)^{2+\alpha}}\end{aligned}$$

It is known that it satisfies F1, F2, F3 and F4.

Then $(\mathcal{A}^2_\alpha, \mathbb{D}, k)$ is the framed Hilbert space which satisfies all the above assumptions. More information about Bergman spaces can be found in [5].

1.4.2 Fock space

For $\alpha > 0$, define $dA_\alpha(z) := \frac{\alpha}{\pi}e^{-\alpha|z|^2}dxdy$,

The Fock space $\mathcal{F}^2_\alpha$ is given by:

$\mathcal{F}^2_\alpha := \{f : \mathbb{C} \to \mathbb{C} | f \text{ is entire and } \int_{\mathbb{C}} |f(z)|^2 dA_\alpha(z) < \infty\}$

$\mathcal{F}^2_\alpha$ is a Hilbert space equipped with the inner product

$$\langle f|g\rangle := \int_{\mathbb{C}} f(z)\overline{g(z)}dA_\alpha(z)$$

The metric d is the Euclidian distance on $\mathbb{C}$ given by $d(z,w) := |z-w|$.

The measure λ is the area measure on $\mathbb{C}$ given by $d\lambda(z) := \frac{\alpha}{\pi}dxdy$.

$(\mathbb{C}, d, \lambda)$ is the metric measure space which satisfies S1 and S2.

The Fock space $\mathcal{F}^2_\alpha$ is a reproducing kernel Hilbert space with reproducing kernel $K_z(w) = e^{\alpha\bar{z}w}$, such that $\langle f|K_z\rangle = f(z)$. Let $k_z = \frac{K_z}{\|K_z\|}$ be the normalized reproducing kernel, we have

$$\begin{aligned}
\int_{\mathbb{C}} |\langle f|k_z\rangle|^2 d\lambda(z) &= \int_{\mathbb{C}} \left|\left\langle f|\frac{K_z}{\|K_z\|}\right\rangle\right|^2 d\lambda(z) \\
&= \int_{\mathbb{C}} \frac{|\langle f|K_z\rangle|^2}{\|K_z\|^2} d\lambda(z) \\
&= \int_{\mathbb{C}} \frac{|f(z)|^2}{\|K_z\|^2} d\lambda(z) \\
&= \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} \frac{\alpha}{\pi} dxdy \\
&= \int_{\mathbb{C}} |f(z)|^2 dA^2_\alpha(z) = \|f\|^2
\end{aligned}$$

So the normalized reproducing kernel k is nothing but the generalized Parseval frame given by

$$\begin{aligned}
k :& \mathbb{C} \longrightarrow \mathcal{F}^2_\alpha \\
& z \longrightarrow k_z(w) := \frac{e^{\alpha\bar{z}w}}{e^{\frac{1}{2}\alpha|z|^2}}
\end{aligned}$$

It is known that it satisfies F1, F2, F3 and F4.

Then $(\mathcal{F}^2_\alpha, \mathbb{C}, k)$ is the framed Hilbert space which satisfies all the above assumptions. More information about Fock spaces can be found in [11].

1.4.3 Paley-Wiener space $\mathcal{PW}^2_\alpha$

Recall, an entire function f has exponential type $\leq \alpha$ if $\limsup\limits_{|z|\to\infty} \frac{\log|f(z)|}{|z|} \leq \alpha$, i.e. $|f(z)|$ grows no faster than $e^{\alpha|z|}$ in each direction.

For $\alpha > 0$, the Paley-Wiener space $\mathcal{PW}^2_\alpha$ is given by:

$\mathcal{PW}^2_\alpha := \{f : \mathbb{C} \to \mathbb{C} | f$ is entire of exponential type $\leq \alpha$, $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\}$

The well-known Paley-Wiener theorem says that $\mathcal{PW}^2_\alpha = \mathcal{F}L^2[-\alpha, \alpha]$, where $\mathcal{F}$ is the Fourier transform.

$\mathcal{PW}^2_\alpha$ is a Hilbert space associated with the inner product

$$\langle f|g\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

Take d to be the metric on $\mathbb{R}$ given by $d(x,y) := |x-y|$.

Take λ to be the Lebesgue measure on $\mathbb{R}$ given by $d\lambda(x) := \frac{\alpha}{\pi}dx$.

$(\mathbb{R}, d, \lambda)$ is the metric measure space which satisfies S1 and S2.

The Paley-Wiener space $\mathcal{PW}^2_\alpha$ is a reproducing kernel Hilbert space with reproducing kernel $K_z(w) = \frac{\sin\alpha(w-\bar{z})}{\pi(w-\bar{z})}$ if $z \neq \bar{w}$, $K_z(w) = \frac{\alpha}{\pi}$ if $z = \bar{w}$. Let $k_z = \frac{K_z}{\|K_z\|}$ be the normalized reproducing kernel. Now only focus on k_x ($x \in \mathbb{R}$) we have

$$\begin{aligned}\int_{-\infty}^{\infty} |\langle f|k_x\rangle|^2 d\lambda(x) &= \int_{-\infty}^{\infty} \left|\left\langle f|\frac{K_x}{\|K_x\|}\right\rangle\right|^2 d\lambda(x) \\ &= \int_{-\infty}^{\infty} \frac{|\langle f|K_x\rangle|^2}{\|K_x\|^2} d\lambda(x) \\ &= \int_{-\infty}^{\infty} \frac{|f(x)|^2}{\|K_x\|^2} d\lambda(x) \\ &= \int_{-\infty}^{\infty} \frac{|f(x)|^2}{\frac{\alpha}{\pi}} \frac{\alpha}{\pi} dx \\ &= \int_{-\infty}^{\infty} |f(x)|^2 dx = \|f\|^2\end{aligned}$$

So the normalized reproducing kernel $\{k_x\}_{x\in\mathbb{R}}$ is nothing but the generalized Parseval frame given by

$$\begin{aligned} k :&\mathbb{R} \longrightarrow \mathcal{PW}^2_\alpha \\ &x \longrightarrow k_x(z) := \frac{\sin\alpha(z-x)}{(\pi\alpha)^{\frac{1}{2}}(z-x)}\end{aligned}$$

It is known that it satisfies F1, F2, F3 and F4.

Then $(\mathcal{PW}^2_\alpha, \mathbb{R}, k)$ is the framed Hilbert space which satisfies all the above assumptions. More information about Paley-Wiener space can be found in [6].

1.5 Other examples

Let G be a locally compact topological group. If G is Hausdoff and first countable, we can find a left invariant metric d which induces the topology on G. It is known that, there is a unique (up to a scalar-multiple) left Haar measure λ on G. Then (G, d, λ) is a metric measure space.

Let H be the Hilbert space, $\mathcal{U}(H)$ be the group which consists of all unitary operators on H. We assume π is square-integrable which means that we can find some $h \in H$ with norm 1 such that $\int_G |\langle h|\pi(g)h\rangle|^2 d\lambda(g) < \infty$. Let $k_g := \pi(g)h$. If π is irreducible unitary representation, then we can scale the left Haar measure λ on G such that $f = \int_G \langle f|k_g\rangle\, k_g d\lambda(g)$ (see [9] [3]). It follows that, for any $f \in H$

$$\begin{aligned} \int_G |\langle f|k_g\rangle|^2 d\lambda(g) &= \left\langle \int_G \langle f|k_g\rangle\, k_g d\lambda(g) | f \right\rangle \\ &= \langle f|f\rangle = \|f\|^2 \end{aligned}$$

So k is the generalized Parseval frame which is given by

$$\begin{aligned} k :& G \longrightarrow H \\ & g \longrightarrow k_g = \pi(g)h \end{aligned}$$

So (H, G, k) is the Farseval framed Hilbert space.

Before we introduce new examples, we will use the following notation for the basic unitary operators on $L^2(\mathbb{R})$:

1. Translation: $T_a f(x) = f(x - a)$,
2. Modulation: $M_a f(x) = e^{2\pi i a x} f(x)$,
3. Dilation: $D_a f(x) = \frac{1}{\sqrt{a}} f(\frac{x}{a})$, $a > 0$.

1.5.1 $L^2(\mathbb{R})$ space and the Weyl-Heisenberg group

It is known that $L^2(\mathbb{R})$ is a Hilbert space with the inner product $\langle f|g\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$ Recall the Weyl-Heisenberg group $(H, \cdot)$ is $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and the group law $\cdot$ is

$$(x_1, \xi_1, t_1) \cdot (x_2, \xi_2, t_2) = (x_1 + x_2, \xi_1 + \xi_2, \frac{x_1\xi_2 - x_2\xi_1}{2})$$

H is a locally compact topological group. Since H is Hausdoff and first countable, we can find a metric d, such that d induces the same topology, and there is a unique (up to scalar-multiple) left Haar measure λ on H.

Let π be the unitary representation of H on $L^2(\mathbb{R})$ given by

$$\pi(x,\xi,t)f = e^{2\pi it+\pi ix\xi}M_\xi T_x f$$

Let h be the Gaussian function given by $h(x) = \frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{2}}$ with norm 1. Let $k_{(x,\xi,t)} := \pi(x,\xi,t)h$. Since π is irreducible unitary representation, we can find a left Haar measure λ on H, such that $f = \int\limits_G \left\langle f|k_{(x,\xi,t)}\right\rangle k_{(x,\xi,t)} d\lambda((x,\xi,t))$ (see [9] [3]). It follows that, for any $f \in H$

$$\begin{aligned}\int\limits_G |\left\langle f|k_{(x,\xi,t)}\right\rangle|^2 d\lambda((x,\xi,t)) &= \left\langle \int\limits_G \left\langle f|k_{(x,\xi,t)}\right\rangle k_{(x,\xi,t)} d\lambda((x,\xi,t))|f\right\rangle \\ &= \langle f|f\rangle = \|f\|^2\end{aligned}$$

Then (H,d,λ) is a metric measure space, $\{k_{(x,\xi,t)}\}_{(x,\xi,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}}$ is a generalized Parseval frame which is given by

$$\begin{aligned}k : &H \longrightarrow L^2(\mathbb{R}) \\ &(x,\xi,t) \longrightarrow k_{(x,\xi,t)} = \pi((x,\xi,t))h\end{aligned}$$

This generalized Parseval frame is often called a Gabor frame. It is useful in the study of pseudo-differential operators (see [2]). And $(L^2(\mathbb{R}), H, k)$ is the framed Hilbert space. In addition it satisfies all the above assumptions.

1.5.2 $L^2(\mathbb{R})$ space and the affine group

Again we let $L^2(\mathbb{R})$ be the Hilbert space with the inner product $\langle f|g\rangle := \int\limits_{\mathbb{R}} f(x)\overline{g(x)}dx$
Recall the affine group $(A,\cdot)$ is $\mathbb{R}^+ \times \mathbb{R}$ and its operation $\cdot$ is given by

$$(a_1,b_1)\cdot(a_2,b_2) = (a_1a_2, a_1b_2 + b_1)$$

.

The left-Haar measure on A is $d\lambda = \frac{dadb}{a^2}$ and the left-invariant metric is the Riemannian metric given by the length element $d(a,b) = \frac{da^2+db^2}{a^2}$. Then (A, λ, d) is a metric measure space.

The unitary representation of the group A on $L^2(\mathbb{R})$ is given by $\pi(a,b)f = D_a T_b f$.

Pick some $h(x) \in L^2(\mathbb{R})$ with norm 1 which satisfies $\int_{\mathbb{R}} \frac{|\hat{h}(x)|^2}{|x|} dx = 1$. Such $h(x)$ is called a wavelet function. Let $k_{(a,b)} := \pi(a,b)h$. By the Calderon reproducing formula, we have $f = \int\limits_G \left\langle f|k_{(a,b)}\right\rangle k_{(a,b)} d\lambda((a,b))$ (see [7]). It follows that, for any $f \in L^2(\mathbb{R})$

$$\begin{aligned}\int\limits_G |\left\langle f|k_{(a,b)}\right\rangle|^2 d\lambda((a,b)) &= \left\langle \int\limits_G \left\langle f|k_{(a,b)}\right\rangle k_{(a,b)} d\lambda((a,b))|f \right\rangle \\ &= \langle f|f\rangle = \|f\|^2\end{aligned}$$

So $\{k_{(a,b)}\}_{(a,b)\in\mathbb{R}^+\times\mathbb{R}}$ is a generalized Parseval frame which is given by

$$\begin{aligned}k : &A \longrightarrow L^2(\mathbb{R}) \\ &(a,b) \longrightarrow k_{(a,b)} = \pi((a,b))h\end{aligned}$$

So $(L^2(\mathbb{R}), A, k)$ is a framed Hilbert space. In addition it satisfies all the above assumptions.

Actually, the first two examples(space Bergman and Fock) can also be obtained in this way by specifying appropriate group G. More information about Parseval frames generated by group representations can be found in [9].

Chapter 2

Multiplier operators

2.1 Definition

To introduce multiplier operators T_μ, we first define μ to be the positive Borel measure on X such that for any $x \in X$ we have

$$\int_X |\langle k_x|k_y\rangle| d\mu(y) = C_x < \infty$$

For any $x \in X$, we define $l_x : H \to \mathbb{C}$ by $l_x(f) = \int\limits_X \langle k_x|k_y\rangle \langle k_y|f\rangle \, d\mu(y)$

Claim: l_x is well-defined bounded anti-linear complex valued functional on H.

Proof. i. Well defined:

$$\begin{aligned}\int_X |\langle k_x|k_y\rangle \langle k_y|f\rangle| d\mu(y) &\leq \int_X |\langle k_x|k_y\rangle|| \langle k_y|f\rangle| d\mu(y) \\ &\leq \int_X |\langle k_x|k_y\rangle| \|f\| d\mu(y) \\ &\leq C_x\|f\| < \infty\end{aligned}$$

So we have $\int_X \langle k_x|k_y\rangle \langle k_y|f\rangle \, d\mu(y) \in \mathbb{C}$

ii. Anti-linear: easy to check

iii. Bounded:

$$\begin{aligned}\left|\int_X \langle k_x|k_y\rangle \langle k_y|f\rangle \, d\mu(y)\right| &\leq \int_X |\langle k_x|k_y\rangle| |\langle k_y|f\rangle| d\mu(y) \\ &\leq \int_X |\langle k_x|k_y\rangle| \|f\| d\mu(y) \\ &\leq C_x \|f\|\end{aligned}$$

□

By Riesz representation theorem, there exists $h_x \in H$ such that for any $f \in H$:

$$\langle h_x|f\rangle = l_x(f)$$

Now we can densely define T_μ on H by

$$\begin{aligned}T_\mu : Span\{k_x\} &\longrightarrow H \\ k_x &\longrightarrow h_x\end{aligned}$$

Notice for any $x \in X$ $f \in H$

$$\langle h_x|f\rangle = \int_X \langle k_x|k_y\rangle \langle k_y|f\rangle \, d\mu(y)$$

So $T_\mu k_x = \int_X \langle k_x|k_y\rangle \, k_y d\mu(y)$

2.2 Basic properties

Theorem 4. *The multiplier operator T_μ given by*

$$\begin{aligned}T_\mu : Span\{k_x\} &\longrightarrow H \\ k_x &\longrightarrow \int_X \langle k_x|k_y\rangle \, k_y d\mu(y)\end{aligned}$$

satisfies:

i. *T_μ is a linear operator.*

ii. T_μ *is symmetric*

iii. T_μ *is positive*

iv. $aT_\mu + bT_\nu = T_{a\mu+b\nu}$

Proof. i: Trivial

ii: $\forall f, g \in Span\{k_x\}$ we have:

$$\begin{aligned}
\langle T_\mu f|g\rangle &= \int_X \langle f|k_y\rangle \langle k_y|g\rangle \, d\mu(y) \\
&= \overline{\int_X \overline{\langle f|k_y\rangle \langle k_y|g\rangle} d\mu(y)} \\
&= \overline{\int_X \langle g|k_y\rangle \langle k_y|f\rangle \, d\mu(y)} \\
&= \overline{\langle T_\mu g|f\rangle} \\
&= \langle f|T_\mu g\rangle
\end{aligned}$$

iii: $\forall f \in H$ we have:

$$\begin{aligned}
\langle T_\mu f|f\rangle &= \int_X \langle f|k_y\rangle \langle k_y|f\rangle \, d\mu(y) \\
&= \int_X |\langle f|k_y\rangle|^2 d\mu(y) \geq 0
\end{aligned}$$

iv: $\forall a, b \in \mathbb{C} \forall f, g \in H$

$$\begin{aligned}
\langle T_{a\mu+b\nu} f|g\rangle &= \int_X \langle f|k_y\rangle \langle k_y|g\rangle \, d(a\mu+b\nu)(y) \\
&= a \int_X \langle f|k_y\rangle \langle k_y|g\rangle \, d\mu(y) + b \int_X \langle f|k_y\rangle \langle k_y|g\rangle \, d\nu(y) \\
&= a \langle T_\mu f|g\rangle + b \langle T_\nu f|g\rangle \\
&= \langle (aT_\mu + bT_\nu) f|g\rangle
\end{aligned}$$

So $T_{a\mu+b\nu} = aT_\mu + bT_\nu$

□

2.3 Notation and definition

We will be interested in when T_μ

i. is bounded

ii. is compact

iii. belong to S_p (Schatten Class S_p)

We will characterize boundedness, compactness and S_p membership of T_μ in terms of the following characteristics of the symbol μ.

2.3.1 Berezin transform

One of the most useful tools in the study of Multiplier operators is the so-called Berezin transform. We begin with the definition of the Berezin symbol of a general operator on H. Suppose T is a linear operator on H(not necessarily bounded) whose domain contains $span\{k_x\}$, then T induces a function $\tilde{T}$ on X given by

$$\tilde{T}(x) = \langle Tk_x | k_x \rangle, \quad x \in X,$$

where $\{k_x\}$ is the generalized Parseval frame. The function $\tilde{T}$ is called the Berezin symbol of the operator T.

If $T = T_\mu$, we will write $\tilde{T_\mu} = \tilde{\mu}$, and we say that $\tilde{\mu}$ is the Berezin transform of μ. It is easy to see that

$$\tilde{\mu}(x) = \int_X | \langle k_x | k_y \rangle |^2 d\mu(y), \quad x \in X$$

2.3.2 Carleson type measures

Definition 2.3.1. *A positive Borel measure μ on X is said to be a Carleson measure if the embedding operator $i_\mu : H \to L^2(X, d\mu)$ given by $i_\mu(f) = \langle f | k_x \rangle$ is bounded. It is called a vanishing Carleson measure if i_μ is compact.*

We will also use the following auxiliary quantities.

Definition 2.3.2. *For any $r > 0$, we define $\bar{\mu}_r : \mathbb{N} \to \mathbb{C}$ by $\bar{\mu}_r(n) = \mu(G_n)$*

where G_n is the r-neighborhood of F_n which is the covering associated to r.

Definition 2.3.3. *For any $r > 0$, we define $\hat{\mu}_r : \mathbb{X} \to \mathbb{C}$ by $\hat{\mu}_r(x) = \mu(B(x, r))$*

Chapter 3

Boundedness

The following result gives a characterization of the boundedness of multiplier operators.

Theorem 5. *Let μ be a positive Borel measure on X that satisfies*

$$\int_X |\langle k_x|k_y\rangle|d\mu(y) < \infty, \quad \textit{for all } x \in X$$

Then the following are equivalent.

(a). $T_\mu \in \mathcal{B}(H,H)$, i.e. T_μ can be boundedly extend on H given by

$$T_\mu(f) = \int_X \langle f|k_y\rangle k_y d\mu(y), \quad \textit{for all } f \in H$$

(b). The Berezin transform $\tilde{\mu} : X \to \mathbb{C}$ is bounded.

(c). $\bar{\mu}_r \in l^\infty(\mathbb{N})$, $\forall r > 0$

(d). $\hat{\mu}_r$ is bounded, $\forall r > 0$

(e). μ is a Carleson measure.

Proof. We prove the theorem in the natural order: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a)$

(a)⇒(b) for $\forall x \in X$

$$\begin{aligned}|\tilde{\mu}(x)| &= |\langle T_\mu k_x | k_x \rangle| \\ &\leq \|T_\mu k_x\| \|k_x\| \\ &\leq \|T_\mu\| \|k_x\| \|k_x\| \\ &= \|T_\mu\| < \infty\end{aligned}$$

(b)⇒(c) $\forall r > 0$, $\{F_n^r\}$ is the covering of X associated to r, x_n is some point in F_n^r for $\forall n \in \mathbb{N}$

$$\begin{aligned}|\bar{\mu}_r(n)| &= |\mu(G_n)| \\ &= \left| \int\limits_{G_n} 1 d\mu(y) \right| \\ &= \int\limits_{G_n} \frac{D_{2r}^2}{D_{2r}^2} d\mu(y) \\ &\leq \frac{1}{D_{2r}^2} \int\limits_{G_n} |\langle k_{x_n} | k_y \rangle|^2 d\mu(y) \\ &\leq \frac{1}{D_{2r}^2} \int\limits_{X} |\langle k_{x_n} | k_y \rangle|^2 d\mu(y) \\ &= \frac{1}{D_{2r}^2} \int\limits_{X} \langle k_{x_n} | k_y \rangle \langle k_y | k_{x_n} \rangle \, d\mu(y) \\ &= \frac{1}{D_{2r}^2} \langle T_\mu k_{x_n} | k_{x_n} \rangle \\ &= \frac{1}{D_{2r}^2} \tilde{\mu}(x_n) \leq C\end{aligned}$$

(c)⇒(d) $\forall r > 0$, for any $x \in X$, by $\{F_n^r\}$ is the covering of X associated to r, $\exists!$ $F_{n_x}^r$ such that $x \in F_{n_x}^r$, notice $B(x, r) \subseteq G_{n_x}^r$, we have

$$\mu(B(x, r)) \leq \mu(G_{n_x}^r) = \bar{\mu}_r(n_x) \leq C$$

(d)⇒(e)

$$
\begin{aligned}
\| \langle f|k_x\rangle \|_{L^2}^2 &= \int_X | \langle f|k_x\rangle |^2 d\mu(x) \\
&= \sum_n \int_{F_n} | \langle f|k_x\rangle |^2 d\mu(x) \\
&\leq \sum_n \int_{F_n} C_r \int_{B(x,r)} | \langle f|k_y\rangle |^2 d\lambda(y) d\mu(x) \text{ (by F2)} \\
&\leq C_r \sum_n \int_{F_n} \int_{G_n} | \langle f|k_y\rangle |^2 d\lambda(y) d\mu(x) \quad (B(x,r) \subseteq G_n, \forall x \in F_n) \\
&\leq C_r \sum_n \int_{G_n} \int_{F_n} | \langle f|k_y\rangle |^2 d\mu(x) d\lambda(y) \\
&= C_r \sum_n \mu(F_n) \int_{G_n} | \langle f|k_y\rangle |^2 d\lambda(y) \\
&\leq C_r \sum_n \mu(B(x_n, r)) \int_{G_n} | \langle f|k_y\rangle |^2 d\lambda(y) \text{ (pick some } x_n \in F_n) \\
&\leq C_r C \sum_n \int_{G_n} | \langle f|k_y\rangle |^2 d\lambda(y) \qquad \text{(by (d))} \\
&\leq C_r C N \int_X | \langle f|k_y\rangle |^2 d\lambda(y) \qquad \text{(by S2)} \\
&= C_0 \|f\|^2
\end{aligned}
$$

(e)⇒(a)

For any $f, g \in H$, we have

$$
\begin{aligned}
| \langle T_\mu f|g\rangle | &= \left| \int_X \langle f|k_x\rangle \langle k_x|g\rangle \, d\mu(x) \right| \\
&\leq \int_X | \langle f|k_x\rangle \langle k_x|g\rangle | d\mu(x) \\
&\leq \left(\int_X | \langle f|k_x\rangle |^2 d\mu(x) \right)^{\frac{1}{2}} \left(\int_X | \langle g|k_x\rangle |^2 d\mu(x) \right)^{\frac{1}{2}} \\
&= C \|f\| \|g\|
\end{aligned}
$$

It follows that T_u is bounded. □

Chapter 4

Compactness

Our next result gives a characterization of the compactness of multiplier operators. In the proof we will use the fact every weakly convergent sequence is bounded which is a consequence of the principle of uniform boundedness.

Theorem 6. *As in Theorem 8, a positive Borel measure μ on X satisfies for any $x \in X$*

$$\int_X |\langle k_x | k_y \rangle| d\mu(y) < \infty$$

Then the following are equivalent.

(a). T_μ is compact operator.

(b). $\tilde{\mu} \in \mathcal{C}_0(X)$ i.e. the Berezin transform $\tilde{\mu}$ "vanishes at infinity".

(c). $\bar{\mu}_r \in c_0(\mathbb{N})$, $\forall r > 0$

(d). $\hat{\mu}_r \in \mathcal{C}_0(X)$, $\forall r > 0$

(e). μ is a vanishing Carleson measure.

Proof. We prove the theorem in the natural order: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a)$

(a)⇒(b). When $d(x, a) \to \infty$ for some $a \in X$, by F4 we have

$$k_x \xrightarrow{w} 0$$

By assumption (a) T_μ is compact which implies

$$T_\mu k_x \to 0$$

On the other hand

$$|\tilde{\mu}(x)| = |\langle T_\mu k_x | k_x \rangle| \leq \|T_\mu k_x\| \|k_x\| = \|T_\mu k_x\|$$

It follows that when $d(x, a) \to \infty$

$$\tilde{\mu}(x) \to 0$$

(b)⇒(c) $\forall r > 0$, $\{F_n^r\}$ is the covering of X associated to r, x_n is some point in F_n^r for $\forall n \in \mathbb{N}$.

By Theorem 1, we have

$$|\bar{\mu}_r(n)| = |\mu(G_n)| \leq \frac{1}{D_{2r}^2} \tilde{\mu}(x_n)$$

Notice $d(x_n, a) \to \infty$ as $n \to \infty$.

By (b), when $n \to \infty$

$$|\bar{\mu}_r(n)| \leq \frac{1}{D_{2r}^2} \tilde{\mu}(x_n) \to 0$$

(c)⇒(d) $\forall r > 0$, for any $x \in X$, by $\{F_n^r\}$ is the covering of X associated to r, $\exists! F_{n_x}^r$ such that $x \in F_{n_x}^r$.

By Theorem 1,

$$\mu(B(x, r)) \leq \mu(G_{n_x}^r)$$

Notice $n_x \to \infty$ as $d(a, x) \to \infty$

By (c), when $d(x, a) \to \infty$

$$\mu(B(x, r)) \leq \mu(G_{n_x}^r) \to 0$$

(d)⇒(e) $\{f_i\}_{i=1}^\infty \subseteq H$ is the sequence which weakly convergence to 0, as i goes to ∞.

Pick some point x_n from F_n, easily see $F_n \subseteq B(x_n, r)$, so we have

$$\mu(F_n) \leq \mu(B(x_n, r))$$

On the other hand $d(x_n, a) \to \infty$ as $n \to \infty$.

By (d), we have when $n \to \infty$

$$\mu(F_n) \leq \mu(B(x_n, r)) \to 0$$

So $\forall \varepsilon > 0$, $\exists N$ such that for all $n > N$

$$\mu(F_n) < \varepsilon$$

$$\begin{aligned}
\int\limits_X |\langle f_i|k_x\rangle|^2 d\mu(x) &= \sum_n \int\limits_{F_n} |\langle f_i|k_x\rangle|^2 d\mu(x) \\
&= \underbrace{\sum_{n \leq N} \int\limits_{F_n} |\langle f_i|k_x\rangle|^2 d\mu(x)}_{\text{I}} + \underbrace{\sum_{n > N} \int\limits_{F_n} |\langle f_i|k_x\rangle|^2 d\mu(x)}_{\text{II}}
\end{aligned}$$

For part I, by $f_i \xrightarrow{w} 0$ as $i \to \infty$ we have

$$|\langle f_i|k_x\rangle|^2 \xrightarrow{p.w.} 0 \text{ as } i \to \infty$$

and by weakly convergent sequence is bounded we have

$$|\langle f_i|k_x\rangle|^2 \leq \|f_i\|^2 \|k_x\|^2 = \|f_i\|^2 \leq C$$

In addition $\int\limits_{F_n} C d\mu(x) = C\mu(F_n) < \infty$

Now we can apply Bounded Convergence Theorem

$$\begin{aligned}
\lim_{i \to \infty} \sum_{n \leq N} \int\limits_{F_n} |\langle f_i|k_x\rangle|^2 d\mu(x) &= \sum_{n \leq N} \lim_{i \to \infty} \int\limits_{F_n} |\langle f_i|k_x\rangle|^2 d\mu(x) \\
&= \sum_{n \leq N} \int\limits_{F_n} \lim_{i \to \infty} |\langle f_i|k_x\rangle|^2 d\mu(x) \\
&= 0
\end{aligned}$$

For part II, by the same process in Theorem 1 we have

$$\begin{aligned}
\sum_{n>N}\int_{F_n} |\langle f_i|k_x\rangle|^2 d\mu(x) &\le \sum_{n>N}\int_{F_n} C_r \int_{B(x,r)} |\langle f_i|k_y\rangle|^2 d\lambda(y) d\mu(x) \\
&= C_r \sum_{n>N}\int_{F_n}\int_{B(x,r)} |\langle f_i|k_y\rangle|^2 d\lambda(y) d\mu(x) \\
&= C_r \sum_{n>N} \mu(F_n) \int_{G_n} |\langle f_i|k_y\rangle|^2 d\lambda(y) \\
&\le C_r\varepsilon \sum_{n>N}\int_{G_n} |\langle f_i|k_y\rangle|^2 d\lambda(y) \quad (\mu(F_n) < \varepsilon,\ n > N) \\
&\le C_r\varepsilon \sum_{n}\int_{G_n} |\langle f_i|k_y\rangle|^2 d\lambda(y) \\
&\le C_r\varepsilon N \int_{X} |\langle f_i|k_y\rangle|^2 d\lambda(y) \\
&= C_r\varepsilon N \|f_i\|^2 \\
&\le C\varepsilon
\end{aligned}$$

It follows that $\forall \varepsilon > 0$

$$\begin{aligned}
\limsup_{i\to\infty} \int_X |\langle f_i|k_x\rangle|^2 d\mu(x) &\le \limsup_{i\to\infty} \sum_{n\le N}\int_{F_n} |\langle f_i|k_x\rangle|^2 d\mu(x) \\
&+ \limsup_{i\to\infty} \sum_{n>N}\int_{F_n} |\langle f_i|k_x\rangle|^2 d\mu(x) \\
&= 0 + C\varepsilon
\end{aligned}$$

So we have

$$\limsup_{i\to\infty} \int_X |\langle f_i|k_x\rangle|^2 d\mu(x) = 0$$

which implies

$$\lim_{i\to\infty} \int_X |\langle f_i|k_x\rangle|^2 d\mu(x) = 0$$

It follows that i_μ is the compact operator.

(e)⇒(a) $\{f_i\}_{i=1}^{\infty} \subseteq H$ is the sequence which weakly convergence to 0, as i goes to ∞.

$$\begin{aligned}
\|T_\mu f_i\| &= \sup_{\|g\|=1} |\langle T_\mu f_i | g\rangle| \\
&\leq \sup_{\|g\|=1} \int_X |\langle f_i | k_x\rangle \langle k_x | g\rangle| d\mu(x) \\
&\leq \sup_{\|g\|=1} \left(\int_X |\langle f_i | k_x\rangle|^2 d\mu(x) \right)^{\frac{1}{2}} \left(\int_X |\langle k_x | g\rangle|^2 d\mu(x) \right)^{\frac{1}{2}} \\
&\leq C \left(\int_X |\langle f_i | k_x\rangle|^2 d\mu(x) \right)^{\frac{1}{2}} \quad (i_\mu \text{ is compact}) \\
&= C \|i_\mu(f_i)\|_{L^2}
\end{aligned}$$

By (e), when $i \to \infty$

$$\|T_\mu f_i\| \leq C \|i_\mu(f_i)\|_{L^2} \to 0$$

It follows that when $i \to \infty$

$$T_\mu f_i \to 0$$

□

Chapter 5

Schatten class membership

Our next result gives a characterization of the Schatten class membership of multiplier operators. In the proof we will use the following result from [10].

Lemma 7. *Suppose T is a positive operator on a Hibert space H and f is a element in H, then*

$\langle T^p f|f\rangle \geq \langle Tf|f\rangle^p$ for all $p \geq 1$.

Theorem 8. *Same as before, a positive Borel measure μ on X satisfies for any $x \in X$*

$$\int_X |\langle k_x|k_y\rangle| d\mu(y) < \infty$$

Then the following are equivalent.

(a). $T_\mu \in S_p$ (Schatten Class $p \geq 1$)

(b). $\tilde{\mu} \in L^p(d\lambda)$

(c). $\bar{\mu}_r \in l^p(\mathbb{N})$, $\forall r > 0$

Proof. We prove the theorem in the natural order: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$

(a)⇒(b). By assumption (a), H has an $O.N.B$ which consists of countably many eigenvectors of T_μ denoted by $\{f_i\}_{i=1}^{\infty}$. In addition $\forall i \in \mathbb{N}$, let λ_i be the eigenvalue of f_i, we have $\sum_i \lambda_i^p < \infty$.

$$
\begin{aligned}
\int\limits_X \tilde{\mu}(x)^p d\lambda(x) &= \int\limits_X \langle T_\mu k_x | k_x \rangle^p \, d\lambda(x) \\
&\leq \int\limits_X \left\langle T_\mu^p k_x | k_x \right\rangle d\lambda(x) \quad \text{(by the lamma)} \\
&= \int\limits_X \left\langle T_\mu^p \sum_i \langle k_x | f_i \rangle \, f_i \middle| \sum_j \langle k_x | f_j \rangle \, f_j \right\rangle d\lambda(x) \\
&= \int\limits_X \left\langle \sum_i \langle k_x | f_i \rangle \, T_\mu^p f_i \middle| \sum_j \langle k_x | f_j \rangle \, f_j \right\rangle d\lambda(x) \\
&= \int\limits_X \left\langle \sum_i \langle k_x | f_i \rangle \, \lambda_i^p f_i \middle| \sum_j \langle k_x | f_j \rangle \, f_j \right\rangle d\lambda(x) \text{ (by spectral theorem)} \\
&= \int\limits_X \sum_i | \langle k_x | f_i \rangle |^2 \lambda_i^p d\lambda(x) \\
&= \sum_i \int\limits_X | \langle k_x | f_i \rangle |^2 \lambda_i^p d\lambda(x) \\
&= \sum_i \int\limits_X | \langle k_x | f_i \rangle |^2 d\lambda(x) \lambda_i^p \\
&= \sum_i \| f_i \| \lambda_i^p \\
&= \sum_i \lambda_i^p < \infty
\end{aligned}
$$

(b)⇒(c) $\forall r > 0$, $\{F_n^r\}$ is the covering of X associated to r, x_n is some point in F_n^r for $\forall n \in \mathbb{N}$.

$$\begin{aligned}
\int\limits_X \tilde{\mu}(x)^p d\lambda(x) &= \int\limits_X \langle T_\mu k_x | k_x \rangle^p \, d\lambda(x) \\
&= \int\limits_X \left(\int\limits_X | \langle k_x | k_y \rangle |^2 d\mu(y) \right)^p d\lambda(x) \\
&= \sum_n \int\limits_{F_n} \left(\int\limits_X | \langle k_x | k_y \rangle |^2 d\mu(y) \right)^p d\lambda(x) \\
&\geq \sum_n \int\limits_{F_n} \left(\int\limits_{G_n} | \langle k_x | k_y \rangle |^2 d\mu(y) \right)^p d\lambda(x) \\
&\geq \sum_n \int\limits_{F_n} \left(\int\limits_{G_n} D_{2r}^2 d\mu(y) \right)^p d\lambda(x) \\
&= D_{2r}^{2p} \sum_n \int\limits_{F_n} \mu(G_n)^p d\lambda(x) \\
&= D_{2r}^{2p} \sum_n \bar{\mu}_r(n)^p \lambda(F_n) \\
&= D_{2r}^{2p} A_r \sum_n \bar{\mu}_r(n)^p \\
&= C \sum_n \bar{\mu}_r(n)^p
\end{aligned}$$

By assumption (b) $\int\limits_X \tilde{\mu}(x)^p d\lambda(x) < \infty$, it follows that

$$\sum_n \bar{\mu}_r(n)^p < \infty$$

(c)⇒(a) $\bar{\mu}_r \in l^p(\mathbb{N})$ implies $\bar{\mu}_r \in \mathcal{C}_0(\mathbb{N})$. We know T_μ is compact from theorem 2. By spectral theorem, T_μ has countable many eigenvalues denoted by $\{\lambda_i\}_{i=1}^{\infty}$. Let f_i be the eigenvector of λ_i with norm 1, we have

$$\begin{aligned}
\sum_i \lambda_i^p &= \sum_i \langle T_\mu f_i | f_i \rangle^p \\
&= \sum_i \left(\int_X |\langle f_i | k_x \rangle|^2 d\mu(x) \right)^p \\
&\leq \sum_i \left(C_1 \sum_n \mu(F_n) \int_{G_n} |\langle f_i | k_y \rangle|^2 d\lambda(y) \right)^p \quad \text{(by theorem } 1(d) \Rightarrow (e)\text{)} \\
&= C_1^p \sum_i \left(\sum_n \mu(F_n) \int_{G_n} |\langle f_i | k_y \rangle|^2 d\lambda(y) \right)^p \\
&\leq C_1^p \sum_i C_2 \sum_n \mu(F_n)^p \int_{G_n} |\langle f_i | k_y \rangle|^2 d\lambda(y) \text{ (by generalized Hölder inequality)} \\
&= C_1^p C_2 \sum_n \mu(F_n)^p \int_{G_n} \sum_i |\langle f_i | k_y \rangle|^2 d\lambda(y) \\
&= C_1^p C_2 \sum_n \mu(F_n)^p \lambda(G_n) \\
&\leq C_1^p C_2 B_r \sum_n \mu(F_n)^p \\
&= C \sum_n \mu(G_n)^p \\
&= C \sum_n \bar{\mu}_r(n)^p < \infty
\end{aligned}$$

□

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