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Robust Parameter Estimation in the Weibull and the Birnbaum-Saunders Distribution

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by
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Abstract

This paper concerns robust parameter estimation of the two-parameter Weibull distribution and the two-parameter Birnbaum-Saunders distribution. We use the proposed method to estimate the distribution parameters from (i) complete samples with and without contaminations (ii) type-II censoring samples, in both distributions. Also, we consider the maximum likelihood estimation and graphical methods to compare the maximum likelihood estimation and graphical method with the proposed method based on quantile. We find the advantages and disadvantages for those three different methods.

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Chapter 1

Introduction

The Weibull distribution is named for its inventor, Waloddi Weibull. Because of its versatility and relative simplicity, the Weibull distribution is widely used in reliability engineering and widely employed as a model in life testing. The Birnbaum-Saunders distribution is derived to model times to failure for metals subject to fatigue. Both distributions are worth considering. The method of the maximum likelihood is one of the most popular technique for estimation. But for the Weibull distribution the calculations involved are not always simple. And for the Birnbaum-Saunders distribution it's hard to derive the closed-form expressions for both the shape and the scale parameter by the maximum likelihood estimation. The proposed method is in a closed-form so it is easier to obtain. In addition it is robust under contamination.

In this paper, we also consider censored samples from the Weibull distribution and the Birnbaum-Saunders distribution. Especially we use the type-II censoring samples. Type-II censoring occurs if an experiment has a set number of subjects or items and stops the experiment when a predetermined number is observed to have failed; the remaining subjects are then right-censored. For more details about censoring, see Lawless (1982). In this paper, we will use the proposed method to estimate both complete and censored observations. Then we compare the proposed method with the maximum likelihood estimation and graphical method and discuss the results.

Chapter 2

Maximum Likelihood Estimation

2.1 Maximum Likelihood Estimation in the Weibull Distribution

2.1.1 Estimate the Weibull Distribution without Censoring

The probability density function of the two-parameter Weibull distribution is

$$f(x; \beta, \delta) = \begin{cases} \frac{\beta}{\delta} \left(\frac{x}{\delta}\right)^{\beta-1} e^{-(x/\delta)^\beta} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The cumulative distribution function of Weibull distribution is

$$F(x; \beta, \delta) = 1 - e^{-(x/\delta)^\beta} \quad for \quad x \geq 0,$$

and $F(x; \delta, \beta) = 0$ for $x < 0$, where β is shape parameter and δ is scale parameter which are positive.

For the pdf given above, we can find the likelihood function

$$
\begin{aligned}
L(\beta, \delta) &= \prod_{i=1}^{n} f(x_i; \beta, \delta) \\
&= (\frac{\beta^n}{\delta^n}) \prod_{i=1}^{n} (\frac{x_i}{\delta})^{\beta-1} e^{-(x_i/\delta)^\beta}, \\
&= (\frac{\beta}{\delta})^n e^{-\sum_{i=1}^{n} (\frac{x_i}{\delta})^\beta} \prod_{i=1}^{n} (\frac{x_i}{\delta})^{\beta-1}.
\end{aligned}
$$

The log-likelihood function is

$$
\begin{aligned}
l(\beta, \delta) &= \ln L(\beta, \delta) = n\ln\beta - n\ln\delta - \sum_{i=1}^{n} (\frac{x_i}{\delta})^\beta + (\beta - 1)\ln(\prod_{i=1}^{n} \frac{x_i}{\delta}), \\
&= n\ln\beta - n\ln\delta - \sum_{i=1}^{n} \frac{x_i{}^\beta}{\delta^\beta} + (\beta - 1)\sum_{i=1}^{n} (\ln x_i) - n(\beta - 1)\ln\delta.
\end{aligned}
$$

Differentiating the above log-likelihood with respect to δ and β, we have

$$
\begin{aligned}
\frac{\partial l}{\partial \delta} &= 0 - \frac{n}{\delta} + \beta \sum_{i=1}^{n} (x_i^\beta \delta^{-\beta-1}) - \frac{n(\beta - 1)}{\delta}, \\
\frac{\partial l}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} (\ln x_i) - n\ln\delta - \frac{\sum_{i=1}^{n} [x_i^\beta \ln(x_i)] - \ln(\delta) \sum_{i=1}^{n} x_i^\beta}{\delta^\beta}.
\end{aligned}
$$

Setting the above equations equal to zero, we obtain

$$
\delta = \frac{(\sum_{i=1}^{n} x_i^\beta)^{\frac{1}{\beta}}}{n^{\frac{1}{\beta}}}, \tag{2.1}
$$

and

$$
0 = \frac{n}{\beta} + \sum_{i=1}^{n} (\ln x_i) - n\ln\delta - \frac{\sum_{i=1}^{n} [x_i^\beta \ln(x_i)] - \ln(\delta) \sum_{i=1}^{n} x_i^\beta}{\delta^\beta}. \tag{2.2}
$$

Substituting (2.1) into (2.2) and solving for β, we obtain β, which is the MLE. And then

we can derive δ easily.

2.1.2   Estimate the Weibull Distribution with Censoring

Now we consider censored samples under the Weibull distribution. We will derive the likelihood function with censored observations.

We denote $d_i = 1$ if the ith observation is not censored and $d_i = 0$ if the ith observation is censored. And n_1 is the predetermined number. (An experiment has a set number of subjects or items and stops the experiment when the predetermined number are observed to have failed.)

The cdf of the Weibull distribution is

$$F(x;\beta,\delta) = 1 - e^{-(x/\delta)^\beta}.$$

Then we can get

$$\overline{F}(X_{n_1};\beta,\delta) = 1 - F(X_{n_1};\beta,\delta) = 1 - (1 - e^{-(X_{n_1}/\delta)^\beta}) = e^{-(X_{n_1}/\delta)^\beta}.$$

Then the log-likelihood in censored cases will be:

$$
\begin{aligned}
l(\beta,\delta) = \ln L(\beta,\delta) \quad &= \quad \ln\left(\prod_{d_i=1} f(x_i;\beta,\delta) \prod_{d_i=0} \overline{F}(x_i;\beta,\delta)\right) \\
&= \quad \sum_{i=1}^{n} d_i\left(\ln\beta - \ln\delta - \frac{x_i{}^\beta}{\delta^\beta} + (\beta-1)\ln x_i - (\beta-1)\ln\delta\right) + \sum_{i=1}^{n}(1-d_i)\left(-\frac{x_i{}^\beta}{\delta^\beta}\right).
\end{aligned}
$$

Then easily we can derive the log-likelihood for type-II censoring. For known n_1 and the order statistics $\{X_{(1)}, X_{(2)}, \ldots, X_{(n_1)}\}$ of the sample $\{X_1, X_2, \ldots, X_n\}$ are given (where $n_1 < n$), and $X_{(j)} \geq X_{(n_1)}$, (where $\{j \in (n_1 + 1, n)\}$). And $d_{(1)}, \ldots, d_{(n_1)} = 1$, $d_{(n_1+1)}, \ldots, d_{(n)} = 0$.

Then the log-likelihood function will be

$$
\begin{aligned}
l(\beta, \delta) &= \ln L(\beta, \delta) = \ln[(\frac{\beta}{\delta})^{n_1} \cdot e^{-\sum_{i=1}^{n_1}(\frac{X_{(i)}}{\delta})^{\beta}} \\
&\quad \cdot \prod_{i=1}^{n_1}(\frac{X_{(i)}}{\delta})^{\beta-1} \cdot (1 - (1 - e^{-(\frac{X_{(n_1)}}{\delta})^{\beta}}))^{n-n_1}], \\
&= n_1\ln\beta - n_1\ln\delta - \frac{\sum_{i=1}^{n_1}X_{(i)}^{\beta}}{\delta^{\beta}} + (\beta - 1)\sum_{i=1}^{n_1}(\ln X_{(i)}) \\
&\quad -n_1(\beta - 1)\ln\delta - (n - n_1)\frac{X_{(n_1)}^{\beta}}{\delta^{\beta}}.
\end{aligned}
$$

Differentiating the above log-likelihood with respect to δ and β, we have

$$
\begin{aligned}
\frac{\partial l}{\partial \delta} &= \beta\frac{\sum_{i=1}^{n_1}\beta X_{(i)}^{\beta}}{\delta^{\beta+1}} - \frac{n_1\beta}{\delta} + \frac{\beta X_{(n_1)}^{\beta}(n - n_1)}{\delta^{\beta+1}}, \\
\frac{\partial l}{\partial \beta} &= \frac{n_1}{\beta} + \sum_{i=1}^{n_1}\ln X_{(i)} - n_1\ln\delta - \frac{\sum_{i=1}^{n_1}(X_{(i)}^{\beta}\ln X_{(i)})}{\delta^{\beta}} + \frac{(\ln\delta)(\sum_{i=1}^{n_1}X_{(i)}^{\beta})}{\delta^{\beta}} \\
&\quad + \frac{X_{(n_1)}^{\beta}\ln(\delta)(n - n_1)}{\delta^{\beta}} - \frac{X_{(n_1)}^{\beta}\ln(X_{(n_1)})(n - n_1)}{\delta^{\beta}}.
\end{aligned}
$$

Setting the above equations equal to zero, we obtain

$$
\delta = \frac{(\sum_{i=1}^{n_1}X_{(i)}^{\beta} + (n - n_1)X_{(n_1)}^{\beta})^{\frac{1}{\beta}}}{n_1^{\frac{1}{\beta}}}, \tag{2.3}
$$

and

$$
0 = \frac{n_1}{\beta} + \sum_{i=1}^{n_1}\ln X_{(i)} - n_1\ln\delta - \frac{\sum_{i=1}^{n_1}(X_{(i)}^{\beta}\ln X_{(i)})}{\delta^{\beta}}
$$

$$
+ \frac{(\ln\delta)(\sum_{i=1}^{n_1}X_{(i)}^{\beta})}{\delta^{\beta}} + \frac{X_{(n_1)}^{\beta}\ln(\delta) \cdot (n - n_1)}{\delta^{\beta}} - \frac{X_{(n_1)}^{\beta}\ln(X_{(n_1)}) \cdot (n - n_1)}{\delta^{\beta}}. \tag{2.4}
$$

Substituting (2.3) into (2.4) and solving for β, we obtain β, which is the MLE. And then we can derive δ.

2.2 The Maximum Likelihood Estimation in the Birnbaum-Saunders Distribution

2.2.1 Estimate the Birnbaum-Saunders Distribution without Censoring

In 1969, Birnbaum and Saunders described a life distribution model that could be derived from a physical fatigue process where crack growth causes failure. Since one of the best ways to choose a life distribution model is to derive it from a physical/statistical argument that is consistent with the failure mechanism, the Birnbaum-Saunders fatigue life distribution is worth considering.

The Birnbaum-Saunders distribution which is fatigue life distribution has several alternative formulations of probability density function. The general formula for the pdf of the Birnbaum-Saunders distribution is

$$f(t; \mu, \alpha, \beta) = \frac{\sqrt{\frac{t-\mu}{\beta}} + \sqrt{\frac{\beta}{t-\mu}}}{2\alpha(t-\mu)} \phi\left(\frac{\sqrt{\frac{t-\mu}{\beta}} - \sqrt{\frac{\beta}{t-\mu}}}{\alpha}\right) \quad (x > \mu; \alpha, \beta > 0),$$

where α is the shape parameter; μ is the location parameter; β is the scale parameter; and ϕ is the probability density function of the standard normal distribution.

We use the two-parameter probability density function of the Birnbaum-Saunders Distribution which is

$$f(t; \alpha, \beta) = \frac{1}{2\alpha\beta} \sqrt{\frac{\beta}{t}} \left(1 + \frac{\beta}{t}\right) \phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right], (t > 0).$$

The cumulative distribution function of the Birnbaum-Saunders distribution is

$$F(t; \alpha, \beta) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right],$$

where Φ is the cumulative distribution function of the standard normal distribution.

For the pdf given above, we can find the maximum likelihood estimator of α and β.

The likelihood function of Birnbaum-Saunders distribution is

$$L(\alpha, \beta) = \prod_{i=1}^{n} f(t_i; \alpha, \beta),$$

$$= \prod_{i=1}^{n} \frac{1}{2\alpha\beta} \sqrt{\frac{\beta}{t_i}} (1 + \frac{\beta}{t_i}) \phi[\frac{1}{\alpha}(\sqrt{\frac{t_i}{\beta}} - \sqrt{\frac{\beta}{t_i}})],$$

$$= \frac{1}{2^n \alpha^n \beta^n} \frac{\beta^{n/2}}{t_1 \cdots t_n} (1 + \frac{\beta}{t_1})(1 + \frac{\beta}{t_2}) \cdots (1 + \frac{\beta}{t_n}) \prod_{i=1}^{n} \phi(\frac{1}{\alpha}(\sqrt{\frac{t_i}{\beta}} - \sqrt{\frac{\beta}{t_i}})).$$

Then we can have the log-likelihood function

$$l(\alpha, \beta) = \ln L(\alpha, \beta),$$

$$= -n\ln 2 - n\ln\alpha - n\ln\beta + \frac{n}{2}\ln\beta - \sum_{i=1}^{n} \ln(t_i) + \sum_{i=1}^{n} \ln(1 + \frac{\beta}{t_i}) + \sum_{i} \ln(\phi(\frac{1}{\alpha}(\sqrt{\frac{t_i}{\beta}} - \sqrt{\frac{\beta}{t_i}}))).$$

Differentiating the above log-likelihood with respect to α and β, we have

$$\frac{\partial l}{\partial \alpha} = -\frac{n}{\alpha} + \frac{1}{\alpha^3} \sum_{i=1}^{n} (\sqrt{\frac{\beta}{t_i}} - \sqrt{\frac{t_i}{\beta}})^2,$$

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{n} \frac{1}{t_i \left(\frac{\beta}{t_i} + 1 \right)} - \frac{n}{2\beta} - \frac{\sum_{i=1}^{n} \left(\frac{1}{2t_i \sqrt{\frac{\beta}{t_i}}} + \frac{t_i}{2\beta^2 \sqrt{\frac{t_i}{\beta}}} \right) \left(\sqrt{\frac{\beta}{t_i}} - \sqrt{\frac{t_i}{\beta}} \right)}{\alpha^2}.$$

Setting the above equations equal to zero, we obtain

$$\alpha = \left(\frac{\sum_{i=1}^{n} \left(\sqrt{\frac{\beta}{t_i}} - \sqrt{\frac{t_i}{\beta}} \right)^2}{n} \right)^{\frac{1}{2}} \tag{2.5}$$

$$0 = \sum_{i=1}^{n} \frac{1}{t_i \left(\frac{\beta}{t_i} + 1 \right)} - \frac{n}{2\beta} - \frac{\sum_{i=1}^{n} \left(\frac{1}{2t_i \sqrt{\frac{\beta}{t_i}}} + \frac{t_i}{2\beta^2 \sqrt{\frac{t_i}{\beta}}} \right) \left(\sqrt{\frac{\beta}{t_i}} - \sqrt{\frac{t_i}{\beta}} \right)}{\alpha^2}. \tag{2.6}$$

Substituting (2.5) into (2.6) and solving for β, we obtain β, which is the MLE. And then we can derive α.

2.2.2 Estimate the Birnbaum-Saunders Distribution with Censoring

Also we can derive the likelihood function of the Birnbaum-Saunders Distribution in type-II censoring.

For known n_1 and the order statistics $\{t_{(1)}, t_{(2)}, \ldots, t_{(n_1)}\}$ of the sample $\{t_1, t_2, \ldots, t_n\}$ are given (where $n_1 < n$), and $t_{(j)} \geq t_{(n_1)}$,(where $\{j \in (n_1 + 1, n)\}$) we can derive the likelihood function of the Birnbaum-Saunders Distribution in type-II censoring.

$$
\begin{aligned}
L(\alpha, \beta) &= \prod_{i=1}^{n_1} f(t_{(i)}) \left(1 - F(t_{(n_1)})\right)^{n - n_1}, \\
&= \prod_{i=1}^{n_1} \frac{1}{2\alpha\beta} \sqrt{\frac{\beta}{t_{(i)}}} (1 + \frac{\beta}{t_{(i)}}) \phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t_{(i)}}{\beta}} - \sqrt{\frac{\beta}{t_{(i)}}}\right)\right] \left(1 - \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t_{(n_1)}}{\beta}} - \sqrt{\frac{\beta}{t_{(n_1)}}}\right)\right]\right)^{n - n_1}, \\
&= \frac{1}{2^{n_1}\alpha^{n_1}\beta^{n_1}} \frac{\beta^{n_1/2}}{t_{(1)} \cdots t_{(n_1)}} (1 + \frac{\beta}{t_{(1)}})(1 + \frac{\beta}{t_{(2)}}) \cdots (1 + \frac{\beta}{t_{(n_1)}}) \prod_{i=1}^{n_1} \phi\left(\frac{1}{\alpha}\left(\sqrt{\frac{t_{(i)}}{\beta}} - \sqrt{\frac{\beta}{t_{(i)}}}\right)\right) \\
&\quad \left(1 - \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t_{(n_1)}}{\beta}} - \sqrt{\frac{\beta}{t_{(n_1)}}}\right)\right]\right)^{n - n_1}.
\end{aligned}
$$

Then we can have the log-likelihood function with censored observations.

$$
\begin{aligned}
l(\alpha, \beta) &= -n_1 \ln 2 - n_1 \ln \alpha - n_1 \ln \beta + \frac{n_1}{2} \ln \beta - \sum_{i=1}^{n_1} \ln\left(t_{(i)}\right) + \sum_{i=1}^{n_1} \ln\left(1 + \frac{\beta}{t_{(i)}}\right) \\
&\quad + \sum_{i}^{n_1} \ln\left(\phi\left(\frac{1}{\alpha}\left(\sqrt{\frac{t_{(i)}}{\beta}} - \sqrt{\frac{\beta}{t_{(i)}}}\right)\right)\right) + (n - n_1)\ln\left(1 - \Phi\left[\frac{1}{\alpha}(\sqrt{\frac{t_{(n_1)}}{\beta}} - \sqrt{\frac{\beta}{t_{(n_1)}}})\right]\right).
\end{aligned}
$$

A root-finding routine is needed to solve for α and β. And we can see that is very difficult to derive the maximum likelihood estimator of α and β in this regular method.

Chapter 3

The Proposed Method Based on Quantile

3.1 Parameter Estimation Based on Quantile in the Weibull Distribution

3.1.1 Quantile Method

In the robust parameter estimation based on quantile method in the Weibull distribution, we use the median $\tilde{X}$ of the sample X and the 63.21% percentile to derive the scale and shape parameters.

In the cdf of the Weibull Distribution, we know that

$$F(X) = 1 - \mathrm{e}^{-(\frac{X}{\delta})^{\beta}}.$$

We let $X = \delta$, then we have

$$F(\delta) = 1 - \mathrm{e}^{-1} \approx 0.6321.$$

Thus, it is easily seen that $\hat{\delta}$ is the 0.6321 quantile. So using the above method, we will use

63.21% sample percentile for the estimate of scale parameter. We denote $\hat{\delta}$ by the 63.21% sample percentile, $\hat{\delta} = 63.21\%$ sample quantile. The breakdown point of $\hat{\delta}$ will be $1 - 63.21\% = 36.79\%$. The breakdown point of an estimator is the proportion of incorrect observations (*i.e.* arbitrarily large observations) an estimator can handle before giving an arbitrarily large result.

And also we can derive

$$\frac{1}{2} = e^{-(\frac{\tilde{X}}{\delta})^{\beta}},$$

$$\ln 2 = (\frac{\tilde{X}}{\delta})^{\beta},$$

$$\ln\ln 2 = \beta(\ln(\tilde{X}/\delta)).$$

Then we can have

$$\hat{\beta} = \frac{log(log(2))}{log(\tilde{X}/\hat{\delta})}, \tag{3.1}$$

to get the estimate of β.

For censored cases in the Weibull distribution, we will apply Kaplan-Meier method into the proposed method then it can solve for survival data of the Weibull distribution. For more about Kaplan-Meier, see E. L. Kaplan and Paul Meier(1958).

3.1.2   Properties of the Proposed Method in the Weibull Distribution

With out considering the sample, we can still get the mean and the variance of $\hat{\delta}$ by the known sample size. We use the pdf of the order statistic to calculate the mean and the variance of the $\hat{\delta}$.

We will easily see that $\hat{\delta} \approx X_{(i)}$, when $i \approx 0.6321 \cdot n$ (rounding).

If we assume $\beta = 2$ and $\delta = 3$ as the true value of the two parameters of the Weibull distribution, then we will have

$$f_{(0.6321 \cdot n)}(x) = \frac{n!}{(i-1)!(n-i)!}(2/9)x(e^{-\frac{x^2}{9}})^{n-i}(1-e^{-\frac{x^2}{9}})^{i-1},$$

$$E_i[x^2] = \int_0^\infty x^2 \frac{n!}{(i-1)!(n-i)!}(2/9)x(e^{-\frac{x^2}{9}})^{n-i}(1-e^{-\frac{x^2}{9}})^{i-1}dx,$$

$$E_i[x] = \int_0^\infty x \frac{n!}{(i-1)!(n-i)!}(2/9)x(e^{-\frac{x^2}{9}})^{n-i}(1-e^{-\frac{x^2}{9}})^{i-1}dx.$$

When n=10000, we have $i = 10000 \cdot 0.6321 = 6321$, $E_i[x^2] = 8.99872$ and $E_i[x] = 2.99972$, then $Var(\hat{\delta}) = 8.99872 - 2.99972^2 = 0.0003999216$ and $E[\hat{\delta}] = 2.99972 \approx 3$.

When n=20, we have $i = 20 \cdot 0.6321 = 12.642 \approx 13$, $E_i[x^2] = 9.04394$ and $E_i[x] = 2.97627$, then $Var(\hat{\delta}) = 9.04394 - 2.97627^2 = 0.18576$ and $E[\hat{\delta}] = 2.97627 \approx 3$.

3.2 Parameter Estimation Based on Quantile in the Birnbaum-Saunders Distribution

From the cumulative distribution function of the Birnbaum-Saunders distribution, we know that

$F(t; \alpha, \beta) = \Phi[\frac{1}{\alpha}(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}})]$.

Then if we assume $t = \beta$:

$$
\begin{aligned}
F(\beta) &= \Phi[\frac{1}{\alpha}(\sqrt{\frac{\beta}{\beta}} - \sqrt{\frac{\beta}{\beta}})], \\
&= \Phi(0), \\
&= \frac{1}{2}.
\end{aligned}
$$

We can see that

$$\hat{\beta} = \text{median}(t_1, \ldots, t_n), \tag{3.2}$$

where $(t_1, \ldots, t_n)$ are iid variables from Birnbaum-Saunders distribution. And the breakdown point of $\hat{\beta}$ is 50%.

By the properties of Birnbaum-Saunders distribution, we know that

$$
Y_i = \frac{1}{2}(\sqrt{\frac{t_i}{\beta}} - \sqrt{\frac{\beta}{t_i}}) \quad \sim \quad N(0, \frac{1}{4}\alpha^2)
$$

11

Since we have obtained $\hat{\beta}$, we can import $\hat{\beta}$ into above equation. Then for known $(t_1, \ldots, t_n)$ we can have $Y = (Y_1, \ldots, Y_n)$.

Since Y follows the normal distribution, we can have $Y \sim N(0, \sigma^2)$ and $\sigma^2 = \frac{1}{4}\hat{\alpha}^2$. Since $1.349\sigma = 2\Phi^{-1}(0.75) = IQR(Y)$, we have

$$1.349\sigma = 1.349\frac{1}{2}\alpha = IQR(Y), \tag{3.3}$$

$$\hat{\alpha} = 2\frac{IQR(Y)}{1.349}, \tag{3.4}$$

where $IQR(Y)$ is the interquartile range of the sample $(Y_1, \ldots, Y_n)$. The interquartile range, also called midspread or middle fifty, is a measure of statistical dispersion, being equal to the difference between the upper and lower quartiles, that is $IQR = Q_3 - Q_1$ and the breakdown point of $IQR(Y)$ is 25%. Then we will get $\hat{\alpha}$ and $\hat{\beta}$.

Chapter 4

Graphical Estimation Method

4.1 Graphical Estimation in the Weibull Distribution

There is a graphical method to estimate the parameters in a population in the Weibull distribution. For more detail see Razali, Salih and Mahdi(2009). In this paper we use the robust regression. Robust regression is an alternative to least squares regression when data are contaminated with outliers or influential observations, and it can also be used for the purpose of detecting influential observations. For more about robust regression see Hampel, Ronchetti, Rousseeuw, and Stahel(1986).

The cdf of the Weibull distribution is

$$F(x; \beta, \delta) = 1 - e^{-(x/\delta)^\beta}.$$

Then we can get the following functions

$$1 - F(x; \beta, \delta) = e^{-(x/\delta)^\beta},$$

$$-\ln(1 - F(x; \beta, \delta)) = \left(\frac{x}{\delta}\right)^\beta,$$

$$\ln(-\ln(1 - F(x; \beta, \delta))) = \beta \ln x - \beta \ln \delta. \tag{4.1}$$

Then we can set

$$y_i \quad = \quad \ln(-\ln(1 - F(x_i; \beta, \delta))); \tag{4.2}$$

$$x_i' \quad = \quad \ln x_i; \tag{4.3}$$

$$b_1 \quad = \quad \beta; \tag{4.4}$$

$$b_0 \quad = \quad -\beta \ln \delta, \tag{4.5}$$

and imported into (4.1). Then we will have a linear function $y_i = b_1 x_i' + b_0$. Then we use the robust linear regression to solve for b_1 and b_0, and by the setting we can have

$$\hat{\beta} = b_1,$$

$$\hat{\delta} = e^{-\frac{b_0}{b_1}}.$$

4.2 Graphical Estimation in the Birnbaum-Saunders Distribution

The cumulative distribution function of the Birnbaum-Saunders distribution is

$$F(t; \alpha, \beta) = \Phi[\frac{1}{\alpha}(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}})],$$

where Φ is the cumulative distribution function of the standard normal distribution.

We can linearize as follows:

$$p_i \quad = \quad \Phi[\frac{1}{\alpha}(\sqrt{\frac{t_i}{\beta}} - \sqrt{\frac{\beta}{t_i}})],$$

$$\Phi^{-1}(p_i) \quad = \quad \frac{1}{\alpha}\frac{1}{\sqrt{\beta}}\sqrt{t_i} - \frac{\sqrt{\beta}}{\alpha}\frac{1}{\sqrt{t_i}},$$

$$\Phi^{-1}(p_i) \cdot \sqrt{t_i} \quad = \quad \frac{1}{\alpha\sqrt{\beta}}t_i - \frac{\sqrt{\beta}}{\alpha},$$

$$= -(\frac{\sqrt{\beta}}{\alpha}) + (\frac{1}{\alpha\sqrt{\beta}})t_i, \qquad (4.6)$$

$$= C_0 + C_1 t_i, \qquad (4.7)$$

where $C_0 = -(\frac{\sqrt{\beta}}{\alpha})$ and $C_1 = (\frac{1}{\alpha\sqrt{\beta}})$.

Then we can estimate α and β as follows:

$$\alpha = -\frac{\sqrt{\beta}}{C_0},$$

also

$$\alpha = \frac{1}{\sqrt{\beta}C_1}.$$

Then we can have

$$\frac{1}{\sqrt{\beta}C_1} = -\frac{\sqrt{\beta}}{C_0}.$$

Solving for α and β, we have $\hat{\alpha}$ and $\hat{\beta}$ which are estimates for α and β.

$$\hat{\beta} = -\frac{C_0}{C_1}, \qquad (4.8)$$

$$\hat{\alpha} = -\frac{\sqrt{-\frac{C_0}{C_1}}}{C_0}. \qquad (4.9)$$

But in the Birnbaum-Saunders distribution, graphical method is not always useful. When the contamination is too large (*e.g.* noise=1000) and the sample size $n = 200$, the graphical method will have both C_0 and C_1 in the same sign. That causes $\hat{\beta} < 0$, then $\sqrt{\hat{\beta}}$ can not be solved now.

Chapter 5

Simulation Study and Results

5.1 Introduction

We start by generating $X = \{X_1, X_2, \ldots, X_n\}$ for a given shape parameter β and scale parameter δ in the Weibull distribution, and $t = \{t_1, t_2, \ldots, t_n\}$ for a given shape parameter α and scale parameter β in the Birnbaum-Saunders distribution.

For those complete data sets, we add noise to $X = \{X_1, X_2, \cdots, X_n\}$, or $t = \{t_1, t_2, \cdots, t_n\}$ by replacing the last value of the sample.

For type-II censoring data sets, we have n_1 order statistics $\{X_{(1)}, X_{(2)}, \ldots, X_{(n_1)}\}$ or $\{t_{(1)}, t_{(2)}, \ldots, t_{(n_1)}\}$ of the samples. Then we can use the methods in Chapter 2, Chapter 3 and Chapter 4 to estimate the two parameters in the observed order statistics. For comparing the advantages and disadvantages of the proposed method, the maximum likelihood estimation method, and graphical method, we repeat the simulation 10,000 times to obtain the MSE (mean square error) and RE (relative efficiency) for every method.

5.2 Comparing Methods in the Weibull Distribution

In this section, we will show the simulations and the results in the Weibull distribution for complete and censoring data.

5.2.1 In Complete Data of the Weibull Distribution

5.2.1.1 Simulation Settings and Process

We set $\beta = 2$, $\delta = 3$, and the sample size $n = 200$. We change the last observation in the sample to the different noises given by $0.01, 0.02, \ldots, 100$ using the equations (2.1) and (2.2), we obtain the estimated parameter through the MLE method.

Using the equations (3.1), we obtain the estimated parameter through the proposed method.

For graphical method, we use the equation (4.1) and use rlm function in R language to have b_0 and b_1, and use (4.6) to obtain $\hat{\delta}$ and $\hat{\beta}$.

5.2.1.2 Results and Conclusion

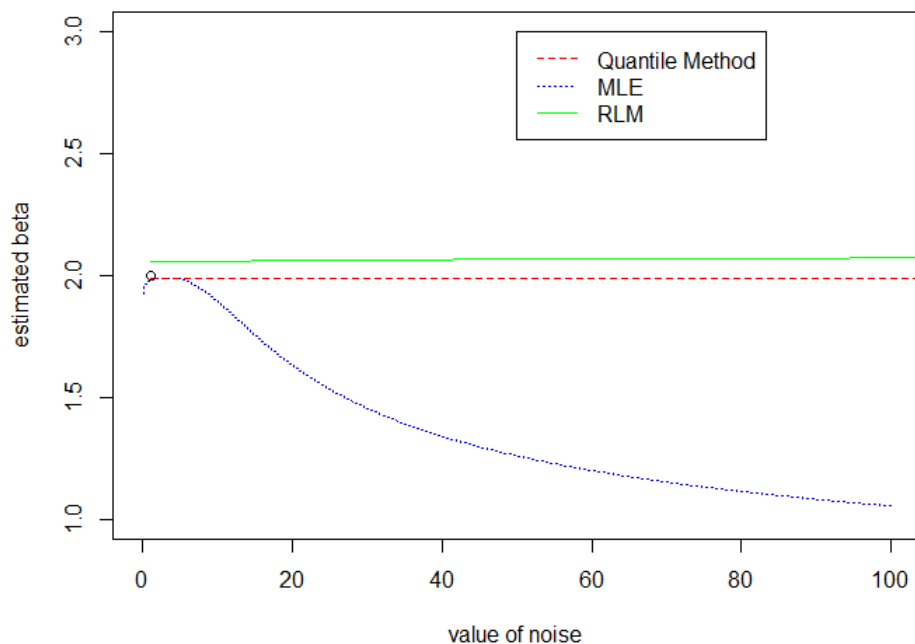**Difference in quantile method, MLE and RLM in the Weibull distribution.**

Figure 5.1: Difference in quantile, MLE and graphical method in estimate β in the Weibull distribution

Difference in quantile method, MLE and RLM in the Weibull distribution.
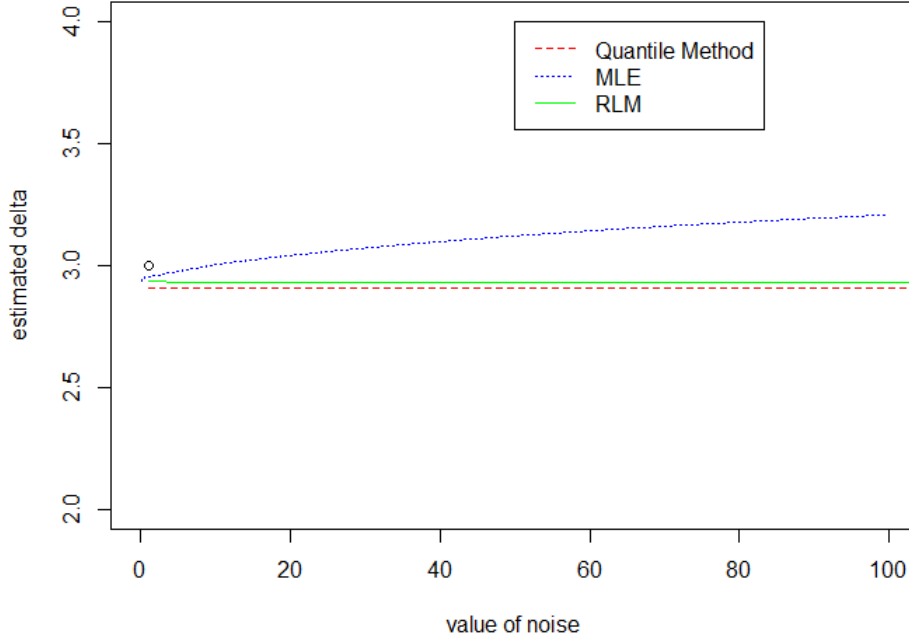

Figure 5.2: Difference in quantile, MLE and graphical method in estimate δ in the Weibull distribution

In Figures 5.1 and 5.2, we can see that for those Weibull samples which contain contaminations, the proposed method and graphical method are much better than the MLE method. We can also see that the proposed method is as good as graphical method.

5.2.2 In Type-II Censoring Data of the Weibull Distribution

5.2.2.1 Simulation Settings and Process

We set $\beta = 2$, $\delta = 3$, for type-II censoring the predetermined number is $n_1 = 190$, 180, 160, and the sample size $n = 200$. For three cases which are $n_1 = 190$, $n_1 = 180$, and $n_1 = 160$ observations in the samples, we will have the value of $X_{(190)}$, $X_{(180)}$, and $X_{(160)}$ as the last failed time. We can get $d_{(1)}, \ldots, d_{(n_1)} = 1$ and $d_{(n_1+1)}, \ldots, d_{(n)} = 0$, then $d = \{d_{(1)} = 1, \ldots, d_{(n_1)} = 1, d_{(n_1+1)} = 0, \ldots, d_{(n)} = 0\}$. We import them into the equations (2.3) and (2.4) to get the estimated parameter through the MLE method.

Using the equation (3.1), we obtain the estimated parameter through the proposed method.

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For survival data, we use the function "survfit(Surv(X,d) 1, type="kaplan-meier")" and "ECDF=c(1-out$surv)" in R language to have the empirical CDF for the sample and "approx(ECDF, out$time,xout=1-exp(-1))" in R language to estimate δ then use (3.1) to estimate β.

For graphical method, we use the equation (4.1) and use rlm function in R language to estimate b_0 and b_1, and use (4.6) to obtain $\hat{\delta}$ and $\hat{\beta}$.

5.2.2.2 Results and Conclusion

Table 5.1: cases for $n_1 = 190$ $n_1 = 180$ $n_1 = 160$ censored data set in repeat times= $10,000$ and sample size $n = 200$ of Weibull(2,3)

n_1	Methods	True value	Mean	Variance	MSE	$RE = \frac{MSE(MLE)}{MSE(Methods)}$
$n_1 = 190$	β_{MLE}		2.014624	0.01426682	0.02334235	
	β_{Robust}	$\beta = 2$	2.082453	0.1589585	0.2549708	0.09154911
	β_{RLM}		2.017383	0.0174336	0.02856898	0.81705227
	δ_{MLE}		2.998914	0.01261977	0.0207021	
	δ_{Robust}	$\delta = 3$	2.997964	0.01927802	0.03168974	0.6532745
	δ_{RLM}		2.990646	0.01295376	0.02126529	0.9735160
$n_1 = 180$	β_{MLE}		2.01915	0.01563721	0.02569902	
	β_{Robust}	$\beta = 2$	2.081754	0.1575484	0.2531769	0.1015062
	β_{RLM}		2.035482	0.02004398	0.03339141	0.7696297
	δ_{MLE}		2.997367	0.01301804	0.02134336	
	δ_{Robust}	$\delta = 3$	2.998802	0.01964424	0.03214544	0.6639622
	δ_{RLM}		2.974105	0.01350316	0.02257922	0.9452656
$n_1 = 160$	β_{MLE}		2.019376	0.01952878	0.03192576	
	β_{Robust}	$\beta = 2$	2.087008	0.1594661	0.2561862	0.1246193
	β_{RLM}		2.091688	0.02559376	0.04654212	0.6859541
	δ_{MLE}		2.997688	0.01388084	0.02265297	
	δ_{Robust}	$\delta = 3$	2.99847	0.0190771	0.03131216	0.723456
	δ_{RLM}		2.916488	0.01447393	0.0285543	0.793330

For those Weibull samples which are censored, we can see that (in table 5.1) the MLE method is better the proposed method and graphical method. But for fixed sample size, the increasing in

the amount of censored data will cause the increasing in accuracy of the proposed method and decreasing in accuracy of graphical method (from RE).

In (table 5.1)the β_{MLE} and δ_{MLE} are estimated by the MLE method,and β_{Robust} and δ_{Robust} are estimated by the proposed method, and β_{RLM} and δ_{RLM} are estimated by graphical method.

5.3 Comparing Methods in the Birnbaum-Saunders Distribution

In this section, we will show the simulations and the results in the Birnbaum-Saunders distribution for complete and censoring data.

5.3.1 In Complete Data of the Birnbaum-Saunders Distribution

5.3.1.1 Simulation Settings and Process

We set $\alpha = 1$, $\beta = 1$ and the sample size $n = 200$. We change the last observation in the sample to the different noises given by $0.01, 0.02, \ldots, 100$ using the equations (2.5) and (2.6), we obtain the estimated parameter through the MLE method.

Using the equations (3.2), we obtain the estimated β through the proposed method then we use (3.3) to obtain estimated α.

For graphical method, we use the equation (4.6) and use rlm function in R language to get C_0 and C_1, then we use (4.8) to obtain $\hat{\alpha}$ and $\hat{\beta}$.

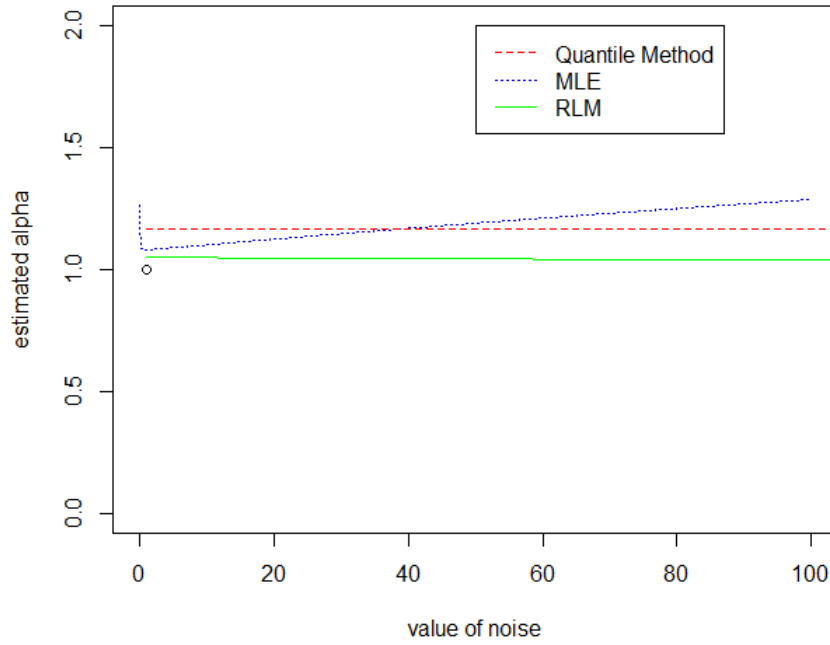**5.3.1.2   Results and Conclusion**

Figure 5.3: Difference in quantile, MLE and graphical method in estimate α in the Birnbaum-Saunders distribution

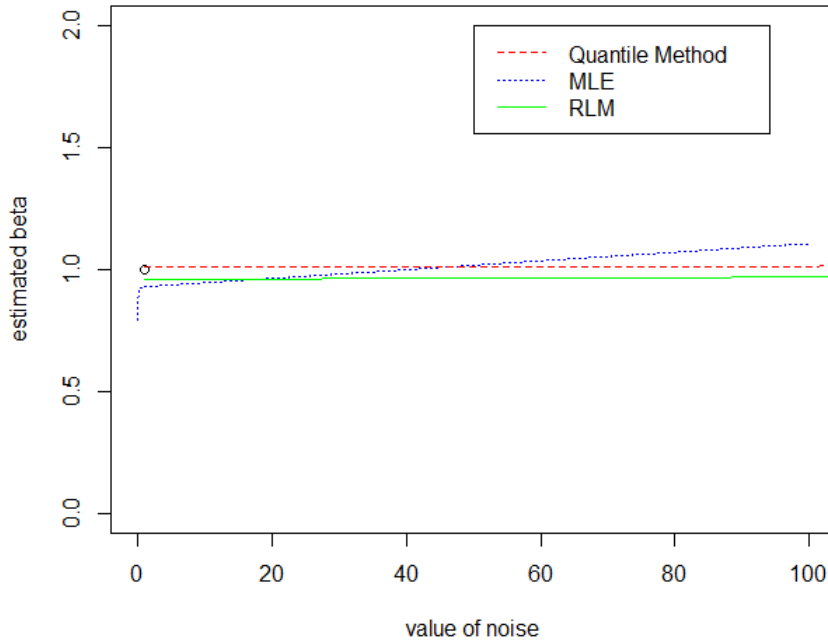**Difference in quantile method, MLE and RLM in BS distribution.**

Figure 5.4: Difference in quantile, MLE and graphical method in estimate β in the Birnbaum-Saunders distribution

In Figure 5.3 and 5.4 we can see that for those Birnbaum-Saunders samples which contain contaminations, the proposed method and graphical method are much better than the MLE method. We can also see that the proposed method is as good as graphical method.

5.3.2 In Type-II Censoring Data of the Birnbaum-Saunders Distribution

5.3.2.1 Simulation Settings and Process

We set $\alpha = 1$, $\beta = 1$, and predetermined number is $n_1 = 190$, 180, 160, and the sample size $n = 200$. For three cases which are $n_1 = 190$, $n_1 = 180$, and $n_1 = 160$ observations in the samples, we will have the value of $X_{(190)}$, $X_{(180)}$, and $X_{(160)}$ as the last failed time. We import them into the log-likelihood function to get the estimated parameter through the MLE method.

Using the equations (3.2 and 3.3), we obtain the estimated parameter through the proposed method.

For graphical method, we use the equation (4.6) and use rlm function in R language to get

C_0 and C_1, then we use (4.8) to obtain $\hat{\alpha}$ and $\hat{\beta}$.

5.3.2.2 Results and Conclusion

Table 5.2: cases for $n_1 = 190$ $n_1 = 180$ $n_1 = 160$ censored data set in repeat times= $10,000$ and sample size $n = 200$ of Birnbaum-Saunders(1,1)

n_1	Methods	True value	Mean	Variance	MSE	$RE = \frac{MSE(MLE)}{MSE(Methods)}$
$n_1 = 190$	α_{MLE}		0.996023	0.002661084	0.004385978	
	α_{Robust}	$\alpha = 1$	0.9946663	0.006107257	0.009972275	0.4398172
	α_{RLM}		0.983849	0.003349655	0.005654915	0.7756046
	β_{MLE}		1.000524	0.003914823	0.006422559	
	β_{Robust}	$\beta = 1$	1.003194	0.007404566	0.01216938	0.5277638
	β_{RLM}		1.000546	0.004536282	0.00740304	0.8675570
$n_1 = 180$	α_{MLE}		0.9946913	0.002986336	0.004926754	
	α_{Robust}	$\alpha = 1$	0.9912729	0.00655287	0.01080154	0.4561159
	α_{RLM}		0.9658971	0.003920155	0.008262377	0.5962877
	β_{MLE}		1.002994	0.004094853	0.006653023	
	β_{Robust}	$\beta = 1$	1.006539	0.007497102	0.01215567	0.5473184
	β_{RLM}		0.9945711	0.004640039	0.009616217	0.6918545
$n_1 = 160$	α_{MLE}		0.9962564	0.003581742	0.005798788	
	α_{Robust}	$\alpha = 1$	0.9938684	0.006774637	0.0112778	0.5141773
	α_{RLM}		0.9096282	0.003322905	0.01204432	0.4814542
	β_{MLE}		0.9995712	0.004518238	0.007374897	
	β_{Robust}	$\beta = 1$	1.004081	0.007381647	0.01212146	0.6084164
	β_{RLM}		0.9672553	0.005586472	0.011304232	0.6524014

For those Birnbaum-Saunders samples which are censored, we can see that (in Table 5.2) the MLE method is better than the proposed method and graphical method. But for fixed sample size, the increasing in the amount of censored data will cause the increasing in accuracy of the proposed method and decreasing in accuracy of graphical method (from RE).

In (table 5.2) α_{MLE} and β_{MLE} are estimated by the MLE method, and α_{Robust} and β_{Robust} are estimated by the proposed method, and α_{RLM} and β_{RLM} are estimated by graphical method.

Chapter 6

Future Work

Because our method is related to quantile, we need to study the Bahadur's representation to get the asymptotic variance of our method and compare it with the simulated variance.

6.1 Bahadur's Representation

Bahadur's representation Let $X_1, \ldots, X_n$ be iid random variables from a CDF F. Suppose that $F'(\xi_p)$ exists and is positive. Then

$$\hat{\xi}_p \quad = \quad \xi_p + \frac{F(\xi_p) - F_n(\xi_p)}{F'(\xi_p)} + o_p(\frac{1}{\sqrt{n}}).$$

Proof: Let $t \in R$, $\xi_{nt} = \xi_p + tn^{-1/2}$, $Z_n(t) = \sqrt{n}[F(\xi_{nt}) - F_n(\xi_{nt})]/F'(\xi_p)$, and $U_n(t) = \sqrt{(n)}[F(\xi_{nt}) - F_n(\hat{\xi}_p)]/F'(\xi_p)$. It can be shown that

$$Z_n(t) - Z_n(0) = o_p(1),$$

note that $|p - F_n(\hat{\xi}_p)| \leq n^{-1}$. Then,

$$
\begin{aligned}
U_n(t) \quad &= \quad \sqrt{n}[F(\xi_{nt}) - p + p - F_n(\hat{\xi}_p)]/F'(\xi_p), \\
&= \quad \sqrt{n}[F(\xi_{nt}) - p]/F'(\xi_p) + O(n^{-1/2}) \to t.
\end{aligned}
$$

Let $\eta_n = \sqrt{n}(\hat{\xi}_p - \xi_p)$. Then for any $t \in R$ and $\epsilon > 0$,

$$
\begin{aligned}
P(\eta_n \le t, Z_n(0) \ge t + \epsilon) &= P(Z_n(t) \le U_n(t), Z_n(0) \ge t + \epsilon), \\
&\le P(|Z_n(t) - Z_n(0)| \ge \epsilon/2) + P(|U_n(t) - t| \ge \epsilon/2) \to 0.
\end{aligned}
$$

Then we can get

$$
P(\eta_n \ge t + \epsilon, Z_n(0) \le t) \to 0.
$$

It follows that $\eta_n - Z_n(0) = o_p(1)$ with Lemma given below, which is the same as the assertion.

Lemma Let $\{X_n\}$ and $\{Y_n\}$ be two sequence of random variables such that X_n is bounded in probability and, for any real number t and $\epsilon > 0$, $\lim_n [P(X_n \le t, Y_n \ge t+\epsilon) + P(X_n \ge t+\epsilon, Y_n \le t)] = 0$. Then $X_n - Y_n \xrightarrow{p} 0$.

Proof. For any $\epsilon > 0$, there exists and $M > 0$ such that $P(|X_n| > M) \le \epsilon$ for any n, since X_n is bounded in probability. For this fixed M, there exists an N such that $2M/N < \epsilon/2$.

Let $t_i = -M + 2Mi/N, i = 0, 1, \cdots, N$. Then,

$$
\begin{aligned}
P(|X_n - Y_n| \ge \epsilon) &\le P(|X_n| \ge M) + P(|X_n| < M, |X_n - Y_n| \ge \epsilon), \\
&\le \epsilon + \sum_{i=1}^{N} P(t_{i-1} \le X_n \le t_i, |X_n - Y_n| \ge \epsilon), \\
&\le \epsilon + \sum_{i=1}^{N} P(Y_n \le t_{i-1} - \epsilon/2, t_{i-1} \le X_n) + P(Y_n \ge t_i + \epsilon/2, X_n \le t_i).
\end{aligned}
$$

This, together with the given condition, implies that

$$
\limsup_{n} P(|X_n - Y_n| \ge \epsilon) \le \epsilon.
$$

Since ϵ is arbitrary, we conclude that $X_n - Y_n \xrightarrow{p} 0$.

Remark Actually, Bahadur gave an a.s. order for $o_p(n^{-1/2})$ under the stronger assumption that F is twice differentiable at ξ_p with $F'(\xi_p) > 0$. The theorem stated here is in the form later given in Ghosh (1971). The exact a.s. order was shown to be $n^{-3/4}(\log \log n)^{3/4}$ by Kiefer (1967) in a landmark paper. However, the weaker version presented here suffices for proving the following

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CLTs.

The Bahadur representation easily leads to the following two joint asymptotic distributions.

Corollary Let $X_1, \ldots, X_n$ be iid random variables from a CDF F having positive derivatives at ξ_{p_j}, where $0 < p_1 < \cdots < p_m < 1$ are fixed constants. Then

$$\sqrt{n}[(\hat{\xi_{p_1}}, \ldots, \hat{\xi_{p_m}}) - (\xi_{p_1}, \ldots, \xi_{p_m})] \xrightarrow{d} N_m(0, D),$$

where D is the $m \times n$ symmetric matrix with element

$$D_{ij} = p_i(1 - p_j)/[F'(\xi_{p_i})F'(\xi_{p_j})], \quad i \leq j.$$

Proof By Bahadur's representation, we know that the $\sqrt{n}[(\hat{\xi_{p_1}}, \ldots, \hat{\xi_{p_m}}) - (\xi_{p_1}, \ldots, \xi_{p_m})]^T$ is asymptotically equivalent to $\sqrt{n}[\frac{F(\xi_{p_1}) - F_n(\xi_{p_1})}{F'(\xi_{p_1})}, \ldots, \frac{F(\xi_{p_m}) - F_n(\xi_{p_m})}{F'(\xi_{p_m})}]^T$ and thus we only need to derive the joint asymptotic distribution of $\frac{\sqrt{n}[F(\xi_{p_i}) - F_n(\xi_{p_i})]}{F'(\xi_{p_i})}$, $\quad i = 1, \ldots, m$. By the definition of ECDF, the sequence of $[F_n(\xi_{p_1}), \ldots, F_n(\xi_{p_m})]^T$ can be represented as the sum of independent random vectors

$$\frac{1}{n}\sum_{i=1}^{n}[I_{\{X_i \leq \xi_{p_1}\}}, \ldots, I_{\{X_i \leq \xi_{p_m}\}}]^T.$$

Thus, the result immediately follows from the multivariate CLT by using the fact that

$$E(I_{\{X_i \leq \xi_{p_k}\}}) = F(\xi_{p_k}), \quad Cov(I_{\{X_i \leq \xi_{p_k}\}}, I_{\{X_i \leq \xi_{p_l}\}}) = p_k(1 - p_l), \quad k \leq l.$$

But due to the time limits, we can't have the simulation and the theory done by this time, and we wish there will be other researchers can complete this part and compare the proposed method with the maximum likelihood method.

Chapter 7

Conclusions and Discussion

This paper introduces the maximum likelihood estimation, robust parameter estimation based on quantile and graphical estimation methods. The proposed robust method is based on the quantile of certain samples. We compared it with the maximum likelihood estimation and graphical methods.

In Chapter 2 and Chapter 3, estimation in the Weibull distribution and the Birnbuam-Saunders distribution with type II censored sample is considered in detail. The maximum likelihood estimators are calculated for both distributions. For both distributions, we carry out the simulation to study their behaviors. The simulation results are compared to that of quantile and graphical method. We can conclude that the proposed method for the shape parameter β and scale δ in the Weibull Distribution, or shape parameter α and scale parameter β in the Birnbaum-Saunders distribution are much more easily to be obtained than the maximum likelihood and graphical estimators. The proposed estimators for the shape and scale parameter is much better than the maximum likelihood estimator and as good as graphical estimators in those samples which contain contaminations. The proposed estimators for the shape and the scale parameter is not as good as the maximum likelihood estimators and graphical estimators for small censoring portion of a sample, but the mean square error of the proposed estimator decreases rapidly as the the amount of censored data increasing.

We can conclude that (i) the proposed method is much robust and this gives the great result on complete data with contain contaminations. (ii) The proposed method would have a decent efficiency property (future work needed). (iii) The proposed method is much simple than the

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maximum likelihood method. Since the closed-form of the proposed method can be derived easily.
(iv) The proposed method will be much more easily applied into the censored samples of the Weibull distribution and the Birnbaum-Saunders distribution as what we have shown in Chapter 5.

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