12-2012

Convex Hull Characterization of Special Polytopes in n-ary Variables

Ruobing Shen
Clemson University, rshen@clemson.edu

Convex Hull Characterization of Special Polytopes in n-ary Variables

---

A Master Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Mathematical Sciences

---

by
Ruobing Shen
Dec 2012

Accepted by:
Dr. Pietro Belotti, Committee Chair
Dr. Warren Adams
Dr. Matthew Saltzman

Abstract

This paper characterizes the convex hull of the set of n-ary vectors that are lexicographically less than or equal to a given such vector. A polynomial number of facets is shown to be sufficient to describe the convex hull. These facets generalize the family of cover inequalities for the binary case. They allow for advances relative to both the modeling of integer variables using base-n expansions, and the solving of n-ary knapsack problems with weakly super-decreasing coefficients.
Key words: convex hull, facets, n-ary optimization, knapsack problem.

Table of Contents

Chapter 1

Introduction

This work describes the convex hull of the set of n-ary vectors that are lexicographically smaller than a given vector $\boldsymbol{\alpha} \in \mathbb{R}^p$. An n-ary vector $\boldsymbol{a}$ is a vector whose components are integer between 0 and $n-1$, i.e., $\boldsymbol{a} \in \{0, 1, \ldots, n-1\}^p$. We say that a vector $\boldsymbol{a}$ is *lexicographically smaller* than a vector $\boldsymbol{b}$, and denote it with $\boldsymbol{a} \preceq \boldsymbol{b}$, if the first nonzero entry of $\boldsymbol{a} - \boldsymbol{b}$ is negative.

Let us define $N = \{0, 1, \ldots, n-1\}$. Given an n-ary vector $\boldsymbol{\alpha} \in N^p$, the task is to compute the convex hull of the set of n-ary vectors that is lexicographically less than or equal to $\boldsymbol{\alpha}$. In other words, the task is to compute the convex hull of the set S, denoted by $\text{conv}(S)$, where

$$S \equiv \{\boldsymbol{x} \in N^p : \ \boldsymbol{x} \preceq \boldsymbol{\alpha}\}, \tag{1.1}$$

and N^p denotes the set of integer p-vectors with components in N. We assume without loss of generality that $\alpha_1 \neq 0$ since otherwise we can fix $x_1 = 0$. Similarly, we assume that $\alpha_p \neq n-1$ since otherwise the only restrictions on x_p would be $0 \leq x_p \leq n-1$.

The set S of (1.1) has relevance in computing base-n expansions of integer variables. Suppose we are given a nonnegative integer variable y that is bounded above by some scalar β, where $n^{p-1} < \beta \leq n^p$. Further suppose that the vector $\boldsymbol{\alpha}$ is subsequently defined in terms of β so that $\beta = \sum_{j=1}^{p} n^{p-j}\alpha_j$. It is trivial to prove that for any β the vector $\boldsymbol{\alpha}$ is unique. Then the n-ary expansion of y computed by the set

$$T \equiv \left\{ (\boldsymbol{x}, y) \in N^p \times \mathbb{Z} : \ \mathbf{0} \leq \boldsymbol{x} \leq (n-1)\mathbf{1}, \ y = \sum_{j=1}^{p} n^{p-j} x_j, \ y \leq \beta \right\}, \tag{1.2}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$, is equivalently expressed by the set

$$W \equiv \left\{ (\boldsymbol{x}, y) \in N^p \times \mathbb{Z} : \ y = \sum_{j=1}^{p} n^{p-j} x_j, \ \boldsymbol{x} \in S \right\}. \tag{1.3}$$

In other words, we study the set of all encodings $\boldsymbol{x}$ of the values of y such that $y \leq \beta$ by imposing that $\boldsymbol{x} \preceq \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is an n-ary encoding of β. Consequently, an explicit representation of $\text{conv}(S)$ gives us $\text{conv}(T)$ as

$$\text{conv}(T) = \text{conv}(W) = \left\{ (\boldsymbol{x}, y) \in N^p \times \mathbb{Z} : \ y = \sum_{j=1}^{p} n^{p-j} x_j, \ \boldsymbol{x} \in \text{conv}(S) \right\}. \tag{1.4}$$

The convex hull representation for the base-2 expansion of integer variables was provided in [1] and [2], but no mention was made of any other base. As we will see, the cuts required to compute $\text{conv}(W)$ are substantially different from those found in [2] and are an extension thereof.

Now, given an n-ary vector $\boldsymbol{\alpha} \in N^p$, an n-ary vector $\boldsymbol{x}$ will have $\boldsymbol{x} \preceq \boldsymbol{\alpha}$ if and only if the following condition is satisfied. For each $i \in \{1, \ldots, p\}$ having $x_i > \alpha_i$, there must exist a $j < i$ so that $x_j < \alpha_j$. Equivalently, we must have

$$\sum_{j=1}^{i-1} (\max\{0, \alpha_j - x_j\}) \geq 1 \ \forall i \in \{1, \ldots, p\} : x_i > \alpha_i. \tag{1.5}$$

Inequality (1.5) is not needed for any i having $\alpha_i = n - 1$. Furthermore, for the special case where each α_i is binary and all x_i are restricted to be binary, inequalities (1.5) simplify to the minimal cover inequalities

$$\sum_{j<i:\ \alpha_j = 1} (1 - x_j) \geq x_i \ \text{ for all } i \in \{1, \ldots, p\} \text{ such that } \alpha_i = 0, \tag{1.6}$$

as found in [1, 2]. The challenge is to find inequalities in the spirit of (1.6) that model (1.5) for n-ary α_i and x_i.

Chapter 2

Facets and Convex Hull Characterization

Given an n-ary vector $\boldsymbol{\alpha} \in \mathbb{R}^p$, we begin by establishing properties of inequalities of the form $\sum_j \gamma_j x_j \leq \beta$ that are valid for S.

Lemma 2.0.1. *Given any inequality of the form $\sum_j \gamma_j x_j \leq \beta$ that is valid for S, the inequality $\sum_j \max\{0, \gamma_j\} x_j \leq \beta$ is also valid for S.*

Proof. Consider any inequality of the form $\sum_j \gamma_j x_j \leq \beta$ that is valid for S, and having an index t with $\gamma_t < 0$. If every $\boldsymbol{x} \in S$ has $x_t = 0$, then $0x_t + \sum_{j \neq t} \gamma_j x_j \leq \beta$ is also valid. Otherwise, for each $\bar{\boldsymbol{x}} \in S$ having $\bar{x}_t > 0$, the vector $\hat{\boldsymbol{x}}$ defined in terms of $\bar{\boldsymbol{x}}$ by $\hat{x}_j = \bar{x}_j$ for all $j \neq t$ and $\hat{x}_t = 0$ has $\hat{\boldsymbol{x}} \in S$ so that $0\bar{x}_t + \sum_{j \neq t} \gamma_j \bar{x}_j = \sum_j \gamma_j \hat{x}_j \leq \beta$ is also valid. □

By virtue of the above Lemma, and provided that $\boldsymbol{x} \geq 0$ is enforced, we need only consider those inequalities $\sum_j \gamma_j x_j \leq \beta$ having $\gamma_j \geq 0$ for all j. This consideration follows because any inequality of the form $\sum_j \gamma_j x_j \leq \beta$ having at least one $\gamma_j < 0$ is implied by $\sum_j \max\{0, \gamma_j\} x_j \leq \beta$ and $\boldsymbol{x} \geq 0$. In the remainder of this paper, we assume that $\gamma_j \geq 0$ for all j. Consider the following result which addresses a special family of these inequalities.

Theorem 2.0.2. *Given any n-ary vector $\boldsymbol{\alpha} \in \mathbb{R}^p$, an inequality of the form*

$$\sum_{j=1}^{p} \gamma_j (\alpha_j - x_j) \geq 0 \tag{2.1}$$

with $\gamma_j \geq 0$ for all $j = 1, \dots, p$, is valid for S if and only if

$$\gamma_j \geq \sum_{k>j} \gamma_k (n - 1 - \alpha_k) \;\; \forall\, j \in \{1, \dots, p-1\} : \; \alpha_j \neq 0. \tag{2.2}$$

Proof. (Only if)

Consider any inequality of the form (2.1) with $\gamma_j \geq 0$ for all $j = 1, \dots, p$ that is valid for S, and select any $t \in \{1, \dots, p-1\}$ having $\alpha_t \neq 0$. The vector $\bar{\boldsymbol{x}} \in S$ given by $\bar{x}_j = \alpha_j$ for $j < t$, $\bar{x}_t = \alpha_t - 1$, and $\bar{x}_j = n - 1$ for $j > t$ has

$$\gamma_t + \sum_{j=t+1}^{p} \gamma_j (\alpha_j - n + 1) \geq 0$$

when inserted into (2.1), verifying (2.2).

(If)

Consider any inequality of the form (2.1) with $\gamma_j \geq 0$ for all $j = 1, \dots, p$ that satisfies (2.2). Arbitrarily select $\bar{\boldsymbol{x}} \in S$. The proof is to show that $\bar{\boldsymbol{x}}$ satisfies (2.1). If $\bar{\boldsymbol{x}} = \boldsymbol{\alpha}$, then (2.1) is trivially satisfied. Otherwise, let t be the first entry of $\bar{\boldsymbol{x}}$ having $\bar{x}_j \neq \alpha_j$, so that $\bar{x}_t < \alpha_t$ as $\bar{\boldsymbol{x}} \in S$. Then

$$\sum_{j=1}^{p} \gamma_j (\alpha_j - \bar{x}_j) = \gamma_t (\alpha_t - \bar{x}_t) + \sum_{j>t} \gamma_j (\alpha_j - \bar{x}_j) \geq \gamma_t + \sum_{j>t} \gamma_j (\alpha_j - n + 1) \geq 0$$

where the equality results from $\bar{x}_j = \alpha_j$ for all $j < t$, and the inequality follows from the nonnegativity of $\boldsymbol{\gamma}$, together with $\bar{x}_t \leq \alpha_t - 1$ and $\bar{x}_j \leq n - 1$ for all $j > t$. This completes the proof. □

In the following, we prove that the only inequalities that are needed for our description are those that satisfy condition (2.2) at equality. To this purpose, we consider a subset of p inequalities of the form (2.1):

$$\sum_{j=1}^{p} \gamma_{ij} (\alpha_j - x_j) \geq 0 \quad \forall i = 1, 2, \dots, p : \alpha_i < n - 1, \tag{2.3}$$

where

$$\gamma_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j > i \text{ or } (j < i \text{ and } \alpha_j = 0) \\ \sum_{k=j+1}^{i} \gamma_{ik}(n-1-\alpha_k) & \text{if } j < i,\ \alpha_j \neq 0. \end{cases} \tag{2.4}$$

Note that (2.4) satisfy (2.2) at equality if $j < i$ and $\alpha_j \neq 0$.

Lemma 2.0.3. *The definition of coefficients γ_{ij} is equivalent to the following:*

$$\gamma_{ij} = \begin{cases} 1 & \textit{if } j = i \\ 0 & \textit{if } j > i \textit{ or } (j < i \textit{ and } \alpha_j = 0) \\ (n-1-\alpha_i)\prod_{j<k<i:\alpha_k \neq 0}(n-\alpha_k) & \textit{if } j < i \textit{ and } \alpha_j \neq 0. \end{cases} \tag{2.5}$$

Proof. The proof is by induction. Suppose that there are only m nonzero elements of $\boldsymbol{\alpha}$, denoted α_{j_l}, $l \in \{1, 2, \dots, m\}$, where m is any integer between 1 and i, and $1 = j_1 < j_2 < \cdots < j_m \leq i$ as per our assumption that $\alpha_1 \neq 0$. For $j = j_m$, $\gamma_{i,j_m} = \gamma_{ii}(n-1-\alpha_i) = (n-1-\alpha_i)$.

Suppose that, for j_l, where $j_1 \leq j_l \leq j_m$,

$$\gamma_{i,j_l} = \sum_{j_l < j \leq i:\alpha_j \neq 0} \gamma_{ij}(n-1-\alpha_j) = (n-1-\alpha_i) \prod_{j_l < j < i:\alpha_j \neq 0} (n-\alpha_j).$$

Then for every preceding coefficient $\gamma_{i,j_{l-1}}$ we have

$$\begin{aligned} \gamma_{i,j_{l-1}} &= \textstyle\sum_{j_{l-1} < j \leq i:\alpha_j \neq 0} \gamma_{ij}(n-1-\alpha_j) \\ &= \gamma_{i,j_l}(n-1-\alpha_{j_l}) + \textstyle\sum_{j_l < j \leq i:\alpha_j \neq 0} \gamma_{ij}(n-1-\alpha_j) \\ &= \gamma_{i,j_l}(n-1-\alpha_{j_l}) + \gamma_{i,j_l} \\ &= \gamma_{i,j_l}(n-\alpha_{j_l}) \\ &= (n-1-\alpha_i)(n-\alpha_{j_l}) \textstyle\prod_{j_l < j < i:\alpha_j \neq 0}(n-\alpha_j) \\ &= (n-1-\alpha_i) \textstyle\prod_{j_{l-1} < j < i:\alpha_j \neq 0}(n-\alpha_j). \end{aligned}$$

The proof is complete. □

Now consider the following polytope F:

$$F \equiv \left\{ \boldsymbol{x} \in \mathbb{R}^p : \ \mathbf{0} \leq \boldsymbol{x} \leq (n-1)\mathbf{1}, \ \sum_{j=1}^{p} \gamma_{ij}(\alpha_j - x_j) \geq 0 \ \forall i = 1, \ldots, p \right\}. \tag{2.6}$$

Note that for each i such that $\alpha_i = n-1$, we have $\gamma_{ij} = 0$ for all $j < i$ so that the associated inequality reduces to the upper bounding restriction $n - 1 - x_i \geq 0$.

We now follow up on Theorem 2.0.2 to further restrict the set of valid inequalities for S. We prove that the p inequalities

$$\sum_{j=1}^{p} \gamma_{ij}(\alpha_j - x_j) \geq 0 \qquad \forall i = 1, 2, \ldots, p$$

are sufficient to describe S and that any other inequality can be obtained as a conic combination of inequalities from this family.

Theorem 2.0.4. *Any inequality of the form*

$$\sum_{j=1}^{p} \beta_j(\alpha_j - x_j) \geq 0, \tag{2.7}$$

with $\beta_j \geq 0$ for all $j = 1, \ldots, p$, that is valid for S can be expressed as a conic combination of the inequalities (2.3).

Proof. Without loss of generality, we assume $\beta_p = 1$. Otherwise, the inequality can be normalized or, if $\beta_p = 0$, we can simply consider the inequality $\sum_{j=1}^{p-1} \beta_j(\alpha_j - x_j) \geq 0$. Let us also recall that

$$\begin{array}{lcll} \gamma_{pj} & = & \sum_{k>j} \gamma_{pk}(n-1-\alpha_k) & \forall j = 1, 2, \ldots, p : \alpha_j \neq 0 \\ \beta_j & \geq & \sum_{k>j} \beta_k(n-1-\alpha_k) & \forall j = 1, 2, \ldots, p : \alpha_j \neq 0, \end{array} \tag{2.8}$$

which implies that $\beta_j \geq \gamma_{pj}$ for all $j = 1, 2, \ldots, p$. Subtracting the p-th inequality of (2.3) from (2.7) yields

$$\sum_{j=1}^{p} (\beta_j - \gamma_{pj})(\alpha_j - x_j) \geq 0. \tag{2.9}$$

Let $\eta_k = \beta_k - \gamma_{pk}$, for $k = 1, 2, \ldots, p$. Note that $\eta_p = 0$ and, for all $j = 1, 2, \ldots, p$ such that $\alpha_j \neq 0$,

$$\eta_j = \beta_j - \gamma_{pj} \geq \sum_{k>j} (\beta_k - \gamma_{pk})(n - 1 - \alpha_k) = \sum_{k>j} \eta_k (n - 1 - \alpha_k)$$

Therefore the η_j's satisfy (2.2). If $\eta_j = 0$, $\forall j = 1, 2, \ldots, p$, then the proof is complete. Otherwise, suppose that $q < p$ is the largest index such that $\eta_q > 0$. Now re-set all η_j's by dividing them by η_q. The new vector $\boldsymbol{\eta}$ is such that $\eta_q = 1$. The procedure can be repeated on the inequality

$$\sum_{j=1}^{q} \eta_j (\alpha_j - x_j) \geq 0$$

by subtracting from it the q-th inequality in (2.3), $\sum_{j=1}^{q} \gamma_{qj}(\alpha_j - x_j) \geq 0$. Because this replicates the conditions described above on β and $q < p$, the procedure terminates in a finite number of steps and allows us to construct (2.7) as a conic combination of (2.3). □

The argument below shows that the inequalities of (2.6), together with integral restrictions on $\boldsymbol{x}$, define S of (1.1). Here, we let $I \equiv \mathbb{Z}^p \cap F$ denote the set of integral vectors in F.

Lemma 2.0.5. *$I = S$.*

Proof. We have $S \subseteq I$ by Theorem 2.0.2 since the inequalities defining (2.6) are valid for S. To show that $I \subseteq S$, consider any $\bar{\boldsymbol{x}} \in I$. If $\bar{\boldsymbol{x}} = \boldsymbol{\alpha}$, then $\bar{\boldsymbol{x}} \in S$. Otherwise, let q be the first entry of $\bar{\boldsymbol{x}}$ having $\bar{x}_j \neq \alpha_j$. Then we have

$$\alpha_q - \bar{x}_q = \gamma_{qq}(\alpha_q - \bar{x}_q) = \sum_{j \leq q} \gamma_{qj}(\alpha_j - \bar{x}_j) \geq 0, \tag{2.10}$$

where the first equality follows from the definition of $\gamma_{qq} = 1$ in (2.5), the second equality follows since $\alpha_j = \bar{x}_j$ for all $j < q$, and the inequality follows from (2.6) with $i = q$. Thus, $\bar{x}_q < \alpha_q$, and the proof is complete. □

Example. Suppose $n = 3$ so that we are considering ternary vectors, and let $p = 5$ with $\boldsymbol{\alpha}^T =$

(1, 1, 0, 2, 1). Further let $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 1 \end{bmatrix}$ denote the lower-triangular 5×5 matrix whose $(i,j)^{th}$ entry for $i \geq j$ records γ_{ij} from (2.5), and whose $(i,j)^{th}$ entry for $i < j$ is set to 0 to denote that the associated terms do not appear in the generated inequalities. Then (2.6) becomes, in matrix notation,

$$F \equiv \left\{ \boldsymbol{x} \in \mathbb{R}^5 : \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{2}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (1 - x_1) \\ (1 - x_2) \\ (0 - x_3) \\ (2 - x_4) \\ (1 - x_5) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Finally, observe that by placing integrality restrictions on the variables $\boldsymbol{x}$ in F, we have $I = (\boldsymbol{x} \in \mathbb{Z}^p \cap F)$ to obtain $I = S$.

Lemma 2.0.5 shows that an integral $\boldsymbol{x}$ has $\boldsymbol{x} \in F$ of (2.6) if and only if $\boldsymbol{x} \in S$. It turns out, however, that F is also an integral polytope so that $\text{conv}(\boldsymbol{x} \in (\mathbb{Z}^p \cap F)) = F$, giving us that $\text{conv}(S) = F$.

We now present our main result.

Theorem 2.0.6. *Given any $p \geq 1$ and any n-ary vector $\boldsymbol{\alpha} \in \mathbb{R}^p$, we have $\text{conv}(S) = F$, where S and F are as given in (1.1) and (2.6) respectively.*

Proof. Since the bounding restrictions $\mathbf{0} \leq \boldsymbol{x} \leq (n-1)\mathbf{1}$ of (2.6) are trivially satisfied for all $\boldsymbol{x} \in S$, and since the remaining inequalities of (2.6) are valid for S by the sufficiency (If) portion of Theorem 2.0.2, we have that $\text{conv}(S) \subseteq F$. We show that $F \subseteq \text{conv}(S)$ by demonstrating that all the extreme points of F are integer, using an inductive argument on the number p of variables $\boldsymbol{x}$.

For $k = 1$, F is simply $F_1 = \{x \in [0, n-1] : x_1 \leq \alpha_1\} = [0, \alpha_1]$, and its extreme point are 0 and α_1, both integer.

For $k = 2$, we have $F_2 = \{(x_1, x_2) : x_1 = 0,\ 0 \leq x_2 \leq \alpha_2\}$ if $\alpha_1 = 0$ (but we remind that this case is excluded by assumption) and $F_2 = \{(x_1, x_2) : 0 \leq x_1 \leq \alpha_1,\ 0 \leq x_2 \leq n-1,\ (n-1-$

$\alpha_2)x_1 + x_2 \leq (n-1-\alpha_2)\alpha_1 + \alpha_2\}$ if $\alpha_1 \neq 0$, both sets of which can be readily shown to have all integral extreme points for $\alpha_1, \alpha_2 \in N^2$.

Consider now the generic k. The set F_k is the polyhedron obtained by intersecting $[0, n-1]^k$ with the inequalities (2.3) for $i = 1, 2, \ldots, k$. By the induction hypothesis, the extreme points of F_k are all integer. An extreme point can be constructed by choosing k linearly independent inequalities from those defining F_k (note that F_k is defined by $3k$ inequalities: k inequalities (2.3), k lower bounds, and k upper bounds on $x_1, x_2, \ldots, x_k$).

Assume that $\alpha_{k+1} < n-1$. Otherwise, $(\boldsymbol{x}_k, x_{k+1}) \preceq (\boldsymbol{\alpha}_k, \alpha_{k+1})$ if and only if $\boldsymbol{x}_k \preceq \boldsymbol{\alpha}_k$, which is implied by the induction hypothesis. F_{k+1} is then a polyhedron in $\mathbb{R}^{k+1}$ constructed with the inequalities defining F_k and amended with the following:

$$\begin{gathered} 0 \leq x_{k+1} \leq n-1 \\ \textstyle\sum_{j \leq k+1} \gamma_{k+1,j}(\alpha_j - x_j) \geq 0. \end{gathered} \tag{2.11}$$

We want to prove that if the above procedure generates a feasible extreme point, then it is integer. We have the following four ways of constructing an extreme point by intersecting $k+1$ linearly independent inequalities.

(a) k inequalities from F_k plus $x_{k+1} \geq 0$ (resp. $x_{k+1} \leq n-1$). Suppose the k inequalities from F_k yield $\boldsymbol{x}$. If $\boldsymbol{x} \notin F_k$, nothing else is needed. Otherwise, this extreme point is obtained as $(\boldsymbol{x}, 0)$ (resp. $(\boldsymbol{x}, n-1)$), which is integer since $\boldsymbol{x}$ is an extreme point of F_k. If $\boldsymbol{x} \in F_k$ but $(\boldsymbol{x}, n-1)$ is infeasible, again the result holds.

(b) k inequalities from F_k plus (2.11). For this case, since all k inequalities have coefficient 0 on x_{k+1}, we can first compute $\boldsymbol{x}_k = (x_1, x_2, \ldots, x_k)$ using the first k inequalities. By the induction hypothesis, if $\boldsymbol{x}_k$ is feasible, then it is integer. If it is infeasible, nothing else is needed. Then we solve for x_{k+1} by substituting the integer $\boldsymbol{x}_k$ into (2.11). Since the coefficient of x_{k+1} in that inequality is 1, we get $x_{k+1} = \sum_{j=1}^{k+1} \gamma_{k+1,j}\alpha_j - \sum_{j=1}^{k} \gamma_{k+1,j}x_j$, which is integer. If $\boldsymbol{x}_{k+1} > n-1$ or $\boldsymbol{x}_{k+1} < 0$, then $\boldsymbol{x} = (\boldsymbol{x}_k, x_{k+1})$ is infeasible. Otherwise, $\boldsymbol{x} = (\boldsymbol{x}_k, x_{k+1})$ provides an integer extreme point.

(c) $k-1$ inequalities from F_k plus $x_{k+1} \leq n-1$ and (2.11).

(d) $k-1$ inequalities from F_k plus $x_{k+1} \geq 0$ and (2.11).

The last two cases need to be treated separately. Note that the case that combines $k-1$ inequalities from F_k plus $x_{k+1} \geq 0$ and $x_{k+1} \leq n-1$ can be omitted since it does not form $k+1$ linearly independent inequalities.

Proof for case (c)**.** Let $\boldsymbol{\alpha}_k$ be the subvector of $\boldsymbol{\alpha}$ with its first k components, and define $\boldsymbol{x}_k$ similarly. Let $\boldsymbol{\alpha}_{k+1} = (\boldsymbol{\alpha}_k, \alpha_{k+1})$, and suppose there are only m nonzero elements in $\boldsymbol{\alpha}_k$, denoted α_{i_j}, $j \in \{1, 2, \ldots, m\}$, where m is any integer between 1 and k, and $1 = i_1 < i_2 < \cdots < i_m \leq k$ as per our assumption that $\alpha_1 \neq 0$. Fix $x_{k+1} = n-1$. Consider the vector $\boldsymbol{\alpha}_k^\star \in N^k$ such that $\boldsymbol{\alpha}_k^\star \prec \boldsymbol{\alpha}_k$ and there is no $\boldsymbol{a} \in N^k : \boldsymbol{\alpha}_k^\star \prec \boldsymbol{a} \prec \boldsymbol{\alpha}_k$, i.e., the vector $\boldsymbol{\alpha}_k^\star$ that is lexicographically one less $\boldsymbol{\alpha}_k$. Then $(\boldsymbol{x}_k, n-1) \preceq (\boldsymbol{\alpha}_k, \alpha_{k+1})$ if and only if (i) $\alpha_{k+1} = n-1$ and $\boldsymbol{x}_k \preceq \boldsymbol{\alpha}_k$ or (ii) $\boldsymbol{x}_k \preceq \boldsymbol{\alpha}_k^\star$. The first case is trivial and we do not discuss it. Because $\alpha_{i_k} \neq 0$ for $k = 1, 2, \ldots, m$, we have $\boldsymbol{\alpha}_k^\star = (\alpha_{i_1}, \ldots, \alpha_{i_2}, \ldots, \alpha_{i_m} - 1, n-1, \ldots, n-1)$. We rewrite all inequalities of F_{k+1}, except for the inequalities written for $\alpha_i = 0$ and $i < i_m$, as follows:

$$\begin{aligned}
\sum_{j=1}^{i_k} \gamma_{i_k,j}(\alpha_j - x_j) &\geq 0 \quad \forall k = 1, 2, \ldots, m-1 \\
\sum_{j=1}^{i_m} \gamma_{i_m,j}(\alpha_j - x_j) &\geq 0 \qquad (2.12) \\
\sum_{j=1}^{i_m} \gamma_{i,j}(\alpha_j - x_j) + (\alpha_l - x_l) &\geq 0 \quad \forall i : i_m < i \leq k \qquad (2.13) \\
\sum_{j=1}^{i_m} \gamma_{k+1,j}(\alpha_j - x_j) + (\alpha_{k+1} - (n-1)) &\geq 0. \qquad (2.14)
\end{aligned}$$

Dividing (2.14) by $\alpha_{k+1} - (n-1)$ yields

$$\sum_{j=1}^{i_m} \bar{\gamma}_{k+1,j}(\alpha_j - x_j) + 1 \geq 0, \tag{2.15}$$

where $\bar{\gamma}_{k+1,j} = \gamma_{k+1,j}/(\alpha_{k+1} - (n-1))$.

Let us write the inequalities for the set $F_k^\star$. Recall that $F_k^\star$ is the set of vectors $\boldsymbol{x}_k$ such that $\boldsymbol{x}_k \preceq \boldsymbol{\alpha}_k^\star$.

$$\sum_{j=1}^{i_k} \gamma_{i_k,j}(\alpha_j^\star - x_j) \qquad \geq 0 \qquad \forall k = 1, 2, \ldots, m-1$$

$$
\begin{aligned}
\sum_{j=1}^{i_m} \gamma_{i_m,j}(\alpha_j^\star - x_j) &\geq 0 \\
x_i &\leq n-1 \quad \forall i : i_m < i \leq k
\end{aligned}
\tag{2.16}
$$

Note that (2.16) and (2.15) are the same, and we want to show that F'_{k+1} and $F^\star_k$ are identical, where $F'_{k+1} := \{\boldsymbol{x} \in F_{k+1} : x_{k+1} = n-1\}$. Note that the first $i_m - 1$ inequalities from the two sets are the same. We see that (2.12) can be constructed by multiplying (2.15) by $(n-1-\alpha_{i_m})/(n-\alpha_{i_m})$ and adding $(x_{i_m} - n + 1 \leq 0)$ multiplied by $1/(n-\alpha_{i_m})$, and is thus redundant. Also, we can show that (2.13) can be constructed by multiplying (2.15) by $(n-1-\alpha_l)$ and adding $(x_l \leq n-1-\alpha_l)$, and the last inequality holds because $\alpha_l = 0$. Thus, (2.13) is redundant for $l = i_m+1, i_m+2, \ldots, k$. Hence, the two sets F'_{k+1} and $F^\star_k$ are identical. Since, by the induction hypothesis, $F^\star_k$ is a polyhedron with integer extreme points only, the feasible extreme points generated by this case are all integer.

Example. Consider the example shown above, where $\boldsymbol{\alpha}^T = (1,1,0,2,1)$. Then $\boldsymbol{\alpha}^\star_4 = (1,1,0,1)$, and

$$
F^\star_4 \equiv \left\{ \boldsymbol{x} \in \mathbb{R}^4 : \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{2}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} (1-x_1) \\ (1-x_2) \\ (0-x_3) \\ (1-x_4) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.
$$

We notice that the first three inequalities of F_5 and $F^\star_4$ are the same and the fifth inequality of F_5, when substituting $x_5 = 2$, is just the fourth inequality of $F^\star_4$, thus $F'_5 = F^\star_4$.

Proof for case (d)**.** If $\alpha_{k+1} > 0$, we want to prove that (2.11) after letting $x_{k+1} = 0$ is redundant, so that this subcase does not generate feasible extreme points.

Suppose there are m nonzeros among the first k elements of $\boldsymbol{\alpha}$, where m is any integer between 1 and k. Let us index them with the set $\{i_j : j = 1, 2, \ldots, m\}$, and denote them as $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m} \neq 0$, where $i_1 < i_2 < \cdots < i_m$. Then after letting $x_{k+1} = 0$, (2.11) becomes:

$$
\sum_{j=1}^{m} \gamma_{k+1,i_j}(\alpha_{i_j} - x_{i_j}) + \alpha_{k+1} \geq 0. \tag{2.17}
$$

since $\gamma_{k+1,j} = 0$ for $j \notin \{i_1, i_2, \dots, i_m\}$. We can prove that (2.17) is redundant by proving that

$$\sum_{j=1}^{m} \gamma_{k+1,i_j}(\alpha_{i_j} - x_{i_j}) \geq 0 \tag{2.18}$$

is redundant. To this purpose, define $\xi_{i_h,j} = \prod_{q=j+1}^{h}(n - \alpha_{i_q})$. This implies that $\xi_{i_m,j} = \gamma_{k+1,j}/(n-1-\alpha_{k+1})$ for all $j \leq i_m$ such that $\alpha_j \neq 0$, and that $\xi_{i_m,i_m} = 1$. Therefore, (2.18) is equivalent to

$$\sum_{j=1}^{m} \xi_{i_m,i_j}(\alpha_{i_j} - x_{i_j}) \geq 0. \tag{2.19}$$

It is easy to show that $\xi_{i_m,j} = \gamma_{i_m,j} + \xi_{i_{m-1},j}$ for $j < i_m$ such that $\alpha_j \neq 0$, while $\xi_{i_{m-1},i_m} = 0$ since $\xi_{i_m,i_m} = \gamma_{i_m,i_m} = 1$. As a result, (2.19) is the sum of the two inequalities

$$\begin{aligned} \textstyle\sum_{j=1}^{m} \gamma_{i_m,i_j}(\alpha_{i_j} - x_{i_j}) &\geq 0 \\ \textstyle\sum_{j=1}^{m-1} \xi_{i_{m-1},i_j}(\alpha_{i_j} - x_{i_j}) &\geq 0, \end{aligned}$$

where $\xi_{i_{m-1},i_{m-1}} = 1$. The first inequality is analogous to (2.19) and hence subject to the same decomposition step. By repeating this procedure on the second inequality, we can prove that (2.18) can be obtained by a conic combination of the m inequalities for $\alpha_{i_j} \neq 0$, $j = 1, 2, \dots, m$, since all multipliers are positive. Because $\alpha_{k+1} > 0$, (2.17) is redundant. If $\alpha_{k+1} = 0$, then let $\boldsymbol{\alpha} = [\alpha_{i_1}, \dots, \alpha_{i_2}, \dots, \alpha_{i_m}, \dots, 0]$, and since $x_{k+1} = 0$, we denote $\boldsymbol{x} = (\boldsymbol{x}_k, 0)$, $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_k, 0)$, and therefore $\boldsymbol{x} \preceq \boldsymbol{\alpha}$ is equivalent to $\boldsymbol{x}_k \preceq \boldsymbol{\alpha}_k$. This completes the inductive step and hence the proof. □

Example. Consider one more time the example described above, where $\boldsymbol{\alpha}^T = (1, 1, 0, 2, 1)$. The fifth inequality of F_5, after fixing $x_5 = 0$, is

$$2(1 - x_1) + (1 - x_2) + (0 - x_3) + (2 - x_4) \geq -1. \tag{2.20}$$

It is easy to see that

$$2(1 - x_1) + (1 - x_2) + (0 - x_3) + (2 - x_4) \geq 0$$

can be constructed by summing the first three inequalities of F_5, which implies that (2.20) is redundant.

Chapter 3

Conclusions

In this paper, we study the convex hull of the set S of n-ary vectors that are lexicographically less than or equal to a given such vector $\boldsymbol{\alpha}$. Because of the nature of lexicographic orderings, we first show that this set is equivalent to a set I defined by a polynomial number of inequalities whose coefficients satisfy a specific property, together with upper and lower bounds on the variables and integrality constraints. Our main result shows that relaxing the integrality constraints yields a set F that is the convex hull of S. We believe that similar results can be made when we introduce the condition that a vector $\boldsymbol{x}$ be lexicographically greater than or equal to another vector $\boldsymbol{\beta}$. Also, these constraints yield a coefficient matrix that is lower triangular and has ones on the diagonal, and is hence unimodular; it is therefore of interest to study generalizations of this type of problem.

Bibliography

[1] R. Gillmann and V. Kaibel. Revlex-initial 0/1-polytopes. *Journal of Combinatorial Theory Series A*, 113(5):799–821, 2006.

[2] F.M. Muldoon, W.P. Adams, and H.D. Sherali. Ideal representations of lexicographic orderings and base-2 expansions of integer variables. *Operations Research Letters*, to appear, 2012.