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Vertex-Edge and Edge-Vertex Parameters in Graphs

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VERTEX-EDGE AND EDGE-VERTEX PARAMETERS IN GRAPHS

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ABSTRACT

The majority of graph theory research on parameters involved with domination, independence, and irredundance has focused on either sets of vertices or sets of edges; for example, sets of vertices that dominate all other vertices or sets of edges that dominate all other edges. There has been very little research on "mixing" vertices and edges. We investigate several new and several little-studied parameters, including vertex-edge domination, vertex-edge irredundance, vertex-edge independence, edge-vertex domination, edge-vertex irredundance, and edge-vertex independence.

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DEDICATION

This dissertation is dedicated to my family. First, I would like to thank my father, Steven, for always encouraging me to pursue my dreams and showing me the importance of a good education and hard work. Next, I would like to thank my late mother, Gail, for all of her love and and kind words throughout the years. I would also like to thank Tasha for sharing this journey with me. Her love, support, and encouragement were what got me through not only the last three years here at Clemson, but also the two years of my masters degree. I would not be where I am today without her walking beside me on this journey! I could not have finished this dissertation without the continuos love and support of the rest of my family; including my twin brother Tim, my sister Darcie, my late brother Matt, and my grandparents Lois and Edwin. I would like to thank each and everyone of them for their love and support as I pursued this life long dream!

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Chapter 1

Introduction

In this dissertation we shall present research of twelve new and several little-studied parameters, including vertex-edge domination, vertex-edge irredundance, vertex-edge independence, edge-vertex domination, edge-vertex irredundance, and edge-vertex independence. Before we proceed, we give some preliminary definitions. Throughout the remainder of this disseration all graphs shall be finite, undirected, loopless, and without multiple edges. Furthermore, we assume that they are non-trival, connected graphs, here after called *ntc graphs*.

Let $G = (V, E)$ be a graph with a vertex set $V(G)$ and edge set $E(G) \subseteq V(G) \times V(G)$ (or simply V and E, respectively, if the graph being considered is clear from the context). Furthermore, we say that a graph G has *order* $n = |V(G)|$ and *size* $m = |E(G)|$. A vertex v is *adjacent* to another vertex u if and only if there exists an edge $e = uv \in E(G)$. Two edges e and f are *adjacent* if they have a vertex in common. A vertex v is *incident* to an edge e and vice versa if $e = uv$ for some vertex $u \in V(G)$.

The *open neighborhood* of a vertex v is $N(v) = \{u \mid uv \in E(G)\}$, and the *closed neighborhood* of a vertex v is $N[v] = N(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V(G)$ is $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood of* S is $N[S] = \cup_{v \in S} N[v] = N(S) \cup S$.

A set S is a *dominating set* if every vertex $v \in V(G)$ is either in S or is adjacent to a vertex in S, that is, S is a dominating set if and only if $N[S] = V(G)$. The minimum cardinality of a dominating set of G is called the *domination number* and is denoted $\gamma(G)$. The maximum cardinality of a minimal dominating set of a graph G is called the *upper domination number* and is denoted $\Gamma(G)$. For a dominating set S, if S has cardinality $\gamma(G)$

then S is called a γ-*set*, and if S has cardinality $\Gamma(G)$ then S is called a Γ-*set*. We use a similar notation for other parameters, that is, for a generic parameter $\mu(G)$, we call a set satisfying the property for the parameter and having cardinality $\mu(G)$, a μ-set.

A set S is called *independent* if no two vertices in S are adjacent. The *independence number* of a graph G is the maximum cardinality of an independent set of vertices and is denoted $\beta_0(G)$. Similarly, a set $F \subseteq E$ of edges is *independent* if no two edges in F have a vertex in common. Independent sets of edges are usually called *matchings*. The *matching number* (or equivalently, the *edge independence number*) $\beta_1(G)$ equals the maximum cardinality of a matching in G. The *lower matching number* $\beta_1^1(G)$ equals the minimum cardinality of a maximal matching in G.

Notice that by definition if G is a graph of order n, then $\beta_1(G) \leq \frac{n}{2}$, and if $\beta_1(G) = \frac{n}{2}$ we say that any matching of cardinality $\frac{n}{2}$ is a *perfect matching*.

If a set S is both independent and dominating, then it is an *independent dominating set* of G. The minimum cardinality of an independent dominating set is the *independent domination number* and is denoted $i(G)$. It is also well known that $i(G)$ also equals the minimum cardinality of a maximal independent set in G.

A vertex v is a *private neighbor* of a vertex u in a set $S \subseteq V(G)$ if $N[v] \cap S = \{u\}$. The *private neighbor set* of u with respect to S is defined as $pn[u, S] = \{v \mid N[v] \cap S = \{u\}\}$. A set S is called *irredundant* if for every vertex u in S, $pn[u, S] \neq \emptyset$, that is, every vertex in S has at least one private neighbor. The *irredundance number* of a graph G is the minimum cardinality of a maximal irredundant set of vertices and is denoted $ir(G)$. The *upper irredundance number* of a graph G is the maximum cardinality of an irredundant set of vertices and is denoted $IR(G)$.

If a set S is irredundant then every vertex $u \in S$ has a private neighbor, say v. From the definition of a private neighbor we know that either $u = v$ or $v \in V - S$. If $u = v$ then we know that u is not adjacent to any vertex in $S - \{u\}$; in this case we say that *u is its own private neighbor*. If $v \in V - S$ then we say that v is an *external private neighbor* of u. We say that an irredundant set S is *open irredundant* if every vertex $u \in S$ has an external private neighbor. The *open irredundance number* $oir(G)$ equals the minimum cardinality of any maximal open irredundant set in G, while the *upper open irredundance number*

$OIR(G)$ equals the maximum cardinality of an open irredundant set in G. Since every open irredundant set is also an irredundant set, it follows that for any graph G, $OIR(G) \leq IR(G)$

Notice also that every open irredundant set $S = \{u_1, u_2, \ldots, u_k\}$ naturally defines a matching, let $\{v_1, v_2, \ldots, v_k\}$ be a set of private neighbors for each of the vertices in S. Then the set of edges $u_1v_2, u_2v_2, \ldots, u_kv_k$ is a matching. It follows therefore, that for any graph G, $OIR(G) \leq \beta_1(G)$.

Similarly, a set $F \subseteq E$ of edges is called *irredundant* if for every edge $uv \in F$ there exists an edge, say vw, that is adjacent to uv but to no other edge in F. In this case we say that vw is a *private edge* of edge uv. Note that if edge uv is not adjacent to any edge in F, then uv is its own private edge.

Let P be a property of sets $S \subseteq V$ of vertices in a graph $G = (V, E)$. For example,

$P_1 : S$ is an independent set,

$P_2 : S$ is a dominating set, or

$P_3 : S$ is an irredundant set.

A set $S \subseteq V$ having property P is called a *P-set*. A property P is called *hereditary* if every subset of a P-set is also a P-set. For example, property P_1: independent set and property P_3: irredundant set are both hereditary properties.

A property P is called *super-hereditary* if every superset of a P-set is also a P-set. The property P_2: dominating is a super-hereditary property.

A set S is a *maximal* P-set if no proper superset of S is a P-set. A set S is a $1-$*maximal P-set* if for every vertex $v \in V - S$, the set $S \cup \{v\}$ is not a P-set.

It follows from these definitions that every maximal P-set is also a 1-maximal P-set, but it can be seen that there are properties P for which a set can be a 1-maximal P-set but not a maximal P-set. However the following was observed in [13].

Proposition 1 If P is a hereditary property then every 1-maximal P-set is a maximal P-set.

This proposition is significant because it implies in order to determine if a given P-set is a maximal P-set for some hereditary property P, all you have to do is verify that it is a 1-maximal P-set, that is, show that for every vertex $v \in V - S$, $S \cup \{v\}$ is not a P-set.

In a similar manner, we say that a set S is a *minimal P-set* if no proper subset of S is a P-set. A set S is a 1-*minimal P-set* if for every vertex $u \in S$, $S - \{u\}$ is not a P-set.

It follows from these definitions that every minimal P-set is a 1-minimal P-set, but the converse statement need not be true. However, as observed in [13]:

Proposition 2 If P is a super-hereditary property then every 1-minimal P-set is a minimal P-set.

Thus, in order to determine if a set S is a minimal P-set for some super-hereditary property P, instead of searching if every possible subset of S is a P-set, one need only check to see if $S - \{u\}$ is a P-set for every vertex $u \in S$.

This research also considers a number of other well-studied graph parameters, including the following:

1. $\alpha_0(G)$, the *vertex covering number*, that is, the minimum number of vertices in a set $S \subseteq V$, called a *vertex cover*, having the property that for every edge $uv \in E$, either $u \in S$ or $v \in S$, or both.

2. $\alpha_1(G)$, the *edge covering number*, the minimum number of edges in a set $F \subseteq E$ having the property that every vertex $u \in V$ is incident to at least one edge in F.

3. $\gamma'(G)$, the *edge domination number*, the minimum number of edges in a set F such that every edge not in F has a vertex in common with at least one edge in F.

4. $i'(G)$, the *independent edge domination number*, the minimum number of edges in an independent edge dominating set F.

5. $\beta_1^1(G)$, the *lower matching number*, the minimum number of edges in a maximal matching.

6. $\beta_1(G)$, the *matching number*, the maximum number of edges in a maximal matching.

7. $\Gamma'(G)$, the *upper edge domination number*, the maximum number of edges in a minimal edge dominating set.

8. $ir'(G)$, the *edge irredundance number*, the minimum number of edges in a maximal irredundant set F, that is, such that every every edge in F has a private edge.

9. $IR'(G)$, the *upper edge irredundance number*, the maximum number of edges in an irredundant set of edges.

The following inequality chain, known as the *domination chain*, was first stated by Cockayne et al. in 1978 [5]. This chain has generated a considerable amount of interest among graph theory researchers, and it has done the same with us.

Theorem 3 [The Domination Chain] [5] For any graph G,

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

A similar edge domination chain exist, as follows:

Theorem 4 [The Edge Domination Chain] For any graph G,

$$ir'(G) \leq \gamma'(G) = i'(G) \leq \beta_1(G) \leq \Gamma'(G) \leq IR'(G).$$

In Chapter Four of [22], Peters introduced two new graph theory concepts: vertex-edge domination and edge-vertex domination. We can informally define vertex-edge domination by saying that a vertex v dominates the edges incident to v as well as the edges adjacent to these incident edges. Edge-vertex domination can informally be defined by saying that an edge $e = uv$ dominates vertices u and v as well as all vertices adjacent to u and v. Theorem 3, along with Chapter Four of [22], motivates much of this dissertation. The research presented here continues the study of this extension of the definition of domination, as well as extending the definitions of irredundance and independence to vertex-edge and edge-vertex variants. We have proved Theorems 5 and 6.

Theorem 5 [The Vertex-Edge Domination Chain] For any graph G,

$$ir_{ve}(G) \leq \gamma_{ve}(G) \leq i_{ve}(G) \leq \beta_{ve}(G) \leq \Gamma_{ve}(G) \leq IR_{ve}(G).$$

Theorem 6 [The Edge-Vertex Domination Chain] For any graph G without isolated vertices,

$$ir_{ev}(G) \le \gamma_{ev}(G) \le i_{ev}(G) \le \beta_{ev}(G) \le \Gamma_{ev}(G) \le IR_{ev}(G).$$

The parameters in Theorems 5 and 6 represent the vertex-edge (*ve*) and edge-vertex (*ev*) variants of the irredundance number, domination number, independent domination number, independence number, upper domination number, and upper irredundance number. We define the vertex-edge parameters in Chapter 3 and the edge-vertex parameters in Chapter 4.

The remainder of this dissertation is organized as follows. In Chapter 2 we present a survey of relevant work that has been published. Of the twelve parameters that we have defined for this dissertation, only two of them, namely γ_{ve} and γ_{ev}, have appeared in the literature. To our knowledge the other ten are new. Hence, this literature survey primarily considers γ_{ve} and γ_{ev}. Following this, Chapters 3 and 4 present respectively the vertex-edge and edge-vertex parameters along with the results that we have obtained. In Chapter 5 we present complexity results for these parameters. Finally in Chapters 6 and 7 we present some concluding remarks along with a list of open problems.

Chapter 2

Literature Review

The majority of research on the parameters involved with domination, independence, and irredundance has focused on either sets of vertices or sets of edges; for example, sets of vertices that dominate other vertices or sets of edges that dominate other edges. There has been little research on "mixing" vertices and edges.

An example where vertices and edges "mix" is in the study of total coverings, which are closely related to our vertex-edge and edge-vertex variants of domination. In the classical definition of covering, a vertex (edge) is said to *cover* all of its incident edges (vertices). In 1977 Alavi et al. introduced a new invariant for both coverings and matchings, which they called *total coverings* and *total matchings* (respectively) [1]. In these new invariants, a vertex *covers* itself, all adjacent vertices, and all incident edges. Also, an edge $e = uv$ covers itself, all adjacent edges, and its two vertices u and v. In Figure 2.1 (a) the vertex u covers the shaded vertices as well as the darkened edges and in Figure 2.1 (b), the edge e covers the vertices u and v as well as the darkened edges. A set $S \subseteq (V \cup E)$ is a *total covering* if every vertex and edge not in S is covered by some vertex or edge in S. The *total covering number* of a graph G, denoted by $\alpha_2(G)$, is the minimum cardinality of a total covering, and the *upper total covering number* of a graph G, denoted $\alpha'_2(G)$, is the maximum cardinality of a minimal total covering. A set $S \subseteq (V \cup E)$ is *independent* with respect to total coverings if for all elements $a \in S$, there does not exist an element $b \in S$ such that b covers a. The *total matching number* of a graph G, denoted by $\beta_2(G)$, is the maximum cardinality of a set $S \subseteq (V \cup E)$ such that S is independent with respect to total coverings, and the *lower total matching number* of a graph G, denoted by $\beta'_2(G)$, is the minimum cardinality of a maximal independent set $S \subseteq (V \cup E)$, with respect to total coverings. Only a handful of papers have been written on total coverings and total matchings. We now summarize the results in these papers.

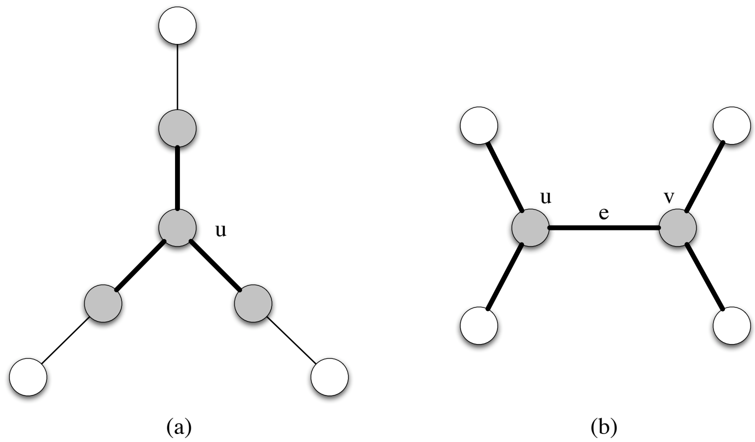

Figure 2.1: Examples of vertex total covers (a) and edge total covers(b)

In [1], Alavi et al. presented exact values and bounds on the total covering number, upper total covering number, total matching number, and lower total matching number for several classes of graphs. Table 2.1 is from [1] and summarizes these results.

From Table 2.1 we can derive Theorem 7.

Theorem 7 [1] If G is a connected graph of order $n \geq 2$, then

$$n \leq \alpha_2(G) + \beta_2(G) \leq \left\lfloor \frac{3n}{2} \right\rfloor - 1.$$

Erdös and Meir presented bounds on the total covering numbers and total matching numbers of graphs and their complements in [8]. We denote the complement of a graph G by $\overline{G}$. The following is a summary of the results of Erdös and Meir.

Theorem 8 [8] For every graph G of order n,

$$2\left\lfloor \frac{n}{2} \right\rfloor \leq \beta_2(G) + \beta_2(\overline{G}) \leq \left\lfloor \frac{3n}{2} \right\rfloor.$$

Theorem 9 [8] For every graph G of order n,

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \alpha_2(G) + \alpha_2(\overline{G}) \leq \left\lfloor \frac{3n}{2} \right\rfloor.$$

Parameters	K_n (complete graphs) $(n > 1)$	$K_{1,n}$ (stars)	$K_{m,n}$ (complete bipartite graphs) $(1 \le m \le n)$	P_n (paths) $(n > 1)$	C_n (cycles) $(n \ge 3)$	T_n^* (saturated stars)	G (general connected graphs) $(n \ge 1)$
α_2	$\left\lfloor \frac{n}{2} \right\rfloor$	1	m	$\left\lfloor \frac{1}{3} * \left\lceil \frac{6n}{5} \right\rceil \right\rfloor$	$\left\lfloor \frac{2n}{5} \right\rfloor$	$\left\lceil \frac{n}{2} \right\rceil$	$1 \le \alpha_2(G) \le \left\lfloor \frac{n}{2} \right\rfloor$
β_2'	$\left\lfloor \frac{n}{2} \right\rfloor$	1	m	$\left\lfloor \frac{1}{3} * \left\lceil \frac{6n}{5} \right\rceil \right\rfloor$	$\left\lfloor \frac{2n}{5} \right\rfloor$	$\left\lceil \frac{n}{2} \right\rceil$	$1 \le \beta_2'(G) \le n - 1$
β_2	$\left\lfloor \frac{n}{2} \right\rfloor$	n	n	$\left\lfloor \frac{2n-1}{3} \right\rfloor$	$\left\lceil \frac{2n}{3} \right\rceil$	$\left\lceil \frac{n}{2} \right\rceil$	$\left\lfloor \frac{n}{2} \right\rfloor \le \beta_2(G) \le n - 1$
α_2'	$n - 1$	n	$m + n - 2$	$\left\lfloor \frac{2n-1}{3} \right\rfloor$	$\left\lceil \frac{2n}{3} \right\rceil$	$\left\lceil \frac{n}{2} \right\rceil$	$\left\lfloor \frac{n}{2} \right\rfloor \le \alpha_2'(G) \le n - 1$

T_n^* denotes a tree of order $n = 2p + 1$ whose p leaves are each at distance two from one center vertex.

Table 2.1: Values of α_2, α_2', β_2, and β_2' for several classes of graphs.

Theorem 10 [8] For every connected graph G of order n,

$$\alpha_2(G) + \beta_2(G) \leq n + \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor.$$

Theorem 11 [8] For every connected graph G of order n,

$$\beta_2(G) \leq n - 2\sqrt{n} + 2, and$$

this bound is best possible.

Following [8], Meir [18] published Theorems 12, 13, and 14.

Theorem 12 [18] For every connected graph G of order $n \geq 2$ that does not contain a triangle,

$$\alpha_2(G) + \beta_2(G) \leq \frac{5n}{4}, and$$

this bound is best possible.

Theorem 13 [18] There exist arbitrarily large integers n and connected graphs G of order n such that

$$\alpha_2(G) + \beta_2(G) > \frac{5n}{4}.$$

Theorem 14 [18] Given $\varepsilon > 0$, there exist arbitrarily large integers n and connected graphs G of order n such that

$$\beta_2'(G) > (1 - \varepsilon)n.$$

In 1992 Alavi co-authored with Liu his second paper on total coverings [2]. They began by presenting the following definitions.

Definition 1 [26] A *matching* in a graph G is a set of edges no two of which have a vertex in common. The vertices incident to the edges of a matching M are said to be *saturated* by M; the others are *unsaturated*. A *perfect matching* (also called a 1-factor) in a graph is a matching that saturates every vertex.

Definition 2 [2] A *near-perfect matching* of G is a perfect matching of $G - v$ for some $v \in V(G)$.

Definition 3 [2] A graph G is *factor-critical* if $G - v$ has a 1-factor for any $v \in V(G)$.

Definition 4 [2] Given a graph G, we define

$D(G) = \{v \in V(G) \mid v \text{ is not saturated by any maximum matching of } G\}$,

$A(G) = \{v \in V(G) - D(G) \mid v \text{ is adjacent to at least one vertex in } D(G)\}$, and

$C(G) = V(G) - A(G) - D(G)$.

Theorem 15 [Galli-Edmonds' Structure Theorem] [2] For any graph G, the following structure properties hold:

1. The components of the subgraph induced by $D(G)$ are factor-critical.

2. The subgraph induced by $C(G)$ has a perfect matching.

3. If M is any maximal matching of G, then it contains a near-perfect matching of each component of $\langle D(G)\rangle$, a perfect matching of $\langle C(G)\rangle$, and matches all vertices of $A(G)$ with vertices in distinct components in $\langle D(G)\rangle$, where $\langle H\rangle$ represents the subgraph induced by the set H of vertices.

4. $\beta_1(G) = \frac{1}{2}[|V(G)| - k(\langle D(G)\rangle) + |A(G)|]$, where $k(G)$ equals the number of components in G.

Using Definitions 2, 3, and 4, and Theorem 15, Alavi and Liu derived the following results.

Lemma 16 [2] If G is a connected graph and $A(G) \neq \emptyset$, then $\alpha_2(G) \leq \beta_1(G)$.

Theorem 17 [2] If G is a connected graph, then $\alpha_2(G) \leq \beta_1(G) + 1$.

Theorem 18 [2] If G is a connected graph of odd order n and $\alpha_2(G) = \left\lceil \frac{n}{2} \right\rceil$, then G is factor-critical and $M_v \cup \{v\}$ is a minimum total cover of G for any $v \in V(G)$, where M_v is a perfect matching of $G - v$.

Theorem 19 [2] If G is a connected graph of even order n and $\alpha_2(G) = \frac{n}{2}$, then G has a perfect matching.

Theorem 20 [2] If G is a connected graph of even order n and $\alpha_2(G) = \frac{n}{2}$, then every minimum total cover of G contains at most one vertex if and only if G is a complete graph.

Theorem 21 [2] If G is a connected graph of odd order n and $\alpha_2(G) = \left\lceil \frac{n}{2} \right\rceil$, then every minimum total cover of G contains at most two vertices if and only if G is complete.

In [28], Zhang et al. present several relationships between the arboricity (not defined here), and the total independence and total covering numbers.

There are two papers on the total covering and total matching numbers for a graph G when G is restricted to a specific family of graphs. The first of these papers [19] is by Olejník and František and concentrates on the total covering and total matching numbers for k-uniform hypergraphs. In [19], they present a bound for the sum of the total covering number of a k-uniform hypergraph H and the total covering number of the complement of H; as well as a bound for the sum of the total matching number of a k-uniform hypergraph J and the total matching number of the complement of J. The second of the two papers is by Peled and Sun, [21], who determine bounds for the total covering number and total matching number for threshold graphs.

Domination is a well studied field of graph theory, but the vertex-edge and edge-vertex domination parameters have received very little attention since their introduction twenty years ago. In the fourth chapter of his 1986 Ph.D. thesis [22], Peters introduced vertex-edge and edge-vertex weak domination, which is what we call vertex-edge and edge-vertex domination. He presented several preliminary results on these two parameters. We now summarize these results.

We must first define vertex-edge and edge-vertex domination.

Definition 5 For a graph $G = (V, E)$, a vertex $u \in V(G)$ *ve-dominates* an edge $vw \in E(G)$ if

1. $u = v$ or $u = w$ (u is incident to vw), or

2. uv or uw is an edge in G (u is incident to an edge that is adjacent to vw).

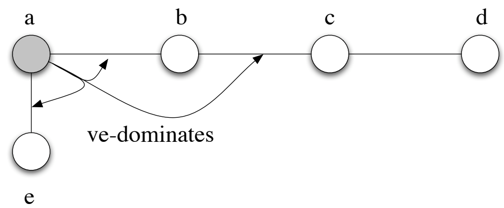

Figure 2.2: Vertex-Edge Domination

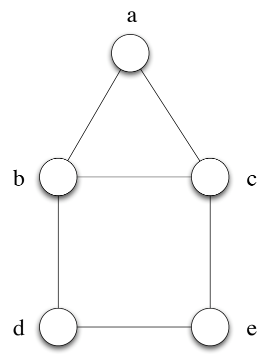

Figure 2.3: The House Graph

Figure 2.2 gives a graphical representation of the definition of *ve*-domination. Notice that the vertex *a* *ve*-dominates the edges *ae*, *ab*, and *bc*, but not the edge *cd*.

Definition 6 A set $S \subseteq V(G)$ is a *vertex-edge dominating set* (or simply a *ve*-dominating set) if for all edges $e \in E(G)$, there exists a vertex $v \in S$ such that v dominates e.

Definition 7 The minimum cardinality of a *ve*-dominating set of G is called the *vertex-edge domination number* (or simply *ve-domination number*), and is denoted by $\gamma_{ve}(G)$.

The graph in Figure 2.3 is known as the *house graph*. From Definition 5 it follows that vertex *a* *ve*-dominates the edges *ab*, *ac*, *bd*, *bc*, and *ce*. Notice that the set $S_1 = \{a\}$ is not a *ve*-dominating set since the edge *de* is not *ve*-dominated by S. But $S_2 = \{b\}$ and $S_3 = \{c\}$ are both *ve*-dominating sets.

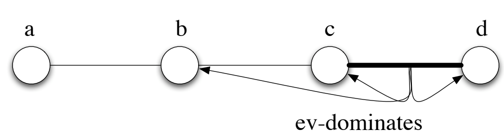

Figure 2.4: Edge-Vertex Domination

Definition 8 For a graph $G = (V, E)$, an edge $e = uv \in E(G)$ *ev-dominates* a vertex $w \in V(G)$ if

1. $u = w$ or $v = w$ (w is incident to e), or

2. uw or vw is an edge in G (w is adjacent to u or v).

Figure 2.4 gives a graphical representation of the definition of *ev*-domination. Notice that the edge *cd* *ev*-dominates the vertices b, c, and d, but not the vertex a.

Definition 9 A set $S \subseteq E(G)$ is an *edge-vertex dominating set* (or simply an *ev*-dominating set) if for all vertices $v \in V(G)$, there exists an edge $e \in S$ such that e dominates v.

Definition 10 The minimum cardinality of an *ev*-dominating set of G is called the *edge-vertex domination number* (or simply *ev-domination number*), and is denoted by $\gamma_{ev}(G)$.

Again, referring to Figure 2.3, from Definition 8 it follows that edge *de* *ev*-dominates the vertices d, e, b, and c but not vertex a. Thus, $F = \{de\}$ is not an *ev*-dominating set, but $F = \{bc\}$ is an *ev*-dominating set.

Proposition 22 [22] For any graph G of order n, $\gamma_{ve}(G) = 1$ if and only if there exists a vertex $x \in V(G)$ such that every vertex of G is within distance two of x and if $Y = \{y \in V(G) : \mathrm{dist}(x, y) = 2\}$ then Y is an independent set of vertices.

Proposition 23 [22] For the complete graph K_n, the complete bipartite graph $K_{m,n}$, and the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$, we have:

1. $\gamma_{ve}(K_n) = \gamma_{ve}(K_{m,n}) = \gamma_{ve}(K_{n_1,n_2,\ldots,n_r}) = 1$,

2. $\gamma_{ve}(P_n) = \left\lfloor \frac{n+2}{4} \right\rfloor$, where P_n is the path on n vertices,

3. $\gamma_{ve}(C_n) = \left\lfloor \frac{n+3}{4} \right\rfloor$, where C_n is the cycle on n vertices.

Proposition 24 [22] For any graph G of order n, $\gamma_{ev}(G) = 1$ if and only if $\gamma_c(G) \leq 2$, where $\gamma_c(G)$ is the minimum cardinality of a set of vertices that both induces a connected subgraph and is a dominating-set (the *connected domination number* of G). Thus:

1. $\gamma_{ev}(K_n) = \gamma_{ev}(K_{m,n}) = \gamma_{ev}(K_{n_1,n_2,\ldots,n_r}) = 1$,

2. $\gamma_{ev}(P_n) = \left\lfloor \frac{n+2}{4} \right\rfloor$,

3. $\gamma_{ev}(C_n) = \left\lfloor \frac{n+3}{4} \right\rfloor$.

Proposition 25 [22] For any graph G of order n,

1. $\gamma_{ve}(G) \leq \gamma(G) \leq \{\alpha_0(G), \alpha_1(G)\}$,

2. $\gamma_{ve}(G) \leq \beta_0(G)$,

3. $\gamma_{ve}(G) \leq \beta_1(G) \leq \frac{n}{2}$,

4. $\gamma_{ve}(G) \leq \gamma'(G)$,

5. $\gamma_{ve}(G) + \gamma_{ve}(\overline{G}) \leq n + 1$.

Proposition 26 [22] For any graph G of order n,

1. $\gamma_{ev}(G) \leq \gamma(G) \leq \{\alpha_0(G), \alpha_1(G)\}$,

2. $\gamma_{ev}(G) \leq \beta_0(G)$,

3. $\gamma_{ev}(G) \leq \beta_1(G) \leq \frac{n}{2}$,

4. $\gamma_{ev}(G) \leq \gamma'(G)$,

5. $\gamma_{ev}(G) + \gamma_{ev}(\overline{G}) \leq n + 1$, if G and $\overline{G}$ have no isolates.

Proposition 27 [22] For any graph G, $\gamma_{ev}(G) \leq \gamma(G) \leq 2\gamma_{ev}(G)$.

Proposition 28 [22] For any graph G with maximum degree $\Delta(G)$, $\gamma_{ev}(G) \leq \Delta(G)\gamma_{ve}(G)$, or $\frac{\gamma_{ev}}{\Delta(G)} \leq \gamma_{ve}$.

Theorem 29 [22] For any graph G of order n, $\gamma_{ve}(G) + \gamma_{ve}(\overline{G}) \leq n - \beta_0(G) + 2 = \alpha_0(G) + 2$.

Proposition 30 [22] For any graph G of size m, maximum degree $\Delta(G)$, and minimum degree $\delta(G)$,

$$\gamma_{ve}(G) \leq m - \Delta(G) - \frac{\Delta(G) * (\delta(G) - 1)}{2} + 1.$$

Definition 11 [22] For any graph G, let $\delta^*(G) = \min(|D| - m(D))$, where D is a minimal dominating set of G and $m(D)$ is the matching number of $\langle D \rangle$.

Theorem 31 [22] For any graph G, $\delta^*(G) \leq ir(G)$.

Proposition 32 [22] For any graph G, $\delta^*(G) = \gamma_{ev}(G)$.

Theorem 33 [22] For any graph G, $\gamma_{ev}(G) \leq ir(G)$.

Chapter 3

Vertex-Edge Parameters

Recall that for a graph $G = (V, E)$, a vertex $u \in V(G)$ *ve*-dominates an edge $vw \in E(G)$ if

1. if u is incident to vw, or

2. u is incident to an edge adjacent to vw.

Also, recall that a set $S \subseteq V(G)$ is a vertex-edge dominating set (or simply a *ve*-dominating set) if for all edges $e \in E(G)$, there exists a vertex $v \in S$ such that v dominates e. The minimum cardinality of a *ve*-dominating set of G is called the *vertex-edge domination number* (or simply *ve*-domination number), and is denoted by $\gamma_{ve}(G)$. A *ve*-dominating set S of size $\gamma_{ve}(G)$ is called a γ_{ve}*-set*.

We now formally define upper vertex edge-domination.

Definition 12 The maximum cardinality of a minimal *ve*-dominating set of a graph G is called the *upper vertex-edge domination number* (or simply the *upper ve-domination number*) and is denoted by $\Gamma_{ve}(G)$.

Traditional (vertex-vertex and edge-edge) domination has been extensively studied and much is known about it. For a thorough survey of the field see [13, 12]. In contrast, very little research has been conducted on vertex-edge domination. We present some preliminary results next.

Definition 13 A set S is an *independent vertex-edge dominating set* (or simply an *independent ve-dominating set*) if S is both an independent set and a minimal *ve*-dominating set.

Definition 14 The *independent vertex-edge domination number* of a graph G is the minimum cardinality of an independent *ve*-dominating set and is denoted $i_{ve}(G)$.

Definition 15 The *upper independent vertex-edge domination number* of a graph G is the maximum cardinality of an independent *ve*-dominating set and is denoted $\beta_{ve}(G)$.

Definition 16 A vertex $v \in S \subseteq V(G)$ has a *private edge* $e = uw \in E(G)$ (with respect to a set S), if :

1. v is incident to e or v is adjacent to either u or w, and

2. for all vertices $x \in S - \{v\}$, x is not incident to e and x is not adjacent to either u or w,

that is, v dominates the edge e and no other vertex in S dominates e.

Definition 17 A set S is a *vertex-edge irredundant set* (or simply a *ve-irredundant set*) if every vertex $v \in S$ has a private edge.

Definition 18 The *vertex-edge irredundance number* of a graph G is the minimum cardinality of a maximal *ve*-irredundant set of vertices and is denoted $ir_{ve}(G)$.

Definition 19 The *upper vertex-edge irredundance number* of a graph G is the maximum cardinality of a *ve*-irredundant set of vertices and is denoted $IR_{ve}(G)$.

Given these definitions, we can now present some basic propositions of *ve*-independence, *ve*-domination, and *ve*-irredundance.

We first observe that property P_4: is a *ve*-dominating set is super-hereditary and property P_5: is a *ve*-irredundant set is hereditary.

Thus from Proposition 2, we know that a set S is a minimal *ve*-dominating set if and only if it is a 1-minimal *ve*-dominating set.

We also know, from Proposition 1, that a set S is a maximal *ve*-irredundant set if and only if it is a 1-maximal *ve*-irredundant set.

We can use this to prove the following:

Proposition 34 Every minimal *ve*-dominating set of an ntc graph G is a maximal *ve*-irredundant set of G.

Proof. Let S be a minimal *ve*-dominating set of an ntc graph G. Suppose to the contrary that S is not a *ve*-irredundant set of G. Therefore, there exists a vertex $v \in S$ such that v does not have a private edge. Notice that if v does not have a private edge with respect to S and S dominates all of the edges of G, then $S - \{v\}$ still dominates all of the edges of G, which contradicts the minimality of S with respect to *ve*-domination. Therefore, every vertex in S must have a private edge with respect to S, and S is *ve*-irredundant.

But S is also maximal *ve*-irredundant. For suppose to the contrary that S is not maximal *ve*-irredundant. This implies that there exists a vertex $v \notin S$ such that $S \cup \{v\}$ is *ve*-irredundant, and in particular v has a private edge e with respect to $S \cup \{v\}$. If e is a private edge of v, then for all $u \in S$, u does not dominate e, which contradicts the assumption that S is a *ve*-dominating set. □

Corollary 35 For any graph G,

$$ir_{ve}(G) \leq \gamma_{ve}(G) \leq \Gamma_{ve}(G) \leq IR_{ve}(G).$$

Recall that an independent *ve*-dominating set by definition is not only a *ve*-dominating set, but a minimal *ve*-dominating set.

Thus, we have the following.

Proposition 36 For any graph G,

$$\gamma_{ve}(G) \leq i_{ve}(G) \leq \beta_{ve}(G) \leq \Gamma_{ve}(G).$$

From Corollary 35 and Proposition 36 we have a vertex-edge variant of the Domination Chain.

Theorem 37 [The Vertex-Edge Domination Chain] For any ntc graph G,

$$ir_{ve}(G) \leq \gamma_{ve}(G) \leq i_{ve}(G) \leq \beta_{ve}(G) \leq \Gamma_{ve}(G) \leq IR_{ve}(G).$$

Now that we have established the Vertex-Edge Domination Chain, we would like to know if it can be expanded to include other parameters.

Proposition 38 Every ve-irredundant set of an ntc graph G is an open irredundant set.

Proof. Let $S = \{u_1, u_2, \ldots, u_k\}$ be a ve-irredundant set, and let $S' = \{v_1w_1, v_2w_2, \ldots, v_kw_k\}$ be a set of k private edges, where for $1 \leq i \leq k$, v_iw_i is a private edge of u_i. Notice that since v_iw_i is a private edge of u_i no vertex u_j, $j \neq i$, can equal or be adjacent to v_i or w_i. Notice also that it is possible that $u_i = v_i$ or $u_i = w_i$, i.e. u_i is incident to edge v_iw_i. In this case assume that $u_i = w_i$. Then v_i is a private neighbor of u_i. If $u_i \neq v_i$ and $u_i \neq w_i$, i.e. u_i ve-dominates edge v_iw_i, then either u_i is adjacent to v_i or u_i is adjacent to w_i. In this case assume that u_i is adjacent to v_i. Then v_i must be is a private neighbor of u_i (with respect to S). Thus, each vertex $u_i \in S$ has a private neighbor v_i in $V - S$. Thus, S is an open irredundant set. $\square$

Corollary 39 For any ntc graph G,

1. $IR_{ve}(G) \leq OIR(G) \leq \beta_1(G) \leq \frac{n}{2}$,
2. $IR_{ve}(G) \leq OIR(G) \leq IR(G)$.

In [22] Peters showed that for any ntc graph G, $\gamma_{ve}(G) \leq \beta_1(G)$. The following chain of inequalities considerably expands this result.

Corollary 40 For any ntc graph G,

$$ir_{ve}(G) \leq \gamma_{ve}(G) \leq i_{ve}(G) \leq \beta_{ve}(G) \leq \Gamma_{ve}(G) \leq IR_{ve}(G) \leq OIR(G) \leq \beta_1(G) \leq \frac{n}{2}.$$

It follows from Corollary 40 that the sum of any two of these parameters is at most n.

We are now in a position to compare the parameters in the domination chain to the corresponding parameters in the ve-domination chain.

Proposition 41 Every dominating set of an ntc graph G is a ve-dominating set of G.

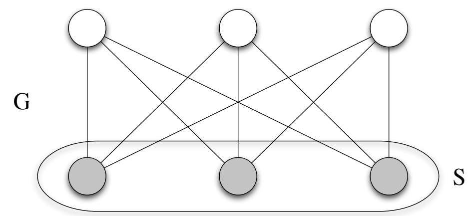

Figure 3.1: Example of an independent dominating set S that is also *ve*-dominating but not minimal *ve*-dominating.

Proof. Let S be a dominating set of an ntc graph G. Recall that every vertex $v \in V$ is either in S or adjacent to a vertex in S. Thus for every edge $uw \in E$, either $u \in S$, $w \in S$ or both $u, w \in V - S$. Clearly S *ve*-dominates uw if u or w is in S. If both u and w are in $V - S$, then without loss of generality, u has a neighbor in S, so S *ve*-dominates uw. Therefore, S is a *ve*-dominating set. □

Corollary 42 [22] For any ntc graph G, $\gamma_{ve}(G) \leq \gamma(G)$.

Proposition 43 Every independent dominating set of an ntc graph G is an independent *ve*-dominating set, but not necessarily a minimal *ve*-dominating set.

Proof. Let S be an independent dominating set of an ntc graph G. From Proposition 41 we know that S is a *ve*-dominating set, but as the example in Figure 3.1 shows, S may not be a minimal *ve*-dominating set. The set S in Figure 3.1 is independent and *ve*-dominating, but not minimal *ve*-dominating. In fact, in this example, $\gamma_{ve}(G) = i_{ve}(G) = 1 < \gamma(G) = 2 < i(G) = 3$. □

Corollary 44 For any ntc graph G, $i_{ve}(G) \leq i(G)$.

Proof. Let S be an i-set of G. From Propositions 41 and 43 we know that S is *ve*-dominating and independent but may not be a minimal *ve*-dominating set. Let S' be any minimal *ve*-dominating subset of S (possibly $S' = S$). Then $i_{ve}(G) \leq |S'| \leq |S| = i(G)$. □

The following theorem summarizes what we know about the inequalities between the parameters in the *ve*-domination chain and the corresponding parameters in the domination chain.

Theorem 45 For any ntc graph G, the following inequalities hold:

1. $ir_{ve}(G) \leq ir(G)$, [Conjectured]

2. $\gamma_{ve}(G) \leq \gamma(G)$, [Corollary 42] [22]

3. $i_{ve}(G) \leq i(G)$, [Corollary 44]

4. $\beta_{ve}(G) \leq \beta_0(G)$, [by definition]

5. $\Gamma_{ve}(G) \leq \Gamma(G)$, [Conjectured]

6. $IR_{ve}(G) \leq OIR(G) \leq IR(G)$. [Corollary 39]

The next result provides a class of graphs G for which $\gamma_{ve}(G) = i_{ve}(G)$. Let us say that a graph G belongs to family $\mathcal{G}$ if for every non-independent γ_{ve}-set S of G, there exists a pair of adjacent vertices u and v in S, one of which has exactly one private edge with respect to S.

Theorem 46 For any graph $G \in \mathcal{G}$, $\gamma_{ve}(G) = i_{ve}(G)$.

Proof. From Proposition 36 we known that $\gamma_{ve}(G) \leq i_{ve}(G)$. Thus, it suffices to show that $i_{ve}(G) \leq \gamma_{ve}(G)$ for every $G \in \mathcal{G}$. Let $G \in \mathcal{G}$ and suppose to the contrary that $i_{ve}(G) > \gamma_{ve}(G)$. Among all γ_{ve}-sets of G, let S be one with a minimum number of edges in the subgraph induced by S. If there are no edges, then we are finished.

Since $G \in \mathcal{G}$, there is a pair of adjacent vertices u and v in S such that v has exactly one private edge, say wx, with respect to S. We may assume that $w \neq v$ and v is adjacent to w. Let $S' = (S - \{v\}) \cup \{w\}$. Note that w has no neighbors in $S - \{v\}$ and that S' is a γ_{ve}-set of G. But the subgraph induced by S' has fewer edges than the one induced by S, contradicting our choice of S. $\square$

Let $S_0 \subseteq V$ be any set of vertices in a graph G. Let S_1 be the set of vertices in $V - S_0$ dominated by S_0, that is, $S_1 = N[S_0] - S_0$. Finally, let $S_2 = V - S_0 - S_1$. Thus any vertex set S_0 naturally defines a partition of $V(G)$ into three sets $\{S_0, S_1, S_2\}$.

We say that $\{S_0, S_1, S_2\}$ is a *ve*-partition, a minimal *ve*-partition, or a γ_{ve}-partition of G, depending on whether S_0 is a *ve*-dominating set, a minimal *ve*-dominating set, or a γ_{ve}-dominating set, respectively.

Theorem 47 Let $\{S_0, S_1, S_2\}$ be a partition of an ntc graph G defined by a set S_0. Then

1. If S_0 is a ve-dominating set, then S_2 is an independent set.

2. If S_0 is a minimal ve-dominating set, then S_1 is a dominating set of G.

3. If S_0 is a ve-irredundant set, then $|S_0| \leq |S_1|$.

4. For any ntc graph G, $\gamma_{ve}(G) \leq \frac{n}{2}$.

Proof. Let $\{S_0, S_1, S_2\}$ be a partition of an ntc graph G defined by a set S_0.

1. Let S_0 be a ve-dominating set of G, and suppose to the contrary that S_2 is not an independent set. Therefore, there exists an edge $uv \in E(G)$ such that $u, v \in S_2$. But then there does not exist a vertex in S_0 that dominates uv, a contradiction.

2. Let S_0 be a minimal ve-dominating set of G, and suppose to the contrary that S_1 is not a dominating set of G. Then there exist a vertex $v \in V - S$, that is not adjacent to (or dominated by) any vertex in S_1. Either $v \in S_0$ or $v \in S_2$.

Case 1 Assume that $v \in S_0$. Since we know that S_0 is a minimal ve-dominating set we know S_0 is a minimal ve-dominating set we know by Proposition 34 that S_0 is a maximal ve-irredundant set. Therefore v has a private edge, either an edge of the form vw for some vertex $w \in S_1$ or an edge of the form wx for some $w \in S_1$. But in this case, v is adjacent to w. Thus, in either case, v is adjacent to a vertex in S_1.

Case 2 Assume that $v \in S_2$. By Theorem 47.1 we know that S_2 is an independent set. And since we are assuming that G is a connected graph, it follows that every vertex in S_2 is adjacent to a vertex in S_1. Thus, S_1 must be a dominating set.

3. Let S_0 be a ve-irredundant set. Therefore, every vertex v in S_0, has a private edge say vw with respect to S_0. Hence, for all $v \in S_0$ there exists a $w \in S_1$ such that $w \notin N[S - \{v\}]$ and v is adjacent to w. Therefore, $|S_0| \leq |S_1|$.

4. Let S be a γ_{ve}-set in a graph G of order m, and assume that $|S| = \gamma_{ve}(G) > \frac{n}{2}$. Since S is a minimal ve-dominating set we know by Proposition 34 that S is also a maximal

ve-irredundant set. By Theorem 47.3 we therefore know that $|S| \leq |S_1|$, where S_1 is the set of vertices dominated by S. Therefore $|S| \leq \frac{n}{2}$, which contradicts our assumption that $|S| > \frac{n}{2}$

□

The following is a well-known, classical result in domination theory due to Ore [20].

Theorem 48 [Ore] [20] Let G be any graph without isolated vertices. Then the complement $V - S$ of any minimal dominating set is a dominating set.

A similar result holds for *ve*-dominating sets.

Theorem 49 Let G be any ntc graph. Then the complement $V - S$ of any minimal *ve*-dominating set is a *ve*-dominating set. In fact, the complement $V - S$ of any minimal *ve*-dominating set is a dominating set.

Proof. This follows immediately from Theorem 47 and Proposition 41. □

Corollary 50 For any ntc graph G,

$$\Gamma_{ve}(G) + \gamma(G) \leq n.$$

Let $\Psi(G)$ denote the maximum number of vertices in a set $S \subseteq V$ that does not contain an *enclave*, that is, a vertex $v \in S$, such that $N[v] \subseteq S$. It is well-known in domination theory (cf. p. 248 of [13]) that for any graph G of order n,

$$\gamma(G) + \Psi(G) = n, \text{ or that}$$

$$\gamma(G) = n - \Psi(G).$$

Corollary 51 For any ntc graph G,

$$\Gamma_{ve}(G) \leq \Psi(G).$$

A converse of Theorem 49 is also true.

Proposition 52 Let G be any ntc graph. Then the complement $V - S$ of every minimal dominating set S is a ve-dominating set.

Proof. This follows immediately from Theorem 48 and Proposition 41. □

Corollary 53 For any ntc graph G,

$$\Gamma(G) + \gamma_{ve}(G) \leq n.$$

A lower bound for $\gamma_{ve}(G)$ is easily obtained in terms of $m = |E|$ and $\Delta(G)$, the maximum degree of a vertex in G.

Proposition 54 For any ntc graph G, $\left\lceil \frac{m}{\Delta(G)^2} \right\rceil \leq \gamma_{ve}(G)$.

Proof. Clearly any vertex can have at most $\Delta(G)$ neighbors. Furthermore, any vertex can ve-dominate at most $\Delta(G)$ edges for each of its neighbors. Therefore, $\left\lceil \frac{m}{\Delta(G)^2} \right\rceil \leq \gamma_{ve}(G)$. □

Corollary 55 For any k-regular graph G, $\left\lceil \frac{n}{2k} \right\rceil \leq \gamma_{ve}(G)$.

Proof. Let G be a k-regular graph and S be a γ_{ve}-set of G. Recall that a k-regular graph on n vertices has $m = \frac{kn}{2}$ edges. Notice that any vertex can ve-dominate at most k^2 edges. Hence, $|S| \geq \left\lceil \frac{m}{k^2} \right\rceil = \left\lceil \frac{n}{2k} \right\rceil$. □

Corollary 56 For any cubic graph G,

$$\left\lceil \frac{n}{6} \right\rceil \leq \gamma_{ve}(G).$$

Proposition 54 can be used to obtain good lower bounds for $\gamma_{ve}(G)$ for a variety of Cartesian product graphs. The *Cartesian product* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \square G_2 = (V_1 \times V_2, E_1 \square E_2)$, where two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $E_1 \square E_2$ if and only if either $u_1 v_1 \in E_1$ and $u_2 = v_2$ or $u_1 = v_1$ and $u_2 v_2 \in E_2$.

Three classes of Cartesian products are of special interest: products of cycles $C_m \square C_n$, products of paths, called *grid graphs*, $P_m \square P_n$, and *prisms* $C_m \square P_2$. For these classes of graphs we can infer that:

Corollary 57 For Cartesian products of cycles and Cartesian products of paths,

$$\left\lceil \frac{mn}{8} \right\rceil \leq \gamma_{ve}(C_m \square C_n) \leq \gamma_{ve}(P_m \square P_n).$$

This corollary follows from Corollary 55 and the simple observation that if G is a spanning subgraph of a graph H, then $\gamma_{ve}(H) \leq \gamma_{ve}(G)$. Corollary 57 is interesting when compared to the simple lower bound for the domination number of a grid graph, namely

$$\left\lceil \frac{mn}{5} \right\rceil \leq \gamma(P_m \square P_n).$$

A prism $C_n \square P_2$ has $3n$ edges, and any vertex can *ve*-dominate at most nine edges. Therefore,

Corollary 58 For any prism $C_n \square P_2$, where $n > 3$ and $n \neq 9$ (cf. Figure 3.2)

$$\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{ve}(C_n \square P_2) \leq \left\lceil \frac{n}{3} \right\rceil + 1.$$

Proof. It follows from Corollary 56 that

$$\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{ve}(C_n \square P_2).$$

It only remains to show that this upper bound can be achieved. This can be proved by a simple induction argument on n using the *ve*-domination partition indicated in Figure 3.3. $\square$

One should note that this lower bound does not hold for $n = 3$ or $n = 9$, since $\left\lceil \frac{3}{3} \right\rceil < \gamma_{ve}(C_3 \square P_2) = 2$ and $\left\lceil \frac{9}{3} \right\rceil < \gamma_{ve}(C_9 \square P_2) = 4$

We can also obtain the exact value of $\gamma_{ve}(G)$ for $2 \times n$ grid graphs.

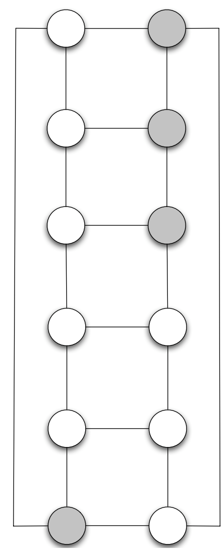

Figure 3.2: A γ_{ve}-set of prism $C_6 \square P_2$

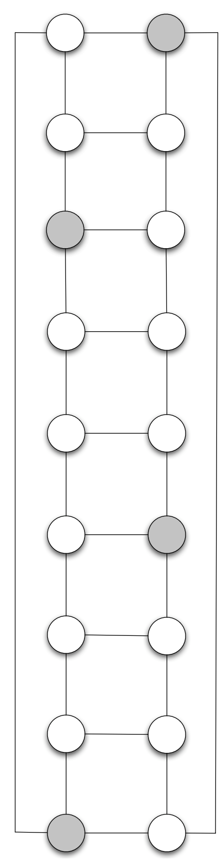

Figure 3.3: A γ_{ve}-set of prism $C_9 \square P_2$

We shall use a standard notation for labeling the vertices of a grid graph $G_{r,c}$, where $r \leq c$. In this notation a vertex $v_{i,j}$ in $V(G)$ is defined to be the vertex in the i^{th} row and the j^{th} column of $G_{r,c}$.

Definition 20 For a $2 \times c$ grid graph $G_{2,c}$ we call the edge incident to both of the vertices in column one and the edge incident to both vertices in column c *end-edges*.

Proposition 59 For any $2 \times c$ grid graph $G_{2,c}$ with $n = 2c$, $\gamma_{ve}(G_{2,c}) = \left\lceil \frac{c}{3} \right\rceil = \left\lceil \frac{n}{6} \right\rceil$.

Proof. It is a simple exercise to show that the formula holds for $c \leq 6$. Hence, let $c \geq 7$. Let S be a γ_{ve}-set of $G_{2,c}$. Note that $G_{2,c}$ has $3c - 2$ edges. In order to dominate the end-edges, S must contain at least one vertex from column 1 or column 2 and at least one from column $c - 1$ or column c. Each vertex from columns 1, 2, $c - 1$, and c can dominate at most 8 edges, and any other vertex can dominate at most 9 edges. Thus $|S| - 2$ vertices must dominate at least $3c - 18$ edges and hence

$$|S| - 2 \geq \frac{3c - 18}{9} = \frac{3\left(\frac{n}{2}\right) - 18}{9} = \left\lceil \frac{n}{6} \right\rceil - 2.$$

Thus $\gamma_{ve}(G_{2,c}) = |S| \geq \left\lceil \frac{n}{6} \right\rceil$.

To see that $\gamma_{ve}(G_{2,c}) \leq \left\lceil \frac{n}{6} \right\rceil$, let $S = \left\{a_{1,6i+2}, a_{2,6j+5} \mid 0 \leq i \leq \left\lfloor \frac{c-2}{6} \right\rfloor \text{ and } 0 \leq j \leq \left\lfloor \frac{c-5}{6} \right\rfloor\right\}$ if $c \equiv 0, 2 \mod 3$, otherwise let $S = \left\{a_{1,6i+2}, a_{2,6j+5}, a_{1,c} \mid 0 \leq i \leq \left\lfloor \frac{c-2}{6} \right\rfloor \text{ and } 0 \leq j \leq \left\lfloor \frac{c-5}{6} \right\rfloor\right\}$. $\square$

Definition 21 [13] A set S of vertices in a graph $G = (V, E)$ is a *distance-2 dominating set* if every vertex in $V - S$ is within distance 2 of at least one vertex in S.

Definition 22 [13] The *distance-2 domination number* $\gamma_{\leq 2}(G)$ of G equals the minimum cardinality of a distance-2 dominating set in G.

Proposition 60 Given any graph G without isolates, $\gamma_{\leq 2}(G) \leq \gamma_{ve}(G) \leq \gamma(G)$.

Proof. Let $G = (V, E)$ be a connected graph, let S_0 be a minimum γ_{ve}-set of G and let $\{S_0, S_1, S_2\}$ be the γ_{ve}-partition. Notice that by definition every vertex in S_1 is distance one from at least one vertex in S_0 and every vertex in S_2 is distance two from at least one vertex in S_0. Thus S_0 is a distance-2 dominating set of G. Therefore, $\gamma_{\leq 2}(G) \leq \gamma_{ve}(G)$. □

Theorem 61 For any tree T, $\gamma_{\leq 2}(T) \leq \gamma_{ve}(T) \leq 2\gamma_{\leq 2}(T)$.

Proof. Recall that from Proposition 60 we know that $\gamma_{\leq 2}(T) \leq \gamma_{ve}(T)$, and therefore it suffices to show that the upper bound holds. Let S be a $\gamma_{\leq 2}$-dominating set for our tree T. Notice that our tree T can be partitioned into k stars, $T_1, T_2, \ldots T_k$ where $k = \gamma_{\leq 2}(T)$, where the root of the star T_i is $r_i \in S$ and the distance from any leaf to r_i is at most two. Furthermore, two stars T_i and T_j are connected by at most one edge $t_x t_y$ between two leaf vertices where $t_x \in V(T_i)$ and $t_y \in V(T_j)$. Notice that the S *ve*-dominates all of the edges of the stars but not any of the edges between two stars. Furthermore, since we can have at most k of these edges, we know that it takes at most k vertices to *ve*-dominate these remaining edges. Let R be the collection of vertices that *ve*-dominate these remaining "inter-star" edges, and let $S' = S \cup R$. Notice that $S' \leq 2\gamma_{\leq 2}(T)$ and S' is a *ve*-dominating set of T. Therefore we have shown that the upper bound of the inequality holds. □

The inequality chains found in Proposition 60 and Theorem 61 are interesting in that it suggests that $\gamma_{ve}(G)$ is something like $\gamma_{\leq \frac{3}{2}}(G)$, that is, a vertex v in a *ve*-dominating set dominates all vertices and edges within distance $\frac{3}{2}$, but not vertices at distance 2. This intuition is helpful in understanding the nature of *ve*-domination.

In [22] Peters shows that the edge domination number and, therefore, the matching number provide upper bounds for the *ve*-domination number.

Proposition 62 [22] For an ntc graph G, $\gamma_{ve}(G) \leq \gamma'(G) \leq \beta_1(G)$.

Before we can present the next result, we need a couple of definitions.

A set S of vertices is called *matchable* if there exists a function $f : S \xrightarrow{1-1} V - S$ such that for every vertex $u \in S$, u is adjacent to $f(u)$. Note that the set of edges $\{uf(u) \mid u \in S\}$ defines a matching. The *matchability number* of G denoted $\mu(G)$, equals the minimum cardinality of a maximal matchable set.

Proposition 63 Let $F = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be the set of edges in a maximal matching of an ntc graph G. Then the edges in F can be so oriented such that the set $U = \{u_1, u_2, \ldots, u_k\}$ is a *ve*-dominating set.

Proof. Let $F = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be the set of edges in a maximal matching of an ntc graph G. Without loss of generality, let $U = \{u_1, u_2, \ldots, u_k\}$ and let $S = \{v_1, v_2, \ldots, v_k\}$. By definition, if an edge $xy \in E(G)$ is not in F, then without loss of generality $x = u_i$ or $x = v_i$ for some $1 \leq i \leq k$. Notice that we only have six classes of edges:

1. Edges in F,
2. Edges between two vertices in U,
3. Edges between two vertices in S,
4. Edges between a vertex in U and a vertex in S, not in F,
5. Edges between vertices in U and vertices in $V(G) - U - S$, and
6. Edges between vertices in S and vertices in $V(G) - U - S$.

It is easy to see that the vertices in U *ve*-dominate all edges in all six classes. Hence, U is a *ve*-dominating set. □

Our next result gives both a descriptive and a constructive characterization of the class of all trees, of order $n \geq 3$, having equal *ve*-domination and domination numbers. But before we present these characterizations, we need to give a few definitions. A *support vertex* in a tree T is any vertex that is adjacent to a leaf. A dominating set S in a graph G is said to be *efficient* if $N[u] \cap N[v] = \emptyset$ for every pair of vertices $u, v \in S$. Finally we define a family $\mathcal{T}$ of trees T that can be obtained from the disjoint union of $k \geq 1$ stars, each of order at least 3, as follows. We refer to the original k stars as the underlying stars of the tree T. Let C be the set of central vertices of the stars, and add $k - 1$ edges between the vertices of $V - C$ so that the resulting graph is a tree and every vertex in C remains a support vertex in T (that is, at least one leaf from each underlying star is not incident to the added edges). (cf. example tree in Figure 3.4

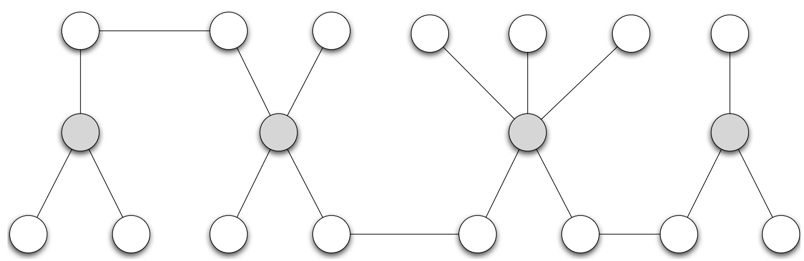

Figure 3.4: A tree $T \in \mathcal{T}$ having 4 underling stars

Theorem 64 For any tree T of order $n \geq 3$, the following statements are equivalent:

1. $\gamma_{ve}(T) = \gamma(T)$.

2. T has an efficient dominating set S, where each vertex in S is a support vertex of T.

3. $T \in \mathcal{T}$.

Proof of the Equivalence of Statements 1 and 2 of Theorem 64

We first give a descriptive characterization of the trees having $\gamma_{ve}(T) = \gamma(T)$.

Theorem 65 For any tree T of order $n \geq 3$, $\gamma_{ve}(T) = \gamma(T)$ if and only if T has an efficient dominating set S, where each vertex in S is a support vertex of T.

Proof. Let T be a tree with an efficient $\gamma(T)$-set $S = \{v_1, v_2, \ldots, v_\gamma\}$, where each vertex in S is a support vertex. Since $\gamma_{ve}(T) \leq \gamma(T)$, it suffices to show that $\gamma_{ve}(T) \geq \gamma(T)$. Let $S_1, S_2, \ldots, S_\gamma$ be the partition of $V(G)$, where $S_i = N[v_i]$, for $1 \leq i \leq \gamma(T)$. Let u_i be a leaf adjacent to v_i. Let D be a $\gamma_{ve}(T)$-set. In order to *ve*-dominate the edge u_iv_i, at least one vertex from $N[v_i]$ is in D, implying that $|D| \geq \gamma(T)$. Hence, $\gamma_{ve}(T) = \gamma(T)$.

For the converse, assume that $\gamma_{ve}(T) = \gamma(T)$. Let S be a $\gamma(T)$-set containing all support vertices of T (such a set is always possible). We first show that S is efficient. Assume that u and v are adjacent vertices in S. Let F be the set of edges incident to a vertex in $N(u)$ but not to u. Now v *ve*-dominates all of the edges incident to u. Note that since T is a tree, each edge in F has a vertex that is not in $N[u]$, so F is *ve*-dominated by $S - \{u\}$. Hence, $S - \{u\}$ is a *ve*-dominating set of T with fewer than $\gamma(T)$ vertices, contradicting the assumption that $\gamma_{ve}(T) = \gamma(T)$. Therefore, S must be an independent set.

Assume that u and v have a common neighbor w in $V - S$. But then w ve-dominates the edges incident to u and those incident to v. The edges incident to vertices in $N(u) \cup N(v)$ have vertices that are dominated by $S - \{u, v\}$, so $S - \{u, v\} \cup \{w\}$ is a ve-dominating set of T having fewer than $\gamma(T)$ vertices, again a contradiction. Thus, S is an efficient dominating set of T. Partition the vertices of T into the closed neighborhood S_i of the vertices in S. To see that every vertex in S must be a support vertex, assume to the contrary that there is a vertex $v_i \in S$ such that no neighbor of v_i is a leaf. Then for each $x \in N(v_i)$, x is adjacent to a vertex in $S_j - \{v_j\}$ for some j. Select one such neighbor of x and call it u_j. Since T is a tree, $j \neq i$ and no pair of vertices in $N(v_i)$ have neighbors in a common S_j (otherwise a cycle is formed). Let U be the collection of these neighbors u_j (cf Figure 3.5). Thus $|U| = \deg(v_i)$. Note further that for each pair of vertices, say u_j and u_k, in U, $j \neq k$, and there are no edges between the vertices of S_j and S_k (because if there, were a cycle would exist). If we let $S = (S - (\{v_i\} \cup \{v_j \mid u_j \in U\})) \cup U$, then S is a ve-dominating set of T with fewer than $\gamma(T)$ vertices; a contradiction. Hence, every vertex of S must be a support vertex. □

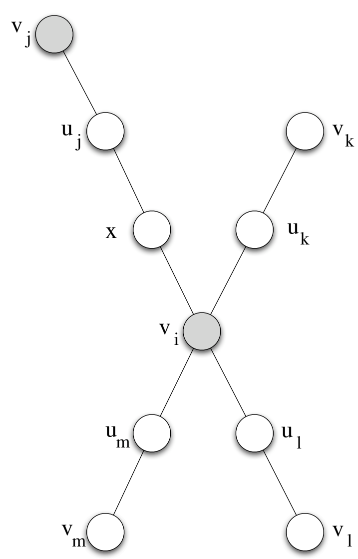

Figure 3.5: A tree T with a collection U of neighbors u_j

Proof of the Equivalence of Statements 1 and 3 of Theorem 64

To complete the proof of Theorem 64, we provide a constructive characterization of the nontrivial trees T for which $\gamma_{ve}(T) = \gamma(T)$. Let $\mathcal{T}$ be the family of trees with k underlying stars as described previously.

Theorem 66 For any tree T of order $n \geq 3$, $\gamma_{ve}(T) = \gamma(T)$ if and only if $T \in \mathcal{T}$.

Proof. Suppose $T \in \mathcal{T}$. The k central vertices of the k underlying stars of T form an efficient dominating set in T, and so $\gamma(T) = k$. Since $\gamma_{ve}(T) \leq \gamma(T) = k$, all we need to show is that $\gamma_{ve}(T) \geq k$. By the way T is constructed, a vertex from an underlying star cannot dominate an edge from another underlying star. Hence, any $\gamma_{ve}(T)$-set must contain at least one vertex from each underlying star implying that $\gamma_{ve}(T) \geq k$.

To prove the converse, let $\gamma_{ve}(T) = \gamma(T) = k$. We show that $T \in \mathcal{T}$. By Theorem 65, T has an efficient γ-set $S = \{v_1, v_2, \ldots, v_k\}$, where each vertex in S is a support vertex of T. Note that S is also a γ_{ve}-set. Let $S_1, S_2, \ldots, S_k$ be the partition of $V(T)$, where $S_i = N[v_i]$, for $1 \leq i \leq k$. Since S is an efficient dominating set and T is a tree, each S_i induces a star with v_i as a center. Hence, T has k underlying stars. Since T is a tree there are exactly $k - 1$ edges in $E(T)$ that are not in the underlying stars. Let F be the set of these edges. Since each v_i is a support vertex in T, it suffices to show that each S_i has at least three vertices. Clearly, S_i has at least two vertices, v_i and its leaf neighbor in T. But since T is connected, at least one vertex of S_i is incident to an edge of F. Since v_i is a support vertex in T at least one of the leaves of the underlying star is a leaf in T. If v_i is incident to an edge of F, then v_i is at most distance two from some v_j, contradicting the fact that S is an efficient dominating set. Hence, v_i must have at least one neighbor that is incident an edge in F and at least one that is not, implying that $|S_i| \geq 3$, and hence $T \in \mathcal{T}$. $\square$

Chapter 4

Edge-Vertex Parameters

Recall that for a graph $G = (V, E)$, an edge $e = uv \in E(G)$ *ev*-dominates a vertex $w \in V(G)$ if

1. $u = w$ or $v = w$, or

2. uw or vw is an edge in G.

Also, recall that a set $F \subseteq E(G)$ is an edge-vertex dominating set (or simply a *ev*-dominating set) if for all vertices $v \in V(G)$, there exists an edge $e \in F$ such that e dominates v. The minimum cardinality of an *ev*-dominating set of G is called the *edge-vertex domination number* (or simply *ev-domination number*), and is denoted by $\gamma_{ev}(G)$.

We now formally define the *ev*-parameters.

Definition 23 The maximum cardinality of a minimal *ev*-dominating set of a graph G is called the *upper edge-vertex domination number* (or simply the *upper ev-domination number*) and is denoted $\Gamma_{ev}(G)$.

Definition 24 A set F is an *independent edge-vertex dominating set* (or simply an *independent ev-dominating set*) if F is both an independent set of edges and a minimal *ev*-dominating set.

Definition 25 The *independent edge-vertex domination number* of a graph G is the minimum cardinality of an independent *ev*-dominating set of vertices and is denoted $i_{ev}(G)$.

Definition 26 The *upper independent edge-vertex domination number* of a graph G is the maximum cardinality of an independent *ev*-dominating set and is denoted $\beta_{ev}(G)$.

Definition 27 An edge $e = uv \in F \subseteq E(G)$ has a *private vertex* $w \in V(G)$ (with respect to a set F), if :

1. e is incident to w or either v or u is adjacent to w, and

2. for all edges $f = xy \in F - \{e\}$, f is not incident to w and neither x nor y is adjacent to w,

that is, e *ev*-dominates the vertex w and no other edge in F *ev*-dominates w.

Definition 28 A set F is an *edge-vertex irredundant set* (or simply an *ev*-irredundant set) if every edge $e \in F$ has a private vertex.

Definition 29 The *edge-vertex irredundance number* of a graph G is the minimum cardinality of a maximal *ev*-irredundant set of edges and is denoted $ir_{ev}(G)$.

Definition 30 The *upper edge-vertex irredundance number* of a graph G is the maximum cardinality of an *ev*-irredundant set of edges and is denoted $IR_{ev}(G)$.

Proposition 67 If F is a minimal *ev*-dominating set of an ntc graph G, then F is a maximal *ev*-irredundant set.

Proof. Let F be a minimal *ev*-dominating set of an ntc graph G. Clearly if F is minimal with respect to *ev*-domination, then every edge in F must have a private vertex. Therefore, we know that F is also an *ev*-irredundant set. Suppose to the contrary that F is not maximal *ev*-irredundant. Since the property of being an *ev*-irredundant set is hereditary, this implies that there exists an edge $e \notin F$, such that $F \cup \{e\}$ is *ev*-irredundant, and e has a private vertex v with respect to F. But if v is a private vertex of e with respect to F, then F does not *ev*-dominate v, which is a contradiction. Therefore, F is maximal with respect to *ev*-domination. □

Corollary 68 For any graph G,

$$ir_{ev}(G) \leq \gamma_{ev}(G) \leq \Gamma_{ev}(G) \leq IR_{ev}(G).$$

Recall that an independent *ev*-dominating set, by definition, is also an *ev*-dominating set. Clearly this implies the following.

Proposition 69 For any ntc graph G,

$$\gamma_{ev}(G) \leq i_{ev}(G).$$

Also, by definition, any independent *ev*-dominating set is also a minimal *ev*-dominating set. Thus, we have the following:

Proposition 70 For any ntc graph G,

$$\gamma_{ev}(G) \leq i_{ev}(G) \leq \beta_{ev}(G) \leq \Gamma_{ev}(G).$$

From Corollary 68 and Propositions 69 and 70 we have Theorem 71.

Theorem 71 [The Edge-Vertex Domination Chain] For any ntc graph graph G,

$$ir_{ev}(G) \leq \gamma_{ev}(G) \leq i_{ev}(G) \leq \beta_{ev}(G) \leq \Gamma_{ev}(G) \leq IR_{ev}(G).$$

As with the Vertex-Edge Domination Chain, we would like to expand the Edge Vertex Domination Chain to include other parameters. It is well-known in domination theory that the edge domination number of any graph equals its independent edge domination number, that is

Theorem 72 For any graph G,

$$\gamma'(G) = i'(G).$$

The same result holds for *ev*-domination and independent *ev*-domination.

Theorem 73 For any ntc graph G,

$$\gamma_{ev}(G) = i_{ev}(G).$$

Proof. From Proposition 69 we know that $\gamma_{ev}(G) \leq i_{ev}(G)$. Thus, it suffices to show that $\gamma_{ev}(G) \geq i_{ev}(G)$. Suppose to the contrary that there exists a graph G for which $\gamma_{ev}(G) < i_{ev}(G)$. This implies that no γ_{ev}-set of G can be independent. Therefore there exists at least two edges that are adjacent in every γ_{ev}-set of G. Let F be a γ_{ev}-set of G with a smallest number of pairs of adjacent edges. If F has zero adjacent edges then we are done. Let $e = uv$ and $f = vw$ be two adjacent edges in F. Since $e, f \in F$ we know that there exists a private vertex x for e and a private vertex y for f. Notice that if we define $F' = F - \{e\} \cup \{ux\}$ we still have a γ_{ev}-set of G, and f still has a private vertex (y) as does the new edge ux (x). Also notice that any vertex adjacent to x, other than u cannot be the sole private vertex of some edge $g \in F$. For if it was, then $F' - \{g\}$ will still dominate all of the vertices of G and $|F' - \{g\}| = |F - \{e\} \cup \{ux\} - \{g\}| < |F|$, a contradiction. But since F' has fewer pairs of adjacent edges than F we have a contradiction, hence $\gamma_{ev}(G) \geq i_{ev}(G)$. □

Theorem 74 For every ev-irredundant set F in an ntc graph G, there exists a matching F' with $|F| = |F'|$.

Proof. Let $F = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be an ev-irredundant set of G. Without loss of generality, we can orient these edges in F such that the $u_i's$ are distinct, but the $v_i's$ don't necessarily have to be distinct. Recall that every edge in F has a private vertex with respect to F. We can define a set of edges $F' = \{u_1w_1, u_2w_2, \ldots, u_kw_k\}$, where w_i is the private vertex for edge u_iv_i. Clearly F' defines a matching and $|F| = |F'|$. □

Corollary 75 For any ntc graph G of order n,

$$IR_{ev}(G) \leq \beta_1(G) \leq \frac{n}{2}.$$

Corollary 76 For any ntc graph G,

$$ir_{ev}(G) \leq \gamma_{ev}(G) = i_{ev}(G) \leq \beta_{ev}(G) \leq \Gamma_{ev}(G) \leq IR_{ev}(G) \leq \beta_1(G) \leq \Gamma'(G) \leq IR'(G).$$

This result considerably improves on the result of Peters [22] that: $\gamma_{ev}(G) \leq \beta_1(G)$.

We can now try to compare the parameters in the edge domination chain with their counterparts in the ev-domination chain.

Proposition 77 For any ntc graph G, every edge dominating set is an ev-dominating set.

Proof. Let $F \subseteq E(G)$ be an edge dominating set of G. Recall that every edge in E is either in F or adjacent to an edge in F. Hence every vertex is either incident to an edge in F or is adjacent to a vertex that is incident to an edge in F. Therefore F is also an *ev*-dominating set of G. □

Corollary 78 [22] For any ntc graph G,

$$\gamma_{ev}(G) \leq \gamma'(G).$$

Proposition 79 Every *ev*-irredundant set of edges in an ntc graph G is an irredundant set of edges.

Proof. Let F be an *ev*-irredundant set of edges in an ntc graph G. This implies that every edge $uv \in F$ has a private vertex w. Without loss of generality, let w be adjacent to u. Since w is a private vertex of uv we know that for all edges $xy \in F - \{uv\}$, w is not adjacent to x or y. Therefore uw is a private edge for the edge uv, with respect to the set F. Hence, F is also an irredundant set of edges. □

Corollary 80 For any ntc graph G,

$$IR_{ev}(G) \leq IR'(G)$$

The following theorem summarizes what we know about the inequalities between the parameters in the *ev*-domination chain and the corresponding parameters in the edge domination chain.

Theorem 81 For any ntc graph G, the following inequalities hold:

1. $ir_{ev}(G) \leq ir'(G)$, [Conjectured]

2. $\gamma_{ev}(G) \leq \gamma'(G)$, [Corollary 78] [22]

3. $i_{ev}(G) \leq i'(G)$, [Corollary 78]

4. $\beta_{ev}(G) \leq \beta_1(G)$, [By Definition]

5. $\Gamma_{ev}(G) \leq \beta_1(G) \leq \Gamma'(G)$, [Theorem 74 and Proposition 67]

6. $IR_{ev}(G) \leq \beta_1(G) \leq \Gamma'(G) \leq IR'(G)$. [Corollary 75]

Proposition 82 For any ntc graph G,

$$\gamma_{\leq 3}(G) \leq ir_{ev}(G).$$

Proof. Let $F = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be an ir_{ev}-set, and let $U = \{u_1, u_2, \ldots, u_k\}$, and $S = \{v_1, v_2, \ldots, v_k\}$, where $|U| = k$ but $|S| \leq k$. This implies that every edge $e = uv \in F$ has a private vertex. Furthermore, if a vertex $w \in V(G)$ is not ev-dominated by an edge in F then w is at most distance two from a vertex on some edge in F else F is not a maximal ev-irredundant set. Notice that U distance-three dominates all of the vertices ev-dominated by F, for these vertices are at most distance two from some vertex in U. The only remaining vertices are those not ev-dominated by F. Recall that these vertices are at most distance two from some vertex on an edge in F, and therefore at most distance three from some vertex in U. Hence, U distance-three dominates all vertices of $V(G)$. □

Proposition 83 For any graph G of order n, without isolates, and maximum degree $\Delta(G)$,

$$\left\lceil \frac{n}{2\Delta(G) - 2} \right\rceil \leq \gamma_{ev}(G).$$

Proof. Any edge uv in $E(G)$, ev-dominates all vertices in $N[u] \cup N[v]$ and $|N[u] \cup N[v]| \leq 2(\Delta(G) - 2) + 2 = 2\Delta(G) - 2$. Therefore, $\left\lceil \frac{n}{2\Delta(G)-2} \right\rceil \leq \gamma_{ev}(G) \leq n - 2\Delta(G) - 1$. □

Again, for cubic graphs we get the following:

Corollary 84 For any cubic graph G of order n,

$$\left\lceil \frac{n}{4} \right\rceil \leq \gamma_{ev}(G).$$

Proposition 85 For any $2 \times c$ grid graph $G_{2,c}$ with $n = 2c$, $\gamma_{ev}(G_{2,c}) = \left\lceil \frac{c}{3} \right\rceil = \left\lceil \frac{n}{6} \right\rceil$.

Proof. Let F be a γ_{ev}-set of $G_{2,c}$. Note that each edge uv can dominate at most 6 vertices, namely the vertices of $N[v] \cup N[u]$. Hence to dominate all n vertices of $G_{2,c}$ our γ_{ev}-set F must contain at least $\left\lceil \frac{n}{6} \right\rceil$ edges.

To see that $\gamma_{ev}(G_{2,c}) \leq \left\lceil \frac{n}{6} \right\rceil$, let $F = \left\{a_{1,3i+2}a_{2,3i+2} \mid 0 \leq i \leq \left\lfloor \frac{c-2}{3} \right\rfloor\right\}$ if $c \equiv 0, 2 \mod 3$, otherwise let $F = \left\{a_{1,3i+2}a_{2,3i+2}, a_{1,c}a_{2,c} \mid 0 \leq i \leq \left\lfloor \frac{c-2}{3} \right\rfloor\right\}$. □

At the end of Chapter 3 we noted that Peters [22] had shown that $\gamma_{ve}(G) \leq \gamma'(G) \leq \beta_1(G)$. The same inequality holds for $\gamma_{ev}(G)$.

Proposition 86 For any ntc graph G, $\gamma_{ev}(G) \leq \gamma'(G) \leq \beta_1(G)$.

Peters [22] also observed the following:

Proposition 87 For any ntc graph G, $\gamma_{ev}(G) \leq \gamma(G) \leq 2\gamma_{ev}(G)$.

But it does not appear that a similar result holds for $\gamma_{ve}(G)$.

A result similar to the classic result, previously mentioned, by Ore can be obtained for ev-dominating sets.

Proposition 88 Let G be any ntc graph. Then the complement $E - F$ of any minimal ev-dominating set is an ev-dominating set.

Proof. Let F be a minimal ev-dominating set for an ntc graph G. By definition we know that every vertex $v \in V(G)$ is either incident to an edge in F or adjacent to a vertex incident to an edge in F. Furthermore, we know that since F is minimal every edge $e \in F$ has a private vertex with respect to F. Thus $E - F$ dominates all of the vertices of G. □

From this we can immediately conclude, as did Peters [22], that

Corollary 89 For any ntc graph G of order n, $\gamma_{ev}(G) \leq \frac{n}{2}$.

Chapter 5

Complexity and Algorithmic Results

In this chapter we investigate the computational complexity and approximability of the *ve*- and *ev*-parameters introduced in this dissertation. A summary of the NP-completeness and approximation results attained so far is given in Tables 6.3 and 6.4. In Section 5.1 we state the decision problems associated with our twelve parameters. In Sections 5.2 and 5.3 we determine which of our parameters are NP-complete for various classes of graphs. We also determine the lower bounds on the approximability of these NP-complete parameters. In Section 5.4 show that each of our parameters are linearly solvable when we restrict their respective decision problems to trees.

First we make the following observation concerning the upper bounds on the approximability of two of our twelve parameters.

SET COVER
INSTANCE: A universe $\mathcal{U}$, a family $\mathcal{S}$ of subsets of $\mathcal{U}$, and a positive integer k.
QUESTION: Does there exists a set $C \subseteq S$ of size at most k, such that $\cup_{s \in C} s = \mathcal{U}$?

It is a well known fact that using a greedy algorithm one can derive a solution to the SET COVER problem that is at most $O(\log m)$ times the optimal, where $m = |\mathcal{U}|$ [14]. Notice that our *ve*- and *ev*-domination parameters are special cases of the set cover problem. Clearly from this we can arrive at Theorem 90.

Theorem 90 There exists a solution that is at most $O(\log n)$ times optimal for VE DOMINATING SET and for EV DOMINATING SET.

	Family of Graphs			
	General	Tree	Bipartite	Chordal
ir_{ve}		Linearly Solvable [Thm. 119]		
γ_{ve}	NP-Complete [Thm. 94]	Linearly Solvable [Thm. 116]	NP-Complete [Thm. 94]	NP-Complete [Thm. 94]
i_{ve}	NP-Complete [Thm. 103]	Linearly Solvable [Thm. 118]	NP-Complete [Cor. 104]	
β_{ve}		Linearly Solvable [Thm. 119]	NP-Complete [Thm. 109]	NP-Complete [Thm. 110]
Γ_{ve}		Linearly Solvable [Thm. 119]		NP-Complete [Cor. 111]
IR_{ve}		Linearly Solvable [Thm. 119]		NP-Complete [Cor. 112]
ir_{ev}		Linearly Solvable [Thm. 119]		
$\gamma_{ev} = i_{ev}$	NP-Complete [Thm. 113]	Linearly Solvable [Thm. 119]	NP-Complete [Thm. 113]	
β_{ev}		Linearly Solvable [Thm. 119]		
Γ_{ev}		Linearly Solvable [Thm. 119]		
IR_{ev}		Linearly Solvable [Thm. 119]		

Table 5.1: Summary of complexity results

	Family of Graphs		
	General	Bipartite	Chordal
ir_{ve}			
γ_{ve}	$c \log n$ [Cor. 98][1, 3]	$c \log n$ [Cor. 98][1, 3]	
i_{ve}	$O\left(n^{1-\varepsilon}\right)$ [Thm. 107][2, 3]	$O\left(n^{1-\varepsilon}\right)$ [Thm. 107][2, 3]	$O\left(n^{1-\varepsilon}\right)$ [Thm. 107][2, 3]
β_{ve}			
Γ_{ve}			
IR_{ve}			
ir_{ev}			
$\gamma_{ev} = i_{ev}$	$c \log n$ [Cor. 114][1, 3]		
β_{ev}			
Γ_{ev}			
IR_{ev}			

[1] There exists a constant $c > 0$ such that it is NP-hard to approximate the given parameter to within a $c \log n$ factor.
[2] For some constant $\varepsilon > 0$.
[3] The above parameter is NP-hard to approximate to within the given factor.

Table 5.2: Summary of inapproximability results

5.1 Statement of our Decision Problems Associated with our Twelve Parameters

The following is the formal statement of the decision problems that are associated with our twelve parameters.

- VE IRREDUNDANT SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

  QUESTION: Does G contain a maximal *ve*-irredundant set of size at most k?

- VE DOMINATING SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

  QUESTION: Does G have a *ve*-dominating set of size at most k?

- INDEPENDENT VE DOMINATING SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

  QUESTION: Does G have an independent *ve*-dominating set of size at most k?

- UPPER INDEPENDENT VE DOMINATING SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

  QUESTION: Does G have an independent *ve*-dominating set of size at least k?

- MAXIMUM MINIMAL VE DOMINATING SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$.

  QUESTION: Does G have a minimal *ve*-dominating set of size at least k?

- UPPER VE IRREDUNDANT SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

  QUESTION: Does G contain a *ve*-irredundant set of size at least k?

- EV IRREDUNDANT SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

  QUESTION: Does G contain a maximal *ev*-irredundant set of size at most k?

- EV DOMINATING SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

  QUESTION: Does G have a ev-dominating set of size at most k?

- INDEPENDENT EV DOMINATING SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

  QUESTION: Does G have an independent ev-dominating set of size at most k?

- UPPER INDEPENDENT EV DOMINATING SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

  QUESTION: Does G have an independent ev-dominating set of size at least k?

- MAXIMUM MINIMAL EV DOMINATING SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$.

  QUESTION: Does G have a minimal ev-dominating set of size at least k?

- UPPER EV IRREDUNDANT SET

  INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

  QUESTION: Does G contain a EV-irredundant set of size at least k?

5.2 Vertex-Edge Parameters

5.2.1 *ve*-Domination

Theorem 91 For any graph G, $\gamma(G) = \gamma_{ve}(G \circ K_1)$.

Proof. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Also let $V(G \circ K_1) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$, and $E(G \circ K_1) = E(G) \cup \{v_1u_1, v_2u_2, \ldots, v_nu_n\}$. We shall proceed first by proving that $\gamma_{ve}(G \circ K_1) \leq \gamma(G)$. Let $S \subseteq V(G)$ be a γ-set of G. Notice that S ve-dominates all of the edges of $E(G)$, thus we are left with showing that S ve-dominates the remaining edges of $G \circ K_1$. If a vertex $v_i \in S$, then clearly v_i ve-dominates the edge v_iu_i. If a vertex $v_j \notin S$, then we know that there exists a vertex $v_k \in S$ such that $v_jv_k \in E(G)$. Notice then that v_k also ve-dominate the edge v_ju_j. Hence all of the edges of $G \circ K_1$ are ve-dominated, and S is a ve-dominating set of $G \circ K_1$.

Conversely we prove that $\gamma(G) \leq \gamma_{ve}(G \circ K_1)$. Let S be a γ_{ve}-set of $G \circ K_1$. Notice that if a vertex $u_i \in S$ is not in $V(G)$, then we can simply replace u_i with v_i and still *ve*-dominate $G \circ K_1$. Therefore, without loss of generality, let $S \subseteq V(G)$. Clearly S dominates all of the vertices of G. For suppose to the contrary that there exist a vertex $v_i \in V(G)$ that is not dominated by S. We know that since S *ve*-dominates $G \circ K_1$, all edges incident to v_i in $G \circ K_1$ must be *ve*-dominated, including the edge $v_i u_i$. To *ve*-dominate $v_i u_i$, a vertex $v_j \in N[v_i]$ must be in S, which is a contradiction. Thus, S dominates $V(G)$, and $\gamma(G) \leq \gamma_{ve}(G \circ K_1)$. □

DOMINATING SET

INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$.

QUESTION: Does G have a dominating set of size at most k?

Theorem 92 [9, cf. GT2] DOMINATING SET is NP-complete, even when restricted to bipartite, chordal, or planar graphs with maximum vertex degree 3 and planar graphs that are regular of degree 4.

Clearly, this implies that DOMINATING SET remains NP-complete for arbitrary planar graphs, since a polynomial algorithm for DOMINATING SET for planar graphs could be used to settle these two restricted problems on planar graphs.

Recall that a graph G is a *circle graph* if G is the intersection graph of chords in a circle.

Theorem 93 [15] DOMINATING SET is NP-complete for circle graphs.

Since the corona reduction found in the proof of Theorem 91 preserves bipartiteness, chordality, and planarity we can derive Theorem 94 and Corollary 95.

Theorem 94 [17] VE DOMINATING SET is NP-complete, even when restricted to bipartite or chordal graphs.

Corollary 95 VE DOMINATING SET is NP-complete, even when restricted to planar graphs.

Notice that if G is a circle graph, then so is the corona $G \circ K_1$. This leads us to Corollary 96.

Corollary 96 VE DOMINATING SET is *NP*-complete, even when restricted to circle graphs.

Before we state our next result we must present several definitions relating to approximation algorithms.

Definition 31 [25] Combinatorial optimization problems are problems of picking the "best" solution from a finite set. An *NP-optimization problem*, Π, consists of:

- A set of *valid instances*, D_Π, recognizable in polynomial time. The *size* of an instance $I \in D_\Pi$, denoted $|I|$, is defined as the number of bits needed to write I under the assumption that all numbers occurring in the instance are written as binary.
- Each instance $I \in D_\Pi$ has a set of *feasible solutions*, $S_\Pi(I)$. We require that $S_\Pi(I) \neq \emptyset$, and that every solution $s \in S_\Pi(I)$ is of length polynomially bounded in $|I|$. Furthermore, there is a polynomial time algorithm that, given a pair (I, s), decides whether $s \in S_\Pi(I)$.
- There is a polynomial time computable *objective function*, obj_Π, that assigns a non-negative rational number to each pair (I, s), where I is an instance and s is a feasible solution for I. The objective function is frequently given a physical interpretation, such as *cost*, *length*, *weight*, etc.
- Finally, Π is specified to be either a *minimization problem* or a *maximization problem*.

The restriction of Π to unit cost instances is called the *cardinality version* of Π.

Definition 32 [25] An *optimal solution* for an instance of a minimization (maximization) problem is a feasible solution that achieves the smallest (largest) objective function value. $\mathrm{OPT}_\Pi(I)$ denotes the objective function value of an optimal solution to instance I. We will shorten to OPT when it is clear that we are referring to a generic instance of the particular problem being studied.

Definition 33 [25] Let Π_1 and Π_2 be two minimization problems (the definition for two maximization problems is quite similar). An *approximation factor preserving reduction* from Π_1 to Π_2 consists of two polynomial time algorithms, f and g such that:

- for any instance I_1 of Π_1, $I_2 = f(I_1)$ is an instance of Π_2 such that $\mathrm{OPT}_{\Pi_2}(I_2) \leq \mathrm{OPT}_{\Pi_1}(I_1)$, and

- for any solution t of I_2, $s = g(I_1, t)$ is a solution of I_1 such that

$$\mathrm{obj}_{\Pi_1}(I_1, s) \leq \mathrm{obj}_{\Pi_2}(I_2, t).$$

In [23] Raz and Safra stated that approximating minimum dominating set to within a logarithmic factor is NP-hard.

Theorem 97 [23] There exists a constant c such that for any graph G DOMINATING SET is NP-hard to approximate to within a $c \log n$ factor, even when restricted to bipartite graphs.

Since the corona reduction found in the proof of Theorem 91 is an approximation-preserving reduction, even when restricted to bipartite graphs, we have Corollary 98.

Corollary 98 There exists a constant c such that for any graph G VE DOMINATING SET is NP-hard to approximate to within a $c \log n$ factor, even when restricted to bipartite.

5.2.2 Independent *ve*-Domination

Theorem 99 For any graph G, $i(G) = i_{ve}(G \circ K_1)$.

Proof. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Also let $V(G \circ K_1) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$, and $E(G \circ K_1) = E(G) \cup \{v_1u_1, v_2u_2, \ldots, v_nu_n\}$. We shall proceed first by showing that $i_{ve}(G \circ K_1) \leq i(G)$. Let $S \subseteq V(G)$ be an independent dominating set of cardinality $i(G)$. By definition S is an independent set of vertices and for all $v_i \notin S$, there exist a $v_j \in S$ such that $v_iv_j \in E(G)$. Clearly S *ve*-dominates all of the edges of $E(G)$. Notice that if $v_i \in S$, then v_i *ve*-dominates the edge $v_iu_i \in E(G \circ K_1)$. Furthermore, S *ve*-dominates the edges $v_ju_j \in E(G \circ K_1)$ where $v_j \notin S$. For if $v_j \notin S$, the there exists a vertex $v_k \in S$ such that $v_jv_k \in E(G)$. Clearly v_k *ve*-dominates the edge v_ju_j. Thus, S is an independent *ve*-dominating set, and therefore $i_{ve}(G \circ K_1) \leq i(G)$.

We now show that $i_{ve}(G \circ K_1) \geq i(G)$. Let S be an independent minimal ve-dominating set of $G \circ K_1$ of cardinality $i_{ve}(G \circ K_1)$. Notice that we have two cases, either:

Case 1: $S \subseteq V(G)$, or

Case 2: $S \nsubseteq V(G)$.

If the latter is the case, then notice that we have two sub-cases.

Case 2a: There exist at least two vertices $u_i, v_j \in S$ such that $v_j \in N(v_i)$. Notice that if $v_j \in S$, then v_j ve-dominates all of the edges ve-dominated by u_i, contradicting the minimality of S with respect to ve-domination. Therefore for all $u_i \in S$, there does not exist a vertex v_j such that $v_j \in S$ and $v_j \in N(v_i)$.

Case 2b: For all vertices $u_i \in S$ where $u_i \in V(G \circ K_1) - V(G)$, there does not exist a vertex $v_j \in S$ where $v_j \in N(v_i)$. If Case 2a holds, then notice that we can define $S' = S - \{u_i\} \cup \{v_i\}$, for all $u_i \in S$. Notice that by construction S' is an independent set of vertices. To show that S' is a minimal ve-dominating set, notice that for all $v_i \in S' - S$ there does not exist a vertex $v_j \in S'$ such that $v_j \in N(v_i)$, and thus v_i has $v_i u_i$ as a private edge. Furthermore, the vertex $v_j \in S$ has as a private edge with respect to S' the edge $v_j u_j$. Therefore we only need to consider Case 1.

Without loss of generality, let $S \subseteq V(G)$. Suppose to the contrary that there exists a vertex $v_i \in V(G)$ that is not dominated by S. We know that since S ve-dominates $G \circ K_1$, all edges incident to v_i in $G \circ K_1$ must be ve-dominated, including the edge $v_i u_i$. To ve-dominate $v_i u_i$, a vertex $v_j \in N[v_i]$ must be in S, which is a contradiction. Thus S dominates $V(G)$. □

INDEPENDENT DOMINATING SET

INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

QUESTION: Does G have an independent dominating set of size at most k?

Theorem 100 [9, cf. GT2] INDEPENDENT DOMINATING SET is NP-complete.

Theorem 101 [6] INDEPENDENT DOMINATING SET is NP-complete, even when restricted to bipartite graphs.

In [4] Chang published the following result.

Theorem 102 [4] Minimum Weight INDEPENDENT DOMINATING SET is NP-complete for chordal graphs.

Using the coronal reduction found in the proof of Theorem 99 we can derive Theorem 103.

Theorem 103 INDEPENDENT VE DOMINATING SET is NP-complete.

Since the corona reduction found in the proof of Theorem 99 preserves bipartiteness and chordality we can derive Corollary 104.

Corollary 104 INDEPENDENT VE DOMINATING SET is NP-complete, even when restricted to bipartite and chordal graphs.

Definition 34 [10] *Zero-Error Probabilistic Polynomial Time* (ZPP) is the class of languages recognized by probabilistic Turing machines with polynomial bounded average run time and zero error probability.

Theorem 105 is from [11], and gives a lower bound on the approximability of a INDEPENDENT DOMINATING SET.

Theorem 105 [11] For any graph G, INDEPENDENT DOMINATING SET is NP-hard to approximate to within a $O\left(n^{1-\varepsilon}\right)$ factor, for any constant $\varepsilon > 0$, unless $NP = ZPP$.

Theorem 106 [7] For any graph G, INDEPENDENT DOMINATING SET is NP-hard to approximate to within a $O\left(n^{1-\varepsilon}\right)$ factor, for any constant $\varepsilon > 0$, even when restricted to bipartite and chordal graphs.

Since the corona reduction found in the proof of Theorem 99 is an approximation-perserving reduction, even when restricted to bipartite and chordal graphs, we have Theorem 107.

Theorem 107 For any graph G, INDEPENDENT VE DOMINATING SET is NP-hard to approximate to within an $O\left(n^{1-\varepsilon}\right)$ factor, for any constant $\varepsilon > 0$, even when restricted to bipartite and chordal graphs.

INDEPENDENT SET

INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$

QUESTION: Does G contain an independent set of size at least k?

Theorem 108 [9, cf. GT20] INDEPENDENT SET is NP-complete.

Theorem 109 UPPER INDEPENDENT VE DOMINATING SET is NP-complete for bipartite graphs.

Proof. The NP-Completeness of UPPER INDEPENDENT VE DOMINATING SET for bipartite graphs follows from a polynomial transformation from EXACT COVER BY THREE SETS.

EXACT COVER BY THREE SETS (X3C)

INSTANCE: Set $X = \{x_1, x_2, \ldots, x_{3q}\}$ of elements, Collection C of 3 element subsets of X.

QUESTION: Is there a subcollection $C' \subseteq C$ of q subsets such that every element of X appears in exactly one subset?

Given an arbitrary instance X and C of X3C, with $|X| = 3q$, and $|C| = m$, create an instance of UPPER INDEPENDENT VE DOMINATING SET for a bipartite graph G, and positive integer k for in this way:

The graph G is made up of three types of components:

1. Element components: For each $x_i \in X$, create a P_3 (Figure 5.1 shows an element component).

2. Subset components: For each subset $c_j \in C$, create the component found in Figure 5.2.

3. Communication edges: Add edges from the subset component $c_{j_1}, c_{j_2}, c_{j_3}$ vertices to the element x_i vertices corresponding to the three elements that appear in the subset - one edge from each c_j. (see the arrows in Figures 5.1 and 5.2)

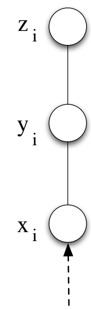

Figure 5.1: Example of an Element Component used for the NP-Completeness proof of β_{ve} when restricted to bipartite graphs.

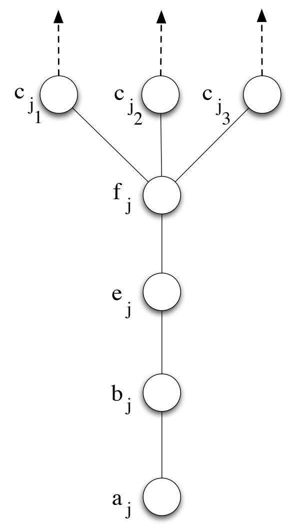

Figure 5.2: Example of a Subset Component used for the NP-Completeness proof of β_{ve} when restricted to bipartite graphs.

Finally Set $k = 3q + m + 3q + m - q = 5q + 2m$

Given an exact cover C' for the X3C instance, find an independent ve-dominating set S in this way: select all the z vertices from the element components. Also select all the a vertices from the subset components. Then, in the subset components that correspond to subsets in C', select all three c vertices. In the subset components that correspond to subsets that are not in C', select the f vertex. It is easy to verify that all edges are dominated, that S is independent, and that each vertex in S has a private edge. In any subset component, (vertices a and f (or a and all the c vertices) dominate the entire subset component as well as the communication edges. The z vertices dominate all the edges in the element components. The a vertices have the ab edges as private edges. The f vertices have the ef

edges as private edges, (the c vertices have their incident communication edges as a private edges), and the z vertices have the yz edges as private edges. Notice there will be $5q + 2m$ vertices in S.

Given an independent ve-dominating set S of cardinality $\geq 5q + 2m$, one can find an exact cover C' in this way: for each subset component that has all of its c vertices in S, add that subset to C'. The following argument shows the only way to have $5q + 2m$ vertices in S is if exactly q of the subset components have all their c vertices in S, and that these are adjacent to all different elements of X.

In any minimal findependent ve-dominating set, there is exactly one representative in S from each element component. Also, there is exactly one representative of the a, b, and e vertices from any one subset component. These representatives account for $3q + m$ of the vertices in S. Since $|S| \geq 5q + 2m$, at least $2q + m$ vertices in S come from the c and f vertices. To account for these vertices, define subsets $S_1 \subset S$ and $S_2 \subset S$ as follows: $S_1 = \{v \mid v \in \{f_j, c_{j_1}, c_{j_2}, c_{j_3}\} \cap S$, and $|S \cap \{f_j, c_{j_1}, c_{j_2}, c_{j_3}\}| = 1\}$ $S_2 = \{v \mid v \in \{f_j, c_{j_1}, c_{j_2}, c_{j_3}\} \cap S$, and $|S \cap \{f_j, c_{j_1}, c_{j_2}, c_{j_3}\}| > 1\}$ In other words, any vertex in S_1 is the only f or c representative in S chosen from its respective subset component. Subset S_2 is made up of the other f and c vertices, from those components with at least two of these vertices in S. Actually since S is an independent set, subset S_2 can contain only c vertices. Let $r = |S_1|$ and $s = |S_2|$. Since no subset component may have more than three vertices in S_2, at least $s/3$ subset components have more than one representative in S. Therefore, $r + s/3 \leq m$. Recall that $r + s \geq 2q + m$. The vertices in S_2 must have private edges. These edges only can be communication edges. Therefore no other vertex in S can be adjacent to the same x vertex as a vertex in S_2. It follows that $s \leq 3q$. Since, $r + 3q \geq r + s \geq 2q + m$, it follows that $r + q \geq m$. From $r + q \geq m \geq r + s/3$, it follows that $s \geq 3q$. So $s = 3q$. Each of these 3q vertices in S_2 must be adjacent to a different x vertex, and these x vertices can have no other neighbors in S. So the vertices in S_1 must come from the f vertices. At least $s/3 = q$ of the subset components have vertices in S_2, so at most $m - q$ of the subset components can have vertices in S_1. Further $2q + m \leq r + s \leq mq + s = m + 2q$. Therefore, r = m - q. Exactly q of the subset components are responsible for the $3q$ vertices in S_2, three from each component. □

Theorem 110 UPPER INDEPENDENT VE DOMINATING SET is NP-complete for chordal graphs.

Proof. For this proof we are using a transformation from X3C to UPPER INDEPENDENT VE DOMINATING SET when restricted to chordal graphs.

Given an arbitrary instance X and C of X3C, with $|X| = 3q$, and $|C| = m$, create an instance of UPPER INDEPENDENT VE DOMINATING SET with a chordal graph G, and a positive integer k in this way:

The graph G is made up of four types of components:

1. Element components: For each $x_i \in X$, create a single vertex x_i (Figure 5.3 has an example of an element component).

2. Subset components: For each subset $c_j \in C$, create the component found in Figure 5.4.

3. Communication edges: Add edges from the subset component $c_{j_1}, c_{j_2}, c_{j_3}$ vertices to the element x_i vertices corresponding to the three elements that appear in the subset - one edge from each c_j. (see the arrows in Figures 5.3 and 5.4)

4. Chordal edges: Form a clique among all $3m$ of the c vertices.

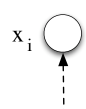

Figure 5.3: Example of an Element Component used for the NP-Completeness proof of β_{ve} when restricted to chordal graphs.

Finally Set $k = 3q + m + (m - q) = 2q + 2m$

Given an exact cover C' for the X3C instance, find an independent ve-dominating set S in this way: select all the x vertices and in the subset components that correspond to subsets in C', select the vertex b. In the subset components that correspond to subsets that are not in C', select the f vertex and the a vertex. You can verify that all edges are dominated (vertices a and f or vertices b and the x's dominate the entire subset component as well as the communication edges), that S is independent, and that each vertex in S has a private

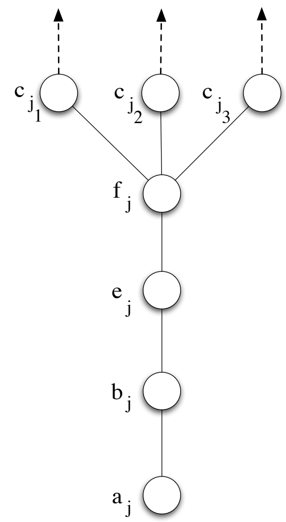

Figure 5.4: Example of a Subset Component used for the NP-Completeness proof of β_{ve} when restricted to chordal graphs.

edge. The a_j vertices have the ab edge as a private edge as the b vertices. The f vertices have the ef edge as a private edge. The x vertices have the communication edge between it and its corresponding c_j vertex as a private edge. Notice there will be $2q + 2m$ vertices chosen. ($3q$ of the x vertices, q of the b vertices, and $2(m - q)$ total of a and f vertices, which equals $3q + q + 2m - 2q = 2q + 2m$.)

Given an independent ve-dominating set S of cardinality $\geq 2q + 2m$, one can find an exact cover C' as follows: for each subset component that does not have its corresponding f vertex in S, add that subset to C'. We must now show that the only way to have $2q + 2m$ vertices in S is if exactly q of the subset components do not have their f vertices in S.

Note that in any independent ve-dominating set S for G, there are exactly one S representative of any subset component from among its a, b, and e vertices. Those vertices account for m of the vertices in S. If $|S| = 2q + 2m$, there needs to be at least m + 2q more vertices selected from the f, c, and x vertices. Because the c vertices form a clique, at most one c vertex could appear in any independent ve-dominating set. Actually no c vertex can appear in an independent ve-dominating set of cardinality $2q + 2m$. Any one c vertex dominates every clique edge and every communication edge, and every edge from a c vertex to an f vertex. The set of edges dominated by any one c vertex is a proper superset of a set of edges dominated by any x vertex. If a c vertex is in S, then no x vertex can be in S because it would not have a private edge. Discounting x vertices, there are not enough vertices left in the graph to get a set of cardinality $2q + 2m$ (there are only m of the f vertices left!) Therefore, the $m + 2q$ vertices must come from the f and x vertices. Since there are only m of the f vertices, at least $2q$ of the x vertices must be chosen.

Let r be the number of f vertices and s be the number of x vertices in S. Each x vertex in S must have a private edge, and that private edge is incident with some c vertex. Clearly that c vertex must not be dominated by another vertex in S, so the f vertex cannot be taken in that component. So taking an x vertex rules out an f vertex. It is possible that up to three of the x vertices would rule out the same f vertex. Since there are only $3q$ of the x vertices in the graph, it is obvious $s \leq 3q$. Therefore, $r \leq ms/3$. It follows $m + 2q \leq r + s \leq ms/3 + s = m + 2s/3 \leq m + 2(3q)/3 = m + 2q$. Therefore, all of the x vertices are in S, and there must be $m - q$ of the f vertices, chosen which happens when only q of the f vertices are ruled out. That means that there are private edges from the x vertices to a limited q of the subset components (those without the f vertices in S), and these q components form an exact cover. □

The exact same construction works for the Γ_{ve} and IR_{ve} decision problems (MAXIMUM MINIMAL VE DOMINATING SET and UPPER VE IRREDUNDANT SET respectively). Note that the argument changes slightly because it would be possible to have more than one c vertex in an irredundant set, but if so, the corresponding cf edges are the only private edges the c vertices could have. So at most m total of the c and f vertices could be chosen, and if any c is in, none of the x's can be in S. The most S could be in this case is $2m$, which is not enough. Therefore, we have Corollaries 111 and 112.

Corollary 111 MAXIMUM MINIMAL VE DOMINATING SET is NP-Complete for chordal graphs.

Corollary 112 UPPER VE IRREDUNDANT SET is NP-Complete for chordal graphs.

5.3 Edge-Vertex Parameters

5.3.1 *ev*-Domination

Theorem 113 EV DOMINATING SET is NP-complete, even when restricted to bipartite graphs.

Proof. We begin with a reduction from DOMINATING SET to EV DOMINATING SET. Given a graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$, we construct a graph $G' = (V', E')$ as follows. Let $V' = V_1 \cup V_2 \cup V_3$, where $V_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,n}\}$, $V_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,n}\}$ and $V_3 = \{v_{3,1}, v_{3,2}, \ldots, v_{3,n}\}$. We add edges between the vertices of V_1 and V_2 so that the subgraph induced by $V_1 \cup V_2$ is a complete bipartite graph $K_{n,n}$. We say that the edge $v_{1,i}v_{2,i}$ represents the vertex $v_i \in V(G)$. Lastly, if $v_iv_j \in E(G)$, then let $v_{2,i}v_{3,j}$ and $v_{3,i}v_{2,j} \in E(G')$. We also add the n edges $v_{2,i}v_{3,i}$. Figure 5.5 has an example of a graph G and the constructed graph G'.

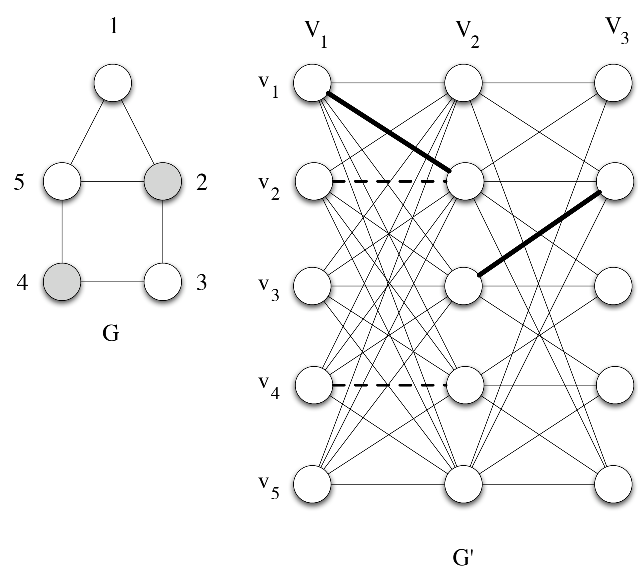

Figure 5.5: Example of the graphs G and G' used for the NP-Completeness proof of γ_{ev} when restricted to bipartite graphs.

Claim: a graph G has a dominating set of size $\leq k$ if and only if the bipartite graph G' has an *ev*-dominating set of size $\leq k$.

Let $S \subseteq V(G)$ be a dominating set of G of size $\leq k$. Then let $F' = \{v_{1,i}v_{2,i} \mid v_i \in S\}$. It is easy to see that F' is an *ev*-dominating set of G' of size $\leq k$.

Conversely, let $F \subseteq E(G')$ be an ev-dominating set of G'. Notice that if $v_{2,i}v_{3,j} \in F$, we can define $F' = F - \{v_{2,i}v_{3,j}\} \cup \{v_{1,i}v_{2,i}\}$ and F' is still an ev-dominating set of G'. Therefore, without loss of generality, let F contain only edges between V_1 and V_2. Furthermore, if $v_{1,i}v_{2,j} \in F$ where $i \neq j$ we can define $F' = F - \{v_{1,i}v_{2,j}\} \cup \{v_{1,j}v_{2,j}\}$ and clearly F' is still an ev-dominating set of G'. Thus, without loss of generality, let F contain only edges $v_{1,i}v_{2,i}$ for $1 \leq i \leq n$.

Recall that the edge $v_{1,i}v_{2,i} \in E(G')$ represents the vertex $v_i \in V(G)$. Notice that if we let $S = \{v_i \mid v_{1,i}v_{2,i} \in F\}$, then S is a dominating set for the graph G. Therefore, if one can find an ev-dominating set F for G' of size at most k, then we have a dominating set S for G of size at most k. Hence, using this reduction we have a transformation from DOMINATING SET to EV DOMINATING SET and from Theorem 92 DOMINATING SET is NP-complete. Since the above transformation preserves bipartiteness, EV DOMINATING SET is NP-complete, even for bipartite graphs. □

Since the 3-partite reduction found in the proof of Theorem 113 is an approximation-perserving reduction, even when restricted to bipartite graphs, we have Corollary 114.

Corollary 114 There exists a constant c such that for any graph G EV DOMINATING SET is NP-hard to approximate to within a $c \log n$ factor, even when restricted to bipartite graphs.

5.4 Complexity Results for Trees

Definition 35 [12] For a positive integer $k \geq 1$, a set S of vertices in a graph $G = (V, E)$ is a *distance-k dominating* set of G if every vertex in $V - S$ is at most distance k from some vertex in S.

Definition 36 [12] The *distance-k domination number* $\gamma_{\leq k}(G)$ of a graph G equals the minimum cardinality of a distance-k dominating set in G.

If we add weights to each of the vertices of $V(G)$, then we can modify Definition 36 by letting the *weighted distance-k domination number* $\gamma^{w}_{\leq k}$ equal the minimum weight of a distance-k dominating set in G. We use this modified definition of distance-k domination below.

WEIGHTED DISTANCE 3 DOMINATION

INSTANCE: Graph $G = (V, E)$, positive integer k.

QUESTION: Does G have a weighted distance-3 dominating set with a weight at most k?

In [24] Slater presented a linear time algorithm for solving the less general problem of finding a minimum cardinality distance-k dominating set (which he called a k-basis) for a forest. We require an algorithm for solving the weighted version of the problem. We present a proof of Theorem 115 using the Wimer method.

Theorem 115 [24] WEIGHTED DISTANCE 3 DOMINATION is solvable in linear time for trees.

Proof. The weighted distance-3 domination problem can be solved on trees using the Wimer method [27]. Let us construct a recurrence system with only seven classes.

Given a tree T with root r and a weighted distance-3 dominating set S, consider how the (tree, set) pair (T, S) can be constructed (or decomposed) using the composition rule for rooted trees. What we must be able to do is to completely characterize the possible classes of (T, S) pairs which can be used to build weighted distance-3 dominating sets in trees. Fortunately, in this case there are only seven possible classes, whose characterizations can be found in Table 5.3 and examples of each class can be found in Figure 5.6.

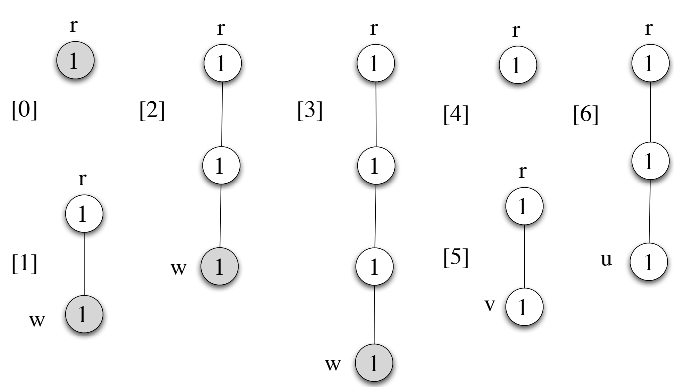

Figure 5.6: Examples of the seven Wimer classes for the $\gamma^w_{\leq 3}$ Algorithm

Class			Definition
[0]	=	$\{(T, S) \mid$	1. r is in S
			2. S is a distance-3 dominating set in $T\}$,
[1]	=	$\{(T, S) \mid$	1. r is not in S
			2. S is a distance-3 dominating set in T
			3. $\min_{w \in S}\{\text{dist}(w, r)\} = 1\}$,
[2]	=	$\{(T, S) \mid$	1. r is not in S
			2. S is a distance-3 dominating set in T
			3. $\min_{w \in S}\{\text{dist}(w, r)\} = 2\}$,
[3]	=	$\{(T, S) \mid$	1. r is not in S
			2. S is a distance-3 dominating set in T
			3. $\min_{w \in S}\{\text{dist}(w, r)\} = 3\}$,
[4]	=	$\{(T, S) \mid$	1. r is not in S
			2. S is a distance-3 dominating set in $T - \{r\}\}$.
[5]	=	$\{(T, S) \mid$	1. r is not in S
			2. S is a distance-3 dominating set in $T - (\{r\} \cup R)$ †
			3. $\max_{w \in R}\{\text{dist}(w, r)\} = 1$,
			4. $R \neq \emptyset\}$,
[6]	=	$\{(T, S) \mid$	1. r is not in S
			2. S is a distance-3 dominating set in $T - (\{r\} \cup R)$ †
			3. $\max_{w \in R}\{\text{dist}(w, r)\} = 2$,
			4. $R \neq \emptyset\}$.

† /∗Vertices in R are not yet distance-3 dominated∗/

Table 5.3: Definitions of the seven possible classes of (T, S) pairs used to build weighted distance-3 dominating sets in trees.

	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[1]	[1]	[1]	[1]	[1]	[1]	X
[2]	[1]	[2]	[2]	[2]	[2]	X	X
[3]	[1]	[2]	[3]	[3]	X	X	X
[4]	[1]	[2]	[3]	[4]	[5]	[6]	X
[5]	[1]	[2]	[5]	[5]	[5]	[6]	X
[6]	[1]	X	X	[6]	[6]	[6]	X

Table 5.4: Composition table for the seven classes of (T, S) pairs used to build weighted distance-3 dominating sets in trees.

$$
\begin{array}{lcl}
[0] & = & [0]\circ[0]\cup[0]\circ[1]\cup[0]\circ[2]\cup[0]\circ[3]\cup[0]\circ[4]\cup[0]\circ[5]\cup[0]\circ[6] \\
[1] & = & [1]\circ[0]\cup[1]\circ[1]\cup[1]\circ[2]\cup[1]\circ[3]\cup[1]\circ[4]\cup[1]\circ[5]\cup[2]\circ[0]\cup \\
 & & [3]\circ[0]\cup[4]\circ[0]\cup[5]\circ[0]\cup[6]\circ[0] \\
[2] & = & [2]\circ[1]\cup[2]\circ[2]\cup[2]\circ[3]\cup[2]\circ[4]\cup[3]\circ[1]\cup[4]\circ[1]\cup[5]\circ[1] \\
[3] & = & [3]\circ[2]\cup[3]\circ[3]\cup[4]\circ[2] \\
[4] & = & [4]\circ[3] \\
[5] & = & [4]\circ[4]\cup[5]\circ[2]\cup[5]\circ[3]\cup[5]\circ[4] \\
[6] & = & [4]\circ[5]\cup[6]\circ[3]\cup[6]\circ[4]\cup[6]\circ[5]\cup[5]\circ[5]
\end{array}
$$

Table 5.5: Closed recurrence system for the seven classes of (T, S) pairs used to build weighted distance-3 dominating sets in trees.

Next, we build a composition table, where 'X' means that this composition never can occur in the process of constructing a weighted distance-3 dominating set. This composition table can be found in Table 5.4.

Since these seven classes are 'closed' under the 'composition' operation, we can build the closed recurrence system found in Table 5.5.

Finally, we need to determine our initial vector, where $W(r)$ is the weight of vertex r:

$$
\begin{array}{ccccccccc}
(& W(r)\,, & +\infty\,, & +\infty\,, & +\infty\,, & 0\,, & 0\,, & 0 &) \\
 & [0] & [1] & [2] & [3] & [4] & [5] & [6] &
\end{array}
$$

That is, we want to compute the smallest weight of a weighted distance-3 dominating set of each class in the graph consisting of a single vertex. We can now write the pseudocode for our algorithm, found in Algorithm 5.1, whose input is the parent array for the input tree T.

```
Procedure MinimumWeightedDistance3DominatingSet(Parent, N);
  for i := 0 to N do
    V(i, ∗) := (W(r), +∞, +∞, +∞, 0, 0, 0);          // V is our vector
  end for
  for j := N to 2 do                                  // Postorder scan through vertices.
    k :=Parent[j];
    Combine(V, k, j);
  end for
  RETURN min(V(0, 0), V(0, 1), V(0, 2), V(0, 3));
END MinimumWeightedDistance3DominatingSet
```

```
Procedure Combine(V, k, j);
  V(k, 0) := min(V(k, 0) + V(j, 0), V(k, 0) + V(j, 1), V(k, 0) + V(j, 2), V(k, 0) + V(j, 3),
               V(k, 0) + V(j, 4), V(k, 0) + V(j, 5), V(k, 0) + V(j, 6));
  V(k, 1) := min(V(k, 1) + V(j, 0), V(k, 1) + V(j, 1), V(k, 1) + V(j, 2), V(k, 1) + V(j, 3),
               V(k, 1) + V(j, 4), V(k, 1) + V(j, 5), V(k, 2) + V(j, 0), V(k, 3) + V(j, 0),
               V(k, 4) + V(j, 0), V(k, 5) + V(j, 0), V(k, 6) + V(j, 0));
  V(k, 2) := min(V(k, 2) + V(j, 1), V(k, 2) + V(j, 2), V(k, 2) + V(j, 3), V(k, 2) + V(j, 4),
               V(k, 3) + V(j, 1), V(k, 4) + V(j, 1), V(k, 5) + V(j, 1));
  V(k, 3) := min(V(k, 3) + V(j, 2), V(k, 3) + V(j, 3), V(k, 4) + V(j, 2));
  V(k, 4) := V(k, 4) + V(j, 3);
  V(k, 5) := min(V(k, 4) + V(j, 4), V(k, 5) + V(j, 2), V(k, 5) + V(j, 3), V(k, 5) + V(j, 4));
  V(k, 6) := min(V(k, 4) + V(j, 5), V(k, 6) + V(j, 3), V(k, 6) + V(j, 4), V(k, 6) + V(j, 5),
               V(k, 5) + V(j, 5));
END Combine;
```

Algorithm 5.1: Algorithm for finding the minimum weight of a weighted distance-3 dominating set for a tree T

Note that our algorithm guarantees that we do not select a final value from classes [4], [5], or [6], for this will leave us with a set S that does not distance-3 dominate the entire tree, in particular the root r is not dominated. $\square$

An implementation of 5.1, in the programming language of C, is given in Appendix A.

We can now use Theorem 115 to arrive at Theorem 116.

Theorem 116 WEIGHTED VE DOMINATING SET is solvable in linear time for trees.

Proof. Given a tree $T = (V, E)$, we first define a new weighted tree $T' = (V', E')$ by subdividing each edge of $E(T)$. Next we place a weight of one on each vertex of $V(T)$ and a weight of $+\infty$ on each vertex of $V(T') - V(T)$. Notice that solving the *ve*-dominating set problem for our tree T is the same as solving the weighted distance-3 dominating set problem for our tree T'. From Theorem 115 we know that the weighted distance-3 dominating problem is solvable in linear time for trees. $\square$

If we require that our set S be independent along with weighted distance-k dominating then we have a *weighted independent distance-k dominating set* and the *weighted independent distance-k domination number* $i^w_{\leq k}$ equals the minimum weight of a weighted independent distance-k dominating set in G.

WEIGHTED INDEPENDENT DISTANCE 3 DOMINATION

INSTANCE: Graph $G = (V, E)$, positive integer j.

QUESTION: Does G have a weighted independent distance-3 dominating set with a weight at most j?

Theorem 117 WEIGHTED INDEPENDENT DISTANCE 3 DOMINATION is solvable in linear time for trees.

Proof. If we modify the Wimer table found in the proof of Theorem 115 to have an 'X' for class [1] $\circ$ class [1] and then make the appropriate changes to the remainder of the proof we have an algorithm for solving the weighted independent distance-3 dominating set problem. $\square$

Using the same argument found in the proof of Theorem 117, we get Theorem 118.

Theorem 118 WEIGHTED INDEPENDENT VE DOMINATING SET is solvable in linear time for trees.

We now state a few definitions and observations from [16] which are used to prove Theorem 119.

A family of graphs is *recursive* if every graph of the family can be generated by a finite number of of applications of *composition operations* starting with a finite set of *basis graphs* [16]. A family of graphs is *k-terminal recursive* if it is a recursive family which has *k* distinguished vertices called *terminals*, and each composition operation is defined in terms of certain primitive operations on terminals [16]. A property is *locally verifiable* if it can be verified by examining a bounded neighborhood of each vertex in a graph [16]. A property *P* is *regular* with respect to a family of graphs if the number of equivalence classes induced by each extended composition operation is finite (and therefore can be represented as a finite binary multiplication table) [16].

Mahajan and Peters [16] note that "the *k*-terminal recursive families of graphs include trees, series-parallel graphs, outerplanar graphs, and partial *k*-trees." Their paper concludes with the following observation:

> "We have shown that any optimal subgraph or vertex partition problem based on a property that is locally verifiable in [amortized] constant time per vertex is regular with respect to all *k*-terminal recursive families of graphs. This implies the existence of a linear-time algorithm for any such problem (when the input is in the form of a parse tree) and we have given an effective procedure for constructing these algorithms. Although we have not proved [it] here, it should be obvious that our results can be extended to problems involving edge-induced bounded partitions by suitable changes to the definitions."

Furthermore, any graph *G* which is a member of some *k*-terminal recursive family can be represented by some parse-tree [16].Notice that each of our twelve parameters are either a "vertex [or an edge] partition problem based on a property which is locally verifiable in [amortized] constant time per vertex [(edge)]." From this observation we clearly have Theorem 119.

Theorem 119 The following decision problems are solvable in linear time for trees:

1. VE IRREDUNDANT SET,
2. VE DOMINATING SET,
3. INDEPENDENT VE DOMINATING SET,
4. UPPER INDEPENDENT VE DOMINATING SET,
5. MAXIMAL MINIMAL VE DOMINATING SET,
6. UPPER VE IRREDUNDANT SET,
7. EV IRREDUNDANT SET,
8. EV DOMINATING SET,
9. INDEPENDENT EV DOMINATING SET,
10. UPPER INDEPENDENT EV DOMINATING SET,
11. MAXIMAL MINIMAL EV DOMINATING SET, and
12. UPPER EV IRREDUNDANT SET.

Chapter 6

Conclusions

Nearly all of domination theory in graphs is concerned with sets S of vertices that dominate all of the vertices in $V - S$, subject to certain conditions, either on S or on $V - S$, or both. A relatively small amount of research has been conducted on edge domination theory, on sets of edges that dominate all other edges, again subject to certain conditions. In this dissertation we have considerably expanded the work of Peters [22]. We have not only continued the study of the vertex-edge and edge-vertex domination numbers, but we have expanded and considered vertex-edge and edge-vertex invariants of classical independence and irredundance.

In Chapter 2 we presented the results that Peters [22] had for the *ve*- and *ev*-domination numbers. In Chapter 3 we defined the six vertex-edge parameters and presented the Vertex-Edge Domination Chain. In this chapter we also presented various bounds on the different parameters. These bounds are summarized in Table 6.1. Furthermore, in Chapter 3 we characterized the family of graphs for which $\gamma_{ve} = i_{ve}$, as well as the family of trees for which $\gamma_{ve} = \gamma$

Parameter	Bound
γ_{ve}	$\geq \left\lceil \frac{m}{\Delta^2} \right\rceil$
	$= \left\lceil \frac{n}{6} \right\rceil$, for $2 \times c$ grid graphs
i_{ve}	$\leq i$
β_{ve}	β_0
Γ_{ve}	$\leq \Psi$
IR_{ve}	$\leq OIR \leq \beta_1 \leq \frac{n}{2}$
	$\leq OIR \leq IR$

Table 6.1: Summary of the bounds on the vertex-edge parameters

In Chapter 4 we defined the six edge-vertex parameters and presented the Edge-Vertex Domination Chain. In this chapter we also presented various bounds on the different parameters. These bounds are summarized in Table 6.2. In both Chapters 3 and 4 we presented a result similar to the classic result by Ore for both *ve*- and *ev*-dominating sets (respectively).

Parameter	Bound
ir_{ev}	$\geq \gamma_{\leq 3}$
γ_{ev}	$= i_{ev}$
	$\leq \gamma' \leq \beta_1$
	$\geq \left\lceil \frac{n}{2\Delta - 2} \right\rceil$
	$= \left\lceil \frac{n}{6} \right\rceil$, for $2 \times c$ grid graphs
i_{ev}	$= \gamma_{ev}$
IR_{ev}	$\leq \beta_1 \leq \frac{n}{2}$
	$\leq \beta_1 \leq \Gamma' \leq IR'$

Table 6.2: Summary of the bounds on the edge-vertex parameters

In Chapter 5 we presented various complexity and algorithm approximation results for our twelve parameters. These results are summarized in Tables **??** and **??**. We also presented linear time algorithms for solving WEIGHTED DISTANCE-3 DOMINATION for trees as well as WEIGHTED INDEPENDENT DISTANCE-3 DOMINATION for trees. These two algorithms were then used to prove that WEIGHTED VE DOMINATION and WEIGHTED INDEPENDENT DOMINATION are solvable in linear time for trees. We then finish Chapter 5 by proving that all twelve of our parameters are solvable in linear time when restricted to trees.

Finally, in Appendix A we present an implementation of the weighted distance-3 algorithm mentioned above.

	Family of Graphs			
	General	Tree	Bipartite	Chordal
ir_{ve}		Linearly Solvable [Thm. 119]		
γ_{ve}	NP-Complete [Thm. 94]	Linearly Solvable [Thm. 116]	NP-Complete [Thm. 94]	NP-Complete [Thm. 94]
i_{ve}	NP-Complete [Thm. 103]	Linearly Solvable [Thm. 118]	NP-Complete [Cor. 104]	
β_{ve}		Linearly Solvable [Thm. 119]	NP-Complete [Thm. 109]	NP-Complete [Thm. 110]
Γ_{ve}		Linearly Solvable [Thm. 119]		NP-Complete [Cor. 111]
IR_{ve}		Linearly Solvable [Thm. 119]		NP-Complete [Cor. 112]
ir_{ev}		Linearly Solvable [Thm. 119]		
γ_{ev} $=$ i_{ev}	NP-Complete [Thm. 113]	Linearly Solvable [Thm. 119]	NP-Complete [Thm. 113]	
β_{ev}		Linearly Solvable [Thm. 119]		
Γ_{ev}		Linearly Solvable [Thm. 119]		
IR_{ev}		Linearly Solvable [Thm. 119]		

Table 6.3: Summary of complexity results

	Family of Graphs		
	General	Bipartite	Chordal
ir_{ve}			
γ_{ve}	$c \log n$ [Cor. 98][1, 3]	$c \log n$ [Cor. 98][1, 3]	
i_{ve}	$O\left(n^{1-\varepsilon}\right)$ [Thm. 107][2, 3]	$O\left(n^{1-\varepsilon}\right)$ [Thm. 107][2, 3]	$O\left(n^{1-\varepsilon}\right)$ [Thm. 107][2, 3]
β_{ve}			
Γ_{ve}			
IR_{ve}			
ir_{ev}			
γ_{ev} = i_{ev}	$c \log n$ [Cor. 114][1, 3]		
β_{ev}			
Γ_{ev}			
IR_{ev}			

[1] There exists a constant $c > 0$ such that it is NP-hard to approximate the given parameter to within a $c \log n$ factor.
[2] For some constant $\varepsilon > 0$.
[3] The above parameter is NP-hard to approximate to within the given factor.

Table 6.4: Summary of inapproximability results

Chapter 7

Open Problems

In conducting this research, virtually any of more than 1,800 papers published in domination theory can be used to develop new or similar *ve* or *ev* results. As expected, quite a few open problems have emerged since beginning this research. Among these are the following:

Let $\square$ represent any of:

$$\left\{ \begin{array}{llllll} ir_{ve}, & \gamma_{ve}, & i_{ve} & \beta_{ve}, & \Gamma_{ve}, & IR_{ve} \\ ir_{ev}, & \gamma_{ev} & i_{ev}, & \beta_{ev}, & \Gamma_{ev}, & IR_{ev} \end{array} \right\}$$

1. What is:
   - $\square(K_n)$?
   - $\square(K_{m,n})$?
   - $\square(K_{n_1,n_2,\ldots,n_r})$?
   - $\square(K_{1,n})$?
   - $\square(P_n)$?
   - $\square(C_n)$?
2. When is $\square(G) = 1$?
3. Is $\square(G) + \square\overline{G}) \leq n + 1$?
4. What is $\square(G_{2\times n})$ and $\square(G_{3\times n})$ equal to?
5. For what trees of order n does $\square(T)$ achieve its minimum/maximum value?
6. If $\gamma_{ve} + \overline{\gamma_{ve}} \leq n - \beta_0 + 1 = \alpha_0 + 2$, what is $\gamma_{ev} + \overline{\gamma_{ev}} \leq$?
7. Is $ir_{ev} \leq ir'$?
8. Is $ir_{ve} \leq ir$?
9. Is $\Gamma_{ve} \leq \Gamma$?
10. We know that $\Gamma_{ve} + \gamma \leq n$ is:

- $IR_{ve} + \gamma \leq n$?
- $\Gamma_{ve} + i \leq n$?

11. We know that $\Gamma + \gamma_{ve} \leq n$ is:

    - $IR + \gamma_{ve} \leq n$?
    - $\Gamma + i_{ve} \leq n$?

12. Is every maximal *ve*-irredundant set also a maximal open irredundant set?

13. Is $\gamma'_{\leq 2} \leq \gamma_{ev} \leq \gamma'$?

Appendix A
An Implementation of Algorithm 5.1 Using the C Programming Language

```c
/*
 *  main.c
 *
 *  Created by Jason Robert Lewis on 4/27/07.
 *  Copyright 2007 Jason Robert Lewis. All rights reserved.
 *
 *  This program will find a minimum weight distance-3 dominating set for a tree.
 *  Input: Order (n) of the tree,
 *         the parent array
 *         the weight array (weights for each of the n vertices)
 */

#include <stdio.h>
#include <stdlib.h>

#define INFTY 100000

int minimumWeightedDistance3DominatingSet(int *parent, int *weights, int order);
void combine(int **V, int k, int j);
int minRootValue(int **V, int vertex);
int minCombine(int **V, int k, int j, int position);

FILE *outputFile;

int main (int argc, const char * argv[]) {

    int *parent;         // pointer to the parent array
                         // index 0 represents the root and it's parent will be 0
    int *weights;        // stores the weights for each of the order vertices
       int i;
    int order;          // number of vertices in tree
    FILE *parentFile;
    FILE *weightFile;


    parentFile = fopen("parentArray.dat", "r");
    weightFile = fopen("weightArray.dat", "r");
    outputFile = fopen("output.dat", "w");
    fscanf(parentFile,"%d", &order);

    parent = (int *) calloc(order, sizeof(int));
    for (i = 0; i < order; i++){
        fscanf(parentFile, "%d", &parent[i]);
    }

    weights = (int *) calloc(order, sizeof(int));
    for (i = 0; i < order; i++){
        fscanf(weightFile, "%d", &weights[i]);
    }
    fprintf(outputFile, "\n\nYour parent array is:\n[");
```

```
    for (i = 0; i < order-1; i++)
        fprintf(outputFile, "%d, ", parent[i]);
    fprintf(outputFile, "%d]", parent[i]);
    fprintf(outputFile, "\n\nYour weights are:\n[");
    for (i = 0; i < order-1; i++)
        fprintf(outputFile, "%d, ", weights[i]);
    fprintf(outputFile, "%d]", weights[i]);
    fprintf(outputFile, "\n\nThe Minimum Weight of a Distance-3 Dominating Set
      for your tree is: %d\n\n", minimumWeightedDistance3DominatingSet(parent,
      weights, order));
    printf("\n\nThe Minimum Weight of a Distance-3 Dominating Set for your
      tree is: %d\n\n", minimumWeightedDistance3DominatingSet(parent, weights, order));
    fclose(parentFile);
    fclose(weightFile);
    return 0;

}

int minimumWeightedDistance3DominatingSet(int *parent, int *weights, int order){
    int i;
    int **V;         // Vector double array, first index is the vertex number
                         // second index is the class number
                         // V[i] = [X,X,X,X,X,X,X]

    // Create space for our vector array V
    V = (int **) calloc (order, sizeof(int *));
    for(i = 0; i < order; i++)
        V[i] = (int *) calloc(7, sizeof(int));

    // initialize our vector, INFTY is used since we don't have +infinity
    for(i = 0; i < order; i++){
        V[i][0] = weights[i];
        V[i][1] = INFTY;
        V[i][2] = INFTY;
        V[i][3] = INFTY;
        V[i][4] = 0;
        V[i][5] = 0;
        V[i][6] = 0;
    }

    // "combine" subtrees
    for(i = order-1; i > 0; i--)
        combine(V, parent[i], i);

    for(i = 0; i < order; i++)
        fprintf(outputFile, "\nminRootValue for vertex %d is: %d\n\n", i,
        minRootValue(V,i));

    return minRootValue(V,0);
}

// This function is used to combine subtree k with subtree j
void combine(int **V, int k, int j){
    int i;
    for(i = 0; i < 7; i++)
        V[k][i] = minCombine(V, k, j, i);

    return;
}
```

```
// This function finds the minimum value in the root's vector, exluding the 5th class
int minRootValue(int **V, int vertex){
    int min=INFTY;          // stores the minimum value from the root vector (V[0])
    int i;

    for(i = 0; i < 4; i++){
        if(min > V[vertex][i])
            min = V[vertex][i];
        fprintf(outputFile, "\nThe value of V[%d][%d]=%d",vertex,i,V[vertex][i]);
    }
    fprintf(outputFile, "\nThe value of V[%d][%d]=%d",vertex,i,V[vertex][i]);
    fprintf(outputFile, "\nThe value of V[%d][%d]=%d",vertex,i+1,V[vertex][i+1]);
    fprintf(outputFile, "\nThe value of V[%d][%d]=%d",vertex,i+2,V[vertex][i+2]);

    return min;
}

// This function is used to find the minimum value for a given position in a vector
// during the combining of the kth and jth subtrees
int minCombine(int **V, int k, int j, int position){
    int min=INFTY;          // stores the minimum value

    switch(position){
        case 0:
                if(min > (V[k][0]+V[j][0]))
                    min = V[k][0]+V[j][0];
                if(min > (V[k][0]+V[j][1]))
                    min = V[k][0]+V[j][1];
                if(min > (V[k][0]+V[j][2]))
                    min = V[k][0]+V[j][2];
                if(min > (V[k][0]+V[j][3]))
                    min = V[k][0]+V[j][3];
                if(min > (V[k][0]+V[j][4]))
                    min = V[k][0]+V[j][4];
                if(min > (V[k][0]+V[j][5]))
                    min = V[k][0]+V[j][5];
                if(min > (V[k][0]+V[j][6]))
                    min = V[k][0]+V[j][6];
                break;
        case 1:
                if(min > (V[k][1]+V[j][0]))
                    min = V[k][1]+V[j][0];
                if(min > (V[k][1]+V[j][1]))
                    min = V[k][1]+V[j][1];
                if(min > (V[k][1]+V[j][2]))
                    min = V[k][1]+V[j][2];
                if(min > (V[k][1]+V[j][3]))
                    min = V[k][1]+V[j][3];
                if(min > (V[k][1]+V[j][4]))
                    min = V[k][1]+V[j][4];
                if(min > (V[k][1]+V[j][5]))
                    min = V[k][1]+V[j][5];
                if(min > (V[k][2]+V[j][0]))
                    min = V[k][2]+V[j][0];
                if(min > (V[k][3]+V[j][0]))
                    min = V[k][3]+V[j][0];
                if(min > (V[k][4]+V[j][0]))
                    min = V[k][4]+V[j][0];
```

```
                if(min > (V[k][5]+V[j][0]))
                    min = V[k][5]+V[j][0];
                if(min > (V[k][6]+V[j][0]))
                    min = V[k][6]+V[j][0];
                break;
        case 2:
                if(min > (V[k][2]+V[j][1]))
                    min = V[k][2]+V[j][1];
                if(min > (V[k][2]+V[j][2]))
                    min = V[k][2]+V[j][2];
                if(min > (V[k][2]+V[j][3]))
                    min = V[k][2]+V[j][3];
                if(min > (V[k][2]+V[j][4]))
                    min = V[k][2]+V[j][4];
                if(min > (V[k][3]+V[j][1]))
                    min = V[k][3]+V[j][1];
                if(min > (V[k][4]+V[j][1]))
                    min = V[k][4]+V[j][1];
                if(min > (V[k][5]+V[j][1]))
                    min = V[k][5]+V[j][1];
                break;
        case 3:
                if(min > (V[k][3]+V[j][2]))
                    min = V[k][3]+V[j][2];
                if(min > (V[k][3]+V[j][3]))
                    min = V[k][3]+V[j][3];
                if(min > (V[k][4]+V[j][2]))
                    min = V[k][4]+V[j][2];
                break;
        case 4:
                if(min > V[k][4] + V[j][3])
                        min = V[k][4]+V[j][3];
                break;
        case 5:
                if(min > (V[k][4]+V[j][4]))
                    min = V[k][4]+V[j][4];
                if(min > (V[k][5]+V[j][2]))
                    min = V[k][5]+V[j][2];
                if(min > (V[k][5]+V[j][3]))
                    min = V[k][5]+V[j][3];
                if(min > (V[k][5]+V[j][4]))
                    min = V[k][5]+V[j][4];
                break;
        case 6:
                if(min > (V[k][4]+V[j][5]))
                    min = V[k][4]+V[j][5];
                if(min > (V[k][5]+V[j][5]))
                    min = V[k][5]+V[j][5];
                if(min > (V[k][6]+V[j][3]))
                    min = V[k][6]+V[j][3];
                if(min > (V[k][6]+V[j][4]))
                    min = V[k][6]+V[j][4];
                if(min > (V[k][6]+V[j][5]))
                    min = V[k][6]+V[j][5];
                break;
        default:
                break;
    }

```

```
    return min;
}
```

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