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ON THE CUSPIDALITY OF MAASS-GRITSENKO AND MIXED LEVEL LIFTS

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Abstract

Representation theoretic results of Schmidt suggest that one should be able to construct a Saito-Kurokawa lifting that has mixed level, i.e., the level is paramodular at some places and congruence level at other places. Adapting Schmidt's proof we show that indeed one can construct such a lifting using representation theory. The goal then becomes to construct such a lifting classically as the more explicit nature of such a construction lends itself nicely to arithmetic applications. To this end we obtain half of the classical construction by giving the cuspidal lifting $\mathcal{G}_M : J^c_{\kappa,t}(\Gamma_0(M)) \to S_\kappa(\Gamma_M[t])$, generalizing the cuspidal lifting of Ibukiyama and the cuspidal lifting of Gritsenko.

Dedication

I dedicate this thesis to my dear husband Haytham and to the most precious joy of our lives, our son Hisham.

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Chapter 1

Introduction

From an arithmetic point of view, it is the Fourier series expansion that modular forms possess that make them highly interesting complex functions to study. There are ample examples of modular forms of degree $n = 1$, called elliptic modular forms, whose Fourier coefficients are numbers of interest that show up in many parts of mathematics. This thesis deals with modular forms of higher degree as developed by Siegel. In particular, the focus is on degree 2 where the modular forms are holomorphic functions of three complex variables that live on the symplectic group Sp_4. These forms are known as Siegel modular forms when they satisfy transformation properties with respect to arithmetic subgroups of $\mathrm{Sp}_4(\mathbb{Q})$. In Chapter 2, we recall the basic properties that these functions satisfy with an emphasis on the Fourier development for any degree n.

One arithmetic family is the family of paramodular groups. In this case, the Siegel modular forms satisfying transformation properties with respect to paramodular groups are called paramodular forms. Historically, paramodular groups arised in geometry due to their connection to abelian surfaces with non-principal polarization. They are not as prevalent in the literature on Siegel modular forms as the discrete subgroups of $\mathrm{Sp}_4(\mathbb{Z})$, however interest in these groups has recently increased due to an extension to the degree 2 of the Taniyama-Shimura-Weil conjecture for abelian surfaces by Brumer and Kramer known as the paramodular conjecture ([15]). Chapter 3 starts with the geometrical connection with the paramodular groups of any degree n, and then discusses the implication of the basic properties of Siegel modular forms in Chapter 2 to the case of paramodular forms of degree 2.

While it is relatively easier to understand the significance of Fourier coefficients for elliptic modular forms, it is usually harder to even compute Fourier coefficients for higher degree modular forms. The reason is that these coefficients are indexed by matrices rather than by integers. Instead, finding connections betweeen the higher degree modular forms and elliptic modular forms proved to be fruitful. These connections, called liftings, are linear mappings between the different spaces of modular forms that provide us with ways to construct examples of higher degree modular forms and, most importantly, they provide relations between the Fourier coefficients. Our focus will be on the Saito-Kurokawa liftings relating elliptic modular forms to Siegel modular forms of degree 2.

Given a normalized eigenform $f \in S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z}))$, Saito and Kurokawa conjectured in 1977 the existence of a cuspidal Siegel eigenform F of weight κ and level $\mathrm{Sp}_4(\mathbb{Z})$ whose Spinor L-function is given by $L(s, F, \mathrm{spin}) = \zeta(s-\kappa+1)\zeta(s-\kappa+2)L(s, f)$. This conjecture was later proven in a series of papers by Andrianov, Maass, Kohnen, and Zagier for the classical set-up and by Piatetski-Shapiro in the language of automorphic forms. The fact that the L-function of F factors like this gives that F is a cusp form that locally looks like an Eisenstein series. In particular, one uses this to show that the Ramanujan-Petersson conjecture fails on GSp_4 for general cuspidal automorphic forms.

The Saito-Kurokawa correspondence is a composition of two explicit isomorphisms involving the space of Jacobi forms of index 1:

$$S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z})) \cong J^c_{\kappa,1} \cong S^*_\kappa(\mathrm{Sp}_4(\mathbb{Z})).$$

The second isomorphism is the full level Maass lifting whose image is characterized by the Maass subspace. In Chapter 5, we review the theory of Jacobi forms of degree 1. In addition, we discuss the connections of Jacobi forms to other spaces of modular forms and in particular the components of the Saito-Kurokawa lifting and its generalizations.

From a classical point of view, there are two explicit constructions that generalize the classical full level Saito-Kurokawa lifting to a higher level. We have the congruence level Saito-Kurokawa lifting

$$S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(M)) \cong J^{c,\mathrm{new}}_{\kappa,1}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_0^{(2)}(M)),$$

where the second map is the Maass lifting $\mathcal{V}$ with level $M \geq 1$ recently proved by Ibukiyama ([31]). It takes a cuspidal Jacobi form of index 1 and level M to the space of cuspidal Siegel modular forms of degree 2 with respect to the congruence subgroup $\Gamma_0^{(2)}(M)$ of $\mathrm{Sp}_4(\mathbb{Z})$. Another generalization is the paramodular level Saito-Kurokawa lifting which is also a composition of two linear maps. The second map

$$\mathcal{G} : J^c_{\kappa,t} \hookrightarrow S_\kappa(\Gamma[t]),$$

called Gritsenko's lifting, takes a cuspidal Jacobi form of index t and full level to the space of cuspidal paramodular forms of level t ([27]).

In the last twenty years such lifted forms have been proven to be very powerful tools to study a host of problems in arithmetic, analysis, and geometry. For instance, a recent conjecture of Brumer and Kramer states that p-adic congruences between a Saito-Kurokawa lift of paramodular level and a non-lifted form should correspond to a p-torsion point on the abelian surface associated to the non-lifted form ([15]). Paramodular Saito-Kurokawa lifts have been used to study the rank of p-adic Selmer groups of 2-dimensional Galois representations by Skinner and Urban ([82]). One can also use Saito-Kurokawa lifts of congruence level to study p-torsion in p-adic Selmer groups of 2-dimensional Galois representations ([1]). As such, there are numerous reasons why one would like to generalize the known class of liftings.

Using representation theoretic methods, Schmidt ([74]) explained that Saito-Kurokawa lifts are predicted by Langlands functoriality. He gives an alternate and unified representation theoretic construction of each of the classical Saito-Kurokawa lifts. His construction is local in nature that allows us in Chapter 7 to construct Saito-Kurokawa lifts with a congruence level and with a paramodular level together with a factorization of their L-functions. We call such Saito-Kurokawa lifts mixed level lifts. The degree 2 Siegel modular forms are in this case mixed level Siegel modular forms satisfying transformation properties with respect to the mixed level paramodular group $\Gamma_M[t]$ defined in Chapter 6.

One disadvantage of the representation theoretic approach is that it does not explicitly describe the linear mappings between the spaces of modular forms and hence does not give a concrete relation between the Fourier coefficients. A remedy to this is to use the lengthier classical approach of constructing the liftings explicitly by defining a linear mapping L between the spaces and show that the lifted form, say $L(f)$ satisfy the properties defining a cuspidal modular form in the degree 2 setting.

This brings up to the main chapter of this thesis, Chapter 6, where we generalize the theory of Maass-Gritsenko cuspidal liftings and construct a mixed level cuspidal lifting $\mathcal{G}_{\mathcal{M}} : J^c_{\kappa,t}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_M[t])$ that is an injective linear map into the space of cuspidal mixed level forms. When $M = 1$, our lifting $\mathcal{G}_{\mathcal{M}}$ recovers Gristenko's lifting $\mathcal{G}$, and when $t = 1$ it gives the Maass lifting with level M. The proof consists of proving the holomorphicity of the lift, its modularity with respect to $\Gamma_M[t]$ and finally that the lift is a cuspidal form. While proving modularity, we also characterize the modularity proofs of the Maass-Gritsenko lifts by presenting a general framework that all modularity proofs of these liftings fit into. We find out that this general framework is basically dictated by the theory of maximal parabolic subgroups of highest type and by presenting generators of the level in degree 2 using the same type maximal parabolic subgroups. In this framework, we reprove the modularity of Ibukiyama's Maass lifting by presenting generators of the Hecke congruence subgroup of degree 2 in terms of its maximal parabolic subgroup. To prove the modularity of our lift, we also give generators for the mixed level $\Gamma_M[t]$. In addition, based on the framework characterizing the theory of Maass-Gritsenko lifts and mixed level lifts, we present ingredients of the modularity of any lifting from the space of Jacobi forms of any integral index with respect to congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

While cuspidality in case of degree 1 is easier to establish as the cusps of a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ are just the zero degree cusps corresponding to points on the rational boudary $\mathbb{Q}\cup\{\infty\}$ of the upper half plane $\mathfrak{h}^1$ fixed by the action of the parabolic elements in Γ, higher degree cusps have a more complicated structure. To understand this structure, we describe in Chapter 4 the Satake compactification of Siegel varieties $\Gamma \setminus \mathfrak{h}^n$. The theory of compactification of Siegel varieties is in turn governed by the theory of parabolic subgroups of general linear algebraic groups. We give a brief review of this in Chapter 4. For the Satake compactification of $\Gamma\setminus\mathfrak{h}^n$, we need the structure of the n representatives of conjugacy classes of maximal parabolic subgroups of $\mathrm{Sp}_{2n}(\mathbb{Q})$, called the standard maximal parabolic subgroups. For each degree $0 \leq r \leq n-1$, we have the standard maximal parabolic subgroup $C_{n,r}(\mathbb{Q})$ of type (n,r) and correspondingly we have a standard rational boundary component (cusp) of degree r associated to it. If the level Γ is different than the Siegel modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$, it turns out the number of the associated rational boundary components of each degree r is equal to the cardinality of the Γ-conjugacy classes of parabolic subgroups $C_{n,r}(\mathbb{Q})$ in $\mathrm{Sp}_{2n}(\mathbb{Q})$. This cardinality is a finite number according to results from Borel's reduction theory of arithmetic groups as reviewed in the Appendix. Constructing a more global Siegel Φ operator whose components are linear operators for each rational boundary allows us to make more sense of the definition of cusp forms found in the literature and which is given in Chapter 2 and in Chapter 3. Most importantly, we get that when proving a form is a cusp form, it suffices to show it vanishes at the maximal degree cusps which correspond bijectively to the double cosets $\Gamma \setminus \mathrm{Sp}_{2n}(\mathbb{Q})/C_{n,n-1}(\mathbb{Q})$, where $C_{n,n-1}(\mathbb{Q})$ is the maximal type standard maximal parabolic subgroup of $\mathrm{Sp}_{2n}(\mathbb{Q})$. Therefore, in our case, the maximal degree cusps that are needed to be computed to prove the cuspidality of the lifting $\mathcal{G}_M$ are representatives of the double cosets space $\Gamma_M[t] \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$. Using the structure of the cusps obtained for the level Γ, where Γ represents $\Gamma_0^{(2)}(M), \Gamma[t]$ or $\Gamma_M[t]$, we characterize the cuspidality of any Siegel modular form of degree 2 of level Γ (not necesarily a lift) using its Fourier-Jacobi expansion.

Chapter 2

Siegel Modular Forms

The roots of the theory of modular functions of many variables go back to the nineteenth century. In the same manner as the theory of elliptic functions leads to elliptic modular forms, the theory of compact Riemann surfaces and abelian functions leads to the theory of modular forms of several variables. These forms arise as examples of theta functions. It was Siegel in 1935 in connection with his investigations of the theory of quadratic forms who provided a systematic function theoretic foundation of the theory of modular forms of many variables. These forms were then given the name Siegel modular forms in his honor. They live on the Siegel upper half space of genus $n > 1$, a generalization of the upper half plane of genus 1. Working with abelian varieties in place of elliptic curves, Siegel modular forms provide the right setting to generalize the Taniyama-Shimura conjecture. The role played by the Siegel upper half plane to Siegel modular forms is the same role played by the upper half plane to elliptic modular forms.

As a generalization of the linear fractional transformation group $\mathrm{SL}_2(\mathbb{Z})$ acting on the upper half plane, there is the notion of an arithmetic group Γ as a discrete subgroup of a real Lie group $G = \mathrm{Aut}(D)$ of biholomorphic automorphisms of a Hermitian bounded symmetric domain D. The quotient space $\Gamma \setminus D$ is a complex analytic space. In this chapter we restrict our presentation to D being the Siegel upper half plane. We recall basic definitions and properties of symplectic matrices and Siegel modular forms. We generously borrow from the standard references [4], [5], and [39].

2.1 Basics of Siegel Modular Forms

2.1.1 The Real Symplectic group

Consider a finite dimensional real vector space and a non-degenerate skew symmetric bilinear form A of degree $2n$. The real automorphism group associated to A is defined as

$$\mathrm{Sp}(A, \mathbb{R}) = \{g \in \mathrm{GL}_{2n}(\mathbb{R}) | \, {}^t g A g = A\}.$$

These symplectic groups in general are known as paramodular groups. In this chapter, we deal with the standard skew symmetric bilinear form which we will denote by J_n and in Chapter 3 we will consider a more general bilinear form leading to the so-called paramodular groups.

The standard skew symmetric bilinear form J_n is given by the square matrix of order $2n$

$$\mathrm{J}_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix},$$

where 1_n is the $n \times n$ identity matrix and 0_n is the $n \times n$ zero matrix. Observe that

$$\mathrm{J}_n^{-1} = -\mathrm{J}_n = {}^t J_n.$$

When $n = 2$ we usually denote J_2 by J.

Definition 2.1.1. A matrix $g \in \mathrm{Mat}_{2n}$ is said to be symplectic if it belongs to the following group:

$$G_n = \mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} : \, {}^t g \, \mathrm{J}_n \, g = \mu_n(g) \, \mathrm{J}_n, \mu_n(g) \in \mathrm{GL}_1\}.$$

The nonzero scalar $\mu_n(g)$ is called the multiplier of g.

One can see that G_n is a subgroup of GL_{2n} due to the fact that μ_n is a group homomorphism called the multiplier homomoprhism.

Definition 2.1.2. Define the general real positive symplectic group of genus n or the group of symplectic similitudes consisting of all real symplectic matrices of order $2n$ with positive mutlipliers:

$$\mathrm{GSp}_{2n}^{+}(\mathbb{R}) = \{g \in \mathrm{GL}_{2n}(\mathbb{R}) : \ {}^{t}g\, \mathrm{J}_n\, g = \mu_n(g)\, \mathrm{J}_n, \mu_n(g) > 0\}.$$

The real symplectic group of genus n

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \ker(\mu_n),$$

is the automorphism group of the bilinear form J_n.

Given an element $g \in \mathrm{GSp}_{2n}(\mathbb{R})$, we will often write $g = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$ with $A_n, B_n, C_n, D_n \in \mathrm{Mat}_n(\mathbb{R})$. If $n = 2$ we will often drop it from the notation. The following proposition characterizes elements in $\mathrm{Sp}_{2n}(\mathbb{R})$ and provides the so called symplectic relations.

Proposition 2.1.3.

1. *The matrix* $g = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R})$ *if and only if the following relations hold*

$$ {}^{t}A_n D_n - {}^{t}C_n B_n = 1_n, \qquad {}^{t}A_n C_n = {}^{t}C_n A_n, \qquad {}^{t}B_n D_n = {}^{t}D_n B_n.$$

In particular, $\mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$.

2. *The transpose of* g, ${}^{t}g = \mathrm{J}_n\, g^{-1}\, \mathrm{J}_n^{-1}$, *is symplectic and hence*

$$ {}^{t}A_n D_n - {}^{t}B_n C_n = 1_n, \qquad {}^{t}A_n B_n = {}^{t}B_n A_n, \qquad C_n^{t} D_n = D_n^{t} C_n.$$

3. *The inverse of g is given by*

$$g^{-1} = \mathrm{J}_n^{-1}\, {}^t g\, \mathrm{J}_n = \begin{pmatrix} {}^tD_n & -{}^tB_n \\ -{}^tC_n & {}^tA_n \end{pmatrix}.$$

For $n = 1$, symplecticity just means $\det(g) = 1$. In general, one can derive from the relations above that for an arbitrary $g \in \mathrm{Sp}_{2n}(\mathbb{R}), \det(g)^2 = 1$ and hence $\det g = \pm 1$. In fact we will see below that $\det(g) = 1$ is true but the converse does not hold for $n > 1$.

Proposition 2.1.4 ([4], Theorem 1.2.2)**.** *The real symplectic group* $\mathrm{Sp}_{2n}(\mathbb{R})$ *is generated by the following elements:*

1. *T(S)=* $\begin{pmatrix} 1_n & S \\ 0_n & 1_n \end{pmatrix}$ *where* S *is symmetric in* $\mathrm{Mat}_n(\mathbb{R})$,
2. *U(V)=* $\begin{pmatrix} {}^tV^{-1} & 0_n \\ 0_n & V \end{pmatrix}$ *where* $V \in \mathrm{GL}_n(\mathbb{R})$,
3. $\mathrm{J}_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$.

Corollary 2.1.5. *If* $g \in \mathrm{Sp}_{2n}(\mathbb{R})$, *then* $\det(g) = 1$.

Proof. Since all generators of $\mathrm{Sp}_{2n}(\mathbb{R})$ given in Proposition 2.1.4 have determinant 1, the conclusion follows immediately. □

2.1.2 The Siegel Upper Half Space

Let $n \geq 1$ be an integer.

Definition 2.1.6. The Siegel upper half space of genus n consists of all symmetric complex $n \times n$ matrices whose imaginary part is positive definite, i.e.,

$$\mathfrak{h}^n = \{Z \in \mathrm{Mat}_n(\mathbb{C}) :\ {}^tZ = Z, Z = X + iY, Y = \mathrm{Im}(Z) > 0\}$$

where $X, Y \in \mathrm{Mat}_n(\mathbb{R})$ are symmetric real matrices of size n and $Y > 0$ means that ${}^t\alpha Y \alpha > 0$ for all vectors $\alpha \in \mathbb{R}^n, \alpha \neq 0$.

Definition 2.1.7. A convex cone C in $\mathbb{R}^n$ is a convex set such that whenever x lies in C, so does the entire ray from the origin to x. It is equivalent to

$$C = \{x \in \mathbb{R}^n | \, tx \in A \ \text{ for all } \ t > 0\}.$$

Proposition 2.1.8. *Consider the subspace*

$$S_n = \{Y \in \mathrm{Mat}_n(\mathbb{R}) : \ {}^tY = Y\}$$

of symmetric matrices in $\mathrm{Mat}_n(\mathbb{R})$*. The imaginary part of the points* $Z \in \mathfrak{h}^n$ *form the subspace*

$$P_n = \{Y \in S_n : Y > 0\}$$

of S_n *in* $\mathbb{R}^{\frac{n(n+1)}{2}}$*. Then* P_n *is a convex cone with vertex at the origin, called the cone of positive definite matrices or the cone of positive definite quadratic forms in* $\mathbb{R}^{\frac{n(n+1)}{2}}$*.*

Proof. Let $Y_1, Y_2 > 0$ be two positive matrices and let $\lambda \in \mathbb{R}$ such that $0 \leq \lambda \leq 1$. We then have $\lambda Y_1 + (1 - \lambda) Y_2 > 0$. Hence

$$P_n = \{Y \in \mathrm{Mat}_n(\mathbb{R}) : \ {}^tY = Y, Y > 0\}$$

is a convex subspace of $\mathbb{R}^{\frac{n(n+1)}{2}}$. Moreover, the ray originating from the point 0 and passing through any point $y \in P_n$ lies completely in P_n. Therefore, P_n is a convex cone with vertex at the origin. □

Proposition 2.1.9. *The Siegel upper half space is an* $\frac{n(n+1)}{2}$ *dimensional complex convex domain.*

Proof. We sketch a proof of this result. We first indicate that it is a subset of $\mathbb{C}^{\frac{n(n+1)}{2}}$. Then we show it is a convex subspace of $\mathbb{C}^{\frac{n(n+1)}{2}}$ and then lastly we indicate why it is an open subset. All of the assertions made conclude the assertion that the Sigel upper half plane is a complex convex domain in $\mathbb{C}^{\frac{n(n+1)}{2}}$. Writing $Z = (z_{kl}) = (x_{kl} + iy_{kl})$ where $1 \le k \le l \le n$, and considering the independent entries z_{kl} $(k \le l)$, $\mathfrak{h}^n$ becomes a subset of $\mathbb{C}^{\frac{n(n+1)}{2}}$. Proposition 2.1.8 implies that $\mathfrak{h}^n$ is a convex subset of $\mathbb{C}^{\frac{n(n+1)}{2}}$ and in particular it is simply connected. For each $Z \in \mathfrak{h}^n$, $Z \in \mathrm{GL}_n(\mathbb{C})$ and $-Z^{-1} \in \mathfrak{h}^n$. Hence $\mathfrak{h}^n$ is an open submanifold of the $\frac{n(n+1)}{2}$ dimensional complex Euclidean space.

□

2.1.3 The Action of the Symplectic Group

Similar to the one variable case where $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{PSp}_2(\mathbb{R})$ is the group of biholomorphic automorphisms of the upper half plane $\mathfrak{h}^1$ due to the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{h}^1$, we have the following.

Proposition 2.1.10. *The real symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ acts on $\mathfrak{h}^n$ as a group of biholomorphic automorphisms. This action is given by*

$$g = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} : Z \mapsto gZ := (A_n Z + B_n)(C_n Z + D_n)^{-1}, \tag{2.1}$$

for $Z \in \mathfrak{h}^n$.

In order to verify this, one has to check that the mapping (2.1) is well defined (i.e. $(C_n Z + D_n)$ is invertible), that it maps the Siegel upper half space to itself and that it satisfies

$$(g_1 g_2)Z = g_1(g_2 Z), \qquad 1_{2n} Z = Z.$$

For a proof, we refer to [5], Lemmas 1.3 and 1.4.

The elements of the form $\begin{pmatrix} 1_n & S \\ 0_n & 1_n \end{pmatrix}$ in the generators of $\mathrm{Sp}_{2n}(\mathbb{R})$ correspond to the translation transformation $Z \to Z + S$ in the Siegel upper half space $\mathfrak{h}^n$. The elements of the form $\begin{pmatrix} {}^tU & 0_n \\ 0_n & U^{-1} \end{pmatrix}$ correspond to the dilation $Z \to^t UZU$ in the Siegel upper half space $\mathfrak{h}^n$. The element $J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$ corresponds to the involution $Z \to -Z^{-1}$ in the Siegel upper half space $\mathfrak{h}^n$.

As a consequence of the action of the real symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ on $\mathfrak{h}^n$, we obtain that for a fixed $g \in \mathrm{Sp}_{2n}(\mathbb{R})$ the map

$$\mathfrak{h}^n \to \mathfrak{h}^n, \qquad Z \mapsto gZ \tag{2.2}$$

is biholomorphic. Assigning for each $g \in \mathrm{Sp}_{2n}(\mathbb{R})$ the automorphism (2.2) we obtain a group homomorphism

$$\mathrm{Sp}_{2n}(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{h}^n). \tag{2.3}$$

Remark 2.1.11. Two symplectic Matrices $g, g' \in \mathrm{Sp}_{2n}(\mathbb{R})$ have the same action on $\mathfrak{h}^n$ if and only if $g = \pm g'$.

Proof. To prove this remark it suffices to show the kernel of the homomorphism (2.3) is $\{\pm 1_{2n}\}$. This is clearly seen from the definition of the action

$$gZ = Z$$

for $Z \in \mathfrak{h}^n$, which is equivalent to

$$A_nZ + B_n = Z(C_nZ + D_n).$$

By specializing to $Z = z1_n$, we obtain

$$A_n = D_n, B_n = C_n = 0_n.$$

This implies that $A_nZ = ZA_n$ and consequently $A_n = a1_n$. It follows from the symplectic relations that $a^2 = 1$, which means that $A_n = D_n = \pm 1_n$. $\square$

This remark shows that the kernel of the homomorphism (2.3) is $\{\pm 1_{2n}\}$. It can be also shown that this homomorphism is surjective ([39], page 3) and hence we obtain that

$$\mathrm{Aut}(\mathfrak{h}^n) \cong \mathrm{Sp}_{2n}(\mathbb{R})/\{\pm 1_{2n}\}.$$

2.1.4 The Siegel Upper Half Space Revisited

Definition 2.1.12. A domain is called homogeneous if the group of biholomorphic automorphisms acts transitively.

Proposition 2.1.13. *The Siegel upper half space $\mathfrak{h}^n$ is a homogeneous space.*

Proof. The proof depends on the crucial fact that every point $Z = X + iY \in \mathfrak{h}^n$ is of the form $gi1_n$ with $g \in \mathrm{Sp}_{2n}(\mathbb{R})$. Since $Y > 0$, we can write $Y =^t AA$ with $A \in \mathrm{GL}_n(\mathbb{R})$. Then we have

$$Z = \begin{pmatrix} 1_n & X \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} A & 0_n \\ 0_n & {}^tA^{-1} \end{pmatrix} (i1_n)$$

It follows that the matrix $g = \begin{pmatrix} A & B \\ 0_n & {}^tA^{-1} \end{pmatrix}$ with $B = X{}^tA^{-1}$ is symplectic. $\square$

Lemma 2.1.14 ([4], Lemma 1.2.5)**.** *Let*

$$K = \{g \in \mathrm{Sp}_{2n}(\mathbb{R}) : gi1_n = i1_n\}$$

be the stabilizer of $i1_n$ in $\mathrm{Sp}_{2n}(\mathbb{R})$. It is given by

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R}) \right\}.$$

The mapping $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \to A + iB$ *is an isomorphism of* K *onto the unitary group of order* n*. In particular* K *is compact.*

Proposition 2.1.13 enables us the following identification of real manifolds

$$\mathfrak{h}^n \cong \mathrm{Sp}_{2n}(\mathbb{R})/K.$$

Proposition 2.1.15. *The mapping* $g \mapsto gi1_n$ *defines a one to one correspondence* $\mathrm{Sp}_{2n}(\mathbb{R})/K \to \mathfrak{h}^n$ *which is compatible with the actions of the group* $\mathrm{Sp}_{2n}(\mathbb{R})$*, where on the left side the group acts by multiplication from the left.*

Definition 2.1.16. Let n be a positive integer. The unit disk of degree n is defined as

$$\mathfrak{D}_n = \{W \in \mathrm{Mat}_n(\mathbb{C}) | {}^tW = W, 1 - W\bar{W} > 0\}.$$

It is a generalization of the unit disk to several complex variables. It is a bounded domain in $\mathbb{C}^{\frac{n(n+1)}{2}}$ and is related to $\mathfrak{h}^n$ by a generalized Cayley transformation.

Proposition 2.1.17 ([39], § 1, Proposition 2)**.** *The Cayley transformation*

$$\begin{aligned} l : \mathfrak{h}^n &\to \mathfrak{D}_n \\ Z &\mapsto W := l(Z) = (Z - i1)(Z + i1)^{-1} \end{aligned}$$

maps $\mathfrak{h}^n$ *biholomorphically onto* $\mathfrak{D}_n$*. The inverse of this map is*

$$Z = l^{-1}(W) = i(1 + W)(1 - W)^{-1}.$$

By this Cayley transformation, the Siegel upper half space $\mathfrak{h}^n$ is realized as a bounded domain. This bounded realization by considering the generalized unit disk model of the Siegel upper half plane has geometrical advantages which will be seen when we later look at the compactification of the Siegel upper half space.

Definition 2.1.18. A domain is called symmetric if to each point there exists an involution in the group of biholomorphic automorphisms with the given point as a single fixed point.

Cartan [17] classified the bounded symmetric domains and showed that each bounded symmetric domain is homogeneous. The unit disk of degree n, $\mathfrak{D}_n$ is one of Cartan's main types.

Proposition 2.1.19. *The space $\mathfrak{h}^n$ is a noncompact Hermitian symmetric space in the sense of Cartan.*

Proof. The involution

$$\mathrm{J}_n : \mathfrak{h}^n \to \mathfrak{h}^n$$

corresponding to the element J_n in $\mathrm{Sp}_{2n}(\mathbb{R})$ which sends Z to $-Z^{-1}$ has $i1_n$ as an isolated fixed point. As the real symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ acts transitively, we obtain the same property for any other point of $\mathfrak{h}^n$. $\square$

There exists an element of volume on $\mathfrak{h}^n$ which is invariant under the group $\mathrm{Sp}_{2n}(\mathbb{R})$. Before we present this volume element we need the following two lemmas.

Lemma 2.1.20 ([4], Lemma 1.2.7)**.** *Let*

$$dZ = dXdY = \prod_{1 \le k \le l \le n} dx_{kl}dy_{kl}, \qquad (Z = X + iY = (z_{kl}) = (x_{kl} + iy_{kl}))$$

be the Euclidean element of volume on $\mathfrak{h}^n$. Then for each matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R})$, the following relation holds:

$$dgZ = |\det(CZ + D)|^{-2n-2}dZ.$$

Lemma 2.1.21 ([4], Lemma 1.2.8)**.** *Let $Z' = X' + iY' = gZ$ with $Z = X + iY$ and*

$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R})$. *Then,*

$$Y' = {}^t(C\overline{Z} + D)^{-1} Y (CZ + D)^{-1};$$

in particular,

$$\det Y' = |\det(CZ + D)|^{-2} \det Y.$$

We can now define the symplectic volume element of $\mathfrak{h}^n$.

Proposition 2.1.22 ([4], Proposition 1.2.9)**.** *Let d^*Z be the element of volume on $\mathfrak{h}^n$ given by*

$$d^*Z = (\det Y)^{-n-1} dZ = (\det y_{kl})^{-n-1} \prod_{1 \le k \le l \le n} dx_{kl} dy_{kl}$$

with $Z = X + iY = (z_{kl}) = (x_{kl}) + i(y_{kl}) \in \mathfrak{h}^n$. Then d^*Z is invariant under all symplectic transformations on $\mathfrak{h}^n$, i.e.

$$d^* g(Z) = d^* Z \qquad (g \in \mathrm{Sp}_{2n}(\mathbb{R})).$$

2.1.5 The Siegel Modular Group

Definition 2.1.23. A subgroup G of the topological group $\mathrm{Sp}_{2n}(\mathbb{R})$ is called discrete if there exists a neighborhood U of the unit element 1_{2n} such that no other element of G is contained in U.

Definition 2.1.24. The symplectic group with coefficients in $\mathbb{Z}$, $\mathrm{Sp}_{2n}(\mathbb{Z})$, is called the Siegel modular group.

Proposition 2.1.25 ([4], Proposition 1.3.7.)**.** *Let $\gamma \in \mathrm{GSp}_{2n}(\mathbb{Q})$ be a symplectic matrix of order $2n$ with entries in $\mathbb{Q}$. Then there exists a matrix $g \in \mathrm{Sp}_{2n}(\mathbb{Z})$ such that*

$$g\gamma = \begin{pmatrix} A & B \\ 0_n & D \end{pmatrix}.$$

Theorem 2.1.26. *Let Γ^n be the subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$ generated by the following matrices:*

1. $\begin{pmatrix} 1_n & S \\ 0_n & 1_n \end{pmatrix}$ *where S is symmetric in $\mathrm{Mat}_{2n}(\mathbb{Z})$,*

2. $\begin{pmatrix} {}^tU & 0_n \\ 0_n & U^{-1} \end{pmatrix}$ *where $U \in \mathrm{GL}_{2n}(\mathbb{Z})$ with $\det(U) = \pm 1$,*

3. $J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$.

Then the Siegel modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$ is equal to Γ^n.

Proof. We prove this theorem using Proposition 2.1.25. Let γ be in $\mathrm{Sp}_{2n}(\mathbb{Z})$. Then there exists a matrix $g \in \Gamma^n$ such that

$$\gamma' = g\gamma = \begin{pmatrix} A & B \\ 0_n & D \end{pmatrix}.$$

Since γ' is symplectic, it follows that ${}^tAD = 1_n$ and ${}^tBD = {}^tDB$. Since γ' is integral, it follows that $A, D \in \mathrm{GL}_n(\mathbb{Z})$ and $S = BD^{-1} = {}^t(BD^{-1})$ is a symmetric matrix with integral entries. Thus $\gamma' = T(S)U(D)$, and $\gamma = g^{-1}\gamma' \in \Gamma^n$. $\square$

Definition 2.1.27. Let $M \geq 1$. The kernel of the natural projection homomorphism (mod M)

$$\Gamma_{2n}(M) := \ker\left(\mathrm{Sp}_{2n}(\mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z}/M\mathbb{Z})\right)$$

is called the principal congruence subgroup of level M. It is a normal subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$ of finite index.

Definition 2.1.28. A subgroup

$$\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{R})$$

is called a congruence subgroup if it contains a principal congruence subgroup $\Gamma_{2n}(M) \subset \Gamma$ as a subgroup of finite index.

The following is true only in the case $n > 1$.

Theorem 2.1.29 ([10], Theorem 1)**.** *Each subgroup $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$ of finite index is a congruence subgroup.*

A congruence subgroup that we shall make use of in this thesis is the generalized Hecke group defined by

$$\Gamma_0^{(n)}(M) := \left\{ g = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}) \middle| \, C_n \equiv 0 \pmod{M} \right\}.$$

Definition 2.1.30. A matrix $g \in \mathrm{Sp}_{2n}(\mathbb{R})$ is called a projectively rational symplectic matrix if there exists a number $t \neq 0$ such that tg has rational entries.

We now define commensurable subgroups of the real symplectic group as defined in [22], Chapter II, Definition 6.3.

Definition 2.1.31. A subgroup G of $\mathrm{Sp}_{2n}(\mathbb{R})$ is commensurable with $\mathrm{Sp}_{2n}(\mathbb{Z})$ if

1. Every element $g \in G$ is projectively rational.
2. The intersection $G \cap \mathrm{Sp}_{2n}(\mathbb{Z})$ is of finite index in both G and $\mathrm{Sp}_{2n}(\mathbb{Z})$.

We will refer to such groups as arithmetic groups.

Important examples of arithmetic groups are the congruence subgroups of $\mathrm{Sp}_{2n}(\mathbb{Z})$. This definition shows that each arithmetic group is a discrete subgroup in $\mathrm{Sp}_{2n}(\mathbb{R})$. Therefore, besides the Siegel modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$ and its congruence subgroups, examples of discrete subgroups are subgroups G of the real symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ commensurable with $\mathrm{Sp}_{2n}(\mathbb{Z})$.

Definition 2.1.32. A subgroup G of $\mathrm{Sp}_{2n}(\mathbb{R})$ acts properly discontinuously on $\mathfrak{h}^n$ if each orbit

$$G(Z) := \{gZ : g \in G\}$$

of each point $Z \in \mathfrak{h}^n$ has no accumulation point in $\mathfrak{h}^n$.

In particular, the stabilizer

$$G_Z := \{g \in G : gZ = Z\}$$

of a point $Z \in \mathfrak{h}^n$ is finite. Moreover, for any $Z \in \mathfrak{h}^n$ there exists a neighborhood U of Z such that

$$\{g \in G : gU \cap U \neq \emptyset\} = G_Z.$$

Proposition 2.1.33 ([39], § 3, Proposition 1)**.** *Let G be any subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$. Then G acts properly discontinuously on $\mathfrak{h}^n$ if and only if G is discrete.*

Proof. It is not hard to show that discontinuity implies discreteness. The other direction follows from Proposition 2.1.15 and the following general fact. □

Lemma 2.1.34. *Let X be a homogeneous space G/K for a real Lie group G and a compact subgroup K. Then any discrete subgroup of G acts properly discontinuously on X.*

Lemma 2.1.34 implies that each of the Siegel modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$, its congruence subroups, and any group G commensurable with $\mathrm{Sp}_{2n}(\mathbb{Z})$ acts properly discontinuously on the Siegel upper half plane $\mathfrak{h}^n$. When $n = 1$, this goes back to an old result of Poincare. Discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ are called Fuchsian groups.

In this thesis, we will be dealing with automorphic froms for subgroups Γ of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$. These are holomorphic functions on the Siegel upper half plane satisfying functional equations connecting its values at points of each Γ-orbit on $\mathfrak{h}^n$ given by

$$\Gamma Z = \{\gamma Z | \gamma \in \Gamma\} \qquad (Z \in \mathfrak{h}^n).$$

Such functions are uniquely determined by their restriction to any subset of $\mathfrak{h}^n$ that meets each Γ-orbit on $\mathfrak{h}^n$. Such subsets are called fundamental domains which are, roughly speaking, irreducible complete sets of representatives for the orbits of the underlying group action. Here we recall more formally the definition of a fundamental domain:

Definition 2.1.35. A closed subset D of a topological space X is called a fundamental domain for a discrete transformation group Γ acting on X if it meets each of the Γ-orbits $\Gamma x = \{\gamma x | \gamma \in \Gamma\}$ with $x \in X$ and has no distinct inner points belonging to the same orbit.

It follows that X can be decomposed in the following way

$$X = \cup_{\gamma \in \Gamma/\Gamma'} \gamma D \qquad \Gamma' = \{\gamma \in \Gamma | \gamma x = x, \text{ for all } x \in X\}$$

such that its components pairwise have no common inner points.

In general fundamental domains need not exist and if they exist their construction is very difficult. Siegel constructed a fundamental domain for the action of $\mathrm{Sp}_{2n}(\mathbb{Z})$ for all n that is called the Siegel fundamental domain. It is denoted by F_n and reduces for $n = 1$ to the well known standard fundamental domain in the upper half plane. For more general arithmetic groups acting on bounded symmetric domains, the existence of fundamental domains is due to Borel ([23]).

We briefly recall the main ideas in the construction of a fundamental domain for the $n = 1$ case. The imaginary part of a point $z = x + iy \in \mathfrak{h}^1$ is called the height of z and is denoted by $h(z)$. Lemma 2.1.21 implies that

$$h(gz) = |cz + d|^{-2} h(z), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Because for a fixed z the following inequality

$$|cz + d|^2 = (cx + d)^2 + (cy)^2 < 1$$

has finitely many solutions in integers c, d, this implies that on each orbit, the height takes only finitely many values greater than a fixed one. Thus, each $\mathrm{SL}_2(\mathbb{Z})z$ orbit on $\mathfrak{h}^1$ contains points of maximal height. These points are characterized by the inequalities $|cz + d| \geq 1$ where (c, d) runs over the set of second rows of all matrices in $\mathrm{SL}_2(\mathbb{Z})$ which is equivalent to say (c, d) runs over all pairs of coprime integers. The translations $z \mapsto z + b$, where $b \in \mathbb{Z}$ do not change the height and hence can be chosen so that $|x + b| \leq 1/2$. This shows that each $\mathrm{SL}_2(\mathbb{Z})$-orbit on $\mathfrak{h}^1$ meets the set

$$F_1' = \{z = x + iy \in \mathfrak{h}^1 : |x| \leq 1/2, |cz + d| \geq 1, \quad c, d \in \mathbb{Z} \ \text{ and } \ \gcd(c, d) = 1\}.$$

In fact, this set F_1' can be defined by a finite number of inequalities (see [4], page 44) and which turns out to be equal to the set

$$F_1 = \{z = x + iy \in \mathfrak{h}^1 : |x| \leq 1/2, |z| = x^2 + y^2 \geq 1\}$$

which is the well-known fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}^1$.

The construction of a fundamental domain for $\mathrm{Sp}_{2n}(\mathbb{Z})$ for arbitrary n is similar to the case $n = 1$ which is based on the following idea: each orbit of $\mathrm{Sp}_{2n}(\mathbb{Z})$ on $\mathfrak{h}^n$ contains points $Z = X + iY$ of maximal height $h(Z) = \det Y$. Better representatives are obtained by applying transformations of $\mathrm{Sp}_{2n}(\mathbb{Z})$ that do not change the height. In the $n = 1$ case, the only such transformations are the translations $z \mapsto z + b$ with $b \in \mathbb{Z}$; but when $n > 1$, besides the translations $Z \mapsto Z + B$ where B is symmetric in $\mathrm{Mat}_n(\mathbb{Z})$ there are also mappings $Z \mapsto {}^tUZU$ with $U \in \mathrm{GL}_n(\mathbb{Z})$.

In general, if we consider the subgroup

$$\left\{ U(V) \in \mathrm{Sp}_{2n}(\mathbb{R}) \middle| U(V) = \begin{pmatrix} {}^tV^{-1} & 0_n \\ 0_n & V \end{pmatrix}, V \in \mathrm{GL}_n(\mathbb{R}) \right\}$$

of $\mathrm{Sp}_{2n}(\mathbb{R})$, which is canonically isomorphic to $\mathrm{GL}_n(\mathbb{R})$, then the action of $\mathrm{Sp}_{2n}(\mathbb{R})$ on $\mathfrak{h}^n$ induces the action of $\mathrm{GL}_n(\mathbb{R})$ on the cone of real positive definite matirces of order n,

$$P_n = \{Y \in \mathrm{Mat}_n(\mathbb{R}) | \, {}^tY = Y, Y > 0\}$$

to which the imaginary parts of elements $Z \in \mathfrak{h}^n$ belong. This action is given by

$$\mathrm{GL}_n(\mathbb{R}) \times P_n \to P_n$$
$$(V, Y) \mapsto {}^tVYV.$$

This discussion shows that there is a connection between the action of $\mathrm{Sp}_{2n}(\mathbb{R})$ on $\mathfrak{h}^n$ and the action of $\mathrm{GL}_n(\mathbb{R})$ on P_n. The same relation holds for arithmetically defined subgroups of $\mathrm{Sp}_{2n}(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{R})$, respectively. For instance, we take $\mathrm{Sp}_{2n}(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Z})$. We call $\mathrm{GL}_n(\mathbb{Z})$, the unimodular group U_n of degree n. It consists of all n-rowed matrices u with integral entries and $\det u = \pm 1$, called unimodular matrices. Siegel's construction of a fundamental domain for the action of $\mathrm{Sp}_{2n}(\mathbb{Z})$ on $\mathfrak{h}^n$ is based on the Minkowski reduction theory of positive definite quadratic forms. This theory, which is concerned with the action of $\mathrm{GL}_n(\mathbb{Z})$ on P_n, is used to determine a fundamental domain for the action of U_n on the cone of real positive definite matrices P_n. The idea is based on finding a nice reduced representative ${}^tUYU$ of the orbit $\mathrm{GL}_n(\mathbb{Z})Y = \{{}^tVYV; V \in \mathrm{GL}_n(\mathbb{Z})\}$ of each point $Y \in \mathfrak{h}^n$. This representative is found by determining U column by column using certain (see [39] page 12 or [4] page 46) minimization conditions. For the full details, the reader can consult the original reference [81], Section VI. The fundamental domain for the action of U_n on P_n turns out to be the following set R_n, called Minkowski's reduced domain.

Definition 2.1.36 ([39], § 2, Definition 1)**.** Minkowski's reduced domain is the set

$$R_n = \{Y \in P_n | \, Y \ \text{ satisfying 1) and 2) }\}$$

where

1. ${}^tgYg \geq y_{kk}$, $1 \leq k \leq n$, for all integral $g = (g_i) \in \mathbb{Z}^n$ where $g_1, g_2, \ldots, g_n$ are relatively prime.

2. $y_{k,k+1} \geq 0$, $1 \leq k \leq n-1$.

Now to construct a fundamental domain for the action of $\mathrm{Sp}_{2n}(\mathbb{Z})$ on $\mathfrak{h}^n$, we first choose representatives Z of $Sp_{2n}(\mathbb{Z})$-orbits satisfying the inequalities $|\det CZ + D| \geq 1$, where (C, D) runs over the set of second rows of all matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{Z})$. Such a pair of matrices is called coprime symmetric pair. We now briefly sketch why it is possible to find such representatives.

Analogous to the case $n = 1$, we call $\det Y$ the height of the point Z in $\mathfrak{h}^n$. Lemma 2.1.21 says that the height satisfies

$$h(gZ) = |\det CZ + D|^{-2} h(Z), \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}),$$

and that for every $g \in Sp_{2n}(\mathbb{Z})$, $h(Z) \geq h(gZ)$ is equivalent to the condition $|\det CZ + D| \geq 1$ where (C, D) is the second raw of the matrix g. For a fixed $Z \in \mathfrak{h}^n$, it turns out (see [4], page 57) that the inequality $|\det CZ + D| < 1$ has only finitely many solutions in non-equivalent coprime symmetric pairs (C, D). This implies that the functions $h(gZ)$ on $\mathrm{Sp}_{2n}(\mathbb{Z})$ takes only finitely many different values greater than $h(Z)$ and that each $\mathrm{Sp}_{2n}(\mathbb{Z})$-orbit contains a point Z of maximal height. This point is characterized by the inequality $|\det CZ + D| \geq 1$ for every coprime symmetric pair (C, D). In addition, the height remains unchanged by applying transformations of $\mathrm{Sp}_{2n}(\mathbb{Z})$ of the form $\begin{pmatrix} U & S\,{}^t\!U^{-1} \\ 0_n & {}^t\!U^{-1} \end{pmatrix}$ where $U \in \mathrm{GL}_n(\mathbb{Z})$ and $S \in \mathrm{Mat}_n(\mathbb{Z})$ is a symmetric matrix. This allows one (see [39], page 29) to obtain a representative Z whose imaginary part Y is in R_n, the Minkowski reduced domain.

Theorem 2.1.37 ([5], Theorem 1.16)**.** *Let F_n be the subset of $\mathfrak{h}^n$ consisting of those $Z = X + iY \in \mathfrak{h}^n$ which statisfy the following conditions:*

1. $|\det CZ + D| \geq 1$ *for all coprime symmetric pairs (C, D) of matrices of order n;*

2. $Y \in R_n$, *where R_n is the Minkowsi reduced domain;*

3. $X \in X_n = \{X = (x_{kl}), {}^tX = X : |x_{kl}| \leq \frac{1}{2}, (1 \leq k, l \leq n)\}$.

Then F_n *is a fundamental domain for the action of* $\mathrm{Sp}_{2n}(\mathbb{Z})$ *on* $\mathfrak{h}^n$*.*

Corollary 2.1.38 ([5], page 16)**.** *If* $Z = X + iY \in F_n$*, then*

$$Y \geq b_n 1_n \quad \textit{and} \quad \mathrm{Tr}(Y^{-1}) \leq n b_n,$$

where b_n *is a positive constant depending only on* n*, and* Tr *denotes the trace.*

It follows from Corollary 2.1.38 that the fundamental domain is closed in the space of all complex symmetric matrices. Siegel proved that F_n is connected and that its boundary consists of a finite number of algebraic hypersurfaces. The images of F_n under the Siegel modular group cover $\mathfrak{h}^n$ without gaps. The domain F_n has only finitely many neighbors and each compact subset of $\mathfrak{h}^n$ is covered by finitely many images of F_n. It is not compact, since $i\lambda 1_n$ belongs to F_n for any arbitrary $\lambda \geq 1$. But as an another consequence of Corollary 2.1.38, it can be shown that the volume of the fundamental domain F_n with respect to the invariant symplectic element of volume is finite ([5], page 16). In fact, Siegel calculated precisely its volume:

$$\mathrm{vol}\, F_n = 2\pi^{-n(n+1)/2} \prod_{k=1}^{n} (k-1)!\zeta(2k).$$

Fundamental domains also exist for congruence subgroups of $\mathrm{Sp}_{2n}(\mathbb{Z})$. Let Γ be a subgroup of finite index n in $\mathrm{Sp}_{2n}(\mathbb{Z})$. We set

$$\Gamma' = \Gamma \cup (-1_{2n})\Gamma.$$

Then, Γ' is again a subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$.

Theorem 2.1.39 ([5], Theorem 1.19)**.** *Each subgroup of the symplectic group* $\mathrm{Sp}_{2n}(\mathbb{R})$ *of the form* $\Gamma^g = g^{-1}\Gamma g$*, where* Γ *is a subgroup of finite index in* $\mathrm{Sp}_{2n}(\mathbb{Z})$ *and* g *is a projectiveliy rational subgroup of* $\mathrm{Sp}_{2n}(\mathbb{R})$*, has a fundamental domain* $F(\Gamma^g)$ *on* $\mathfrak{h}^n$ *given by*

$$F(\Gamma^g) = \cup_{\gamma \in \Gamma' \backslash \Gamma} (g^{-1}\gamma) F_n,$$

where F_n is a fundamental domain for $\mathrm{Sp}_{2n}(\mathbb{Z})$, and γ ranges over a system of representatives of different left cosets of Γ modulo Γ'.

Proposition 2.1.40. *Let $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$ be a subgroup of finite index. Let $F(\Gamma)$ be a fundamental domain for Γ on $\mathfrak{h}^n$. Then the volume*

$$\mathrm{vol}(F(\Gamma)) = \int_{F(\Gamma)} d^*Z,$$

does not depend on the choice of fundamental domain and is finite.

Proof. Proposition 2.1.22 shows that $\mathrm{vol}(F(\Gamma))$ is independent of the choice of $F(\Gamma)$. Taking $F(\Gamma) = \cup_{i=1}^n \gamma_i F_n$, where $\{\gamma_i : i = 1, \dots n\}$ is a set of representatives for the left cosets of $\mathrm{Sp}_{2n}(\mathbb{Z})$ modulo Γ' as defined above, we then have

$$\begin{aligned} \mathrm{vol}(F(\Gamma)) &= \sum_{i=1}^n \int_{\gamma_i F_n} d^*Z = \sum_{i=1}^n \int_{F_n} d^*\gamma_i Z \\ &= \sum_{i=1}^n \int_{F_n} d^*Z = [\mathrm{Sp}_{2n}(\mathbb{Z}) : \Gamma'] \, \mathrm{vol}(F_n). \end{aligned}$$

This finishes the proof since we know $\mathrm{vol}(F_n)$ is finite. □

2.2 Siegel Modular Forms

Definition 2.2.1 ([4], Lemma 1.4.1)**.** Let S be a semigroup acting on a set H. This action is given by $g : h \mapsto g(h)$ for $g \in S$ and $h \in H$. A mapping

$$\phi : S \times H \to T$$

where T is a group, is called a T-valued factor of automorphy of S on H if it satisfies

$$\phi(g_1 g_2, h) = \phi(g_1, g_2(h))\phi(g_2, h)$$

for all $g_1, g_2 \in S$ and $h \in H$.

Lemma 2.2.2. *Let $\phi : S \times H \to T$ be a T-valued factor of automorphy of S on H and let $F : H \to V$, where V is a left $T-$module, and $g \in S$, be a mapping, then the mapping $F|g : H \to V$ given by*

$$(F|g)(h) = (F|_{\phi}g)(h) = \phi(g, h)^{-1}F(g(h))$$

satisfies

$$F|g_1|g_2 = F|g_1g_2$$

for all $g_1, g_2 \in S$.

Proof. It follows immediately from the definition that

$$\begin{aligned}(F|g_1|g_2)(h) &= \phi(g_2, h)^{-1}(F|g_1)(g_2(h)) \\ &= \phi(g_2, h)^{-1}\phi(g_1, g_2(h))^{-1}F(g_1(g_2h)) \\ &= \phi(g_1g_2, h)^{-1}F(g_1g_2(h)) \\ &= (F|g_1g_2)(h).\end{aligned}$$

□

In our treatment of Siegel modular forms of integral weight κ,

$$j(\gamma, Z) = \det(CZ + D)^{\kappa}$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R})$, is a $\mathbb{C}^*$-valued factor of automorphy of the real symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ on the Siegel upper half plane $\mathfrak{h}^n$. The slash operator of integral weight κ action on complex valued functions F on $\mathfrak{h}^n$, given by

$$F \mapsto F|_{\kappa}\gamma : \; (F|_{\kappa}\gamma)(Z) = \det(CZ + D)^{-\kappa}F(\gamma Z),$$

sends holomorphic functions to holomorphic functions. By the lemma above, we have the relations

$$F|_\kappa \gamma_1\gamma_2 = (F|_\kappa \gamma_1)|_\kappa \gamma_2 \tag{2.4}$$

for every $\gamma_1, \gamma_2 \in \mathrm{Sp}_{2n}(\mathbb{R})$.

We are now ready to define a Siegel modular form for a general arithmetic group.

Definition 2.2.3. A character χ of a group G is a group homomorphism of G into the group of roots of unity with kernel of finite index in G.

Definition 2.2.4. Let Γ be an arithmetic group, χ a character of Γ and κ an integer. A complex valued function F on $\mathfrak{h}^n$ is called a Siegel modular form of degree n, weight κ and character χ for the group Γ if it satisfies the following conditions:

1. F is holomorphic on $\mathfrak{h}^n$,

2. for every matrix $\gamma \in \Gamma$ we have that

$$F|_\kappa \gamma = \chi(\gamma) F,$$

3. for every projective rational symplectic matrix $g \in \mathrm{Sp}_n(\mathbb{R})$, $F|_\kappa g$ is bounded in subsets of $\mathfrak{h}^n$ of the form $\mathfrak{h}^n(\epsilon) = \{Z = X + iY \in \mathfrak{h}^n |\, Y \geq \epsilon 1_{2n}\}$, with $\epsilon > 0$.

Remark 2.2.5. 1. The set of all modular forms of weight κ and character χ for the group Γ is a vector space over $\mathbb{C}$ that we denote by $M_\kappa(\Gamma, \chi)$. If χ is the trivial character, then we write $M_\kappa(\Gamma, \chi)$ as $M_\kappa(\Gamma)$.

2. For $n > 1$, we shall see that the third condition in the definition of a Siegel modular form follows immediately from the first two conditions due to Koecher principle, which we will state below. However for $n = 1$, we require that every function $F|_\kappa g$ for $g \in \mathrm{SL}_2(\mathbb{Q})$ is bounded in subsets of $\mathfrak{h}^1$ of the form $\{z = x + iy \in \mathfrak{h}^1 | y \geq \epsilon\}$, with $\epsilon > 0$.

Definition 2.2.6. A character of a congruence subgroup Γ is called a congruence character if it is trivial on a principal congruence subgroup contained in Γ.

Example 2.2.7. Let χ be a Dirichlet character modulo M. Then for an element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(M)$, the mapping

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto [\chi]\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \chi(\det D)$$

is clearly a congruence character of $\Gamma_0^{(n)}(M)$.

Proposition 2.2.8 ([5], Proposition 1.21)**.** *Assume that $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{R})$ is a congruence subgroup, χ a congruence character of Γ and $g \in \mathrm{Sp}_{2n}(\mathbb{R})$ is a projectively rational symplectic matrix. Then*

$$\Gamma^g = g^{-1}\Gamma g$$

is also a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$. For every $g' \in g^{-1}\Gamma g$, the character $\chi^g(g') = \chi(gg'g^{-1})$ of this group is a congruence character of Γ^g. Moreover,

$$|_\kappa : F \in M_\kappa(\Gamma, \chi) \mapsto F|_\kappa g \in M_\kappa(\Gamma^g, \chi^g).$$

Proof. It suffices to consider $g \in \mathrm{Sp}_{2n}(\mathbb{Q})$. Let $q \in \mathbb{N}$ be such that $\Gamma^n(q) \subset \Gamma$ and χ trivial on $\Gamma^n(q)$. Let $W \in \Gamma^n(q)$. We would like to show that $gWg^{-1} \in \Gamma^n(q)$. We write

$$\begin{aligned} gWg^{-1} &= g(1_{2n} + (W - 1_{2n}))g^{-1} \\ &= 1_{2n} + qg(q^{-1}(W - 1_{2n}))g^{-1}. \end{aligned}$$

Hence gWg^{-1} is congruent to 1_{2n} modulo q and $\chi^g(W) = \chi(gWg^{-1}) = 1$. It remains to show that $F|_\kappa g \in M_\kappa(\Gamma^g, \chi^g)$ for any degree n. The function $F|_\kappa g$ is holomorphic because the matrix $(CZ + D)$ is invertible (implicit in the definition of the action of $\mathrm{Sp}_{2n}(\mathbb{R})$ on $\mathfrak{h}^n$)

where (C, D) is the lower row of block matrices defining g. Let R be in Γ^g, then $R = g^{-1}S_1g$ with $S_1 \in \Gamma$. Now relations (2.4) imply that

$$(F|_\kappa g)|_\kappa R = (F|_\kappa g)|_\kappa g^{-1}S_1N = F|_\kappa S_1g = \chi(S_1)F|_\kappa g = \chi^g(S)F|_\kappa g.$$

For $n = 1$, let W be in $\mathrm{SL}_2(\mathbb{Z})$, then we have

$$F|_\kappa g|_\kappa W = F|_\kappa gW.$$

Propostion 2.1.25 implies that we can write $gW = W'g'$ where $W' \in \mathrm{SL}_2(\mathbb{Z})$ and $g' = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $ad = \mu(N) > 0$. Hence

$$(F|_\kappa g)|_\kappa W = F|_\kappa (W'g') = d^{-\kappa}(F|_\kappa W')(az/d + b/d).$$

Since $F|_\kappa W'$ is bounded on $\mathfrak{h}^1(a\epsilon/d)$ for any $\epsilon > 0$, it follows that $(F|_\kappa g)W$ is bounded on $\mathfrak{h}^1(\epsilon)$ and this proves that $F|_\kappa g \in M_\kappa(\Gamma^g, \chi^g)$ for any $n \geq 1$. □

2.2.1 Fourier Series of a Siegel Modular Form

Arithmetically, we are interested in expanding a Siegel modular form into a Fourier series. In the case $n = 1$ we require F to be holomorphic at the cusps (equivalent to the third condition in the definition of a Siegel modular form) so that F has a Fourier expansion. However, for $n > 1$, a Siegel modular form of degree n has a Fourier expansion automatically as a consequence of being holomorphic and satisfying the transformation properties due to the Koecher principle. This phenomenon is absent in the $n = 1$ case and is only true in several complex variables. The Koecher principle was proved by Max Koecher in 1954 [40] and says that the boundedness of Siegel modular forms in subsets of $\mathfrak{h}^n$ of the form $\mathfrak{h}^n(\epsilon)$ is automatically satisfied in the case $n > 1$. A corresponding principle for Hilbert modular forms was already known in 1928 by Fritz Götzky [24].

Theorem 2.2.9. *(Koecher principle) Let F be a Siegel modular form of degree $n \geq 2$, weight κ and level Γ that satisfies only the first two conditions in the defintion of a Siegel modular form. Then F and all its conjugate functions $F|_\kappa g$ are bounded on any subset of $\mathfrak{h}^n$ of the form $\mathfrak{h}^n(\epsilon) = \{Z = X + iY \in \mathfrak{h}^n | Y \geq \epsilon 1_{2n}\}$ with $\epsilon > 0$.*

Theorem 2.2.10. *Let $F \in M_\kappa(\Gamma, \chi)$ be a modular form of integral weight κ and a congruence character χ for a congruence subgroup Γ of $\mathrm{Sp}_{2n}(\mathbb{R})$, then for every $g \in \mathrm{Sp}_{2n}(\mathbb{Q})$, the following expansion holds*

$$(F|_\kappa g)(Z) = \sum_{N \in \mathbb{E}^n, N \geq 0} a_g(N) e^{\frac{\pi i}{h} \mathrm{Tr}(NZ)}, \tag{2.5}$$

with constant coefficients $a_g(N)$, where

$$\mathbb{E}^n = \{N = (n_{ij}) \in \mathrm{Mat}_n(\mathbb{Z}) |\, {}^tN = N, n_{ii} \in 2\mathbb{Z}\}$$

is the set of even matrices of order n, Tr denotes the trace and where $h = h(\Gamma, \chi)$ is the least positive integer such that the group Γ contains a subgroup of the form

$$\mathbb{T}(h) = \mathbb{T}^n(h) = \left\{ \begin{pmatrix} 1_n & hS \\ 0 & 1_n \end{pmatrix} \middle| S = {}^tS \in \mathrm{Mat}_n(\mathbb{Z}) \right\}. \tag{2.6}$$

The series converges absolutely on $\mathfrak{h}^n$ and uniformly on subsets of $\mathfrak{h}^n$ of the form $\mathfrak{h}^n(\epsilon)$; in particular, each function $F|_\kappa g$ is bounded on each of the subsets $\mathfrak{h}^n(\epsilon)$.

The coefficients $a_g(N)$ satisfy

$$a_g({}^tV\, NV) = (\det V)^\kappa \chi(A) e^{-\frac{\pi i}{h} \mathrm{Tr}(NVU)} a_g(N), \quad (\text{ for every } N \in \mathbb{E}^n), \tag{2.7}$$

for every matrix A of the group Γ of the form

$$A = A(U, V) = \begin{pmatrix} V^{-1} & U \\ 0_n & {}^tV \end{pmatrix}. \tag{2.8}$$

If $\kappa \geq 0$, then

$$|(F|_\kappa g)(Z)| \leq c \det(Y)^\kappa, \qquad Z = X + iY \in \mathfrak{h}^n,$$

and

$$a_g(N) \leq c'(\det N)^\kappa$$

where c and c' are constants depending on F and on g.

Proof. It suffices to prove all the above for the modular form F; we obtain the corresponding results for the conjugate functions $F|_\kappa g$ due to Proposition 2.2.8. In the case $n = 1$, an elliptic modular form f is periodic with period h

$$f(z+h) = f(z) \quad \text{ for every } \; z = x + iy \in \mathfrak{h}^1$$

where h is the smallest positive integer such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$. As such, it has a Laurent series of a holomorphic function in the variable $q = e^{\frac{2\pi i z}{h}}$ when $|q| = e^{\frac{-2\pi y}{h}} < 1$, except, possibly, at $q = 0$. This Laurent series in the variable q looks like the Fourier expansion in the original variable z. This expansion is of the form

$$f(z) = \sum_{n \in \mathbb{Z}} a(n) q^n = \sum_{n \in \mathbb{Z}} a(n) e^{\frac{2n\pi i z}{h}}.$$

The boundedness of $f(z)$ for $y \geq \epsilon$, the third condition in the definition of a modular form which is also referred to as holomorphicity of f at the cusp ∞, shows that f is bounded in

$$|q| = e^{\frac{-2\pi y}{h}} < e^{\frac{-2\pi \epsilon}{h}} \quad \text{ with } \; \epsilon > 0.$$

This, together with Riemann's criterion of removable singularities imply that f becomes holomorphic at $q = 0$. Hence, $a(n) = 0$ for negative n. It follows that the series converges uniformly on any $\mathfrak{h}^1(\epsilon)$ with $\epsilon > 0$.

For $n \geq 2$, the transformation formula that F satisfies applied to the type of matrices of order $2n$ in $\mathbb{T}(h) \subset \Gamma$ of the form $\begin{pmatrix} 1_n & hS \\ 0 & 1_n \end{pmatrix}$ becomes

$$F(Z + hS) = F(Z) \qquad (Z = (z_{kl}) \in \mathfrak{h}^n, S = {}^tS \in \mathrm{Mat}_n(\mathbb{Z})). \tag{2.9}$$

This means that F is a periodic function of period h in its $n(n+1)/2$ variables z_{kl}, $k \leq l$. Since F is also holomorphic, it can be expanded in a Fourier series of the form

$$F(Z) = \sum_{N'} a'(N'; Y) e^{\left(\frac{2\pi i}{h} \sum_{1 \leq k \leq l \leq n} n_{kl} x_{kl}\right)},$$

where $Z = X + iY$ with $X = (x_{kl})$, and $N' = (n_{kl})$ ranges over the set of all upper triangular matrices of order n with integral entries n_{kl}. Due to holomorphicity, the series can also be differentiated with respect to all variables. Now taking even matices $N = N' + {}^tN'$. of order n running in the set $\mathbb{E}^n$ and since

$$2 \sum_{1 \leq k \leq l \leq n} n_{kl} x_{kl} = \mathrm{Tr}(NZ),$$

the expansion can be rewritten in the form

$$F(Z) = \sum_{N \in \mathbb{E}^n} a(N; Y) e^{\frac{2\pi i}{h} \mathrm{Tr}(NZ)},$$

where $a(N, Y) = a'(N'; Y) e^{\left(\frac{-\pi}{h} \mathrm{Tr}(NY)\right)}$. Since $F(Z)$ is holomorphic in each of the variables z_{kl}, it satisfies the Cauchy Riemann equations with respect to z_{kl}, which can be written in the form

$$\frac{\partial F}{\partial \bar{z}_{kl}} = 0, \qquad \text{where } \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right).$$

Since the last expansion can be differentiated term by term, it follows that

$$\frac{\partial a(N, Y)}{\partial \bar{z}_{kl}} = \frac{i}{2} \frac{\partial a(N, Y)}{\partial y_{kl}} = 0, \qquad 1 \leq k \leq l \leq n.$$

Hence the coefficients $a(N,Y)=a(N)$ are independent of Y, and we get the Fourier expansion of F

$$F(Z)=\sum_{N\in\mathbb{E}^n} a(N)e^{\frac{\pi i}{h}\operatorname{Tr}(NZ)}$$

with constant coefficients.

As in the $n=1$ case, this expansion can be considered as a Laurent expansion of the holomorphic Function F in the $n(n+1)/2$ variables $q_{kl}=e^{\frac{2\pi i z_{kl}}{h}}$, and so it converges absolutely on $\mathfrak{h}^n$.

The transformation property of a Siegel modular form for a matrix A as given in the form (2.8), gives the relation

$$(\det(V))^{-\kappa}\sum_{N\in\mathbb{E}^n} a(N)e^{\frac{\pi i}{h}\operatorname{Tr}(NV^{-1}Z\,{}^tN^{-1}+NU\,{}^tV^{-1})}=\chi(A)\sum_{N\in\mathbb{E}^n} a(N)e^{\frac{\pi i}{h}\operatorname{Tr}(NZ)}. \qquad (2.10)$$

Replacing N by ${}^tVNV$ on the left and comparing the coefficients, we get the relation (2.7). It remains to show that $a(N)=0$ unless $N\geq 0$ and that the series converges uniformly on subsets of the form $\mathfrak{h}^n(\epsilon)$. The proof of this is what goes into Koecher's argument. We just give a brief sketch of this.

Restricting to matrices $A(0,V)$ in the kernel of the character χ, for $V\in\mathrm{GL}_n(\mathbb{Z})$, relation (2.7) becomes $a({}^tV\,NV)=(\det(V))^{\kappa}a(N)$ and in particular for $V\in\mathrm{SL}_n(\mathbb{Z})$, we have that $a({}^tV\,NV)=a(N)$. Now for such $V's$ and if $a(N)\neq 0$, we sum over matrices $N'={}^tV\,NV$ and the series $a(N)\sum_{N'} e^{\frac{\pi i}{h}\operatorname{Tr}(N'Z)}$ converges absolutely for every $Z\in\mathfrak{h}^n$ (that is, $\sum_{N'} e^{\frac{-\pi}{h}\operatorname{Tr}(N'Z)}$ is convergent) because it is a partial sum of an absolutely convergent series and in particular the series $\sum_{N'} e^{\frac{-\pi}{h}\operatorname{Tr}(N'1_n)}$ is convergent. Now the rest of the proof is based on proving if N does not satisfy $N\geq 0$, then this last series diverges and this proves the assertion that $a(N)=0$ unless $N\geq 0$, which proves the expansion

$$F(Z)=\sum_{N\in\mathbb{E}^n,N\geq 0} a(N)e^{\pi i\frac{\operatorname{Tr}(NZ)}{h}}$$

as a consequence of the first two conditions in the definition of a Siegel modular form.

The uniform convergence on subsets of $\mathfrak{h}^n$ of the form $\mathfrak{h}^n(\epsilon)$ is established first by using the fact

$$\operatorname{Tr}(AR) \geq \operatorname{Tr}(BR) \quad \text{if} \;\; A \geq B \;\; \text{and} \;\; R \geq 0.$$

Now when $Z = X + iY \in \mathfrak{h}^n(\epsilon)$, we have that

$$\operatorname{Tr}(NA) \geq \epsilon \operatorname{Tr}(N),$$

for every $N \in \mathbb{E}^n$. Next, we show that the Fourier expansion of F,

$$F(Z) = \sum_{N \in \mathbb{E}^n, N \geq 0} a(N) e^{\pi i \frac{\operatorname{Tr}(NZ)}{h}},$$

is majorized on each set $\mathfrak{h}^n(\epsilon)$ by a convergent series with nonnegative constant coefficients. For $Y \geq \epsilon 1_n$, we obtain

$$\begin{aligned} |F(Z)| &\leq \sum_{N \in \mathbb{E}^n, N \geq 0} |a(N)| e^{-\frac{\pi}{h} \operatorname{Tr}(NY)} \\ &\leq \sum_{N \in \mathbb{E}^n, N \geq 0} |a(N)| e^{-\frac{\pi}{h} \epsilon \operatorname{Tr}(N)} < \infty. \end{aligned}$$

Finally, for the proof on the bounds of $F|_\kappa g$ and its Fourier coefficients, we refer to [4], page 70.

□

Remark 2.2.11. 1. The expansion (2.5) is called the Fourier expansion of $F|_\kappa g$ and the numbers $a_g(N)$ with $N \in \mathbb{E}^n, N \geq 0$ are the Fourier coefficients of $F|_\kappa g$.

2. Instead of summing over even matrices in $\mathbb{E}^n$, it is also possible to consider summing over half integral symmetric matrices $T = (t_{lk})$, where half integral is meant to indicate the linear form

$$X \mapsto \operatorname{Tr}(TX) = \sum_{k=1}^{n} t_{ll} x_{ll} + 2 \sum_{k<l=1} t_{kl} x_{kl}$$

has integral coefficients, i.e., that the integers t_{kk} are integers and for $l < k$, $2t_{lk}$ are integers. In this case, the Fourier expansion of F can be written as

$$F(Z) = \sum_{T \in \mathbb{S}_n^{\geq 0}(\mathbb{Z})} a(T) e^{\frac{2\pi i}{h} \operatorname{Tr}(TZ)}.$$

The notation $\mathbb{S}_n^{\geq 0}(\mathbb{Z})$ means that T ranges over positive semidefinite semi-integral symmetric $n \times n$ matrices T.

3. In case the congruence subgroup Γ is the full level $\mathrm{Sp}_{2n}(\mathbb{Z})$ or the congruence subgroup $\Gamma_0^n(M)$, then Γ will contain the subgroup $\mathbb{T}(h)$ as given in (2.6) above with $h = 1$. In particular, modular forms for such levels Γ will satisfy

$$F(Z + S) = F(Z) \qquad (Z = (z_{kl}) \in \mathfrak{h}^n, S = {}^tS \in \mathrm{Mat}_n(\mathbb{Z})),$$

and hence becomes a periodic function of period 1 in each of its $n(n+1)/2$ variables z_{kl}, $k \leq l$ and has a Fourier expansion of the form $F(Z) = \sum_{N \in \mathbb{E}^n, N \geq 0} a(N) e^{\pi i \operatorname{Tr}(NZ)}$.

2.3 Cusp Forms

If Γ is a congruence subgroup, χ a congruence character of Γ, and g is a projectively rational symplectic matrix, then Theorem 2.2.10 tells us that the function $F|_\kappa g$ has a Fourier expansion of the form

$$(F|_\kappa g)(Z) = \sum_{N \in \mathbb{E}^n, N \geq 0} a_{F,g}(N) e^{\frac{\pi i}{h} \operatorname{Tr}(NZ)}$$

with a positive integer h depending on Γ, χ and g.

Definition 2.3.1. The modular form F is called a cusp form if the coefficients $a_{F,g}(N)$ satisfy the conditions

$$a_{F,g}(N) = 0 \quad \text{for every } g \in \mathrm{Sp}_{2n}(\mathbb{Q}), N \in \mathbb{E}^n \quad \text{with} \quad \det N = 0.$$

The subspace of cusp forms of $M_\kappa(\Gamma,\chi)$ will be denoted $S_\kappa(\Gamma,\chi)$.

Proposition 2.3.2. *Let Γ be a congruence subgroup and χ a congruence character of Γ. Then for each cusp form $F \in S_\kappa(\Gamma,\chi)$ and for each $g \in \mathrm{Sp}_{2n}(\mathbb{Q})$, the modular form $F|_\kappa g \in M_\kappa(\Gamma^g,\chi^g)$ is a cusp form with a Fourier expansion of the form*

$$(F|_\kappa g)(Z) = \sum_{N\in\mathbb{E}^n, N>0} a_g(N) e^{\pi i \frac{\mathrm{Tr}(NZ)}{h}},$$

where h is a positive integer. If $\kappa \geq 0$, then the form $F|_\kappa g$ satisfies

$$|(F|_\kappa g)(X+iY)| \leq c(\det Y)^{\kappa/2}, \qquad Z = X+iY \in \mathfrak{h}^n,$$

and its Fourier coefficients satisfy

$$|a_{F,g}(N)| \leq c'(\det N)^{\kappa/2}$$

for all $N \in \mathbb{E}^n$ with $N > 0$, where c, c' are constants depending only on F and g.

Proof. The Fourier expansion and the inclusion $F|_\kappa g \in S_\kappa(\Gamma^g,\chi^g)$ follow from Proposition 2.2.8 and the definition of cusp forms. For the rest of the proof, the reader can consult the proof of Proposition 1.25 on page 2 of [5]. $\square$

2.3.1 The Siegel Φ operator

Siegel was able to connect modular forms of many variables to modular forms of fewer variables. The Siegel operator Φ is a linear map relating modular forms on $\mathfrak{h}^n$ to modular forms on $\mathfrak{h}^{n-1}$ with the weight κ being fixed. The relation is based on the properties of Fourier expansions of modular forms described in the previous section. Let $\mathbb{F}_n$ denote the set of all Fourier series of the form (2.5) which converge absolutely on $\mathfrak{h}^n$ and uniformly on subsets of $\mathfrak{h}^n$ of the form $\mathfrak{h}^n(\epsilon)$ with $\epsilon > 0$. Let $F \in \mathbb{F}_n$. If $Z' \in \mathfrak{h}^{n-1}(\epsilon)$ and $\lambda \geq \epsilon$, then

$$Z'_\lambda = \begin{pmatrix} Z' & 0_{n-1} \\ 0 & i\lambda \end{pmatrix} \in \mathfrak{h}^n(\epsilon).$$

Since the series F converges uniformly, we can take the following limit term by term and we obtain

$$\lim_{\lambda \to +\infty} F(Z'_\lambda) = \sum_{N \in \mathbb{E}^n} a(N) \lim_{\lambda \to +\infty} e^{\frac{\pi i}{h} \operatorname{Tr}(NZ'_\lambda)}.$$

If $N = \begin{pmatrix} N' & * \\ * & N_{nn} \end{pmatrix}$, then $\operatorname{Tr}(NZ'_\lambda) = \operatorname{Tr}(N'Z') + i\lambda N_{nn}$. Hence

$$\lim_{\lambda \to +\infty} e^{\frac{\pi i}{h} \operatorname{Tr}(NZ'_\lambda)} = \lim_{\lambda \to +\infty} e^{\frac{-\pi}{h}\lambda N_{nn}} e^{\frac{\pi i}{h} \operatorname{Tr}(N'Z')} = \begin{cases} e^{\frac{\pi i}{h} \operatorname{Tr}(N'Z')}, & \text{if } N_{nn} = 0, \\ 0, & \text{if } N_{nn} > 0. \end{cases}$$

Since $N \geq 0$, the equality $N_{nn} = 0$ implies that $N_{1n} = N_{n1} = \ldots = N_{n-1,n} = N_{n,n-1} = 0$, i.e. $N = \begin{pmatrix} N' & 0_{n-1} \\ 0 & 0 \end{pmatrix}$. Thus, for all $Z' \in \mathfrak{h}^{n-1}$, we have

$$(F|\Phi)(Z') = \lim_{\lambda \to +\infty} F(Z'_\lambda) = \sum_{N' \in \mathbb{E}^{n-1}, N' \geq 0} a\left(\begin{pmatrix} N' & 0_{n-1} \\ 0 & 0 \end{pmatrix}\right) e^{\frac{\pi i}{h} \operatorname{Tr}(N'Z')}.$$

The last series is a partial series for the Fourier expansion of F, and so it converges absolutely on $\mathfrak{h}^{n-1}$ and uniformly on subsets of $\mathfrak{h}^n$ of the form $\mathfrak{h}^n(\epsilon)$ with $\epsilon > 0$. Thus we have $F|\Phi \in \mathbb{F}_{n-1}$.

If $n = 1$, we set

$$(F|\Phi)(i\infty) = \lim_{\lambda \to +\infty} F(i\lambda).$$

Arguing as above, the limit always exists and is equal to the constant term of the Fourier expansion of F. Taking $\mathbb{F}_0 = \mathbb{C}$, we obtain that Φ, called the Siegel operator, is a linear operator

$$\Phi : \mathbb{F}_n \to \mathbb{F}_{n-1},$$

for all $n \in \mathbb{N}$.

Now, we consider the action of the Siegel operator on modular forms for congruence subgroups of $\mathrm{Sp}_{2n}(\mathbb{R})$. Let $n > 1$. For a matrix

$$g' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{Sp}_{2(n-1)}(\mathbb{R})$$

with square blocks A', B', C', and D' of order $n-1$, we set

$$i(g') = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = \begin{pmatrix} A' & 0_{n-1} \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} B' & 0_{n-1} \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} C' & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} D' & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that i is an injective homomorphism of $\mathrm{Sp}_{2(n-1)}(\mathbb{R})$ into $\mathrm{Sp}_{2n}(\mathbb{R})$. Its image is the subgroup of the real symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$

$$S^{n,n-1} = \left\{ g \in \mathrm{Sp}_{2n}(\mathbb{R}) |\, g = i(g'), g' \in \mathrm{Sp}_{2(n-1)}(\mathbb{R}) \right\}.$$

The corresponding subgroups in the rational symplectic group $\mathrm{Sp}_{2n}(\mathbb{Q})$ and in the Siegel modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$ are denoted by $S^{n,n-1}(\mathbb{Q})$ and $S^{n,n-1}(\mathbb{Z})$ respectively.

For a subgroup $K \subset \mathrm{Sp}_{2n}(\mathbb{R})$, we set

$$K^{n,n-1} = K \cap S^{n,n-1} \quad \text{and} \quad K^{(n-1)} = i^{-1}(K^{n,n-1}) \subset \mathrm{Sp}_{2(n-1)}(\mathbb{R}).$$

Lemma 2.3.3. *Let K be a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ and let χ be a congruence character of K. Then the group $K^{(n-1)}$ is a congruence subgroup of $\mathrm{Sp}_{2(n-1)}(\mathbb{R})$ and the map $\chi^{(2n-2)}$ given by*

$$\chi^{(2n-2)}(g') = \chi(i(g'))$$

is a congruence character of the group $K^{(n-1)}$.

Proof. We have that K contains a principal congruence subgroup $\Gamma_{2n}(q)$. We have $\Gamma^{n,n-1}(q) = \Gamma_{2n}(q) \cap S^{n,n-1} \subset K^{n,n-1}$, hence

$$\Gamma_{2n-2}(q) = \Gamma_{2n}(q)^{(n-1)} = i^{-1}(\Gamma^{n,n-1}(q)) \subset K^{(n-1)},$$

and the character $\chi^{(2n-2)}$ is trivial on $\Gamma_{2n-2}(q)$. □

Theorem 2.3.4. *Let K be a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ and let χ be a congruence character of K. Then the Siegel operator Φ maps the space $M_\kappa(K, \chi)$ into the space $M_\kappa(K^{(n-1)}, \chi^{(2n-2)})$:*

$$\Phi : M_\kappa(K, \chi) \mapsto M_\kappa(K^{(n-1)}, \chi^{(2n-2)}),$$

and for $n = 1$, we set $M_\kappa(K^{(n-1)}, \chi^{(2n-2)}) = M_\kappa(K^{(0)}, \chi^{(0)}) = \mathbb{C}$.

Proof. Let $F \in M_\kappa(K, \chi)$, $Z' \in \mathfrak{h}^{n-1}$ and $g' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{Sp}_{2n-2}(\mathbb{Q})$, then for $\lambda > 0$, we have

$$Z'_\lambda = \begin{pmatrix} Z' & 0_{n-1} \\ 0 & i\lambda \end{pmatrix} \in \mathfrak{h}^n; \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = i(g') \in S^{n,n-1}(\mathbb{Q}),$$

$$gZ'_\lambda = \begin{pmatrix} A'Z' + B' & 0 \\ 0 & i\lambda \end{pmatrix} \begin{pmatrix} C'Z' + D' & 0 \\ 0 & 1 \end{pmatrix}^{-1} = g'Z'_\lambda,$$

$$\det(CZ'_\lambda + D) = \det \begin{pmatrix} C'Z' + D' & 0 \\ 0 & 1 \end{pmatrix} = \det(C'Z' + D').$$

It follows that

$$(F|\Phi)|_\kappa g'(Z') = \lim_{\lambda\to+\infty} \det(C'Z'+D')^{-\kappa} F(g'Z'_\lambda) \tag{2.11}$$

$$= \lim_{\lambda\to+\infty} \det(CZ'_\lambda+D)^{-\kappa} F(gZ'_\lambda) = (F|_\kappa g)|\Phi(Z'). \tag{2.12}$$

In particular, if $g' \in K^{(n-1)}$, we have

$$(F|\Phi)|_\kappa g' = \chi(g)F|\Phi = \chi^{(2n-2)}(g')F|\Phi.$$

Also, equation (2.11) tells us that any function $(F|\Phi)|_\kappa g'$ with $g' \in \mathrm{Sp}_{2n-2}(\mathbb{Q})$ is bounded on each $\mathfrak{h}^{n-1}(\epsilon)$ with $\epsilon > 0$ since the function $F|_\kappa i(g')$ is bounded on $\mathfrak{h}^n(\epsilon)$. Finally, since $F|\Phi \in \mathbb{F}_{n-1}$, it follows that $F|\Phi$ is holomorphic on $\mathfrak{h}^{n-1}$ and this finishes the proof. □

We now give useful characterizations of cusp forms in terms of the Siegel operator.

Proposition 2.3.5. *Let K be a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$, χ a congruence character of K, and $F \in M_\kappa(K,\chi)$. Then the following three conditions are equivalent:*

1. *F is a cusp form;*
2. *F satisfies*

$$(F|_\kappa g)|\Phi = 0 \qquad \textit{for all} \quad g \in \mathrm{Sp}_{2n}(\mathbb{Q}),$$

   where Φ is the Siegel operator;

3. *F satisfies*

$$(F|_\kappa \gamma)|\Phi = 0 \qquad \textit{for all} \quad \gamma \in \mathrm{Sp}_{2n}(\mathbb{Z}),$$

Proof. It follows from the definition of cusp forms and formula (2.3.1) that condition 1 implies condition 2. Condition 3 is a special case of condition 2. Therefore, it remains to show that condition 3 implies $F \in S_\kappa(K,\chi)$. Proposition 2.1.25 implies that every $g \in \mathrm{Sp}_{2n}(\mathbb{Q})$ can be written in the form $g = \gamma g'$, where $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})$ and g' has the form

$\begin{pmatrix} A' & B' \\ 0_n & D' \end{pmatrix}$. Since $F|_\kappa g = F|_\kappa \gamma g' = F|_\kappa \gamma|_\kappa g'$, the Fourier expansion of the function $F|_\kappa g$ can be rewritten in the form

$$\begin{aligned}\sum_{N \in \mathbb{E}^n, N \geq 0} a_g(N) e^{\frac{\pi i}{h} \operatorname{Tr}(NZ)} &= (F|_\kappa g)(Z) = (\det D')^{-\kappa} (F|_\kappa \gamma)(A'ZD'^{-1} + B'D'^{-1}) \\ &= (\det D')^{-\kappa} \sum_{N' \in \mathbb{E}^n, N' \geq 0} a_\gamma(N') e^{\frac{\pi i}{h'} \operatorname{Tr}(N'B'D'^{-1})} e^{\frac{\pi i}{h'} \operatorname{Tr}(D'^{-1}N'A'Z)},\end{aligned}$$

where $a_\gamma(N')$ are the coefficients of the Fourier expansion (2.5) of the function $F|_\kappa \gamma$. Hence if $a_\gamma(N') = 0$ for all $N' \in \mathbb{E}^n$ with $\det N' = 0$, then $a_g(N) = 0$ for all $N \in \mathbb{E}^n$ with $\det N = 0$. $\square$

Proposition 2.3.5 allows us to say that $F \in M_\kappa(\Gamma, \chi)$ is a cusp form if for every $g \in \mathrm{Sp}_{2n}(\mathbb{Z})$, the Fourier expansion of $F|_\kappa g \in M_\kappa(\Gamma^g, \chi^g)$ is of the form

$$(F|_\kappa g)(Z) = \sum_{N \in \mathbb{E}^n, N > 0} a_g(N) e^{\frac{\pi i}{h} \operatorname{Tr}(NZ)},$$

i.e. $a_g(N) = 0$, unless $N > 0$. Also, by Proposition 2.3.5, we give the following characterization of the subspace of cusp forms.

Let K be a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$. Let ν be the index $[\mathrm{Sp}_{2n}(\mathbb{Z}) : K]$ of K in $\mathrm{Sp}_{2n}(\mathbb{Z})$. Let $M_1, \ldots, M_\nu$ be a system of representatives for the cosets. For a $F \in M_\kappa(K)$, we consider the functions

$$F_1 = F|_\kappa M_1, \ldots, F_\nu = F|_\kappa M_\nu.$$

Using Proposition 2.2.8, we have that $F_i \in M_\kappa(K^{M_i})$ and using Theorem 2.3.4 we have that $F_i|\Phi \in M_\kappa((K^{M_i})^{(n-1)})$. Denoting $M_\kappa(K^{M_i,(n-1)}) := M_\kappa((K^{M_i})^{(n-1)})$, we define the vector valued Siegel operator

$$\vec{\Phi} : M_\kappa(K, \chi) \to M_\kappa(K^{M_1,(n-1)}) \times \ldots \times M_\kappa(K^{M_\nu,(n-1)})$$

by

$$F|\vec{\Phi} = (F_1|\Phi, \ldots, F_\nu|\Phi).$$

Proposition 2.3.5 implies that the kernel of $\vec{\Phi}$ coincides with the subspace $S_\kappa(K)$ of cusp forms. This characterization of cusp forms means that $F(Z)$ tends to zero as Z approaches any rational boundary component of $\mathfrak{h}^n$. We will explain this statement in Chapter 4.

Definition 2.3.6. Let $F \in M_\kappa(K, \chi)$, where K is a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$. The constant term $a_g(0)$ of the Fourier expansion of $F|_\kappa g$ with $g \in \mathrm{Sp}_{2n}(\mathbb{Z})$ is called a boundary value of F and it is given by

$$a_g(0) = \lim_{\lambda \to +\infty} (F|_\kappa g)(i\lambda 1_n) = (F|_\kappa g)|\Phi| \ldots |\Phi,$$

where the Siegel operator Φ is applied n-times. For a fixed g in case $n = 1$, the constant term $a_g(0)$ is the image of $F|_\kappa g$ under the Siegel Φ operator which happens to be its boundary value at its zero-degree rational boundary component.

2.3.2 Spaces of Modular Forms

Theorem 2.3.7 ([5], Theorem 1.30)**.** *Let K be a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$, χ a congruence character of K, and κ a nonnegative inetger. Then the space $M_\kappa(K, \chi)$ of modular forms of weight κ and character χ for the group K is a finite dimensional vector space over the field $\mathbb{C}$.*

The proof of this theorem is based on the following important lemma.

Lemma 2.3.8 ([5], Lemma 1.31)**.** *Let*

$$F(Z) = \sum_{N \in \mathbb{E}^n, N > 0} a(N) e^{\pi i \operatorname{Tr}(NZ)}$$

be a cusp form of a nonnegative integral weight κ and the unit character for the Siegel *modular group* $\mathrm{Sp}_{2n}(\mathbb{Z})$*. Suppose that the Fourier coefficients satisfy the conditions*

$$a(N) = 0 \quad \text{if} \;\; \mathrm{Tr}(N) \leq \frac{\kappa}{2\pi b_n},$$

where b_n *is a constant such that if* $Z = X + iY \in F_n$ *(the fundamental domain of* $\mathrm{Sp}_{2n}(\mathbb{Z})$ *on* $\mathfrak{h}^n$*), then* $Y \geq b_n 1_n$*. Then the form* F *is identically equal to zero.*

Theorem 2.3.9 ([4], Theorem 2.4.5)**.** *Let* $\Gamma^n(q) \subset K$ *be a congruence subgroup of* $\mathrm{Sp}_{2n}(\mathbb{Z})$*,* χ *a congruence character of* K*, and* κ *an integer. Then the dimension of* $M_\kappa(K, \chi)$ *over* $\mathbb{C}$ *satisfies*

1. $\dim(M_\kappa(K, \chi)) \leq d_n(\kappa\nu(K)q)^{n(n+1)/2}$*, if* $\kappa > 0$*, where* d_n *depends only on* n*, and* $\nu(K)$ *is the index of* K *in* $\mathrm{Sp}_{2n}(\mathbb{Z})$*.*
2. $\dim(M_0(K, \chi)) = \begin{cases} 1, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 \end{cases}$
3. $\dim(M_\kappa(K, \chi)) = 0$*, if* $\kappa < 0$*.*

The proof is based on Theorem 2.3.7 and on induction arguments using the Siegel Φ operator. For the full details, we refer to page 95 in [4].

Proposition 2.3.10. *The Space of modular forms of full level and trivial character*

$$M_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z})) = \{0\}$$

if $n\kappa$ *is odd.*

Proof. The coefficients $a(N)$ of a modular form $F \in M_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$ satisfy the relation (2.7) given by

$$a({}^tVNV) = (\det V)^\kappa e^{-\pi i\,\mathrm{Tr}(NVU)} a(N), \quad (\text{for all } N \in \mathbb{E}^n).$$

In particular for $U = 0_n$, the coefficients $a(N)$ satisfy

$$a({}^tVNV) = (\det V)^\kappa a(N), \quad (\text{for all } N \in \mathbb{E}^n).$$

Now taking $V = -1_n$, then the coefficients satisfy $a(N) = (-1)^{n\kappa} a(N)$ so that if F does not vanish identically the integer $n\kappa$ must be even. Hence any modular form of weight κ vanishes identically if $kn \equiv 1 \mod 2$.

$\square$

Remark 2.3.11. 1. In case of $n > 1$ and full level $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$, it is noted on page 181 of [47] that is reasonable to assume the weight κ to be an integer as there are no nonidentically zero modular forms with weight $\kappa \notin \mathbb{N}$.

2. Finding explicit formulas for the dimension of $M_k(Sp_{2n}(\mathbb{Z}))$ is still an ongoing program in which the Rankin Selberg trace formula is a key tool (See the references in [39] on page 54).

3. We have seen so far that there are no modular forms different from zero if $n\kappa$ is odd or $\kappa < 0$. If κ is even and $\kappa > n+1$ nontrivial forms do exist and are given by Eisenstein series. These examples will be seen in Chapter 4, § 4.4.1.1.

2.3.3 Petersson Scalar Product

Every space of cusp forms $S_\kappa(K, \chi)$ of integral weight κ and a congruence character χ for a congruence subgroup K in $\mathrm{Sp}_{2n}(\mathbb{Z})$ can be endowed with the structure of a Hilbert space by means of a scalar product called the Petersson scalar product. For two functions F and F' on $\mathfrak{h}^n$, we consider the differential form on $\mathfrak{h}^n$ defined by

$$w_\kappa(F, F') = F(Z)\overline{F'}(Z)h(Z)^\kappa d^*Z,$$

where $h(Z) = h(X + iY) = \det Y$ is the height of Z and d^*Z is the invariant volume defined in (2.1.22). It follows from Lemmas 2.1.20 and 2.1.21 that for each matrix $g \in \mathrm{GSp}_{2n}^+(\mathbb{R})$, the form satisfies the relation (see [5] page 35)

$$w_\kappa(F, F')(gZ) = \mu(g)^{n(n+1-\kappa)} w_\kappa(F|_\kappa g, F'|_\kappa g)(Z).$$

In particular if $F, F' \in M_\kappa(K,\chi)$ and $g \in K$, we then have

$$w_\kappa(F,F')(gZ) = w_\kappa(\chi(g)F, \chi(g)F')(Z) = w_\kappa(F,F')(Z).$$

It follows that the integral

$$\int_{F(K)} w_\kappa(F,F')(Z)$$

converges absolutely on a fundamental domain $F(K)$ of K on $\mathfrak{h}^n$ provided at least one of the forms is a cusp form ([5], Lemma 1.37) and does not depend on the choice of fundamental domain. This justifies the following definition.

For two modular forms $F, F' \in M_\kappa(K,\chi)$, such that at least one of the forms is a cusp form, we define their scalar product (F,F'), called the Petersson scalar product, by

$$(F,F') = \nu(K') \int_{F(K)} w_\kappa(F,F')(Z),$$

where $K' = K \cup (-1_{2n})K, \nu(K') = [\mathrm{Sp}_{2n}(\mathbb{Z}) : K']$ and $F(K)$ is a fundamental domain for K on $\mathfrak{h}^n$.

Theorem 2.3.12 ([5], Theorem 1.38)**.** *The Petersson scalar product has the following properties:*

1. (F,F') *converges absolutely and is independent of the choice of fundamental domain* $F(K)$;
2. (F,F') *is independent of the choice of the group* K *with* $F, F' \in M_\kappa(K,\chi)$,
3. (F,F') *is linear in* F *and is conjugate linear in* F';
4. $(F',F) = \overline{(F,F')}$;
5. $(F,F) \geq 0$, *and* $(F,F) = 0$ *if and only if* $F = 0$;

6. *If* $g \in \mathrm{GSp}_{2n}^{+}(\mathbb{Q})$, *then*

$$(F|_{\kappa}g), F'|_{\kappa}g) = \mu(g)^{-n\kappa}(F, F').$$

Definition 2.3.13. Let $E_{\kappa}(K, \chi)$ be the subspace of all modular forms of $M_{\kappa}(K, \chi)$ which are orthogonal to the subspace of cusp forms $S_{\kappa}(K, \chi)$ with respect to the Petersson scalar product:

$$E_{\kappa}(K, \chi) = \{F \in M_{\kappa}(K, \chi) : (F, G) = 0 \text{ for all } G \in S_{\kappa}(K, \chi)\}.$$

From the properties of the Petersson scalar product and linear algebra we obtain the following.

Proposition 2.3.14. 1. *The space* $M_{\kappa}(K, \chi)$ *is a direct sum of* $E_{\kappa}(K, \chi)$ *and* $S_{\kappa}(K, \chi)$:

$$M_{\kappa}(K, \chi) = E_{\kappa}(K, \chi) \oplus S_{\kappa}(K, \chi).$$

2. *If* $F \in E_{\kappa}(K, \chi)$ *and* g *a projectively rational symplectic matrix, then* $F|_{\kappa}g \in E_{\kappa}(K^{g}, \chi^{g})$.

The second part of this proposition follows from Proposition 2.3.2.

Chapter 3

Paramodular Forms

3.1 Introduction

This chapter deals with paramodular forms. These are modular forms that live on the Siegel upper half plane $\mathfrak{h}^n$ and satisfy a transformation formula with respect to paramodular groups (certain arithmetic subgroups of $\mathrm{Sp}_4(\mathbb{Q})$). Paramodular groups have been studied in geometry for their connection to polarized abelian varieties. Arithmetic interest in the degree two case of these groups has increased in the past years due to a conjecture by Brumer and Kramer known as the paramodular conjecture. This conjecture is the analogue of the Taniyama-Shimura conjecture for abelian surfaces along the lines of Langlands philosophy which suggests that the L-series of an abelian surface A over $\mathbb{Q}$ should be associated to a cuspidal Siegel eigenform of weight 2 with rational eigenvalues for some arithmetic group. Brumer and Kramer's search for a modularity conjecture for abelian surfaces over $\mathbb{Q}$ whose endomorphism ring is $\mathbb{Z}$ showed that the right group to be considered for the modular forms should be the paramodular group of degree two. In addition, Roberts and Schmidt obtained a local newform theory for paramodular forms in [64]. Gristenko ([27]) obtained a lifting from Jacobi cusp forms of index $t \geq 1$ to the space of paramodular cusp forms. This lifting will be discussed in Chapters 5 and 6. The newforms of weight 2 which are perpendicular to the Gritsenko lifts are called non-lifts. Here we state the paramodular conjecture:

Conjecture 3.1.1 ([15], Conjecture 1.1)**.** There is a one-to-one correspondence between isogeny classes of abelian surfaces A over $\mathbb{Q}$ of conductor N with $\mathrm{End}_{\mathbb{Q}} A = \mathbb{Z}$ and weight 2 non-lifts f of level $\Gamma[N]$, the paramodular group, with rational eigenvalues, up to scalar multiplication such that the L-series of A and f should agree.

Ibukiyama ([30]) computed a dimension formula for degree two paramodular cusp forms of weight $\kappa \geq 5$ with respect to the paramodular group of level p for every prime p. Also, Ibukiyama and Onodera in [32] described the ring structure of the graded ring of paramodular forms of degree two with respect to the paramodular group of level two. Gritsenko lifts played a crucial role in constructing paramodular cusp forms. Poor and Yuen ([60]) used dimension formulas for paramodular cusp forms and dimension of spaces of Gritsenko lifts to obtain the desired paramodular nonlifts for the paramodular conjecture. In fact, they found the following.

Theorem 3.1.2 ([60], Theorem 1.2)**.** *For primes $p < 600$ and not in the set*

$$\{277, 349, 353, 389, 461, 523, 587\},$$

the space of paramodular cusp forms of degree 2 and of paramodular level p is spanned by Gritsenko lifts.

In this chapter, we first introduce geometrically the paramodular groups for any degree as they have been known historically. Later, we describe their elements and present the paramodular forms.

3.2 Geometric Introduction to Paramodular Groups

Our goal in this section is to introduce the paramodular groups through the moduli space of polarized abelian varieties. In the process, we briefly discuss complex tori that are abelian varieties. We keep our discussion elementary. We do not go into the formal language of defining polarization as a cetain class of a line bundle on the complex torus, instead we just describe a polarization by a Riemann form. In order not to make this exposition very lengthy, we will not provide proofs. In addition, some theorems presented here are not trivial to prove as they originate from groundbreaking research done by Cartan ([18]), Lefschetz ([46]), and Serre ([76]). For a general and a comprehensive reference for this section we recommend [11].

Let V be a real vector space of finite dimension n. In Chapter 2, we have defined discrete subgroups of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$. We now define discrete subgroups of a real vector space and highlight the difference.

Definition 3.2.1. A subgroup L of V is discrete if for any compact subset K of V, $K \cap L$ is finite.

For instance, $\mathbb{Z}^n$ is a discrete subgroup of $\mathbb{R}^n$.

Proposition 3.2.2 ([19], Theorem 1.2)**.** *A subgroup L of V is discrete if and only if there exists an integer $r \leq n$, and vectors $e_1, \ldots, e_r$ that are linearly independent over $\mathbb{R}$, such that*

$$L = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_r.$$

The integer r is uniquely determined: it is the dimension of the subspace $L \otimes_{\mathbb{Z}} \mathbb{R}$ of V. It is called the rank of L.

Definition 3.2.3. If the rank r of L which occurs in Proposition 3.2.2 is equal to the dimension of V, then L is called a lattice in V. In other words, a lattice L in V is a discrete subgroup of maximal rank, the dimension of V.

We consider the canonical map $\pi : V \to V/L$ where L is a lattice of V. The quotient topology on V/L makes it connected. Moreover, since L is of maximal rank as a discrete subgroup of V, thus V/L is the image of a bounded subset of V and due to Heine-Borel theorem we conclude that it is compact.

Definition 3.2.4. Let X be a topological space and let G be a discrete group acting on X. The action of G is said to be discontinuous if for any point $x \in X$, there exists a neighborhoood U_x of x such that

$$g(U_x) \cap U_x = \emptyset, \quad \text{for all non trivial } \ g \in G.$$

Definition 3.2.5. We say that the action is free if the stabilizer of every point $x \in X$ is the trivial group.

Recalling the definition of a properly discontinuous action from Chapter 2, we have the following.

Proposition 3.2.6. *An action is discontinuous if and only if it is properly discontinuous and free.*

Now, given a lattice $L \subset V$, then L acts by translations of V and defines an equivalence relation on V in the following way:

$$v \sim v' \Leftrightarrow v - v' \in L.$$

In fact, the action of L on V is discontinuous. Moreover, the set of equivalence classes $\{[v]\} = \{v + l : l \in L\}$ is a compact Hausdorff space homeomorphic to a product of n circles and is called the torus V/L associated to L.

From now on, we consider the case of a complex torus

$$X = \mathbb{C}^n/L,$$

where $L \subset \mathbb{C}^n$ is a lattice. Or more generally, we consider a finite dimensional complex vector space V of dimension n and L a lattice in V. Thus, L is a free abelian group of rank $2n$ and we consider the complex torus

$$X = V/L.$$

A generalization of the concept of a complex manifold as defined by Cartan is an analytic space that allows singularities. For the formal definition that requires the definition of a ringed space using the language of sheafs, we refer to Cartan [18]. A properly discontinuous action turns out to be the right action required in order to put an analytic structure on a quotient space as given by the following.

Theorem 3.2.7. *(Cartan) Let X be an analytic space, G a group acting properly discontinuously on X by biholomorphic maps, and let $\pi : X \to X/G$ be the quotient map. The sheaf of rings $\mathcal{O}$ on X/G defined for any open set U of X/G by*

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} \big| f \circ \pi \ \ \textit{is holomorphic in} \quad \pi^{-1}(U)\},$$

determines an analytic space structure on X/G.

A discontinuous action is a nicer property than a properly discontinuous action. In the special case when X is a complex manifold and the action of G is discontinuous, then the following gives that we obtain a smooth structure on the corresponding quotient space.

Corollary 3.2.8 ([11], Corollary A.7.)**.** *Let X be a complex manifold and G is a group acting discontinuously on X. Then X/G is a complex manifold.*

Since the action of L on V is discontinuous, then the complex torus $X = V/L$ endowed with the quotient topology admits a complex manifold structure by defining a sheaf of holomorphic functions on it. This is achieved as in Theorem 3.2.7 by declaring that if U is open X, a function $f : U \to \mathbb{C}$ is holomorphic if and only if $f \circ \pi$ is holomorphic over the open set $\pi^{-1}(U)$ in V. In other words, the holomorphic functions on U correspond to L-periodic holomorphic functions on $\pi^{-1}(U)$.

Remark 3.2.9. This is the difference between a lattice as a discrete subgroup of a real vector space and in a general a discrete subgroup G of the real symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$. For the latter, Proposition 2.1.33 in Chapter 2 together with Theorem 3.2.7 say that the quotient space $\mathrm{Sp}_{2n}(\mathbb{R})/G$ has an analytic structure which is not necessarily a manifold structure due to the existence of points in X with nontrivial stabilizer. These points are the singularities in the quotient space.

An application of Liouville's theorem says that the holomorphic functions on a complex torus are the constant functions. The meromorphic functions on X are defined in an analogous manner as in Theorem 3.2.7. These are the meromorphic functions on V which are L-periodic. They are known as the abelian functions and they form the field of functions of the complex torus.

Definition 3.2.10. An abelian function f is a meromorphic function on $\mathbb{C}^n$ or on V that is periodic with respect to L:

$$f(z+w) = f(z) \text{ for all } w \in L.$$

Definition 3.2.11. An abelian function has $2n$ periods which are linearly independent over $\mathbb{R}$ and lives on the complex torus $\mathbb{C}^n/L$. In the case $n = 1$, an abelian function is an elliptic function.

The addition on the vector space V induces the structure of an abelian complex Lie group on V/L. In fact, any connected compact complex Lie group X of dimension n is a complex torus. (See [11], Lemma 1.1.1).

3.2.1 Complex Tori as Abelian Varieties

Definition 3.2.12. Over any field K, an abelian variety X is a projective nonsingular algebraic variety together with an algebraic group law $m : X \times X \to X$ such that m and the inverse map $X \to X$ given by $x \mapsto x^{-1}$ are morphisms of varieties, both defined over K.

To any algebraic variety X over the complex numbers one can associate a complex analytic space X^{an} following Serre's GAGA functorial construction in [76]. We thus have

Proposition 3.2.13. *Let X be an abelian variety over $\mathbb{C}$. Then $X \cong V/L$, as complex manifolds, where V is a $\mathbb{C}$ vector space of dimension $= \dim A = n$ (say) and L is a lattice in V of rank $2n$.*

An abelian variety of dimension n is a higher dimensional generalization of an elliptic curve ($n = 1$). Over any field of characteristic different from 2 or 3, an elliptic curve may be described by an equation of the form

$$y^2 = x^3 + ax + b.$$

The set of points satisfying the equation of such an elliptic curve together with a point at infinity form an abelian group. Using the theory of elliptic functions, one can show that for an elliptic curve over the complex numbers $\mathbb{C}$, the set of points can be identified with a complex torus $\mathbb{C}/L$, where L can be taken in the form $L = \mathbb{Z} \oplus \tau\mathbb{Z}$ for some $\tau \in \mathfrak{h}^1$. We will see below why a complex torus of dimension 1 is an abelian variety. A complex abelian variety of dimension n is a complex torus of dimension n that is also a projective algebraic variety over $\mathbb{C}$. A theorem of Chow ([25], page 167) states that any compact complex subvariety of a projective space is an algebraic projective subvariety and hence defined by homogeneous polynomial equations. Thus, in order for a complex torus to be an abelian variety, it is necessary and sufficient that it admits a holomorphic embedding in projective space. When $n > 1$, not every complex torus can be embedded in a projective space. The tori that are also abelian varieties are, according to a theorem of Lefschetz, exactly those complex tori equipped with a Riemann form w. In fact, Lefschetz's theorem shows how to construct a projective embedding of a complex torus X into projective space given the Riemann form w.

An elementary matrix T is a diagonal matrix of the form

$$T := \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix}, \quad t_i \in \mathbb{N}, \quad t_i | t_{i+1} \quad (1 \leq i < n).$$

We now consider the classification of integral alternating bilinear forms which comes as a consequence of the elementary divisor formula.

Proposition 3.2.14 ([19], Proposition 6.1)**.** *Let L be a lattice and let w be an nondegenerate alternating bilinear form which is integral on L, i.e.,*

1. $w(a+b,c) = w(a,c) + w(b,c)$;
2. $w(a,b) = -w(b,a)$;
3. $w(a,x) = 0$ *for all* $x \in L \Rightarrow a = 0$;
4. $w(x,y) \in \mathbb{Z}$ *for all* $x, y \in L$.

Then there exists positive integers (elementary divisors) $t_1, \ldots, t_n$ that satisfy $t_1 | \ldots | t_n$, and a $\mathbb{Z}$-basis $w_1, w_2, \ldots, w_{2n}$ of L in which the matrix of w is given by

$$\begin{pmatrix} 0_n & T \\ -T & 0_n \end{pmatrix},$$

where T is the elementary matrix with entries $t_1 | \ldots | t_n$ on the diagonal uniquely determined by the nondegenerate alternating bilinear form and the lattice L.

Remark 3.2.15. Another formulation of the result obtained in Proposition 3.2.14 can be given in this way: for every integral alternating matrix w whose determinant is different from zero, there exists a unimodular matrix $U \in \mathrm{GL}_n(\mathbb{Z})$ and a uniquely determined elementary matrix T with the property

$$^{t}UwU = \begin{pmatrix} 0_n & T \\ -T & 0_n \end{pmatrix}.$$

The matrices of w in different bases have the same determinant. Proposition 3.2.14 shows the determinant of w is positive. In fact, it is the square of a natural number. We call its positive square root

$$\sqrt{\det w} = t_1.t_2.\dots t_n$$

the Pfaffian of w.

Definition 3.2.16. A Hermitian form H on V is a map

$$H : V \times V \to \mathbb{C}$$

satisfying the following properties:

1. $H(x, y)$ is linear on x and antilinear on y.

2. $H(x, y) = \overline{H(y, x)}$.

Let H be a Hermitian form. Decomposing H into a real part and an imaginary part

$$H(x, y) = S(x, y) + iA(x, y),$$

then

$$S(x, y) = A(ix, y).$$

In addition the real part and the imaginary parts of H are uniquely determined.

Proposition 3.2.17. *Given a Hermitian form H as above, then*

1. *the real part $S(x, y)$ is a symmetric, $\mathbb{R}$-valued and $\mathbb{R}$-bilinear form on V satisfying*

$$S(ix, iy) = S(x, y)$$

for all $x, y \in V$ and such a symmetric form is the real part of a unique Hermitian form.

2. *The imaginary part $A(x, y)$ is an $\mathbb{R}$-valued, $\mathbb{R}$-bilinear alternating form on V satisfying satisfying*

$$A(ix, iy) = A(x, y)$$

for all $x, y \in V$. Conversely any such real alternating bilinear form is the imaginary part of a unique Hermitian form, given by

$$H(x, y) = A(ix, y) + iA(x, y)$$

Remark 3.2.18. Proposition 3.2.17 shows that there is a bijection between Hermitian forms H on a complex vector space V and real alternating bilinear forms A on V satisfying $A(ix, iy) = A(x, y)$. This correspondence is given by

$$H \mapsto A = \mathrm{Im}(H),$$

and

$$A \mapsto H(x, y) = A(ix, y) + iA(x, y).$$

Definition 3.2.19. A Riemann form on a complex vector space V with a lattice L is a Hermitian form H on V with the following properties:

1. $\mathrm{Im}\, H(\lambda_1, \lambda_2) \in \mathbb{Z}$ for all λ_1, λ_2 in L.

2. H is positive definite, i.e.

$$H(x, x) > 0 \qquad \text{for all} \quad x \neq 0.$$

The Riemann form H is also sometimes called nondegenerate to imply H is positive definite.

Alternatively and due to Proposition 3.2.17 or Remark 3.2.18, one can also define a Riemann form in terms of the corresponding alternating form in the following way:

Definition 3.2.20. A Riemann form on a complex vector space V with a lattice L is an alternating $\mathbb{R}$-bilinear form $w : V \times V \to \mathbb{R}$ satisfying

1. $w(ix, iy) = w(x, y)$ for all $x, y \in V$.
2. w takes values in $\mathbb{Z}$ on $L \times L$, that is $w(\lambda_1, \lambda_2) \in \mathbb{Z}$ for all λ_1, λ_2 in L.
3. $w(ix, x) > 0$ for all $x \neq 0 \in V$ (equivalently the corresponding Hermitian form is positive definite).

We also say that w is a polarization for V/L. A complex torus equipped with a polarization is called a polarized abelian variety.

Theorem 3.2.21. *(Riemann Conditions) In order for the complex torus V/L to embed in a projective space, it is necessary and sufficient that there exists a basis B of the complex vector space V, positive inetgers $t_1, \ldots, t_n$, and a complex symmetric square matrix Z of order n with positive definite imaginary part, such that, in the basis B,*

$$L = Z\mathbb{Z}^n \oplus T\mathbb{Z}^n,$$

where T is the diagonal matrix with diagonal coefficients $t_1, \ldots, t_n$.

Definition 3.2.22. The vector $(t_1, \ldots, t_n)$, as well as the matrix T, are called the type of the polarization. An important special case of polarization is when the Riemann form w is unimodular over the lattice L, that is its Pfaffian equals 1 and it is of the type $(1, \ldots, 1)$. In this case, one says the polarization is prinicipal and (X, w) is a principally polarized abelian variety.

We now sketch why an elliptic curve is an abelian variety.

Example 3.2.23. Suppose $X = \mathbb{C}/L$ is an elliptic curve. We may take $\{1, \tau\}$ with $\tau \in \mathfrak{h}^1$ as a basis for L. Define

$$\begin{array}{rcl} H : \mathbb{C} \times \mathbb{C} & \to & \mathbb{C} \\ H(x, y) & = & \frac{x.\bar{y}}{\mathrm{Im}(\tau)}. \end{array}$$

Then H is a Hermitian form, $\mathrm{Im}(H(L, L)) \subset \mathbb{Z}$ and H is positive definite.

3.2.2 Moduli Space of Polarized Abelian Varieties

Our goal is to describe the moduli space of polarized abelian varieties. We work with the notion of moduli space as viewed by [11], Chapter 8. A moduli space for a set of abelian varieties means a complex analytic space or a complex manifold whose points are in one to one correspondence with the elements of the set.

Let $\Gamma = (\gamma_1, \ldots, \gamma_{2n})$ be a complex $n \times 2n$ matrix. If the columns of Γ are linearly independent over $\mathbb{R}$, then they span a lattice $L_\Gamma \subset \mathbb{C}^n$:

$$L_\Gamma := \sum_{i=1}^{2n} \mathbb{Z}\gamma_i.$$

If H is a Hermitian $n \times n$ matrix, we can consider the Hermitian form on $\mathbb{C}^n$ given by $H(z, z') := {}^t z H \bar{z}'$. It is interesting to know what matrix Γ and what Hermitian matrix will make the form H a Riemann form.

Suppose we have an integral $n \times n$ matrix T whose determinant is different from zero, and $Z \in \mathfrak{h}^n$. The columns of the matrix $\Gamma = (T, Z)$ generate a lattice $L(T, Z)$ in $\mathbb{C}^n$, the quotient space $\mathbb{C}^n/L(T, Z)$ is a complex torus. Define a Hermitian form $H(T, Z)$ by the matrix $(\mathrm{Im}\, Z)^{-1}$ with respect to the standard basis of $\mathbb{C}^n$. Then $H(T, Z)$ is a Riemann form on $\mathbb{C}^n/L(T, Z)$ for the following reasons: as $Z \in \mathfrak{h}^n$, it is clear that $H(T, Z)$ is positive definite. Consider the $\mathbb{R}$-linear isomorphism $\mathbb{R}^{2n} \to \mathbb{C}^n$ defined by the matrix (T, Z). Then the columns of the matrix (T, Z) with respect to the standard basis of $\mathbb{C}^n$ are the images of the standard basis of $\mathbb{R}^{2n}$ in $\mathbb{C}^n$. This means the columns of the matrix (T, Z) form a basis of $L(T, Z)$. With respect to this basis $\mathrm{Im}(H(T, Z))|L(T, Z) \times L(T, Z)$ is given by the matrix

$$\operatorname{Im}({}^{t}(T,Z)(\operatorname{Im}(Z))^{-1}\overline{(T,Z)}) = \begin{pmatrix} 0_n & T \\ -T & 0_n \end{pmatrix}.$$

This shows that for every $Z \in \mathfrak{h}^n$, one can associate a polarized abelian variety of fixed type T.

Conversely, we consider triples (V, L, H), where L is a lattice in a finite-dimensional complex vector space V and H is a nondegenerate Riemannian form up to isomorphism. Two triples are isomorphic

$$(V, L, H) \cong (V', L', H'),$$

if there exists an isomorphism

$$\sigma : V \to V'$$

with the property

$$L' = \sigma(L), \qquad H'(\sigma(z), \sigma(z')) = H(z, z') \;\text{ for all}\, z, z' \in V.$$

To select a nice representative of each isomorphism class, we choose a lattice basis $\{w_1, \ldots, w_{2n}\}$ as provided by Theorem 3.2.21 such that $w := \operatorname{Im} H$ is of the form $\begin{pmatrix} 0_n & T \\ -T & 0_n \end{pmatrix}$, with an elementary matrix T. The lattice basis contains a $\mathbb{C}$-basis for V. More precisely, let W be the real vector space generated by $w_1, \ldots, w_n$. To show that $V = W \oplus iW$, it suffices to show that $W \cap iW = \{0\}$ since $w_1, \ldots, w_n$ are linearly independent over $\mathbb{R}$. If $z = iz'$ is an element of the intersection, then we have $H(z, z) = w(iz, z) = w(-z', z) = 0$ because $w(w_i, w_j) = 0$. The positive definiteness of H shows that $z = 0$. Hence $\{w_1, \ldots, w_n\}$ is a $\mathbb{C}$-basis for the complex vector space V. Now we consider the isomorphism which transforms the basis to the standard basis. Hence we can assume that $V = \mathbb{C}^n$ and $(w_1, \ldots w_n) = T$ where T is the diagonal matrix with diagonal elements $t_1, \ldots, t_n$. We collect the remaining lattice vectors together in an $n \times n$ matrix $Z := (w_{n+1}, \ldots, w_{2n})$ and hence we denote the lattice L by (T, Z). It turns out that $Z \in \mathfrak{h}^n$ and the Riemann form H is given by

$H(z, z') =^t (\operatorname{Im} Z)^{-1} \bar{z'}$. So we have shown that if we consider the set of all triples (V, L, H) up to isomorphism, a nice representative of the isomorphism class is given by the following proposition.

Proposition 3.2.24 ([11], Proposition 8.1.1)**.** *For each nondegenerate Riemann form H on a lattice L in a complex vector space V, there exists*

1. *an elementary matrix T,*

2. *a matrix $Z \in \mathfrak{h}^n$,*

such that

$$(V, L, H) \cong (\mathbb{C}^n, L(T, Z), H(T, Z))$$

where $L(T, Z)$ is the lattice

$$L(T, Z) = T\mathbb{Z}^n \oplus Z\mathbb{Z}^n.$$

Summing up, we have that the assignment

$$Z \mapsto (\mathbb{C}^n, L(T, Z), H(T, Z))$$

gives a bijection between the Siegel upper half space $\mathfrak{h}^n$ and the set of isomorphism classes of polarized abelian varieties of type T. This can be expressed as follows.

Proposition 3.2.25 ([11], Proposition 8.1.2)**.** *Given a fixed type T, the Siegel upper half space $\mathfrak{h}^n$ is a moduli space for polarized abelian varieties of type T.*

Next, we would like to put an analytic structure on the moduli space of polarized abelian varieties of type T. Recall that by Theorem 3.2.21 each abelian variety (V, L, H) of type T is isomorphic to one of the form $X_Z = (\mathbb{C}^n, L(Z, T), H(Z, T))$ where the lattice L is generated by the columns of the matrix $L = L(Z, T)$ for a uniquely determined elementary matrix T and Z is in the Siegel upper half space $\mathfrak{h}^n$. In order for two varieties X_Z and $X_{Z'}$ of type T to be isomorphic, we seek an isomorphism u of $\mathbb{C}^n$ that takes the lattice

$L_Z = Z\mathbb{Z}^n \oplus T\mathbb{Z}^n$ to $L_{Z'} = Z'\mathbb{Z}^n \oplus T\mathbb{Z}^n$. Let $R \in \mathrm{GL}_n(\mathbb{C})$ be the matrix of the $\mathbb{C}$-linear map u in the canonical basis of $\mathbb{C}^n$, then we seek a matrix R such that

$$RL(Z,T) = L(Z',T).$$

This condition can be formulated as follows (refer to Section 1.2 in [11]): there is a matrix $N \in \mathrm{GL}_{2n}(\mathbb{Z})$ with the property

$$(Z',T) = R(T,Z)^t N.$$

Decomposing N into four $n \times n$ blocks

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then the condition reads as

$$\begin{aligned} Z' &= R(Z\,{}^tA + T\,{}^tB), \\ T &= R(Z\,{}^tC + T\,{}^tD). \end{aligned}$$

Therefore, the matrix R is determined by

$$R = T(Z\,{}^tC + T\,{}^tD)^{-1},$$

and $T^{-1}Z' = (Z\,{}^tC + T\,{}^tD)^{-1}(Z\,{}^tA + T\,{}^tB)$. Now using the fact that Z' is symmetric, we can rewrite this last equation in the form

$$Z' = (AZ + BT)(CZ + DT)^{-1}T.$$

This is equivalent to saying that the integral matrix $N \in \mathrm{GL}_{2n}(\mathbb{Z})$ preserves the polarization, that is it satisfies

$$
{}^{t}N \begin{pmatrix} 0_n & T \\ T & 0_n \end{pmatrix} N = \begin{pmatrix} 0_n & T \\ T & 0_n \end{pmatrix}.
$$

The set of all matrices with this property is in fact a group, and is called the integral paramodular group of level T.

Definition 3.2.26. The integral paramodular group of level T, $\Gamma^i[T]$, consists of all integral N of order $2n$ given by

$$
\Gamma^i[T] = \left\{ N \in \mathrm{GL}_{2n}(\mathbb{Z}) \,\middle|\, {}^{t}N \begin{pmatrix} 0_n & T \\ T & 0_n \end{pmatrix} N = \begin{pmatrix} 0_n & T \\ T & 0_n \end{pmatrix} \right\}.
$$

Therefore, an isomorphism between two abelian varieties with fixed polarization T creates an equivalence relation on elements in $\mathfrak{h}^n$ given by the following.

Definition 3.2.27. Two points $Z, Z' \in \mathfrak{h}^n$ are equivalent modulo $\Gamma^i[T]$ if there exists a matrix $N \in \Gamma^i[T]$ such that

1. the matrix $CZ + DT$ is invertible.

2. $$Z' = (AZ + BT)(CZ + DT)^{-1}T.$$

The equivalence classes of elements in $\mathfrak{h}^n$ correspond to the orbits of the action of the integral paramodular group of $\mathfrak{h}^n$.

Proposition 3.2.28. *The action of* $\Gamma^i[T]$ *on* $\mathfrak{h}^n$ *is given by*

$$
\left(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z \right) \mapsto \gamma Z = (AZ + BT)(CZ + DT)^{-1}T.
$$

Summing up, we have have proved the implication (1) $\Rightarrow$ (2) of the following proposition.

Proposition 3.2.29 ([11], Proposition 8.1.3)**.** *For $Z, Z' \in \mathfrak{h}^n$, the following statements are equivalent.*

1. *The polarized abelian varieties X_Z and X'_Z of the same type T are isomorphic.*

2. *$Z' = \gamma Z$ for $\gamma \in \Gamma^i[T]$.*

Consider the automorphism σ_T

$$P \mapsto \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix}^{-1} P \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix}$$

of $\mathrm{GL}_{2n}(\mathbb{Q})$ and consider the matrix

$$M = \sigma_T(N) = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix} = \begin{pmatrix} A & BT \\ T^{-1}C & T^{-1}DT \end{pmatrix}.$$

We can rewrite equation (2) in Definition 3.2.27 in the form

$$Z' = (A'Z + B')(C'Z + D')^{-1}.$$

Hence, we obtain that the matrix $M := \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ is rational symplectic and satisfies

$${}^tM \begin{pmatrix} 0_n & 1_n \\ 1_n & 0_n \end{pmatrix} M = \begin{pmatrix} 0_n & 1_n \\ 1_n & 0_n \end{pmatrix}.$$

Thus, we have obtained an injective homomorphism

$$\sigma_T : \Gamma^i[T] \to \mathrm{Sp}_{2n}(\mathbb{Q}),$$

mapping

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto M = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}.$$

The image of $\Gamma^i[T]$ by this homomorphism is called the paramodular group $\Gamma[T]$ of level T.

Definition 3.2.30. The paramodular group of level T is defined as

$$\begin{aligned} \Gamma[T] &= \sigma_T(\mathrm{Mat}_{2n}(\mathbb{Z})) \cap \mathrm{Sp}_{2n}(\mathbb{Q}) \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Q}) \middle| A, BT^{-1}, TC, TDT^{-1} \text{ with integer coefficients} \right\}. \end{aligned}$$

The advantage of this definition of the paramodular group is that it is now a subgroup of the rational symplectic group $\mathrm{Sp}_{2n}(\mathbb{Q})$ and hence acts on the Siegel upper half space of genus n using the usual action as defined in Chapter 2.

Proposition 3.2.31. $\Gamma[T]$ *acts on* $\mathfrak{h}^n$ *and this action is given by*

$$\begin{gathered} \Gamma[T] \times \mathfrak{h}^n \to \mathfrak{h}^n \\ \left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z \right) \mapsto MZ = (AZ+B)(CZ+D)^{-1}. \end{gathered}$$

In fact, we have the following:

Lemma 3.2.32 ([65], page 78)**.** *The paramodular group* $\Gamma[T]$ *is commensurable with the Siegel modular group* $\mathrm{Sp}_{2n}(\mathbb{Z})$.

Proof. Every element M in $\Gamma[T]$ is clearly projective rational.

$$\Gamma[T] \cap \mathrm{Sp}_{2n}(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Q}) \middle| A, B, C, D, BT^{-1}, TC, TDT^{-1} \in \mathrm{Mat}_n(\mathbb{Z}) \right\}$$

contains a principal congruence subgroup $\Gamma_{2n}(M)$ for any integer M with MT^{-1} integral. $\square$

As an arithmetic group, the paramodular group is a discrete subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ and hence using Proposition 2.1.33 we have.

Proposition 3.2.33. *The paramodular group acts properly discontinuously on the Siegel upper half plane $\mathfrak{h}^n$.*

Theorem 3.2.7 implies that the quotient space $\mathfrak{h}^n/\Gamma[T]$ with its natural quotient structure is an analytic space of dimension $n(n+1)/2$. Also, because $\sigma_T(\Gamma^i[T]) = \Gamma[T]$, then $\mathfrak{h}^n/\Gamma^i[T]$ is an analytic space of the same dimension $n(n+1)/2$. The identity on $\mathfrak{h}^n$ induces an isomorphism

$$\mathfrak{h}^n/\Gamma^i[T] \cong \mathfrak{h}^n/\Gamma[T].$$

Applying Prpositions 3.2.25 and 3.2.29, we have:

Theorem 3.2.34. *The analytic space $\mathfrak{h}^n/\Gamma[T]$ is a moduli space for polarized abelian varieties of type T.*

Remark 3.2.35. If T is the unit matrix or a multiple of it, that is T is the fixed principal polarization on the set of abelian varieties, then the isomorphism σ_{1_n} is the identity on $\mathrm{Sp}_{2n}(\mathbb{R})$ and the paramodular group $\Gamma[T]$ coincides with the Siegel modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$ and hence $\mathfrak{h}^n/\mathrm{Sp}_{2n}(\mathbb{Z})$ is the moduli space of principally polarized abelian varieties.

We say a point y in an analytic space Y is smooth if there exists a neighborhood of y in Y which is a complex manifold. Otherwise we say y is a singular point of Y. With the properly discontinuous action of G on a complex manifold X, not all points in X have a trivial stabilizer. In the neighborhood of these points, the quotient space X/G looks like the quotient of a manifold by finite group, hence the points with nontrivial stablizer are the singular points of the quotient. It is nice to note that because every point Z in $\mathfrak{h}^n$ corresponds to an isomorphism class of polarized abelian varities of dimension n and polarization T, the stabilzer $\Gamma[T]_Z$ of such a point under the action of $\Gamma[T]$ corresponds to the automorphism group of the abelian variety X_Z consisting of automorphims that preserve the polarization. Hence the singular points are those corresponding to polarized abelian varieties whose automorphism group strictly contains $\{\pm\text{identity}\}$. In fact we have the following result due to Tai.

Theorem 3.2.36 ([19], Theorem 7.5)**.** *The smooth points of $\mathfrak{h}^n/\Gamma[T]$ correspond exactly, for $n \geq 3$, to polarized abelian varieties with automorphism group $\{\pm identity\}$.*

3.3 Paramodular Groups

In this section, we keep the notation as in the previous section. We have seen how paramodular groups arise when parametrizing isomorphism classes of polarized abelian varieties of type T and that when $T = 1_n$, we get back the Siegel modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$. Now, we present paramodular groups from the group theoretic point of view and give a generalization of the symplectic groups defined in Chapter 2.

Definition 3.3.1. Let

$$P_T := \begin{pmatrix} 0_n & T \\ -T & 0_n \end{pmatrix}.$$

The general symplectic group asscociated to T is

$$\mathrm{GSp}_{2n}(T, \mathbb{R}) = \{g \in \mathrm{GL}_{2n}(\mathbb{R}), {}^t g P_T g = \mu(g) P_T\},$$

for a multiplier homomorphism $\mu : \mathrm{GL}_{2n}(\mathbb{R}) \to \mathrm{GL}_1(\mathbb{R})$. The subgroup of matrices of positive multiplier is

$$\mathrm{GSp}^+_{2n}(T, \mathbb{R}) = \{g \in \mathrm{GSp}_{2n}(T, \mathbb{R}), \mu(g) > 0\}.$$

The real symplectic group associated to T,

$$\mathrm{Sp}_{2n}(T, \mathbb{R}) = \{g \in \mathrm{GSp}_{2n}(T, \mathbb{R}), \mu(g) = 1\},$$

is the automorphism group of the alternating form associated to T.

The integral paramodular group of type T is then defined as

$$\Gamma^i[T] = \mathrm{Sp}_{2n}(T,\mathbb{Z}) = \mathrm{Sp}_{2n}(T,\mathbb{R}) \cap \mathrm{Mat}_{2n}(\mathbb{Z}).$$

Note that when $T = 1_n$ we drop 1_n from the notation and we get back the groups defined in Chapter 2.

Using this definition, we can can easily obtain the following relations on elements in $\mathrm{Sp}_{2n}(T,\mathbb{R})$.

Proposition 3.3.2. *The matrix* $g = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in \mathrm{Sp}_{2n}(T,\mathbb{R})$ *if and only if the following relations hold*

1. $$^{t}A_nTD_n - {}^{t}C_nTB_n = T, \quad {}^{t}A_nTC_n = {}^{t}C_nTA_n, \quad {}^{t}B_nTD_n = {}^{t}D_nTB_n.$$

   Especially, $\mathrm{Sp}_2(T,\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$.

2. *The inverse* $g^{-1} \in \mathrm{Sp}_{2n}(T,\mathbb{R})$ *for* ${}^{t}gP_Tg = P_T$ *is equivalent to* ${}^{t}g^{-1}P_Tg^{-1} = P_T$. *It is given by*
   $$g^{-1} = P_T^{-1}\,{}^{t}gP_T = \begin{pmatrix} T^{-1}\,{}^{t}D_nT & -T^{-1}\,{}^{t}B_nT \\ -T^{-1}\,{}^{t}C_nT & T^{-1}\,{}^{t}A_nT \end{pmatrix}.$$

3. *The transpose* ${}^{t}g = P_Tg^{-1}P_T^{-1}$ *is symplectic and hence*
   $$D_nT^{-1}\,{}^{t}A_n - C_nT^{-1}\,{}^{t}B_n = T^{-1}, \quad D_nT^{-1}\,{}^{t}C_n, B_nT^{-1}\,{}^{t}A_n \quad \textit{are symmetric.}$$

Using these symplectic relations and recalling the conjugation map σ_T relating elements of the integral paramodular group $\Gamma^i[T]$ and the paramodular group $\Gamma[T]$ in $\mathrm{Sp}_{2n}(\mathbb{Q})$

in the following way

$$\Gamma[T] = \sigma_T(\Gamma^i[T]) = \begin{pmatrix} 1_n & 0_n \\ 0_n & T^{-1} \end{pmatrix} \Gamma^i[T] \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix},$$

or

$$\Gamma^i[T] = \sigma_T^{-1}(\Gamma[T]) = \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix} \Gamma[T] \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix}^{-1},$$

we obtain:

Corollary 3.3.3.

$$\Gamma^i[T] = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{Z}) \middle| {}^tATD - {}^tCTB = T,\ {}^tATC,\ {}^tBTD \ \ \textit{symmetric} \right\}$$

Remark 3.3.4. For notational preference, from now on we would rather deal with the conjugate of the map σ_T that we will also call σ_T. That is,

$$\sigma_T \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix}^{-1}.$$

Consequently,

$$\Gamma[T] = \sigma_T(\Gamma^i[T]) = \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix} \Gamma^i[T] \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix}^{-1}, \tag{3.1}$$

and

$$[\Gamma^i[T] = \sigma_T^{-1}(\Gamma[T]) = \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix}^{-1} \Gamma[T] \begin{pmatrix} 1_n & 0_n \\ 0_n & T \end{pmatrix}. \tag{3.2}$$

We would like to determine the shape of the elements in the paramodular group $\Gamma[T]$. For this we use

$$\Gamma[T] = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Q}) \middle| \sigma_T^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^i[T] \right\}.$$

Consequently, we have

1. Using the usual symplectic relations (Proposition 2.1.3), $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Q})$ implies that

   $$ {}^tAD - {}^tCB = 1_n, {}^tAC, {}^tBD \text{ symmetric.}$$

2. $$\sigma_T^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & BT \\ T^{-1}C & T^{-1}DT \end{pmatrix} \in \Gamma^i[T]$$

   means that

   $$A, BT, T^{-1}C, T^{-1}DT \in \mathrm{Mat}_n(\mathbb{Z}),$$

   and also using the symplectic relations in Proposition 3.3.2, we get that

   $$ {}^tADT - {}^tCBT = T, \quad {}^tAC, T\,{}^tBDT \text{ symmetric.}$$

3. $g^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix} \in \Gamma[T]$ also implies that

   $$ {}^tD, {}^tBT, T^{-1}\,{}^tC, T^{-1}\,{}^tAT \in \mathrm{Mat}_n(\mathbb{Z}).$$

Putting all this together and working on the level of the elements of the matrices in consideration where $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij})$, we get that

Proposition 3.3.5. *The element* $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Q})$ *is in* $\Gamma[T]$ *if and only if the following* *holds*

$$a_{ij} \in \begin{cases} \frac{t_j}{t_i}\mathbb{Z} & j \geq i \\ \mathbb{Z} & i > j \end{cases}, \quad b_{ij} \in \begin{cases} t_i^{-1}\mathbb{Z} & j \geq i \\ t_j^{-1}\mathbb{Z} & i > j \end{cases}, \quad b_{ij} \in t_{\min(i,j)}^{-1}\mathbb{Z},$$

$$c_{ij} \in \begin{cases} t_j\mathbb{Z} & j \geq i \\ t_i\mathbb{Z} & i > j \end{cases}, \quad d_{ij} \in \begin{cases} \mathbb{Z} & j \geq i \\ \frac{t_i}{t_j}\mathbb{Z} & i > j \end{cases}.$$

In particular, when $n = 2$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ for $t \geq 1$, then

Definition 3.3.6.

$$\Gamma[T] = \left\{ \begin{pmatrix} * & t* & * & * \\ * & * & * & t^{-1}* \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Q}) \right\},$$

$$\Gamma^i[T] = \left\{ \begin{pmatrix} * & t* & * & t* \\ * & * & * & * \\ * & t* & * & t* \\ * & * & * & * \end{pmatrix} \cap \mathrm{Mat}_4(\mathbb{Z}) \right\}.$$

Remark 3.3.7. There is no containment between the paramodular groups $\Gamma[t]$ and $\Gamma[t']$ if $t, t' \neq 1$. To see this, note the element

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^{-1} \\ 0 & 0 & -1 & 0 \\ t & 0 & 0 & 0 \end{pmatrix}$$

is contained in $\Gamma[t']$ with $t' \neq 1$ if and only if $t' = t$. This is in contrast to the situation of congruence levels as there one has $\Gamma_0^{(2)}(N) \subset \Gamma_0^{(2)}(M)$ if $M \mid N$.

It is easy to verify that

$$J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \in \Gamma^i[T].$$

Applying the conjuguation σ_T, we get that

$$J_T := \sigma_T(J_n) = \begin{pmatrix} 0_n & T^{-1} \\ -T & 0_n \end{pmatrix} \in \Gamma[T].$$

Rosati gave generators for the paramodular groups.

Theorem 3.3.8 ([65], Theorem 2.1)**.** *Consider the following subgroup*

$$\Omega(T) = \{U \in \mathrm{GL}_n(\mathbb{Z}) | \, T^{-1}UT \in \mathrm{Mat}_n(\mathbb{Z})\}$$

of $\mathrm{GL}_n(\mathbb{Z})$*. The group* $\Gamma[T]$ *is generated by*

$$J_T = \begin{pmatrix} 0_n & T^{-1} \\ -T & 0_n \end{pmatrix}$$

and the two subgroups

$$\left\{ \begin{pmatrix} {}^tU^{-1} & 0_n \\ 0_n & U \end{pmatrix} \middle| \, U \in \Omega(T) \right\},$$

and

$$\left\{ \begin{pmatrix} 1_n & S \\ 0_n & 1_n \end{pmatrix} \middle| \, ST \in \mathrm{Mat}_n(\mathbb{Z}), S =^t S \right\}.$$

Also in his thesis Kapler has shown that

Lemma 3.3.9 ([50], Lemma 2.1.3)**.** *let* $t_{i,j} := t_j t_i^{-1}$ *and* E_{ij} *be the matrix with the* (i,j)*-entry is* 1 *and all others are zero.* $\Gamma^i[T]$ *is then generated by* J_n *and matrices* $\begin{pmatrix} 1_n & S_{ij} \\ 0_n & 1_n \end{pmatrix}$

where

$$S_{ij} = \begin{cases} E_{ij} & \textit{if} \quad i = j, \\ \\ t_{i,j} E_i j + E_{ji} & \textit{if} \quad i < j. \end{cases}$$

Since all generators of the paramodular groups have determinant 1, we deduce the following

Corollary 3.3.10. *If* $g \in \Gamma[T]$*, then* $\det g = 1$.

3.4 Paramodular Forms

Definition 3.4.1. Let $n \geq 2$ and $\kappa \in \mathbb{Z}$. A function f: $\mathfrak{h}^n \to \mathbb{C}$ is a paramodular form of degree n, weight κ if it satisfies the following:

1. f is holomorphic,
2. for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma[T]$, $Z \in \mathfrak{h}^n$ we have

$$f(\gamma Z) = \det (CZ + D)^\kappa f(Z).$$

We denote the complex vector space of paramodular forms by $M_\kappa(\Gamma[T])$ and denote by $M(T) = \oplus M_\kappa(\Gamma[T])$ the ring of paramodular forms of polarization T.

Remark 3.4.2 ([65], page 77)**.** In greater generality, paramodular groups are defined using a symmetric rational matrix $\Delta \in \mathrm{GL}_n(\mathbb{Q})$ instead of an elementary matrix T. But,

1. For any $r \in \mathbb{Q}$ we have the isomorphism

$$\Gamma[r\Delta] = \begin{pmatrix} 1_n & 0_n \\ 0_n & r1_n \end{pmatrix} \Gamma[\Delta] \begin{pmatrix} 1_n & 0_n \\ 0_n & r1_n \end{pmatrix}^{-1},$$

which induces the isomorphism

$$f(Z) \in M(\Delta) \mapsto f(rZ) \in M(r\Delta).$$

Hence, we may assume that the matrix Δ is integral and primitive.

2. Every paramodular group $\Gamma[\Delta]$ is conjugate to a paramodular group $\Gamma[T]$ where T is an elementary matrix as decribed in §1 of this chapter. This is due to the well known fact that for any integral symmetric matrix Δ, there exists unimodular matrices U and V in $\mathrm{GL}_n(\mathbb{Z})$, such that $\Delta = {}^tUTV$ for a diagonal matrix T given by the elementary divisors of Δ. The equality $\Delta = {}^tUTV$ induces the isomorphim

$$\begin{array}{rcl} \Gamma[T] & \to & \Gamma[\Delta] \\ M & \mapsto & M' := \begin{pmatrix} U^{-1} & 0_n \\ 0_n & {}^tU \end{pmatrix} M \begin{pmatrix} U^{-1} & 0_n \\ 0_n & {}^tU \end{pmatrix}^{-1}. \end{array}$$

If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$M' = \begin{pmatrix} U^{-1}AU & U^{-1}B\,{}^tU^{-1} \\ {}^tUCU & {}^tUDU^{-1} \end{pmatrix} \in \Gamma[\Delta]$$

follows from the fact that $M \in \Gamma[T]$ and the symplectic relations. This is equivalent to say $M' \in \mathrm{Sp}_{2n}(\mathbb{Q})$ and that,

$$A' = U^{-1}AU, \quad B'\Delta = U^{-1}BTV, \quad \Delta^{-1}C' = V^{-1}T^{-1}CU, \quad \Delta^{-1}D'\Delta = V^{-1}T^{-1}DTV$$

all belong to $\mathrm{Mat}_n(\mathbb{Z})$. The isomorphism

$$\Gamma(\Delta) = \begin{pmatrix} U^{-1} & 0_n \\ 0_n & {}^tU \end{pmatrix} \Gamma[T] \begin{pmatrix} U^{-1} & 0_n \\ 0_n & {}^tU \end{pmatrix}^{-1}$$

also induces an isomorphism

$$f(Z) \in M(\Delta) \mapsto f(UZ{}^tU) \in M(T).$$

3. Therefore for the interest in paramodular forms, it is enough to work with an elementary matrix T. We can also choose T such that $t_1 = 1$ and $t_i/t_{i-1} \in \mathbb{N}$ for all $2 \le i \le n$. In this case we call the polarization T to be minimal polarization.

4. When the degree n equals to 2, we will always deal with the minimal polarization

$$T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

and denote $\Gamma[T], \Gamma^i[T], P_T$ and J_T by $\Gamma[t], \Gamma^i[t], P_t$ and J_t respectively.

Applying the transformation formula of a paramodular form f with respect to generators of the paramodular groups $\Gamma[T]$, we get that f satisfies the following

$$\begin{aligned} f({}^tUZU) &= (\det U)^{\kappa} f(Z) \quad \text{for all } \ U \in \Omega(T), \\ f(Z+S) &= f(Z) \quad \text{for all } \ S = {}^tS \ \text{ with } \ ST \in \mathrm{Mat}_n(\mathbb{Z}), \\ f(-T^{-1}Z^{-1}T^{-1}) &= (\det(-TZ))^{\kappa} f(Z). \end{aligned}$$

Since $\Gamma[T]$ is a commensurable group, a paramodular form has then an absolutely convergent Fourier development (we refer the reader to the details in Chapter 2). It is periodic because of the relation $f(Z+S) = f(Z)$ for all $S =^t S$ with $ST \in \mathrm{Mat}_n(\mathbb{Z})$. Denoting the entries of ST by $s_{ij}t_j$, we have then $s_{ii} \in t_i^{-1}\mathbb{Z}$ and $s_{ij} \in t_j^{-1}\mathbb{Z}$ for all $i > j$. Following the same arguments in the Fourier development of a Siegel modular form as in Theorem 2.2.10, we obtain that a paramodular form $f \in M_\kappa(\Gamma[T])$ has a Fourier development of the form

$$f(Z) = \sum_{N \in \mathbb{E}(T)} a(N) e^{\pi i \operatorname{Tr}(NZ)}$$

where

$$\mathbb{E}(T) = \left\{ N = {}^t N \,\middle|\, N = \begin{pmatrix} 2x_{11}t_1 & \dots & x_{1n}t_n \\ \vdots & \ddots & \vdots \\ x_{1n}t_n & \dots & 2x_{nn}t_n \end{pmatrix} \quad \text{with} \;\; x_{ij} \in \mathbb{Z}. \right\}$$

Similar to the relations that the Fourier coefficents of a Siegel modular form hold (see Theorem 2.2.10), one can also show that

$$a({}^t UNU) = (\det U)^{-\kappa} a(N) \quad \text{for all} \quad N \in \mathbb{E}(T), U \in \Omega(T).$$

The Koecher principle also says that if N is not positive semidefinite then

$$\sum_{N \in \mathbb{E}(T)} |a(N)| e^{-\pi \operatorname{Tr}(NZ)}$$

diverges. In other words, $a(N) = 0$ for all N not positive semidefinite. That is, we have shown that

Theorem 3.4.3. *Let $f \in M_\kappa(\Gamma[T])$, then f has an absolutely convergent Fourier series*

$$f(Z) = \sum_{N \in \mathbb{E}(T), N \geq 0} a(N) e^{\pi i \operatorname{Tr}(NZ)}.$$

Summing over half integral matrices N in the set

$$\mathbb{S}_n^{\geq 0}(\mathbb{Z}, T) = \left\{ N = {}^t N \,\middle|\, N = \begin{pmatrix} x_{11}t_1 & \dots & x_{1n}/2t_n \\ \vdots & \ddots & \vdots \\ x_{1n}/2t_n & \dots & x_{nn}t_n \end{pmatrix} \;\; \text{with} \;\; x_{ij} \in \mathbb{Z}, N \geq 0 \right\},$$

we can instead write the Fourier expansion of a paramodular form $f \in M_\kappa(\Gamma[T])$ in the following form

$$f(Z) = \sum_{N \in \mathbb{S}_n^{\geq 0}(\mathbb{Z}, T)} a(N) e^{2\pi i \operatorname{Tr}(NZ)}.$$

For the degree $n = 2$ case, using the minimal polarization $T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$, a paramodular form satisfies the following equation

$$f(Z + S) = f(Z),$$

for all $S = \begin{pmatrix} a & b \\ b & \frac{d}{t} \end{pmatrix}$ with $a, b, d \in \mathbb{Z}$. In particular taking $S = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{t} \end{pmatrix}$ we have that f satisfies

$$f(\tau + 1, z + 1, \tau' + 1/t) = f(\tau, z, \tau').$$

Let $\mathbb{S}_2^{\geq 0}(\mathbb{Z}, t)$ be defined by

$$\mathbb{S}_2^{\geq 0}(\mathbb{Z}, t) = \left\{ \begin{pmatrix} n & r/2 \\ r/2 & mt \end{pmatrix} \in \mathbb{S}_2^{\geq 0}(\mathbb{Z}) : m, n, r \in \mathbb{Z} \right\}.$$

We have that $f \in M_\kappa(\Gamma[t])$ has a Fourier expansion of the form

$$f(Z) = \sum_{N \in \mathbb{S}_2^{\geq 0}(\mathbb{Z}, t)} a_f(N) e^{2\pi i \operatorname{Tr}(NZ)}.$$

3.5 Paramodular Cusp Forms

Let T be an elementary matrix of order n and let $F \in M_\kappa(\Gamma[T])$. We have shown that such F has a Fourier expansion of the form

$$F(Z) = \sum_{N \in \mathbb{E}(T), N \geq 0} a(N) e^{\pi i \operatorname{Tr}(NZ)}.$$

Applying the Siegel Φ operator,

$$
\begin{aligned}
(F|\Phi)(Z^*) &= \lim_{\lambda\to\infty} F\begin{pmatrix} Z^* & 0 \\ 0 & i\lambda \end{pmatrix} \\
&= \lim_{\lambda\to\infty} \sum_{N\in\mathbb{E}(T),N\geq 0} a(N) e^{\pi i \operatorname{Tr}\left(N\begin{pmatrix} Z^* & 0 \\ 0 & i\lambda \end{pmatrix}\right)} \\
&= \sum_{\begin{pmatrix} N_1 & * \\ * & 2x_{nn}t_n \end{pmatrix},N\geq 0} a(N) \lim_{\lambda\to\infty} e^{\pi i \operatorname{Tr}(N_1Z^*)-2\pi\lambda x_{nn}t_n} \\
&= \sum_{N_1\in\mathbb{E}(T_1),N_1\geq 0} a\begin{pmatrix} N_1 & * \\ * & 2x_{nn}t_n \end{pmatrix} e^{\pi i \operatorname{Tr}(N_1Z^*)}.
\end{aligned}
$$

One can verify that

$$
\begin{aligned}
(F|\Phi)(M^*Z^*) &= \lim_{\lambda\to\infty} F\begin{pmatrix} M^*Z^* & 0 \\ 0 & i\lambda \end{pmatrix} \\
&= \det(C^*Z^*+D^*)^{\kappa}(F|\Phi)(Z^*).
\end{aligned}
$$

Proposition 3.5.1. *We have*

$$
\begin{aligned}
\Phi : M_\kappa(\Gamma[T]) &\to M_\kappa(\Gamma[T_1]) \\
F &\mapsto F|\Phi.
\end{aligned}
$$

That is, the Siegel Φ operator takes paramodular forms of degree n to paramodular forms of degree $n-1$.

To define cusp forms, we have the following criterion.

Lemma 3.5.2. *Let $F \in M_\kappa(\Gamma[T])$, then F is a cusp form if $(F|_\kappa M)|\Phi = 0$ for all $M \in \mathrm{Sp}_{2n}(\mathbb{Q})$.*

We will show in Chapter 4 § 4.4 that it suffices to check Lemma 3.5.2 for finitely many matrices. More precisely, we will show the following:

Theorem 3.5.3. *$F \in M_\kappa(\Gamma[T])$ is a paramodular cusp form if $F|M|\Phi$ for all representatives M in $\Gamma[T] \setminus \mathrm{Sp}_{2n}(\mathbb{Q})/C_{n,n-1}(\mathbb{Q})$.*

Chapter 4

Compactification of Siegel varieties $\Gamma \setminus \mathfrak{h}^n$

4.1 Introduction

Let Γ be an arithmetic group. Quotients of the Siegel upper half space $\mathfrak{h}^n$ by Γ, denoted $\Gamma \setminus \mathfrak{h}^n$, are called Siegel varieties. The ultimate goal of this chapter is to describe the cusps of Siegel varieties. Cusps are related to the geometry at infinity of these varieties and can be studied through compactifications. Compactifications of noncompact quotients of the upper half plane $\mathfrak{h}^1$ have played an important role in the theory of modular forms in one variable. It is natural to study compactifications of noncompact quotients of $\mathfrak{h}^n$ to better understand the theory of modular forms in many variables.

It turns out that several different compactifications have been constructed in the literature. Each is built for a specific purpose and is motivated by a certain problem in mathematics that imposes a different structure on the type of boundary components that should be used.

In order to relate modular forms with respect to $\mathrm{Sp}_{2n}(\mathbb{Z})$ to meromorphic functions on $\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n$ (a problem posed by Siegel), Satake ([66]) defined a compactification $(\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n)^c$ of $\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n$ by constructing a topological space obtained by adjoining the disjoint union of the lower dimensional spaces $\mathrm{Sp}_{2r}(\mathbb{Z}) \setminus \mathfrak{h}^r$ of the same type, $0 \leq r \leq n-1$, with a certain topology called the Satake topology that turns the quotient space $(\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n)^c$ into a compact Hausdorff topological space:

$$(\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n)^c = \mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n \cup \mathrm{Sp}_{2(n-1)}(\mathbb{Z}) \setminus \mathfrak{h}^{n-1} \cup \ldots \cup \mathrm{Sp}_{2(0)}(\mathbb{Z}) \setminus \mathfrak{h}^0;$$

where $\mathfrak{h}^0$ denotes a space of one point and $\mathrm{Sp}_{2(0)}(\mathbb{Z})$ denotes a group composed of the neutral element, that is $\mathrm{Sp}_{2(0)}(\mathbb{Z}) \setminus \mathfrak{h}^0 = \{i\infty\}$, the unique standard cusp of $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{h}^n$. This compactification is not in general a complex manifold but Satake also showed in [66] that it has a complex analytic structure and conjectured that with this analytic structure, it is a projective variety.

It was Baily who showed the following.

Theorem 4.1.1 ([8], Theorem 4)**.** *The compactification* $(\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n)^c$ *is a normal analytic space which extends the analytic structure on* $\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n$*.*

Moreover he showed that a projective embedding of $(\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n)^c$ into a complex projective space as a normal projective subvariety may be obtained by means of automorphic forms. A corollary of such a result is:

Theorem 4.1.2 ([8], Theorem 5)**.** *Let* $n \geq 2$*. Every meromorphic function on* $\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n$ *is a quotient of two modular forms of the same weight.*

More generally, Satake was able to provide a compactification of quotients of the upper half space $\mathfrak{h}^n$ by an arithmetic group Γ. We will describe the compactification of $\Gamma \setminus \mathfrak{h}^n$ in § 4.3.3.

4.2 Parabolic Subgroups of Algebraic Groups

The compactification of Siegel varieties $\Gamma \setminus \mathfrak{h}^n$, where Γ is an arithmetic group, is governed by the theory of parabolic subgroups of $\mathrm{Sp}_{2n}(\mathbb{Q})$. In this section, we give a brief description of the structure of the set of parabolic subgroups of an algebraic group G without explaining thoroughly every term involved in the description. Any detailed exposition with proofs would divert our purpose too far. We start with some known facts in the theory of linear algebraic groups following the helpful summary presented in [54]. For a general reference, the reader can consult [52] and for the general algebraic geometry facts we refer to [29]. We will also give examples of parabolic subgroups but even proving the examples would mean one has to invoke the whole theory, so we just limit ourselves to the description of their structure.

Given a group G, the commutator subgroup is defined as

$$G^{(1)} = [G, G] = \langle xyx^{-1}y^{-1} | x, y \in G \rangle.$$

The derived series of G is found by computing iteratively the commutator subgroups. That is, a derived series of G is given by the following series of subgroups

$$G^{(0)} \geq G^{(1)} \geq \dots$$

where $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.

Example 4.2.1. An abelian group A has the derived series

$$A \geq \{1\} \geq \{1\}...$$

since the commutator $xyx^{-1}y^{-1}$ of any two elements is the identity.

Definition 4.2.2. A group G is called solvable if the derived series eventually terminates to the trivial group.

Example 4.2.3. The subgroup of $\mathrm{GL}_n(\mathbb{C})$ consisting of upper triangular matrices is solvable.

For the rest of this section we assume that we are working over an algebraically closed field F unless otherwise noted.

Definition 4.2.4. An algebraic group G defined over a field F is an algebraic variety over F and a group such that the multiplication map $m : G \times G \to G$, $m(a, b) = ab$, and the inverse map $i : G \to G$, $i(a) = a^{-1}$ are morphisms of algebraic varieties.

Morphisms between algebraic groups, called homomorphisms, are morphisms of algebraic varieties and group homomorphisms. If G is an algebraic set in F^n for some natural number n (the variety G is affine), we say G is a linear algebraic group. Another characterization of linear algebraic groups is the following.

Definition 4.2.5. A linear algebraic group G is a Zariski closed subgroup of the general linear group $\mathrm{GL}_n(F)$ for some natural number n:

$$G = \{g = (g_{ij}) \in \mathrm{GL}_n(F) | P_a(g_{ij}) = 0, a \in A\},$$

where each P_a is a polynomial in g_{ij}, A a parameter space. If the polynomials P_a have coefficients in a subfield K, G is called a linear algebraic group defined over K.

If K is a subfield of F, A linear algebraic group G is said to be defined over K if the polynomials which define G as a subvariety of $\mathrm{GL}_n(F)$ have coefficients in K. In this case, the set $G(K) := G \cap K^n$ is called the K-rational points of G.

Example 4.2.6. The first example of a linear algebraic group is $\mathrm{GL}_n(\mathbb{C})$ which is a connected linear algebraic group of dimension n^2. It is contained in the afffine space $\mathrm{Mat}_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$. It is a variety because of the embedding

$$\begin{aligned} \mathrm{GL}_n(\mathbb{C}) &\to \mathrm{Mat}_n(\mathbb{C}) \times \mathbb{C} = \mathbb{C}^{n^2+1} \\ (g_{ij}) &\mapsto ((g_{ij}), (\det(g_{ij}))^{-1}. \end{aligned}$$

Let X_{ij}, Z be the coordinates of $\mathrm{Mat}_n(\mathbb{C}) \times \mathbb{C}$. Then the image is the affine hypersurface $\det(X_{ij})Z = 1$, which is a polynomial in X_{ij}, Z.

Other examples of linear algebraic groups often occur as the automorphism group of some structures such as determinant and bilinear forms. For example, $\mathrm{SL}_n = \mathrm{SL}_n(\mathbb{C}) =$

$\{g \in \mathrm{GL}_n(\mathbb{C}) | \det g = 1\}$, and the symplectic groups $\mathrm{Sp}_{2n} = \mathrm{Sp}_{2n}(\mathbb{C})$. It is clear that SL_n and Sp_{2n} are defined over $\mathbb{Q}$.

Definition 4.2.7. Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a linear algebraic group defined over $\mathbb{Q}$. Let $G(\mathbb{Q}) \subset \mathrm{GL}_n(\mathbb{C})$ be the set of its rational points, and $G(\mathbb{Z}) \subset \mathrm{GL}_n(\mathbb{Z})$ be the set of its elements with integral entries. A subgroup $\Gamma \subset \mathrm{GL}_n(\mathbb{Q})$ is called an arithmetic subgroup if it is commensurable to $G(\mathbb{Z})$, i.e., $\Gamma \cap G(\mathbb{Z})$ has finite index in both Γ and $G(\mathbb{Z})$.

This definition depends on the embedding $G \subset \mathrm{GL}_n(\mathbb{C})$ and the integral subgroup $\mathrm{GL}_n(\mathbb{Z})$. However, we have the following.

Proposition 4.2.8 ([36], Proposition 4.2)**.** *Let G, G' be two linear algebraic groups defined over $\mathbb{Q}$ and $\phi : G \to G'$ is an isomorphism defined over $\mathbb{Q}$. Then $\phi(G(\mathbb{Z}))$ is commensurable to $G'(\mathbb{Z})$.*

As consequence of this, we get.

Corollary 4.2.9 ([36], Corollary 4.3)**.** *If Γ is an arithmetic subgroup of G, then for any $g \in G(\mathbb{Q})$, $g\Gamma g^{-1}$ is also an arithmetic subgroup.*

Definition 4.2.10. An element g of a linear algebraic group G is unipotent if $(g - I)^m = 0$ for some integer m. A linear algebraic group G is said be unipotent if every element $g \in G$ is unipotent.

Clearly, the subgroup U of $\mathrm{GL}_n(\mathbb{C})$ consisting of upper triangular matrices with ones on the diagonal is unipotent and clearly any subgroup of U is also unipotent. The converse is also true; any connected unipotent algebraic group is isomorphic to a subgroup of U. A unipotent group is always solvable.

Definition 4.2.11. A linear algebraic group T is called a torus if it is isomorphic to a product of $F^* = \mathrm{GL}_1(F)$.

Another characterization is that a linear algebraic group T is a torus if and only if T is connected and abelian, and every element of T is diagonalizable. Let K be a subfield of F. A torus T is a K-torus if T is defined over K. Let T be a K-torus. We say T splits over K if T is K-isomorphic (isomorphism defined over K) to a product of $\mathrm{GL}_1(K)$.

Example 4.2.12. Let T be the subgroup of $\mathrm{GL}_n(\mathbb{C})$ consisting of diagonal matrices. Then T is a K-split torus for any subfield K of $\mathbb{C}$.

Example 4.2.13. Consider

$$T_1 = \left\{ g \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, {}^t g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Now if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T_1$ then one can show that $b = c = 0$ and $d = a^{-1}$ and hence

$$T_1 = \left\{ g \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}.$$

This implies that $T_1 \cong \mathrm{GL}_1(\mathbb{C})$ over $\mathbb{Q}$ under the map $g \mapsto a$.

A character of a torus T is a homomorphism of algebraic groups in $X(T) = \mathrm{Hom}\,(T, \mathrm{GL}_1(F))$. The group $X(T)$ is the set of characters of T, called the character group. The Lie algebra of G is denoted by $\mathfrak{g}$. There is an adjoint representation $Ad : G \to \mathrm{GL}(\mathfrak{g})$ which is a homomorphism of algebraic groups such that $Ad(T)$ consists of commuting diagonalizable elements, and so it is diagonalizable.

Now, let G be a connected linear algebraic group which contains at least one torus. Then there is partial order on the set of tori in G given by inclusion and the set of tori has maximal elements called maximal tori of G. All maximal tori are conjuguate and the rank of G is defined to be the dimension of a maximal torus in G.

We define an abstract root system, we then associate to G and to a maximal torus T a root system.

Definition 4.2.14. An abstract root system in a finite dimensional real vector space V is a subset R of V that satisfies the following axioms:

1. R is finite, R spans V and $0 \notin R$.

2. If $\alpha \in R$, there exists a reflection s_α relative to α such that $s_\alpha(R) \subset R$. A reflection relative to α is a linear transformation sending α to $-\alpha$ that restricts to the identity map on a subspace of codimension one.

3. If $\alpha, \beta \in R$, then $s_\alpha(\beta) - \beta$ is an integer multiple of α.

The rank of R is defined to be $\dim(V)$. The Weyl group $W(R)$ is the subgroup of $\mathrm{GL}(V)$ generated by the set $\{s_\alpha | \alpha \in R\}$. A base of R is a subset $\delta = \{\alpha_1, \ldots, \alpha_l\}$, $l = \mathrm{rank}(R)$, such that δ is a basis of V and each $\alpha \in R$ is uniquely expressed in the form $\alpha = \sum_{i=1}^{l} c_i \alpha_i$, where the $c_i^{\prime}s$ are all integers, no two of which have different signs. The elements of δ are called simple roots. The set of positive roots R^+ is the set of $\alpha \in R$ such that the coefficients of the simple roots in the expression for α, as a linear combination of simple roots are all nonnegative. Similarly, R^- consists of those $\alpha \in R$ such that the coefficients are all nonpositive. R is the disjoint union of R^+ and R^-.

Given $\alpha \in X(T)$, let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid Ad(t)X = \alpha(t)X, \text{ for all}\, t \in T\}$.

Definition 4.2.15. The nonzero $\alpha \in X(T)$ such that $g_\alpha \neq 0$ are the roots of G relative to T. The set of roots of G relative to T are denoted by $\Phi_t = \Phi(G, T)$.

Let $\langle \Phi_t \rangle$ be the subgroup of $X(T)$ generated by Φ_t and let $V = \langle \Phi_t \rangle \otimes_{\mathbb{Z}} \mathbb{R}$. Then Φ_t is a root system for V. Since any two maximal tori are conjugate, then Φ_t depends only on G up to conjugation.

A connected algebraic group G contains a unique maximal normal connected solvable subgroup that we call it the radical $R(G)$ of G. The set $R_u(G)$ of unipotent elements in $R(G)$ is a normal closed subgroup of G and is called the unipotent radical of G.

Definition 4.2.16. A connected linear algebraic group G is called semisimple if the radical $R(G) = \{e\}$.

Equivalently, G is semisimple if and only if G has no nontrivial connected abelian normal subgroups.

Definition 4.2.17. A connected linear algebraic group G is called reductive if the unipotent radical $R_u(G) = \{e\}$.

The general linear group GL_n is reductive but not semisimple, since its center is the set of scalar matrices and hence GL_n has nontrivial radical. A maximal torus is given by the diagonal matrices which is isomorphic to GL_1^n. The group of upper triangular matrices is not reductive because of the subgroup of upper triangular matrices with ones on the diagonal.

We have that $G/R(G)$ is semisimple and $G/R_u(G)$ is reductive.

4.2.1 Borel and Parabolic Subgroups

The set of connected closed solvable subgroups of G, ordered by inclusion, contains maximal elements.

Definition 4.2.18. A Borel subgroup B of an algebraic group G is a maximal closed and connected solvable algebraic subgroup.

Fixing a Borel subgroup B, then every other Borel subgroup is conjugate to B. In otherwords, there is a single conjugacy class of Borel subgroups. An example of a Borel subgroup is the subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$.

Closed subgroups between a Borel subgroup B and the group G are called parabolic subgroups. Hence, the Borel subgroups are the minimal parabolic subgroups. Another way to characterize a parabolic subgroup is: P is a parabolic subgroup if and only if G/P is a projective variety. If P is a parabolic subgroup, then P is connected and normal. If P and P' are parabolic subgroups containing a Borel subgroup B, and P and P' are conjugate, then $P = P'$.

From now on, assume G is connected reductive linear algebraic group. Let T be a maximal torus in G. Then T lies inside some Borel subgroup B of G. The set of Borel subgroups of G which contain T is in one to one correspondence with the set of bases of Φ_t. Let δ be the base of Φ_t corresponding to B. For every subset I of δ, there is a corresponding parabolic subgroup P_I of G containing B and every subgroup of G containing B is equal to some some parabolic subgroup P_J for some subset J of δ. We note that the parabolic subgroup P_I associated to I is obtained through the Bruhat decomposition of an algebraic group relative to its Borel subgroup B. (For a brief description of the construction of the parabolic subgroups associated to subsets of simple roots using the Bruhat decomposition, one can check [54], page 386.) If I and J are subsets of δ, then $I \subset J$ if and only if $P_I \subset P_J$. A parabolic subgroup P_I is conjugate to P_J if and only if $I = J$. A parabolic subgroup is called standard if it contains B. Any parabolic subgroup P is conjugate to some standard parabolic subgroup. Hence, the set of representatives of conjugacy classes of parabolic subgroups of G is the set of standard parabolic subgroups which is in bijection with the set of all subsets of the set of simple roots δ. The Borel subgroup correspond to the empty set and the group G correspond to the set of all simple roots. When $\delta - I$ consists of one element, then P_I is a proper maximal parabolic subgroup and every maximal parabolic subgroup of G is conjugate to one of the P_I. In fact, the number of simple roots is equal to the number of conjugacy classes of proper maximal parabolic subgroups. If a parabolic subgroup P is defined over $\mathbb{Q}$, it is called a rational parabolic subgroup.

4.2.2 Description of the Levi Decomposition of Parabolic Subgroups with Examples

Let G, B and T be as above. Let $I \subset \delta$. There is a corresponding parabolic subgroup P_I according to the discussion above. The set

$$\Phi_I = \left\{ \alpha \in \Phi_t | \, \alpha = \sum_i c_i \alpha_i : \alpha_i \in I, c_i \in \mathbb{Z} \right\}$$

forms a root system. Let $N_I = R_u(P_I)$ and T_I be the identity component of $(\cap_{\alpha \in I} \ker \alpha)$. Now, we let M_I be the centralizer of T_I in G. The set Φ_I coincides with the set of roots in Φ_t which are trivial on T_I. The group M_I is reductive such that $\Phi(M_I, T) = \Phi_I$ and normalizes $N_I = R_u(P_I)$ such that

$$P_I = M_I \ltimes N_I. \tag{4.1}$$

We call M_I the Levi component or the Levi factor of P_I and the decomposition given by equation 4.1 is called a Levi decomposition of P_I.

Since any other parabolic subgroup P of G is conjugate to a standard parabolic subgroup P_I for some I, it then has also a Levi decomposition via conjugation from a Levi decomposition of P_I.

Example 4.2.19. Let $G = \mathrm{GL}_n(\mathbb{C})$, $n \geq 2$. G is a reductive group but not semisimple. The group

$$T = \{\mathrm{diag}\,(t_1, t_2, \ldots, t_n) : t_i \in \mathbb{C}^*\}$$

is a maximal torus in G. The set of simple roots δ consists of $n-1$ elements. The corresponding Borel subgroup is the subgroup of G consisting of upper triangular matrices. For any subset I of δ, there exists a partition $(n_1, n_2, \ldots, n_r)$ of n such that

$$T_I = \{\mathrm{diag}\,(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_r, \ldots, a_r)| \;\; a_1, a_2, \ldots, a_r \in \mathbb{C}^*\},$$

where a_i appears n_i times, for $1 \leq i \leq n$.

$$M_I \cong \mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C}) \times \ldots \times \mathrm{GL}_{n_r}(\mathbb{C})$$

and

$$N_I = \left\{ \begin{pmatrix} 1_{n_1} & * & * & * \\ & 1_{n_2} & * & \vdots \\ & & \ddots & * \\ & & & 1_{n_r} \end{pmatrix} \right\}$$

such that $P_I = M_I \ltimes N_I$.

Example 4.2.20. $G = \mathrm{SL}_2(\mathbb{C})$. The only conjugacy class of parabolic subgroups is the minimal parabolic subgroups. The standard minimal parabolic subgroup which coincides with the Borel subgroup is

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

P is defined over $\mathbb{Q}$. Its real locus is the stabilizer of $i\infty$ in $\mathrm{SL}_2(\mathbb{R})$. Its unipotent radical is

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{R} \right\},$$

and its Levi factor is

$$M = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{R}^* \right\}.$$

Example 4.2.21. $G = \mathrm{Sp}_4(\mathbb{C})$. The group

$$T = \{\mathrm{diag}\,(a, b, a^{-1}, b^{-1}) : a, b \in \mathbb{C}^*\}$$

is a maximal torus in G. The set of simple roots consists of two roots $\delta = \alpha, \beta$. Hence the set of conjugacy classes of parabolic subgroups consists of a single conjugacy class of Borel subgroups (minimal parabolic subgroups) and two conjugacy classes of maximal parabolic subgroups. The standard Borel subgroup which corresponds to the set of simple roots is the subgroup B of upper triangular matrices in G. The standard maximal parabolic subgroups will be denoted by $C_{2,0}$ and $C_{2,1}$ where 0 corresponds to α and 1 corresponds to β.

For $C_{2,0}$,

$$T_0 = \{\mathrm{diag}\,(a, a, a^{-1}, a^{-1}) | a \in \mathbb{C}^*\},$$

$$M_0 = \left\{ \begin{pmatrix} A & 0_2 \\ 0_2 & {}^tA^{-1} \end{pmatrix} \middle| A \in \mathrm{GL}_2(\mathbb{C}) \right\},$$

$$N_0 = \left\{ \begin{pmatrix} 1_2 & B \\ 0_2 & 1_2 \end{pmatrix} \middle| B \in \mathrm{Mat}_2(\mathbb{C}), {}^tB = B \right\},$$

and

$$C_{2,0} = M_0 \ltimes N_0.$$

For $C_{2,1}$

$$T_1 = \{\mathrm{diag}\,(1, a, 1, a^{-1}) | a \in \mathbb{C}^*\},$$

$$M_1 = \left\{ \begin{pmatrix} a' & 0 & b' & 0 \\ 0 & a & 0 & 0 \\ c' & 0 & d' & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^*, a'd' - b'c' = 1 \right\} \cong \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^*,$$

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & z \\ x & 1 & z & y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

and

$$C_{2,1} = M_1 \ltimes N_1.$$

4.2.3 Maximal Parabolic Subgroups of $\mathrm{Sp}_{2n}(\mathbb{R})$

The maximal torus is the set of diagonal matrices $\mathrm{diag}(t_1, t_2, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1})$. The corresponding Borel subgroup or the standard minimal parabolic subgroup is the subgroup of upper triangular matrices. The set of simple roots consists of n roots. Hence, we have 2^n conjugacy classes of parabolic subgroups containing n conjugacy classes of maximal parabolic subgroups. For each r, $0 \le r \le n-1$, the standard maximal parabolic subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ is given by:

$$C_{n,r} = \left\{ \begin{pmatrix} A_1 & 0 & B_1 & B_{12} \\ A_{21} & u & B_{21} & B_2 \\ C_1 & 0 & D_1 & D_{12} \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R}) \middle| \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}_{2r}(\mathbb{R}), u \in \mathrm{GL}_{n-r}(\mathbb{R}) \right\}.$$

Its unipotent radical is

$$N_r = \left\{ \begin{pmatrix} 1_r & 0 & 0 & n \\ {}^t m & 1_{n-r} & {}^t n & b \\ 0 & 0 & 1_r & -m \\ 0 & 0 & 0 & 1_{n-r} \end{pmatrix} \middle| {}^t nm + b = {}^t mn + {}^t b \right\},$$

and its Levi factor M_r is $(G_r \times G_{n-r})$ where

$$G_r := \left\{ \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ C_1 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{pmatrix} \middle| \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}_{2r}(\mathbb{R}) \right\} \cong \mathrm{Sp}_{2r}(\mathbb{R}),$$

and

$$G_{n-r} := \left\{ \begin{pmatrix} 1_r & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \middle| u \in \mathrm{GL}_{n-r}(\mathbb{R}) \right\} \cong \mathrm{GL}_{n-r}(\mathbb{R}).$$

Hence the Levi decomposotion says that

$$C_{n,r} = (G_r \times G_{n-r}) \ltimes N_r. \tag{4.2}$$

We now describe how we obtain the standard maximal parabolic subgroups $C_{n,r}$ as subgroups of $\mathrm{Sp}_{2n}(\mathbb{R})$. The set $C_{n,r}$ can be defined as the set of matrices $M \in \mathrm{Sp}_{2n}(\mathbb{R})$ whose elements in the first $2n-r$ columns and last $n-r$ rows vanish. That is,

$$C_{n,r} = \left\{ M \in \mathrm{Sp}_{2n}(\mathbb{R}) \middle| M = \begin{pmatrix} * & * \\ 0_{n-r,2n-r} & * \end{pmatrix} \right\}.$$

If we decompose each $n \times n$ block in an element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $\mathrm{Sp}_{2n}(\mathbb{R})$ into $r \times (n-r)$ blocks in the following way:

$$A = \begin{pmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_{12} \\ B_{21} & B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_{12} \\ C_{21} & C_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & D_{12} \\ D_{21} & D_2 \end{pmatrix},$$

then $M \in C_{n,r}$ implies that $A_{12} = 0, C_{12} = 0, C_{21} = 0, C_{22} = 0, D_{21} = 0$ and

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}_{2r}(\mathbb{R}).$$

That is,

$$C_{n,r} = \left\{ M \in \mathrm{Sp}_{2n}(\mathbb{R}) \middle| M = \begin{pmatrix} A_1 & 0 & B_1 & B_{12} \\ A_{21} & A_2 & B_{21} & B_2 \\ C_1 & 0 & D_1 & D_{12} \\ 0 & 0 & 0 & D_2 \end{pmatrix} \right\}.$$

Remark 4.2.22. Due to the symplectic relations, it suffices to say that $M \in \mathrm{Sp}_{2n}(\mathbb{R})$ is in $C_{n,r}$ if and only if $A_{12} = 0$, $C_{12} = 0$, and $C_2 = 0$, (or equivalently, $M \in \mathrm{Sp}_{2n}(\mathbb{R}) \cap C_{n,r}$ if and only if $C_{21} = 0$, $C_2 = 0$, and $D_{21} = 0$).

When $r = 0$,

$$C_{n,0} = \left\{ M \in \mathrm{Sp}_{2n}(\mathbb{R}) \middle| M = \begin{pmatrix} A & B \\ 0_n & {}^tA^{-1} \end{pmatrix} \right\}.$$

When $r = n-1$, $C_{n,n-1}$ consists of matrices whose last row is of the form $(0, 0, \dots, 0, *)$ with the first $2n-1$ elements are zeros. That is,

$$C_{n,n-1} = \left\{ M \in \mathrm{Sp}_{2n}(\mathbb{R}) \middle| M = \begin{pmatrix} A_1 & 0 & B_1 & B_{12} \\ A_{21} & A_2 & B_{21} & B_2 \\ C_1 & 0 & D_1 & D_{12} \\ 0 & 0 & 0 & D_2 \end{pmatrix} \right\},$$

where

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}_{2(n-1)}(\mathbb{R}).$$

4.3 Concrete Description of the Satake Compactification of $\Gamma \setminus \mathfrak{h}^n$

In this section, we give a concrete description of the rational boundary components (cusps) that are added to the space $\Gamma \setminus \mathfrak{h}^n$ to obtain its Satake compactification $\Gamma \setminus (\mathfrak{h}^n)^*$. We follow closely the presentation of [55] that is based on the work of Piatetski-Shapiro in [58].

We first start by considering two important projections. For any fixed r, we have the following maps:

$$\begin{array}{rcl} * : \mathfrak{h}^n & \to & \mathfrak{h}^r \\ Z = \begin{pmatrix} Z^* & * \\ * & * \end{pmatrix} & \mapsto & Z^* \end{array},$$

where $Z^* = Z \begin{bmatrix} 1_r & \\ & 0_{n-r} \end{bmatrix}$, and

$$\begin{aligned} \omega_r : C_{n,r} & \rightarrow \operatorname{Sp}_{2r}(\mathbb{R}) \\ M & \mapsto M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}. \end{aligned}$$

It can be easily verified that the map ω_r is a group homomorphism. On the other side, we consider the injection of $\operatorname{Sp}_{2r}(\mathbb{R})$ in $\operatorname{Sp}_{2n}(\mathbb{R})$ defined by

$$\iota : M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \mapsto M = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ C_1 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{pmatrix},$$

then $\omega_r \circ \iota = 1$, which means that ω_r is surjective.

Remark 4.3.1. Every element $M \in C_{n,r}$ acts on $\mathfrak{h}^r$ because $\omega_r(M) \in \operatorname{Sp}_{2r}(\mathbb{R})$.

Both projections $*$ and ω_r are called projections onto the standard boundary component of degree r (this will be defined below). In fact, they are compatible with each other

$$MZ^* = \omega_r(M) Z^*$$

for every $M \in C_{n,r}$ and $Z \in \mathfrak{h}^n$.

4.3.1 Description of the Rational Boundary Components

To describe the rational boundary components of the space $\Gamma \setminus \mathfrak{h}^n$ and form its Satake compactification, we first start with the natural compactification of $\mathfrak{h}^n$ obtained from the realization of the Siegel upper half space $\mathfrak{h}^n$ as analytically isomorphic to the bounded symmetric domain $\mathfrak{D}_n$ in the space of symmetric complex matrices of degree n

using the Cayley transformation

$$Z \to (Z - i1_n)(Z + i1_n)^{-1}.$$

This realization is known as the Harish-Chandra realization of a homogeneous space. We then express this natural compactification in terms of boundary components. To obtain the Satake compactification of the space $\Gamma \setminus \mathfrak{h}^n$, we instead need to consider the so-called rational boundary components of the Siegel upper half space $\mathfrak{h}^n$. This is how general compactifications of Siegel varieties are described. To make sense of this, the reader can consult the appendix in which a general approach of compactifying spaces like Siegel varieties is outlined.

Let $\overline{\mathfrak{D}}_n = \{Z \in \mathrm{Mat}_n(\mathbb{C}) : Z = {}^tZ, 1_n - Z\bar{Z} \geq 0\}$ be the topological closure of $\mathfrak{D}_n$ in the space of complex symmetric matrices of order n. $\overline{\mathfrak{D}}_n$ is therefore a compact space and provides what is called the natural compactification of $\mathfrak{h}^n$.

For two points p, q in the topological closure $\overline{\mathfrak{D}}_n$ of $\mathfrak{D}_n$, there is an equivalence relation defined in the following manner

$$p \sim q \Leftrightarrow \text{ there exists } \ \xi_i : \mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\} \to \mathfrak{D}_n, \quad i = 1, \dots m,$$

holomorphic maps, such that $\xi_i(0) = p$, $\xi_m(0) = q$, and $\xi_i(D) \cap \xi_{i+1}(D) \neq \emptyset$. Loosely speaking, $p \sim q$ if they can be connected by a finite number of holomorphic curves.

Definition 4.3.2. A maximal subset in $\overline{\mathfrak{D}}_n$ of mutually equivalent points is called a boundary component of $\mathfrak{D}_n$.

The space $\overline{\mathfrak{D}}_n$ is then equal to the disjoint union of all of its boundary components. For $n = 1$, the boundary components of the unit circle are the single points of the circumference.

Example 4.3.3 ([55], Proposition 4.4, Claim 4.4.1 and Lemma 4.4.2). 1. The set $F_r =$

$$\left\{ \begin{pmatrix} Z^* & 0_n \\ 0_{n-r} & 1_{n-r} \end{pmatrix} \middle| Z^* \in \mathfrak{D}_r \right\} \cong \mathfrak{D}_r \text{ is a boundary component for every } 0 \le r \le n.$$

2. In particular, $\mathfrak{D}_n$ itself is a boundary component.

We know that the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ acts on the Siegel upper half plane $\mathfrak{h}^n$. It turns out that the corresponding action of $\mathrm{Sp}_{2n}(\mathbb{R})$ on $\overline{\mathfrak{D}}_n$ induced from the Cayley transformation extends to the boundary.

Proposition 4.3.4 ([55], Proposition 4.3)**.** *The action of* $\mathrm{Sp}_{2n}(\mathbb{R})$ *on* $\mathfrak{D}_n$ *induced from the Cayley transformation*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \to ((A-iC)(Z+1_n)+(B-iD)i(Z-1_n))\,((A+iC)(Z+1_n)+(B+iD)i(Z-1_n))^{-1}$$

extends to the closure of $\mathfrak{D}_n$*.*

By this action, a boundary component is transformed into another. Therefore, the division of $\overline{\mathfrak{D}}_n$ into boundary components is invariant under the action of $\mathrm{Sp}_{2n}(\mathbb{R})$.

Proposition 4.3.5 ([55], Proposition 4.4)**.** *1. We have a bijective correspondence between the set of boundary components* $\{F\}$ *and the set of real subspaces* $\{U\}$ *in* $\mathbb{R}^{2n}$ *of dimension* $r \le n$ *defined in the following way: Let* W *be the subspace in* $\mathbb{C}^{2n}$ *spanned by the columns of* $\begin{pmatrix} Z+1_n \\ i(Z-1_n) \end{pmatrix}$ *for* $Z \in \overline{\mathfrak{D}}_n$*, then*

$$F = F(U) = \{Z \in \overline{\mathfrak{D}}_n \mid U_{\mathbb{C}} = U \otimes \mathbb{C} = W \cap \bar{W}\}.$$

2. For F_r*, the corresponding real subspace (see Remark 4.3.7)* W *is of dimension* $n-r$ *and* $F_n = F(\{0\})$*.*

3. For a boundary component $F = F(U)$ *and* $M \in \mathrm{Sp}_{2n}(\mathbb{R})$ *we denote the action of*

$M \in \mathrm{Sp}_{2n}(\mathbb{R})$ on the set of boundary components by $M.F$ and we have

$$M.F = F(M.U)$$

where M acts on $\mathbb{R}^{2n}$ (considered as a set of column vectors) as a $2n$-matrix.

4. *Any boundary component of $\mathfrak{D}_n$ has the form $M.F_r$ for some $M \in \mathrm{Sp}_{2n}(\mathbb{R})$ and $0 \leq r \leq n$.*

5. *Consider the compact space $\overline{\mathfrak{D}}_n$ which is the disjoint union of all the boundary components. Because of the action of $\mathrm{Sp}_{2n}(\mathbb{R})$ on the boundary as in Proposition 4.3.4, we can then write $\overline{\mathfrak{D}}_n$ as the disjoint union of all $\mathrm{Sp}_{2n}(\mathbb{R})$ orbits of the boundary components F_r for $0 \leq r \leq n$:*

$$\overline{\mathfrak{D}}_n = \bigcup_{0 \leq r \leq n} \mathrm{Sp}_{2n}(\mathbb{R}).F_r.$$

6. *If $F = M.F_r$ for $M \in \mathrm{Sp}_{2n}(\mathbb{R})$, we say F is of degree r and we denote this by $\deg(F_r) = r$. In general, for $F = F(U)$ we have:*

$$\deg(F) + \dim(U) = n.$$

That is, if F has degree r it is then called an $n-r$ dimensional boundary component.

Definition 4.3.6. For $0 \leq k \leq n$, a k-th boundary component of $\mathfrak{D}_n$ is a boundary component $F(U)$ with $\dim U = k$ and of degree $n-k$.

Remark 4.3.7. We note that $F_r = F(U_r)$ where U_r is the space generated by the columns of

$$\begin{pmatrix} 0_r \\ 1_{n-r} \\ 0_n \end{pmatrix}.$$

4.3.2 Structure of the Parabolic Subgroup Associated with a Boundary Component

To each boundary component F, the stabilizer of F in $\mathrm{Sp}_{2n}(\mathbb{R})$ is called a parabolic subgroup.

Definition 4.3.8. Let F be a boundary component of $\mathfrak{D}_n$. We define the following:

1. $P(F) = \{M \in \mathrm{Sp}_{2n}(\mathbb{R}) \mid M.F = F\}$ is the parabolic subgroup associated to F,

2. $W(F)$ is the unipotent radical of $P(F)$.

Lemma 4.3.9 ([55], Lemma 4.7)**.** *Let F_1 and F_2 be two boundary components. If $M.F_1 = F_2$ for an element $M \in \mathrm{Sp}_{2n}(\mathbb{R})$, then $P(F_2) = MP(F_1)M^{-1}$. Similarly $W(F_2) = MW(F_1)M^{-1}$.*

Proof. It suffices to show that $P(F_2) = MP(F_1)M^{-1}$. Using the fact that $P(F_1)$ stabilizes F_1, we then have

$$P(F_2).F_2 = MP(F_1)M^{-1}.F_2 = MP(F_1).F_1 = M.F_1 = F_2.$$

Hence $MP(F_1)M^{-1}$ is the stablizer of the boundary component F_2. $\square$

Remark 4.3.10. Because every boundary component $F = M.F_r$ for some $0 \leq r \leq n-1$ and due to Lemma 4.3.9, it is enough to know the structure of the groups given in Definition 4.3.8 for F_r.

Proposition 4.3.11 ([55], Proposition 4.8)**.** *For a boundary component $F = F_r$, $0 \leq r \leq n$, the subgroups of $\mathrm{Sp}_{2n}(\mathbb{R})$ given as in Definition 4.3.8 can be expressed explicitly as follows.*

1. $$P_r = P(F_r) = C_{n,r} = \left\{ \begin{pmatrix} A_1 & 0 & B_1 & B_{12} \\ A_{21} & u & B_{21} & B_2 \\ C_1 & 0 & D_1 & D_{12} \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R}) \;\middle|\; \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}_{2r}(\mathbb{R}), u \in \mathrm{GL}_{n-r}(\mathbb{R}) \right\}.$$

2. $W_r = W(F_r) = N_r = \left\{ \begin{pmatrix} 1_r & 0 & 0 & n \\ {}^t m & 1_{n-r} & {}^t n & b \\ 0 & 0 & 1_r & -m \\ 0 & 0 & 0 & 1_{n-r} \end{pmatrix} \middle| \, {}^t nm + b = {}^t mn + {}^t b \right\}.$

Proof. It suffices to prove (1). Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in P(F_r)$. Decompose every $n \times n$ block in the following way:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

such that the matrices with index 1 are $r \times r$ matrices, with index 2 are $r \times (n-r)$, with index 3 are $(n-r) \times r$ and with index 4, $(n-r) \times (n-r)$ respectively.

By Propositon 4.3.5, (3), we have

$$P(F) = P(F(U)) = \{g \in \mathrm{Sp}_{2n}(\mathbb{R}) \mid g.U = U\}.$$

Applying this to $U = U_r$ as in Remark 4.3.7, we see that

$$A_2 = C_2 = 0, C_4 = 0.$$

Also using the symplectic relations for g, we get

$$C_3 = D_3 = 0, \quad {}^t A_4 D_4 = 1_n$$

and

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}_{2r}(\mathbb{R}).$$

To see why for example $C_3 = 0$ and ${}^t A_4 D_4 = 1_n$, we rewrite the symplectic relation

${}^tAD - {}^tCB = 1$ in terms of block matrices, we obtain

$$\begin{aligned}\begin{pmatrix} 1_r & 0_{r,n-r} \\ 0_{n-r,r} & 1_{n-r} \end{pmatrix} &= {}^t\begin{pmatrix} A_1 & 0_{r,n-r} \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} - {}^t\begin{pmatrix} C_1 & 0_{r,n-r} \\ C_3 & 0_{n-r,n-r} \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \\ &= \begin{pmatrix} {}^tA_1D_1 + {}^tA_3D_3 - {}^tC_1B_1 - {}^tC_3B_3 & {}^tA_1D_2 + {}^tA_2D_4 - {}^tC_1B_2 - {}^tC_3B_4 \\ {}^tA_4D_3 & {}^tA_4D_4 \end{pmatrix}.\end{aligned}$$

The lower half of the last matrix shows that

$${}^tA_4D_4 = 1_{n-r} \quad \text{and} \quad {}^tA_4D_3 = 0_{n-r,r},$$

hence

$$D_3 = 0_{n-r,r}.$$

□

Proposition 4.3.11 and equation (4.2) imply that $P(F_r)$ is the semidirect product of $(G_r(F_r) \times G_{n-r}(F_r))$ with $W(F_r)$.

$$P(F_r) = P_r = (G_r(F_r) \times G_{n-r}(F_r)) \ltimes W(F_r),$$

where

$$G_n(F_r) := \left\{ \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ C_1 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{pmatrix} \middle| \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}_{2r}(\mathbb{R}) \right\} \cong \iota(\mathrm{Sp}_{2r}(\mathbb{R})).$$

$$G_{n-r}(F_r) := \left\{ \begin{pmatrix} 1_r & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \middle| \, u \in \mathrm{GL}_{n-r}(\mathbb{R}) \right\} \cong \mathrm{GL}_{n-r}(\mathbb{R}).$$

Remark 4.3.12 ([55], Remark 4.11)**.** For any boundary component F, the parabolic subgroup $P(F)$ acts transitively on $\mathfrak{D}_n$.

We now investigate the relationships between boundary components.

Proposition 4.3.13 ([55], Proposition 4.12)**.** *Let W be the vector space in $\mathbb{C}^{2n}$ spanned by the columns of $\begin{pmatrix} Z + 1_n \\ i(Z - 1_n) \end{pmatrix}$. Take the boundary component $F = F(U)$, for $U \subset \mathbb{R}^{2n}$. Then, the topological closure of F in $\overline{\mathfrak{D}}_n$ is*

$$\overline{F} = \{Z \in \overline{\mathfrak{D}}_n, U_{\mathbb{C}} \subset W \cap \overline{W}\}.$$

Fact 4.3.14. If F is a boundary component, then its closure $\overline{F}$ in $\overline{\mathfrak{D}}_n$ may be indentified with the natural compactification of F and its boundary components are also boundary components of $\mathfrak{D}_n$.

For two boundary components F, F' of $\mathfrak{D}_n$, we write

$$F < F' \quad \text{if} \quad F \subset \overline{(F')}.$$

This defines a partial order on the set of boundary components. Proposition 4.3.13 says that for $F = F(U)$ and $F' = F(U')$,

$$F < F' \quad \text{if and only if} \quad U \supset U'.$$

Theorem 4.3.15 ([55], Theorem 4.14)**.** *For two boundary components F and F' of $\mathfrak{D}_n$ with $F < F'$, we have*

1. $G_{n-r}(F) \supset G_{n-r}(F')$*,*

2. $G_r(F) \subset G_r(F')$*.*

3. There is an element $M \in \mathrm{Sp}_{2n}(\mathbb{R})$ *such that* $F_{r_1} = M.F$ *and* $F_{r_2} = M.F'$ *with* $r_1 \leq r_2$*.*

4.3.2.1 Rationality of Boundary Components

Definition 4.3.16. A boundary component F of $\mathfrak{D}_n$ is called rational if it satisfies one of the following conditions.

1. $P(F)$ is defined over $\mathbb{Q}$.

2. $F = F(U)$ and U is defined over $\mathbb{Q}$ (i.e. a basis is chosen in $\mathbb{Q}^{2n}$).

3. There exists $M \in \mathrm{Sp}_{2n}(\mathbb{Q})$ with $M.F = F_r$.

In particular, the boundary components F_r for $0 \leq r \leq n-1$ are rational boundary components. The rational closure of $\mathfrak{D}_n$ which is defined to be the space $\mathfrak{D}_n$ enlarged by its rational boundary is called the partial compactification of $\mathfrak{D}_n$.

Definition 4.3.17. The rational closure of $\mathfrak{D}_n$ is given by

$$\mathfrak{D}_n^* = \bigcup_{F:\text{rational}} F = \bigcup_{0 \leq r \leq n} \mathrm{Sp}_{2n}(\mathbb{Q}) F_r \subset \overline{\mathfrak{D}}_n.$$

Remark 4.3.18 ([55], Remark 4.16)**.** If F is a rational boundary component of $\mathfrak{D}_n$, then there is $M \in \mathrm{Sp}_{2n}(\mathbb{Z})$ such that $M.F = F_r$.

Definition 4.3.19. A k-rational boundary component F ($\dim F = k$) is said to be a k-th cusp of degree $n-k$ or cusp of depth k and degree $n-k$. The rational boundary components F_r for $0 \leq r \leq n-1$ are called the standard cusps of degree r or the $(n-r)$-cusps.

Definition 4.3.20. The partial order defined on the boundary components implies that the degree $n-1$ cusps are the maximal cusps of depth 1 that correspond to $1-$dimensional subspaces U. The minimal cusps are the cusps of zero degree and of depth n that correspond to n-dimensional subspaces U.

Now to define a topology on the partial compactification, Satake provided a topology which induces a topology on the quotient space $\Gamma \setminus \mathfrak{D}_n^*$ that makes this space Hausdorff and compact and contains $\Gamma \setminus \mathfrak{D}_n$ as an open dense subset. Note that Γ acts on $\mathfrak{D}_n$ and its closure $\overline{\mathfrak{D}}_n$, then Γ acts in a natural way on $\mathfrak{D}_n^*$. In another way, Piatetski-Shapiro has equipped the rational closure $\mathfrak{D}_n^*$ with a certain topology called the cylindrical topology (see [58]) which is not the same as the Satake topology, but Kierna and Kobayashi [37] have proved that both topologies define the same quotient topology on the compactification $(\Gamma \setminus \mathfrak{D}_n)^c$ of $\Gamma \setminus \mathfrak{D}_n$. A description of the cylindrical topology put on the rational closure $\mathfrak{D}_n^*$ is found in [55] on page 34.

Here we reach the main theorem which says the the topological space $\Gamma \setminus \mathfrak{D}_n^*$ is the Satake compactification of $\Gamma \setminus \mathfrak{D}_n$.

Theorem 4.3.21 ([55], Theorem 5.10). *Let $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Q})$ be an arithmetic group acting on $\mathfrak{D}_n$. We consider the rational closure $(\mathfrak{D}_n)^*$ of $\mathfrak{D}_n$ equipped with the cylindrical topology. Then*

1. *Γ acts on $(\mathfrak{D}_n)^*$ properly discontinuously.*
2. *$(\Gamma \setminus \mathfrak{D}_n)^c := \Gamma \setminus (\mathfrak{D}_n)^*$ with its quotient topology is compact and contains $\mathfrak{D}_n/\Gamma$ as an open dense subset.*
3. *$(\Gamma \setminus \mathfrak{D}_n)^c$ admits a canonical structure of a normal analytic space such that $\Gamma \setminus \mathfrak{D}_n$ is an analytic open subset. Moreover, it can be given a projective algebraic variety structure using explicitly constructed modular forms.*

We call $\Gamma \setminus (\mathfrak{D}_n)^*$ the Satake-Baily-Borel compactification of $\Gamma \setminus \mathfrak{D}_n$ when referring to its projective structure and the Satake compactification when referring to its topological structure.

4.3.3 Concrete Description of the Compact Space $\Gamma \setminus (\mathfrak{h}^n)^*$ in terms of its Rational Boundary Components.

We give a description of the rational boundary attached to $\Gamma \setminus X$ following [18], Exposés 12 and 13. To understand the topology defined on the compactified space and for other useful details on normal analytic spaces, we refer the reader to the same reference. Let Γ be the Siegel modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$. According to the discussion above, we have that

$$\mathfrak{D}_n^* = \bigcup_{F:\text{rational}} F = \bigcup_{0 \le r \le n} \mathrm{Sp}_{2n}(\mathbb{Z}) F_r.$$

Going back to the Siegel upper half plane $\mathfrak{h}^n$ through the inverse of the Cayley transofrmation, we then have

$$(\mathfrak{h}^n)^* = \bigcup_{0 \le r \le n} \mathrm{Sp}_{2n}(\mathbb{Z}) \mathfrak{h}^r.$$

Considering the action of $\mathrm{Sp}_{2n}(\mathbb{Z})$ from the left, we get

$$\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus (\mathfrak{h}^n)^* = \bigcup_{0 \le r \le n} \mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathrm{Sp}_{2n}(\mathbb{Z}) \mathfrak{h}^r.$$

The stabilizer of $\mathfrak{h}^r$ in $\mathrm{Sp}_{2n}(\mathbb{Z})$ is $\mathrm{Sp}_{2n}(\mathbb{Z}) \cap C_{n,r}$ whose action on $\mathfrak{h}^r$ is equal to that of $\omega_r(\mathrm{Sp}_{2n}(\mathbb{Z}) \cap C_{n,r}) = \mathrm{Sp}_{2r}(\mathbb{Z})$. Therefore $\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathrm{Sp}_{2n}(\mathbb{Z}) \mathfrak{h}^r$ is identified with $\mathrm{Sp}_{2r}(\mathbb{Z}) \setminus \mathfrak{h}^r$ and consequently we obtain the following decomposition of the Satake compactification of $\Gamma \setminus \mathfrak{h}^n$:

$$\begin{aligned} \mathrm{Sp}_{2n}(\mathbb{Z}) \setminus (\mathfrak{h}^n)^* &= \bigcup_{0 \le r \le n} \mathrm{Sp}_{2r}(\mathbb{Z}) \setminus \mathfrak{h}^r \\ &= \mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n \cup \mathrm{Sp}_{2(n-1)}(\mathbb{Z}) \setminus \mathfrak{h}^{n-1} \cup \ldots \cup \mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{h}^1 \cup \mathrm{SL}_{2(0)}(\mathbb{Z}) \setminus \mathfrak{h}^0, \end{aligned} \tag{4.3}$$

where $\mathrm{SL}_{2(0)}(\mathbb{Z}) \setminus \mathfrak{h}^0$ is just the infinity cusp $\{i\infty\}$. Hence, in case of the full Siegel modular group, we have a single cusp for each degree $0 \le r \le n-1$ given by the Siegel variety of degree r.

In particular, when $n = 2$, we have that

$$\mathrm{Sp}_4(\mathbb{Z}) \setminus (\mathfrak{h}^2)^* = \mathrm{Sp}_4(\mathbb{Z}) \setminus \mathfrak{h}^2 \cup \mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{h}^1 \cup \{\infty\}.$$

The cusps in this case are the open modular curve $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{h}^1$ of degree 1 and the single cusp ∞ of degree zero.

Now let $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Q})$ be a commensurable subgroup with $\mathrm{Sp}_{2n}(\mathbb{Z})$. We adjoin all rational boundary components of $\mathfrak{h}^n$ in the following manner

$$(\mathfrak{h}^n)^* = \bigcup_{0 \le r \le n} \mathrm{Sp}_{2n}(\mathbb{Q})\mathfrak{h}^r = \mathfrak{h}^n \cup \bigcup_{0 \le r \le n-1} \mathrm{Sp}_{2n}(\mathbb{Q})\mathfrak{h}^r.$$

For each $0 \le r \le n-1$, we decompose $\mathrm{Sp}_{2n}(\mathbb{Q})$ from the right by $\mathrm{Sp}_{2n}(\mathbb{Q}) \cap C_{n,r}$ and from the left by Γ, we obtain

$$\mathrm{Sp}_{2n}(\mathbb{Q}) = \bigcup_{\lambda} \Gamma M_{r,\lambda}(\mathrm{Sp}_{2n}(\mathbb{Q}) \cap C_{n,r})$$

for finitely many representatives $M_{r,\lambda}$ for each r. We then have the corresponding decomposition of $(\mathfrak{h}^n)^*$,

$$(\mathfrak{h}^n)^* = \mathfrak{h}^n \cup \bigcup_{0 \le r \le n-1} \bigcup_{\lambda} \Gamma M_{r,\lambda}(\mathrm{Sp}_{2n}(\mathbb{Q}) \cap C_{n,r})\mathfrak{h}^r.$$

But $\mathrm{Sp}_{2n}(\mathbb{Q}) \cap C_{n,r}$ is the stabilizer of $\mathfrak{h}^r$ in $\mathrm{Sp}_{2n}(\mathbb{Q})$, consequently we get

$$(\mathfrak{h}^n)^* = \mathfrak{h}^n \cup \bigcup_{r} \bigcup_{\lambda} \Gamma M_{r,\lambda}\mathfrak{h}^r.$$

Taking the quotient of $(\mathfrak{h}^n)^*$ by Γ from the left,

$$\Gamma \setminus (\mathfrak{h}^n)^* = \Gamma \setminus \mathfrak{h}^n \cup \bigcup_{r} \bigcup_{\lambda} \Gamma \setminus \Gamma M_{r,\lambda}\mathfrak{h}^r.$$

For fixed r, the stabilizer of a rational boundary component of degree r, $\Gamma M_{r,\lambda}\mathfrak{h}^r$, in the commensurable group $M_{r,\lambda}^{-1}\Gamma M_{r,\lambda}$ is $M_{r,\lambda}^{-1}\Gamma M_{r,\lambda}\cap C_{n,r}$. Its action on $\mathfrak{h}^r$ is given by the group

$$\Gamma(M_{r,\lambda}.F_r) := \omega_r(M_{r,\lambda}^{-1}\Gamma M_{r,\lambda} \cap C_{n,r}).$$

Now $\Gamma(M_{r,\lambda}.F_r)$ as subgroup of $\mathrm{Sp}_{2r}(\mathbb{R})$ is commensurable with $\mathrm{Sp}_{2r}(\mathbb{Z})$ and the quotient space $\Gamma \setminus \Gamma M_{r,\lambda}\mathfrak{h}^r$ is identified with the quotient space $\Gamma(M_{r,\lambda}.F_r) \setminus \mathfrak{h}^r$. Hence each rational boundary component of the quotient space $\Gamma\setminus\mathfrak{h}^n$ has the structure of a Siegel variety of lower degree than n and we obtain the following concrete decomposition of the Satake compact space $\Gamma \setminus (\mathfrak{h}^n)^*$ in the following way:

$$\Gamma \setminus (\mathfrak{h}^n)^* = \Gamma \setminus \mathfrak{h}^n \cup \bigcup_{0\leq r\leq n-1} \bigcup_{\lambda} \Gamma(M_{r,\lambda}.F_r) \setminus \mathfrak{h}^r.$$

The elements $M_{r,\lambda}$ for fixed r are the finitely many representatives of double cosets $\Gamma \setminus \mathrm{Sp}_{2n}(\mathbb{Q})/(C_{n,r} \cap \mathrm{Sp}_{2n}(\mathbb{Q}))$. That is, for each degree r, we have finitely many cusps corresponding to representatives $M_{r,\lambda}$ of a double coset space. This is why we give the cusps a group theoretic description through double cosets and it is with these double cosets that we work.

We note that it would be interesting to compute the groups $\Gamma(M_{r,\lambda}.F_r)$ for specific arithmetic groups of interest like the Hecke congruence subgroup $\Gamma_0^{(2)}(N)$ or the paramodular group $\Gamma[t]$ in degree two. Doing so will help identify geometrically the modular curves of degree 1 and the cusps of degree zero. Poor and Yuen have recently geometrically identified the cusps in case $\Gamma = \Gamma[t]$ in their joint work [62].

4.3.4 Cusps in Degree $n = 1$

In this part, we describe the cusps when the degree $n = 1$. Plenty of references can be found for this section, we refer to [20], [35], and [77].

One model of the hyperbolic plane is the Poincaré upper half plane

$$\mathfrak{h}^1 \cong \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2)$$

endowed with the Poincaré metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ where

$$K = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$$

is the special orthogonal group of order two. The group $\mathrm{SL}_2(\mathbb{R})$ acts isometrically and transitively on $\mathfrak{h}^1$ by fractional linear transformations which makes $\mathfrak{h}^1$ the simplest example of a symmteric space of noncompact type.

Another model of the hyperbolic plane is the Poincaré disc

$$\mathfrak{D}_1 = \{z \in \mathbb{C} \mid |z| < 1\} \cong \mathrm{SU}(1,1)/U(1)$$

endowed with the Poincaré metric $ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}$, where

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}$$

is the special unitary group of order two that acts transitively and isometrically on $\mathfrak{D}_1$ by fractional linear transformations. The two models are identified by the Cayley transformation

$$\mathfrak{h}^1 \to \mathfrak{D}_1, \qquad z \mapsto \frac{z-i}{z+i}.$$

The disc $\mathfrak{D}_1$ can be compactified by adding the unit circle and the $\mathrm{SU}(1,1)$ action on $\mathfrak{D}_1$ extends continuously to the boundary. Transporting this to the upper half plane, the natural compactification of $\mathfrak{h}^1$ is homeomorphic to

$$(\mathfrak{h}^1)^c = \mathfrak{h}^1 \cup \mathbb{R} \cup \{i\infty\}$$

as a subset of $P^1(\mathbb{C}) = \mathbb{C} \cup \{i\infty\}$. Therefore we view $\mathfrak{h}^1$ as a subset of $P^1(\mathbb{C})$ whose natural boundary is $P^1(\mathbb{R})$.

We now consider the group $\mathrm{GL}_2(\mathbb{C})$. It acts on $P^1(\mathbb{C}) = \mathbb{C} \cup \{i\infty\}$ by fractional linear transformations. Let $g \in \mathrm{GL}_2(\mathbb{C})$ be such that g is not the identity transformation i.e., g is not a scalar matrix. Form the theory of Jordan canonical forms, the matrix g is conjugate to a matrix of either the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ or of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \neq \mu$. When we restrict to elements in $\mathrm{SL}_2(\mathbb{C})$ we have the following classification of elements.

Proposition 4.3.22 ([77], Proposition 1.12)**.** *Let $g \neq \pm 1_2 \in \mathrm{SL}_2(\mathbb{C})$. Then,*

1. *g is parabolic if and only if $\mathrm{Tr}(g) = \pm 2$.*

2. *g is elliptic if and only if $\mathrm{Tr}(g)$ is real and $|\,\mathrm{Tr}(g)| < 2$.*

3. *g is hyperbolic if and only if $\mathrm{Tr}(g)$ is real and $|\,\mathrm{Tr}(g)| > 2$.*

4. *g is loxodromic if and only if $\mathrm{Tr}(g)$ is not real.*

Evidently, if we restrict to $\mathrm{SL}_2(\mathbb{R})$, then there would be no loxodromic elements. For every $s \in P^1(\mathbb{R}) = \mathbb{R} \cup \{i\infty\}$, define

$$\begin{aligned} F(s) &= \{g \in \mathrm{SL}_2(\mathbb{R}) |\, gs = s\} \\ P(s) &= \{g \in F(s) |\, g \text{ is parabolic or } g = \pm 1_2\}. \end{aligned}$$

Then

$$\begin{aligned} F(\infty) &= \left\{ \begin{pmatrix} a & n \\ 0 & a^{-1} \end{pmatrix} \middle|\, a \in \mathbb{R}^*, n \in \mathbb{R} \right\}, \\ P(\infty) &= \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \middle|\, h \in \mathbb{R} \right\} \cong \mathbb{R} \times \{\pm\}. \end{aligned}$$

Since $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $P^1(\mathbb{R})$, hence for any $s \in P^1(\mathbb{R})$, there is an element $g \in \mathrm{SL}_2(\mathbb{R})$ such that $g\infty = s$. This allows us to relate the groups $F(s)$ and $P(s)$ in $\mathrm{SL}_2(\mathbb{R})$ to the groups $F(\infty)$ and $P(\infty)$. We clearly see that,

$$\begin{aligned} F(s) &= gF(\infty)g^{-1}, \\ P(s) &= gP(\infty)g^{-1}. \end{aligned}$$

This shows that if an element in $\mathrm{SL}_2(\mathbb{R})$ has a fixed point in $P^1(\mathbb{R})$, then it is either parabolic or hyperbolic. Parabolic elements in $\mathrm{SL}_2(\mathbb{R})$ are characterized by being the elements that have a unique fixed point in $P^1(\mathbb{R}) = \mathbb{R} \cup \{i\infty\}$ and no fixed points in $\mathfrak{h}^1$.

Proposition 4.3.23 ([77], Proposition 1.13)**.** *Let $g \neq \pm 1_2, g \in \mathrm{SL}_2(\mathbb{R})$, then g is parabolic if and only if g has only one fixed point on $P^1(\mathbb{R})$.*

Definition 4.3.24. Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. An element $s \in P^1(\mathbb{R})$ is called a cusp of Γ if there exists a parabolic element $g \in \Gamma$ such that $gs = s$.

Proposition 4.3.25 ([77], Proposition 1.17)**.** *Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ and let s be a cusp of Γ and $\Gamma_s = \{g \in \Gamma |\, gs = s\}$, then every element of Γ_s is either $\pm 1_2$ or parabolic, i.e., $\Gamma_s = \Gamma \cap P(s)$.*

A cusp s of Γ is then a point in $P^1(\mathbb{R})$ fixed by a parabolic element in $\Gamma \cap P(s)$ where $P(s)$ is the real parabolic subgroup associated to s. For example, we have

$$\Gamma_\infty = P(\infty) \cap \Gamma = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle|\, n \in \mathbb{Z} \right\}.$$

Similar to above, since $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $P^1(\mathbb{R})$, then for any other cusp s

$$\Gamma_s = (gP(\infty)g^{-1}) \cap \Gamma.$$

4.3.5 Cusps of $\mathrm{SL}_2(\mathbb{Z})$

First, let us describe the cusps geometrically. We consider the open modular curve

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}^1 = \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}(2),$$

which is the moduli space of complex elliptic curves.

The well known j function maps bijectively the open modular curve $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{h}^1$ to $\mathbb{C}$ and hence provides the complex projective space $P^1(\mathbb{C}) = \mathbb{C} \cup \{i\infty\}$ as a compactification of this open modular curve by adding a single cusp $\{i\infty\}$. We now explain the single cusp $i\infty$ added in a group theoretic manner.

Proposition 4.3.26. *Let Γ be the special linear group $\mathrm{SL}_2(\mathbb{Z})$. The set of cusps of Γ is exactly the points in $P^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$.*

Proof. First it is clear that $i\infty$ is a cusp as it is fixed by the parabolic element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{Z})$. For any $\frac{p}{q} \in \mathbb{Q}$ with $p \in \mathbb{Z}, q \in \mathbb{Z}$ and $\gcd(p,q) = 1$, we take integers t and u so that $pt - qu = 1$. Then, $g = \begin{pmatrix} p & u \\ q & t \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $g(i\infty) = \frac{p}{q}$. Conversely, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a parabolic element of Γ, it has only one fixed point s. If s is finite, it satisfies

$$cs^2 + (d-a)s - b = 0, \quad c \neq 0.$$

The discriminant of this equation vanishes, hence s must be contained in $\mathbb{Q}$. This shows that all points of $\mathbb{Q} \cup \{i\infty\}$ are cusps of $\mathrm{SL}_2(\mathbb{Z})$. $\square$

We have also shown that all cusps of $\mathrm{SL}_2(\mathbb{Z})$ are equivalent under an element of $\mathrm{SL}_2(\mathbb{Z})$ to the cusp $i\infty$. Let $(\mathfrak{h}^1)^* = \mathfrak{h}^1 \cup P^1(\mathbb{Q})$, then

$$\mathrm{SL}_2(\mathbb{Z}) \setminus (\mathfrak{h}^1)^* = \mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{h}^1 \cup \{i\infty\}.$$

$$P(\infty) = \mathrm{SL}_2(\mathbb{Z})_\infty = \left\{ \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m \middle| m \in \mathbb{Z} \right\}.$$

Now if Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then the cusps of Γ are Γ-equivalence classes of $P^1(\mathbb{Q})$. When $\Gamma = \Gamma(N)$ the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then

$$\Gamma(N)_\infty = \Gamma(N) \cap \mathrm{SL}_2(\mathbb{Z})_\infty \left\{ \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}^n \middle| n \in \mathbb{Z} \right\}.$$

To find an explicit set of representatives for the cusps modulo $\Gamma(N)$-equivalence, we use the following.

Lemma 4.3.27 ([77], Lemma 1.41)**.** *Let* $a, b, c, d \in \mathbb{Z}$ *such that* $\gcd(a, b) = 1$*,* $\gcd(c, d) = 1$ *and* $\begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix} \pmod{N}$*. Then there exists an element* g *of* $\Gamma(N)$ *such that* $\begin{pmatrix} a \\ b \end{pmatrix} = g \begin{pmatrix} c \\ d \end{pmatrix}$*.*

Thus, the cusps of $\Gamma(N)$ are $\Gamma(N)s$, $s = \frac{a}{b}$ for all pairs $\pm \begin{pmatrix} a \\ b \end{pmatrix} \pmod{N}$ such that $\gcd(a, b) = 1$.

However, we can express the cusps of any congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with a purely group-theoretic description.

Proposition 4.3.28 ([20], Proposition 3.8.5)**.** *Let* Γ *be a congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$*. Let* $P = P(\infty)$ *be the parabolic subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *and let* S *be the set of cusps of* Γ*. Then the map*

$$\begin{aligned} \Gamma \setminus \mathrm{SL}_2(\mathbb{Z})/P &\to S \\ \Gamma g P &\mapsto \Gamma g(i\infty) \end{aligned}$$

is a bijection. Specifically, the map is $\Gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} P \mapsto \Gamma(\frac{a}{c})$*.*

Proof. The map is well defined since if $\Gamma g P = \Gamma g' P$ then $g' = \gamma g p$ for some $\gamma \in \Gamma$ and $p \in P$, so that $\Gamma g'(i\infty) = \Gamma \gamma g p(i\infty) = \Gamma(i\infty)$ since $\gamma \in \Gamma$ and p fixes ∞. The map is injective since the condition $\Gamma g'(i\infty) = \Gamma g(i\infty)$ is equivalent to $g'(i\infty) = \gamma g(i\infty)$ for some $\gamma \in \Gamma$. Or, $g^{-1}\gamma^{-1}g' \in P$ for some $\gamma \in \Gamma$. This means that $\Gamma g' P = \Gamma g P$. Finally, it is clear that the map is surjective. □

Remark 4.3.29. 1. Note that the proof of Proposition 4.3.28 is essentially identical to showing first that $\mathrm{SL}_2(\mathbb{Z})/P$ identifies with $P^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$ through the map that sends every $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to its first column $\begin{pmatrix} a \\ c \end{pmatrix}$. The double coset space $\Gamma \setminus \mathrm{SL}_2(\mathbb{Z})/P$ identifies then with the cusps of $\Gamma \setminus P^1(\mathbb{Q})$.

2. The identification $\Gamma \setminus \mathrm{SL}_2(\mathbb{Z})/P$ with $\Gamma \setminus P^1(\mathbb{Q})$ generalizes to higher degree n when using the maximal parabolic subgroup $C_{n,n-1}(\mathbb{Q})$. That is the maximal degree cusps of an arihmetic group Γ of $\mathrm{Sp}_{2n}(\mathbb{Q})$ are given by the double coset space $\Gamma \backslash \mathrm{Sp}_{2n}(\mathbb{Q})/C_{n,n-1}(\mathbb{Q})$ (See Proposition 6.5.7 in Chapter 6).

3. The group theoretic description of maximal degree cusps in all degrees n, which in particular when $n = 1$ gives all the cusps (not just maximal) because of the single conjugacy of parabolic subgroups, is in line with the geometric description of the boundary components of a Siegel variety $\Gamma \setminus \mathfrak{h}^n$ as done in § 4.3.3 above.

One can view compactifications of $\Gamma \setminus \mathfrak{h}^1$ in the following manner. There is a close relation between the compactifications of $\mathfrak{h}^1$ and $\Gamma \setminus \mathfrak{h}^1$. Instead of adding the full boundary $\mathbb{R} \cup \{i\infty\}$ of the compactification $(\mathfrak{h}^1)^c$ of $\mathfrak{h}^1$, we add only a partial boundary consisting of the so-called rational boundary components $\mathbb{Q} \cup \{i\infty\}$ and we call

$$\mathfrak{h}^1 \cup \mathbb{Q} \cup \{i\infty\}$$

the partial compactification of $\mathfrak{h}^1$. A certain topology called the Satake topology is defined by means of neighborhood basis such that for the finite part we have the ususal neighborhoods and if $s \neq i\infty$, a typical open neighborhood of s is the set $\{s\} \cup \{$interior of a circle in $\mathfrak{h}^1$ tangent to $\mathbb{R}$ a If $s = i\infty$, a neighborhood is given by the set $\{i\infty\} \cup \{z \in \mathfrak{h}^1 \mid \Im z > c\}$ for any $c > 0$, c in $\mathbb{R}$. The Γ action extends properly to $\mathfrak{h}^1 \cup \mathbb{Q} \cup \{i\infty\}$ so that the quotient $\Gamma \setminus \mathfrak{h}^1 \cup \mathbb{Q} \cup \{i\infty\}$ is a compact Hausdorff space that provides a compactification of $\Gamma \setminus \mathfrak{h}^1$.

In fact, the rational boundary $P^1(\mathbb{Q})$ added was not a choice. It is directly related to a fundamental domain for Γ. The closure of the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ in the Riemann sphere $P^1(\mathbb{C})$ meets the natural boundary of $\mathfrak{h}^1$ at the point $i\infty$. Taking the fundamental domain of a congruence subgroup of Γ which is a union of translates of the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$, then its closure in $P^1(\mathbb{C})$ meets $P^1(\mathbb{R})$ in points in $\mathbb{Q} \cup \{i\infty\}$ which are the Γ-translates of the point $i\infty$. For a general description of this procedure, we refer to § 4.5.

4.4 The Siegel Φ Operator Revisited

The Siegel Φ operator was introduced in Chapter 2 as a linear operator that reduces the number of variables when acting on modular forms. Understanding the Satake compactification of Siegel varieties $\Gamma \setminus \mathfrak{h}^n$ enables us to view the Siegel Φ operator in a more meaningful way. In this section, we define a global linear operator that projects functions living on the Siegel upper half space of degree n to all the rational boundaries of lower degrees. This global operator reduces to the Siegel Φ operator when considering only the maximal standard cusps of degree $n-1$. One advantage of this generalization is that it allows us to give a general notion of cusp forms and to bridge this new notion with the classical one found in the literature. More importantly, it helps to explain why it suffices to check the conditions given in Proposition 2.3.5 for only finitely many matrices (See Remark 4.4.27). We keep all notations as in the previous section.

4.4.1 Cusp Forms for The Siegel Modular Group

In this part, we explain why it is enough to show a Siegel modular form with respect to the Siegel modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$ is a cuspidal form by showing it belongs to the kernel of the Siegel Φ operator. When $n = 1$, we usually say a modular $f \in M_\kappa(\mathrm{SL}_2(\mathbb{Z}))$ is a cusp form if it vanishes at the cusp $i\infty$. It is then natural to extend this definition to higher degree n and say that a Siegel modular $F \in M_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$ is a cusp form if it vanishes at all cusps. We now make sense of this statement.

Let $\Gamma_n = \mathrm{Sp}_{2n}(\mathbb{Z})$ and let $M_\kappa(\Gamma_n)$ be the space of modular forms of weight κ with respect to the Siegel modular group Γ_n. The compactification of $\Gamma_n \setminus \mathfrak{h}^n$ is the space

$$\Gamma_n \setminus (\mathfrak{h}^n)^* = \Gamma_n \setminus \mathfrak{h}^n \cup \Gamma_{n-1} \setminus \mathfrak{h}^{n-1} \cup \ldots \cup \Gamma_0 \setminus \mathfrak{h}^0.$$

The lower dimensional spaces

$$\Gamma_{n-1} \setminus \mathfrak{h}^{n-1}, \ldots, \Gamma_0 \setminus \mathfrak{h}^0$$

are called the cusps or the rational boundary components of $\Gamma \setminus (\mathfrak{h}^n)^*$.

Definition 4.4.1. With respect to the full modular group Γ_n, we have a unique cusp $F_r := \Gamma_r \setminus \mathfrak{h}^r$ for each degree $0 \leq r \leq n-1$ which is called the standard cusp of degree r. The cusp $F_{n-1} := \Gamma_{n-1} \setminus \mathfrak{h}^{n-1}$ is called the standard maximal cusp.

On the other hand, we have seen that the Siegel Φ operator is a linear operator such that

$$\Phi : M_\kappa(\Gamma_n) \to M_\kappa(\Gamma_{n-1}).$$

Therefore, we can now give the following definition from the point of view of compactification.

Definition 4.4.2. For every $F \in \mathrm{M}_\kappa(\Gamma_n)$, $F|\Phi$ is a projection of F to the standard maximal cusp of degree $n-1$.

Equivalently, this definition says that the Siegel Φ operator is the operator associated with the maximal degree standard cusp $F_{n-1} = \Gamma_{n-1} \setminus \mathfrak{h}^{n-1}$. That is,

$$\Phi := \Phi_{F_{n-1}}.$$

It associates to a Siegel modular form $F \in M_\kappa(\Gamma_n)$ its projection $F|\Phi$ to the maximal degree standard rational boundary component. This projection is a modular form for

$$\Gamma_{n-1} = \omega_{n-1}(\Gamma_n \cap C_{n,n-1}).$$

Now for any $1 \leq j \leq n$, the j-fold Siegel operator Φ^j is defined as follows

Definition 4.4.3. For $1 \leq j \leq n$,

$$F|\Phi^j(Z^*) = \lim_{\lambda\to\infty} F \begin{pmatrix} Z^* & 0_{n-r} \\ 0_{n-r} & i\lambda 1_{n-r} \end{pmatrix}, \qquad (Z^* \in \mathfrak{h}^{n-j}).$$

One can show that

Proposition 4.4.4. *The projection of F, $F|\Phi^j$, belongs to $M_k(\Gamma_{n-j})$.*

Moreover, the Siegel Φ operator enjoys the following transitivity property:

Proposition 4.4.5 ([67], page 14-17)**.** *For $0 \leq s < r < n$, the Siegel operator satisfies*

$$\Phi^s(\Phi^r) = \Phi^{r+s}.$$

Using this transitivity property, it is easy to verify that the j-fold operator Φ^j means applying the Siegel Φ operator j-times. This brings us to the following conclusion:

Proposition 4.4.6. *For each $1 \leq j \leq n$, applying the Siegel Φ operator j-times gives a projection $F|\Phi^j$ of $F \in M_\kappa(\Gamma_n)$ to the standard cusp $F_{n-j} := \Gamma_{n-j} \setminus \mathfrak{h}^{n-j}$ of degree $n-j$. Applying it $n-$times gives the projection of F to the zero dimensional cusp $F_0 := i\infty$. That is,*

$$\Phi^j := \Phi_{F_{n-j}}$$

is the Siegel operator associated to the standard cusp of degree $n-j$.

We are now ready to give a more general definition of the Siegel operator. This operator associates to each $F \in M_\kappa(\Gamma_n)$ the collection of automorphic forms $(F|\Phi^j)_{1\leq j\leq n}$ which are the projections of F to all the standard rational boundary components or to all standard cusps (which are in this case the only cusps) for $\Gamma_n \setminus \mathfrak{h}^n$.

Definition 4.4.7. The Siegel operator $\overrightarrow{\Phi}$ is the linear operator which projects a Siegel modular form $F \in M_\kappa(\Gamma_n)$ to the standard cusps of $\Gamma_n \setminus (\mathfrak{h}^n)^*$ of all degrees $0 \leq r \leq n-1$.

$$\begin{aligned} \overrightarrow{\Phi} : M_\kappa(\Gamma_n) &\rightarrow M_\kappa(\Gamma_{n-1}) \times \ldots \times M_\kappa(\Gamma_{n-j}) \times \ldots \times M_\kappa(\Gamma_1) \times M_\kappa(\Gamma_0) \\ F &\mapsto (F|\Phi, \ldots, F|\Phi^j, \ldots, F|\Phi^{n-1}, F|\Phi^n), \end{aligned}$$

where

$$\Gamma_{n-j} = \omega_{n-j}(\Gamma_n \cap C_{n,n-j})$$

for $1 \leq j \leq n$.

This allows us to generalize the classical definition of cusp forms to a one that take cares of all the cusps in case of the Siegel modular group.

Definition 4.4.8. A modular form $F \in M_\kappa(\Gamma_n)$ is called a cusp form if

$$F|\overrightarrow{\Phi} = 0$$

where 0 is a zero vector of order n.

In other words, F is a cusp form if

$$F|\Phi^j = 0 \quad \text{for all } \ 0 \leq j \leq n-1.$$

Remark 4.4.9. 1. This definition says that a modular form $F \in M_\kappa(\Gamma_n)$ is a cusp form if its projections to all of the standard rational boundary components, or to all of standard cusps, vanish. Implicitly, we can say that $F \in S_\kappa(\Gamma_n)$ if it vanishes at all the cusps. This makes the refined definition of a cusp form of degree $n \geq 2$ more in parallel to the definition of cusp form when $n = 1$ for the full level. For in the latter case, if $F \in \mathrm{M}_\kappa(\mathrm{SL}_2(\mathbb{Z}))$, the projection of F to the unique cusp $i\infty$ is defined as

$$F(\infty) = F|\Phi(i\infty) = \lim_{\lambda \to \infty} F(i\lambda),$$

and that's why we say F is a cusp form with respect to $\mathrm{SL}_2(\mathbb{Z})$ if it vanishes at the unique cusp $i\infty$.

2. Due to the transitivity property 4.4.5 of the Siegel Φ operator, it suffices to say that $F \in M_\kappa(\Gamma_n)$ is a cusp form if it vanishes at the maximal degree cusp (or the $n-1$ degree rational boundary component), and hence this brings us back to the usual definition of cusp forms for $n \geq 2$ which is more commonly found in the literature.

We now discuss the surjectivity of the Siegel Φ operator and we follow [39] and the references listed there. We recall that $M_\kappa(\Gamma_0) = S_\kappa(\Gamma_0) = \mathbb{C}$.

4.4.1.1 Eisenstein Series

The surjectivity of

$$\Phi : M_k(\Gamma_n) \to M_k(\Gamma_{n-1})$$

is guaranteed for some κ if

$$S_k(\Gamma_r) \subset \Phi^{n-r}(M_k(\Gamma_n))$$

is valid for $0 \leq r \leq n$, because then we can deduce by induction (see [39], page 59) that

$$M_k(\Gamma_r) \subset \Phi^{n-r}(M_k(\Gamma_n)).$$

To achieve this, we must lift cusp forms to modular forms of higer degree with respect to Φ , but this is not easy as we must increase the number of variables. The main instrument that will provide a positive answer when κ is even and sufficiently large is the construction of Eisenstein series associated to boundary components by which modular forms can be lifted from a boundary component to all of $\mathfrak{h}^n$.

Definition 4.4.10. Let $n \geq 1$ and let r be a natural number with $0 \leq r \leq n$. Suppose that $F \in S_\kappa(\Gamma_r)$ with even weight $\kappa > 0$. We attach to F the Eisenstein Series associated to the standard rational boundary component of degree r defined by

$$E^{\kappa}_{n,r}(Z;F) := \sum_{\gamma \in C_{n,r}\cap\Gamma_n\backslash\Gamma_n} F(\gamma Z^*)\det(CZ+D)^{-\kappa}$$

where $Z \in \mathfrak{h}^n$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

In case $r = 0$, F is constant, say $F = 1$, we get the classical Eisenstein series

$$E^{\kappa}_{n,r}(Z;1) = \sum_{(C,D)} \det(CZ+D)^{-\kappa}$$

where the summation runs over a full set of representatives for the left cosets $C_{n,0}\cap\Gamma_n\backslash\Gamma_n$. Also in case $r = n$, $C_{n,n} = \Gamma_n$ the Eisenstein series consists of only one term which is $F(Z)$.

Theorem 4.4.11 ([39], Chapter II, Theorem 1)**.** *Let $n \geq 1$, $0 \leq r \leq n$, $\kappa > n+r+1$ and κ is even. For any cusp form $F \in S_\kappa(\Gamma_r)$, the Eisenstein series $E^{\kappa}_{n,r}(Z;F) \in M_\kappa(\Gamma_n)$.*

We now affirm that the Eisenstein series is the right procedure to lift cusp forms with respect to Φ.

Proposition 4.4.12 ([39], Chapter II, Proposition 5)**.** *Let $n \geq 1$, $0 \leq r \leq n$, $\kappa > n+r+1$ and κ is even. For any cusp form $F \in S_\kappa(\Gamma_r)$,*

$$E^{\kappa}_{n,r}(Z;F)|\Phi^{n-r} = F(Z).$$

Corollary 4.4.13. *The Siegel Φ operator*

$$\Phi : M_\kappa(\Gamma_n) \to M_\kappa(\Gamma_{n-1})$$

is surjective for even $\kappa > 2n$.

Remark 4.4.14. From the surjectivity of Φ we deduce the isomorphism $M_\kappa(\Gamma_{n-1}) \cong M_\kappa(\Gamma_n)/S_\kappa(\Gamma_n)$ for even $\kappa > 2n$. This formula helps in obtaining results on $M_\kappa(\Gamma_n)$ from the subspaces of cusp forms by an induction argument. Also any information on the dimension of $M_\kappa(\Gamma_n)$ can be reduced to cusp forms in this way.

Another consequence of the surjectivity of the Siegel Φ operator and Remark 4.4.14, we have

Proposition 4.4.15 ([39], Chapter II, Proposition 6)**.** *For even $\kappa > 2n$ the Eisenstein series $\{E^\kappa_{n,r}(Z;F) \mid F \in S_\kappa(\Gamma_r), 0 \le r \le n\}$ span the vector space of all modular forms of degree n and weight κ.*

For a generalization of this result and the surjectivity of the Siegel Φ operator which is used in the argument that the quotient space $\Gamma \setminus (\mathfrak{h}^n)^*$ can be embedded in a projective space, the reader can consult [18], Exposé 16.

4.4.2 Cusp Forms for Arithmetic Groups

Let $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Q})$ be a commensurable group with $\mathrm{Sp}_{2n}(\mathbb{Z})$ and let $F \in M_\kappa(\Gamma)$. In this section, we give a general definition of cusp forms with respect to Γ. Using the maximal parabolic subgroups $C_{n,r}(\mathbb{Q}) = C_{n,r} \cap \mathrm{Sp}_{2n}(\mathbb{Q})$ of $\mathrm{Sp}_{2n}(\mathbb{Q})$ and the homomorphism ω_r, we can now restate using a better notation the image of the space of modular forms $M_\kappa(\Gamma)$ under the Siegel Φ operator. In addition, by forming a more global map than the Siegel Φ operator we will explain why it is not enough to show that F belongs to the kernel of the Siegel Φ operator to prove it is a cusp form but rather we require also that all its conjugates $F|_\kappa M$ to belong to the kernel of the Siegel Φ operator.

We start by clarifying the image space of the usual Siegel Φ operator.

Proposition 4.4.16. *The image of the Siegel Φ operator is given by*

$$\begin{array}{rcl} \Phi : M_\kappa(\Gamma) & \to & M_\kappa(\omega_{n-1}(\Gamma \cap C_{n,n-1}(\mathbb{Q}))) \\ F & \mapsto & F|\Phi \end{array}.$$

Also the image the j-fold Siegel Φ for $1 \leq j \leq n-1$ is

$$\begin{array}{rcl} \Phi^j : M_\kappa(\Gamma) & \rightarrow & M_\kappa(\omega_{n-j}(\Gamma \cap C_{n,n-j}(\mathbb{Q}))) \\ F & \mapsto & F|\Phi^j \end{array}.$$

To verify this one needs to just rewrite the old notation in terms of the projection map ω_r.

Definition 4.4.17. For any $M \in \mathrm{Sp}_{2n}(\mathbb{Q})$, the conjugates of a Siegel modular form $F \in M_\kappa(\Gamma)$ are $F|_\kappa M \in M_\kappa(\Gamma^M)$ where

$$\Gamma^M := M^{-1}\Gamma M.$$

Applying the Siegel Φ operator on the conjugates of $F \in M_\kappa(\Gamma)$, we get

Proposition 4.4.18. *For $F \in M_\kappa(\Gamma)$,*

$$\begin{array}{rcl} \Phi : M_\kappa(\Gamma^M) & \rightarrow & M_\kappa(\omega_{n-1}(\Gamma^M \cap C_{n,n-1}(\mathbb{Q}))) \\ F|_\kappa M & \mapsto & (F|_\kappa M)|\Phi \end{array}.$$

Also extending the definition of the j-fold Siegel Φ operator we get that for $1 \leq j \leq n-1$,

$$\begin{array}{rcl} \Phi^j : M_\kappa(\Gamma^M) & \rightarrow & M_\kappa(\omega_{n-j}(\Gamma^M \cap C_{n,n-j}(\mathbb{Q}))) \\ F|_\kappa M & \mapsto & (F|_\kappa M)|\Phi^j \end{array}.$$

We now recall the structure of the boundary of compactification of the quotient space $\Gamma \setminus \mathfrak{h}^n$,

$$\Gamma \setminus (\mathfrak{h}^n)^* = \bigcup_{0 \leq r \leq n} \bigcup_{\lambda} \Gamma(M_{r,\lambda}.F_r) \setminus \mathfrak{h}^r,$$

such that

1. $M_{r,\lambda}$ for fixed r are the finitely many representatives of double cosets

$$\Gamma \setminus \mathrm{Sp}_{2n}(\mathbb{Q})/(C_{n,r}(\mathbb{Q}) \cap \mathrm{Sp}_{2n}(\mathbb{Q})),$$

2. $\Gamma(M_{r,\lambda}.F_r) := \omega_r(M_{r,\lambda}^{-1}\Gamma M_{r,\lambda} \cap C_{n,r})$,

3. $\Gamma(F_r) := \omega_r(\Gamma \cap C_{n,r})$.

4. For each degree $0 \leq r \leq n-1$, we have the standard cusp of degree r

$$F_r := \Gamma(F_r) \setminus \mathfrak{h}^r,$$

and all its Γ-conjugates which we call the nonstandard cusps of degree r

$$M_{r,\lambda}.F_r := \Gamma(M_{r,\lambda}.F_r) \setminus \mathfrak{h}^r$$

which correspond to the double cosets representatives of $\Gamma \setminus \mathrm{Sp}_{2n}(\mathbb{Q})/(C_{n,r} \cap \mathrm{Sp}_{2n}(\mathbb{Q}))$.

Unlike the case of the Siegel modular group $\Gamma_n = \mathrm{Sp}_{2n}(\mathbb{Z})$ where the quotient space $\Gamma_n \setminus \mathfrak{h}^n$ has a unique cusp (the standard cusp) for each degree $0 \leq r \leq n-1$, with respect to a commensurable group $\Gamma \in \mathrm{Sp}_{2n}(\mathbb{Q})$, the quotient space $\Gamma \setminus \mathfrak{h}^n$ has finitely many cusps for each degree $0 \leq r \leq n-1$ corresponding to double cosets $\Gamma \setminus \mathrm{Sp}_{2n}(\mathbb{Q})/(\mathrm{Sp}_{2n}(\mathbb{Q}) \cap C_{n,r})$.

From this point of view, we will now attach to each cusp F_i of the quotient space $\Gamma \setminus (\mathfrak{h}^n)^*$ an associated linear operator Φ_{F_i}, but we always reserve the notation Φ and Φ^j to be the usual Siegel Φ operator and j-fold Siegel operator respectively. When $F = M.F_r$, then we denote Φ_F by $\Phi_{M.F_r}$.

Proposition 4.4.19. *Let $F \in M_\kappa(\Gamma)$, then*

1. *The Φ operator associated to the maximal (degree $n-1$) standard cusp $F_{n-1} := \Gamma(F_{n-1}) \setminus \mathfrak{h}^{n-1}$ is*

$$\Phi_{F_{n-1}} := \Phi.$$

$\Phi_{F_{n-1}}$ projects the Siegel modular form F to the standard maximal cusp F_{n-1} such that

$$F|\Phi \in M_\kappa(\Gamma(F_{n-1})).$$

2. *For the remaining nonstandard finitely many cusps of maximal degree (degree $n-1$) that are Γ-conjugate to F_{n-1} and that correpsond to the double cosets representatives $M_{n-1,\lambda}$, the associated linear Φ operator denoted by $\Phi_{M_{n-1,\lambda}.F_{n-1}}$ is given by*

$$F|\Phi_{(M_{n-1,\lambda}.F_{n-1})} = (F|_k M_{n-1,\lambda})|\Phi.$$

   It projects the Siegel modular form F to a maximal nonstandard cusp such that

$$F|\Phi_{(M_{n-1,\lambda}.F_{n-1})} \in M_\kappa(\Gamma(M_{n-1,\lambda}.F_{n-1})).$$

3. *For any standard lower degree cusp F_r for $0 \leq r \leq n-2$, the associated Φ operator is given by*

$$\Phi_{F_r} := \Phi^{n-r}$$

   that projects F to the standard cusp of degree r, $\Gamma(F_r) \setminus \mathfrak{h}^r$, such that

$$F|\Phi \in M_\kappa(\Gamma(F_r)).$$

4. *For all the remaining nonstandard finitely many cusps of degree r that are Γ-conjugate to F_r and that correspond to the double coset representatives $M_{r,\lambda}$, the associated Φ operator denoted by $\Phi_{M_{r,\lambda}.F_r}$ is given by*

$$F|\Phi_{(M_{r,\lambda}.F_r)} = (F|_k M_{r,\lambda})|\Phi^{n-r}.$$

   It projects the Siegel modular form F to the cusp of degree r, $M_{r,\lambda}.F_r := \Gamma(M_{r,\lambda}.F_r)\setminus\mathfrak{h}^r$ such that

$$F|\Phi_{(M_{r,\lambda}.F_r)} \in M_\kappa(\Gamma(M_{r,\lambda}.F_r))$$

We now give a definition of a linear for each degree r which associates to each $F \in M_\kappa(\Gamma)$ its extensions to all cusps of fixed degree r.

Definition 4.4.20. To each degree r, $0 \leq r \leq n-1$, we define a linear operator

$$\overrightarrow{\Phi_r} = (\Phi^{n-r}, (\Phi_{(M_{r,\lambda}.F_r)})_\lambda)$$

where $M_{r,\lambda}.F_r$ are the nonstandard cusps of degree r that correspond to the double cosets representatives $M_{r,\lambda} \in \Gamma \backslash \mathrm{Sp}_{2n}(\mathbb{Q})/(C_{n,r} \cap \mathrm{Sp}_{2n}(\mathbb{Q}))$. It is then clear that the linear operator $\overrightarrow{\Phi_r}$ associates to F its projections to all cusps of degree r such that

$$\begin{array}{rcl} \overrightarrow{\Phi_r} : M_\kappa(\Gamma) & \to & M_{k,r} := \prod_\lambda M_\kappa(\Gamma(M_{r,\lambda}.F_r)) \\ F & \mapsto & F|\overrightarrow{\Phi_r} = (F|\Phi^{n-r}, \left(F|\Phi_{(M_{r,\lambda}.F_r)}\right)_\lambda) = (F|\Phi^{n-r}, ((F|_k M_{r,\lambda})|\Phi^{n-r})_\lambda) \end{array}$$

Remark 4.4.21. When $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$, we have one cusp for each degree $0 \leq r \leq n-1$. In this case, $\overrightarrow{\Phi_r} = \Phi^{n-r}$.

Having defined $\overrightarrow{\Phi_r}$ for all cusps of same degree, we now define the general linear operator that takes care of cusps of all degrees.

Definition 4.4.22. The general operator for all cusps (of all degrees) is defined by

$$\overrightarrow{\Phi} = (\overrightarrow{\Phi_r})_{0 \leq r \leq n-1}.$$

such that

$$\begin{array}{rcl} \overrightarrow{\Phi} : M_\kappa(\Gamma) & \to & \prod_r M_{k,r} \\ F & \mapsto & F|\overrightarrow{\Phi} = (F|\overrightarrow{\Phi_r})_r. \end{array}$$

Remark 4.4.23. When $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$, $\overrightarrow{\Phi}$ reduces to the operator defined in 4.4.7.

Definition 4.4.24. A Siegel modular form $F \in M_\kappa(\Gamma)$ is a cusp form if it vanishes at all cusps. Equivalently, $F \in S_\kappa(\Gamma)$ if and only if

$$F|\overrightarrow{\Phi} = 0.$$

Remark 4.4.25. 1. Going back to Definition 4.4.20 of the general operators we defined,

$F \in S_\kappa(\Gamma)$ if and only if for all double cosets representatives $M_{r,\lambda}$ of the double coset space $\Gamma \setminus \mathrm{Sp}_{2n}(\mathbb{Q})/(\Gamma^{M_{r,\lambda}} \cap C_{n,r}(\mathbb{Q}))$ for every $0 \leq r \leq n-1$, $F|_\kappa M_{r,\lambda}$ vanishes at the cusps.

2. When $F \in M_\kappa(\Gamma)$, for Γ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we have only degree zero cusps. That is $\overrightarrow{\Phi} = \overrightarrow{\Phi_0} = (\Phi, (\Phi_{(M_\lambda . i\infty)})_\lambda)$, where $i\infty$ is the standard cusp and $M_{\lambda . i\infty}$ are the nonstandard cusps which are Γ-conjugate to $i\infty$.

One can transfer the transitivity property 4.4.5 of the Siegel Φ operator to become a transitivity property for the operator $\overrightarrow{\Phi}_r$. Consequently, we have

Proposition 4.4.26. $F \in S_\kappa(\Gamma)$ *if and only if it vanishes at all maximal cusps of degree* $n-1$*, which is equivalent to say* $F \in S_\kappa(\Gamma)$ *if and only if*

$$F|\overrightarrow{\Phi}_{n-1} = 0.$$

Remark 4.4.27. Proposition 4.4.26 is equivalent to say $F \in S_\kappa(\Gamma)$ if and only if

$$(F|_\kappa M_\lambda)|\Phi = 0,$$

for all $M_\lambda \in \Gamma \setminus \mathrm{Sp}_{2n}(\mathbb{Q})/(\mathrm{Sp}_{2n}(\mathbb{Q}) \cap C_{n,n-1})$.

4.5 Satake Compactifications of Locally Symmetric Spaces

The goal of this section is to generally describe Satake's procedure in compactifying more general spaces than the Siegel varieties. These general spaces are called locally symmetric spaces (see below). In the appendix, we present a more general approach given by Borel and Ji to compactifying these spaces and give a brief description of the rational boundary components in this setting.

Definition 4.5.1. Let G be a connected semisimple Lie group and K a maximal compact subgroup of G. If the homogeneous space

$$X = G/K$$

is endowed with a G-invariant metric, then X is said to be a symmetric space of noncompact type. In this setting, $G = \mathrm{Aut}\,(X)$. Let Γ be an arithmetic subgroup of G acting properly discontinuously on X, then

$$\Gamma \setminus X = \Gamma \setminus G/K$$

is called the associated noncompact locally symmetric space.

We have seen in Chapter 2 that the Siegel upper half space $\mathfrak{h}^n$ can be realized as the homogeneous space $\mathrm{Sp}_{2n}(\mathbb{R})/K$ where K is the unitary group of order n, a maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$. Moreover, we can endow $\mathfrak{h}^n$ with an invariant Riemannian metric (check [39], Chapter 1). This makes $\mathfrak{h}^n$ a symmetric space of noncompact type and Siegel varieties $\Gamma \setminus \mathfrak{h}^n$ locally symmetric spaces. Locally symmetric spaces are important for classification problems in geometry. They arise for example as moduli spaces of elliptic curves or of abelian varities with a polarization. These spaces are not compact in general.

Satake ([68]) started the modern theory of topological compactifications of general symmetric spaces and locally symmetric spaces. In fact, Satake provided finitely many nonisomorphic compactifications on these spaces that form a partially ordered set. The minimal one is the one which corresponds to the compactification of Siegel varieties $\Gamma \setminus \mathfrak{h}^n$. To get a brief outline of how Satake proceeded we refer the reader to the appendix of this thesis.

One class of symmetric spaces consists of the so called Hermitian symmetric spaces. These are symmetric spaces with a G-invariant complex structure. When X is a Hermitian symmetric space of noncompact type, $\Gamma\backslash X$ is a noncompact Hermitian locally symmetric space and hence equipped with a complex structure. When X is a Hermitian symmetric space, $\Gamma \setminus X$ has a complex structure and hence once can talk about meromorphic functions. Siegel raised the question whether the transcendental degree of the field of meromorphic functions of $\Gamma \setminus X$, $M(\Gamma \setminus X)$, is equal to $\dim_{\mathbb{C}}(\Gamma \setminus X)$. Now if $\Gamma \setminus X$ is compact, then Kodaira's embedding theorem implies that $\Gamma \setminus X$ is a projective variety and hence the transcendental degree of $M(\Gamma \setminus X)$ is equal to the complex dimension of $\Gamma \setminus X$. If $\Gamma \setminus X$ is noncompact but admits a compactification $(\Gamma \setminus X)^c$ which is a normal projective variety and the codimension of the boundary $(\Gamma \setminus X)^c - \Gamma \setminus X$ is of complex dimension at least two, or equivalently $\dim_{\mathbb{C}}((\Gamma \setminus X)^c - \Gamma \setminus X) \leq \dim_{\mathbb{C}}(\Gamma \setminus X) - 2$, then facts about normal spaces imply that every meromorphic function on $\Gamma \setminus X$ extends to a meromorphic function on the projective variety $(\Gamma \setminus X)^c$. Consequently, $M(\Gamma \setminus X)$ is an algebraic function field and each meromorphic function $F \in M(\Gamma \setminus X)$ is a quotient of two modular forms of the same weight.

Motivated to solve this question which was solved by Baily in case of $\mathrm{Sp}_{2n}(\mathbb{Z}) \setminus \mathfrak{h}^n$ (see Theorem 4.1.2), Baily and Borel ([9]) used the minimal Satake topology to construct the compactification of locally Hermitian symmetric spaces $\Gamma \setminus X$. Then they were able to prove that such compactification is a normal projective variety and hence answering Siegel's question. We refer to the compactification of $\Gamma \setminus X$ with the normal projective structure as the Satake-Baily-Borel compactification. Here are the general three steps used to achieve the Satake-Baily-Borel compactification.

1. Apply the general procedure in [68] to construct a topological compactification of $\Gamma \setminus X$ using the minimal Satake compactification of X.

2. Define a sheaf of analytic functions on the topological compactification to turn it into a compact normal analytic space.

3. Embed the compactification into a complex projective space as a normal projective variety by using automorphic forms.

Remark 4.5.2. We note that in [9], Baily and Borel did not use the minimal Satake compactification of X as exactly described by Satake in [69] but used the fact that if X is a Hermitian symmetric space of noncompact type then X is biholomorphic to a bounded symmetric domain. This is the so-called Harish-Chandra realization of X as a bounded domain D. Realizing X as a bounded symmetric domain provides a natural compactification of X by taking the closure of D. Moore showed ([53]) that this natural compactification of X is isomorphic to the minimal Satake compactification of X.

Chapter 5

Jacobi Forms

Jacobi forms are the automorphic forms on the Jacobi group. The classical Jacobi forms of degree 1 are the functions $\phi : \mathfrak{h}^1 \times \mathbb{C} \to \mathbb{C}$ which are holomorphic in both variables $\tau \in \mathfrak{h}^1$ and $z \in \mathbb{C}$ and invariant under the action of the Jacobi group. They are a mixture of elliptic modular forms and elliptic functions. Eichler and Zagier have developed a systematic theory of Jacobi forms along the line of the theory of elliptic modular forms in their joint book [21], which is the standard reference for Jacobi forms. Understanding Jacobi forms is crucial to the understanding of other types of more complicated modular forms like the Siegel modular forms of one degree higher. In fact, there are a number of well-known isomorphisms between spaces of Jacobi forms and other types of modular forms. In more general settings, Jacobi forms can be defined for higher degree n as in [87]. We limit our exposition here to the degree 1 case, which serves as a bridge between the space of elliptic modular forms and Siegel modular forms of degree two.

5.1 Basics of Jacobi forms

In this section we give the basics of Jacobi forms as presented in [21]. We reserve the notations τ and z for the variables in $\mathfrak{h}^1$ and $\mathbb{C}$ respectively.

We start with the real Jacobi group acting on the domain of definition $\mathfrak{h}^1 \times \mathbb{C}$

of Jacobi forms under which these forms satisfy transformation formulas. To discuss the arithmetic theory, we then consider the action of discrete Jacobi groups.

5.1.1 The Real Jacobi Group

We start by defining the Heisenberg group over any commutative ring R with identity.

Definition 5.1.1. Define the Heisenberg group $H(R)$ by

$$H(R) = \{(\lambda, \mu; \varkappa) : (\lambda, \mu) \in R^2, \varkappa \in R\}$$

with the group law

$$(\lambda, \mu; \varkappa)(\lambda', \mu'; \varkappa') = (\lambda + \lambda', \mu + \mu'; \varkappa + \varkappa' + \lambda\mu' - \lambda'\mu).$$

We sometimes denote the elements of the Heisenberg group as $(X, \varkappa)$ where $X = [\lambda, \mu]$.

The subgroup

$$Z_{H(R)} = \{(0, 0; \varkappa), \varkappa \in R\}$$

is the center of $H(R)$. We have

$$H(R)/Z_{H(R)} \cong R^2.$$

By identifying $(0, 0; \varkappa)$ with $\varkappa$, the center $Z_{H(R)}$ can be identified with R and we get the following short exact sequence

$$1 \to R \to H(R) \to R^2 \to 1$$

giving the Heisenberg group $H(R)$ as the central extension of R^2 by R. As a set we have $H(R) = R^2 \times R$.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $H(\mathbb{R})$ on the right by

$$(X, \varkappa)\gamma = (X\gamma, \varkappa), \qquad \gamma \in \mathrm{SL}_2(\mathbb{R}), \tag{5.1}$$

where $X\gamma$ means a 1×2 row vector multiplied by a matrix in $\mathrm{SL}_2(\mathbb{R})$.

We now define the real Jacobi group of degree one, $\mathrm{SL}_2(\mathbb{R})^J$, to be the semidirect product of $\mathrm{SL}_2(\mathbb{R})$ with the Heisenberg group $H(\mathbb{R})$.

Definition 5.1.2. The real Jacobi group of degree one $\mathrm{SL}_2(\mathbb{R})^J$ is defined to be the set

$$\mathrm{SL}_2(\mathbb{R})^J := \mathrm{SL}_2(\mathbb{R}) \ltimes H(\mathbb{R}) = \{\eta := (\gamma, X, \varkappa) : \gamma \in \mathrm{SL}_2(\mathbb{R}), (X = [\lambda, \mu], \varkappa) \in H(\mathbb{R})\}$$

with the group multiplication

$$(\gamma, X, \varkappa).(\gamma', X', \varkappa') = \left(\gamma\gamma', X\gamma' + X', \varkappa + \varkappa' + \det\begin{pmatrix} X\gamma' \\ X' \end{pmatrix}\right).$$

The real Jacobi group $\mathrm{SL}_2(\mathbb{R})^J$ acts on $\mathfrak{h}^1 \times \mathbb{C}$ by

$$\eta(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right)$$

where $\eta = (\gamma, X, \varkappa)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Fixing integers κ and m, the action of the real Jacobi group $\mathrm{SL}_2(\mathbb{R})^J$ on functions $\{\phi : \mathfrak{h}^1 \times \mathbb{C} \to \mathbb{C}\}$ is given by the following slash operators:

$$(\phi|_{\kappa,m}\gamma)(\tau, z) := (c\tau + d)^{-\kappa} e^{2\pi i m\left(\frac{-cz^2}{c\tau+d}\right)} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \tag{5.2}$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, and

$$(\phi|_m(\lambda, \mu, \varkappa))(\tau, z) := e^{2\pi i m(\lambda^2\tau + 2\lambda z + \lambda\mu + \varkappa)}\phi(\tau, z + \lambda\tau + \mu) \tag{5.3}$$

for every $(\lambda, \mu, \varkappa) \in H(\mathbb{R})$.

Remark 5.1.3. Note that for the later goal of defining certain index shifting operators on Jacobi forms, we can actually define a larger action of $\mathrm{GL}_2^+(\mathbb{R})$ on functions living on $\mathfrak{h}^1 \times \mathbb{C}$ by

$$(\phi|_{\kappa,m}\gamma)(\tau, z) := (c\tau + d)^{-\kappa} e^{2\pi i m l\left(\frac{-cz^2}{c\tau+d}\right)} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{lz}{c\tau + d}\right) \tag{5.4}$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ with $\det \gamma = l > 0$.

Given the action of $\mathrm{SL}_2(\mathbb{R})$ on the Heisenberg group as in equation (6.2.4), the following relations

$$(\phi|_{\kappa,m}\gamma)|_{\kappa,m}\gamma' = \phi|_{\kappa,m}(\gamma\gamma'),$$

$$(\phi|_m X)|_m X' = \phi|_m(X + X'),$$

$$(\phi|_{\kappa,m}\gamma)|_m X\gamma = (\phi|_m X)|_{\kappa,m},$$

for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{R})$ and $X, X' \in \mathbb{R}^2$ show that the slash operators jointly define an action of a general element $\eta := \left(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (X = (\lambda, \mu), \varkappa)\right)$ of the real full Jacobi group $\mathrm{SL}_2(\mathbb{R})^J$ on functions $\{\phi : \mathfrak{h}^1 \times \mathbb{C} \to \mathbb{C}\}$. A general element of the Jacobi group acts first by the matrix and then by the vector. We define the following factor of automorphy:

$$j(\eta, (\tau, z)) = (c\tau + d)^{-\kappa} e^{2\pi i m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu + \varkappa\right)}. \tag{5.5}$$

The joint action of $\mathrm{SL}_2(\mathbb{R})^J$ on $\{\phi : \mathfrak{h}^1 \times \mathbb{C} \to \mathbb{C}\}$ is then given by

$$(\phi|_{\kappa,m}\eta)(\tau, z) := j(\eta, (\tau, z))\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right) \tag{5.6}$$

for every $\eta := \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (X = (\lambda, \mu), \varkappa)\right) \in \mathrm{SL}_2(\mathbb{R})^J$.

5.1.2 Jacobi Forms

To discuss the theory of Jacobi forms, we need to consider discrete subgroups of the real Jacobi group.

Definition 5.1.4. The full Jacobi group is the semidirect product $\mathrm{SL}_2(\mathbb{Z})^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$. For any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ of finite index, we call $\Gamma^J = \Gamma \ltimes H(\mathbb{Z})$ a general Jacobi group. In particular, when Γ is the congruence subgroup $\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c \equiv 0 \pmod{M} \right\}$, the Jacobi group $\Gamma_0(M)^J = \Gamma_0(M) \ltimes H(\mathbb{Z})$ is called the Jacobi group with level $M \geq 1$.

We now consider holomorphic functions $\phi : \mathfrak{h}^1 \times \mathbb{C} \to \mathbb{C}$ invariant under the actions given by equations (5.2) and (5.3) above. That is,

$$\begin{aligned} \phi|_{\kappa,m}\gamma &= \phi, \quad \text{for every } \gamma \in \mathrm{SL}_2(\mathbb{Z}) \\ \phi|_m\eta &= \phi, \quad \text{for every } \eta = (X, \varkappa) \in H(\mathbb{Z}). \end{aligned} \tag{5.7}$$

For the smallest positive integer h such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ where Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, if we apply the transformation formulas (5.7) with $\gamma = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ and $\eta = (0, 1, 0)$ to ϕ we get that

$$\phi(\tau + h, z) = \phi(\tau, z + 1) = \phi(\tau, z).$$

Hence ϕ is periodic with period h with respect to the variable τ and with period 1 with respect to the variable z and therefore possesses a Fourier expansion with respect to both τ and z

$$\phi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n/h, r)e^{2\pi i(n/h)\tau}e^{2\pi irz}.$$

Remark 5.1.5. If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ or $\Gamma_0(M)$, then $h = 1$ and consequently a holomorphic function on $\mathfrak{h}^1 \times \mathbb{C}$ satisfying equations (5.7) has a Fourier expansion of the form $\phi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n, r)e^{2\pi in\tau}e^{2\pi irz}$.

We now define Jacobi forms for congruence subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 5.1.6. Let κ, m be integers. A Jacobi form of weight κ and index m on a congruence subgroup Γ is a holomorphic function $\phi : \mathfrak{h}^1 \times \mathbb{C} \to \mathbb{C}$ such that

1. $\phi|_{\kappa,m}\gamma = \phi$, for every $\gamma \in \Gamma$,
2. $\phi|_m X = \phi$, for every $X \in \mathbb{Z}^2$,
3. for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $\phi|_{\kappa,m}\gamma$ has the Fourier expansion of the form

$$\phi(\tau, z)|_{\kappa,m}\gamma = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn/h-r^2\geq 0}} c_\gamma(n/h, r)e^{2\pi i(n/h)\tau}e^{2\pi i(rz)}$$

where h is defined as above.

Remark 5.1.7. The condition $r^2 \leq 4mn/h$ is a condition to make the function analytic as $\tau \to i\infty$.

Definition 5.1.8. If ϕ satisfies the additional condition that $c_\gamma(n/h, r) = 0$ whenever $4mn/h = r^2$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ then ϕ is a cusp form.

The vector space of Jacobi forms with respect to Γ^J is denoted by $J_{\kappa,m}(\Gamma^J)$. When $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we simply denote it by $J_{\kappa,m}$. The space of cusp forms with respect to Γ^J is denoted by $J^c_{\kappa,m}(\Gamma^J)$. When $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, it is simply denoted by $J^c_{\kappa,m}$.

Remark 5.1.9. Note that because $e^{2\pi i(\lambda\mu+\varkappa)} = 1$ for $(\lambda,\mu,\varkappa) \in H(\mathbb{Z})$, equation (5.3) reduces to the action

$$(\phi|_m[\lambda,\mu])(\tau,z) := e^{(2\pi im)(\lambda^2\tau+2\lambda z)}\phi(\tau, z+\lambda\tau+\mu)$$

for every $(\lambda,\mu) \in \mathbb{Z}^2$. Consequently, the definition of Jacobi forms with respect to the Jacobi group $\Gamma_0(M)^J := \Gamma_0(M) \ltimes H(\mathbb{Z})$ is equivalent to the definition of Jacobi forms with respect to the group $\Gamma_0(M) \ltimes \mathbb{Z}^2$. The latter is consistently used in the literature. To discuss the arithmetic theory, it is enough to use $\Gamma_0(M) \ltimes \mathbb{Z}^2$. We will do the same below.

Remark 5.1.10.

1. Restricting the variable z to $z = 0$ and considering the first transformation formula of Jacobi forms, we get the classical elliptic modular transformation formula
$$\phi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^\kappa\phi(\tau).$$
Hence the form $\phi(\tau, 0)$ is a classical modular form of the same weight κ.

2. The second transformation fromula in (5.7) expresses an invariance under translations by the integer lattice spanned by $\{1,\tau\}$.

3. Interesting Jacobi forms occur for positive index only. For fixed $\tau \in \mathfrak{h}^1$, a Jacobi form $\phi(\tau,z)$ as a function of z is an elliptic function on $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ whose number of zeros (counting multiplicity) in any fundamental domain of the action of the lattice $\mathbb{Z}\tau+\mathbb{Z}$ on $\mathbb{C}$ is equal to $2m$ (See [21], Theorem 1.2). Hence there are no holomorphic Jacobi forms of negative index. Also, a Jacobi form of index $m = 0$ is independent of z and becomes an elliptic modular form in τ of weight κ.

4. Let λ,μ be rational numbers. We can associate to the Jacobi form ϕ the function $f(\tau) = e^{2\pi im\lambda^2\tau}\phi(\tau,\lambda\tau+\mu)$ which is a modular form of weight κ with respect to some subgroup Γ' of finite index depending on Γ and on λ,μ. (For details see [21], Theorem 1.3).

5. The interesting weight of Jacobi forms is nonnegative as $J_{\kappa,m}$ is the zero vector space for $\kappa \leq 0$ unless $\kappa = m = 0$ in which case it reduces to constants. ([21], page 11).

The first examples of Jacobi forms are Jacobi theta series (see § 5.2.1 below). These were first studied by Jacobi and is the reason the forms defined above are referred to as Jacobi forms. They are given by

$$\theta_{m,\delta}(\tau, z) := \sum_{r \in \mathbb{Z}, r \equiv \delta \pmod{2m}} e^{2\pi i \left(\frac{r^2}{4m}\tau + rz\right)},$$

where m is a positive integer and δ an integer modulo $2m$. These forms are elements in $J_{\frac{1}{2},m}(\Gamma_0(4m)^J)$. Jacobi forms of half integral weight will not be defined in this thesis but one can check the work of Shimura [78] on half integeral weight modular forms for a reference. These Jacobi theta series turn out to be not only the first examples of Jacobi forms but in fact they span the space of Jacobi forms (See § 5.2.1). For more general examples of theta series, the reader can consult [21], § 7.

5.1.3 Structural Theorems

Theorem 5.1.11 ([21], Theorem 1.1)**.** *For fixed κ, m, the space $J_{\kappa,m}(\Gamma^J)$ is finite dimensional as a vector space over $\mathbb{C}$.*

The product of two Jacobi forms of weight κ_1 and κ_2 and index m_1 and m_2 is a Jacobi form of weight $\kappa = \kappa_1 + \kappa_2$ and index $m = m_1 + m_2$. In fact we have the following theorem.

Theorem 5.1.12 ([21], Theorem 1.5)**.** *The Jacobi forms form a bigraded ring $J_{*,*} = \bigoplus_{\kappa,m} J_{\kappa,m}$.*

Let $E_4(\tau) = 1 + 240 \sum_{n>0} \sigma_3(n) e^{2\pi i(n\tau)}$ and $E_6(\tau) = 1 - 504 \sum_{n>0} \sigma_5(n) e^{2\pi i(n\tau)}$ be the classical elliptic Eisenstein series of weight 4 and 6 respectively with respect to $\mathrm{SL}_2(\mathbb{Z})$, where $\sigma_d(n)$ is the sum of positive divisors function. Let $M_\kappa(\mathrm{SL}_2(\mathbb{Z}))$ denote the complex vector space of elliptic modular forms of weight κ and level 1. The difference between the bigraded ring $J_{*,*}$ and the graded ring $M_* := \bigoplus_\kappa M_\kappa(\mathrm{SL}_2(\mathbb{Z}))$ of elliptic modular forms is that while the latter is a finite dimensional graded ring over $\mathbb{C}$ with the well-known classical isomorphism $M_* \cong \mathbb{C}[E_4, E_6]$, the latter is not finitely generated over $\mathbb{C}$ (see Theorem 5.1.17).

Consider the subgroup $\mathrm{SL}_2(\mathbb{Z})^J_\infty$ of $\mathrm{SL}_2(\mathbb{Z})^J$ defined by:

$$\mathrm{SL}_2(\mathbb{Z})^J_\infty = \left\{ \eta = \left[\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right] \middle| \, n, \mu \in \mathbb{Z} \right\}.$$

Definition 5.1.13. Given integers $m \geq 0$ and $\kappa \geq 4$, we define the Jacobi-Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$, $E_{\kappa,m}$, by

$$\begin{aligned} E_{\kappa,m}(\tau, z) &:= \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})^J_\infty \backslash \mathrm{SL}_2(\mathbb{Z})^J} (1|_{\kappa,m}\gamma)(\tau, z) \\ &= \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-\kappa} e^{2\pi i m \left(\lambda^2 \frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d}\right)}. \end{aligned}$$

Theorem 5.1.14 ([21], page 17 and Theorem 2.1)**.** *The Jacobi-Eisenstein series $E_{\kappa,m} \in J_{k,m}$.*

Remark 5.1.15. We note that when $m = 0$, the definition of the Jacobi-Eisenstein series $E_{\kappa,m}$ reduces to the well-known elliptic Eisenstein series

$$E_\kappa(\tau) := \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (1|_\kappa \lambda)(\tau) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} (c\tau + d)^{-\kappa},$$

where $\mathrm{SL}_2(\mathbb{Z})_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| \, n \in \mathbb{Z} \right\}$ is the parabolic subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Since elliptic modular forms can be considered Jacobi forms of zero index, if we multiply a Jacobi form by an elliptic modular form we obtain a Jacobi form with same index but different weight. It turns out that the space $J_{*,m} := \oplus_{\kappa \in \mathbb{Z}} J_{\kappa,m}$ is a module over the ring $M_* = \mathbb{C}[E_4, E_6]$. In fact, it is a free module of rank $2m$ (see [21], page 90) and reduces to M_* if $m = 0$. In case $m = 1$, the two generators are given as follows.

Theorem 5.1.16 ([21], Theorem 3.5)**.** *The space of Jacobi forms of index 1 on* $\mathrm{SL}_2(\mathbb{Z})$*,* $J_{*,1}$*, is a free module of rank 2 over* M_* *with generators the Jacobi-Eisenstein series* $E_{4,1}$ *and* $E_{6,1}$*.*

The structure theorem of the bigraded ring of Jacobi forms is given by:

Theorem 5.1.17 ([21], Theorem 8.4)**.** *The ring of Jacobi forms* $J_{*,*}$ *is free as a module over* M_* *of transcendence degree* 2 *over* M_* *and is an infinitely generated ring over* $\mathbb{C}$ *of transcendence degree 4.*

5.1.4 Petersson Scalar Product

Writing $\tau = u + iv$ $(v > 0)$ and $z = x + iy$ we have a volume element dV on $\mathfrak{h}^1 \times \mathbb{C}$ given by

$$dV := v^{-3} dx dy du dv.$$

This is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})^J$ on $\mathfrak{h}^1 \times \mathbb{C}$.

Definition 5.1.18. If ϕ and ψ transform like Jacobi forms of weight κ and index m then the expression

$$v^{\kappa} e^{-4\pi m y^2/v} \phi(\tau, z) \overline{\psi(\tau, z)}$$

is invariant under $\mathrm{SL}_2(\mathbb{Z})^J$. The Petersson scalar product of ϕ and ψ is defined by

$$\langle \phi, \psi \rangle := \int_{\mathrm{SL}_2(\mathbb{Z})^J \backslash \mathfrak{h}^1 \times \mathbb{C}} v^k e^{-4\pi m y^2/v} \phi(\tau, z) \overline{\psi(\tau, z)} dV$$

Theorem 5.1.19 ([21], Theorem 2.5)**.** *The Petersson scalar product of* ϕ *and* ψ *is well defined and converges when at least one of* ϕ *and* ψ *is a cusp form. It is positive definite on the space of Jacobi cusp forms.*

The proof actually follows from a certain expansion of a Jacobi form in terms of elliptic modular forms h_δ called the theta expansion of ϕ (refer to § 5.2.1) and from the corresponding statements for those elliptic modular forms (see Theorem 5.2.3).

5.1.5 Linear operators

For $l > 0$ Eichler and Zagier ([21], page 41) define linear operators U_l, V_l on functions $\phi : \mathfrak{h}^1 \times \mathbb{C} \to \mathbb{C}$ as follows:

1. $$U_l : J_{\kappa,m} \to J_{\kappa,ml^2}$$

   given by

   $$(\phi|_{\kappa,m} U_l)(\tau, z) = \phi(\tau, lz)$$

2. $$V_l : J_{\kappa,m} \to J_{\kappa,ml}$$

   given by

   $$(\phi|_{k,m} V_l)(\tau, z) = l^{\kappa-1} \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \backslash M_2^l} (\phi|_{k,t}\gamma)(\tau, z),$$

where
$$M_2^l = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}), ad - bc = l \right\}.$$

Theorem 5.1.20 ([21], Theorem 4.2)**.** *Let $\phi \in J_{\kappa,m}$ with Fourier expansion $\phi(\tau, z) = \sum_{n,r} c(n,r) e^{2\pi i(n\tau)} e^{2\pi i(rz)}$. Then,*

$$\phi | U_l(\tau, z) = \sum_{n,r} c(n, r/l) e^{2\pi i(n\tau)} e^{2\pi i(rz)}$$

with the convention $c(n, r/l) = 0$ *if* $l \nmid r$ *and*

$$\phi|V_l(\tau, z) = \sum_{n,r} \left(\sum_{a|\gcd(n,r,l)} a^{\kappa-1} c\left(\frac{nl}{a^2}, \frac{r}{a}\right) \right) e^{2\pi i(n\tau)} e^{2\pi i(rz)}.$$

In addition, the operators U_l *and* V_l *satisfy the following relations:*

$$U_l \circ U_{l'} = U_{ll'},$$

$$U_l \circ V_{l'} = V_{l'} \circ U_l,$$

$$V_l \circ U_{l'} = \sum_{d|\gcd(l,l')} d^{\kappa-1} U_d \circ V_{ll'/d^2}.$$

Proof. The proof of the Fourier expansion $\phi|U_l$ is clear from the definition. The action of the V_l operators on the Fourier coefficients is obtained by taking the standard set of representatives

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, a, d, > 0, 0 \le b \le d - 1, ad = l$$

for the action of $\mathrm{SL}_2(\mathbb{Z})$ on M_2^l. For the remaining details, one can find them on page 43 in [21]. $\square$

Remark 5.1.21. The index shifting operators V_l ($l \ge 1$) on $J_{\kappa,m}$ play a crucial role in establishing the full level Maass lifting and Gritsenko lifting (see Section 5.3 below).

5.2 Connections of Jacobi Forms to Other Types of Modular Forms

5.2.1 Theta expansion

The second transformation formula for Jacobi forms breaks the Fourier coefficients into congruence classes modulo $2m$.

Theorem 5.2.1 ([21], Theorem 2.2)**.** *Let ϕ be a Jacobi form of weight κ and index m in $J_{\kappa,m}$ with Fourier development $\sum_{n,r,4mn\geq r^2} c(n,r)e^{(2\pi i n\tau)}e^{(2\pi i r z)}$. Then the coefficients $c(n,r)$ depend only on the discriminant r^2-4nm and on the value of $r \pmod{2m}$. That is, $c(n,r)=c_r(4nm-r^2)$ where $c_{r'}(N)=c_r(N)$ for $r'\equiv r \mod 2m$. If κ is even and $m=1$ or m is prime, then $c(n,r)$ depends only on $4nm-r^2$. If $m=1$ and κ is odd then ϕ is identically zero.*

Proof. For any $X=[\lambda,\mu]\in\mathbb{Z}^2$,

$$\begin{aligned}
\phi(\tau,z) &= \sum_{n,r,4mn\geq r^2} c(n,r)e^{2\pi i n\tau}e^{2\pi i r z} \\
&= e^{(2\pi i m(\lambda^2\tau+2\lambda z))}\phi(\tau,z+\lambda\tau+\mu) \\
&= e^{2\pi i(\lambda^2 m)\tau}e^{2\pi i(2\lambda m)z}\sum_{n,r,4mn\geq r^2} c(n,r)e^{(2\pi i n\tau)}\left(e^{(2\pi i z)}e^{2\pi i\lambda\tau}\right)^r \\
&= \sum_{n,r,4mn\geq r^2} c(n,r)e^{2\pi i(n+\lambda^2 m+\lambda r)\tau}e^{2\pi i(r+2\lambda m)z}.
\end{aligned}$$

Hence,

$$c(n,r)=c(n+\lambda^2 m+\lambda r, r+2\lambda m).$$

That is, $c(n,r)=c(n',r')$ whenever $r'\equiv r \mod (2m)$ and $4n'm-r'^2=4m(\lambda^2 m+\lambda r+n)-(r+2\lambda m)^2=4nm-r^2$. Consequently, if we write

$$c(n,r)=c_r(4mn-r^2),$$

then

$$c_r(N)=c_{r'}(N) \quad \text{for} \;\; r\equiv r' \pmod{2m}. \tag{5.8}$$

Equation (5.8) gives us coefficients $c_\delta(N)$ for all $\delta\in\mathbb{Z}/2m\mathbb{Z}$ and all integers $N\geq 0$ satisfying $N\equiv -\delta^2 \pmod{4m}$. Namely,

$$c_\delta(N):=c\left(\frac{N+r^2}{4m},r\right), \quad (r\in\mathbb{Z},\;\; r\equiv\delta \pmod{2m}). \tag{5.9}$$

This proves the first statement in the theorem. If κ is even, then by applying the first transformation law of Jacobi forms to $-I_2 \in \mathrm{SL}_2(\mathbb{Z})$, we get that $\phi(\tau,-z) = (-1)^\kappa \phi(\tau,z)$ which implies that $c(n,-r) = (-1)^\kappa c(n,r) = c(n,r)$. If m is 1 or a prime, then

$$4n'm - r'^2 = 4nm - r^2 \Rightarrow r' \equiv \pm r \pmod{2m} \Rightarrow c(n,r) = c(n',r').$$

Finally, if $m = 1$ and κ is odd then $\phi \equiv 0$ because $c(n,-r) = -c(n,r)$ but $4nm - (-r)^2 = 4nm - r^2$ and $-r \equiv r \pmod{2m}$ in this case. $\square$

We extend the definition of coefficients given in equation (5.9) to all N (see [21], page 58) by setting $c_\delta(N) = 0$ if $N \not\equiv -\delta^2 \pmod{4m}$. Then,

$$\begin{aligned}
\phi(\tau,z) &= \sum_{n,r,4mn\geq r^2} c(n,r) e^{2\pi i n\tau} e^{2\pi i r z} \\
&= \sum_{\delta \pmod{2m}} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\delta \pmod{2m}}} \sum_{n\geq r^2/4m} c_\delta(4nm - r^2) e^{2\pi i n\tau} e^{2\pi i r z} \\
&= \sum_{\delta \pmod{2m}} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\delta \pmod{2m}}} \sum_{N\geq 0} c_\delta(N) e^{2\pi i (\frac{N+r^2}{4m})\tau} e^{2\pi i r} \\
&= \sum_{\delta \pmod{2m}} \sum_{N\geq 0} c_\delta(N) e^{2\pi i (\frac{N}{4m})\tau} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\delta \pmod{2m}}} e^{2\pi i (\frac{r^2}{4m})\tau} e^{2\pi i r}.
\end{aligned}$$

We now set

$$h_\delta(\tau) := \sum_{N\geq 0} c_\delta(N) e^{2\pi i (\frac{N}{4m})\tau}, \qquad (\delta \in \mathbb{Z}/2m\mathbb{Z}),$$

and for each congruence class modulo $2m$ we have the theta series of weight $1/2$ and index m given by

$$\theta_{m,\delta}(\tau,z) := \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\delta \pmod{2m}}} e^{2\pi i \left(\frac{r^2}{4m}\tau + rz\right)}.$$

The above calculations give the theta expansion of a Jacobi form ϕ as linear combination of fixed theta functions as follows:

$$\phi(\tau, z) = \sum_{\delta \pmod{2m}} h_\delta(\tau)\theta_{m,\delta}(\tau, z). \tag{5.10}$$

From this expansion, we conclude that knowing the $(2m)$-tuple $(h_\delta)_{\delta \pmod{2m}}$ of functions of one variable is equivalent to knowing ϕ. Reversing the above calculations, we obtain a function ϕ with Fourier coefficients satisfying equation (5.14) and which transforms like a Jacobi form with respect to $z \mapsto z + \lambda\tau + \mu$, $(\lambda, \mu \in \mathbb{Z})$ and with the right expansion at infinity. Now, since ϕ is a Jacobi form of weight κ and index m, then the functions h_δ must be modular forms of weight $\kappa - 1/2$. To specify their transformation properties, we consider the transformation law of the theta series with respect to the generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{Z})$:

$$\theta_{m,\delta}(\tau + 1, z) = e^{2\pi i \frac{\delta^2}{4m}} \theta_{m,\delta}(\tau, z),$$

and using Poisson summation formula,

$$\theta_{m,\delta}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \left(\frac{\tau}{2mi}\right)^{\frac{1}{2}} e^{2\pi i m \frac{z^2}{\tau}} \sum_{\nu \mod 2m} \theta_{m,\nu}(\tau, z) e^{\frac{2\pi i(-\nu\delta)}{2m}}.$$

Using these transformation formulas together with the transformation law of ϕ, we obtain the following transformation of $h_\delta(\tau)$

$$h_\delta(\tau + 1) = e^{-2\pi i \frac{\delta^2}{4m}} h_\delta(\tau), \tag{5.11}$$

and

$$h_\delta\left(-\frac{1}{\tau}\right) = \frac{\tau^\kappa}{\sqrt{\frac{2m\tau}{i}}} \sum_{\nu \mod 2m} e^{\frac{2\pi i(\nu\delta)}{2m}} h_\nu(\tau). \tag{5.12}$$

We have shown

Theorem 5.2.2 ([21], Theorem 5.1)**.** *Equation (5.10) gives an isomorphism between $J_{\kappa,m}$ and the space of vector valued modular forms $(h_\delta)_{\delta \pmod{2m}}$ on $\mathrm{SL}_2(\mathbb{Z})$ satisfying the transformation laws (5.11) and (5.12) and bounded as $\operatorname{Im}\tau$ tends to ∞.*

Another consequence of the theta expansion (5.10) of a Jacobi form is that we can express the Petersson scalar product of two Jacobi forms ϕ and ψ in terms of the Petersson product of the modular forms $(h_\delta)_\delta$.

Theorem 5.2.3 ([21], Theorem 5.3)**.** *Let*

$$\phi = \sum_{\delta \pmod{2m}} h_\delta(\tau)\theta_{m,\delta}$$

and

$$\psi = \sum_{\delta \pmod{2m}} g_\delta(\tau)\theta_{m,\delta}$$

be two Jacobi forms in $J_{\kappa,m}$. Then,

$$\langle \phi, \psi \rangle = \frac{1}{\sqrt{2m}} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}^1} \sum_{\delta \pmod{2m}} h_\delta(\tau)\overline{g_\delta(\tau)}v^{\kappa-5/2}dudv.$$

5.2.1.1 Involutions on $J_{\kappa,m}$

Using the theta expansion of a Jacobi form given in equation (5.10) together with the equations

$$\theta_{m,-\delta}(\tau, z) = \theta_{m,\delta}(\tau, -z),$$

and

$$\phi(\tau, -z) = (-1)^\kappa \phi(\tau, z),$$

we obtain for each congruence class δ modulo $2m$

$$h_{-\delta} = (-1)^\kappa h_\delta.$$

Hence every $2m$ tuple $(h_\delta)_\delta$ reduces to an $(m+1)$-tuple of forms $(h_\delta + h_{-\delta})_{0\le\delta\le m}$ if κ is even and to an $(m-1)$-tuple of forms $(h_\delta - h_{-\delta})_{0<\delta<m}$ if κ is odd.

If m is composite then there is a finer splitting on the $(h_\delta)_\delta$. Let m' be a divisor of m such that $\gcd(m', m/m') = 1$. There are $2^{\nu(m)}$ such divisors where $\nu(m)$ is the number of distinct prime factors of m. Using the Chinese Remainder Theorem, we can find a unique integer $\xi = \xi_{m'}$ modulo $2m$ such that

$$\xi \equiv 1 \pmod{2m/m'}, \quad \xi \equiv -1 \pmod{2m'}.$$

The set

$$\{\xi_{m'} : m' \mid m, \gcd(m', m/m') = 1\}$$

is precisely the set

$$\Xi(m) = \{\xi \pmod{2m} | \xi^2 \equiv 1 \pmod{4m}\} \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(m)}.$$

We can map the collection of $(2m)$-tuples $(h_\delta)_{\delta \pmod{2m}}$ into itself by the permutation

$$(h_\delta)_{\delta \pmod{2m}} \mapsto (h_{\xi\delta})_{\delta \pmod{2m}}.$$

Because $\xi^2 \equiv 1$, $\pmod{4m}$, equations (5.11) and (5.12) are preserved. Hence we have shown the following:

Theorem 5.2.4 ([21], Theorem 5.2)**.** *For each divisor m' of m such that* $\gcd(m', m/m') =$ 1*, there is an operator $W_{m'}$ from $J_{k,m}$ to itself given by*

$$\phi(\tau, z) = \sum_{n,r} c(n,r) e^{2\pi i(n\tau)} e^{2\pi i(rz)} \mapsto (\phi|W_{m'})(\tau, z) = \sum_{n',r'} c(n', r') e^{2\pi i(n'\tau)} e^{2\pi i(r'z)}$$

where $r' \equiv -r \pmod{2m'}$, $r' \equiv r \pmod{2m/m'}$, and $4n'm - r'^2 = 4nm - r^2$. These operators are all involutions that form a group isomorphic to $\Xi(m)$ and generated by the $W_{p_i^{\nu_i}}$ with $m = \prod_{i=1}^{t} p_i^{\nu_i}$.

5.2.1.2 Connection with Modular Forms of Degree 1

In this section we identify the space of Jacobi forms of index 1 or prime p, with the space of half integral weight modular forms. Before we do so we review some features of elliptic modular forms which we take from [51], Chapter 4. For all the details, skipped definitions, and arguments, the reader should consult the reference.

Let $f \in S_{2\kappa}(\Gamma_0(N))$ be a cusp form. We have seen in Chapter 2 that f has a Fourier expansion of the form $f(z) = \sum_{n\geq 1} a(n)e^{2\pi inz}$. Attached to f is the Dirichlet series or L-function, $L(s,f) = \sum_{n\geq 1} \frac{a(n)}{n^s}$, $s \in \mathbb{C}$, which converges absolutely for $\Re(s) > \kappa + 1$.

Let $\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$. It is easy to verify that conjugation by α_N preserves $\Gamma_0(N)$. Define W_N to be the operator (called Fricke involution) that acts by α_N on f

$$(W_N f)(z) = (\sqrt{Nz})^{-2\kappa} f(-1/Nz).$$

Then W_N is an involution ($W_N^2 = 1$) and preserves the space $S_{2\kappa}(\Gamma_0(N))$. Therefore the only possible eigenvalues for W_N are $\epsilon_N = \pm 1$, and consequently we obtain a splitting of the space of cusp forms into ± 1-eigenspaces

$$S_{2\kappa}(\Gamma_0(N)) = S_{2\kappa}(\Gamma_0(N))^{+1} \oplus S_{2\kappa}(\Gamma_0(N))^{-1}.$$

Hecke showed the following:

Theorem 5.2.5 ([51], Theorem 4.4)**.** *Let $f \in S_{2\kappa}(\Gamma_0(N))$ be a cusp form in the ϵ_N eigenspace. Then $L(s,f)$ can be analytically continued to a meromorphic function on the whole complex plane, and satisfies the functional equation*

$$\Lambda(s,f) = \epsilon_N(-1)^\kappa \Lambda(\kappa - s, f) \tag{5.13}$$

where

$$\Lambda(s,f) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(s,f),$$

and $\Gamma(s)$ *is the Gamma function.*

Remark 5.2.6. 1. We call the term $\epsilon_N(-1)^\kappa$ the sign of the functional equation satisfied by the Dirichlet series $L(s,f)$.

2. When we use the minus sign in the notation of the subspace of cusp forms $S^{-}_{2\kappa}(\Gamma_0(N))$, we mean the subspace of modular forms f whose functional equation sign $\epsilon_N(-1)^\kappa = -1$. It is not to be confused with the subspace $S_{2\kappa}(\Gamma_0(N))^{-1}$ of forms with $\epsilon_N = -1$.

3. For a prime divisor p of N such that $\gcd(p, N/p) = 1$, we can define an operator called the Atkin-Lehner involution at p, W_p, given by the matrix

$$\begin{pmatrix} p & a \\ N & pb \end{pmatrix}$$

with determinant p and whose eigenvalue $\epsilon_p = \pm 1$.

4. When $N = 1$, $W_N \equiv I$, where I is the identity map, since $\alpha_N \in \mathrm{SL}_2(\mathbb{Z})$ and $S^{-1}_{2\kappa} = \{0\}$.

5. When $N = 1$, then $L(s,f)$ satisfies the functional equation

$$\Lambda(s,f) = (-1)^\kappa \Lambda(\kappa - s, f).$$

The Dirichlet series $L(s,f)$ is called an L-function if and only if it satifies a functional equation and has an Euler product factorization.

Proposition 5.2.7 ([51], Proposition 4.5)**.** *The Dirichlet series $L(s,f)$ has an Euler product expansion of the form*

$$L(s,f) = \prod_{p|N} \frac{1}{1 - a(p)p^{-s}} \prod_{\gcd(p,N)=1} \frac{1}{1 - a(p)p^{-s} + p^{2\kappa-1-s}}$$

if (and only if)

$$\begin{cases} a(mn) = a(m)a(n), & \text{if } \gcd(m,n) = 1 \\ a(p)a(p^r) = a(p^{r+1}) + p^{2\kappa-1}a(p^{r-1}) & \text{if } r \geq 1, \gcd(p,N) = 1 \\ a(p^r) = a(p)^r & \text{if } r \geq 1, p \mid N. \end{cases}$$

Hecke defined linear maps for each positive integer $n \geq 1$

$$T(n) : S_{2\kappa}(\Gamma_0(N)) \to S_{2\kappa}(\Gamma_0(N))$$

called the Hecke operators, and proved the following theorems.

Theorem 5.2.8 ([51], Theorem 4.7)**.** *The linear maps $T(n)$ have the following properties:*

1. $T(mn) = T(m)T(n)$ *if* $\gcd(m,n) = 1$,

2. $T(p)T(p^r) = T(p^{r+1}) + p^{2\kappa-1}T(p^{r-1})$ *if* $\gcd(p,N) = 1$,

3. $T(p^r) = T(p)^r$ *if* $p \mid N$,

4. *all $T(n)$ commute.*

Theorem 5.2.9 ([51], Theorem 4.8)**.** *Let $f(z) = \sum_{n\geq 1} a(n)e^{2\pi inz} \in S_{2\kappa}(\Gamma_0(N))$ be an eigenform for all the Hecke operators $T(n)$ with eigenvalues $\lambda(n)$. Then*

$$a(n) = \lambda(n)a(1). \tag{5.14}$$

Corresponding to f in Theorem 5.2.9, we define the Dirichlet series $Z(s,f) = \sum_{n\geq 1} \frac{\lambda(n)}{n^s}$, which is absolutely convergent for $\Re(s) > \kappa + 1$ ([2], Theorem 1.2.2). We have the identity

$$L(s,f) = a(1)Z(s,f).$$

As a consequence of equation (5.14), if f is an eigenform of all Hecke operators, then its Fourier coefficients inherit properties of the Hecke operators. Applying Proposition 5.2.7 we obtain:

Corollary 5.2.10. *Let $f \in S_{2\kappa}(\Gamma_0(N))$ be an eigenform for all Hecke operators normalized so that $a(1) = 1$, then*

$$L(s, f) = \prod_{p|N} \frac{1}{1 - \lambda(p)p^{-s}} \prod_{\gcd(p,N)=1} \frac{1}{1 - \lambda(p)p^{-s} + p^{2\kappa-1-s}}.$$

The problem of finding cusp forms f whose Dirichlet series have Euler product expansions is then a problem of finding simultaneous eigenforms for the Hecke operators $T(n) : S_{2\kappa}(\Gamma_0(N)) \to S_{2\kappa}(\Gamma_0(N))$. The answer is in the spectral theorem from Hilbert theory on finite dimensional complex vector spaces.

Theorem 5.2.11. *(Spectral Theorem) Let V be a finite dimensional complex vector space equipped with a positive definite Hermitian form $\langle , \rangle$.*

1. *Let $t : V \to V$ be a linear map which is Hermitian. Then V has a basis consisting of eigenvectors for t; i.e., t is diagonalizable.*

2. *Let $t_1, t_2, \ldots$ be a sequence of Hermitian operators sending V to V which commute with each other. Then V has a basis consisting of vectors which are eigenvectors for all the t_i; i.e., t_i are simultaneously diagonalizable.*

A linear map $T : V \to V$ is self-adjoint (or hermitian) relative to the form $\langle , \rangle$ if $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all $v, w \in V$.

The Petersson inner product defines a positive-definite hermitian form on the vector space of cusp forms $S_{2\kappa}(\Gamma_0(N))$. The Hecke operators $T(n)$ are self-adjoint for all $\gcd(n, N) = 1$. Applying the spectral theorem to the sequence of Hecke operators $T(n)$ with $\gcd(n, N) = 1$ on the complex vector space $S_{2\kappa}(\Gamma_0(N))$, we obtain a decomposition

$$S_{2\kappa}(\Gamma_0(N)) = \oplus V_i$$

into a direct sum of orthogonal subspaces V_i each of which is a simultaneous eigenspace for all $T(n)$ with $\gcd(n, N) = 1$.

The Fricke involution W_N is self-adjoint with respect to the Petersson product, and commutes with $T(n)$ for $\gcd(n, N) = 1$, and so each V_i decomposes into orthogonal eigenspaces

$$V_i = V_i^{+1} \oplus V_i^{-1}$$

for W_N. However, W_N does not commute with the $T(p)$'s for $p \mid N$. Hence we can not have a single f that is simultaneously an eigenvector for W_N (and hence its L-function satisfies a functional equation) and for all $T(n)$ (and hence its L-function has an Euler product factorization). The obstacle to obtaining such a form is due to the existence of a space of old forms. If M is a proper divisor of N, then $\Gamma_0(N)$ is a subgroup of $\Gamma_0(M)$ and $S_{2\kappa}(\Gamma_0(M)) \subset S_{2\kappa}(\Gamma_0(N))$. For every divisor m of N/M, $f(mz) \in S_{2\kappa}(\Gamma_0(N))$. The subspace of $S_{2\kappa}(\Gamma_0(N))$ spanned by all forms $f(m\tau)$ with $f \in S_{2\kappa}(\Gamma_0(M))$, $mM \mid N$, $M \neq N$ is called the space of old forms and is denoted by $S_{2\kappa}^{\text{old}}(\Gamma_0(N))$. The orthogonal complement $S_{2\kappa}^{\text{new}}(\Gamma_0(N))$ is called the space of new forms. It is stable under all the operators $T(n)$ and W_N and so it decomposes into a direct sum of orthogonal subspaces W_i

$$S_{2\kappa}^{\text{new}}(\Gamma_0(N)) = \oplus W_i,$$

each of which is a simultaneous eigenspace for all $T(n)$ with $\gcd(n, N) = 1$. The $T(p)$ for $p \mid N$ and W_N stabilize each W_i.

Theorem 5.2.12 ([51], Theorem 26.21)**.** *The spaces W_i in the above decomposition all have dimension* 1*.*

Because they have dimension 1, each W_i is also an eigenspace for W_N and $T(p)$ for $p \mid N$. Consequently, there exists exactly one normalized cusp form in each W_i that is an eigenform for all the Hecke operators and for the Fricke involution. For this form, its L-function $L(s, f)$ has an Euler product factorization as in (5.2.7) and satisfies a functional equation as in (5.13). Such cusp form is called a newform. The newforms form a basis for the space of new forms $S_{2\kappa}^{\text{new}}(\Gamma_0(N))$.

Finally there is a direct sum decomposition of the space $S_{2\kappa}(\Gamma_0(N))$ as

$$S_{2\kappa}(\Gamma_0(N)) = \bigoplus_{mM|N} \{f(m\tau) | f \in S_{2\kappa}^{\text{new}}(\Gamma_0(M))\}.$$

A basis of $S_{2\kappa}(\Gamma_0(N))$ consists of the functions $f(m\tau)$ where $m \mid N$ and f a newform of level dividing N/m.

Half integral weight modular forms are similar to modular forms of integral weight, except that the automorphy factor describing the action of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ involves the Legendre symbol $\left(\frac{c}{d}\right)$. For a definition of this symbol and for an extensive treatment of modular forms of half integral weight, we refer to [80].

Shimura showed that one can also define Hecke operators $T(p)$ on the space of modular forms $M_{\kappa-1/2}(\Gamma_0(4m))$ for all primes $p \nmid 4m$ and that $M_{\kappa-1/2}(\Gamma_0(4m))$ is spanned by simultaneous eigenforms of these operators. He defined a Hecke equivariant lifting that associates to a Hecke eigenform in $M_{\kappa-1/2}(\Gamma_0(4m))$ a Hecke eigenform of weight $2k-2$ with the same set of eigenvalues but its level was unclear. This lifting goes under the name of Shimura correspondence. Later Niwa [56] showed that the right level of the eigenform of integral weight obatined under the Shimura lifting is $2m$. For the case $m = 1$ and later for the case of m odd squarefree ([41], [42], [43]) Kohnen showed that one could get all the way down to level m by restricting the Shimura correspondence to the subspace

$$M_{k-1/2}^{+}(4m) := \left\{ f \in M_{k-1/2}(\Gamma_0(4m)) \middle| f(\tau) = \sum_{\substack{n \geq 0 \\ (-1)^{\kappa-1} n \equiv 0,1 \pmod 4}} c_f(n) e^{2\pi i n \tau} \right\} \quad (5.15)$$

consisting of forms in $M_{k-1/2}(\Gamma_0(4m))$ whose n^{th} Fourier coefficient vanishes for all n with $(-1)^{\kappa-1} n \equiv 2, 3 \pmod 4$. This subspace is called "Kohnen's +-space. " Kohnen also showed that one can define commuting Hecke operators $T(p)$ on $M_{k-1/2}^{+}(4m)$ for all p

agreeing with Shimura's Hecke operators if $p \neq 2$ and that under a refined vesion of Shimura correspondence, $M^{+}_{k-1/2}(4m)$ becomes isomorphic to $M_{2\kappa-2}(\Gamma_0(m))$ as modules over the ring of Hecke operators. (There is a 1-1 correspondence between eigenforms $h \in M^{+}_{k-1/2}(4m)$ and $f \in M_{2\kappa-2}(\Gamma_0(m))$ such that the eigenvalues of h and f agree for all p). To be more precise, this correspondence is between certain subspaces of newforms. All the details can be found in the papers [42], [43].

Now for the connection of Jacobi forms with half integral weight modular forms, Theorem 5.2.2 is the main key to understand the connection found by Eichler and Zagier. In case $m = 1$, let $\phi(\tau, z) \in J_{\kappa,1}$ where κ is even. Then using the theta expansion, we have

$$\begin{aligned}\phi(\tau, z) &= \sum_{\delta \pmod 2} h_\delta(\tau)\theta_{1,\delta}(\tau, z) \\ &= h_0(\tau)\theta_{1,0}(\tau, z) + h_1(\tau)\theta_{1,1}(\tau, z).\end{aligned}$$

Define

$$h(\tau) = h_0(4\tau) + h_1(4\tau).$$

Now, ϕ is a Jacobi form if and only if h_0 and h_1 satisfy the transformations (5.11) and (5.12) above. This translates to

$$\begin{aligned}h(\tau + 1) &= h(\tau) \\ h\left(\tfrac{\tau}{4\tau+1}\right) &= (4\tau + 1)^{\kappa-1/2} h(\tau).\end{aligned}$$

Since $\Gamma_0(4)$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, this shows that

$$h(\tau) = \sum_{N \equiv 0 \pmod 4} c_0(N)e^{2\pi i(N\tau)} + \sum_{N \equiv 3 \pmod 4} c_1(N)e^{2\pi i(N\tau)} \in M^{+}_{\kappa-1/2}(\Gamma_0(4)).$$

One can reverse the calculation and show that h_0 and h_1 satisfy the proper transformation properties given in (5.11) and (5.12). We have sketched the following result.

Theorem 5.2.13 ([21], Theorem 5.4)**.** *Let κ be an even integer. Then*

$$M^+_{k-1/2}(4) \cong J_{\kappa,1}$$

with the isomorphism given by

$$\sum_{\substack{N \geq 0 \\ N \equiv 0,3 \pmod 4}} c(N) e^{2\pi i(N\tau)} \mapsto \sum_{\substack{n,r \in \mathbb{Z} \\ 4n-r^2 \geq 0}} c(4n-r^2) e^{2\pi i(n\tau + rz)}.$$

This isomorphism is in fact an isomorphism of Hecke modules.

Using Kohnen's work as discussed above together with Theorem 5.2.13, we have:

Corollary 5.2.14 ([21], Corollary 3, page 66)**.** *If $\phi \in J_{\kappa,1}$ is an eigenfunction of all Hecke operators $T_J(l)$ (see § 5.3), then there is a Hecke eigenform in $M_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ with the same eigenvalues. Moreover, this correspondence is bijective.*

Remark 5.2.15. We note that using the map

$$\phi(\tau, z) \mapsto h(\tau) = \sum_{\delta \pmod{2m}} h_\delta(4m\tau),$$

which generalizes the map $\phi(\tau, z) \mapsto h_0(4\tau) + h_1(4\tau)$ above, Eichler and Zagier proved an analogue of Theorem 5.2.13 when the index m is prime.

One can generalize Kohnen's work for arbitary m not necessarily odd square free but technical difficulties occur as mentioned in [84]. Skoruppa and Zagier overcame these difficulties by replacing the space of half integral weight modular forms by Jacobi forms and established the Shimura correspondence for Jacobi forms, bypassing the space of half integral weight forms. One has the following result.

Theorem 5.2.16 ([85], Theorem 5)**.** *For each pair of integers $\kappa, m \in \mathbb{Z}$, $m > 0$, the space of Jacobi forms $J_{\kappa,m}$ is Hecke-equivariantly isomorphic to the subspcae $M^-_{2\kappa-2}(\Gamma_0(m))$ of the space $M_{2\kappa-2}(\Gamma_0(m))$ of all elliptic modular forms of weight $2\kappa - 2$ on $\Gamma_0(m)$.*

Moreover, Skoruppa and Zagier in [85] defined a notion of new forms for the space of $J^c_{\kappa,m}$ and obtained a Hecke equivariant isomorphism

$$S^{\text{new}}_{2\kappa-2}(\Gamma_0(m)) \cong J^{c,\text{new}}_{\kappa,m}. \tag{5.16}$$

5.2.2 Connection with Siegel Modular Forms of Degree 2

5.2.2.1 Some Preliminaries

For all the definitions below, we follow [2]. For Siegel modular forms of degree $n \geq 2$, there are more Hecke operators than there are for the degree $n = 1$. For each prime p we no longer have just one corresponding Hecke operator $T(p)$. An additional complication is the theory of L-functions which are not defined through Fourier coefficients since these coefficients are attached to matrices. Let F be a Siegel modular form in $M_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$. Suppose F is an eigenfunction of all the Hecke operators (see § 5.3) $T_S(m) \in \mathbb{T}^S_{\mathbb{Z}}$ (for $m = 1, 2, \ldots$):

$$T_S(m)F = \lambda_F(m)F.$$

For every prime p, let

$$Q_{p,F}(t) = 1 - \lambda_F(p)t + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2\kappa-4})t^2 - \lambda_F(p)p^{2\kappa-3}t^3 + p^{4k-6}t^4.$$

For some right half plane $\Re(s) > \sigma$, let

$$Z(s, F) = \prod_p (Q_{p,F}(p^{-s}))^{-1}. \tag{5.17}$$

Then

$$Z(s, F) = \zeta(2s - 2\kappa + 4) \sum_{m \geq 1} \frac{\lambda(m)}{m^s},$$

where $\zeta(s)$ is the Riemann zeta-function ([2], Theorem 2.2.1).

5.2.2.2 The Fourier-Jacobi Development of a Siegel Modular Form of Degree 2

We saw in Chapter 2 that for any genus $n \geq 1$, a Siegel modular form F has a Fourier expansion. But in case $n > 1$ there are other developments that provide additional information, like the so-called Fourier-Jacobi development of a Siegel modular form. We limit our exposition to the case $n = 2$.

For $n = 2$, we can write $Z \in \mathfrak{h}^2$ as $\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $\tau, \tau' \in \mathfrak{h}^1$ and $z \in \mathbb{C}$ and the condition $\operatorname{Im}(Z) > 0$ translates to $\operatorname{Im}(z)^2 < \operatorname{Im}(\tau)\operatorname{Im}(\tau')$. We summarize this into the definition of the Siegel upper half space of genus 2 as

$$\mathfrak{h}^2 = \left\{ Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \middle| \, z \in \mathbb{C}, \tau, \tau' \in \mathfrak{h}^1, \operatorname{Im}(z)^2 < \operatorname{Im}(\tau)\operatorname{Im}(\tau') \right\}.$$

Let F be a Siegel modular form of weight κ on $\mathfrak{h}^2$. We know that F has a Fourier expansion of the form

$$F(Z) = \sum_{T \in \mathbb{S}_2^{\geq 0}(\mathbb{Z})} a(T) e^{2\pi i \operatorname{Tr}(TZ)}$$

where $T \in \mathbb{S}_2^{\geq 0}(\mathbb{Z})$ means that T ranges over positive semidefinite semi-integral symmetric 2×2 matrices T. For every $T \in \mathbb{S}_2^{\geq 0}(\mathbb{Z})$, we set $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ where $n, r, m \in \mathbb{Z}, n, m \geq 0, r^2 \leq 4nm$, and $\operatorname{Tr}(TZ) = n\tau + rz + m\tau'$. We write $F(\tau, z, \tau')$ instead of $F(Z)$ and we also write $a(n, r, m)$ for $a(T)$, so the Fourier development of F becomes

$$F(\tau, z, \tau') = \sum_{\substack{n,m,r \in \mathbb{Z} \\ m,n,4nm - r^2 \geq 0}} a(n, r, m) e^{2\pi i (n\tau + rz + m\tau')}.$$

Rearranging the Fourier expansion of F and writing it in terms of τ', we obtain the Fourier-

Jacobi expansion of F:

$$F(\tau, z, \tau') = \sum_{\substack{m,n,r \in \mathbb{Z} \\ m, 4mn - r^2 \geq 0}} a(n,r,m) e^{2\pi i(n\tau + rz + m\tau')} \tag{5.18}$$

$$= \sum_{m \geq 0} \sum_{\substack{n,r \in \mathbb{Z} \\ 4mn - r^2 \geq 0}} a(n,r,m) e^{2\pi i(n\tau + rz)} e^{2\pi i(m\tau')} \tag{5.19}$$

$$= \sum_{m \geq 0} \phi_m(\tau, z) e^{2\pi i(m\tau')}. \tag{5.20}$$

We now extend Theorem 6.1 in [21] and we also prove it for cusp forms.

Theorem 5.2.17. *Let F be a Siegel modular form of weight κ with respect to the Hecke congruence subgroup $\Gamma_0^{(2)}(M)$. Consider the Fourier-Jacobi development of F:*

$$F(\tau, z, \tau') = \sum_{m \geq 0} \phi_m(\tau, z) e^{2\pi i(m\tau')},$$

then for each m, ϕ_m is a Jacobi form of weight κ and index m with respect to the Jacobi group $\Gamma_0(M)^J$. Moreover, if $F \in S_\kappa(\Gamma_0^{(2)}(M))$, then ϕ_m is a cusp form with respect to $\Gamma_0(M)^J$ for each m.

Proof. Consider the mapping

$$\iota : \left(\begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \right) \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ for all } \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \in \Gamma_0(M).$$

Then $\iota : \Gamma_0(M) \to \Gamma_0^{(2)}(M)$ is an injective group homomorphism for $ad - bMC = 1$ and the image satisfies the symplectic relations.

Similarly, the map

$$h : (\lambda, \mu) \mapsto \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for all} \quad (\lambda, \mu) \in \mathbb{Z}^2$$

is an injective homomorphism. The action of the matrices

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.21}$$

on $\mathfrak{h}^2$ is given by

$$(\tau, z, \tau') \to \left(\frac{a\tau + b}{Mc\tau + d}, \frac{z}{Mc\tau + d}, \tau' - \frac{Mcz^2}{Mc\tau + d} \right),$$

$$\left(\tau, z, \tau'\right) \to \left(\tau, z + \lambda\tau + \mu, \tau' + 2\lambda z + \lambda^2\tau\right).$$

Now applying the transformation formula for Siegel modular forms we get the two transformation laws of Jacobi forms for the Fourier coefficients ϕ_m, and the Fourier expansion of the Jacobi forms is given from the Fourier development of the Siegel modular form equation (5.18) where the condition $4nm \geq r^2$ is coming from the condition on the matrices T. That is, we have shown that for every m, $\phi_m \in J_{\kappa,m}(\Gamma_0(M)^J)$.

For the rest of the proof, we know (see Chapter 2) that every Siegel modular form F of degree two has a Fourier expansion

$$(F|_\kappa\gamma)(Z) = \sum_{T \in \mathbb{S}_2^{\geq 0}(\mathbb{Z})} a_\gamma(T) e^{2\pi i \operatorname{Tr}(TZ)},$$

for every $\gamma \in \mathrm{Sp}_4(\mathbb{Z})$. When $F \in S_\kappa(\Gamma_0^{(2)}(M))$, then for all $\gamma \in \mathrm{Sp}_4(\mathbb{Z})$, $a_\gamma(T) = 0$ for every $T \in \mathbb{S}_2^{\geq 0}(\mathbb{Z})$ such that $\det T = 0$. In particular, taking $\gamma \in \mathrm{Sp}_4(\mathbb{Z})$ in the image $\iota(\mathrm{SL}_2(\mathbb{Z}))$ and working with the same notation used above where $\det T = 0$ translates to $r^2 = 4mn$, we have

$$\begin{aligned} F(\tau, z, \tau')|_\kappa \gamma &= \sum_{\substack{m,n,r \in \mathbb{Z} \\ m, 4mn - r^2 > 0}} a_\gamma(n, r, m) e^{2\pi i(n\tau + rz + m\tau')} \\ &= \sum_{m \geq 1} \sum_{\substack{n,r \in \mathbb{Z} \\ 4mn - r^2 > 0}} a_{\gamma'}(n, r, m) e^{2\pi i(n\tau + rz)} e^{2\pi i(m\tau')} \\ &= \sum_{m \geq 0} \phi_m(\tau, z)|_{\gamma'} e^{2\pi i(m\tau')}, \end{aligned}$$

where γ' is the inverse image of γ in $\mathrm{SL}_2(\mathbb{Z})$. The condition $a_\gamma(T) = 0$ for every $T \in \mathbb{S}_2^{\geq 0}(\mathbb{Z})$ such that $\det T = 0$ translates to

$$\phi_m(\tau, z)|_{\gamma'} = \sum_{\substack{n,r \in \mathbb{Z} \\ 4mn - r^2 > 0}} a_{\gamma'}(n, r, m) e^{2\pi i(n\tau + rz)},$$

which proves that $\phi_m \in J^c_{\kappa,m}(\Gamma_0(M)^J)$. □

Using Theorem 5.2.17, we have an injective map

$$\mathcal{H} : M_\kappa(\Gamma_0^{(2)}(M)) \to \prod_{m \geq 0} J_{\kappa,m}(\Gamma_0(M)^J),$$

and a corresponding map for the space of cusp forms

$$\mathcal{H}^c : S_\kappa(\Gamma_0^{(2)}(M)) \to \prod_{m \geq 1} J^c_{\kappa,m}(\Gamma_0(M)^J).$$

In case $M = 1$, these maps reduce to

$$\mathcal{H} : M_\kappa(\mathrm{Sp}_4(\mathbb{Z})) \to \prod_{m \geq 0} J_{\kappa,m},$$

and a corresponding map for the space of cusp forms

$$\mathcal{H}^c : S_\kappa(\mathrm{Sp}_4(\mathbb{Z})) \to \prod_{m\geq 1} J^c_{\kappa,m}.$$

5.2.2.3 The Maass Lifting

Liftings in the study of modular forms are of considerable importance. We study an important linear map from the space of Jacobi forms to the space of Siegel modular forms of degree 2 called the Maass lifting.

Maass identified a subspace $M^*_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ in $M_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ called the Maass subspace consisting of forms F whose Fourier coefficients $a(T)$ for $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ depend only on the discriminant $D(T) = 4nm - r^2$ and the content $e(T) = \gcd(n, r, m)$. It is explicitly defined by

$$M^*_\kappa(\mathrm{Sp}_4(\mathbb{Z})) = \left\{ F \in M_\kappa(\mathrm{Sp}_4(\mathbb{Z})) \,\middle|\, a(n,r,m) = \sum_{d>0, d|\gcd(n,r,m)} d^{\kappa-1} a(1, r/d, mn/d^2) \right\}.$$

The corresponding space in the space of cusp forms is denoted by $S^*_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$.

The inverse of the map $F \mapsto \phi_1$ that sends a Siegel modular form in $M_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ to its first Fourier-Jacobi coefficient $\phi_1 \in J_{\kappa,1}$ is the full level Maass lifting due to [48]. We present its proof exactly as given in [21].

Theorem 5.2.18 ([21], Theorem 6.2)**.** *Let ϕ be a Jacobi form of weight κ and index 1 with respect to the full Jacobi group $\mathrm{SL}_2(\mathbb{Z})^J$. Then the Siegel modular form $\mathcal{V}(\phi)$ whose Fourier Jacobi coefficients are $\phi|V_m$ $(m \geq 0)$ is a Siegel modular form of weight κ in $M_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$.*

Proof. The proof consists of verifying that the form $\mathcal{V}(\phi)$ transforms like a Siegel modular form under the generators of $\mathrm{Sp}(4,\mathbb{Z})$. When $\phi \in J_{\kappa,1}$, $\phi|V_m \in J_{\kappa,m}$. The map $\mathcal{V}(\phi)$ is given by

$$\begin{aligned}\mathcal{V}(\phi)(Z) &= \sum_{m\geq 0} V_m(\phi)(\tau,z)e^{2\pi i(m\tau')} \\ &= \sum_{m\geq 0}\sum_{n,r,4mn-r^2\geq 0} a(n,r,m)e^{2\pi i(n\tau+rz)}e^{2\pi i(m\tau')}\end{aligned}$$

where according to the action of V_m we have that

$$a(n,r,m) = \left(\sum_{a|\gcd(n,r,m)} a^{\kappa-1}c\left(\frac{nm}{a^2},\frac{r}{a}\right)\right) \tag{5.22}$$

given that $\phi(\tau,z) = \sum_{n,r,4mn-r^2\geq 0} c(n,r)e^{2\pi i(n\tau+rz)}$ is the Fourier expansion of ϕ.

In general, a function defined by the series $\sum_{m\geq 0}\phi_m(\tau,z)e^{2\pi i(m\tau')}$ transforms like a Siegel modular form under the action of matrices in $\mathrm{Sp}_4(\mathbb{Z})$ in the image subspace $\iota(\mathrm{SL}_2(\mathbb{Z}))$ of the map ι given in equation (5.2.2.2) when $M = 1$ and in the image of $h(\mathbb{Z}^2)$. In particular this holds for the function

$$\mathcal{V}(\phi)(Z) = \sum_{m\geq 0} V_m(\phi)(\tau,z)e^{2\pi i(m\tau')}.$$

On the other hand, the Fourier expansion of ϕ is symmetric in n and m. From this, we deduce that $\mathcal{V}(\phi)$ is symmetric in τ and τ'. Equivalently, this says that it transforms like a Siegel modular form with respect to the matrix

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since the matrix R together with matrices of the form $\iota(\mathrm{SL}_2(\mathbb{Z}))$ and $h(\mathbb{Z}^2)$ generate $\mathrm{Sp}_4(\mathbb{Z})$, this shows that $\mathcal{V}(\phi) \in M_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$. $\square$

Recall that by associating to a Siegel modular form its Fourier-Jacobi coefficients we have the injective map

$$\mathcal{H} : M_\kappa(\mathrm{Sp}_4(\mathbb{Z})) \to \prod_{m \geq 0} J_{\kappa,m}.$$

The full level Maass lifting shows that we also have a map in the other direction

$$\mathcal{V} : J_{\kappa,1} \to M_\kappa(\mathrm{Sp}_4(\mathbb{Z})),$$

such that the composition

$$J_{\kappa,1} \to M_\kappa(\mathrm{Sp}_4(\mathbb{Z})) \to \prod_{m \geq 0} J_{\kappa,m} \to J_{\kappa,1}$$

is the identity (where the last map is the projection to the first coefficient). Thus $\mathcal{V}$ is injective and the image is exactly the set of F with $F = \mathcal{V}(\mathcal{H}(F))$, i.e of Siegel modular forms whose Fourier Jacobi expansion has the property $\phi_m = \phi_1 | V_m$ for all m. This means that the Fourier coefficients defined by (5.22) satisfy the following relation

$$a(n,r,m) = \left(\sum_{d | \gcd(n,r,m)} d^{\kappa-1} c\left(\frac{4nm - r^2}{d^2} \right) \right), \qquad ((n,r,m) \neq (0,0,0)),$$

where

$$c(N) = \begin{cases} a(n,0,1) & \text{if } N = 4n \\ a(n,1,1) & \text{if } N = 4n-1. \end{cases}$$

Equivalently, we can characterize the image by the subspace of Siegel modular forms whose Fourier coefficients satisfy the relations

$$a(n,r,m) = \sum_{d | (n,r,m)} d^{k-1} a\left(\frac{nm}{d^2}, \frac{r}{d}, 1 \right)$$

for all (n,r,m). These relations are exactly the Maass relations satisfied by forms in the Maass subspace $M_\kappa^*(\mathrm{Sp}_4(\mathbb{Z}))$.

Summarizing, the full level Maass lifting implies the isomorphism

$$J_{\kappa,1} \cong M^*_\kappa(\mathrm{Sp}_4(\mathbb{Z})). \tag{5.23}$$

5.3 The Classical Saito Kurokawa lifting

Andrianov [3] has shown that the Maass subspace $S^*_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ is invariant under the action of Hecke operators. Using the fact that the Maass subspace is the image of the full level Maass lifting, it is shown in [21] that the full level Maass lifting $\mathcal{V} : J^c_{\kappa,1} \to S^*_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ is compatible with the action of Hecke operators in $J_{\kappa,1}$ and $M_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$. This implies that $\mathcal{V}$ maps Hecke eigenforms to Hecke eigenforms. We now give the Hecke operators used for Jacobi forms of index t and for Siegel modular forms with general level $\Gamma_0(M)^J$ and $\Gamma_0^{(2)}(M)$ respectively. The Hecke operators in the symplectic case are defined as in [2] using double coset operators.

The Hecke algebra $\mathbb{T}^J_{\mathbb{Z}}$ for the Jacobi group $\Gamma_0(M)^J$ is generated over $\mathbb{Z}$ by the operators $T_J(p), p \nmid tM$ and $U_J(p), p \mid M$ which are defined as in [49] and their actions on Fourier coefficients are given by

$$(\phi | T_J(p))(\tau, z) = \sum_{\substack{D \le 0, r \in \mathbb{Z} \\ D \equiv r^2 \pmod{4t}}} c^*(D, r) e^{2\pi i \left(\frac{r^2 - D}{4t}\tau + rz\right)}, \quad (p \nmid tM) \tag{5.24}$$

where

$$c^*(D, r) = c(p^2 D, pr) + p^{\kappa-2} \left(\frac{D}{p}\right) c(D, r) + p^{2\kappa-3} c\left(\frac{D}{p^2}, \frac{r}{p}\right)$$

and $c(D, r) = 0$ if D or r is not an integer,

$$(\phi | U_J(p))(\tau, z) = \sum_{\substack{D \le 0, r \in \mathbb{Z} \\ D \equiv r^2 \pmod{4t}}} c(p^2 D, pr) e^{2\pi i \left(\frac{r^2 - D}{4t}\tau + rz\right)}, \quad p | M. \tag{5.25}$$

The Hecke algebra $\mathbb{T}^S_{\mathbb{Z}}$ for the symplectic case is generated over $\mathbb{Z}$ by $T_S(p)$ and

$T'_S(p)$ when $p \nmid M$ and by $U_S(p)$ when $p \mid M$ ([31], page 157). For $p \nmid M$,

$$T_S(p) = \Gamma_0^{(2)}(M)\,\mathrm{diag}\,(1,1,p,p)\Gamma_0^{(2)}(M).$$

Define

$$\begin{aligned}
T_{12}(p^2) &= \Gamma_0^{(2)}(M)\,\mathrm{diag}\,(1,p,p^2,p)\Gamma_0^{(2)}(M),\\
T_{02}(p^2) &= \Gamma_0^{(2)}(M)\,\mathrm{diag}\,(1,1,p^2,p^2)\Gamma_0^{(2)}(M),\\
T_{11}(p^2) &= \Gamma_0^{(2)}(M)\,\mathrm{diag}\,(p,p,p,p)\Gamma_0^{(2)}(M).
\end{aligned}$$

Let

$$T_S(p^2) := T_{12}(p^2) + T_{02}(p^2) + T_{11}(p^2)$$

and

$$T_S^{'}(p) = T_S(p)^2 - T_S(p^2) = (p^3+p^2+p)T_{11}(p^2) + pT_{12}(p^2).$$

For $p \mid M$,

$$U_S(p) = \Gamma_0^{(2)}(M)\,\mathrm{diag}\,(1,1,p,p)\Gamma_0^{(2)}(M). \tag{5.26}$$

Theorem 5.3.1 ([21], Theorem 6.3)**.** *The full level Maass lifting $\mathcal{V} : J_{\kappa,1} \to M_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ is Hecke equivariant with respect to the Hecke algebra homomorphism $\iota' : \mathbb{T}_{\mathbb{Z}}^S \to \mathbb{T}_{\mathbb{Z}}^J$ defined on the generators of the Hecke algebra by*

$$\begin{aligned}
\iota'(T_S(p)) &= T_J(p) + p^{\kappa-1} + p^{\kappa-2},\\
\iota'(T'_S(p)) &= (p^{\kappa-1} + p^{\kappa-2})T_J(p) + 2p^{2\kappa-3} + p^{2\kappa-4}.
\end{aligned}$$

Equivalently, the Maass lifting satisfies

$$(\mathcal{V}(\phi))|T = \mathcal{V}(\phi|\iota'(T))$$

for any Hecke operator $T \in \mathbb{T}_{\mathbb{Z}}^{S}$.

If $F \in M_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ is a Hecke eigenform such that $T_S(p)F = \gamma_p F$ and $T'_S(p)F = \gamma'_p F$, then we can associate a zeta function as the one given in equation (5.17)

$$Z(s,F) = \prod_p (1 - \lambda_p p^{-s} + (\gamma'_p - p^{2\kappa-4})p^{-2s} - \gamma_p p^{2\kappa-3-3s} + p^{4k-6-4s})^{-1}.$$

If $F = \mathcal{V}(\phi)$ with $T_J(p)\phi = \lambda_p \phi$, then it follows from Theorem 5.3.5 that

$$\gamma_p = \lambda_p + p^{\kappa-1} + p^{\kappa-2}, \quad \gamma'_p = (p^{\kappa-1} + p^{\kappa-2})\lambda_p + 2p^{2\kappa-3} + p^{2\kappa-4}$$

and hence

$$1 - \gamma_p t + (\gamma'_p - 2p^{2\kappa-4})t^2 - \gamma_p p^{2\kappa-3} t^3 + p^{\kappa-6} t^4 = (1 - p^{\kappa-1}t)(1 - p^{\kappa-2}t)(1 - \lambda_p t + p^{2\kappa-3}t^2).$$

Corollary 5.2.14 says that there is a 1-1 correspondence between eigenforms in $M_{2\kappa-2}$ and $J_{\kappa,1}$, the eigenvalues being the same. This, together with the Hecke equivariance of the Maass lifting whose image is the Maass subspace allows us to deduce the following:

Theorem 5.3.2 ([21], Corollary 1, § 6)**.** *The space $S_\kappa^*(\mathrm{Sp}_4(\mathbb{Z}))$ is spanned by Hecke eigenforms. These are in 1-1 correspondence (when κ is even) with normalized Hecke eigenforms $f \in S_{2\kappa-2}$, the correspondence being such that*

$$Z(s,F) = \zeta(s-\kappa+1)\zeta(s-\kappa+2)L(s,f). \tag{5.27}$$

The result obtained in Theorem 5.3.2 asserts the existence of a lifting from $S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) \to S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ such that equation (5.27) relating their L-functions holds when κ is even. This lifting is called the classical Saito-Kurokawa lifting (of full level). As constructed above when κ is even, the Saito-Kurokawa lifting is constructed as the composition of three linear maps

$$S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z})) \cong S^{+}_{\kappa-1/2}(\Gamma_0(4)) \cong J^c_{\kappa,1} \to S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$$

where $S^+_{\kappa-1/2}(\Gamma_0(4))$ is Kohen's +-space defined by equation (5.15) using the corresponding space of cusp forms. The first isomorphism is obtained via the Shimura isomorphism as adjusted by Kohnen ([41]), the second isomorphism is as obtained by Zagier (Theorem 5.2.13) and the third one is the full level Maass lifting (Theorem 5.2.18).

5.3.1 Congruence Level Saito-Kurokawa Lifting

Given a Siegel modular form $F \in M_\kappa(\Gamma_0^{(2)}(M))$, we have obtained an injective map

$$\mathcal{H} : S_\kappa(\Gamma_0^{(2)}(M)) \to \prod_{m \geq 1} J^c_{\kappa,m}(\Gamma_0(M)^J)$$

using the Fourier-Jacobi expansion of F.

The inverse of the map $F \mapsto \phi_1$ projecting F to its first component induces the Maass lifting with level. As far as the author knows, its proof had never appeared in the literature before a very recent paper of Ibukiyama [31], with most authors just citing Eichler-Zagier [21]. Ironically, the Eichler-Zagier book never contained a real proof of the Maass lifting with level except the statement that says the same arguments used in the proof of the full level Maass lifting work also for the Maass lifting with level.

We state the Maass lifting with level as done by Ibukiyama, we will include its proof in Chapter 6.

Theorem 5.3.3 ([31], Theorem 3.2). *Let ϕ be a Jacobi cusp form in $J^c_{\kappa,1}(\Gamma_0(M)^J)$. Then the function defined by*

$$\mathcal{V}(\phi)(\tau, z, \tau') := \sum_{m \geq 1} V_m(\phi)(\tau, z) e^{2\pi i m t \tau'}$$

belongs to $S_\kappa(\Gamma_0^{(2)}(M))$.

Remark 5.3.4. The V_m operators used in the Maass lifting with level are a generalized version of the V_m operators defined in equation (5.1.5) and used in the full level Maass lifting.

The Maass lifting with level is compatible with the following Hecke algebra homomorphism.

Theorem 5.3.5 ([31], Theorem 4.1)**.** *The Maass lifting with level is Hecke-equivariant with respect to the Hecke algebra homomorphism $\iota' : \mathbb{T}_{\mathbb{Z}}^{S} \to \mathbb{T}_{\mathbb{Z}}^{J}$ defined on the generators of the Hecke algebra by*

$$\begin{aligned}
\iota'(T_S(p)) &= T_J(p) + (p^{k-1} + p^{k-2}) \quad (p \nmid M), \\
\iota'(T'_S(p)) &= (p^{k-1} + p^{k-2})T_J(p) + (2p^{2k-3} + p^{2k-4}) \quad (p \nmid M), \\
\iota'(U_S(p)) &= U_J(p) \quad (p \mid M).
\end{aligned}$$

Equivalently, the Maass lifting satisfies

$$(\mathcal{V}(\phi))|T = \mathcal{V}(\phi|\iota'(T))$$

for any operator $T \in \mathbb{T}_{\mathbb{Z}}^{S}$.

Using a generalization of the Eichler-Zagier map when M is odd and squarefree and κ even, we have that $J_{\kappa,1}^{c}(\Gamma_0(M)^J) \cong S_{\kappa-1/2}^{+}(\Gamma_0(4M))$ from [44]. Combining with Kohnen's isomorphism ([43])

$$S_{2\kappa-2}^{\text{new}}(\Gamma_0(M)) \cong S_{\kappa-1/2}^{+,\text{new}}(\Gamma_0(4M))$$

we obtain the congruence level Saito-Kurokawa lifting

$$S_{2\kappa-2}^{\text{new}}(\Gamma_0(M)) \cong S_{\kappa-1/2}^{+,\text{new}}(\Gamma_0(4M)) \cong J_{\kappa,1}^{c,\text{new}}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_0^{(2)}(M)),$$

where $J_{\kappa,1}^{c,\text{new}}(\Gamma_0(M)^J)$ is defined as in [49].

5.3.2 Paramodular level Saito-Kurokawa lifting

Let $\kappa \geq 2$ and $t \geq 1$ be integers. Let $S_{2\kappa-2}^{\text{new},-}(\Gamma_0(t))$ be the subspace of cusp forms of weight $2\kappa - 2$ with level $\Gamma_0(t)$ whose $L-$functions satisfy the functional equation (5.13) with sign -1. Skoruppa and Zagier showed in [85] that

$$S_{2\kappa-2}^{\mathrm{new},-}(\Gamma_0(t)) \cong J_{\kappa,t}^{c,\mathrm{new}} \tag{5.28}$$

as Hecke modules for the Hecke operators $T(n)$, $\gcd(n,t)=1$. Equivalently, starting with $f \in S_{2\kappa-2}^{\mathrm{new},-}(\Gamma_0(t))$ a normalized Hecke eigenform, one can associate to it a unique (up to scalar) Jacobi cusp form ϕ_f having the same eigenvalues of f under the Hecke operators $T(n)$, $\gcd(n,t)=1$.

On the other hand, Gritsenko obtained a lifting [26] from the space of Jacobi forms $J_{\kappa,t}$ to the space of paramodular forms $M_\kappa(\Gamma[t])$. Here we state Gritsenko's lifting for cusp forms:

Theorem 5.3.6 ([27], Theorem 3)**.** *Let ϕ_t be a Jacobi cusp form of weight $\kappa \geq 2$, index $t \geq 1$ and level $\mathrm{SL}_2(\mathbb{Z})^J$ with the Fourier expansion*

$$\phi_t(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z},n\geq 0\\ 4nt\geq r^2}} c(n,r)e^{2\pi i(n\tau+rz)}.$$

Then

$$\mathcal{G}(\phi_t)(\tau,z,\tau') := \sum_{m\geq 1} V_m(\phi_t)(\tau,z)e^{2\pi i m t\tau'}$$

is a paramodular cusp form of weight κ with respect to the paramodular group $\Gamma[t]$.

Remark 5.3.7.

1. The map $\mathcal{G}$ that associates to a Jacobi form of weight κ, index $t \geq 1$, and level $\mathrm{SL}_2(\mathbb{Z})^J$ a paramodular form with respect to the paramodular group $\Gamma[t]$ is injective.

2. Gritsenko's lifting is a generalization of the full level Maass lifting $\mathcal{V} : J_{\kappa,1}^c \to S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$.

3. The V_m operators used in the definition of $\mathcal{G}$ are the same as the operators used in the definition of the full level Maass lifting.

Combining Gritsenko's lifting $\mathcal{G}$ with the Skoruppa-Zagier lifting given in equation

(5.16), we obtain the Saito-Kurokawa lifting of paramodular level t:

$$S^{\mathrm{new},-}_{2\kappa-2}(\Gamma_0(t)) \cong J^{c,\mathrm{new}}_{\kappa,t} \to S_\kappa(\Gamma[t]). \tag{5.29}$$

Chapter 6

Generalized Maass-Gritsenko Cuspidal Liftings

6.1 Introduction

We recall from Chapter 5 that Gritsenko obtained a generalization of the full level Maass lifting $\mathcal{V} : J^c_{\kappa,1} \to S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$. He constructed an injective map $\mathcal{G}$ that associates to a Jacobi cusp form of weight κ, index $t \geq 1$, and level $\mathrm{SL}_2(\mathbb{Z})^J$ a paramodular cusp form with respect to the paramodular group $\Gamma[t]$ of level t :

$$\mathcal{G} : J^c_{\kappa,t} \to S_\kappa(\Gamma[t]).$$

In this chapter we give a cuspidal lifting that generalizes both the Maass lifting with level and Gritsenko's lifting starting from a Jacobi cusp form of level $\Gamma_0(M)^J$. This amounts to proving the modularity of the lift and the cuspidality of the lift. In proving modularity, we obtain a general characterization of the modularity of Maass-Gritsenko liftings. This characterization is based on the maximal standard proper parabolic subgroup $C_{2,1}$ of type $(2, 1)$ of the Hecke congruence subgroups of degree two or the paramodular groups. We recall that this standard parabolic subgroup is associated with the maximal degree standard cusp of the compactified space $(\mathfrak{h}^2)^*/\Gamma$, where Γ is either $\mathrm{Sp}_4(\mathbb{Z})$, its Hecke congruence subgroup $\Gamma_0^{(2)}(M)$, the paramodular group $\Gamma[t]$, or the mixed level group $\Gamma_M[t]$, a subgroup of $\Gamma[t]$ that we will introduce later in this chapter. We also obtain a more general characterization of modularity of liftings from Jacobi forms of level Γ^J for any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. The conclusion is that the theory of liftings from Jacobi forms is governed by the theory of maximal parabolic subgroups of $\mathrm{Sp}_4(\mathbb{Q})$.

To obtain the cuspidality of our lift, we have to give a description of the double cosets associated to the maximal degree cusps of the compact space $(\mathfrak{h}^2)^*/\Gamma_M[t]$. The structure of these cusps says that the lifting is cuspidal if and only if the Jacobi form is cuspidal. We also notice that the structure of the maximal degree cusps of the compact space $(\mathfrak{h}^2)^*/\Gamma_0^{(2)}(M)$ can give a shorter proof of the cuspidality of the Maass lifting with level than that given by Ibukiyama ([31]). Another consequence of the structure of the maxiaml degree cusps is that we obtain a characterization of cuspidal Siegel modular forms (not necessarily lifts) with respect to the Hecke congruence subgroup $\Gamma_0^{(2)}(M)$ and with respect to the mixed level paramodular group $\Gamma_M[t]$ in terms of Fourier-Jacobi expansions. Finally, the description of the image of the cuspidal mixed level lifting as forms whose Fourier coefficients satisfy linear relations similar to the classical Maass relations enables us to define a subspace of forms similar to the Maass subspace that we hope to characterize in the future. Towards that goal, we attempt to describe Fourier-Jacobi expansions of mixed level paramodular forms. We note that Ibukiyama, Poor and Yuen ([33]) are interested in such characterizations and have been successful in the case of paramodular forms $M_\kappa(\Gamma[t])$ when $t = 1, 2, 3, 4$.

We start by presenting the ingredients of the modularity of the Maass lifting with level $M \geq 1$. We note that during our work on this topic and finding out the general framework of liftings from Jacobi forms, we have proved the modularity of the Maass lifting with level M independently of Ibukiyama's recent proof in [31]. We then work out modularity ingredients of a lifting that generalizes both the Maass lifting with level and Gritsenko's lifting. To prove cuspidality, we review the cusps given by Ibukiyama and Gritsenko and present the cusps for our generalized lifting.

6.2 Modularity of the Maass lifting with Level M

6.2.1 Maximal Parabolic Subgroups

Fix n the degree of the Siegel upper half plane $\mathfrak{h}^n$. We have seen in Chapter 4 that we have n proper standard maximal parabolic subgroups of $\mathrm{Sp}_{2n}(\mathbb{Q})$, $C_{n,r}(\mathbb{Q})$, for $0 \le r \le n-1$, associated with the standard cusp of corresponding degree r. For each $0 \le r \le n-1$, the subgroup $C_{n,r}(\mathbb{Q})$ is of the form

$$C_{n,r}(\mathbb{Q}) = \left\{ M \in \mathrm{Sp}_{2n}(\mathbb{Q}) \,\middle|\, M = \begin{pmatrix} A_{11} & 0 & B_{11} & B_{12} \\ A_{21} & U & B_{21} & B_{22} \\ C_{11} & 0 & D_{11} & D_{12} \\ 0 & 0 & 0 & U^{-1} \end{pmatrix} \right\},$$

where $\begin{pmatrix} A_{11} & B_{11} \\ C_{11} & D_{11} \end{pmatrix} \in \mathrm{Sp}_{2r}(\mathbb{Q})$ and $U \in \mathrm{GL}_{n-r}(\mathbb{Q})$. We call $C_{n,r}(\mathbb{Q})$ the standard maximal parabolic subgroup of type (n, r).

For genus $n = 2$, we have two standard maximal parabolic subgroups: $C_{2,1}(\mathbb{Q})$ of type $(2, 1)$ associated with the standard cusp of degree 1 (maximal degree) and $C_{2,0}(\mathbb{Q})$ of type $(2, 0)$ associated with the standard cusp $i\infty$ of degree zero (see Chapter 4). For our purpose of proving lifting theorems, we will be interested in $C_{2,1}(\mathbb{Q})$:

$$C_{2,1}(\mathbb{Q}) = \left\{ \gamma \in \mathrm{Sp}_4(\mathbb{Q}) \,\middle|\, \gamma = \begin{pmatrix} a & 0 & b & z' \\ x & u & z & y \\ c & 0 & d & -x' \\ 0 & 0 & 0 & u^{-1} \end{pmatrix} \right\}, \tag{6.1}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q})$ and $u \in \mathrm{GL}_1(\mathbb{Q})$. We give a characterization of standard maximal subgroups $C_{2,1}(\mathbb{Q})$ of type $(2, 1)$. The same characterization holds for any degree n (See

Chapter 4).

Remark 6.2.1. An element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q})$, where $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and $D = (d_{ij})$ with $1 \leq i, j \leq 2$, belongs to $C_{2,1}(\mathbb{Q})$ if and only if $c_{12} = c_{21} = c_{22} = 0$ and $c_{11} \neq 0$, or equivalently $a_{12} = c_{12} = c_{22} = 0$.

Proof. Since a general element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q})$ satisfies the symplectic relations, we then have C^tD and ${}^tAC$ are symmetric matrices. This implies that we must have $d_{21} = a_{12} = 0$. $\square$

The Levi decomposition (see Chapter 4) of $C_{2,1}(\mathbb{Q})$ allows us to write elements of $C_{2,1}(\mathbb{Q})$ uniquely in the following way:

$$C_{2,1}(\mathbb{Q}) = \left\{ \gamma \in \mathrm{Sp}_4(\mathbb{Q}) \,\middle|\, \gamma = \begin{pmatrix} a & 0 & b & 0 \\ 0 & u & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & z \\ x & 1 & z & y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

For any arithmetic subgroup Γ of $\mathrm{Sp}_4(\mathbb{Q})$, the corresponding standard maximal parabolic subgroups are $\Gamma \cap C_{2,1}(\mathbb{Q})$ of highest type $(2,1)$ and $\Gamma \cap C_{2,0}(\mathbb{Q})$ of type $(2,0)$. When Γ is the Hecke congruence subgroup $\Gamma_0^{(2)}(M)$, we will denote the standard maximal parabolic subgroup $\Gamma_0^{(2)}(M) \cap C_{2,1}(\mathbb{Q})$ of type $(2,1)$ by $\Gamma_{M,\infty}(\mathbb{Z})$. When $M = 1$, the corresponding standard maximal parabolic subgroup of type $(2,1)$ of the Siegel modular group $\mathrm{Sp}_4(\mathbb{Z})$ will be denoted by $\Gamma_\infty(\mathbb{Z})$.

Proposition 6.2.2. *The standard maximal parabolic subgroup of $\Gamma_0^{(2)}(M)$ of type $(2,1)$ is given by*

$$\Gamma_{M,\infty}(\mathbb{Z}) = \left\{ \gamma \in \Gamma_0^{(2)}(M) \,\middle|\, \gamma = \begin{pmatrix} a & 0 & b & z' \\ x & \pm 1 & z & y \\ Mc & 0 & d & -x' \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \right\}.$$

Proof. Since $\Gamma_{M,\infty}(\mathbb{Z}) = \Gamma_0^{(2)}(M) \cap C_{2,1}(\mathbb{Q})$, it is clear from equation (6.1) that

$$\Gamma_{M,\infty}(\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & 0 & b & z' \\ x & u & z & y \\ Mc & 0 & d & -x' \\ 0 & 0 & 0 & u^{-1} \end{pmatrix} \right\},$$

where $\begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \in \Gamma_0(M)$ and $u \in \mathrm{GL}_1(\mathbb{Z})$. Because $\Gamma_{M,\infty}(\mathbb{Z})$ is a subgroup of $\Gamma_0^{(2)}(M)$, then every element $\gamma \in \Gamma_{M,\infty}(\mathbb{Z})$ must have determinant equal to 1. But $\det(\gamma) = \det \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} uu^{-1} = 1$. This proves the final form of elements of $\Gamma_{M,\infty}(\mathbb{Z})$. $\square$

Remark 6.2.3. By considering the projective parabolic subgroup $\Gamma_{M,\infty}(\mathbb{Z})/\pm 1_4$, we can always assume $u = 1$. Since we can do so we will instead abuse notation and denote

$$\Gamma_{M,\infty}(\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & 0 & b & z' \\ x & 1 & z & y \\ Mc & 0 & d & -x' \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

6.2.2 Jacobi Group with Level M as a Maximal Parabolic Subgroup of $\Gamma_0^{(2)}(M)$ of Type $(2,1)$

We recall from Chapter 4 the surjective group homomorphism

$$\begin{aligned} \omega_1 : \Gamma_{M,\infty}(\mathbb{Z}) &\to \Gamma_0(M) \\ \gamma &\mapsto \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \end{aligned}$$

which is a projection onto the SL_2 part. The kernel of ω_1 is the set of all matrices

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \lambda, \mu, \kappa \in \mathbb{Z} \right\},$$

which is isomorphic to the integral Heisenberg group $H(\mathbb{Z})$ through the embedding

$$\begin{aligned} h : H(\mathbb{Z}) &\longrightarrow \Gamma_0^{(2)}(M) \\ (\lambda, \mu, \kappa) &\mapsto \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

We also recall the embedding of $\Gamma_0(M)$ into $\Gamma_0^{(2)}(M)$ defined by

$$\begin{aligned} i : \Gamma_0(M) &\longrightarrow \Gamma_0^{(2)}(M) \\ \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} &\mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Identifying $H(\mathbb{Z})$ and $\Gamma_0(M)$ with their corresponding images in $\Gamma_0^{(2)}(M)$, we obtain an exact sequence

$$1 \to H(\mathbb{Z}) \to \Gamma_{M,\infty}(\mathbb{Z}) \to \Gamma_0(M) \to 1$$

which splits because of the injection i. Consequently, we have that $\Gamma_{M,\infty}(\mathbb{Z}) \cong \Gamma_0(M) \ltimes$

$H(\mathbb{Z})$, where the action of $\Gamma_0(M)$ on $H(\mathbb{Z})$ is given by

$$g \mapsto ((X, \kappa) \mapsto (Xg, \kappa)),$$

for $(X = (\lambda, \mu), \kappa) \in H(\mathbb{Z})$, $g \in \Gamma_0(M)$ and Xg means row multiplication by a matrix. Thus, we have shown that:

Theorem 6.2.4. *The Jacobi group $\Gamma_0(M) \ltimes H(\mathbb{Z})$, where $H(\mathbb{Z})$ is the integral Heisenberg group, is isomorphic to $\Gamma_{M,\infty}(\mathbb{Z})$.*

Consequently, every element γ of the parabolic subgroup $\Gamma_{M,\infty}(\mathbb{Z})$ is written uniquely as

$$\Gamma_{M,\infty}(\mathbb{Z}) = \left\{ \gamma \in \Gamma_0^{(2)}(M) \,\middle|\, \gamma = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & z \\ x & 1 & z & y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

for $\begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \in \Gamma_0(M)$ and $x, y, z \in \mathbb{Z}$.

6.2.3 Jacobi Forms as Forms with respect to Maximal Parabolic Subgroup of Type $(2, 1)$

With this realization of the Jacobi group $\Gamma_0(M)^J$ as the maximal parabolic subgroup $\Gamma_{M,\infty}(\mathbb{Z})$ of $\Gamma_0^{(2)}(M)$, a Jacobi form can be viewed as a $\Gamma_{M,\infty}(\mathbb{Z})$-invariant function on $\mathfrak{h}^2$.

Definition 6.2.5. A holomorphic function ϕ on $\mathfrak{h}^1 \times \mathbb{C}$ is called a Jacobi form of weight κ, index $t \geq 1$ and level $\Gamma_0(M)^J$ if the corresponding $\widetilde{\phi}$ living on $\mathfrak{h}^2$ defined by

$$\widetilde{\phi}(\tau, z, \tau') := \phi(\tau, z) e^{2\pi i t \tau'}$$

for $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ is a modular form of weight κ with respect to $\Gamma_{M,\infty}(\mathbb{Z})$. That is,

1. $\widetilde{\phi}|_\kappa \gamma = \widetilde{\phi}$ for all $\gamma \in \Gamma_{M,\infty}(\mathbb{Z})$,

2. for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ it has a Fourier expansion

$$\left(\widetilde{\phi}|_\kappa i(\gamma)\right)(\tau, z, \tau') = \sum_{\substack{n,r\in\mathbb{Z}, n\geq 0 \\ 4nt-r^2\geq 0}} c_\gamma(n,r) e^{2\pi i(n\tau+rz+t\tau')}.$$

We say ϕ is a Jacobi cusp form if $c_\gamma(n,r) = 0$ when $4nt - r^2 = 0$ for every γ in $\mathrm{SL}_2(\mathbb{Z})$.

This definition is equivalent to the usual definition of Jacobi forms presented in Chapter 5. We now show why. First, we have seen that $\Gamma_{M,\infty}(\mathbb{Z})$ is generated by its subgroup $i(\Gamma_0(M))$ and the embedding of the integral Heisenberg group consisting of matrices $h(\lambda, \mu, \kappa) := \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Now writing an element $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathfrak{h}^2$ as (τ, z, τ'), we compute the action $(AZ+B)(CZ+D)^{-1}$ of $i(\Gamma_0(M))$ on Z by decomposing a matrix $\gamma \in i(\Gamma_0(M))$ as a block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} Mc & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$, and $ad - Mbc = 1$. We obtain

$$i(\Gamma_0(M))Z = \left(\frac{a\tau+b}{Mc\tau+d}, \frac{z}{Mc\tau+d}, \tau' - \frac{Mcz^2}{Mc\tau+d}\right).$$

The functional equation $\widetilde{\phi}|_\kappa\gamma = \widetilde{\phi}$, where $\widetilde{\phi}|_\kappa\gamma = \det(CZ+D)^{-\kappa}\widetilde{\phi}(\gamma Z)$ becomes

$$(Mc\tau+d)^{-\kappa}\phi\left(\frac{a\tau+b}{Mc\tau+d}, \frac{z}{Mc\tau+d}\right) e^{2\pi it\left(\tau' - \frac{Mcz^2}{Mc\tau+d}\right)} = \phi(\tau, z)e^{2\pi it\tau'}.$$

Cancelling the factor $e^{2\pi it\tau'}$ from both sides of the equation, we obtain the usual transformation formula for a Jacobi form $\phi(\tau, z)$ with respect to the $\Gamma_0(M)$ elements in the Jacobi group $\Gamma_0(M) \ltimes H(\mathbb{Z})$

$$\phi\left(\frac{a\tau+b}{Mc\tau+d}, \frac{z}{Mc\tau+d}\right) e^{2\pi it\left(-\frac{Mcz^2}{Mc\tau+d}\right)} = (Mc\tau+d)^{\kappa}\phi(\tau, z).$$

Similarly, computing the action of $h(\lambda, \mu, \kappa)$ on $Z = (\tau, z, \tau')$, we get

$$\begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} Z = (\tau, z+\lambda\tau+\mu, \lambda^2\tau + 2z\lambda + \mu\lambda + \kappa + \tau').$$

The functional equation $\widetilde{\phi}|_{\kappa} h(\lambda, \mu, \kappa) = \widetilde{\phi}$ becomes

$$\phi(\tau, z+\lambda\tau+\mu)e^{2\pi it(\lambda^2\tau+2z\lambda+\mu\lambda+\kappa+\tau')} = \phi(\tau, z)e^{2\pi it\tau'}.$$

Again, cancelling the common factor $e^{2\pi it\tau'}$ from both sides of the equations gives the right transformation formula for a Jacobi form with respect to the Heisenberg elements in the Jacobi group $\Gamma_0(M) \ltimes H(\mathbb{Z})$.

Remark 6.2.6. Using Definition 6.2.5, we get a one to one correspondence between the set of Jacobi forms $\{\phi\}$ on $\mathfrak{h}^1 \times \mathbb{C}$ with respect to $\Gamma_0(M) \ltimes H(\mathbb{Z})$ and the set of functions $\{\widetilde{\phi}\}$ on $\mathfrak{h}^2$ invariant under the action of the standard maximal parabolic subgroup $\Gamma_{M,\infty}(\mathbb{Z})$ of $\Gamma_0^{(2)}(M)$ of type $(2, 1)$.

$$\{\phi \in J_{\kappa,t}(\Gamma_0(M)^J)\} \leftrightarrow \{\widetilde{\phi} \in M_{\kappa}(\Gamma_{M,\infty}(\mathbb{Z}))\}. \tag{6.2}$$

The beauty of this correspondence is that it now translates the complicated transformation formulas of Jacobi forms (see Chapter 5) to the nice invariance transformation formula given in Definition 6.2.5.

As another consequence of Definition 6.2.5 we have

Proposition 6.2.7. *The Jacobi group $\Gamma_0(M) \ltimes H(\mathbb{Z})$ stabilizes the Fourier-Jacobi expansion of a Siegel modular form $F \in M_\kappa(\Gamma_0^{(2)}(M))$.*

Proof. We recall from Chapter 5 that the Fourier expansion of a Siegel modular form $F \in M_\kappa(\Gamma_0^{(2)}(M))$ can be rearranged as a Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \sum_{m \geq 0} \phi_m(\tau, z) e^{2\pi i m \tau'}$$

with $\phi_m \in J_{\kappa,m}(\Gamma_0(M)^J)$.

Using correspondence (6.2), we can rewrite this Fourier-Jacobi expansion in the following manner:

$$\begin{aligned} F(\tau, z, \tau') &= \sum_{m \geq 0} \phi_m(\tau, z) e(2\pi i m \tau') \\ &= \sum_{m \geq 0} \widetilde{\phi}_m(\tau, z, \tau'). \end{aligned}$$

From this series, it is clear that standard maximal parabolic subgroup $\Gamma_{M,\infty}(\mathbb{Z})$ stabilizes each of its terms. But since F is the uniform limit of this series, we have then that $\Gamma_{M,\infty}(\mathbb{Z})$ stabilizes the Fourier-Jacobi expansion of F. This together with Theorem 6.2.4 finishes the proof. □

6.2.4 Generators

We give generators for the Hecke congruence subgroup $\Gamma_0^{(2)}(M)$. We note that Aoki and Ibukiyama in [6] have obtained generators for $\Gamma_0^{(2)}(M)$ that are in fact equivalent to the generators we will give below. Since our goal is to characterize modularity of liftings from Jacobi forms with index 1 or index $t > 1$, we would like to present generators that fit into the general framework we are presenting. We also note that our proof will start with similar steps as theirs but will be slightly modified afterwards. We start with a classical lemma.

Lemma 6.2.8 ([4], Lemma 1.3.8.). *Let* $\mathbf{t} = {}^t(t_1, \ldots, t_m)$ *be a non-zero integral column with* $m \geq 2$. *Then there exists a matrix* $V \in \mathrm{SL}_m(\mathbb{Z})$ *such that*

$$V\mathbf{t} = {}^t(d, 0, \ldots, 0),$$

where $d = \gcd(t_1, \ldots, t_m)$.

Proof. The proof goes by induction on m. For the base case $m = 2$ the proof is very simple. For any nonzero integers t_1, t_2, we have that $\gcd\left(\frac{t_1}{d}, \frac{t_2}{d}\right) = 1$. Then there exists integers α, β such that $\alpha(\frac{t_1}{d}) - \beta(\frac{-t_2}{d})) = 1$, where $d = \gcd(t_1, t_2)$. Thus $\alpha, \frac{t_1}{d}, \beta, \frac{t_2}{d}$ fit into a matrix V in $\mathrm{SL}_2(\mathbb{Z})$ and that $\alpha t_1 + \beta t_2 = d$. Equivalently,

$$\begin{pmatrix} \alpha & \beta \\ \frac{-t_2}{d} & \frac{t_1}{d} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix}.$$

For the rest of the details, the reader can consult [4]. $\square$

Let

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Theorem 6.2.9. *The Hecke congruence subgroup* $\Gamma_0^{(2)}(M)$ *is generated by* $\Gamma_{M,\infty}(\mathbb{Z})$ *and* R, *i.e.,*

$$\Gamma_0^{(2)}(M) = \langle \Gamma_{M,\infty}(\mathbb{Z}), R \rangle.$$

Proof. We have seen that $\Gamma_{M,\infty}(\mathbb{Z})$ has elements of the following form

$$S(a,b,c,d) := \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h(\lambda,\mu,\kappa) := \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For an element $S(a,b,c,d)$, we have

$$RS(a,b,c,d)R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & Mc & 0 & d \end{pmatrix},$$

and for an element $h(\lambda,\mu,\kappa)$, we have

$$Rh(\lambda,\mu,\kappa)R = \begin{pmatrix} 1 & \lambda & \kappa & \mu \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 \end{pmatrix}.$$

Now, we take a general element $\gamma = \begin{pmatrix} A & B \\ MC & D \end{pmatrix} \in \Gamma_0^{(2)}(M)$, where

$$A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij}),$$

for $i,j \in \{1,2\}$. The first column of γ is then $\begin{pmatrix} a_{11} \\ a_{21} \\ Mc_{11} \\ Mc_{21} \end{pmatrix}$. Assuming $a_{11}, a_{21} \geq 0$ (we can always do so since we can always mulitply it from the left by $S(-1,0,0,-1)$ or $RS(-1,0,0,-1)R$

if we need to). Multiplying γ from the left by $h(\lambda, 0, 0)R$ we get $\begin{pmatrix} a_{21} \\ \lambda a_{21} + a_{11} \\ Mc_{21}\lambda Mc_{11} \\ Mc_{11} \end{pmatrix}$, so we can always start with a matrix γ whose a_{21} entry is zero. Let $m = \gcd(a_{11}, c_{11})$, then $\gcd(\frac{a_{11}}{m}, \frac{c_{11}}{m}) = 1$. Moreover, since $\det(\gamma) = 1$, we get that $\gcd(a_{11}, M) = 1$. This implies that $\gcd(\frac{a_{11}}{m}, M\frac{c_{11}}{m}) = 1$ and consequently there exists integers α and β such that $\alpha\frac{a_11}{m} - \beta M\left(\frac{-c_{11}}{m}\right) = 1$. Applying the matrix $S\left(\alpha, \beta, -M\frac{c_{11}}{m}, \frac{a_{11}}{m}\right)$, we obtain

$$S\left(\alpha, \beta, -M\frac{c_{11}}{m}, \frac{a_{11}}{m}\right)\begin{pmatrix} a_{11} \\ 0 \\ Mc_{11} \\ Mc_{21} \end{pmatrix} = \begin{pmatrix} m \\ 0 \\ 0 \\ Mc_{21} \end{pmatrix}.$$

Now applying the matrix $h(-1, 0, 0)$, we have

$$h(1, 0, 0)\begin{pmatrix} m \\ 0 \\ 0 \\ Mc_{21} \end{pmatrix} = \begin{pmatrix} m \\ m \\ -Mc_{21} \\ Mc_{21} \end{pmatrix}.$$

Since any row or any column of a modular matrix consists of coprime numbers, we have $\gcd(m, Mc_{21}) = 1$. Arguing similarly to the previous step, there are integers α' and β' such that $\alpha' m + \beta' Mc_{21} = 1$. Multiplying by $RS(\alpha', \beta', -Mc_{21}, m)R$, we obtain

$$RS(\alpha', \beta', -Mc_{21}, m)R\begin{pmatrix} m \\ m \\ -Mc_{21} \\ Mc_{21} \end{pmatrix} = \begin{pmatrix} m \\ 1 \\ -Mc_{21} \\ 0 \end{pmatrix}.$$

Similarly because $\gcd(m, -Mc_{21}) = 1$, Multiplying by a suitable matrix $S(a,b,c,d)$ we get

$$S(a,b,c,d)\begin{pmatrix} m \\ 1 \\ -Mc_{21} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, multiplying by $h(-1,0,0)$, we have

$$h(-1,0,0)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using that γ is a symplectic matric in $\mathrm{Sp}_4(\mathbb{R})$, it follows that that $c_{12} = d_{12} = 0$ and that $d_{11} = 1$. This means that $\begin{pmatrix} a_{22} & b_{22} \\ Mc_{22} & d_{22} \end{pmatrix} \in \Gamma_0(M)$ and we can multiply γ by $RS(a_{22}, -b_{22}, -Mc_{22}, a_{22})R$ to get that $C = 0$ and $d_{12} = 0$. Summing up, so far we have multiplied γ by a sequence of matrices, say T, that transformed it into

$$T\gamma = \begin{pmatrix} 1 & a_{12} & b_{11} & b_{22} \\ 0 & a_{22} & b_{21} & b_{22} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & d_{21} & d_{22} \end{pmatrix}.$$

But since a symplectic matrix should have determinant 1, we get that

$$T\gamma = \begin{pmatrix} 1 & a_{12} & b_{11} & b_{22} \\ 0 & 1 & b_{21} & b_{22} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & d_{21} & 1 \end{pmatrix}.$$

Lastly, we notice that $T\gamma$ is an element of the form $Rh(\lambda, \mu, \kappa)R$. This finishes the proof. $\square$

6.2.5 The Maass Lifting with Level M

We now have the ingredients to prove the modularity of the Maass lifting with level.

6.2.5.1 Index shifting operators

We need to consider a suitable definition of the V_m operators

$$V_m : J_{\kappa,t}(\Gamma_0(M)^J) \to J_{\kappa,mt}(\Gamma_0(M)^J)$$

that were already presented in Chapter 5 for Jacobi forms with respect to the full Jacobi group $\mathrm{SL}_2(\mathbb{Z})^J$. We use the same notation for these operators, but it will be clear from the context which ones we mean. We let

$$M_{2,M}^m = \left\{ \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) : ad - Mbc = m, (a, M) = 1 \right\},$$

and then define

$$V_m(\phi_t)(\tau, z) = m^{\kappa-1} \sum_{\gamma \in \Gamma_0(M) \backslash M_{2,M}^m} (\phi_t|_\kappa \gamma)(\tau, z).$$

Taking the standard set of representatives

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d > 0, \gcd(a, M) = 1, ad = m, b \pmod{d} \right\}$$

for $\Gamma_0(M) \setminus M_{2,M}^m$, we have

$$V_m(\phi_t)(\tau, z) = m^{\kappa-1} \sum_{\substack{ad=m \\ \gcd(a,M)=1}} \sum_{b=0}^{d-1} d^{-k} \phi_t \left(\frac{a\tau + b}{d}, \frac{mz}{d} \right).$$

Using the Fourier expansion of $\phi_t(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\le 4tn}} c(n,r)e^{2\pi i(n\tau+rz)}$ and calculations exactly like the ones done in [21] on page 44, we obtain the Fourier expansion of $V_m(\phi_t)(\tau,z)$ in the following form

$$V_m(\phi_t)(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\le 4mtn}} \left(\sum_{\substack{a|(n,r,m)\\ \gcd(a,M)=1}} a^{\kappa-1} c\left(\frac{nm}{a^2},\frac{r}{a}\right)\right) e^{2\pi i(n\tau+rz)}. \tag{6.3}$$

Lemma 6.2.10 ([31], Lemma 3.1)**.** *If $\phi_t \in J_{\kappa,t}(\Gamma_0(M)^J)$, then $V_m(\phi_t) \in J_{\kappa,mt}(\Gamma_0(M)^J)$ and if $\phi_t \in J^c_{\kappa,t}(\Gamma_0(M)^J)$, then $V_m(\phi_t) \in J^c_{\kappa,mt}(\Gamma_0(M)^J)$.*

6.2.6 Modularity of the Maass Lift with Level

Theorem 6.2.11 ([31], Theorem 3.2)**.** *Let $M \ge 1$ be an integer. Let $\phi \in J^c_{\kappa,1}(\Gamma_0(M)^J)$. The map*

$$\mathcal{V} : J^c_{\kappa,1}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_0^{(2)}(M))$$

given by sending ϕ to

$$\mathcal{V}\phi(\tau,z) := \sum_{m\ge 1} V_m(\phi)(\tau,z)e^{2\pi i(m\tau')}$$

is an injective linear map.

We now prove the modularity of the Maass lift $\mathcal{V}(\phi)$, i.e. that $\mathcal{V}(\phi) \in M_\kappa(\Gamma_0^{(2)}(M))$.

Proof. The Maass lift $\mathcal{V}(\phi)$ is given by a Fourier-Jacobi expansion whose Fourier-Jacobi coefficients given by $V_m(\phi)(\tau,z)$ are Jacobi cusp forms (Lemma 6.2.10) of indices m with associated functions $\widetilde{V_m(\phi)}(\tau,z,\tau') = V_m(\phi)(\tau,z)e(2\pi i m\tau')$ on the Siegel upper half plane $\mathfrak{h}^2$. The correspondence given by equation (6.2) implies that $\widetilde{V_m(\phi)}(\tau,z,\tau')$ is $\Gamma_{M,\infty}(\mathbb{Z})$-invariant. This implies that $\mathcal{V}\phi$ is $\Gamma_{M,\infty}(\mathbb{Z})$-invariant. Using Theorem 6.2.9, it remains to show the invariance of $\mathcal{V}\phi$ with respect to R. The action of R on $\mathfrak{h}^2$ is given by

$$(\tau,z,\tau') \to (\tau',z,\tau).$$

To show that $\mathcal{V}(\phi)$ transforms properly under the action of R, which in turn is equivalent to showing that it is symmetric in τ and τ', it suffices to show that its Fourier coefficients are symmetric with respect to n and m. The Fourier coefficients of $\mathcal{V}\phi(\tau, z)$ are given in equation (6.2.10) which clearly shows that they are symmetric with respect to n and m. $\square$

Remark 6.2.12. We note that within our general framework, we present the ingredients of proving the modularity of the full level Maass lifting which are a special case ($M = 1$) of the ingredients above and which turn out equivalent to the proof given in Chapter 5.

1. We consider the standard maximal parabolic subgroup, $\Gamma_\infty(\mathbb{Z})$ of $\mathrm{Sp}_4(\mathbb{Z})$ of type $(2, 1)$.
2. The Jacobi group $\mathrm{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z}) \cong \Gamma_\infty(\mathbb{Z})$.
3. Correspondence (6.2) implies that the forms $V_m(\phi)$ are $\Gamma_\infty(\mathbb{Z})$-invariant.
4. One can show similarly that $\mathrm{Sp}_4(\mathbb{Z})$ is generated by $\Gamma_\infty(\mathbb{Z})$ and R.
5. Invariance of the lift with respect to R becomes clear from the form of the lift which is given by its Fourier-Jacobi expnasion $\sum_{m\geq 1} V_m(\phi)(\tau, z)e^{2\pi i m\tau'}$.

6.3 Mixed Level Lifting

Throughout this section we fix positive integers M and t such that $\gcd(t, M) = 1$. Our goal is to generalize Gritsenko's lifting $\mathcal{G} : J^c_{\kappa,t} \to S_\kappa(\Gamma[t])$ and the Maass lifting with level and hence provide a lifting

$$\mathcal{G}_M : J^c_{\kappa,t}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_M[t])$$

from the space of Jacobi forms of weight κ, index $t \geq 1$ and level $\Gamma_0(M)^J$ to a space of cuspidal paramodular forms with mixed level. In this section, we show we have a map

$$\mathcal{G}_M : J^c_{\kappa,t}(\Gamma_0(M)^J) \to M_\kappa(\Gamma_M[t])$$

and show that it commutes under certain involutions operators that we will define below.

6.3.1 Mixed Level $\Gamma_M[t]$

We start by defining the mixed level our lifted forms will have.

Definition 6.3.1. We define the subgroup of the paramodular group of paramodular level t with congruence level M to be

$$\Gamma_M[t] = \mathrm{Sp}_4(\mathbb{Q}) \bigcap \left\{ \begin{pmatrix} \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t^{-1}\mathbb{Z} \\ M\mathbb{Z} & Mt\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ Mt\mathbb{Z} & Mt\mathbb{Z} & t\mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}. \tag{6.4}$$

We show that indeed $\Gamma_M[t]$ is a subgroup of $\Gamma[t]$. The identity element $1_4 \in \Gamma_M[t]$. Decomposing every $g \in \Gamma_M[t]$ into a block matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A = \begin{pmatrix} * & t* \\ * & * \end{pmatrix}$, $B = \begin{pmatrix} * & * \\ * & t^{-1}* \end{pmatrix}$, $C = \begin{pmatrix} M* & Mt* \\ Mt* & Mt* \end{pmatrix}$, $D = \begin{pmatrix} * & * \\ t* & * \end{pmatrix}$, and applying the inverse formula for a matrix in block form given by $g^{-1} = \begin{pmatrix} {}^tD & -B \\ -C & {}^tA \end{pmatrix}$, we indeeed obtain that $g^{-1} \in \Gamma_M[t]$. It remains to show that if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, g' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \Gamma_M[t]$, then $gg' = \begin{pmatrix} AA' + bC' & AB' + BD' \\ CA' + DC' & CB' + DD' \end{pmatrix} \in \Gamma_M[t]$. Performing the multiplication of elements of each block matrix in gg' and analyzing elementarily the elements, we find out that gg' is indeed in $\Gamma_M[t]$.

Remark 6.3.2. 1. The group $\Gamma_M[t]$ has mixed levels in it; it is of paramodular level t and of congruence level M. We will call $\Gamma_M[t]$ a mixed level paramodular subgroup or congruence level paramodular group.

2. For $M = 1$, $\Gamma_M[t] = \Gamma[t]$; hence we can say that the paramodular group $\Gamma[t]$ is of congruence level 1.

3. The group $\mathrm{Sp}_4(\mathbb{Z})$ is recovered from the paramodular group $\Gamma[t]$ of level t by restricting to $t = 1$. Similarly, for $t = 1$, the congruence subgroup $\Gamma_M[t]$ of $\Gamma[t]$ is the congruence subgroup $\Gamma_0^{(2)}(M)$ of $\mathrm{Sp}_4(\mathbb{Z})$.

6.3.2 Standard Maximal Parabolic Subgroup of Type $(2,1)$ in $\Gamma_M[t]$

The standard maximal parabolic subgroup of type $(2,1)$ in $\Gamma_M[t]$ is $\Gamma_M[t] \cap C_{2,1}(\mathbb{Q})$. We will denote it by $\Gamma_{M,\infty}[t]$.

Lemma 6.3.3. *The standard maximal parabolic subgroup, $\Gamma_{M,\infty}[t]$, of $\Gamma_M[t]$ of type $(2,1)$ consists of elements in $\Gamma_M[t]$ of the following form*

$$\Gamma_{M,\infty}[t] = \left\{ \gamma \in \Gamma_M[t] \,\middle|\, \gamma = \begin{pmatrix} a & 0 & b & z' \\ x & \pm 1 & z & yt^{-1} \\ Mc & 0 & d & -x' \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \right\}.$$

Proof. Considering the form of the elements of the mixed level paramodular group in equation (6.4) and the form of the elements of $C_{2,1}(\mathbb{Q})$ as given in equation (6.5.13), we can clearly see that

$$\Gamma_{M,\infty}[t] = \left\{ \gamma = \begin{pmatrix} a & 0 & b & z' \\ x & u & z & yt^{-1} \\ Mc & 0 & d & -x' \\ 0 & 0 & 0 & u^{-1} \end{pmatrix} \right\},$$

where $\begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \in \Gamma_0(M)$ and $u \in \mathrm{GL}_1(\mathbb{Z})$. Because $\Gamma_{M,\infty}[t]$ is a subgroup of $\Gamma_M[t]$, every element $\gamma \in \Gamma_{M,\infty}[t]$ must have determinant equal to 1. But $\det(\gamma) = \det \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} uu^{-1} =$

1. This proves the final form of the elements of $\Gamma_{M,\infty}[t]$. $\square$

Let

$$\Delta(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 6.3.4. *The standard maximal parabolic subgroup of $\Gamma_M[t]$ of type $(2,1)$, $\Gamma_{M,\infty}[t]$, is generated by $\Gamma_{M,\infty}(\mathbb{Z})$ (the standard maximal parabolic subgroup of $\Gamma_0^{(2)}(M)$ of type $(2,1)$) and $\Delta(t)$, i.e.,*

$$\Gamma_{M,\infty}[t] = \langle \Gamma_{M,\infty}(\mathbb{Z}), \Delta(t) \rangle.$$

Proof. The inclusion $\supseteq$ is obvious. To prove the reverse inclusion, we start with a general element

$$\gamma = \begin{pmatrix} a & 0 & b & z' \\ x & \pm 1 & z & yt^{-1} \\ Mc & 0 & d & -x' \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}$$

of $\Gamma_M[t]$ for some $a, b, c, d, x, y, x \in \mathbb{Z}$ such that $ad - Mbc = 1$. Let,

$$\alpha := \Delta(t)^{\pm 1 y(t-1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \pm 1 \frac{y(t-1)}{t} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$\alpha\gamma = \begin{pmatrix} a & 0 & b & z \\ x & \pm 1 & z & \frac{y}{t} + \left(y - \frac{y}{t}\right) \\ c & 0 & d & -x \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} a & 0 & b & z \\ x & \pm 1 & z & y \\ c & 0 & d & -x \\ 0 & 0 & 0 & \pm 1 \end{pmatrix},$$

which is clearly an element $\beta \in \Gamma_{M,\infty}(\mathbb{Z})$. That is, we have shown that

$$\gamma = \Delta(t)^{-(\pm 1 y(t-1))}\beta \in \langle \Gamma_{M,\infty}(\mathbb{Z}), \Delta(t) \rangle.$$

$\square$

Remark 6.3.5. We note that we can prove Proposition 6.3.4 similar to how we proved Theorem 6.2.4. We use the same projection map

$$\begin{aligned} \omega_1 : \Gamma_{M,\infty}[t] &\rightarrow \Gamma_0(M) \\ \gamma &\mapsto \begin{pmatrix} a & b \\ Mc & d \end{pmatrix}, \end{aligned}$$

which is a surjective group homorphism onto the $\Gamma_0(M)$-part. However its kernel is now the set of all matrices

$$H := \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa t^{-1} \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Using the same injective map i

$$\begin{aligned} i : \Gamma_0(M) &\longrightarrow \Gamma_{M,\infty}[t] \\ \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} &\mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

we obtain an exact sequence

$$1 \to H \to \Gamma_{M,\infty}[t] \to \Gamma_0(M) \to 1,$$

which splits because of the injection i. Consequently, we have that

$$\Gamma_{M,\infty}[t] \cong \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \ltimes \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \varkappa t^{-1} \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

where the action of $\Gamma_0(M)$ on H is given by

$$g \mapsto ((X, \varkappa) \mapsto (Xg, \varkappa)),$$

for $(X = (\lambda, \mu), \varkappa) \in H(\mathbb{Z})$, $g \in \Gamma_0(M)$ and Xg means row multiplication by a matrix.

This shows that every element γ of the parabolic subgroup $\Gamma_{M,\infty}[t]$ is written uniquely as a product of an element in $i(\Gamma_0(M))$ and an element of H. We have

$$\Gamma_{M,\infty}[t] = \left\{ \gamma \in \Gamma_M[t] \middle| \gamma = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & z \\ x & 1 & z & yt^{-1} \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \in \Gamma_0(M) \quad x, y, z \in \mathbb{Z} \right\}.$$

However, every element in H can be decomposed as

$$\begin{pmatrix} 1 & 0 & 0 & z \\ x & 1 & z & y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{y-yt} .$$

Remark 6.3.6. We note that the set of elements

$$H := \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa t^{-1} \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

is not isomorphic to the Heisenberg group $H(\mathbb{Z})$ because multiplying two elements in H yields the following multiplication

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa t^{-1} \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu' \\ \lambda' & 1 & \mu' & \kappa' t^{-1} \\ 0 & 0 & 1 & -\lambda' \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu + \mu' \\ \lambda + \lambda' & 1 & \mu + \mu' & \lambda\mu' - \mu\lambda' + t^{-1}(\kappa + \kappa') \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

This shows that H corresponds to the following group

$$H(t, \mathbb{Z}) = \{(\lambda, \mu, \kappa) \mid \lambda, \mu, \kappa \in \mathbb{Z}\}$$

with the group law

$$(\lambda,\mu,\kappa).(\lambda',\mu',\kappa') = \left(\lambda+\lambda', \mu+\mu', \begin{vmatrix} \lambda & \mu \\ \lambda' & \mu' \end{vmatrix} + t^{-1}(\kappa+\kappa')\right).$$

6.3.3 Mixed Level Paramodular Forms

Since $\Gamma_M[t]$ is a subgroup of $\mathrm{Sp}_4(\mathbb{Q})$, it also acts on the Siegel upper half plane $\mathfrak{h}^2$ by fractional linear transformations. We can use this to define modular forms of mixed level.

Definition 6.3.7. Fix an integer κ. A holomorphic function $F : \mathfrak{h}^2 \to \mathbb{C}$ is said to be a mixed level paramodular form of weight κ if it is a paramodular form with respect to $\Gamma_M[t]$, i.e., it satisfies $F|_\kappa\gamma = F$ for all $\gamma \in \Gamma_M[t]$. We denote the space of such paramodular forms by $M_\kappa(\Gamma_M[t])$. The space of cuspidal mixed level paramodular forms is denoted by $S_\kappa(\Gamma_M[t])$.

Proceeding as in the case of Siegel modular forms $F \in M_\kappa(\Gamma_0^{(2)}(M))$ in Chapter 5, § 5.2.2.2, mixed level paramodular forms have a Fourier-Jacobi expansion

$$\begin{aligned} F(Z) &= \sum_{m'\geq 0}\left(\sum_{\substack{n\geq 0, r\in\mathbb{Z}\\ 4nm'\geq r^2}} a_F(n,r,m')e^{2\pi i(n\tau+rz)}\right)e^{2\pi i m'\tau'} \\ &= \sum_{m'\geq 0}\phi_{m'}(\tau,z)e^{2\pi i m'\tau'}. \end{aligned}$$

However, when dealing with paramodular forms, we will show below that the Fourier-Jacobi coefficients of such forms are Jacobi forms with indices divisible by t.

Theorem 6.3.8. *Let $F \in M_\kappa(\Gamma_M[t])$ be nonzero. We can rewrite its Fourier-Jacobi expansion as*

$$F(\tau,z,\tau') = \sum_{m\geq 0}\phi_{mt}(\tau,z)e^{2\pi i(mt)\tau'}.$$

Then, for each m, its Fourier-Jacobi coefficient ϕ_{mt} belongs to $J_{\kappa,mt}(\Gamma_0(M)^J)$. Moreover, if $F \in S_\kappa(\Gamma_M[t])$, then $\phi_{mt} \in J^c_{\kappa,mt}(\Gamma_0(M)^J)$ for each $m \geq 1$.

Proof. We have that the Fourier-Jacobi expansion of $F \in M_\kappa(\Gamma_M[t])$ is $F(\tau, z, \tau') = \sum_{m' \geq 0} \phi_{m'}(\tau, z) e^{2\pi i m' \tau'}$. We use the invariance of F with respect to the standard maximal parabolic subgroup $\Gamma_{M,\infty}[t]$. We have seen in Proposition 6.3.4 that $\Gamma_{M,\infty}[t] = \langle \Gamma_{M,\infty}(\mathbb{Z}), \Delta(t) \rangle$. Applying the invariance of F with respect to $\Gamma_{M,\infty}(\mathbb{Z})$ to particular matrices of the following type

$$g_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and using the same arguments as in the proof of Theorem 5.2.17, we get that $\phi_{m'} \in J_{\kappa,m'}(\Gamma_0(M)^J)$. Also, using the same cuspidality argument as in the proof of the same theorem, we have that if $F \in S_\kappa(\Gamma_M[t])$, then $\phi_{m'} \in J^c_{\kappa,m'}(\Gamma_0(M)^J)$.

It remains to show that in case of paramodular forms of level t, then the Fourier-Jacobi coefficients have indices divisible by t. The invariance of F with respect to $\Delta(t)$,

$$(F|_\kappa \Delta(t))(\tau, z, \tau') = F(\tau, z, \tau'),$$

gives the following relation:

$$(F|_\kappa \Delta(t))(\tau, z, \tau') = F(\tau, z, \tau' + 1/t) = \sum_{m' \geq 0} \phi_{m'}(\tau, z) e^{2\pi i m' \tau} e^{2\pi i m'/t} = F(\tau, z, \tau') = \sum_{m' \geq 0} \phi_{m'}(\tau, z) e^{2\pi i m' \tau}.$$

This implies that

$$\phi_{m'}(\tau, z) e^{2\pi i m'/t} = \phi_{m'}(\tau, z),$$

which is only true when $\phi_{m'}$ is zero or when $t | m'$. Therefore, the Fourier-Jacobi coefficients of a nonzero paramodular modular form F with paramodular level t must have their indices divisble by t. Hence, we write the Fourier-Jacobi expansion of F as

$$F(\tau, z, \tau') = \sum_{m \geq 0} \phi_{mt}(\tau, z) e^{2\pi i (mt)\tau'}$$

and its Fourier-Jacobi coefficients belong to $J_{\kappa,mt}(\Gamma_0(M) \ltimes H(\mathbb{Z}))$. $\square$

The same proof works when we substitute $M = 1$ and we get:

Corollary 6.3.9. *Let $F \in M_\kappa(\Gamma[t])$. For each m, the Fourier-Jacobi coefficient ϕ_{mt} of F belongs to $J_{\kappa,mt}(\mathrm{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z}))$. If $F \in S_\kappa(\Gamma[t])$, then for each m, ϕ_{mt} belongs to $J^c_{\kappa,mt}(\mathrm{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z}))$.*

The Fourier-Jacobi expansion of a mixed level paramodular form (Theorem 6.3.8) gives an injective map

$$\mathcal{H} : M_\kappa(\Gamma_M[t]) \to \prod_{m \geq 0} J_{\kappa,mt}(\Gamma_0(M)^J),$$

and a corresponding injective map for the space of cusp forms

$$\mathcal{H}^c : S_\kappa(\Gamma_M[t]) \to \prod_{m \geq 1} J^c_{\kappa,mt}(\Gamma_0(M)^J).$$

In case $M = 1$, these maps reduce to

$$\mathcal{H} : M_\kappa(\Gamma[t]) \to \prod_{m \geq 0} J_{\kappa,mt},$$

and a corresponding map for the space of cusp forms

$$\mathcal{H}^c : S_\kappa(\Gamma[t]) \to \prod_{m \geq 1} J^c_{\kappa,mt}.$$

Proposition 6.3.10. *The standard maximal parabolic subgroup $\Gamma_{M,\infty}[t]$ of type $(2,1)$ stabilizes the Fourier-Jacobi expansion of a mixed level paramodular form $F \in M_\kappa(\Gamma_M[t])$.*

Proof. Let $F \in M_\kappa(\Gamma_M[t])$ with Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \sum_{m \geq 1} \phi_{mt}(\tau, z) e^{2\pi i m t \tau'} = \sum_{m \geq 1} \widetilde{\phi_{mt}}(\tau, z, \tau').$$

Correspondence (6.2) implies that $\Gamma_{M,\infty}(\mathbb{Z})$ stabilizes the Fourier-Jacobi expansion of F. Now,

$$\begin{aligned}
\left(\widetilde{\phi_{mt}}|_\kappa \Delta(t)\right)(\tau, z, \tau') &= \widetilde{\phi_{mt}}(\tau, z, \tau' + \kappa t^{-1}) \\
&= \phi_{mt}(\tau, z) e^{2\pi i m t(\tau' + \kappa t^{-1})} \\
&= \phi_{mt}(\tau, z) e^{2\pi i m t \tau'} e^{2\pi i m \kappa} \\
&= \phi_{mt}(\tau, z) e^{2\pi i m t \tau'} \\
&= \widetilde{\phi_{mt}}(\tau, z, \tau').
\end{aligned}$$

Using Proposition 6.3.4, we have that $\Gamma_{M,\infty}[t]$ stabilizes the Fourier-Jacobi expansion of $F \in M_\kappa(\Gamma_M[t])$. In other words, the Fourier-Jacobi coefficients of a mixed level paramodular form F of level $\Gamma_M[t]$ are $\Gamma_{M,\infty}[t]$-invariant on $\mathfrak{h}^2$. $\square$

6.3.4 The Generalized Gritsenko Lifting

Before we give our generalization of Gritsenko's lifting, we give a theorem giving the generators of the mixed level group $\Gamma_M[t]$. The proof of this theorem will be given later in section § 6.5.3.

Let $V_t \in \mathrm{Sp}_4(\mathbb{R})$ be the transformation given by

$$V_t = \begin{pmatrix} {}^tU_t & 0 \\ 0 & U_t \end{pmatrix}$$

where

$$U_t = \begin{pmatrix} 0 & (\sqrt{t})^{-1} \\ \sqrt{t} & 0 \end{pmatrix}.$$

Theorem 6.3.11. *The generators of the mixed level $\Gamma_M[t]$ are given by*

$$\Gamma_M[t] = \langle \Gamma_{M,\infty}[t], V_t \Gamma_{M,\infty}[t] V_t \rangle.$$

Theorem 6.3.12. *Let ϕ_t be a Jacobi cusp form of weight $\kappa \geq 2$, index t, and level $\Gamma_0(M)^J$ with Fourier expansion*

$$\phi_t(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, n \geq 0 \\ 4nt \geq r^2}} c(n,r) e^{2\pi i (n\tau + rz)}.$$

Then, the series $\mathcal{G}_M(\phi_t)$ given by

$$\mathcal{G}_M(\phi_t)(\tau, z, \tau') := \sum_{m \geq 1} V_m(\phi_t)(\tau, z) e^{2\pi i m t \tau'}$$

lies in $M_\kappa(\Gamma_M[t])$.

Proof. We assume that the series defines a holomorphic function and proceed to prove modularity of the lift. We will come back and prove the holomorphicity of the lift in a separate part. Since $V_m : J_{\kappa,t}(\Gamma_0(M)^J) \to J_{\kappa,mt}(\Gamma_0(M)^J)$, we have that $V_m(\phi_t) \in J^c_{\kappa,mt}(\Gamma_0(M)^J)$. The function $\mathcal{G}_M(\phi_t)$ is then clearly $\Gamma_{M,\infty}(\mathbb{Z})$-invariant function since it is the uniform limit of $\Gamma_{M,\infty}(\mathbb{Z})$-invariant functions. The lift $\mathcal{G}_M(\phi_t)$ is given by a Fourier-Jacobi expansion such that every Fourier-Jacobi coefficient is a Jacobi form of index divisible by t. Proposition 6.3.10 implies that $\mathcal{G}_M(\phi_t)$ is also invariant under $\Delta(t)$ and hence according to Proposition 6.3.4, it is invariant under $\Gamma_{M,\infty}[t]$.

We calculate the Fourier expansion of $\mathcal{G}_M(\phi_t)$ by applying the action of the linear operators V_m on the Fourier expansion of

$$\phi_t(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 < 4nt}} c(n,r) e^{2\pi i (n\tau + rz)}$$

given by

$$V_m(\phi_t)(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ r^2<4nmt}} c_m(n,r)e^{2\pi i(n\tau+rz)}$$

where

$$c_m(n,r) = \sum_{\substack{d|(n,r,m)\\ (d,M)=1}} d^{\kappa-1}c\left(\frac{nm}{d^2},\frac{r}{d}\right).$$

Therefore,

$$\mathcal{G}_M(\phi_t)(\tau,z,\tau') = \sum_{m\geq 1}\sum_{\substack{n,r\in\mathbb{Z},n\geq 0\\ 4nmt>r^2}}\sum_{\substack{d|(n,r,m)\\ (d,M)=1}} d^{k-1}c\left(\frac{nm}{d^2},\frac{r}{d}\right)e^{2\pi i(n\tau+rz)}e^{2\pi imt\tau'}.$$

It is clear that this Fourier expansion is invariant under the substitution $n\to m$ and $m\to n$. This shows that $\mathcal{G}_M(\phi_t)(\tau,z,\tau')$ is invariant under the change of variables

$$\begin{aligned}\tau &\mapsto t\tau',\\ \tau' &\mapsto t^{-1}\tau.\end{aligned}$$

The element $V_t\in \mathrm{Sp}_4(\mathbb{R})$ realizes this transformation as

$$V_t(\tau,z,\tau') = (t\tau',z,t^{-1}\tau).$$

Computing the action of V_t on $\mathcal{G}_M(\phi_t)$, we obtain

$$\mathcal{G}_M(\phi_t)(\tau,z,\tau')|_\kappa V_t = (-1)^\kappa\mathcal{G}_M(\phi_t)(t\tau',z,t^{-1}\tau) = (-1)^\kappa\mathcal{G}_M(\phi_t)(\tau,z,\tau'). \tag{6.5}$$

Now,

$$\begin{aligned}\mathcal{G}_M(\phi_t)|_\kappa V_t\Gamma_{M,\infty}[t]V_t &= (-1)^\kappa\mathcal{G}_M(\phi_t)|_\kappa\Gamma_{M,\infty}[t]V_t\\ &= (-1)^\kappa\mathcal{G}_M(\phi_t)|_\kappa V_t\\ &= (-1)^{2\kappa}\mathcal{G}_M(\phi_t)\\ &= \mathcal{G}_M(\phi_t).\end{aligned}$$

The modularity of the lift with respect to the group $\Gamma_M[t]$ follows now from Theorem 6.3.11. $\square$

6.3.4.1 Holomorphicity of the Lift

To prove that the lift $\mathcal{G}_M$ is holomorphic, we need to show that the series defining it absolutley and uniformly converges on subsets of the Siegel upper half space of the form $Y = \mathrm{Im}\,(Z) > c1_2$ (see Chapter 2). Let $c > 0$ be a real number. The proof is similar to the proof given in the Maass and Gritsenko lifting. We note that Gritsenko's proof is based on an estimate on Jacobi forms in a fundamental domain of the action of the Jacobi group on $\mathfrak{h}^1 \times \mathbb{C}$. This estimate on the Jacobi forms and the fundamental domain was already proven by Klingen in [38]. We also note that Choie had pointed us to another proof following the theta expansion of Jacobi forms with respect to a congruence subgroup. The theta expansion of Jacobi forms with respect to a congruence subgroup which can be found in the proof of Theorem 2 in [34] and also in [83] is exactly similar to the theta expansion of Jacobi forms developed in Chapter 5. The Fourier coefficients of a Jacobi form ϕ_t of index t depend only on $r \pmod{2t}$ and on $4tn - r^2$, so we define $c_\delta(N = 4tn - r^2) = c\left(\frac{N+r^2}{4t}, r\right)$, where $r \equiv \delta \pmod{2t}$ and $c_\delta(N) = 0$ if $N \not\equiv -\delta^2 \pmod{2t}$. We then have

$$\phi(\tau, z) = \sum_{\delta \pmod{2t}} h_\delta(\tau)\theta_{t,\delta}(\tau, z), \tag{6.6}$$

where

$$h_\delta(\tau) := \sum_{N \geq 0} c_\delta(N) e^{2\pi i(\frac{N}{4t})\tau}, \qquad (\delta \in \mathbb{Z}/2t\mathbb{Z}),$$

are modular forms in $M_{\kappa-1/2}(\Gamma_0(4t), \mu)$ with μ a certain character and

$$\theta_{t,\delta}(\tau, z) \qquad \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \delta \pmod{2m}}} e^{2\pi i\left(\frac{r^2}{4t}\tau + rz\right)}$$

are theta functions independent of the Jacobi form ϕ and are exactly the ones used in Chapter 5.

There is an estimate on the Fourier coefficients of modular forms with a certain character as in [57], which was cited in Shimura's article [79]. This estimate is given by

$$|c_\delta(N)| \leq C_\delta(\phi_t) N^{\kappa/2-1/4},$$

$C_\delta(\phi_t)$ is a constant depending on ϕ_t and δ. Now taking $C = \max_\delta C_\delta(\phi_t)$ and using equation (6.6), we obtain a bound on the coefficients of the Jacobi form ϕ_t

$$|c(n,r)| \leq C N^{\kappa/2-1/4}.$$

The Fourier expansion of the lift is

$$\begin{aligned}
\mathcal{G}_M(\phi_t)(\tau, z, \tau') &= \sum_{m\geq 1} \sum_{\substack{n,r\in\mathbb{Z}, n\geq 0 \\ 4nmt>r^2}} \sum_{\substack{d|(n,r,m) \\ (d,M)=1}} d^{k-1} c\left(\frac{nm}{d^2}, \frac{r}{d}\right) e^{2\pi i(n\tau+rz)} e^{2\pi i m t\tau'} \\
&= \sum_{T\in\mathbb{S}_{2,t}^{>0}(\mathbb{Z})} a_{\mathcal{G}_M(\phi_t)}(T) e^{2\pi i \operatorname{Tr}(TZ)},
\end{aligned}$$

where the summation is taken over $\mathbb{S}_{2,t}^{>0}(\mathbb{Z}) = \left\{ \begin{pmatrix} n & r/2 \\ r/2 & mt \end{pmatrix} \,\middle|\, n, r, m \in \mathbb{Z}, r^2 < 4nmt \right\}$.

Therefore, the Fourier coefficients of the lift satisfy

$$\begin{aligned}
|a_{\mathcal{G}_M(\phi_t)}(T)| &= \left|\sum_{\substack{d|(n,r,m)\\(d,M)=1}} d^{\kappa-1}c\left(\frac{nm}{d^2},\frac{r}{d}\right)\right| \\
&\leq \sum_{\substack{d|(n,r,m)\\(d,M)=1}} d^{\kappa-1}\left|c\left(\frac{4nmt}{d^2}-\frac{r^2}{d^2}\right)\right| \\
&\leq C\sum_{\substack{d|(n,r,m)\\(d,M)=1}} d^{\kappa-1}\left(\frac{4nmt}{d^2}-\frac{r^2}{d^2}\right)^{\kappa/2-1/4} \\
&\leq C(4nmt-r^2)^{\kappa/2-1/4}\sum_{\substack{d|(n,r,m)\\(d,M)=1}} d^{\kappa-1-\kappa+1/2} \\
&\leq C(\det T)^{\kappa/2-1/4}\sum_{d=1}^{n} d^{-1/2} \\
&\leq C(\det T)^{\kappa/2-1/4}n,
\end{aligned}$$

where the last inequality follows because $d > 1$. Using the formula $\det T = e^{\mathrm{Tr}(\log\det(T))}$ together with the inequality $\log(n) < n$, we can say that there is a constant C_1 such that

$$|a(T)| \leq C_1 e^{\pi c\,\mathrm{Tr}(T)}.$$

Now,

$$\begin{aligned}
|\mathcal{G}_M(\phi_t)(\tau,z,\tau')| &\leq \sum_{T\in\mathbb{S}_{2,t}^{\geq 0}(\mathbb{Z})} |a_{\mathcal{G}_M(\phi)}(T)|e^{-2\pi\,\mathrm{Tr}(T\,\mathrm{Im}(Z))} \\
&\leq C_1\sum_{\mathbb{S}_{2,t}^{\geq 0}(\mathbb{Z})} e^{-2\pi\,\mathrm{Tr}\,(T(Y-\frac{c}{2}1_2))}.
\end{aligned}$$

That is, for $\mathrm{Im}(Z) > \frac{c}{2}1_2$,

$$|\mathcal{G}_M(\phi_t)(\tau,z,\tau')| \leq C_1\sum_{T>0} e^{-\pi c\,\mathrm{Tr}\,(T)} < \infty.$$

The proof of the convergence of the majorant series is very classical and is shown as in the proof of the convergence of the Fourier series of a Siegel modular form as in Chapter 2.

Corollary 6.3.13. *The map* $\mathcal{G}_M : J^c_{\kappa,t}(\Gamma_0(M)^J) \to M_\kappa(\Gamma_M[t])$ *is linear and injective.*

Proof. The holomorphicity of the lift together with the form of the lift shows clearly that it is linear. For injectivity, since the lifting is decribed by its Fourier-Jacobi expansion we see that if we start with a Jacobi cusp form $\phi_t \neq 0$, then $\mathcal{G}_M(\phi_t) \neq 0$. □

6.3.4.2 Involutions V_d

For each divisor $d \mid t$ such that $\gcd(t, t/d) = 1$, we can find integers $x, y \in \mathbb{Z}$ such that $xd - y\frac{t}{d} = 1$. We have involutions as defined in [28] on page 4 and on page 11, given by

$$V_d = \begin{pmatrix} \sqrt{d}x & -\frac{1}{\sqrt{d}}t & 0 & 0 \\ -\frac{1}{\sqrt{d}}y & \sqrt{d} & 0 & 0 \\ 0 & 0 & \sqrt{d} & \frac{1}{\sqrt{d}}y \\ 0 & 0 & \frac{1}{\sqrt{d}}t & \sqrt{d}x \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R}).$$

We note that $xd + y\frac{t}{d} = -1 \pmod{2d}$ and $xd + y\frac{t}{d} = 1 \pmod{2\frac{t}{d}}$. For all V_d, one can verify that

$$V_d^2 \in \Gamma_M[t], \qquad V_d\Gamma_M[t]V_d = \Gamma_M[t].$$

Thus the V_d are involutions for the group $\Gamma_M[t]$. In particular, when $d = t$, we can write V_t in the following way:

$$V_t = \begin{pmatrix} 0 & \sqrt{t} & 0 & 0 \\ \frac{1}{\sqrt{t}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{t}} \\ 0 & 0 & \sqrt{t} & 0 \end{pmatrix},$$

which is the involution that occured in the proof of Theorem 6.3.12 above.

We have seen in Chapter 5, §5.2.1.1, that there is a decomposition for the space of Jacobi forms due to the involutions W_d for each divisor $d \mid t$ such that $\gcd(t, t/d) = 1$. We have in the exact same way as proven in Chapter 5 that

$$J^c_{\kappa,t}(\Gamma_0(M)^J) = \bigoplus_{\epsilon_d} J^{c,\epsilon_d}_{\kappa,t}(\Gamma_0(M)^J).$$

The subspace $J^{c,\epsilon_d}_{\kappa,t}(\Gamma_0(M)^J)$ is the eigenspace under the action of the involution W_d, namely

$$J^{c,\epsilon_d}_{\kappa,t}(\Gamma_0(M)^J) = \{\phi \in J^c_{\kappa,t}(\Gamma_0(M)^J) : \phi|_\kappa W_d = \epsilon_d \phi\},$$

and ϵ_d is the eigenvalue at the involution W_d such that

$$\prod_{\substack{d|t \\ \gcd(d,\frac{t}{d})=1}} \epsilon_d = \epsilon_t = (-1)^\kappa.$$

We show that the lifting $\mathcal{G}_M$ commutes with these involutions. In other words, we have:

Theorem 6.3.14. *Let ϕ_t be a Jacobi cusp form in $J^c_{\kappa,t}(\Gamma_0(M)^J)$ and $\mathcal{G}_M(\phi_t)$ be its lift. For each divisor d of t with $\gcd(t, \frac{t}{d}) = 1$, we have*

$$\mathcal{G}_M(\phi_t)|_\kappa V_d = \mathcal{G}_M(\phi_t|_\kappa W_d).$$

Proof. The proof goes exactly like the proof of Theorem 2.1 given by Gritsenko and Hulek in [28]. Writing each matrix V_d in a block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the action of V_d on the lift $\mathcal{G}_M(\phi_t)$ is given by

$$(\mathcal{G}_M(\phi_t)|_\kappa V_d)(Z) = \det(CZ+D)^{-\kappa} \mathcal{G}_M(\phi_t)(V_d Z).$$

Because we have the relation $xd - y\frac{t}{d} = 1$, then $\det(CZ+D) = 1$. It remains to compute the action of V_d on an element $Z \in \mathfrak{h}^2$. Taking

$$A_d = \begin{pmatrix} dx & -t \\ -y & d \end{pmatrix},$$

we observe that

$$V_d Z = d^{-1} A_d Z^t A_d.$$

Hence

$$\begin{aligned}
(\mathcal{G}_M(\phi_t)|_\kappa V_d)(Z) &= \mathcal{G}_M(\phi_t)(d^{-1} A_d Z\,{}^t A_d) \\
&= \sum_{T \in \mathbb{S}_{2,t}^{>0}(\mathbb{Z})} a_{\mathcal{G}_M(\phi_t)}(T) e^{2\pi i \operatorname{Tr}(d^{-1}\,{}^t A_d T A_d Z)} \\
&= \sum_{T \in \mathbb{S}_{2,t}^{>0}(\mathbb{Z})} a_{\mathcal{G}_M(\phi_t)}(d^{-1}\,{}^t\tilde{A}_d T \tilde{A}_d) e^{2\pi i \operatorname{Tr}(TZ)},
\end{aligned}$$

where $\tilde{A}_d = dA_d^{-1} = \begin{pmatrix} d & t \\ y & dx \end{pmatrix}$. Let $\tilde{T} = d^{-1}\,{}^t\tilde{A}_d T \tilde{A}_d = \begin{pmatrix} \tilde{n} & \tilde{r}/2 \\ \tilde{r}/2 & \tilde{m}t \end{pmatrix}$. Then, $\det T = \det \tilde{T}$ and

$$\begin{aligned}
\tilde{n} &= dn + yr + y^2 m \tfrac{t}{d}, \\
\tilde{m} &= n + xr + dx^2 m,
\end{aligned}$$

and

$$\tilde{r} = r(y\frac{t}{d} + dx) + 2(nt + xymt) \quad \text{and} \quad \tilde{r} \equiv \begin{cases} -r & \pmod{2d} \\ r & \pmod{2\frac{t}{d}}. \end{cases}$$

Hence the elements n, r, m and $\tilde{n}, \tilde{m}, \tilde{r}$ have a common set of divisors, and the Fourier coefficients of $\mathcal{G}_M(\phi_t)|_\kappa V_d$ are given by:

$$a_{\mathcal{G}_M(\phi_t)}(d^{-1}\,{}^t\tilde{A}_d T \tilde{A}_d) = \sum_{d | \gcd(n,r,m)} d^{\kappa-1} c\left(\frac{\tilde{n}\tilde{m}}{d^2}, \frac{\tilde{r}}{d}\right)$$

which are the Fourier coefficients of $\phi_t|_\kappa W_d$ (see Theorem 5.2.4 in Chapter 5). That is, we have shown that

$$\mathcal{G}_M(\phi_t)|_\kappa V_d = \mathcal{G}_M(\phi_t|_\kappa W_d) = \mathcal{G}_M(\epsilon_d \phi_t) = \epsilon_d \mathcal{G}_M(\phi_t),$$

and that the lifting preserves the eigenvalue at the involution W_d for each $d \mid t$ such that $\gcd(d, \frac{d}{t}) = 1$. This finishes the proof. $\square$

In particular, we have seen during the proof of Theorem 6.3.12 that

$$\mathcal{G}_M(\phi_t)|_\kappa V_t = (-1)^\kappa \mathcal{G}_M(\phi_t).$$

6.3.5 Gritsenko Liftings

Since we are characterizing modularity of lifts between Jacobi forms and Siegel modular forms of degree 2, we now state Gritsenko's lifting in order to present the ingredients that went in the proof of its modularity.

Theorem 6.3.15. *([26], Theorem 3) Let ϕ_t be a Jacobi cusp form of weight $\kappa \geq 2$, index $t \geq 1$ and level $\mathrm{SL}_2(\mathbb{Z})^J$ with the Fourier expansion*

$$\phi_t(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, n \geq 0 \\ 4nt \geq r^2}} c(n,r) e^{2\pi i(n\tau + rz))}.$$

Then

$$\mathcal{G}(\phi_t)(\tau, z, \tau') := \sum_{m \geq 1} V_m(\phi_t)(\tau, z) e^{2\pi i m t \tau'}$$

is a paramodular cusp form of weight κ with respect to the paramodular group $\Gamma[t]$. The map $\mathcal{G} : J^c_{\kappa,t} \to S_\kappa(\Gamma[t])$ is injective.

Remark 6.3.16.

1. For index $t = 1$, the lifting $\mathcal{G} : \phi_1 \to \mathcal{G}(\phi_1)$ is the full level Maass lifting.
2. Gritsenko's theorem says that the Maass lifting is only the first member in an infinite

series of liftings. In particular, given $F \in S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ with Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \sum_{m \geq 1} \phi_m(\tau, z) e(2\pi i m \tau'),$$

we get an infinite series of lifted functions $\mathcal{G}(\phi_m) \in S_\kappa(\Gamma[m])$. Thus, we have a map $S_\kappa(\mathrm{Sp}_4(\mathbb{Z})) \to \prod_{m\geq 1} S_\kappa(\Gamma[m])$ given by

$$\begin{array}{ccccc} S_\kappa(\mathrm{Sp}_4(\mathbb{Z})) & \longrightarrow & \prod_{m\geq 1} J^c_{\kappa,m} & \longrightarrow & \prod_{m\geq 1} S_\kappa(\Gamma[m]) \\ & & & & \\ F & \longmapsto & (\phi_m)_{m\geq 1} & \longmapsto & (\mathcal{G}(\phi_m))_{m\geq 1}. \end{array}$$

We now give the ingredients of the proof of the modularity of the Gritsenko lift.

1. Starting from a Jacobi cusp form ϕ_t of index t, the forms $V_m(\phi_t)$ have all indices divisible by t. Hence forming the lift in degree 2 by its Fourier-Jacobi expansion whose Fourier-Jacobi coefficients are $V_m(\phi_t)$ shows that the lift is invariant with respect to $\Delta(t)$.

2. Correspondence (6.2) implies that all forms $\widetilde{V_m(\phi_t)}$ are invariant under the standard maximal parabolic subgroup $\Gamma_\infty(\mathbb{Z})$ of $\mathrm{Sp}_4(\mathbb{Z})$ of type $(2, 1)$.

3. Now, $\Gamma_\infty(\mathbb{Z})$ together with $\Delta(t)$ generate the standard maximal parabolic subgroup $\Gamma_\infty[t]$ of the paramodular group $\Gamma[t]$ of type $(2, 1)$.

4. The Fourier-Jacobi expansion of the lift show that it belongs to the eigenspace of the involution V_t. (Note that when $t = 1$, V_t reduces to R.)

5. One can show that the paramodular group is generated by $\Gamma_\infty[t]$ and by $V_t\Gamma_\infty[t]V_t$ due to Theorem 6.5.13 and Remark 6.5.16.

6.4 Characterization of Modularity of Liftings from Jacobi Forms to Siegel Modular Forms of degree 2

Based on the general framework we provided above for proving the Maass lifting (full level and level M), Gritsenko's lifting and mixed level liftings, we now present modularity ingredients of any lifting from the space of Jacobi forms of integral weight and index to Siegel modular forms of degree 2. Starting with a Jacobi cusp form of index 1 with respect to a Jacobi group $\Gamma \ltimes H(Z)$, where Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we consider the embedding $i(\Gamma)$ and $h(H(\mathbb{Z}))$ in $\mathrm{Sp}_4(\mathbb{Z})$. Then, $i(\Gamma) \ltimes h(H(\mathbb{Z}))$ is the maximal standard parabolic subgroup of type $(2,1)$ of the subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ generated by $i(\Gamma) \ltimes h(H(\mathbb{Z}))$ and the involution R. We consider the correct notion of V_m operators:

$$V_m(\phi)(\tau, z) = m^{\kappa-1} \sum_{\gamma \in \Gamma \backslash M_{2,M}^m} (\phi|_{\kappa,1}\gamma)(\tau, z) \tag{6.7}$$

and form the lift F_ϕ by its Fourier-Jacobi expansion.

For any congruence subgroup Γ of $\mathrm{Sp}_4(\mathbb{Z})$, we take its intersection $\Gamma \cap C_{2,1}(\mathbb{Q})$ to obtain the standard maximal parabolic subgroup of type $(2,1)$ in Γ. We denote it by $\Gamma_{2,1}$. Consider the projection w_1 of $\Gamma_{2,1}$ onto the SL_2 part. Then $w_1(\Gamma_{2,1})$ is a congruence subgorup of $\mathrm{SL}_2(\mathbb{Z})$. Moreover,

$$w_1(\Gamma_{2,1}) \ltimes H(Z) \cong \Gamma_{2,1}.$$

Hence, $w_1(\Gamma_{2,1}) \ltimes H(Z)$ is the level of the Jacobi form that we should consider. The generators of Γ should be in terms of $\Gamma_{2,1}$ and a certain involution. This involution depends on the index of the Jacobi form we are lifting. If the index is 1, then the involution is R. Next, we find a suitable definition of the V_m operators. These operators should be defined in the following way:

$$V_m(\phi)(\tau, z) = m^{\kappa-1} \sum_{\gamma \in w_1(\Gamma_{2,1})\backslash M_{2,M}^m} (\phi|_{\kappa,1}\gamma)(\tau, z).$$

We then form the lift by a Fourier-Jacobi expansion

$$F_\phi(\tau, z, \tau') = \sum_{m\geq 1} V_m(\phi)(\tau, z)e^{2\pi m\tau'}.$$

The correspondence (6.2) of Jacobi forms show that each term of the Fourier-Jacobi expansion is invariant under $\Gamma_{2,1}$. The holomorphicity of the lift, which can be proved as we did above, implies that F_ϕ is holomorphic. The symmetry of the Fourier-Jacobi expansion suggested naturally by the structure of the Fourier expansion of $V_m(\phi)$ then shows that the lift is modular with respect to Γ.

Now when the Jacobi form ϕ_t has index $t > 1$ with respect to the Jacobi group $\Gamma' \ltimes H(\mathbb{Z})$, where Γ' is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we consider the right notion of the V_m operators as in equation 6.7. The forms $V_m(\phi_t)$ have all indices multiple of t. Hence forming the lift in degree 2 by its Fourier-Jacobi expansion whose Fourier-Jacobi coefficients are $V_m(\phi_t)$ show that the lift is invariant with respect to $\Delta(t)$. In addition, using the embeddings $i(\Gamma')$ and $h(H(\mathbb{Z}))$ show that their semidirect product is the maximal standard parabolic subgroup of type $(2,1)$ of the subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ generated by $i(\Gamma') \ltimes h(H(\mathbb{Z}))$ and the involution R that we denote by Γ'_∞. Correspondence (6.2) implies that all forms of the lift are invariant under Γ'_∞. But Γ'_∞ and $\Delta(t)$ generate the standard maximal parabolic subgroup $\Gamma'_\infty[t]$ of a certain subgroup $\Gamma'[t]$ of the paramodular group $\Gamma[t]$ of level t generated by $\Gamma'_\infty[t]$ and $V_t\Gamma'_\infty[t]V_t$. The Fourier-Jacobi expansion of the lift shows its invariance under the second set of generators $V_t\Gamma'_\infty[t]V_t$.

Conversely, starting with a congruence subgroup $\Gamma'[t]$ of the paramodular group $\Gamma[t]$ such that $\Gamma'[1]$ is a congruence subgroup of $\mathrm{Sp}_4(\mathbb{Z})$. We consider the standard maximal parabolic subgroup $\Gamma'_\infty[t] = \Gamma'[t] \cap C_{2,1}(\mathbb{Q})$ of $\Gamma'[t]$ such that $\Gamma'[t]$ is generated by $\Gamma'_\infty[t]$ and

$V_t\Gamma'_\infty[t]V_t$ and $\Gamma'_\infty[t]$ is generated by $\Gamma'_\infty(\mathbb{Z})$ (the standard maximal parabolic subgroup of type $(2,1)$ of $\Gamma'[1]$) and $\Delta(t)$. We consider the projection $w_1(\Gamma'_\infty[t])$ onto the SL_2 part. Then $w_1(\Gamma'_\infty[t]) \times H(\mathbb{Z})$ is isomorphic to $\Gamma'_\infty(\mathbb{Z})$. We then should consider to lift a Jacobi form ϕ_t with index t with respect to the Jacobi group $w_1(\Gamma'_\infty[t]) \times H(\mathbb{Z})$ using the V_m operators defined by

$$V_m(\phi)(\tau, z) = m^{\kappa-1} \sum_{\gamma \in w_1(\Gamma'_\infty[t]) \backslash M^m_{2,M}} (\phi|_{\kappa,1}\gamma)(\tau, z).$$

This shows then that that the lift defined by $\sum_{m\geq 1} V_m(\phi_t)e^{2\pi i m t\tau'}$ is then a Siegel modular form with respect to $\Gamma'[t]$.

Note that Poor and Yuen in [61] have obtained a lifting from Jacobi forms of index $t/2 > 1/2$ to Siegel modular forms with respect to the Igusa paramodular subgroup and their lifting fits exactly into the characterization described above even though they lifted a Jacobi form of half integral index $t/2$.

We also believe one can characterize lifting Jacobi forms of degree $n-1$ to Siegel modular forms of degree n using the same method described above by working with the standard maximal parabolic subgroup of type $(n, n-1)$. However, in that setting we believe it will be difficult to compute the corresponding notion of the V_m operators.

6.5 Cuspidality of Lifts

To prove that a Siegel modular form $F \in M_\kappa(\Gamma)$ of degree n is a cusp form, one needs to show that it vanishes at all the rational boundary components (cusps) of every degree $0 \leq r \leq n-1$. Chapter 4 together with the appendix give a group theoretic description of all the rational boundary components in terms of double cosets spaces $\Gamma \backslash \mathrm{Sp}_{2n}(\mathbb{Q})/C_{n,r}(\mathbb{Q})$. However, it turns out according to the discussion in Chapter 4, § 4.4 that it suffices to show it vanishes at the maximal degree cusps of degree $n-1$. These maximal degree cusps correspond to the coset space $\Gamma \backslash \mathrm{Sp}_{2n}(\mathbb{Q})/C_{n,n-1}(\mathbb{Q})$. Therefore, to prove the cuspidality of all the lifting theorems for genus 2 presented in the previous sections, we need to compute the cusps $\Gamma \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$, where $\Gamma = \Gamma_0^{(2)}(M)$, the paramodular group $\Gamma[t]$, the mixed level paramodular group $\Gamma_M[t]$. This will be the task of this section.

6.5.1 Maximal Degree Cusps when $\Gamma = \Gamma_0^{(2)}(M)$

To prove the cuspidality of the Maass lift with level $M \geq 1$, Ibukiyama computed the double coset representatives of

$$\Gamma_0^{(2)}(M) \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q}).$$

Theorem 6.5.1 ([31], Lemma 3.4)**.** *The representatives of the double cosets* $\Gamma_0^{(2)}(M) \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$ *are chosen from the elements in* $C_{2,1}(\mathbb{Q})R$ *where* R *is given by* $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

We give Ibukiyama's proof and follow his notation.

Proof. Let $P_1(\mathbb{Q}) := C_{2,1}(\mathbb{Q})$, and consider $P_1' = R^{-1}P_1(\mathbb{Q})R = RP_1(\mathbb{Q})R$. Then

$$P_1' = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q}) \right\},$$

where $*$ runs over rational numbers.

Since $\mathrm{Sp}_4(\mathbb{Q}) = \mathrm{Sp}_4(\mathbb{Q})R$ and $P_1'R = RP_1(\mathbb{Q})$, it suffices to show that the representatives of $\Gamma_0^{(2)}(M) \setminus \mathrm{Sp}_4(\mathbb{Q})/P_1'$ are taken in $P_1(\mathbb{Q})$. We do this because if $\mathrm{Sp}_4(\mathbb{Q}) = \bigcup_i \Gamma_0^{(2)}(M)g_iP_1'$, then $\mathrm{Sp}_4(\mathbb{Q}) = \bigcup_i \Gamma_0^{(2)}(M)g_iP_1'R = \bigcup_i \Gamma_0^{(2)}(M)g_iRP_1(\mathbb{Q})$ and hence the representatives $g_iR \in P_1(\mathbb{Q})R$.

We decompose any $g \in Sp_4(\mathbb{Q})$, including the coset representative of the form $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

where $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and $D = (d_{ij})$ with $1 \le i, j \le 2$. To show that g can be taken from $P_1(\mathbb{Q})$, we need to show that $c_{12} = c_{21} = c_{22} = 0$. We can always assume the block C in the representative g is such that c_{11} or c_{21} is nonzero, since if otherwise both elements are zeros then a_{11} or a_{21} is nonzero or otherwise g would be such that $\det g = 0$. Multiplying g by an element $\begin{pmatrix} 1_2 & 0 \\ M1_2 & 1_2 \end{pmatrix} \in \Gamma_0^{(2)}(M)$ from the left makes the lower left block of g look like $\begin{pmatrix} Ma_{11} & Ma_{12} + c_{12} \\ Ma_{21} & Ma_{22} + c_{22} \end{pmatrix}$ which implies that we can always make c_{11} or c_{21} nonzero.

Now using Lemma 6.2.8, we can find $U \in \mathrm{SL}_2(\mathbb{Z})$ such that for any $x, y \in \mathbb{Q}^*$, $U\,{}^t(x, y) = (z, 0)$. So multiplying g from the left by $\begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix}$ for some $U \in \mathrm{SL}_2(\mathbb{Z})$, we can assume that $c_{21} = 0$ and $c_{11} \neq 0$. Taking $V = (v_{ij}) \in \mathrm{SL}_2(\mathbb{Z})$ such that $(c_{22}, d_{22})V = (0, *)$ and multiplying by $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & v_{11} & 0 & v_{12} \\ 0 & 0 & 1 & 0 \\ 0 & v_{21} & 0 & v_{22} \end{pmatrix} \in P_1'$ from the right, we can take $c_{22} = 0$. Also, multiplying by $\begin{pmatrix} 1 & -c_{11}^{-1}c_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c_{11}^{-1}c_{12} & 1 \end{pmatrix} \in P_1'$ from the right, we can take $c_{12} = 0$. By the characterization given in Remark 6.2.1, the result is obtained. $\square$

To show the Maass lift is cuspidal, it is sufficent to show that $F|_\kappa\gamma|\Phi = 0$ for every $\gamma \in \Gamma_0^{(2)}(M) \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$, where Φ is the Siegel Φ operator. According to Theorem 6.5.1, it suffices to show that

$$(F|_\kappa pR)|\Phi = 0,$$

for every $p \in C_{2,1}(\mathbb{Q})$. We give a shorter proof of Ibukiyama's cuspidality proof of the Maass lift with level according to the following proposition.

Proposition 6.5.2. *Fix $n \geq 1$. For each*

$$\gamma = \begin{pmatrix} A_1 & 0 & B_1 & * \\ * & u^{-1} & * & * \\ C_1 & 0 & D_1 & * \\ 0 & 0 & 0 & u \end{pmatrix} \in C_{n,n-1}(\mathbb{Q}),$$

we have

$$F|_\kappa \gamma|\Phi = u^{-\kappa}(F|\Phi)|_\kappa w_{n,n-1}(\gamma)$$

where $w_{n,n-1}(\gamma) = \gamma_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}_{2(n-1)}(\mathbb{Q})$, $u \in \mathbb{Q} \setminus \{0\}$ *and F is a Siegel modular form of degree n.*

Proof. Before we start, we note that the following computations follow from more detailed computations done in the proof of Theorem 2.3.4 and from the definition of the Siegel Φ operator on Fourier expansions. We have

$$\begin{aligned} (F|_\kappa \gamma|\Phi)(Z') &= \lim_{\lambda \to \infty}(F|_\kappa \gamma) \begin{pmatrix} Z' & 0 \\ 0 & i\lambda \end{pmatrix} \\ &= \lim_{\lambda \to \infty} \det(CZ'_\lambda + D)^{-\kappa} F(\gamma Z'_\lambda) \\ &= u^{-\kappa} \det(C_1 Z' + D_1)^{-\kappa} \lim_{\lambda \to \infty} F(\gamma_1 Z'_\lambda) \\ &= u^{-\kappa}(F|\Phi|_\kappa \gamma_1)(Z'), \end{aligned}$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $Z'_\lambda = \begin{pmatrix} Z' & 0 \\ 0 & i\lambda \end{pmatrix}$. □

We now show that the Maass lift with level M (Theorem 6.2.11) is cuspidal. The

Fourier expansion of the Maass lift,

$$
\begin{aligned}
F_\phi(\tau, z, \tau') &= \sum_{m \geq 1} V_m(\phi)(\tau, z) e^{2\pi i m \tau'} \\
&= \sum_{m \geq 1} \sum_{\substack{n, r \in \mathbb{Z} \\ 4nm - r^2 > 0}} \sum_{\substack{d \mid \gcd(n,r,m) \\ (d,M)=1}} d^{\kappa-1} c\left(\frac{nm}{d^2}, \frac{r}{d}\right) e^{2\pi i (n\tau + rz + m\tau')} \\
&= \sum_{T \in \mathbb{S}_2^{>0}(\mathbb{Z})} a_\phi(T) e^{2\pi i \operatorname{Tr}(TZ)},
\end{aligned}
$$

where $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} > 0$, shows that $F|\Phi = 0$. To show F is a cuspidal Siegel modular form, we need to show that $(F|_\kappa pR)|\Phi = 0$. However,

$$
(F|_\kappa pR)|\Phi = (F|_\kappa p|_\kappa R)\Phi.
$$

Since the action of R only exchanges τ and τ', $(F|_\kappa pR)|\Phi = 0$ will follow from $(F|_\kappa p)|\Phi = 0$. Writing $p \in C_{2,1}(\mathbb{Q})$ as

$$
p = \begin{pmatrix} a & 0 & b & * \\ * & u^{-1} & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & u \end{pmatrix},
$$

and using Propostion 6.5.7 we get

$$
(F|_\kappa p)|\Phi = u^{-\kappa} (F|\Phi)|_\kappa \begin{pmatrix} a & b \\ c & d \end{pmatrix}
$$

which clearly vanishes since $F|\Phi = 0$.

Remark 6.5.3. In the proof of the cuspidality of the Maass lift, we have implicitly shown due to the structure of the maximal degree cusps (the double cosets) that the Maass lift is cuspidal because the Jacobi form we are lifting is cuspidal.

6.5.2 Maximal Degree Cusps when $\Gamma = \Gamma[t]$

The maximal degree cusps of the compact space $\Gamma[t] \setminus (\mathfrak{h}^2)^*$ correspond bijectively to representatives of double cosets $\Gamma[t] \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$. Reefschläger in his PhD thesis [63] showed that the number of maximal degree cusps for the paramodular group of degree two is equal to the number of divisors of t.

Theorem 6.5.4 ([63], Satz p.19)**.** *Let $t = \prod_{i=1}^{n} p_i^{\alpha_i}$ be the prime factor decomposition of t, then the number of degree 1 (maximal) cusps is given by*

$$|\Gamma[t] \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})| = \prod_{i=1}^{n}(\alpha_i + 1).$$

Gritsenko also computed representatives of the double cosets in [27]. We now present Gritsenko's work in [27] for finding representatives of the double cosets $\Gamma[t]\setminus\mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$ with detailed proofs omitted in [27].

Definition 6.5.5. A vector $X = {}^t(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$ is said to be primitive if $\gcd(x_1, x_2, x_3, x_4) = 1$.

Definition 6.5.6 ([27], p.97)**.** Let $T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$. Consider the lattice

$$L_t = \begin{pmatrix} 1_2 & 0 \\ 0 & T \end{pmatrix} \mathbb{Z}^4 = \{X \in \mathbb{Z}^4 \mid {}^tX = (x_1, x_2, x_3, tx_4)\}.$$

A vector ${}^tX = (x_1, x_2, x_3, tx_4) \in L_t$ is said to be primitive if $\gcd(x_1, x_2, x_3, x_4) = 1$.

We denote the set of primitive integral vectors in L_t by L_t'.

Proposition 6.5.7 ([50], Corollary 2.5.8)**.** *The quotient $\mathrm{Sp}_{2n}(\mathbb{Q})/C_{n,n-1}(\mathbb{Q})$ is isomorphic to $P^{2n-1}(\mathbb{Q}) = (\mathbb{Q}^{2n}\setminus\{0\})/\mathbb{Q}^*$ via the transformation which maps $\gamma C_{n,n-1}(\mathbb{Q})$ onto the n-th column of γ.*

Proof. We prove that the map is well-defined and injective first. Let γ_1, γ_2 be in $\mathrm{Sp}_{2n}(\mathbb{Q})$ such that their n-th column are the same up to a constant multiple. Consider $\gamma_1^{-1}\gamma_2$. Then its n-th column corresponds to a constant multiple $a \neq 0$ of the unit vector. Writing $\gamma_1^{-1}\gamma_2$ in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and using the symplectic relations $A\,{}^t\!C = C\,{}^t\!A$ and $A\,{}^t\!D - C\,{}^t\!B = 1_n$, we obtain that $c_{n1} = c_{n2} = \ldots = c_{n(n-1)} = 0$ and $d_{n1} = d_{n2} = \ldots = d_{n(n-1)} = 0$, $d_{nn} = 1/a$. Therefore, $\gamma_1^{-1}\gamma_2 \in C_{n,n-1}(\mathbb{Q})$. To prove surjectivity we proceed as follows. We take a representative x of an element in $P^{2n-1}(\mathbb{Q})$ to be a column vector $x = {}^t(x_1, \ldots, x_{2n})$ of order $2n$ consisting of primitive integral entries. Using Lemma 6.2.8, we can find $U \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$U\,{}^t(x_1, \ldots, x_n) = {}^t(0, \ldots, 1) = e,$$

and

$$\begin{pmatrix} U & 0_n \\ 0_n & {}^t U^{-1} \end{pmatrix} {}^t(x_1, \ldots, x_{2n}) = {}^t(e, y_1, \ldots y_n).$$

Let $y := {}^t(y_1, \ldots, y_n)$, $A = \begin{pmatrix} 1_{n-1} & 0_{n-1} \\ 0 & 1 \end{pmatrix}$, $B = 0_n, C = \begin{pmatrix} 0_{n-1} & y \\ {}^t y & y_n \end{pmatrix}$, and $D = 1_n$. Set $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Q})$ by verifying that A, B, C, and D satisfy the symplectic relations. In addition, we have

$$\begin{pmatrix} U & 0_n \\ 0_n & {}^t U^{-1} \end{pmatrix}^{-1} \gamma$$

has x as its n-th column. □

In particular, we have

$$\mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q}) \cong P^3(\mathbb{Q}).$$

Since elements of $P^3(\mathbb{Q})$ are equivalence classes, we can choose a representative in

$\mathbb{Z}^4$ and since for every vector X in $\mathbb{Z}^4$ we can write it as $X = (\gcd X)X'$ for some primitive vector X' in $\mathbb{Z}^4$. We have that

Lemma 6.5.8.

$$P^3(\mathbb{Q}) \cong L'_t/\pm 1,$$

where L'_t is the set of primitve vectors in L_t.

Remark 6.5.9. The integral paramodular group $\Gamma^i[t]$ stabilizes the primitive lattice in $\mathbb{Z}^4$ and since $\Gamma^i[t] = \sigma_t^{-1}(\Gamma[t]) = \begin{pmatrix} 1_2 & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma[t] \begin{pmatrix} 1_2 & 0 \\ 0 & T \end{pmatrix}$, we see that the action of $\Gamma^i[t]$ on the primitive lattice in $\mathbb{Z}^4$ is equivalent to the action of $\Gamma[t]$ on the primitive lattice L'_t. This is because for every $\gamma \in \Gamma[t]$ and every primitive vector $X = {}^t(x_1, x_2, x_3, x_4)$ in $\mathbb{Z}^4$, we have

$$\gamma \begin{pmatrix} 1_2 & 0 \\ 0 & T \end{pmatrix} X = \begin{pmatrix} 1_2 & 0 \\ 0 & T \end{pmatrix} \left(\begin{pmatrix} 1_2 & 0 \\ 0 & T^{-1} \end{pmatrix} \gamma \begin{pmatrix} 1_2 & 0 \\ 0 & T \end{pmatrix} X \right).$$

Therefore, $\Gamma[t]$ stabilizes L_t. Because $\pm 1_4 \in \Gamma[t]$, we have then

$$\Gamma[t] \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q}) \cong \Gamma[t] \setminus L'_t.$$

Lemma 6.5.10 ([27], Lemma 2.3')**.** *For a primitive integral vector $X \in L'_t$,*

$$\Gamma[t]X = \Gamma[t]\, {}^t(d_t(X), 1, 0, 0),$$

where $d_t(X)$ is a divisor of t.

In fact, Lemma 6.5.10 comes as a consequence of the following lemma and Remark 6.5.9.

Lemma 6.5.11 ([27], Lemma 2.3)**.** *For a primitve integral vector $X \in \mathbb{Z}^4$, the orbit $\Gamma^i[t]X$ contains an element of the form ${}^t(d_t(X), 1, 0, 0)$, where $d_t(X)$ is a divisor of t.*

Remark 6.5.12. The proof of this lemma is done by showing that an element X of the primitive lattice in $\mathbb{Z}^4$ can be reduced to the form $(d_t(X), 1, 0, 0)$ using the action of the generators of the integral paramodular group, which we will show below in the proof of Theorem 6.5.17. In fact, it is given implicitly in the proof of this theorem.

Therefore, to find representatives of double cosets of $\Gamma[t] \setminus \mathrm{Sp}_4(\mathbb{Q})/P(\mathbb{Q})$, we need to find convenient generators of $\Gamma[t]$.

Theorem 6.5.13 ([27], Lemma 2.2)**.** *The paramodular group $\Gamma[t]$ is generated by $\Gamma_\infty[t]$ and by the element J_t.*

The proof of this theorem follows from proving a corresponding set of generators of the integral paramodular group. We first give examples of elements in $\Gamma^i[t]$. Recall the involution $V_t = \begin{pmatrix} {}^tU_t & 0 \\ 0 & U_t \end{pmatrix}$ where $U_t = \begin{pmatrix} 0 & (\sqrt{t})^{-1} \\ \sqrt{t} & 0 \end{pmatrix}$. The element

$$J_2 = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$$

belongs to $\Gamma^i[t]$. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ and $I_t = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & T \end{pmatrix}$, consider

$$J_t = \sigma_t(J_2) = I_t J_2 I_t^{-1}.$$

We know (see §3 in the special case when $M = 1$) that the standard maximal parabolic subgroup of $\Gamma[t]$ of type $(2,1)$, denoted by $\Gamma_\infty[t]$, is the set of elements

$$\Gamma_\infty[t] = \left\{ \begin{pmatrix} a & 0 & b & x' \\ y & 1 & x & zt^{-1} \\ c & 0 & d & -y' \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & x \\ y & 1 & x & zt^{-1} \\ 0 & 0 & 1 & -y \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

To prove that the group $\Gamma_\infty[t]$ and J_t generate the paramodular group $\Gamma[t]$, we equivalently prove that the integral paramodular group $\Gamma^i[t]$ is generated by $\sigma_t^{-1}(\Gamma_\infty[t])$ and $\sigma_t^{-1}(J_t) = J_2$.

Definition 6.5.14. The parabolic subgroup $\Gamma^i_\infty[t]$ of $\Gamma^i[t]$ is $\sigma_t^{-1}(\Gamma_\infty[t])$. It is given by:

$$\Gamma^i_\infty[t] := \sigma_t^{-1}(\Gamma_\infty[t]) = \left\{ \begin{pmatrix} a & 0 & b & x't \\ y & 1 & x & z \\ c & 0 & d & -y't \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & xt \\ y & 1 & x & z \\ 0 & 0 & 1 & -yt \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Proposition 6.5.15. *We provide typical elements in the integral paramodular group $\Gamma^i[t]$.*

1. *The set of elements*

$$i\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^i[t]$$

   for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. *In particular,*

$$I := i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.

$$H := \{h(x,y,z)\} = \sigma_t^{-1}\left(\left\{\begin{pmatrix} 1 & 0 & 0 & x \\ y & 1 & x & zt^{-1} \\ 0 & 0 & 1 & -y \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}\right) = \left\{\begin{pmatrix} 1 & 0 & 0 & xt \\ y & 1 & x & z \\ 0 & 0 & 1 & -yt \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\},$$

for $x, y, z \in \mathbb{Z}$.

3.

$$J = \sigma_t^{-1}(J_t),$$

4.

$$JHJ^{-1} = \left\{ Jh(x,y,z)J^{-1} = \begin{pmatrix} 1 & -yt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -xt & 1 & 0 \\ -x & -z & y & 1 \end{pmatrix} \,\middle|\, (x,y,z) \in \mathbb{Z}^3 \right\},$$

5.

$$j\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma_t^{-1}\left(V_t i\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) V_t^{-1}\right) = \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}\right\}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. It is easy to verify that if we apply the geometric definition of the integral paramodular group as given in Chapter 3, then each of the type of elements presented above preserve the bilinear form given by the matrix $J(t)$. $\square$

Remark 6.5.16. If we consider the involution V_t of the paramodular group $\Gamma[t]$, then we can express J_t in another way

$$J_t = V_t i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} V_t i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We now prove the following theorem which is a more detailed proof than the one given by Gritsenko. This proof gives immediately the proof of Theorem 6.5.13.

Theorem 6.5.17. *The integral paramodular group $\Gamma^i[t]$ is generated by $\Gamma^i_\infty[t]$ and J. That is,*

$$\Gamma^i[t] = \langle \Gamma^i_\infty[t], J \rangle.$$

Proof. Let G^i_t be the group generated by $\Gamma^i_\infty[t]$ and J. Using Proposition 6.5.15, we have that $G^i_t \subset \Gamma^i[t]$. We now show the other inclusion. For this end, take an element $g \in \Gamma^i[t]$, and consider the second column $g_2 = {}^t(tx_1, x_2, tx_3, x_4)$. Then ${}^tg_2$ is a primtive vector of $\mathbb{Z}^4$ since g is an integral matrix with $\det g = 1$. We can always assume $x_4 \neq 0$ because we know that at least one of the elements of the vector g_2 should be nonzero since otherwise $\det g = 0$ and because we can always multiply g by an element $Jh(x, y, z)J^{-1}$ of JHJ^{-1}. For example if $x_3 \neq 0$ then multiplying by $Jh(0, 1, 0)J^{-1}$ transforms g into a column with nonzero x_4. Next we can find $x, y, z \in \mathbb{Z}$ such that

$$h(x, y, z)g_2 =^t (ty_1, y_2, ty_3, y_4)$$

with $\gcd(y_2, y_4) = 1$. For this part, we follow exactly the argument given in [61] Proposition 11, Step 2. Let $w = \gcd(x_2, x_4)$, then $w \neq 0$ because $x_4 \neq 0$ and x_2 can never be zero since otherwise $\det g = 0$. Consider $w_2 = t\gcd(x_1, x_3)$ and $w_3 = \gcd(x_4/w, w^{|x_4|})$, then $\gcd(x_4/(ww_3), w) = 1$ and there exists integers $a, b, \in \mathbb{Z}$ such that $w_2 = atx_1 + btx_3$. Now mulitplying g_2 by $h\left(\frac{bx_4}{ww_3}, \frac{ax_4}{ww_3}, z\right)$ from the left, we obtain that $h\left(\frac{bx_4}{ww_3}, \frac{ax_4}{ww_3}, z\right) g_2 =^t (ty_1, y_2, ty_3, y_4)$ where $y_4 = x_4$ and $y_2 = x_2 + w_2\frac{x_4}{ww_3} + zx_4$. Let $p|y_4$ be a prime. If $p|w$ then $p|x_2$ and p does not divide w_2 since $\gcd(tx_1, x_2, tx_3, x_4) = 1$. Also, p does not divide

$\frac{x_4}{ww_3}$ since $\gcd(x_4/(ww_3), w) = 1$. This implies p does not divide y_2 and so in this case $\gcd(y_2, y_4) = 1$. The other case is when p does not divide w. Then p does not divide x_2 and $p|\frac{x_4}{w}$ since $p|x_4$ and p does not divide w. In addition, p does not divide w_3 or else $p|w$ so that $p|\frac{x_4}{ww_3}$. Then p does not divide y_2 and we also obtain that $\gcd(y_2, y_4) = 1$.

Now using Lemma 6.2.8, we can find a matrix $V \in \mathrm{SL}_2(\mathbb{Z})$ such that $V \begin{pmatrix} y_2 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Consequently multiplying g_2 by $\alpha = j(V)$ from the left we obtain that

$$\alpha h\left(\frac{bx_4}{ww_3}, \frac{ax_4}{ww_3}, z\right) g_2 = {}^t(ty_1, 1, ty_3, 0).$$

Applying $\beta = h(0, y_1, 0)$ we have then

$$\beta\alpha h\left(\frac{bx_4}{ww_3}, \frac{ax_4}{ww_3}, z\right) g_2 = {}^t(0, 1, ty_3, ty_1y_3).$$

Using $\gamma = j\begin{pmatrix} 1 & 0 \\ -ty_1y_3 & 1 \end{pmatrix}$ we get

$$\gamma\beta\alpha h\left(\frac{bx_4}{ww_3}, \frac{ax_4}{ww_3}, z\right) g_2 = {}^t(0, 1, ty_3, 0).$$

Multiplying by I we get

$$I\gamma\beta\alpha h\left(\frac{bx_4}{ww_3}, \frac{ax_4}{ww_3}, z\right) g_2 = {}^t(ty_3, 1, 0, 0).$$

Finally multiplying by $Jh(0, y_3, 0)J^{-1}$ we reach that

$$Jh(0, y_3, 0)J^{-1}I\gamma\beta\alpha h\left(\frac{bx_4}{ww_3}, \frac{ax_4}{ww_3}, z\right) g_2 = {}^t(0, 1, 0, 0).$$

This shows that we transformed $g \in \Gamma^i[t]$ to a matrix whose second column is ${}^t(0, 1, 0, 0)$.

Equivalently, $Jh(0,y_3,0)J^{-1}I\gamma\beta\alpha h\left(\frac{bx_4}{ww_3},\frac{ax_4}{ww_3},z\right)g \in \Gamma^i_\infty[t] \subset G^i_t$ and consequently $g \in G^i_t$. $\square$

6.5.3 Maximal Degree Cusps When $\Gamma = \Gamma_M[t]$

Definition 6.5.18. For $t, M \in \mathbb{N}$, $\gcd(t, M) = 1$, we define the group G_{Mt} to be the group generated by $\Gamma_{M,\infty}[t]$ and $V_t\Gamma_{M,\infty}[t]V_t$, i.e.,

$$G_{Mt} = \langle \Gamma_{M,\infty}[t], V_t\Gamma_{M,\infty}[t]V_t\rangle\,.$$

Our goal is to show that the mixed level paramodular group is given by

$$\Gamma_M[t] = G_{Mt}.$$

To achieve this, we prefer to work with the integral mixed level paramodular group.

$$\Gamma^i_M[t] = \sigma_t^{-1}(\Gamma_M[t]) = \left\{\begin{pmatrix} * & t* & * & t* \\ * & * & * & * \\ M* & Mt* & * & t* \\ M* & M* & * & * \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{Z})\right\}.$$

We recall the standard maximal parabolic subgroup of the congruence level paramodular group of type $(2,1)$, $\Gamma_{M,\infty}[t] = \Gamma_M[t] \cap C_{2,1}(\mathbb{Q})$. It is given by

$$\Gamma_{M,\infty}[t] = \left\{\begin{pmatrix} a & 0 & b & x' \\ y & 1 & x & zt^{-1} \\ Mc & 0 & d & -y' \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & x \\ y & 1 & x & zt^{-1} \\ 0 & 0 & 1 & -y \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}.$$

Definition 6.5.19. The corresponding parabolic subgroup of the integral mixed level paramodular group is

$$\Gamma^i_{M,\infty}[t] := \sigma_t^{-1}(\Gamma_{M,\infty}[t]) = \left\{ \begin{pmatrix} a & 0 & b & x't \\ y & 1 & x & z \\ Mc & 0 & d & -y't \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & xt \\ y & 1 & x & z \\ 0 & 0 & 1 & -yt \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Consider the involution of the integral paramodular group corresponding to the involution V_t:

$$V_t^i = \sigma_t^{-1}(V_t) = \left\{ \begin{pmatrix} 0 & \sqrt{t} & 0 & 0 \\ t^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{t} \\ 0 & 0 & t^{-1/2} & 0 \end{pmatrix} \right\}.$$

We now provide a set of typical elements in the integral mixed level paramodular group.

Lemma 6.5.20. *The following are elements of the mixed level integral paramodular group* $\Gamma^i_M[t]$.

1. *The set of elements*

$$i\left(\begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ Mc & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^i_M[t]$$

   for all $\begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \in \Gamma_0(M)$.

2.

$$H_M^i := \{h(x,y,z)\} = \sigma_t^{-1}\left(\left\{\begin{pmatrix}1 & 0 & 0 & x\\ y & 1 & x & zt^{-1}\\ 0 & 0 & 1 & -y\\ 0 & 0 & 0 & 1\end{pmatrix}\right\}\right) = \left\{\begin{pmatrix}1 & 0 & 0 & xt\\ y & 1 & x & z\\ 0 & 0 & 1 & -yt\\ 0 & 0 & 0 & 1\end{pmatrix}\right\},$$

for $x, y, z \in \mathbb{Z}$.

3.

$$V_t^i H_M^i V_t^i = \left\{\begin{pmatrix}1 & -yt & z & xt\\ 0 & 1 & xt & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & yt & 0\end{pmatrix} \mid (x,y,z) \in \mathbb{Z}^3\right\}.$$

4.

$$j\left(\left(\begin{pmatrix}a & b\\ Mc & d\end{pmatrix}\right)\right) = V_t^i i\left(\left(\begin{pmatrix}a & b\\ Mc & d\end{pmatrix}\right)\right)V_t^i = \left\{\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & a & 0 & b\\ 0 & 0 & 1 & 0\\ 0 & Mc & 0 & d\end{pmatrix}\right\},$$

for all $\begin{pmatrix}a & b\\ Mc & d\end{pmatrix} \in \Gamma_0(M)$.

Proof. Using the injective homomorphism i, we clearly see that $i\left(\left(\begin{pmatrix}a & b\\ Mc & d\end{pmatrix}\right)\right) \in \Gamma_{M,\infty}[t] \subset \Gamma_M[t]$. The corresponding set of elements in $\Gamma^i_{M,\infty}[t]$ is $\sigma_t^{-1}\left(i\left(\left(\begin{pmatrix}a & b\\ Mc & d\end{pmatrix}\right)\right)\right) = i\left(\left(\begin{pmatrix}a & b\\ Mc & d\end{pmatrix}\right)\right)$.

The elements $\left\{\begin{pmatrix}1&0&0&x\\y&1&x&zt^{-1}\\0&0&1&-y\\0&0&0&1\end{pmatrix}\right\} \in \Gamma_{M,\infty}[t]$ and hence $h(x,y,z) \in \Gamma^i_{M,\infty}[t]$. Now $V_t^i H_M^i V_t^i \in \Gamma^i_M[t]$ since V_t^i is an involution of $\Gamma^i_M[t]$. Lastly, $j\begin{pmatrix}a&b\\Mc&d\end{pmatrix} = \sigma_t^{-1}\left(V_t i\left(\begin{pmatrix}a&b\\Mc&d\end{pmatrix}\right)V_t\right)$. $\square$

Definition 6.5.21. Let $T = \begin{pmatrix}1&0\\0&t\end{pmatrix}$ be the diagonal matrix with diagonal entries 1 and t and let $\mathbb{M} = \begin{pmatrix}M&0\\0&M\end{pmatrix}$ where $\gcd(t,M) = 1$.

1. We consider the lattice

$$L_{M,t} := \begin{pmatrix}1_2&0\\0&T\end{pmatrix}\begin{pmatrix}1_2&0\\0&\mathbb{M}\end{pmatrix}\mathbb{Z}^4 = \{X \in \mathbb{Z}^4 \mid X = {}^t(x_1,x_2,Mx_3,tMx_4)\}.$$

2. The dual lattice of $L_{M,t}$ is

$$L^*_{M,t} = \begin{pmatrix}1_2&0\\0&\mathbb{M}^{-1}\end{pmatrix}\begin{pmatrix}1_2&0\\0&T^{-1}\end{pmatrix}\mathbb{Z}^4.$$

3. We also consider the lattice

$$L_M : \begin{pmatrix}1_2&0\\0&\mathbb{M}\end{pmatrix}\mathbb{Z}^4 = \{X \in \mathbb{Z}^4 \mid X = {}^t(x_1,x_2,Mx_3,Mx_4)\}.$$

4. The dual lattice of L_M is

$$L^*_M = \begin{pmatrix}1_2&0\\0&\mathbb{M}^{-1}\end{pmatrix}\mathbb{Z}^4.$$

Definition 6.5.22. A vector $X = {}^t(x_1, x_2, Mx_3, tMx_4) \in L_{M,t}$ is said to be primitive if ${}^tXL^*_{M,t} = \mathbb{Z}$. We denote the set of primitive integral vectors in $L_{M,t}$ by $L'_{M,t}$. Similarly, a vector $X = {}^t(x_1, x_2, Mx_3, Mx_4) \in L_M$ is said to be primitive if ${}^tXL^*_M = \mathbb{Z}$.

Equivalently, a vector $X = {}^t(x_1, x_2, Mx_3, tMx_4) \in L_{M,t}$ is primitive if $\gcd(x_1, x_2, x_3, x_4) = 1$ and a vector $X = {}^t(x_1, x_2, Mx_3, Mx_4) \in L_M$ is primitive if $\gcd(x_1, x_2, x_3, x_4) = 1$.

Remark 6.5.23. The mixed level paramodular group $\Gamma_M[t]$ stabilizes $L_{M,t}$. We see this because the integral mixed level paramodular group $\Gamma^i_M[t]$ stabilizes the lattice L_M and for every $\gamma \in \Gamma_M[t]$ we have:

$$\gamma \begin{pmatrix} 1_2 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} 1_2 & 0 \\ 0 & \mathbb{M} \end{pmatrix} \mathbb{Z}^4 = \begin{pmatrix} 1_2 & 0 \\ 0 & T \end{pmatrix} \left(\begin{pmatrix} 1_2 & 0 \\ 0 & T^{-1} \end{pmatrix} \gamma \begin{pmatrix} 1_2 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} 1_2 & 0 \\ 0 & \mathbb{M} \end{pmatrix} \mathbb{Z}^4 \right).$$

Theorem 6.5.24. *The mixed level paramodular group $\Gamma_M[t]$ is generated by $\Gamma_{M,\infty}[t]$ and by $V_t\Gamma_{M,\infty}[t]V_t$, i.e.,*

$$\Gamma_M[t] = G_{Mt} = \langle \Gamma_{M,\infty}[t], V_t\Gamma_{M,\infty}[t]V_t \rangle .$$

Proof. To prove this statement, we will proceed with the integral version of the groups. It is clear from the definition of the integral parabolic group $\Gamma^i_{M,\infty}[t]$ and from Lemma 6.5.20 that $G^i_{Mt} \subset \Gamma^i_M[t]$. For the other direction, take an element $g \in \Gamma^i_M[t]$. Its second column $g_2 = {}^t(tx_1, x_2, Mtx_3, Mx_4)$ is primitive since $\det g = 1$. Repeating a similar first step as discussed in the proof 6.5.2, we can always multiply g_2 from the left by an element $h(x, y, z) \in H^i_M$ such that

$$h(x, y, z)g_2 = {}^t(ty_1, y_2, Mty_3, My_4)$$

such that $\gcd(y_2, My_4) = 1$ for some $x, y, z \in \mathbb{Z}$. Now there exists integers a and b in $\mathbb{Z}$

such that $ay_2 + bMy_4 = 1$ and hence using elements of type $j\left(\begin{pmatrix} a & b \\ Mc & d \end{pmatrix}\right)$, we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & -My_4 & 0 & y_2 \end{pmatrix} \begin{pmatrix} ty_1 \\ y_2 \\ Mty_3 \\ My_4 \end{pmatrix} = \begin{pmatrix} ty_1 \\ 1 \\ Mty_3 \\ 0 \end{pmatrix}.$$

Let $\gcd(ty_1, tMy_3) = t\gcd(y_1, My_3) := tm$. There are integers α and β such that $t(\alpha y_1 + \beta My_3) = tm$ and using elements of type $i\left(\begin{pmatrix} a & b \\ Mc & d \end{pmatrix}\right)$ we obtain

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ My_3 & 0 & -y_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ty_1 \\ 1 \\ Mty_3 \\ 0 \end{pmatrix} = \begin{pmatrix} tm \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, multipying by an element of type $V_t^i H_M^i V_t^i$ with $x = z = 0$ and $y = -m$, we get

$$\begin{pmatrix} 1 & -mt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & mt & 0 \end{pmatrix} \begin{pmatrix} tm \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Summing up, we have multiplied a matrix $g \in \Gamma_M^i[t]$ from the left by a sequence of matrices in G_{Mt}^i and we have obtained a matrix whose second column is $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Using the characterization of the standard maximal parabolic subgroups (see Remark 6.2.1), this shows that

$$\begin{pmatrix} 1 & -mt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & mt & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ My_3 & 0 & -y_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & -My_4 & 0 & y_2 \end{pmatrix} h(x,y,z)g \in \Gamma^i_{M,\infty}[t] \subset G^i_{Mt}.$$

□

We now give representatives of the double cosets $\Gamma_M[t] \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$. With similar reasoning as in case of the paramodular group $\Gamma[t]$, we get that $P^3(\mathbb{Q}) \cong L'_{M,t}/\pm 1_4$. Because $\pm 1_4 \in \Gamma_M[t]$, we have

$$\Gamma_M[t] \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q}) \cong \Gamma_M[t] \setminus L'_{M,t}.$$

We select a primitive vector $X \in L_{M,t}$ such that ${}^tXL^*_{M,t} = \mathbb{Z}$. We write ${}^tX := (x_1, x_2, Mx_3, Mtx_4)$ where $x_1, x_2, x_3, x_4 \in \mathbb{Z}$ with $\gcd(x_1, x_2, x_3, x_4) = 1$. Let $g := \gcd(x_2, Mx_4)$. Then there exists integers $(a, b) \in \mathbb{Z}^2$ such that $ax_2 + bMx_4 = g$. Consequently,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & bt^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & Mtx_4/g & 0 & x_2/g \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ Mx_3 \\ Mtx_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ g \\ Mx_3 \\ 0 \end{pmatrix}.$$

Similarly, we take $m := \gcd(x_1, Mx_3)$. Then there exists integers $(\alpha, \beta) \in \mathbb{Z}^2$ such that

$\alpha x_1 + \beta M x_3 = m$. Consequently,

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ Mx_3/m & 0 & -x_1/m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ g \\ Mx_3 \\ 0 \end{pmatrix} = \begin{pmatrix} m \\ g \\ 0 \\ 0 \end{pmatrix}.$$

Since ${}^tXL^*_{M,t} = \mathbb{Z}$, we have $\gcd(m, g) = 1$. So we can find integers c and d such that $cm + dg = 1$ and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & ctM & 1 & 0 \\ ctM & dtM & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ g \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \\ g \\ cgtM \\ Mt \end{pmatrix}.$$

We repeat the first step by finding integers c', d' such that $c'g + d'M = 1$ and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c' & 0 & d't^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & -Mt & 0 & g \end{pmatrix} \begin{pmatrix} m \\ g \\ ctMg \\ Mt \end{pmatrix} = \begin{pmatrix} m \\ 1 \\ ctMg \\ 0 \end{pmatrix}.$$

Lastly, taking $d_{tM} := \gcd(m, ctMg)$ and repeating the second step we obtain a representative of the form

$$\begin{pmatrix} d_{tM} \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Proposition 6.5.25. *Each representative of a double coset in* $\Gamma_M[t] \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$ *is* *of the form*

$$\mathrm{Rot}(a) = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 1 \end{pmatrix} : a \in \mathbb{Z}$$

Proof. Let α be an element in $\mathrm{Sp}_4(\mathbb{Q})$. Then the discussion above implies that there a $\gamma \in \Gamma_M[t]$ such that

$$\gamma\alpha = \begin{pmatrix} * & \beta d_{tM} & * & * \\ * & \beta & * & * \\ * & 0 & * & * \\ * & 0 & * & * \end{pmatrix}$$

for some $\beta \in \mathbb{Q}^*$. Then,

$$\mathrm{Rot}(d_{tM})^{-1}\gamma\alpha = \begin{pmatrix} * & 0 & * & * \\ * & \beta & * & * \\ * & 0 & * & * \\ * & 0 & * & * \end{pmatrix} \in C_{2,1}(\mathbb{Q})$$

by Remark 6.2.1. $\square$

To prove the lift $\mathcal{G}_M(\phi_t)$ is cuspidal, we need to show that $\mathcal{G}_M(\phi_t)|_\kappa(\mathrm{Rot}(a))|\Phi = 0$, or equivalently we need to show that the Fourier expansion of $\mathcal{G}_M(\phi_t)|_\kappa(\mathrm{Rot}(a))$ is restricted to postive definite matrices $T \in \mathbb{S}_{2,t}^{>0}(\mathbb{Z})$. Before we do so, we prove a more general proposition.

Proposition 6.5.26. *Let F be a Siegel modular form of degree 2 and weight κ. Then,*

$$F|\Phi = 0 \quad \textit{if and only if} \quad F|_\kappa \mathrm{Rot}(a)|\Phi = 0.$$

Proof. Assume $F|\Phi = 0$. Equivalently, this means that the Fourier coefficients $a(T)$ in the Fourier series development $F(Z) = \sum_{T\in\mathbb{S}_2^{\geq 0}} a(T)e^{2\pi i\,\mathrm{Tr}(TZ)}$ are equal to zero unless T is positive definite for all $T \in \mathbb{S}_2^{\geq 0}$. Writing $\mathrm{Rot}(a)$ as $\begin{pmatrix} A & 0_2 \\ 0_2 & {}^tA^{-1} \end{pmatrix}$, where $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, we have

$$\begin{aligned}
(F|_\kappa \mathrm{Rot}(a))(Z) &= \det(0_2Z +^t A^{-1})^{-\kappa}F(AZ) \\
&= (\det A)^\kappa F(AZ^tA) \\
&= (\det A)^\kappa \sum_{\substack{T\in\mathbb{S}_2^{\geq 0}\\ T>0}} a(T)e^{2\pi i\,\mathrm{Tr}(TAZ^tA)} \\
&= \sum_{\substack{T\in\mathbb{S}_2^{\geq 0}\\ T>0}} a(T)e^{2\pi i\,\mathrm{Tr}({}^tATAZ)} \\
&= \sum_{\substack{N\in\mathbb{S}_2^{\geq 0}\\ N>0}} a({}^tA^{-1}NA^{-1})e^{2\pi i\,\mathrm{Tr}(NZ)}.
\end{aligned}$$

This shows that the Fourier coefficients of $F|_\kappa \mathrm{Rot}(a)$ vanish unless they are restricted to positive definite and hence $F|_\kappa \mathrm{Rot}(a)|\Phi = 0$. The other direction is immediate. $\square$

Theorem 6.5.27. *Let $\phi_t \in J^c_{\kappa,t}(\Gamma_0(M)^J)$. Then $\mathcal{G}_M(\phi_t) \in S_\kappa(\Gamma_M[t])$.*

Proof. It remains to show the cuspidality of the lift. Because $V_m(\phi_t)$ is a cusp form (Lemma 6.2.10), if we write

$$V_m(\phi_t)(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z},n\geq 0\\ 4nmt>r^2}} c_{V_m(\phi_t)}(n,r)e^{2\pi i(n\tau+rz)},$$

we have that $c_{V_m(\phi_t)}(n,r) = 0$ unless $4mnt > r^2$. The Fourier coefficients of the lift $\mathcal{G}_M(\phi_t)$ are given by

$$a_{\mathcal{G}_M(\phi_t)}(T) = c_{V_m(\phi_t)}(n,r)$$

where $T \in \mathbb{S}_{2,t}^{\geq 0}(\mathbb{Z}) = \left\{ \begin{pmatrix} n & r/2 \\ r/2 & mt \end{pmatrix} > 0 \,\middle|\, n, r, m \in \mathbb{Z} \right\}$. Therefore, we have that $\mathcal{G}_M(\phi_t)|\Phi =$

0. Using Proposition 6.5.26, we get

$$\mathcal{G}_M(\phi_t)|_\kappa \operatorname{Rot}(a)|\Phi = 0.$$

This proves that the lift is cuspidal. $\square$

Remark 6.5.28. 1. For $t = 1$ the lifting

$$\mathcal{G}_M : \phi_1 \to \mathcal{G}_M(\phi_1)$$

is the Maass lifting with level

$$\mathcal{V} : J^c_{\kappa,1}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_0^{(2)}(M)).$$

2. For $M = 1$ the lifting

$$\mathcal{G}_M : \phi_t \to \mathcal{G}_M(\phi_t)$$

is Gritsenko's lifting

$$\mathcal{G} : J^c_{\kappa,t} \to S_\kappa(\Gamma[t]).$$

Using the Fourier-Jacobi expansion of $F \in S_\kappa(\Gamma_M[t])$ together with first projection map gives us a map

$$L : S_\kappa(\Gamma_M[t]) \to \prod_{l\in\mathbb{N}} J^c_{\kappa,lt}(\Gamma_0(M)^J) \to J^c_{\kappa,t}(\Gamma_0(M)^J).$$

The inverse of this map gives the mixed level lifting $\mathcal{G}_M$ which gives us a form $F \in S_\kappa(\Gamma_M[t])$.

Corollary 6.5.29. *The linear map $\mathcal{G}_M : J^c_{\kappa,t}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_M[t])$ is injective.*

Proof. The composition of the mixed level lifting $\mathcal{G}_M$ together with the map L

$$J^c_{\kappa,t}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_M[t]) \to \prod_{l\in\mathbb{N}} J^c_{\kappa,lt}(\Gamma_0(M)^J) \to J^c_{\kappa,t}(\Gamma_0(M)^J)$$

is the identity. Hence

$$\mathcal{G}_M : J^c_{\kappa,t}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_M[t])$$

is injective since otherwise, if we start with a nonzero Jacobi cusp form ϕ_t, such that its mixed level lift $\mathcal{G}_M(\phi_t)$ is zero, then all its Fourier-Jacobi coefficients are zero and in particular $L(\mathcal{G}_M(\phi_t)) = \phi_t = 0$. A contradiction. $\square$

Ibukiyama and Skoruppa in [34] showed that $J_{1,t}(\Gamma_0(M)^J) = 0$ for all $\gcd(t, M) = 1$. In fact, Schmidt removed the restriction on the integers t and M and showed that $J_{1,t}(\Gamma_0(M)^J) = 0$ for all integers t and M ([75], Theorem 2). As an easy corollary to this result, we have

Corollary 6.5.30. *There are no mixed level lifts $\mathcal{G}_M(\phi_t)$ of weight 1.*

The Fourier-Jacobi expansion of a mixed level paramodular cusp form $F \in S_\kappa(\Gamma_M[t])$ given by

$$F(\tau, z, \tau') = \sum_{l \geq 1} \phi_{lt}(\tau, z) e^{2\pi i l t \tau'}$$

combined with the mixed level lifting $\mathcal{G}_M$ gives us a map

$$S_\kappa(\Gamma_M[t]) \to \prod_{l \in \mathbb{N}} J^c_{\kappa,lt}(\Gamma_0(M)^J) \to \prod_{l \in \mathbb{N}} S_\kappa(\Gamma_M[lt]).$$

Thus, as in the original work of Gritsenko, from our mixed level lifting one obtains an infinite family of liftings given by

$$\begin{array}{ccccc} S_\kappa(\Gamma_M[t]) & \longrightarrow & \prod_{l\geq 1} J^c_{\kappa,lt}(\Gamma_0(M)^J) & \longrightarrow & \prod_{l\geq 1} S_\kappa(\Gamma_M[lt]) \\ & & & & \\ F & \longmapsto & (\phi_{lt})_{l\geq 1} & \longmapsto & (\mathcal{G}(\phi_{lt}))_{l\geq 1}. \end{array}$$

In particular, for each $l \geq 1$ using the lth projection map (projection map to the lth Fourier-Jacobi coefficient), we get a paramodular level raising lifting between mixed level paramodular cusp forms

$$S_\kappa(\Gamma_M[t]) \to J^c_{\kappa,lt}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_M[lt]).$$

Therefore combining Fourier-Jacobi expansions giving us Jacobi forms with the mixed level lifting applied iteratively gives an infinite sequence of paramodular level raising liftings between mixed level paramodular cusp forms starting from paramodular level $t \geq 1$ to all possible paramodular levels multiples of t:

$$\begin{aligned} S_\kappa(\Gamma_M[t]) &\to \prod_{l\in\mathbb{N}} J^c_{\kappa,lt}(\Gamma_0(M)^J) \to \prod_{l\in\mathbb{N}} S_\kappa(\Gamma_M[lt]) \\ &\to \prod_{l\in\mathbb{N}}\prod_{m\in\mathbb{N}} J^c_{\kappa,m(lt)}(\Gamma_0(M)^J) \to \prod_{l\in\mathbb{N}}\prod_{m\in\mathbb{N}} S_\kappa(\Gamma_M[mlt]) \\ &\to \prod_{l\in\mathbb{N}}\prod_{m\in\mathbb{N}}\prod_{n\in\mathbb{N}} J^c_{\kappa,n(mlt)}(\Gamma_0(M)^J) \to \prod_{l\in\mathbb{N}}\prod_{m\in\mathbb{N}}\prod_{n\in\mathbb{N}} S_\kappa(\Gamma_M[nmlt]) \to \dots \end{aligned}$$

Also if we start with $F \in S_\kappa(\Gamma_0^{(2)}(M))$ and combining Fourier-Jacobi expansion of Siegel modular forms together with the general Gritsenko's lifting $\mathcal{G}_M$ we get infinitely many liftings to paramodular cusp forms of all possible paramodular levels l

$$S_\kappa(\Gamma_0^{(2)}(M)) \to \prod_{l\in\ N} J^c_{\kappa,l}(\Gamma_0(M)^J) \to \prod_{l\in\ N} S_\kappa(\Gamma_M[l]).$$

Combining this with the infinitely many paramodular level raising liftings between paramodular cusp forms we get an infinite sequence of liftings starting from Siegel modular forms with level $\Gamma_0^{(2)}(M)$ to mixed level paramodular cusp forms of paramodular level any natural number.

6.5.4 Image of the Mixed Level Lifting and a Certain Maass Subspace

Let $S_\kappa^{\mathrm{SK}}(\Gamma_M[t]) \subset S_\kappa(\Gamma_M[t])$ be the image of the lifting $\mathcal{G}_M$. Note that for $F \in S_\kappa^{\mathrm{SK}}(\Gamma_M[t])$, the Fourier-Jacobi coefficients of F are of the form $V_m(\phi_t)$ for some $\phi_t \in$

$J^c_{\kappa,t}(\Gamma_0(M)^J)$. Recall the action of the V_m operators

$$V_m(\phi_t)(\tau,z) = \sum_{\substack{n\geq 1, r\in\mathbb{Z}\\ r^2<4mnt}} \sum_{\substack{d|(n,r,m)\\ \gcd(d,M)=1}} d^{\kappa-1} c_{\phi_t}\left(\frac{nm}{d^2},\frac{r}{d}\right) e^{2\pi i(n\tau+rz)}.$$

Using this, we see that the lifted forms $\mathcal{G}_M(\phi_t)$ satisfy the following linear relations in terms of the Fourier coefficients of the Jacobi form that we are lifting.

Proposition 6.5.31. *The subspace $S^{\mathrm{SK}}_\kappa(\Gamma_M[t])$ consists of modular forms whose Fourier coefficients satisfy the following relations*

$$a_{\mathcal{G}_M(\phi_t)}(n,r,mt) = \sum_{\substack{d|(n,r,m), r^2<4nmt\\ \gcd(d,M)=1}} d^{\kappa-1} c_{\phi_t}\left(\frac{nm}{d^2},\frac{r}{d}\right).$$

We note that the linear relations of the forms in $S^{\mathrm{SK}}_\kappa(\Gamma_M[t])$ as given in 6.5.31 are similar to the classical Maass relations (see Chapter 5) in case $t=M=1$. This naturally suggests that that we give a description of a certain subspace of mixed level paramodular forms whose Fourier coefficients satisfy the following linear relations.

Remark 6.5.32. When $t=1$, the space $S^{\mathrm{SK}}_\kappa(\Gamma_M[t]) = S^{\mathrm{SK}}_\kappa(\Gamma^{(2)}_0(M))$ is the image of the Maass lifting with level (see Theorem 6.2.11) that consists of Siegel modular forms $\mathcal{V}(\phi)$ of degree two in $S_\kappa(\Gamma^{(2)}_0(M))$ whose Fourier coefficients satisfy the relations

$$a_{\mathcal{V}(\phi)}(n,r,m) = \sum_{\substack{d|(n,r,m), r^2<4nm\\ \gcd(d,M)=1}} d^{\kappa-1} c_{\phi}\left(\frac{nm}{d^2},\frac{r}{d}\right).$$

Definition 6.5.33. We say $F\in S_\kappa(\Gamma_M[t])$ is in the Maass subspace $S^*_\kappa(\Gamma_M[t])$ if and only if the Fourier coefficients of F satisfy

$$a_F(n,r,mt) = \sum_{\substack{d|(n,r,m)\\ \gcd(d,M)=1}} d^{\kappa-1} a_F\left(\frac{mn}{d^2},\frac{r}{d},t\right) \tag{6.8}$$

for every $m,n,r\in\mathbb{Z}$ with $m,n,4mnt-r^2>0$.

In particular, $c_{\phi_t}\left(\frac{nm}{d^2}, \frac{r}{d}\right) = a_{\mathcal{G}_M(\phi_t)}\left(\frac{mn}{d^2}, \frac{r}{d}, t\right)$, we have then:

Corollary 6.5.34. *With the notation as above we have the following isomorphism of vector spaces*

$$J^c_{\kappa,t}(\Gamma_0(M)^J) \cong S^{\mathrm{SK}}_\kappa(\Gamma_M[t]) \subseteq S^*_k(\Gamma_M[t]).$$

Note that for $t = M = 1$, $S^*_\kappa(\Gamma_M[t])$ is the classical Maass subspace $S^*_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ (see Chapter 4). While in case $t = M = 1$, the Maass subspace completely characterizes the image of the full level Maass lifting, we don't yet have this when $M \geq 1$ or $t \geq 1$. While we have that the subspace generated by the mixed level lifts sits inside $S^*_\kappa(\Gamma_M[t])$, it would be very nice for computational purposes to obtain the other inclusion and give a complete characterization of the image of the mixed level lifts. Writing the forms in the subspace $S^*_\kappa(\Gamma_M[t])$ in terms of their Fourier-Jacobi expansion, the problem resides in the choice of the Fourier-Jacobi coefficient to which we should project to so that when we lift we get back again the form we start it with. There are certain operators $U_S(p)$ for $(p \mid M)$ on Fourier expansions given by

$$(F|U_S(p))(\tau, z, \tau') = \sum_{\substack{m,n,r\in\mathbb{Z} \\ m,n,4mtn-r^2>0}} a_F(np, rp, mtp)e(n\tau + rz + mp\tau'). \tag{6.9}$$

Given $F \in S^*_\kappa(\Gamma_M[t])$, calculating the Fourier coefficient of $F|U_S(p)$, we have that

$$a_{F|U_S(p)}(T) = a_F(np, rp, mtp) = \sum_{\substack{d|(np,rp,mp) \\ \gcd(d,M)=1}} d^{\kappa-1} a_F\left(\frac{mnp^2}{d^2}, \frac{rp}{d}, t\right)$$

for every $m, n, r \in \mathbb{Z}$ with $m, n, 4mtn - r^2 > 0$. But since $\gcd(d, p) = 1$, we get that

$$a_{F|U_S(p)}(T) = a_F(np, rp, mtp) = \sum_{\substack{d|(n,r,m) \\ \gcd(d,M)=1}} d^{\kappa-1} a_F\left(\frac{mnp^2}{d^2}, \frac{rp}{d}, t\right).$$

Hence the operators $U_S(p)$ for $(p \mid M)$ preserve $S^*_k(\Gamma_M[t])$. The Fourier coefficients of the form $\mathcal{G}_M(\phi_t)|U_S(p)$ are given by

$$a_{\mathcal{G}_M(\phi_t)|U_S(p)}(T) = a_{\mathcal{G}_M(\phi_t)}(np, rp, mtp) = \sum_{\substack{d|(n,r,m)\\ \gcd(d,M)=1}} d^{k-1} c_{\phi_t}\left(\frac{nmp^2}{d^2}, \frac{rp}{d}\right),$$

for every $m, n, r \in \mathbb{Z}$ such that $m, n, 4mtn - r^2 > 0$. Comparing this with the Fourier coefficient of

$$a_{\mathcal{G}_M(\phi_t)}(T) = \sum_{\substack{d|(n,r,m)\\ \gcd(d,M)=1}} d^{k-1} c_{\phi_t}\left(\frac{nm}{d^2}, \frac{r}{d}\right),$$

for every $m, n, r \in \mathbb{Z}$ such that $m, n, 4mtn - r^2 > 0$, we observe that it is not clear how the Maass relations as given in Definition 6.5.33 describe the relationship between $a_{\mathcal{G}_M(\phi_t)|U_S(p)}$ and $a_{\mathcal{G}_M(\phi_t)}$.

6.6 Cuspidality Characterized by Fourier-Jacobi coefficients

We consider the Fourier-Jacobi expansion of a Siegel modular form $F \in M_\kappa(\Gamma_0^{(2)}(M))$,

$$\begin{aligned} F(\tau, z, \tau') &= \sum_{T=\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in \mathbb{S}_2^{\geq 0}(\mathbb{Z})} a_F(T) e^{2\pi i \operatorname{Tr}(TZ)} \\ &= \sum_{m\geq 0} \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\leq 4nm}} a_F(n,r,m) e^{2\pi i(n\tau + rz + m\tau')} \\ &= \sum_{m\geq 0} \phi_m(\tau, z) e^{2\pi i m\tau'} \\ &= \sum_{m\geq 0} \left(\sum_{\substack{n,r\\ r^2\leq 4mn}} c_{\phi_m}(n,r) e^{2\pi i(n\tau+rz)} \right) e^{2\pi i m\tau'}. \end{aligned} \tag{6.10}$$

We have seen (Chapter 5, Theorem 5.2.17) that if F is a cusp form then each of its Fourier-Jacobi coefficients is a cusp form. We now prove that the converse is also true due to the structure of the maximal degree cusps.

Theorem 6.6.1. *A modular form $F \in M_\kappa(\Gamma_0^{(2)}(M))$ is a cusp form if and only if each of its Fourier-Jacobi coefficients ϕ_m is a Jacobi cusp form.*

Proof. Suppose each ϕ_m is a Jacobi cusp form. This implies that the coefficients $c_{\phi_m}(n,r)$ vanish unless $r^2 < 4mn$. But, since $a_F\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} = c_{\phi_m}(n,r)$, then $a_F\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} = 0$ for all T such that $\det(T) = 0$. In other words, we have $F|\Phi = 0$, where Φ is the Siegel Φ operator. Now to show $F \in S_\kappa(\Gamma_0^{(2)}(M))$, we have to show that $F|_\kappa\gamma|\Phi$ for all $\gamma \in \Gamma_0^{(2)}(M)) \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$. According to Theorem 6.5.1, the representatives γ come form $\mathbb{C}_{2,1}(\mathbb{Q})R$. Using Propostion 5.4 and the same argument as in the proof of Theorem 5.5, we obtain the cuspidality of F. $\square$

Given $F \in M_\kappa(\Gamma_M[t])$, we have seen (Theorem 6.3.8) that its Fourier-Jacobi coefficients all have indices divisible by t.

$$\begin{aligned} F(\tau,z,\tau') &= \sum_{T=\begin{pmatrix} n & r/2 \\ r/2 & mt \end{pmatrix} \in \mathbb{S}_{2,t}^{\geq 0}(\mathbb{Z})} a_F(T)e^{2\pi i \operatorname{Tr}(TZ)} \\ &= \sum_{m\geq 0} \sum_{\substack{n,r\in\mathbb{Z} \\ r^2 \leq 4nmt}} a_F(n,r,mt)e^{2\pi i(n\tau+rz+mt\tau')} \\ &= \sum_{m\geq 0} \phi_{mt}(\tau,z)e^{2\pi imt\tau'} \\ &= \sum_{m\geq 0} \left(\sum_{\substack{n,r \\ r^2\leq 4nmt}} c_{\phi_{mt}}(n,r)e^{2\pi i(n\tau+rz)} \right) e^{2\pi imt\tau'}. \end{aligned} \tag{6.11}$$

We have also seen (Theorem 6.3.8) that if $F \in S_\kappa(\Gamma_M[t])$ is a cusp form then, each of it is Fourier-Jacobi coefficients ϕ_{mt} is a cusp form. We now prove that the converse is also true due to the structure of the maximal degree cusps.

Theorem 6.6.2. *A paramodular form $F \in M_\kappa(\Gamma_M[t])$ is a cusp form if and only if each of its Fourier-Jacobi coefficients ϕ_{mt} is a Jacobi cusp form.*

Proof. Similar to the proof of Theorem 6.6.1 and as a consequence of the cuspidalilty of ϕ_{mt} for each m, we have

$$a_F(n, r, mt) = a_F(T) = c_{\phi_{mt}}(n, r)$$

vanish for all $T = \begin{pmatrix} n & r/2 \\ r/2 & mt \end{pmatrix}$ such that $\det(T) = 0$. Equivalently, $F|\Phi = 0$. To show $F \in S_\kappa(\Gamma_M[t])$, we need to show $F|_\kappa \gamma|\Phi$ for all $\gamma \in \Gamma_M[t] \setminus \mathrm{Sp}_4(\mathbb{Q})/C_{2,1}(\mathbb{Q})$. According to Theorem 6.5.25, the representatives γ are of the form $\mathrm{Rot}(a)$. But using Propostion 6.5.26, we immediately get the cuspidality of the F. $\square$

Remark 6.6.3. We note that in general we can't claim that if the Fourier-Jacobi coefficients of a Siegel modular form are cusp forms then the form is a cusp form. But, we have been able to obtain such results for the Siegel modular forms with respect to the Hecke congruence subgroup $\Gamma_0^{(2)}(M)$ and the mixed level paramodular group $\Gamma_M[t]$ and in particular with respect to $\mathrm{Sp}_4(\mathbb{Z})$ and $\Gamma[t]$ due to the structure of the cusps.

6.7 When Is the Inverse of the Fourier-Jacobi Expansion Map a Mixed Level Paramodular Form

The action of the involution $V_t = \begin{pmatrix} {}^tU_t & 0_2 \\ 0_2 & U_t \end{pmatrix}$ on paramodular forms is given by

$$(F|_\kappa V_t)(Z) = (\det(U_t))^{-\kappa} F(V_t Z).$$

We have $V_t Z = {}^t U_t Z U_t^{-1}$, $U_t^{-1} = U_t$, and $\det(U_t) = -1$, hence

$$\begin{aligned}
(F|_\kappa)V_t(Z) &= (-1)^\kappa \sum_{T \in \mathbb{S}_{2,t}^{\geq 0}(\mathbb{Z})} a_F(T) e^{2\pi i \operatorname{Tr}(T {}^t U_t Z U_t)} \\
&= (-1)^\kappa \sum_{T \in \mathbb{S}_{2,t}^{\geq 0}(\mathbb{Z})} a_F(T) e^{2\pi i \operatorname{Tr}(U_t T {}^t U_t Z)} \\
&= (-1)^\kappa \sum_{T \in \mathbb{S}_{2,t}^{\geq 0}(\mathbb{Z})} a_F(U_t T {}^t U_t) e^{2\pi i \operatorname{Tr}(TZ)}.
\end{aligned}$$

That is,

$$a_{F|_\kappa V_t}(T) = a_F(U_t T {}^t U_t).$$

If $T = \begin{pmatrix} n & r/2 \\ r/2 & mt \end{pmatrix} \in \mathbb{S}_{2,t}^{\geq 0}(\mathbb{Z})$, then

$$U_t T {}^t U_t = \begin{pmatrix} m & r/2 \\ r/2 & nt \end{pmatrix}.$$

Also, since V_t is an involution, it splits the space of paramodular forms in $M_\kappa(\Gamma_M[t])$ into plus and minus eigenspaces. Consequently, we have

$$(F|_\kappa V_t)(Z) = \epsilon_t F(Z),$$

and on the level of the Fourier coefficients we have

$$(-1)^\kappa a_F(U_t T {}^t U_t) = \epsilon_t a_F(T).$$

Writing this relationship in terms of the Fourier coefficients $c_{\phi_{mt}}(n, r)$ of the Fourier-Jacobi coefficients ϕ_{mt} we get the relationship that the Fourier-Jacobi coefficients of a paramodular form of level t should satisfy the following relationship:

$$c_{\phi_{mt}}(n, r) = \epsilon c_{\phi_{nt}}(m, r), \tag{6.12}$$

where $\epsilon = \pm 1$.

For each $d \mid t$, we have an involution V_d such that

$$a_{F|_\kappa V_d}(T) = a_F(\tilde{T}),$$

where the description of the matrices $\tilde{T}$ can be found in § 6.3.4.2.

Also since each V_d is an involution, it splits the space of paramodular into plus and minus eigenspaces. Let ϵ_d be the eigenvalue at V_d, where $\prod_d \epsilon_d = \epsilon_t$, then

$$(F|_\kappa V_d)(Z) = \epsilon_d F(Z).$$

Comparing the Fourier coefficients, we get

$$a_F(\tilde{T}) = \epsilon_d a_F(T)$$

Writing this relationship in terms of the Fourier coefficients of the Fourier-Jacobi coefficients of the Fourier-Jacobi expansion of F is

$$c_{\phi_{\tilde{m}t}}(\tilde{n}, \tilde{r}) = \epsilon_d c_{\phi_{mt}}(n, r),$$

where $\tilde{m}, \tilde{n}$ and $\tilde{r}$ are as defined in § 6.3.4.2.

We recall that the Fourier-Jacobi expansion of a Siegel modular form gives a map

$$\mathcal{H} : M_\kappa(\Gamma_M[t]) \to \prod_{m \geq 0} J_{\kappa,mt}(\Gamma_0(M)^J).$$

Restricting this map to the eigenspace $M_\kappa^\epsilon(\Gamma_M[t])$ and denoting this restriction by $\mathcal{H}^\epsilon$, the image of $\mathcal{H}^\epsilon$ sits in the subspace of $\prod_{m \geq 0} J_{\kappa,mt}(\Gamma_0(M)^J)$ consisting of Jacobi forms whose Fourier coefficients satisfy the relationship given by equation (6.12). We denote this subspace by

$$\prod_{m\geq 0} J^{\epsilon}_{\kappa,mt}(\Gamma_0(M)^J.$$

That is, we have an injective map

$$\mathcal{H}^{\epsilon}: M^{\epsilon}_{\kappa}(\Gamma_M[t]) \to \prod_{m\geq 0} J^{\epsilon}_{\kappa,mt}(\Gamma_0(M)^J).$$

Theorem 6.7.1. *Taking a collection of Jacobi forms* $\{\phi_{mt}\} \in \prod_{m\geq 0} J^{\epsilon}_{\kappa,mt}(\Gamma_0(M)^J)$ *such that the series* $\sum_{m\geq 0} \phi_{mt}(\tau, z)e^{2\pi i mt\tau'}$ *converges on compact subsets of the Siegel upper half space* $\mathfrak{h}^2$*, then it defines a mixed level paramodular form in* $M^{\epsilon}_{\kappa}(\Gamma_M[t])$*. Moreover, if* $\{\phi_{mt}\} \in \prod_{m\geq 0} J^{c,\epsilon}_{\kappa,mt}(\Gamma_0(M)^J)$*, then it defines a mixed level paramodular cusp form in* $S^{\epsilon}_{\kappa}(\Gamma_M[t])$*.*

Proof. Since the series $\sum_{m\geq 0} \phi_{mt}(\tau, z)e^{2\pi i mt\tau'}$ with $\{\phi_{mt}\} \subset \prod_{m\geq 0} J^{\epsilon}_{\kappa,mt}(\Gamma_0(M)^J)$ is convergent on compact subsets of the Siegel upper half space $\mathfrak{h}^2$, hence it defines a holomorphic function F on $\mathfrak{h}^2$. We now show that $F \in M^{\epsilon}_{\kappa}(\Gamma_M[t])$. Since each of the Jacobi forms have index divisible by t, then using Proposition 6.3.10, we get that each of the Jacobi forms is invariant under $\Delta(t)$. Also, using Definition 6.2.5, we see that each of the term of the series $\sum_{m\geq 0} \phi_{mt}e^{2\pi i mt\tau'}$ is invariant under $\Gamma_{M,\infty}(\mathbb{Z})$. But since $\Gamma_{M,\infty}[t]$ is generated by $\Gamma_{M,\infty}(\mathbb{Z})$ and $\Delta(t)$, we have that each of the term of the series is invariant under $\Gamma_{M,\infty}[t]$. Since the series converges on compact subsets of the Siegel upper half space $\mathfrak{h}^2$, we conculde that F is invariant under $\Gamma_{M,\infty}[t]$. Working backwards, the relationship given by equation (6.12) on the Fourier coefficients of the Jacobi forms, imply that the function F defined by the series satisfies $F|_{\kappa}V_t = \epsilon F$. Therefore,

$$F|_{\kappa}V_t\Gamma_{M,\infty}[t]V_t = \epsilon F|_{\kappa}\Gamma_{M,\infty}[t]V_t = \epsilon F|_{\kappa}V_t = F.$$

In other words, we have shown that $F \in M^{\epsilon}_{\kappa}(\Gamma_M[t])$. In addition, if we require all Jacobi forms in the Fourier series $\sum_{m\geq 0} \phi_{mt}e^{2\pi i mt\tau'}$ belong to $J^{c}_{\kappa,mt}(\Gamma_0(M)^J)$, we get using Theorem 6.6.2 that $F \in S^{\epsilon}_{\kappa}(\Gamma_M[t])$. $\square$

Ibukiyama, Poor and Yuen in [33] working with full level paramodular forms have weakened the requirement of the holomorphicity of the series by requiring the Fourier coefficients of the Jacobi forms satisfy a certain growth condition that suffices to show the holomorphicity of the function F ([33], Theorem 2.2). We believe the same result also works for mixed level paramodualr forms. We also add that their result should also work for cusp forms due to Theorem 6.6.2 in the particular case when $M = 1$.

Chapter 7

Mixed Level Saito-Kurokawa Lifts From a Representation Theoretic Point of View

The goal of this chapter is to construct the mixed level Saito-Kurokawa lifts via representation theory.

7.1 Introduction

From a classical point of view, we have seen in § 5.3 of Chapter 5 that there are two constructions that generalize the classical Saito-Kurokawa lifting. When M is odd squarefree and κ a positive even integer, we have the construction that we called the congruence level Saito-Kurokawa lifting. It is given by

$$S_{2\kappa-2}^{\text{new}}(\Gamma_0(M)) \cong S_{\kappa-1/2}^{+}(4M) \cong J_{\kappa,1}^{c,\text{new}}(\Gamma_0(M)^J) \to S_\kappa^{\text{SK}}(\Gamma_0^{(2)}(M)).$$

We recall that the third linear map is the Maass lifting with level M that is generally true for any κ and for M not necessarily squarefree.

The other classical construction that generalizes the classical Saito-Kurokawa lifting, the paramodular level Saito-Kurokawa lifting, works without assuming κ to be an even integer. Given positive integers κ and t, we have

$$S_{2\kappa-2}^{\text{new},-}(\Gamma_0(t)) \cong J_{\kappa,t}^c \to S_\kappa(\Gamma[t]).$$

This construction involves the middle space of Jacobi forms of index $t \geq 1$ with respect to the full Jacobi group and produces a paramodular form with respect to the paramodular level $\Gamma[t]$ for any $t \geq 1$ from an elliptic newform in the minus subspace.

We note that if $f \in S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z}))$, the functional equation satisfied by its L-function as discussed in Remark 5.2.6 of Chapter 5 is given by

$$\Lambda(s, f) = (-1)^{\kappa-1}\Lambda(\kappa - s, f).$$

It is then clear that the condition that κ be an even positive integer agrees with the condition of being in the minus subspace when considering the full level $\mathrm{SL}_2(\mathbb{Z})$.

Assume κ is an even integer and m squarefree. Starting with a newform $f \in S_{2\kappa-2}^{\text{new},-}(\Gamma_0(m))$, one can classically construct two different Saito-Kurokawa lifts $F_1 \in S_\kappa(\Gamma_0^{(2)}(m))$ and $F_2 \in S_\kappa(\Gamma[m])$ associated to f. Our goal is to construct mixed level Saito-Kurokawa liftings, where we impose some congruence conditions at some primes dividing the level and some paramodular condition at the other primes dividing the level. Our method is based on Schmidt's approach in [74] which frames the Saito-Kurokawa lifiting under Langlands functoriality.

We note that the Saito-Kurokawa liftings have been considered and generalized in many different points of view by different authors, one instance is in the general framework of automorphic representations. In [59], Piatetski-Shapiro looks at these liftings from the point of view of dual reductive pairs and the Weil representation.

Using representation theoretic methods and exploiting the connection between automorphic forms and representations, one can reformulate the classical theory of forms for a certain underlying algebraic group G in terms of local and global automorphic representations of G. One can then apply results obtained from the representation theoretic point of view to get results on classical modular forms.

In general, given a linear algebraic group G viewed over the base field $\mathbb{Q}$ behind the definition of a classical cuspidal modular form f, then one can view f as an automorphic form Φ_f on G which generates a global automorphic representation π_f of the adelized group $G(\mathbb{A})$ where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. This global representation π_f, if irreducible, can be decomposed as the restricted tensor product of local components π_p which are representations of the corresponding local group $G(\mathbb{Q}_p)$. If f is a modular form with respect to some level Γ, then the local component π_p contains a non-zero vector invariant under the local group corresponding to Γ.

This procedure of associating an automorphic representation to a classical modular form is well known in the elliptic case ($G = \mathrm{GL}_2$). Here we review it very briefly following [16], Section 3.6.

7.1.1 $G = GL_2$

For the degree $n = 1$, the $\mathbb{Q}$-algebraic group G behind the definition of elliptic modular forms is GL_2. Let $f \in S_\kappa(\Gamma_0(N))$ be an elliptic cusp form. One can associate a cuspidal automorphic representation $\pi_f = \otimes \pi_p$ of $\mathrm{GL}_2(\mathbb{A})$. Here is a brief summary of the steps involved:

First we associate to f a function $\phi_f : \mathrm{GL}_2^+(\mathbb{R}) \to \mathbb{C}$. Let Z be the center of GL_2 and $\mathrm{SO}_2(\mathbb{R}) := K_\infty^+$ be the special orthogonal group which is a maximal compact subgroup of $\mathrm{GL}_2^+(\mathbb{R})$. For every $z \in Z(\mathbb{R}), k \in K_\infty^+, g \in \mathrm{GL}_2^+(\mathbb{R})$, let the associated holomorphy factor be $j(g, z) := \det(g)^{-1/2}(cz + d)$. Define a function

$$\phi_f(g) := j(g, i)^{-\kappa} f(g \cdot i).$$

Let $G := \mathrm{GL}_2$, $G_\infty := G(\mathbb{R}) \supset K_\infty := O_2(\mathbb{R})$, $K_f := \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$ and $K_0(N) =$

$\prod_{p\nmid N}\mathrm{GL}_2(\mathbb{Z}_p)\times\prod_{p|N}K_p(N)$, where $K_p(N) := \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Z}_p)\,\middle|\, c\equiv 0 \pmod N\right\}$. Take $K := K_\infty K_f$.

Using a strong approximation theorem ([16], Theorem 3.3.1), we have that $G(\mathbb{A}) = G(\mathbb{Q})G^+(\mathbb{R})K_f$. Hence, we transport ϕ_f to a function on $G(\mathbb{A})$ by decomposing $g\in G(\mathbb{A})$ as $g = g_{\mathbb{Q}}g_\infty k_\infty k_f$ and defining

$$\phi_f(g) = j(g_\infty k_\infty, i)^{-\kappa} f(g_\infty k_\infty \cdot i).$$

Since f is a cusp form, ϕ_f is an element of $L^2(G(\mathbb{Q})\setminus G(\mathbb{A})/Z(\mathbb{A}))$ ([16], Theorem 3.6.1). We let G act on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z(\mathbb{A}))$ by right translation and let π_f be the unitary $\mathrm{PGL}_2(\mathbb{A})$-subrepresentation of this L^2-space generated by ϕ_f.

In case f is an eigenform for the Hecke operators, then using strong multiplicity one for GL_2, one has that the associated automorphic representation π_f is irreducible according to the following theorem.

Theorem 7.1.1 ([71], Theorem 1.1)**.** *The representation π_f is irreducible if and only if f is an eigenform for the Hecke operators $T(p)$ for almost all primes p. If this is the case, then f is automatically an eigenform for $T(p)$ for all $p\nmid N$.*

Thus to each eigenform f, there is a corresponding automorphic representation

$$\pi_f = \otimes\pi_{f,p}.$$

If f has no nebentypus, then π_f will have trivial central character and descends to a representation of $\mathrm{PGL}_2(\mathbb{A})$. Identifying the local representations $\pi_{f,p}$ is well known at the archimidean place ∞ and for all finite primes p not dividing N. These are described by the weight κ and the Hecke eigenvalues λ_p of f. At the archimidean place, $\pi_{f,\infty}$ is the discrete series representations of $\mathrm{PGL}_2(\mathbb{R})$ with a lowest weight vector κ. For finite primes $p\nmid N$, $\pi_{f,p}$ is an unramified principal series representation (an infinite dimensional representation containing a non-zero $\mathrm{GL}_2(\mathbb{Z}_p)$-fixed vector) characterized by its Satake parameter α whose relationship to its Hecke eigenvalue λ_p is given by $\lambda_p = p^{(\kappa-1)/2}(\alpha + \alpha^{-1})$ ([71], page 3). Identifying the local components at the bad places is not as easy.

If N is squarefree and $f \in S_\kappa^{\mathrm{new}}(\Gamma_0(N))$ a newform, then Schmidt ([71]) shows that the local representation $\pi_{f,p}$ at a place $p \mid N$ contains non-trivial vectors invariant under an Iwahori subgroup I, where

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \quad : \quad c \in p\mathbb{Z}_p \right\}.$$

It is determined in the following theorem.

Theorem 7.1.2 ([71], Theorem 1.2)**.** *Assume N is a squarefree positive integer, and let $f \in S_\kappa(\Gamma_0(N))$ be a newform. Then the local component $\pi_{f,p}$ of the associated automorphic representation π_f at a place $p \mid N$ is given as follows:*

$$\pi_{f,p} = \left\{ \begin{array}{ll} \xi\,\mathrm{St}_{\mathrm{GL}(2)} & \textit{if } \epsilon_p = 1 \\ \mathrm{St}_{\mathrm{GL}(2)} & \textit{if } \epsilon_p = -1, \end{array} \right\}$$

where ϵ_p is the eigenvalue of the Atkin-Lehner involution at p, $\mathrm{St}_{\mathrm{GL}(2)}$ is the Steinberg representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and ξ is the unique non-trivial unramified quadratic character of $\mathbb{Q}_p^*$.

Moreover, the local L-factor at $p \mid N$ is given by

$$L_p(s, \pi_{f,p}) = (1 + \epsilon_p p^{-s-1/2})^{-1}.$$

The archimedean and unramified factors $L_p(s, \pi_{f,p})$ are such that

$$L(s - 1/2 + \kappa/2, f) = L(s, \pi_f).$$

7.1.2 $G = \mathrm{GSp}_4$

In this section, we refer the reader to Section 3 in [7] for all the details concerning the construction of cuspidal automorphic representations associated to Siegel cusp forms. For the degree $n = 2$, the $\mathbb{Q}$-algebraic group behind the definition of Siegel modular forms is GSp_4. Using a strong approximation theorem one can regard a cuspidal eigenform F of weight κ with no character as an automorphic function Φ_F on $\mathrm{GSp}_4(\mathbb{A})$ in the cuspidal subspace $L^2_0(Z(\mathbb{A})\,\mathrm{GSp}_4(\mathbb{Q})\setminus\mathrm{GSp}_4(\mathbb{A}))$ where $Z(\mathbb{A})$ is the center of $\mathrm{GSp}_4(\mathbb{A})$. If we consider the subrepresentation Π generated by Φ_F in this space then it may not be irreducible because there is no strong multiplicity one result for GSp_4. We decompose Π into a direct sum of irreducibles $\Pi = \oplus\Pi_i$. When F is an eigenform, then all irreducible components of Π will turn out to be isomorphic. This isomorphism class is the cuspidal automorphic representation Π_F attached to F, which is in fact a representation of $\mathrm{PGSp}_4(\mathbb{A})$. We can decompose it into local components $\Pi_F = \otimes\Pi_{F,p}$ with $\Pi_{F,p}$ a representation of $\mathrm{PGSp}_4(\mathbb{Q}_p)$. In case $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$, it is shown in ([70], Proposition 4.1.3) that the local components $\Pi_{F,p}$ of the cuspidal automorphic representaiton Π_F are also described in terms of the weight and the Hecke eigenvalues of F. In fact, the local component $\Pi_{F,\infty}$ is the discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ of weight κ and the $\Pi_{F,p}$ for finite p is the spherical (containing a non-zero vector fixed by the maximal compact subgroup $\mathrm{GSp}_4(\mathbb{Z}_p)$ of $\mathrm{GSp}_4(\mathbb{Q}_p)$) principal series representation determined by the Hecke eigenvalues of F.

If $F \in S_\kappa(\Gamma_0^{(2}(N))$, determining the local components $\Pi_{F,p}$ for all $p \mid N$ even for N squarefree is more complicated because of the longer list of Iwahori-spherical representations of $\mathrm{GSp}_4(\mathbb{Q}_p)$ (14 of them!) containing non-trivial vectors fixed under the local analogue of the Siegel congruence subgroup $\Gamma_0^{(2)}(N)$ (see Table 3 in [72]). We will focus below on those that arise as components of the Saito-Kurokawa lifts so we do not comment further here.

7.1.2.1 L-Functions

Consider a Siegel modular form $F \in S_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$ that is an eigenform for all the Hecke operators $T \in \mathbb{T}^S_{\mathbb{Z}}$ with eigenvalues $\lambda_F(T)$. Associated to F are the Satake p-

parameters which are the complex numbers of the $n+1$-tuple $(\alpha_{0,p},\ldots,\alpha_{n,p})$ for each prime p. The $n+1$-tuple is in $(\mathbb{C}^*)^{n+1}/W$ where W is the Weyl group. For more details, the reader can consult [86]. The L-function of F can be expressed in terms of the Satake p-parameters as follows. We restrict to the case $n=2$.

Definition 7.1.3. Let $F \in S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$ be a simultaneous eigenform for all the Hecke operators $T \in H_2$. Define the spinor L-function

$$L(s,F,\mathrm{spin}) = \prod_p L_p(s,F)$$

where

$$L_p(s,F) = [(1-\alpha_{0,p}p^{-s})(1-\alpha_{0,p}\alpha_{1,p}p^{-s})(1-\alpha_{0,p}\alpha_{2,p}p^{-s})(1-\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}p^{-s})]^{-1},$$

and $\alpha_{0,p},\alpha_{1,p},\alpha_{2,p}$ are the Satake p-parameters of F.

Remark 7.1.4. According to Andrianov [2], we have associated a zeta function

$$Z(s,F) = \prod_p (Q_{p,F}(p^{-s}))^{-1}$$

to a Siegel modular form F of degree two. It turns out ([2], Theorem 1.3.2) that this zeta function is the same as the spinor L-function for

$$\begin{aligned} Q_{p,F}(t) &= 1-\lambda_F(p)t+(\lambda_F(p)^2-\lambda_F(p^2)-p^{2\kappa-4})t^2-\lambda_F(p)p^{2\kappa-3}t^3+p^{4k-6}t^4 \\ &= (1-\alpha_{0,p}t)(1-\alpha_{0,p}\alpha_{1,p}t)(1-\alpha_{0,p}\alpha_{2,p}t)(1-\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}t). \end{aligned}$$

Asgari and Schmidt in [7] explained the definition of the spinor L-function within the general framework of automorphic representations. We have seen above that we can associate an automorphic representation $\Pi_F = \otimes \Pi_{F,p}$ to $F \in S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$. At the finite places, the p-adic representations $\Pi_{F,p}$ are spherical denoted by $\pi(\chi_0,\chi_1,\chi_2)$ where χ_i are certain unramified characters of $\mathbb{Q}_p^*$ with Satake parameters $b_{i,p} = \chi_i(p)$. We refer to [7], section 2.2 and section 3.4 for all the details. We associate to each spherical representation $\Pi_{F,p}$ a local L-factor as follows.

Lemma 7.1.5 ([7], Lemma 2.4.2)**.** *The local L-factor corresponding to the spherical principal series representation $\Pi_{F,p}$ is given by:*

$$L(s,\Pi_{F,p}) = \left[(1-b_{0,p}p^{-s})(1-b_{0,p}b_{1,p}p^{-s})(1-b_{0,p}b_{2,p}p^{-s})(1-b_{0,p}b_{1,p}b_{2,p}p^{-s})\right]^{-1},$$

where the complex numbers $b_{0,p},\ldots,b_{n,p}$ are the Satake parameters associated to $\Pi_{F,p}$.

This shows that there is a relation between the classical Satake p-parameters $\alpha_{i,p}$ of F and the Satake parameters $b_{i,p}$ associated to the local component $\Pi_{F,p}$ of the automorphic representation Π_F associated to F. Here we give a lemma for general degree n that says these Satake parameters are essentially the same.

Lemma 7.1.6 ([7], Lemma 3.4.1)**.** *If $\alpha_{0,p},\ldots,\alpha_{n,p}$ are the classical Satake p-parameters of $F\in S_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$, then*

$$p^{n(n+1)/4-n\kappa/2}\alpha_{0,p},\alpha_{1,p},\ldots,\alpha_{n,p}$$

are the Satake parameters $b_{0,p},\ldots,b_{n,p}$ of the spherical principal series representation $\Pi_{F,p}$.

Using Langlands Theory of Euler products, one can associate to Π_F a global L-function $L(s,\Pi_F)$ as an Euler product of the local L-factors corresponding to the spherical prinicipal series representation $\Pi_{F,p}$ as given in Lemma 7.1.5.

Definition 7.1.7. Let Π_F be the associated cuspidal automorphic representation of a Siegel modular form $F\in S_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$. The Langlands L-function of Π_F is given by

$$L(s,\Pi_F) := \prod_p L(s,\Pi_{F,p}).$$

The product is taken over all finite places. Sometimes, this is referred to as the incomplete Langlands L-function in the literature.

Moreover, Asgari and Schmidt gave the following relation to the classical L-function.

Theorem 7.1.8 ([7], Theorem 3.6.1). *Let Π_F be the associated cuspidal automorphic representation of a Siegel modular form $F \in S_\kappa(\mathrm{Sp}_4(\mathbb{Z}))$, then for the spinor L-function of F we have*

$$L(s', F, \mathrm{spin}) = L(s, \Pi_F),$$

where $s' = s + \kappa - 3/2$. For general degree n, $s' = s + n\kappa/2 - n(n+1)/4$.

7.2 Saito-Kurokawa Lifting As a Functorial Lifting

Langlands ([45], § 3) interpreted Saito-Kurokawa liftings in terms of Langland's principle of functoriality. This principle predicts the existence of a lifting of automorphic forms from

$$\mathrm{PGL}_2(\mathbb{A}) \times \mathrm{PGL}_2(\mathbb{A}) \to \mathrm{PGSp}_4(\mathbb{A})$$

coming from the embedding of

$$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$$

as a homomorphism of complex Lie groups given by as in [73] § 4, Equation (20)

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \mapsto \begin{pmatrix} a' & 0 & 0 & b' \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ c' & 0 & 0 & d' \end{pmatrix}.$$

Schmidt in [70] shows that the classical Saito-Kurokawa lifting proves Langlands functoriality for the representations of the algebraic groups $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ of the form (π_f, π_2). The first factor is the cuspidal automorphic representation π_f of $\mathrm{PGL}_2(\mathbb{A})$ attached to $f \in S_{2\kappa-2}(\mathrm{SL}_2(\mathbb{Z}))$ as above and π_2 is a noncuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$ not depending on f whose archimedean component $\pi_{2,\infty}$ is the lowest discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ denoted by $D(1)$ and each of the local components for finite p is the trivial representation. To such a pair (π_f, π_2), the automorphic PGSp_4 representation Π_{F_f} associated to the Saito-Kurokawa lift F_f of f is then the representation obtained by the expected lifting of $\pi_f \otimes \pi_2$ predicted by Langlands functoriality. For more details on this, we refer the reader to [70].

7.2.1 Schmidt's Main Lifting Theorem

Working within the framework of Langlands functoriality in [73], Schmidt unified both constructions of the Saito-Kurokawa liftings described in § 7.1. From the local point of view, $\mathrm{GSp}_4(\mathbb{Q}_p)$ has two conjugacy classes of maximal open compact subgroups: the standard maximal open compact subgroup $\mathrm{GSp}_4(\mathbb{Z}_p)$ and the local analogue of the paramodular group $K(p)$. Consequently, a unified construction makes sense to exist and it explains the existence of more than one Saito-Kurokawa lift.

For each finite set S of places of $\mathbb{Q}$, we have an automorphic representation $\pi_S = \otimes \pi_{S,p}$ of $\mathrm{PGL}_2(\mathbb{A})$ defined by

$$\pi_{S,p} = \begin{cases} 1_{\mathrm{GL}_2} & \text{if } p \notin S \\ \mathrm{St}_{\mathrm{GL}_2} & \text{if } p \in S, \end{cases}$$

such that 1_{GL_2} is the trivial representation and $\mathrm{St}_{\mathrm{GL}_2}$ is the Steinberg representation for PGL_2. The global representation π_S is a noncuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$.

Let $\pi_f = \otimes \pi_{f,p}$ be the cuspidal irreducible automorphic representation on $\mathrm{PGL}_2(\mathbb{A})$ attached to $f \in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(m))$ with m squarefree. We consider the pair (π_f, π_S), where S is a finite set of places of $\mathbb{Q}$ such that for each $p \in S$, the local component $\pi_{f,p}$ is square integrable. These places are some of the finite places p dividing the level m or $p = \infty$. If the set S satisfies certain conditions, Schmidt shows using local and global theta liftings that the lifting $\Pi(\pi_f \otimes \pi_S) = \otimes \Pi_p$, where $\Pi_p = \otimes \Pi(\pi_p \otimes \pi_{S,p})$ is a global cuspidal automorphic representation of $\mathrm{PGSp}_4(\mathbb{A})$.

We now state Schmidt's main lifting theorem given in [73] based on the lifting predicted by Langlands functoriality.

Theorem 7.2.1 ([73], Theorem 3.1)**.** *Assume m is squarefree. Let π_f be the cuspidal automorphic representation of* $\mathrm{PGL}_2(\mathbb{A})$ *corresponding to a classical newform $f \in S^{\mathrm{new}}_{2k-2}(\Gamma_0(m))$. Let S be a finite set of places p of $\mathbb{Q}$ such that $\pi_{f,p}$ is square integrable (i.e $p \mid m$ or $p = \infty$). Let $\pi_S = \otimes \pi_{S,p}$ be the non-cuspidal automorphic representation of* $\mathrm{PGL}_2(\mathbb{A})$ *as defined above. Assume that the sign condition $(-1)^{|S|} = \epsilon(1/2, \pi_f)$ is fulfilled. Then*

1. *The global lifting $\Pi(\pi_f \otimes \pi_S) := \otimes \Pi(\pi_{f,p} \otimes \pi_{S,p})$ is an automorphic representation of* $\mathrm{PGSp}_4(\mathbb{A})$ *which appears discretely in the space of automorphic forms.*

2. *If $L(1/2, \pi_f) = 0$ or if $S \neq \emptyset$, then $\Pi(\pi_f \otimes \pi_S)$ is a cuspidal automorphic representation.*

The term $\epsilon(1/2, \pi_f)$ is the sign of the functional equation satisfied by the L-function associated to f. We now describe this representation locally as is done in § 5 of [74].

7.2.1.1 Local Components of $\Pi(\pi_f \otimes \pi_S)$

We note that the choice of S will always contain the archimedean place ∞ in order to obtain holomorphic forms. Therefore when it exists, the global lifting will be a cuspidal automorphic representation of $\mathrm{PGSp}_4(\mathbb{A})$ whose local component at ∞ is given by $\Pi(\pi_{f,\infty} \otimes \mathrm{St}_\infty)$. Since f has weight $2\kappa - 2$, the archimedean component $\pi_{f,\infty}$ is the discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ with a lowest weight vector $2\kappa - 2$ denoted by $D(2k-3)$ and the archimedean component $\pi_{S,\infty} := \mathrm{St}_\infty$ is the lowest discrete series representation $D(1)$ of $\mathrm{PGL}_2(\mathbb{R})$. It is shown in [7] that $\Pi(D(2\kappa - 3) \otimes D(1))$ is the archimedean component of an automorphic representation corresponding to a holomorphic Siegel modular form of weight κ. In other words, $\Pi(D(2k-3) \otimes D(1))$ is the holomorphic discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with weight κ.

For places $p \nmid m$, the local representations $\Pi(\pi_{f,p} \otimes \pi_{S,p})$ are unramified of the form $\Pi(\pi_{f,p} \otimes 1_{\mathrm{GL}_2})$, where $\pi_{f,p}$ is the unramified principal series representation containing non-zero vectors fixed by $\mathrm{GL}_2(\mathbb{Z}_p)$.

It remains to describe the local representations of the global lifting for finite places $p \mid m$. For such a place $p|m$ when m is squarefree, the local component $\pi_{f,p}$ is given by Theorem 7.1.2 and the local component $\pi_{S,p}$ was determined in equation (7.2.1). Consequently, there is a total of four possible lifts for such a p:

1. $\Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes 1)$,
2. $\Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)})$,
3. $\Pi(\xi\, \mathrm{St}_{\mathrm{GL}(2)} \otimes 1)$,
4. $\Pi(\xi\, \mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)})$.

However, for the choices of the set of places S, the fourth representation will be avoided because it turns out to be supercuspidal. A supercuspidal local representation cannot occur as a local component of an automorphic representation associated to a Siegel modular form when the level m is squarefree.

7.2.1.2 Atkin-Lehner Eigenvalues

We follow Schmidt in [74], page 4 and define the local Atkin-Lehner element is the matrix

$$\eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \end{pmatrix}.$$

We will call the local analogue of the Siegel congruence subgroup, P_1, and the local analogue of the paramodular group, P_2, Iwahori subgroups of $\mathrm{GSp}_4(\mathbb{Q}_p)$. To see an explicit description of these, we refer the reader to the same paper [74], page 4. Let H denote one of these groups. Then η acts on the space of H-invariant vectors for any representation (π, V) of $\mathrm{GSp}_4(\mathbb{Q}_p)$. If we assume that π has trivial central character, then $\pi(\eta)$ acts as an involution.

We call these operators Atkin-Lehner involutions. They split the space of V^H of H invariant vectors into ± 1-eigenspaces V_+^H and V_-^H. If v is an eigenvector, we call its eigenvalue the Atkin-Lehner eigenvalue of v. One has via table (30) of [74]:

The Atkin-Lehner eigenvalue of $\Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes 1)$ is -1 and the Atkin-Lehner eigenvalue of $\Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)})$ and $\Pi(\xi\, \mathrm{St}_{\mathrm{GL}(2)} \otimes 1)$ is 1.

7.2.1.3 Local L-factors

The global L-function of π_S is given by ([73], § 1)

$$\begin{aligned} L(s, \pi_S) &= \left(\prod_{p \notin S} L_p(s, \mathbf{1})\right) \left(\prod_{p \in S} L_p(s, \mathrm{St}_{\mathrm{GL}(2)})\right) \\ &= \left(\prod_p L_p(s, \mathbf{1})\right) \left(\prod_{p \in S} \frac{L_p(s, \mathrm{St}_{\mathrm{GL}(2)})}{L_p(s, \mathbf{1})}\right) \\ &= Z(s + \tfrac{1}{2}) Z(s - \tfrac{1}{2}) \left(\prod_{p \in S} \frac{L_p(s, \mathrm{St}_{\mathrm{GL}(2)})}{L_p(s, \mathbf{1})}\right). \end{aligned}$$

Here $Z(s)$ is the completed Riemann zeta function, and

$$\frac{L_p(s, \mathrm{St}_{\mathrm{GL}(2)})}{L_p(s, \mathbf{1})} = \begin{cases} \frac{1}{4\pi}(s - \frac{1}{2}), & \text{if } p \text{ is archimedean} \\ 1 - p^{-s+1/2}, & \text{if } p \text{ is finite.} \end{cases}$$

We have seen that in Theorem 7.1.8 that if F is a Siegel modular form for the full modular group, and an eigenform for the Hecke operators, then the classically defined spinor L-function coincides up to a shift in the argument with the Langlands L-function $L(s, \Pi_F)$. Now when F is a Siegel modular form for a congruence subgroup, it is not clear how to classically associate an L-function because there is no a well-defined notion of a space of newforms. We can associate an L-function to F to be the Langlands L-function associated to the cuspidal automorphic representation Π_F induced from F:

$$L(s + \kappa - 3/2, F, \mathrm{spin}) := L(s, \Pi_F),$$

where $L(s, \Pi_F) = \prod_p L(s, \Pi_{F,p})$. For finite places $p \nmid N$, the Langlands L-factors are given as the Langlands L-factors attached to Π_F for F Siegel modular form for the full Siegel modular group. As for the finite places $p \mid N$, the local L-factors are given using Local Langlands correspondence as given in Table 2 in [72] for all types of Iwahori-spherical representations of GSp_4.

Now Schmidt's main theorem gives a cuspidal automorphic representation $\Pi(\pi_f \otimes \pi_S) := \otimes \Pi(\pi_{f,p} \otimes \pi_{S,p})$ of GSp_4. We are interested in computing the L-function of this representation so that when we extract from it a Siegel modular form F with level, we can talk about its L-function. In the squarefree situation, the only Langlands L-factors that need to be determined are for the finite places dividing the level. That is, we only need to determine the local L-factors for the three local liftings $\Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes 1)$, $\Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)})$, and $\Pi(\xi\, \mathrm{St}_{\mathrm{GL}(2)} \otimes 1)$. These are given in Table 33 in [74] and taken from Table 2 in [72]:

$$\begin{aligned} L(s, \Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1})) &= (1 - p^{-s-1/2})^{-2}(1 - p^{-s+1/2})^{-1}, \\ L(s, \Pi(\xi\, \mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1})) &= (1 - p^{-s-1/2})^{-1}(1 - p^{-s+1/2})^{-1}(1 + p^{-s-1/2})^{-1}, \\ L(s, \Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)})) &= (1 - p^{-s-1/2})^{-2}. \end{aligned} \tag{7.1}$$

7.2.2 The Relationship to Classical Saito-Kurokawa Lifts

The relationship between the representation theoretic Saito-Kurokawa liftings and the classical Saito-Kurokawa liftings comes from two different (non-empty) choices of the set of places S. The conditions κ even and the -1 sign are the conditions that the choice of the set S has to meet in order for the global lifting to occur. This is equivalent to saying that the two different classical Saito-Kurokawa lifts corresponding to the two different choices of S are vectors in different automorphic representations of $\mathrm{PGSp}_4(\mathbb{A}_{\mathbb{Q}})$. Here we give a lemma given by Brown and Pitale that summarizes what we said.

Proposition 7.2.2 ([14], Lemma 2.4)**.** *Let* $f \in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(m))$.

1. *If m is odd, squarefree and κ is even, let F_f be the congruence level Saito-Kurokawa lift associated to f. If one chooses $S = \{\infty\} \cup \{p|m : \epsilon_p = -1\}$, then $\Pi(\pi_f \otimes \pi_S)$ is the cuspidal automorphic representation of* $\mathrm{GSp}_4(\mathbb{A})$ *corresponding to F_f.*

2. *If the sign of the functional equation of $L(s,f)$ is -1, let F_f be the paramodular level Saito-Kurokawa lift associated to f. If one chooses $S=\{\infty\}$, then $\Pi(\pi_f\otimes\pi_S)$ is the cuspidal automorphic representation of* $\mathrm{GSp}_4(\mathbb{A})$ *corresponding to F_f.*

7.3 Mixed Level Representation Theoretic Saito-Kurokawa Lifts

Let t, $M\in\mathbb{N}$ be such that $\gcd(M,t)=1$. Recall, the congruence paramodular level subgroup

$$\Gamma_M[t]=\mathrm{Sp}_4(\mathbb{Q})\bigcap\left\{\begin{pmatrix}\mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z}\\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t^{-1}\mathbb{Z}\\ M\mathbb{Z} & Mt\mathbb{Z} & \mathbb{Z} & \mathbb{Z}\\ Mt\mathbb{Z} & Mt\mathbb{Z} & t\mathbb{Z} & \mathbb{Z}\end{pmatrix}\right\}.$$

In the previous chapter, we have defined mixed level Siegel modular forms with respect to the congruence paramodular level $\Gamma_M[t]$ and we classically described a mixed level lifting from the space of $J^c_{\kappa,t}(\Gamma_0(M)^J)$ to the space of cusp forms $S_\kappa(\Gamma_M[t])$. In this section, we will show that we can choose a set S that will produce mixed level Saito-Kurokawa lifts from a representation theoretic point of view when tM is squarefree.

Corresponding to a cusp form $F\in S_\kappa(\Gamma_M[t])$, one can attach a cuspidal automorphic representation of PGSp_4 as is done in [7]. To do so, one has to define compact open subgroups of $\mathrm{GSp}_4(\mathbb{Q})$ as a local analogue of $\Gamma_M[t]$.

Let m, M, and t be square-free integers with $\gcd(M,t)=1$. Given a prime p, define

$$K_{0,p}(m)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Z}_p): c\equiv 0\pmod{m\mathbb{Z}_p}\right\},$$

$$K^{(2)}_{0,p}(M)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{GSp}_4(\mathbb{Z}_p): c\equiv 0\pmod{M\mathbb{Z}_p}\right\},$$

and

$$K_p^{(2)}[t] = \left\{ \gamma = \begin{pmatrix} a_1 & ta_2 & b_1 & b_3 \\ a_3 & a_4 & b_3 & t^{-1}b_4 \\ c_1 & tc_2 & d_1 & d_2 \\ tc_3 & tc_4 & td_3 & d_4 \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{Q}_p) : a_i, b_i, c_i, d_i \in \mathbb{Z}_p, \det(\gamma) \in \mathbb{Z}_p^\times \right\}.$$

Note that in the case that $p \nmid M$ we have $K_{0,p}^{(2)}(M) = \mathrm{GSp}_4(\mathbb{Z}_p)$ and if $p \nmid t$ we have $K_p^{(2)}[t] = \mathrm{GSp}_4(\mathbb{Z}_p)$. Define

$$K_M^{(2)}[t] = \prod_{p|M} K_{0,p}^{(2)}(M) \prod_{p|t} K_p^{(2)}[t] \prod_{p \nmid Mt} \mathrm{GSp}_4(\mathbb{Z}_p).$$

One should note if $M = p$, there is the possibility of confusion between the local group $K_p^{(2)}[t]$ and the global group denoted the same way, but it will always be clear from context what is meant.

One can check that

$$\Gamma_M[t] = \mathrm{GSp}_4(\mathbb{Q}) \cap \mathrm{GSp}_4^+(\mathbb{R}) K_M^{(2)}[t]$$

where $\mathrm{GSp}_4^+(\mathbb{R})$ is the group of elements in $\mathrm{GSp}_4(\mathbb{R})$ with positive multiplier.

We consider the following subspace of cusp forms.

Definition 7.3.1. Let $t, M \in \mathbb{N}$ be squarefree such that $\gcd(M, t) = 1$. We define the following subspace of elliptic cusp forms:

$$S_{2\kappa-2}^t(\Gamma_0(Mt)) = \{f \in S_{2\kappa-2}(\Gamma_0(Mt)) : f|W_t = (-1)^\kappa f\},$$

where W_t is the Atkin-Lehner involution at t. Note that in case $M = 1$, $S_{2\kappa-2}^t(\Gamma_0(Mt)) = S_{2\kappa-2}^-(\Gamma_0(t))$ and in case $t = 1$, $S_{2\kappa-2}^t(\Gamma_0(Mt)) = S_{2\kappa-2}(\Gamma_0(M))$.

Definition 7.3.2. We will call a set S of places of $\mathbb{Q}$ that satisfies the conditions given in

Schmidt's main lifting theorem an admissible set. We note that such a set always exists as long as π_f is square integrable and will always be nonempty for we always assume it contains the archimedean place.

Notation 7.3.3. We denote the global lifting associated to $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(m))$ as given in Schimdt's main lifting theorem by

$$\Pi_f := \Pi(\pi_f \otimes \pi_S) = \otimes_{p\leq\infty}\Pi_{f,p}.$$

We now give all possible admissible sets S.

Lemma 7.3.4. *Let $t, M \in \mathbb{N}$ be square-free such that* $\gcd(M,t) = 1$*. Fix f a newform in $S^{\text{new}}_{2\kappa-2}(\Gamma_0(tM))$. All the possible choices of admissible sets S are given below.*

1. *If $f \in S^{\text{new},-}_{2\kappa-2}(\Gamma_0(Mt))$, then $S = \{\infty\}$.*
2. *If κ is even , then $S = \{\infty\} \cup \{p \mid tM : \epsilon_p = -1\}$.*
3. *If $f \in S^{t,\text{new}}_{2\kappa-2}(\Gamma_0(Mt))$, $S = \{\infty\} \cup \{p \mid M : \epsilon_p = -1\}$.*

Proof. 1. When $S = \{\infty\}$, $(-1)^{|S|} = -1$. The sign of the functional equation satisfied by the L-function corresponding to f is $\epsilon(1/2, \pi_f)$. When $f \in S^{\text{new},-}_{2\kappa-2}(\Gamma_0(Mt))$, $\epsilon(1/2, \pi_f) = -1$. This proves the admissibility of S.

2. Let $S = \{\infty\} \cup \{p \mid tM; \epsilon_p = -1\}$. The sign of the functional equation is $\epsilon(1/2, \pi_f) = (-1)^{\kappa-1}\epsilon_{tM}$ where ϵ_{tM} is the Atkin-Lehner eigenvalue at tM. Because $\gcd(M,t) = 1$, we then have that $\prod_{p|tM}\epsilon_p = \left(\prod_{p|t}\epsilon_p\right)\left(\prod_{p|M}\epsilon_p\right)$. But $\epsilon_p = \pm 1$, therefore $\prod_{p|t}\epsilon_p = (-1)^{|\{p|t:\epsilon_p=-1\}|}$. Similarly, $\prod_{p|M}\epsilon_p = (-1)^{|\{p|M:\epsilon_p=-1\}|}$. According to our choice of the set S, we have $\prod_{p|t}\epsilon_p\prod_{p|M}\epsilon_p = (-1)^{|S|-1}$. Consequently, $\epsilon(1/2, \pi_f) = (-1)^{\kappa-1}(-1)^{|S|-1}$ which is clearly equal to $(-1)^{|S|}$ when assuming κ is even.

3. If $f \in S^{t,\text{new}}_{2\kappa-2}(\Gamma_0(Mt))$, arguing as in the second case, we have that the sign of the functional equation is $\epsilon(1/2, \pi_f) = (-1)^{\kappa-1}\epsilon_{tM}$ where $\epsilon_{tM} = \left(\prod_{p|t}\epsilon_p\right)\left(\prod_{p|M}\epsilon_p\right) =$

$\epsilon_t \left(\prod_{p|M} \epsilon_p\right) = \epsilon_t \left((-1)^{|\{p|M:\epsilon_p=-1\}|}\right)$. But $|\{p|M : \epsilon_p = -1\}| = (-1)^{|S|-1}$ and $\epsilon_t = (-1)^\kappa$. Hence, $\epsilon(1/2, \pi_f) = ((-1)^{\kappa-1})((-1)^\kappa(-1)^{|S|-1}) = (-1)^{2\kappa-1}(-1)^{|S|-1} = (-1)^{-2}(-1)^{|S|} = (-1)^{|S|}$.

Conversely, let S be an admissible set containing the archimedean place. Then the condition $(-1)^{|S|} = \epsilon(1/2, \pi_f)$ could be satisfied when $\epsilon(1/2, \pi_f) = -1$. This gives that $|S| = 1$ and hence $S = \{\infty\}$. More generally, $\epsilon(1/2, \pi_f) = (-1)^{\kappa-1}\epsilon_{tM} = (-1)^{|S|}$ can be satisfied either when $\epsilon_{tM} = (-1)^{|S|-1}$ and κ being even which means $S = \{\infty\} \cup \{p|tM : \epsilon_p = -1\}$, or when $\epsilon_{tM} = \epsilon_t\epsilon_M$ and S contains some of the finite places. Since tM is such that $\gcd(t, M) = 1$, then $S = \{\infty\} \cup \{p|M : \epsilon_p = -1\}$ or $S = \{\infty\} \cup \{p|t : \epsilon_p = -1\}$. Suppose, $S = \{\infty\} \cup \{p|M : \epsilon_p = -1\}$, then $(-1)^{\kappa-1}\epsilon_{tM} = (-1)^{\kappa-1}\epsilon_t\epsilon_M = (-1)^{\kappa-1}\epsilon_t(-1)^{|S|-1} = (-1)^{|S|}$ implies that $\epsilon_t = (-1)^\kappa$. This explains the third choice of the set S. Reversing the roles of t and M explains the possibility of the choice of the set $S = \{\infty\} \cup \{p|t : \epsilon_p = -1\}$.

□

Theorem 7.3.5. *The mixed Saito-Kurokawa lifts from a representation theoretic point of view occur by considering the set* $S = \{\infty\} \cup \{p \mid M; \epsilon_p = -1\}$.

In other words, given $f \in S^{t,\mathrm{new}}_{2\kappa-2}(\Gamma_0(Mt))$, *there exists a unique (up to scalars) Siegel modular form* $F_f \in S_\kappa(\Gamma_M[t])$ *extracted from* $\Pi_f := \Pi(\pi_f \otimes \pi_S)$ *that we call the mixed Saito-Kurokawa lift associated to* f.

Proof. We will show that for the other choices of admissible sets listed in Lemma 7.3.4, the Saito-Kurokawa lifts are the classical ones.

If $f \in S^{\mathrm{new},-}_{2\kappa-2}(\Gamma_0(Mt))$, Lemma 7.3.4 with $S = \{\infty\}$ says that there exists a cuspidal lifting Π_f to $\mathrm{GSp}_4(\mathbb{A})$ as described by Schmidt's main lifting theorem. Its local components at the bad primes ($p \mid tM$) are given by $\Pi(\mathrm{St}_{\mathrm{GL}_2} \otimes \mathbf{1})$ or $\Pi(\xi\, \mathrm{St}_{\mathrm{GL}_2} \otimes 1)$ since for all $p \mid tM$, $p \notin S$. Following the proof of Theorem 5.2 in [74], we can extract a unique up to scalars $F_f \in S_\kappa(\Gamma[tM])$.

If κ is even and $f \in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(Mt))$, Lemma 7.3.4 with $S = \{\infty\} \cup \{p \mid tM : \epsilon_p = -1\}$ says that there exists a cuspidal lifting Π_f to $\mathrm{GSp}_4(\mathbb{A})$ as described by Schmidt's main lifting theorem. For $p \mid tM$ ($p \mid t$ or $p \mid M$), if $\epsilon_p = 1$ then $p \notin S$ and hence the corresponding local component is $\Pi(\xi \, \mathrm{St}_{\mathrm{GL}_2} \otimes 1)$. And if $\epsilon_p = -1$, whether $p \mid t$ or $p \mid M$, then $p \in S$. The corresponding local component is then $\Pi(\mathrm{St}_{\mathrm{GL}_2} \otimes \mathrm{St}_{\mathrm{GL}_2})$. Following the proof of Theorem 5.2 in [74], we can then extract a unique up to scalars $F_f \in S_\kappa(\Gamma_0^{(2)}(tM))$.

For the last choice of the set S, the proof will be included in the proof of Theorem 7.3.7. □

Remark 7.3.6. Note that the definition of the space $S^t_{2\kappa-2}(\Gamma_0(Mt))$ is symmetric in t and M. We could have defined instead the space $S^M_{2\kappa-2}(\Gamma_0(Mt))$ and accordingly chose the set $S = \{\infty\} \cup \{p \mid t : \epsilon_p = -1\}$ in place of the third choice above. The corresponding mixed Saito Kurokawa lift F_f then belongs to $S_\kappa(\Gamma_t[M])$.

We now give the main theorem describing the mixed Saito-Kurokawa lift. One can see Theorem 5.3 of [74] for the result in the case M or t is 1.

Theorem 7.3.7. *Let M and t be square-free integers,* $\gcd(M,t) = 1$, $\kappa \geq 2$ *an even integer, and $f \in S^{t,\mathrm{new}}_{2\kappa-2}(\Gamma_0(Mt))$ a newform. Let ϵ_p be the eigenvalue of f under the Atkin-Lehner involution at p and let η_p be the Atkin-Lehner involution of degree 2 at p. There exists an eigenform $F_f \in S_\kappa(\Gamma_M[t])$, unique up to constant multiples, whose Spinor L-function is given by*

$$L(s, F_f, \mathrm{spin}) = \left(\prod_{\substack{p \mid M \\ \epsilon_p = -1}} (1 - p^{-s+\kappa-1}) \right) \zeta(s-\kappa+1)\zeta(s-\kappa+2)L(s,f).$$

Moreover, for each $p \mid t$ we have $\eta_p F_f = \epsilon_p F_f$ and for each $p \mid M$ we have $\eta_p F_f = F_f$.

Proof. Let π_f be the cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ corresponding to f. We have seen from Lemma 7.3.4 and from Theorem 7.3.5 that the set $S = \{\infty\} \cup \{p \mid M : \epsilon_p = -1\}$ is an admissible set and that we obtain a cuspidal lifting

$$\Pi_f = \otimes \Pi_{f,p} = \otimes \Pi(\pi_{f,p} \otimes \pi_{S,p})$$

whose local components $\Pi_{f,p}$ are given by

1. for $p \mid t$,
$$\Pi_{f,p} = \begin{cases} \Pi(\xi\,\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}) & \text{if } \epsilon_p = 1 \\ \Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}) & \text{if } \epsilon_p = -1; \end{cases}$$

2. for $p \mid M$,
$$\Pi_{f,p} = \begin{cases} \Pi(\xi\,\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathbf{1}) & \text{if } \epsilon_p = 1 \\ \Pi(\mathrm{St}_{\mathrm{GL}(2)} \otimes \mathrm{St}_{\mathrm{GL}(2)}) & \text{if } \epsilon_p = -1; \end{cases}$$

3. for $p \nmid tM$, $\Pi_{f,p}$ is the unramified representation of $\mathrm{PGSp}_4(\mathbb{R})$ that contain a vector Φ_p fixed by the open compact subgroup $\mathrm{GSp}_4(\mathbb{Z}_p)$ of $\mathrm{GSp}_4(\mathbb{Q})$. The vector Φ_p is called a spherical vector;

4. for $p = \infty$, $\Pi_{f,\infty} = \Pi(\pi_{f,\infty} \otimes \mathrm{St}_\infty) = \Pi(D(2\kappa-2) \otimes D(1))$, where $D(2\kappa-2)$ is the holomorphic discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ of weight (κ,κ). It contains a distinguished lowest weight vector Φ_∞.

Now for the other components at the bad primes, table (30) in [74] says that in the components $\Pi_{f,p}$ for $p \mid t$, the space of vectors invariant under $K_p^{(2)}[t]$ is one dimensional and hence each of the components in part 1) contains a unique paramodular invariant vector Φ_p. Similarly, in the components $\Pi_{f,p}$ for $p \mid M$, from the same table (30), the space of vectors invariant under $K_{0,p}^{(2)}(M)$ is one dimensional and each of the components in part 2) contains unique vector Φ_p invariant under the local congruence subgroup $K_{0,p}^{(2)}(M)$. Piecing all these vectors together, $\Phi = \otimes\Phi_p$ and following the procedure on page 21 in [74], we see that Φ corresponds to a classical holomorphic Siegel cuspform of weight κ with respect to $\Gamma_M[t] = \mathrm{GSp}_4(\mathbb{Q}) \cap \mathrm{GSp}_4^+(\mathbb{R})K_M^{(2)}[t]$. This shows how to extract a Siegel modular form $F \in S_\kappa(\Gamma_M[t])$ from the representation Π_f corresponding to $f \in S_{2\kappa-2}^{t,\mathrm{new}}(\Gamma_0(tM))$.

As for the uniqueness proof (up to scalar factors), it goes exactly as in the proof of Theorem 5.2 in [74]. We include it for the convenience of the reader. Assume F' is another cusp form with the L-function as in the statement of the theorem. Let Φ' be the corresponding adelic function, generating a multiple of an automorphic representation Π'. From the form of Euler factors at good primes, we see that the local components of Π_f and Π' coincide almost everywhere. Using a theorem of Piatetski-Shapiro ([59], Theorem 2.2), the representation Π' is also a lift of the form $\Pi(\pi' \otimes \pi'_S)$ for some automorphic representation π' of $\mathrm{GL}_2(\mathbb{A})$ and some set of places S'. The local components Π'_p being Iwahori-spherical must therefore be amongst the ones occuring in table (30) of [74]. Looking at the Euler factors ([74], table 33), it shows that $\Pi_{f,\infty} = \Pi'_\infty$ and that $\Pi_{f,p} = \Pi'_p$ for $p \mid tM$. Therefore the global representations are isomorphic. Using the multiplicity one result (Theorem 6.2 in [59]) for lifts from $\mathrm{SL}_2(\mathbb{A})$, the representations Π_f and Π' coincide as spaces of automorphic forms. Hence Φ_f and Φ' are elements of the same irreducible space of automorphic forms. The uniqueness now follows from the local uniqueness of vectors expressed by the one-dimensionality of the spaces of fixed vectors in table (30).

The result on the L-functions now follow from the local L-functions defined above in 7.1 and from $L(s, \pi_S)$. It is exactly as given in Theorem 5.3 in [74] upon switching from the automorphic normalizations of the L-functions to the modular normalizations used here as described in Theorem 7.1.8. The result on the Atkin-Lehner eigenvalues can be read off from the information in 7.2.1.2. $\square$

Remark 7.3.8. If $f \in S^{t,-,\mathrm{new}}_{2\kappa-2}(\Gamma_0(Mt))$ and κ is even, then we can associate to f three Saito-Kurokawa liftings based on the prime factorization of Mt. We list the possible three Saito-Kurokawa lifts attached to f.

1. If $S = \{\infty\}$, then we have a Saito-Kurokawa lift $F_1 \in S_\kappa(\Gamma[tM])$.
2. If $S = \{\infty\} \cup \{p \mid tM : \epsilon_p = -1\}$, we get a Saito-Kurokawa lift $F_2 \in S_\kappa(\Gamma_0^{(2)}(Mt))$.
3. If $S = \{\infty\} \cup \{p \mid M : \epsilon_p = -1\}$, we get a Saito-Kurokawa lift $F_3 \in S_\kappa(\Gamma_M[t])$.

Appendix A

On Compactifications of Locally Symmetric Spaces $\Gamma \setminus X$

In Chapter 4, we gave a concrete description of the cusps attached to the Siegel variety $\Gamma \setminus \mathfrak{h}^n$ for any arithmetic group Γ in order to form the Satake compactification of this variety. These cusps are the rational boundary components whose stabilizers are the standard rational maximal parabolic subgroups $C_{n,r}(\mathbb{Q})$ and their conjugates. In this description, we have seen how the boundary varieties representing the cusps are dictated by these maximal rational parabolic subgroups. Our first goal in this appendix is to briefly outline the general ideas of Satake compactifications of locally symmetric spaces $\Gamma \setminus X$. According to Satake ([13]), the compactification of the locally symmetric space $\Gamma \setminus X$ depends on the compactification of the symmetric space X. Borel and Ji ([13]) found a uniform approach to the many different types of compactifications present in the literature. Their approach is based on the fact that parabolic subgroups play a crucial role in understanding the geometry at infinity of symmetric and locally symmetric spaces. Instead of working with each compactification separately by building geometrically the boundary components suitable for the purpose of the compactification, they used the correspondence between those components and the type of parabolic subgroups associated to them. They have been able to reconstruct all compactifications using group-theoretic methods by making a choice on the type of parabolic subgroups of Γ to be used and then associating boundary components to the symmetric space X using the Langlands decomposition of the parabolic subgroups. This Langlands decomposition induces a decomposition of the space X and shows how to attach boundary spaces to X. Our second goal in this appendix is to outline Borel and Ji's approach just to give the reader a global picture of why we worked with maximal parabolic subgroups when dealing with the Satake compactification of $\Gamma \setminus \mathfrak{h}^n$ in Chapter 4. We can't provide the details because unfortunately Borel and Ji's approach relies a lot on big theories of mathematics like the theory of Lie groups, Lie algebras, algebraic groups and Borel reduction theory of arithmetic groups. In the last part, we come back to Satake's approach and give a concrete description of the boundary of the Satake compactification of $\Gamma \setminus X$ that was described in general terms in the first part and that gives a generalization of the description given in Chapter 4 in case of $\Gamma \setminus \mathfrak{h}^n$.

1 Satake Compactifications of $\Gamma \setminus X$

We mentioned in Chapter 4 that Satake started the modern theory of compactifications of symmetric spaces X and locally symmetric spaces $\Gamma\setminus X$ which gives in particular the topological construction of $(\mathrm{Sp}_{2n}(\mathbb{Z})\setminus\mathfrak{h}^n)^c$. Unlike the case of the Siegel variety $\mathrm{Sp}_{2n}(\mathbb{Z})\setminus\mathfrak{h}^n$, there are no obvious choices of lower dimensional locally symmetric spaces which can be attached at infinity to obtain a topological compactification of $\Gamma \setminus X$. To obtain the desired boundary spaces and the required topology to attach these spaces to X in order to form the compactification of $\Gamma \setminus X$, Satake in [68] proceeded in two steps. He first constructed compactifications X^c of X. In fact, he provided finitely many non-isomorphic compactifications of X which form a partially ordered set. The minimal Satake compactification is the one used in the Satake compactification of the Siegel upper half space $\mathfrak{h}^n$ which is usually known as the Satake compactification in the literature. The real parabolic subgroups needed for a Satake compactification of X are then used to decompose the boundary of the compactification X^c into lower dimensional symmetric spaces, called boundary components. We will provide an explanation of this process below.

Second, to get the compactification of the locally symmetric space $\Gamma \backslash X$, only certain rational boundary components (similar to what we we described in Chapter 4 as rational boundary components) are needed to form the rational compactification X^* of X. To obtain these rational boundary components, Satake needed a suitably chosen fundamental set Σ of Γ. This fundamental set is constructed using reduction theory of arithmetic groups as laid out by Borel [12] which gives Siegel sets associated to rational parabolic subgroups. Taking the closure $\overline{\Sigma}$ of Σ in X^c and using facts from reduction theory, $\overline{\Sigma}$ intersects X^c with finitely many boundary components X_I. These boundary components will be the standard rational ones and all other rational boundary components are Γ conjugate to them. Γ then acts properly discontinuously on the partial compactification X^* with a compact Hausdorff quotient $\Gamma \setminus X^*$ which defines a Satake compactification of $\Gamma \setminus X$.

Implicitly in his constructions, Satake showed that his compactifications of $\Gamma \setminus X$ depend on the his compactifications X^c of X and provided a general method of passing from a compactification of the symmetric space X to a compactification of $\Gamma \setminus X$.

Remark 1.1. 1. As mentioned above, Satake created a family of compactifications of $\Gamma \backslash X$ based on a family of compactifications of the symmetric space X equipped with a partial order. What we refer to in this thesis as the Satake compactification and which is used to compactify the Siegel variety $\Gamma \backslash \mathfrak{h}^n$ in [18] corresponds to the minimal Satake compactification of the locally symmetric space $\Gamma \setminus X$ as decribed in [68].

2. The compactification of a fundamental domain of Γ in X is different from compactification of $\Gamma \setminus X$, and there are difficulties in passing from one to other (see [13]).

2 The Approach of Borel and Ji

The discussion above shows that the construction of Satake compactifications of locally symmetric spaces $\Gamma \setminus X$ depends on compactifications of the symmetric space X and on forming a partial compactification of X with a suitable choice of a fundamental set from compactifications of X. It turns out that to every Γ-rational boundary component F in the boundary of the compactified locally symmetric space corresponds a rational parabolic subgroup $P(F)$. Moreover, the map $F \mapsto P(F)$ induces a bjiection of the set of rational boundary components onto the set of proper rational parabolic subgroups used. From this point of view, Borel and Ji in [13] reframed the compactification of symmetric spaces $X = G/K$ and of their corresponding locally symmetric spaces using the theory of parabolic subgroups of G by attaching boundary components of real parabolic subgroups in case of compactification of symmetric spaces and rational boundary components attached to rational parabolic subgroups in case of compactification of locally symmetric spaces. This uniform approach is based on the reduction theory of arithmetic groups [12] which aims at constructing suitable fundamental domains of arithmetic groups Γ based on certain sets called Siegel sets or fundamental sets associated to parabolic subgroups. The benefit of this uniform approach is that the compactification of $\Gamma \setminus X$ is done intrinsically and avoids obtaining the rationality of the compactification of X from its compactification X^c which was a key step to to pass from a compactification of the symmetric space of X to a compactification of $\Gamma\backslash X$ as done by Satake. From now on, we closely follow the presentation of [13] in providing the general setting of how to associate to parabolic subgroups boundary components. Every single fact or result listed below can be found in this reference. We leave most of the details and all the proofs for the reader to consult in this reference.

2.1 Symmetric Spaces

Suppose $X = G/K$ is a symmetric space. The uniform approach given by Borel and Ji for the construction of the compactifications of a symmetric space X depends on the structure of real parabolic subgroups of G.

Given a connected reductive linear group G, we have seen in Chapter 4 that associated to every subset I of the set of simple roots δ we associate a conjuguacy class of parabolic subgroups whose representative P_I is called a standard parabolic subgroup. Each standard (and hence every) parabolic subgroup enjoys a Levi decomposition

$$P_I = M_I \ltimes N_I$$

with M_I its Levi factor and N_I its unipotent radical. We refer the reader to Chapter 4 for the details. The Levi factor $M_I = Z(T_I)$ is the centralizer of T_I, the identity component of $(\cap_{\alpha \in I} \ker \alpha)$. T_I is called the split component in G and M_I is the maximal reductive subgroup of P_I. Now M_I can be written as the following product

$$M_I = A_I \times T_I$$

for some semisimple group A_I. As an example of this, we have expressed the parabolic subgroups $C_{n,r}$ in Chapter 4 § 4.2.3. as

$$C_{n,r} = (G_r \times G_{n-r}) \ltimes N_r.$$

As a consequence of this finer Levi decomposition of the parabolic subgroup P_I, we can write

$$P_I = T_I A_I N_I$$

and obtain the well known Langlands decomposition of P_I:

$$P = T_I A_I N_I \cong T_I \times A_I \times N_I \tag{A.1}$$

because the map

$$(t, a, n) \mapsto tan \quad (t \in T_I, a \in A_I, n \in N_I)$$

is an analytic isomorphism of analytic manifolds.

Let $K_I = A_I \cap K$ be a maximal compact subgroup in A_I. It is also maximal compact in P_I. The quotient

$$X_I = A_I / K_I = P_I / K_I T_I N_I$$

is a symmetric space of noncompact type for A_I called the boundary symmetric space associated to P_I attached at the infinity of X. The reason for naming it this way is that it often appears in the boundary of compactification of X.

The Iwasawa decomposition $G = NAK$ which decomposes a semisimple group G as the product of nilpotent, abelian, and compact subgroups respectively such that $NA \subset P$ implies that

$$G = PK.$$

Consequently, P acts transitively on $X = G/K$ and the Langlands decomposition of P as in equation (A.1) induces a decomposition of X associated to P, called the horospherical decomposition of X. It is given by

$$X \cong N_I \times T_I \times X_P.$$

It can be shown that $K \cap P_I = K \cap A_I$ and the Langlands decomposition of P_I also induces the following horospherical decomposition of G:

$$G = N_I T_I A_I K \cong N_I \times A_I \times A_I K.$$

Consider the roots $\Phi(P_I, T_I)$. For any $t > 0$, define a cone in T_I by

$$T_{I,a} = \{t \in T_I \mid t^\alpha > a, \alpha \in \Phi(P_I, T_I)\}.$$

Let $U \subset N_I$, $V \subset X_I$ be bounded subsets. Then the subset

$$S_{P_I,U,V,a} = U \times T_{I,a} \times V \subset X$$

is called the Siegel set associated with P_I.

Example 2.1. Let Γ be $\mathrm{SL}_2(\mathbb{Z})$. A good fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ is found using reduction theory (discussed below) and is well-known and simple.

In Chapter 2, we have seen that a fundamental domain F for $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}^1$ is given by the following region

$$F = \left\{ z = x + iy \in \mathfrak{h}^1 \middle| |z| > 1, -\frac{1}{2} < x < \frac{1}{2} \right\}.$$

We have seen that we can view $\mathfrak{h}^1$ as a subset of $P^1(\mathbb{C})$ whose boundary is $P^1(\mathbb{R})$ which contains $P^1(\mathbb{Q})$. Taking the closure of F in $P^1(\mathbb{C})$, it meets $P^1(\mathbb{R})$ at exactly one point $i\infty$.

Corollary 2.2 ([36], Corollary 2.8)**.** *The quotient* $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{h}^1$ *has finite area.*

Proof. The fundamental domain F is contained in the subset

$$S = \left\{ z = x + iy \in \mathfrak{h}^1 \mid y > \frac{\sqrt{3}}{2}, -\frac{1}{2} < x < \frac{1}{2} \right\}.$$

Then

$$\mathrm{Area}(\Gamma \setminus \mathfrak{h}^1) = \int_F \frac{dxdxy}{y^2} \leq \int_S \frac{dxdxy}{y^2} = \int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{dy}{y^2} < \infty.$$

□

Remark 2.3. The set S is called a Siegel set for $\mathrm{SL}_2(\mathbb{Z})$ associated with the cusp $i\infty$. It plays an important role in the general reduction theory.

In general the region

$$S_t = \left\{ x + iy \mid -\frac{1}{2} \leq x \leq \frac{1}{2}, y > t \right\}$$

can be expressed as

$$S_t = U \times T_{P,t}$$

where $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| -\frac{1}{2} \leq b \leq \frac{1}{2} \right\}$ and $T_{P,t} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a > t^{1/2} \right\}$.

A subset of $\mathfrak{h}^1$ of the form S_t is called a Siegel set associated with the parabolic subgroup P given by

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

The parabolic subgroup P is defined over $\mathbb{Q}$. Its real locus is the stabilizer of $i\infty$ in $\mathrm{SL}_2(\mathbb{R})$. The components of its Langlands decomposition are:

1. Its unipotent radical is
$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{R} \right\},$$

2. its split component is
$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{R}^* \right\},$$

3. and
$$A = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

2.2 Uniform approach

In the description of the Satake compactification of a symmetric space X, the stabilizers of boundary components are given by real parabolic subgroups. This shows that parabolic subgroups are naturally associated with the geometry at infinity of symmetric spaces. To develop a uniform intrinsic approach to constructing a compactification of a noncompact bounded symmetric spaces X without embedding X into some compact space, boundary points should be given in terms of points of the symmetric space X and the topology of the compactification should be given in terms of convergent sequences of interior points to the boundary points. All this can be done using Siegel sets associated to real parabolic subgroups. Using Siegel sets, the G-action extends continuously to the boundary of the compactification and a certain property of these Siegel sets called the strong separation property helps in proving the Hausdorff property of the compactification. This strong separation property is related to Minkowski reduction theory. Another benefit of this approach is that an explicit description of the neighborhoods of the boundary points in the compactification of X can be also described using the Siegel sets. Here are the three basic steps of the intrinsinc method.

1. A suitable collection of parabolic subgroups of G which is invariant under conjugation by elements of G.

2. For every parabolic subgroup P in the collection, define a boundary component $e(P)$ by making use of the Langlands decomposition of P or the induced horospherical decomposition of X.

3. For every parabolic subgroup in the collection, attach the boundary component $e(P)$ to X via the horospherical decomposition of X to obatin $X \cup \coprod_P e(P)$ and show that the induced topology is compact and Hausdorff and the $G-$action on X extends continuously to the compactification.

Different compactifications can be constructed using this intrinsic approach by varying the choices of the collection of parabolic subgroups. For the Satake compactification the collection of parabolic subgroups that is used is the collection of maximal parabolic subgroups.

We now, as an example, outline how we form the boundary components $e(P)$ associated to maximal parabolic subgroups in order to form the Satake (minimal one) compactification X^c of X. Let Q be a maximal standard parabolic subgroup, and let δ be the set of simple roots. The associated boundary symmteric space X_Q has the following canonical splitting:

$$X_Q = F_I \times F_{I'},$$

where $I \subset \delta - \{\alpha\}$, $\alpha \in \delta$, $I' = \delta - \{\alpha\} \setminus I$. Define the boundary component of $e(Q)$ of Q by

$$e(Q) = F_I.$$

Now when we consider the Satake compactification or the Baily-Borel compactification, we assume that X is also a Hermitian space. We are interested that the compactification X^c to also be Hermitian, and hence it was essential to split the boundary symmetric space X_Q for it is not Hermitian. Instead, the boundary component $e(Q)$ is a Hermitian space of lower rank.

For any parabolic subgroup Q, the boundary symmetric space X_Q is a homogeneous space of Q. Now when Q is a maximal parabolic subgroup, the boundary component $e(Q) = F_I$ is also a homogenous space of Q. Here is how the action of Q on F_I is defined. Let

$$\pi : X_Q = F_I \times F_{I'} \to F_I$$

be the projection onto the first factor. For any $q \in Q$ and $z = aK_Q \in F_I$, define the action

$$q.z = \pi(q.aK_Q),$$

where aK_Q is a point in X_Q and $q.aK_Q$ is given by the action of Q on X_Q.

Once the boundary components are defined, we are ready to describe the compactification X^c in the following manner. Let $\mathcal{Q}$ be the collection of maximal real parabolic subgroups,

$$X^c = X \cup \coprod_{Q \in \mathcal{Q}} e(Q) = X \cup \coprod_{Q \in \mathcal{Q}} F_I.$$

To describe the convergence of interior points of X to the boundary points in X^c, the horospherical decomposition of X is used. The neighborhoods of boundary points in boundary components are explicitly described in [13]. Moreover, using properties of the Siegel sets, it is shown that the topology on X^c is Hausdorff.

2.3 Locally Symmetric Spaces

The advantage of this intrinsic description of the compactification of symmetric spaces of noncompact type is that similar steps are taken to construct compactification of locally symmetric spaces $\Gamma \backslash X$ but in the latter only boundary components associated with rational parabolic subgroups will be used.

Compactifications of locally symmetric spaces depend on a suitably chosen fundamental set for the action of an arithmetic group Γ of a semisimple linear algebraic group G on a symmetric space X. The problem of finding a fundamental set is called reduction theory of G. The goal of reduction theory is construct fundamental sets of arithmetic groups in terms of the Siegel sets of rational parabolic subgroups.

2.4 Borel Reduction Theory of Arithmetic Subgroups

Let G be a connected linear algebraic group defined over $\mathbb{Q}$. Let $G(\mathbb{Q}) \subset \mathrm{GL}_n(\mathbb{Q})$ be the set of its rational points, and $G(\mathbb{Z}) \subset \mathrm{GL}_n(\mathbb{Z})$ the set of its elements with integral entries.

Assume from now on that G is a connected semisimple linear algebraic group defined over $\mathbb{Q}$ and Γ an arithmetic subgroup of G.

Definition 2.4. A subset S of $G(\mathbb{R})$ is called a fundamental set for Γ if

1. $S = SK$, where K is some maximal compact subgroup of $G(\mathbb{R})$.

2. $S.\Gamma = G(\mathbb{R})$

3. For any $g \in G(\mathbb{R})$, the set $\{\gamma \in \Gamma \mid gS \cap \gamma S \neq \emptyset\}$ is finite.

   To define the fundamental sets of a fixed arithmetic group Γ, condition (2) can be replaced by a weaker condition:

4. The set $\{\gamma \in \Gamma \mid S \cap \gamma S \neq \emptyset\}$ is finite.

Condition (2) is called the Siegel finiteness property which plays an important role in defining the topology of compactifications of $\Gamma \backslash X$ and showing that it is Hausdorff. Due to condition (1), the projection $S' = \pi(S)$ of a fundamental set in $G(\mathbb{R})$ into $X = G(\mathbb{R})/K$ satisfies the conditions

$$\Gamma.S' = X$$

and condition (3). A subset of X satisfying these conditions is called a fundamental set for Γ in X. Thus S is a fundamental set for Γ in $G(\mathbb{R})$ if and only if $S' = \pi(S)$ is one in X.

We now state a classical reduction theory which gives fundamental sets for Γ in terms of Siegel sets of rational parabolic subgroups.

Proposition 2.5 ([36], Proposition 6.14)**.** *Let G be a reductive linear algebraic group defined over $\mathbb{Q}$, and $\Gamma \subset G(Q)$ an arithmetic subgroup. Let $S_{u,t} = N_{P,u} \times A_{P,t}$ be a fundamental set for $\mathrm{GL}_n(\mathbb{Z})$. Then there exists finitely many elements $b_1, \dots, b_r \in \mathrm{GL}_n(\mathbb{Q})$ and an element $a \in G$ such that*

$$\Gamma((\cup_{i=1}^r b_i^{-1} S_{u,t} a) \cap X) = X.$$

Proposition 2.6 ([36], Proposition 6.15)**.** *Let G be a reductive linear algebraic group defined over $\mathbb{Q}$ and Γ an arithmetic subgroup. If P is a minimal rational parabolic subgroup, then $\Gamma \backslash G(\mathbb{Q})/P(\mathbb{Q})$ is finite, i.e., there are only finitely many Γ-conjugacy classes of rational parabolic subgroups. Let C be a finite set of representatives of this double coset space. Then there exists a Siegel set $S_{P,t}$ such that*

$$(\cup_{i=1}^r b_i^{-1} S_{u,t} a) \cap X \subset C S_{P,t}.$$

Hence $CS_{P,t}$ is a fundamental set for Γ in X.

The finite set C can be identified with the $\Gamma-$conjugacy classes of minimal rational parabolic subgroups. So Proposition 2.6 can be reformulated as follows. Let $P_1, \dots P_m$ be a set of representatives of Γ-conjugacy classes of minimal rational parabolic subgroups of G. Then for each P_i there exists a Siegel set S_{P_i,t_i} such that

$$X = \Gamma(\cup_{i=1}^m S_{P_i,t_i}).$$

This is the classical reduction theory of an arithmetic group Γ. It only gives a fundamental set for Γ.

Remark 2.7 ([36], page 46)**.** Let Γ be an arithemtic group. The finiteness of Γ-conjuguacy classes of minimal rational parabolic subgroups P is equivalent to the finiteness of Γ-conjuguacy classes of rational parabolic subgroups. That is, if R is any rational parabolic subgroup, then the double coset space $\Gamma\backslash G(\mathbb{Q})/R(\mathbb{Q})$ is finite.

Example 2.8. When $G = \mathrm{SL}_2$, we have seen in Chapter 4 that every proper rational parabolic subgroup is minimal and conjugate to the standard one given by P. We have also seen that there is a one-to-one correspondence between Γ-conjugacy classes of rational parabolic subgroups and the cusps of $\Gamma\backslash\mathfrak{h}^1$. That is the set of cusps for the Riemann surface $\Gamma\backslash\mathfrak{h}^1$ is given by $\Gamma\backslash\mathrm{SL}_2(\mathbb{Q})/P(\mathbb{Q})$, where $P(\mathbb{Q})$ is the rational locus of the parabolic subgroup P associated to the cusp $i\infty$.

There is a refined version of reduction theory which allows one to get a fundamental domain of Γ or an exact fundamental set without overlap under Γ-translates. Such a theory is called the precise reduction theory. To state this precise version, one needs to define Siegel sets slightly differently. We will denote them by $S_{P,T}$.

Proposition 2.9 ([36], Proposition 6.17)**.** *Let $P_1, \ldots, P_k$ be a set of representatives of $\Gamma-$conjugacy classes of all proper (not only minimal) rational parabolic subgroups. There exists a bounded set Ω_0 in X. For each P_i define a corresponding Siegel set S_{P_i,T_i} such that each is mapped injectively into $\Gamma \setminus X$ under the projection $\pi : X \to \Gamma \setminus X$. The space X admits the decomposition*

$$X = \Omega_0 \cup \coprod_{i=1}^{k} S_{P_i,T_i},$$

and

$$\Gamma \setminus X = \pi(\Omega_0) \cup \coprod_{i=1}^{k} \pi(S_{P_i,T_i}).$$

2.5 Uniform Construction of Compactification of $\Gamma \setminus X$

While one can think that a compactification of $\Gamma \setminus X$ can be achieved using Proposition 2.9, it turns out that the reduction theory plays a crucial role in showing Step (4) below. We now list the steps used in the general approach of compactifying locally symmetric spaces.

1. Choose a Γ-invariant collection $\mathbf{P}$ of rational parabolic subgroups.

2. For every rational parabolic subgroup P of G in $\mathbf{P}$, define a boundary component $e(P)$ using the Langlands decomposition of P.

3. Form a partial compactification of X by attaching all the rational boundary components in $\mathbf{P}$.
$$X^* = X \cup \coprod_{P \in \mathbf{P}} e(P),$$
using the rational horospherical coordinate decomposition of X with respect to P.

4. Show that X^* is a Hausdorff space and Γ acts continuously with a compact Hausdorff quotient. Then $\Gamma \backslash X^*$ is a compactification of $\Gamma \backslash X$.

Remark 2.10. 1. Different choices of the collection of rational parabolic subgroups and their boundary components $e(P)$ lead to different compactifications.

2. For the Satake-Baily-Borel compactification, $\mathbf{P}$ is the collection of maximal rational parabolic subgroups.

For the Satake-Baily-Borel compactification, X is a Hermitian space and the goal is to make the compactification of $\Gamma \setminus X$ a complex space. Therefore, each boundary component needs to be a complex space. The boundary symmetric space X_P associated to a rational parabolic subgroup is in general not Hermitian. However, it admits a certain decomposition

$$X_P = X_{P,h} \times X_{P,l},$$

where $X_{P,h}$ is Hermitian and $X_{P,l}$ is a linear symmetric space. Accordingly, we take the boundary component to be

$$e(P) = X_{P,h},$$

$$X^* = X \cup \coprod_{P \in \mathbf{P}} e(P)$$

with a suitable topology, where $\mathbf{P}$ is the collection of maximal rational parabolic subgroups.

The benefit of this uniform approach is that it provides an intrinsic description of the compactification of $\Gamma \setminus X$ rather than obtaining it from a compactification of X as done by Satake.

3 Concrete Boundary Components of the Satake Compactification of $\Gamma \setminus X$

We briefly review the procedure of constructing the Satake (minimal one) compactification of $\Gamma \setminus X$ from the Satake compactification of X as given in [68]. The reader should compare with § 4.3.3. The main idea as discussed in § 1 is to get a partial compactification of X from a compactification X^c of X by adding only rational boundary components of X^c. For each maximal real parabolic subgroup Q, the associated symmetric boundary space X_Q splits as $X_{Q,I} \times X_{Q,I'}$. Define the boundary component $e(Q)$ of Q to be $e(Q) = X_I$. Then

$$X^c = X \cup \coprod_{Q_i} e(Q_i) = X \cup \coprod_{Q_i} X_{Q_i,I}.$$

Let Σ be a fundamental set constructed from Siegel sets of rational parabolic subgroups as discussed in § 2.4. We then take the closure $\overline{\Sigma}$ of Σ in X^c that meets X^c with only finitely many boundary components X_I of X^c. The action of the subgroup $\Gamma \cap Q$ on X_I induces a discrete group Γ_{X_I} in the automorphism group of X_I. The boundary of X^* is given by

$$X^* \setminus X = \bigcup_{i=1}^{m} \Gamma X_{Q_i,I}.$$

Taking the quotient of X^* by Γ, we obtain

$$\Gamma \setminus X^* = \Gamma \setminus X \cup \bigcup_{i=1}^{m} \Gamma \setminus \Gamma X_{Q_i,I}.$$

The quotient space $\Gamma \setminus \Gamma X_{Q_i,I}$ identifies with $\Gamma_{X_{Q_i,I}} \setminus X_{Q_i,I}$. Hence each boundary of $\Gamma \setminus X^*$ is a quotient of the boundary space $X_{Q_i,I}$ and

$$\Gamma \setminus X^* = \Gamma \setminus X \cup \bigcup_{i=1}^{m} \Gamma_{X_{Q_i,I}} \setminus X_{Q_i,I}.$$

defines a compactification of $\Gamma \setminus X$, where Q_i are representatives of Γ-conjugacy classes of maximal rational parabolic subgroups. The boundary of X^* is a union of what is called $\Gamma-$rational boundary component of X^c. They are also called the cusps.

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