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Linear Programming Methods for Identifying Solvable Cases of the Quadratic Assignment Problem

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Abstract

This research effort is concerned with identifying and characterizing families of polynomially solvable instances of the celebrated NP-hard quadratic assignment problem (qap). The approach is novel in that it uses polyhedral methods based on an equivalent mixed 0-1 linear reformulation of the problem. The continuous relaxation of this mixed 0-1 form yields a feasible region having extreme points that are both binary and fractional. The solvable instances of concern essentially possess objective function structures that ensure a binary extreme point must be optimal, so that the linear program solves the qap. The ultimate contribution of this work is the unification and subsumption of a variety of known solvable instances of the qap, and the development of a theoretical framework for identifying richer families of solvable instances.

The qap was introduced over 50 years ago in the context of facility layout and location. The underlying mathematical structure, from which the problem draws its name, consists of the minimization of a quadratic function of binary variables over an assignment polytope. Since its inception, this structure has received considerable attention from various researchers, both practitioners and theoreticians alike, due to the diversity of practical applications and the resistance to exact solution procedures. Unfortunately, the combinatorial explosion of feasible solutions to the qap, in terms of the number of binary variables, creates a significant gap between the sizes of the motivating applications and the instances that can be solved by state-of-the-art solution algorithms. The most successful algorithms rely on linear forms of the qap to compute bounds within enumerative schemes.

The inability to solve large qap instances has motivated researchers to seek special

objective function structures that permit polynomial solvability. Various, seemingly unrelated, structures are found in the literature. This research shows that many such structures can be explained in terms of the linear reformulation which results from applying the level-1 reformulation-linearization technique (RLT) to the qap. In fact, the research shows that the level-1 RLT not only serves to explain many of these instances, but also allows for simplifications and/or generalizations. One important structure centers around instances deemed to be linearizable, where a qap instance is defined to be linearizazble if it can be equivalently rewritten as a linear assignment problem that preserves the objective function value at all feasible points. A contribution of this effort is that the constraint structure of a relaxed version of the continuous relaxation of the level-1 RLT form gives rise to a necessary and sufficient condition for an instance of the qap to be linearizable. Specifically, an instance of the qap is linearizable if and only if the given relaxed level-1 RLT form has a finite optimal solution. For all such cases, an optimal solution must occur at a binary extreme point. As a consequence, all linearizable qap instances are solvable via the level-1 RLT. The converse, however is not true, as the continuous relaxation of the level-1 RLT form can have a binary optimal solution when the qap is not linearizable. Thus, the linear program available from the level-1 RLT theoretically identifies a richer family of solvable instances. Notably, and as a consequence of this study, the level-1 RLT serves as a unifying entity in that it integrates the computation of linear programming-based bounds with the identification of polynomially solvable special cases, a relationship that was previously unnoticed.

Dedication

I dedicate this work to my grandfathers, Alfonso D. Brigante and John A. Waddell, who both strongly believed in the value of education, and would have been proud of this achievement.

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Chapter 1

Introduction

The *quadratic assignment problem* (qap) is an NP-hard discrete, nonlinear optimization program that seeks to minimize a quadratic function of 0-1 variables over an assignment polytope. The qap was first introduced in the context of facility location over 50 years ago in [37], where it was desired to situate n facilities in a one-to-one fashion on n location sites so as to minimize a total cost of construction and material handling. The construction costs are associated with the individual facility-site pairs in such a manner that a cost is incurred for each pair selected. The material handling costs are computed using interactions between facilities on a product per-unit distance basis. Specifically, the material handling cost between each pair of facilities is computed as the product of the known material flow interaction with the distance between the selected location sites. The construction costs give rise to linear objective terms while the material handling costs require quadratic terms. The assignment restrictions ensure a one-to-one allocation of facilities to sites.

Since its introduction, the qap has been extensively studied. These studies include both the above-described instances where the quadratic objective coefficients consist of the products of flows with distances, known as Koopmans-Beckmann forms, and also more general instances having arbitrary quadratic objective terms that need not be represented as products of flows and distances. Applications abound in facility layout and location, including college campus planning [24], hospital layout [26], and the assigning of gates at airport terminals [31]. Other applications include backboard wiring [52], turbine manufacturing [39], typewriter keyboard design [17], dartboard design [25], scheduling parallel production lines [30], analyzing chemical reactions [53], ranking archaeological data [38], and designing the layout of letters on touchscreen devices [23]. Additionally, classical optimization problems such as the traveling salesman problem and the bandwidth reduction problem are special cases of the qap. Surveys are found in [14, 18, 42, 45].

The qap has proven itself extremely difficult to solve, with exact solution procedures generally limited to instances having up to $n = 30$ facilities [1, 8, 33, 34]. The underlying difficulty is combinatorial in nature, as the qap with n facilities has n-factorial feasible solutions. The most successful exact methods attempt to implicitly enumerate these solutions by employing branch-and-bound algorithms that use linear programming relaxations to obtain bounds for pruning branches of the binary search tree.

Motivated by the need to obtain effective bounding mechanisms for branch-and-bound algorithms, considerable historic effort has been given to computing equivalent mixed 0-1 linear reformulations. These linearizations use additional variables to substitute for the nonlinear terms, and additional constraints to ensure that the substituted variables realize their intended values. A multitude of such forms are available in the literature [2, 9, 11, 29, 36, 41], and these forms boast different sizes and relaxation strengths. Some forms are fairly compact in terms of the numbers of auxiliary variables and constraints, while others are larger in size. Generally speaking, the larger forms give rise to tighter bounds. An ongoing challenge is to obtain forms that balance the sizes of the linearized problems with the strengths of the relaxations.

A method that has proven itself extremely effective for constructing favorable linear reformulations of the qap is the reformulation-linearization technique (RLT) of [4, 5, 6]. The RLT is a general procedure for constructing mixed 0-1 linear reformulations of 0-1 programs. When applied to the qap with n facilities, an n-level hierarchy of mixed 0-1 linear reformulations results, with each level yielding a relaxation at least as tight as the previous level, and with the highest level affording the convex hull representation. Unfortunately, as one progresses up the hierarchical ladder, the formulations increase in size. The level-1 and level-2 forms have produced state-of-the-art computational results [1, 2], and the level-3 form is under study [34].

Due to the difficulty associated with solving the qap, researchers have focused attention on finding special instances that are polynomially solvable. The idea here is to identify objective function structures that permit efficient solution strategies. Many, seemingly unrelated, such identifications are available in the literature (see, for example, [12, 13, 15, 18, 19, 20, 21, 22, 27, 28, 35, 40, 46]). These works deal with both the Koopmans-Beckmann form and with more general objective structures. Depending on the specific problem of concern, these works either provide closed-form solutions or provide equivalent, polynomially solvable reformulations.

This research poses a novel polyhedral approach for identifying polynomially solvable instances of the qap. In the process, it affords a unifying framework for characterizing various known solvable instances, and also serves to form a natural linkage between bounding strategies and the objective function structures of solvable instances. The key insight is a previously unnoticed relationship between the polyhedral structure of the continuous relaxation of the level-1 RLT representation and various classes of readily solvable instances.

The level-1 RLT form of the qap is a mixed 0-1 linear representation that was first studied in [2]. The level-1 RLT augments the binary variables inherent to the qap with a family of continuous variables in such a manner that a continuous variable is defined for each distinct nonlinear term. Auxiliary constraints are strategically constructed to ensure that each continuous variable is equal to its substituted product at every binary solution, as well as to afford a tight linear programming approximation. The feasible region to the continuous relaxation, which is obtained by replacing the binary restrictions on the original variables with nonnegativity, is a polytope having a highly specialized structure. Every binary solution to the qap is associated with an extreme point of this polytope, and the objective function value is preserved at each such point. However, there exist extreme points that do not correspond to binary solutions. Strikingly, we show that a variety of apparently unrelated solvable cases of the qap can be categorized in the following sense: each such case has an objective function which ensures that an optimal solution to the continuous relaxation of the level-1 RLT form occurs at a binary extreme point. Interestingly, there exist instances that are solvable by the level-1 RLT form which do not satisfy the conditions of these cases, so that the level-1 form theoretically identifies a richer family of solvable instances.

The contributions of this research are grouped into two main emphases, with one emphasis found in each of Chapters 2 and 3.

In Chapter 2, we examine four different polynomially solvable cases of the qap found in the literature [13, 18, 27, 28], and show how each can be directly explained in terms of the continuous relaxation of the level-1 RLT form. Included within the first three forms are symmetric flow and skew symmetric distances as in [13, 18] and the cost coefficient decompositions of [13, 18, 28]. The fourth case involves path structures in the flow data, together with grid structures in the distance data [27]. These explanations allow for simplifications and/or generalizations of the conditions defining these cases. The arguments are based on the Karush-Kuhn-Tucker optimality conditions for linear programs.

The organization of Chapter 2 is as follows. Section 2.2 reviews the level-1 RLT form that serves as the backbone of the study. It highlights the derivation from the qap, and identifies instances in which the continuous relaxation affords an optimal binary solution. A variation of this linear form allows for a much simpler identification criterion, and emphasis is placed on this new form and criterion. The section also identifies special objective transformations for which the optimal solution to the qap and the continuous relaxations of these forms remain unaffected. Section 2.3 addresses the four readily solvable special cases of the qap referenced above, detailing each case within a separate subsection. Section 2.4 provides concluding remarks.

Chapter 3 is dedicated to instances of the qap known in the literature as *linearizable*. An instance of the qap is defined to be linearizable if and only if the problem can be equivalently written as a linear assignment problem that preserves the objective function value at all feasible solutions. This chapter provides an entirely new perspective on the concept of linearizable by showing that an instance of the qap is linearizable if and only if a relaxed version of the continuous relaxation of the level-1 RLT form is bounded. It also shows that the level-1 RLT form can identify a richer family of solvable instances than those deemed linearizable by demonstrating that the continuous relaxation of the level-1 RLT form can have an optimal binary solution for instances that are not linearizable. A matrix decomposition approach of [35] that provides an alternate set of necessary and sufficient conditions for an instance of the qap to be linearizable is thus encompassed by the level-1 RLT form.

Chapter 3 is organized as follows. Section 3.2 briefly summarizes the level-1 RLT form, presents a more compact representation that is obtained via a substitution of variables, and provides a roadmap for identifying a maximal set of linearly independent equations that is implied by the resulting constraints. Section 3.3 identifies the linearly independent equations, with the identification applying the RLT process to a basis of the assignment polytope, and also constructing a second set of constraints that has a network substructure. Section 3.4 provides the main result, showing that an instance of the qap is linearizable if and only if a relaxed version of the continuous relaxation of the level-1 RLT form is bounded. This is accomplished in two steps. The first step shows that all equations that are valid for the level-1 RLT form are implied by the linearly independent equations of Section 3.3, verifying this set as maximal. The second step establishes the relationship to the notion of linearizable. A numeric example is given to show that the continuous relaxation of the level-1 RLT form can provide a binary solution for instances of the qap that are not linearizable. A byproduct of the first step of Section 3.4 is the characterization of the dimensions of the level-1 RLT form and various relaxations. These dimensions are given in closed form in Section 3.5. Section 3.6 gives concluding remarks.

Chapters 2 and 3 are written independently of each other so that the reader can choose to read either work alone. Thus, each chapter contains its own introduction. However, an encompassing bibliography is found at the end.

Chapter 2

Linear Programming Insights into Solvable Cases of the Quadratic Assignment Problem

2.1 Introduction

The quadratic assignment problem is a discrete, nonlinear optimization problem that can be formulated as

$$\text{QAP:} \quad \text{minimize} \quad \sum_{k=1}^{n}\sum_{\ell=1}^{n} c_{k\ell}x_{k\ell} + \sum_{\substack{i=1\\i\neq k}}^{n}\sum_{\substack{j=1\\j\neq \ell}}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} C_{ijk\ell}x_{ij}x_{k\ell}$$

$$\text{subject to} \quad \boldsymbol{x} \in X,\ \boldsymbol{x} \text{ binary},$$

where

$$X \equiv \left\{ \begin{array}{ll} \boldsymbol{x} \in \mathbb{R}^{n^2}: & \sum_{j=1}^{n} x_{ij} = 1 \ \ \forall\, i = 1,\ldots,n, \\ & \sum_{i=1}^{n} x_{ij} = 1 \ \ \forall\, j = 1,\ldots,n, \\ & x_{ij} \geq 0 \ \ \forall\,(i,j), i = 1,\ldots,n,\ j = 1,\ldots,n \end{array} \right\}. \tag{2.1}$$

It is so named because the objective function is quadratic in the n^2 binary variables x_{ij} and the set X defines an assignment polytope. (We henceforth assume that all indices and summations run from 1 to n unless otherwise stated.)

This problem was originally introduced within the context of facility location by Koopmans and Beckmann [37]. In this setting, there exist n facilities and n location sites upon which the facilities are to be situated. The objective is to assign the facilities to the sites in a one-to-one fashion so that a total cost is minimized. The cost includes, for each pair (k, ℓ), a fixed charge $c_{k\ell}$ associated with assigning facility k to site ℓ, and a "quadratic" material handling cost incurred between pairs of assigned facilities. Here, each pair (i, k) has some known material flow f_{ik} from facility i to facility k, and each pair (j, ℓ) has some known distance from site j to site ℓ. (In general, the flows and distances need not be symmetric so that we can have $f_{ik} \neq f_{ki}$ and $d_{j\ell} \neq d_{\ell j}$.) The material handling cost incurred for shipment from facility i on site j to facility k on site ℓ is computed as $C_{ijk\ell} = f_{ik} d_{j\ell}$ in terms of flow times distance. For each (i, j) pair, the decision variable $x_{ij} = 1$ if facility i is assigned to site j, and 0 otherwise. In this manner, (2.1) enforces the one-to-one assignment of facilities to sites. The double-sum in the objective function records the assignment cost, and the quadruple-sum records the cost of material flow. As that version of Problem QAP having $C_{ijk\ell} = f_{ik} d_{j\ell}$ was introduced in [37], it has come to be known as the "Koopmans-Beckmann" form.

Problem QAP has been intensely studied over the past 50 years, both for the Koopmans-Beckmann form and for more general cases having arbitrary $C_{ijk\ell}$ quadratic cost coefficients. Surveys are available in [14, 18, 42]. Applications arise in various contexts, including backboard wiring [52], campus planning [24], typewriter keyboard design [17], hospital layout [26], scheduling parallel production lines [30], analyzing chemical reactions [53], and ranking archaeological data [38]. However, it is extremely difficult to solve, and falls under the class of NP-hard problems. Over the years, test-beds of problems have emerged [16, 44]. Generally speaking, exact solution strategies are available only for problems having n as large as 30 [1, 8, 33, 34]. The latest, most successful methods are enumerative in nature, and primarily use linear programming reformulations in higher-variable spaces for computing bounds to curtail the binary search tree.

Since optimal solutions to instances of Problem QAP are not generally computable for large values of n, researchers have considered different avenues for attacking the problem. One avenue is to devise heuristic procedures for generating good-quality solutions. These solutions are potentially useful, but suffer from not being provably optimal. A second avenue, and the focus of this chapter, is to identify special objective function structures that make the problem much simpler to solve.

Different approaches for characterizing objective function structures are available in the literature. One approach is to identify coefficients $C_{ijk\ell}$ that reduce Problem QAP to a *linear* assignment problem. In this vein, a size n instance of Problem QAP is defined to be *linearizable* if there exists a size n linear assignment problem having the same objective function value as Problem QAP at all n-factorial feasible binary solutions. The paper [28] gives, in the form of consistency to a specified linear system of $O(n^3)$ variables in $O(n^4)$ equations, sufficient conditions for recognizing an instance of Problem QAP as linearizable. It also shows, for Problem QAP rewritten to have the objective coefficients $c_{k\ell}$ subsumed within the quadratic terms by $C_{k\ell k\ell} = c_{k\ell}$ for all (k, ℓ) that, when the coefficients $C_{ijk\ell}$ can be expressed as $C_{ijk\ell} = v_{ij}v_{k\ell}$ for all (i, j, k, ℓ) with $i \neq k, j \neq \ell$ or with $i = k, j = \ell$, for some $n \times n$ matrix V whose $(i, j)^{th}$ entry is v_{ij}, and when all feasible solutions to Problem QAP have either nonnegative or nonpositive objective value, then an optimal solution to Problem QAP can be obtained by solving a linear assignment problem. The work of [35] builds upon the first contribution by providing conditions that are both necessary and sufficient for recognizing a linearizable instance of Problem QAP, together with an $O(n^4)$ algorithm for determining whether these conditions hold true. Later, [46] shows that the special structure of the Koopmans-Beckmann form of Problem QAP allows the conditions to be checked in $O(n^2)$ time. A second approach for characterizing objective function structure, separate from the concept of linearizable, identifies special flow and distance structures of the Koopmans-Beckmann form of QAP. Efforts in this regard include certain instances having an anti-Monge-Toeplitz objective structure [15], a simple block structure [20], a Kalmanson-circulant structure [22], a Robinsonian-Toeplitz structure [40], a variation [19] of the work of [15] motivated by a scheduling problem of [54], and instances having a path structure in the flow data and a grid structure in the distance data [27]. For each of these cases, an optimal solution is immediately available, without solving an optimization problem. The paper [21] shows that a "Wiener maximum quadratic assignment problem," while NP-hard, can be solved in pseudo-polynomial time, with a special case solvable in polynomial time. References [13, 18] survey solvable cases, including classic instances found in [12].

We expose an unnoticed relationship between a mixed 0-1 linear reformulation of Problem QAP and four special cases that permit QAP to be solved in polynomial time. Included within these forms are symmetric flow and skew symmetric distances as in [13, 18], the flow and/or distance decomposition of [13, 18], the consistency check of the linear system of equations by [28], and the above-cited work of [27]. The feasible region to the continuous relaxation of the mixed 0-1 linear form, obtained by relaxing the $\boldsymbol{x}$ binary restrictions to $\boldsymbol{x} \geq \mathbf{0}$, is a polytope having both binary and fractional extreme points. Each of the n-factorial feasible binary solutions to X is associated with an extreme point, but the relaxation also has additional, fractional such points. This chapter exploits the structure of the polytope to make three contributions. The first contribution is to demonstrate that the linear form is a unifying entity for the four special cases in the sense that each can be interpreted as restricting the objective coefficients so that a binary extreme point is optimal. Thus, a necessary condition for any of these special cases to hold true is that the continuous relaxation of the mixed 0-1 linear form has an optimal binary solution. The second contribution is to show that the linear form identifies a larger family of solvable instances of Problem QAP than the four cases combined. The third contribution is to substantially simplify the third case. All three contributions are based on the Karush-Kuhn-Tucker (KKT) optimality conditions for linear programs.

The chapter is organized as follows. The next section briefly reviews the linear form that serves as the backbone of this study. It highlights the derivation from Problem QAP, and identifies instances in which the continuous relaxation affords an optimal binary solution. A variation of this linear form allows for a much simpler identification criterion, and this form and criterion are studied. The section finishes by showing that certain objective transformations do not affect the optimal solution set to either QAP or the continuous relaxation of either form. Section 2.3 addresses the four readily solvable special cases of Problem QAP reported in the literature, where each case is encompassed within the framework of Section 2.2 in a separate subsection. Section 2.4 provides concluding remarks.

2.2 Mixed 0-1 Linear Reformulation: Review and Structure

A variety of mixed 0-1 linear reformulations of Problem QAP exist in the literature (see, for example, [2, 9, 11, 29, 36, 41]). These forms differ in terms of the numbers of variables and constraints employed, as well as the strengths of the continuous relaxations. The particular program of interest to us in this study is the level-1 form [4, 5] resulting from the reformulation-linearization-technique (RLT) of [50, 49, 51], as specialized for the quadratic assignment problem in [2]. This section reviews the construction of the level-1 form, and then uses the KKT conditions to identify special cases for which the continuous relaxation affords optimal binary solutions to Problem QAP. Two versions of this relaxation are considered separately in two theorems, with the second version obtained from the first by removing a family of constraints. A third theorem provides objective transformations that preserve the optimal solution sets of Koopmans-Beckmann instances. These theorems are collectively used in Section 2.3 to subsume and generalize four published special cases.

The RLT is a general methodology for recasting pure and mixed 0-1 polynomial programs into higher-variable spaces through the defining of auxiliary continuous variables and constraints. Given a discrete program, the idea is to construct mixed 0-1 linear forms whose continuous relaxations form tight polyhedral approximations of the convex hull of feasible solutions. There are different implementations of the RLT, yielding a hierarchy of relaxation levels, and these levels provide successively tighter approximations until the convex hull is achieved at the highest level. For Problem QAP, there are n levels, with each level obtained by multiplying the constraints defining X in (2.1) by special monomials in the variables $x_{k\ell}$. A detailed discussion is found in [49, pp. 104-105]. The paper [2] shows that, for QAP, even the weakest level-1 form affords a tight approximation of the convex hull.

The level-1 RLT form of Problem QAP is obtained by conducting the two distinct steps of *reformulation* and *linearization* in the following manner. The reformulation step multiplies every constraint defining the set X in (2.1) by each of the n^2 nonnegative variables $x_{k\ell}$, and then substitutes $x_{k\ell} = x_{k\ell}^2$ for all (k, ℓ). For each (k, ℓ), multiplication of the equation $\sum_i x_{i\ell} = 1$ and the $n-1$ inequalities $x_{i\ell} \geq 0$ for $i \neq k$ by $x_{k\ell}$ yields $x_{i\ell}x_{k\ell} = 0$ for all $i \neq k$. Similarly for each (k, ℓ), multiplication of the equation $\sum_j x_{kj} = 1$ and the $n-1$ inequalities $x_{kj} \geq 0$ for $j \neq \ell$ by $x_{k\ell}$ yields $x_{kj}x_{k\ell} = 0$ for all $j \neq \ell$. These products that are set to 0, and the associated constraints, are not included within the problem. The resulting (redundant) $2(n-1)n^2$ equality and $n^2(n-1)^2$ nonnegativity constraints are appended to the $\boldsymbol{x} \in X$, $\boldsymbol{x}$ binary, restrictions of Problem QAP. The *linearization* step then substitutes a continuous variable $y_{ijk\ell}$ for each resulting $x_{ij}x_{k\ell}$ product; that is, for each (i, j, k, ℓ), $i \neq k$, $j \neq \ell$. It also enforces that $y_{ijk\ell} = y_{k\ell ij}$ for all (i, j, k, ℓ), $i < k$, $j \neq \ell$. The level-1 RLT form of Problem QAP then becomes the following, as provided in [2].

$$\text{RLT1:} \quad \text{minimize} \quad \sum_k \sum_\ell c_{k\ell}x_{k\ell} + \sum_{i\neq k} \sum_{j\neq \ell} \sum_k \sum_\ell C_{ijk\ell}y_{ijk\ell}$$

$$\text{subject to} \quad \sum_{j\neq\ell} y_{ijk\ell} = x_{k\ell} \qquad \forall\ (i,k,\ell),\ i \neq k \tag{2.2}$$

$$\sum_{i\neq k} y_{ijk\ell} = x_{k\ell} \qquad \forall\ (j,k,\ell),\ j \neq \ell \tag{2.3}$$

$$y_{ijk\ell} = y_{k\ell ij} \qquad \forall\ (i,j,k,\ell),\ i < k,\ j \neq \ell \tag{2.4}$$

$$y_{ijk\ell} \geq 0 \qquad \forall\ (i,j,k,\ell),\ i \neq k,\ j \neq \ell \tag{2.5}$$

$$\boldsymbol{x} \text{ binary},\ \boldsymbol{x} \in X$$

The mixed 0-1 *linear* representation RLT1 is equivalent [2] to QAP in that, given a feasible solution to either problem, there exists a feasible solution to the other problem with the same objective value. In fact, Problem RLT1 enforces that, for $\boldsymbol{x}$ binary, we must have that $y_{ijk\ell} = x_{ij}x_{k\ell}$ for all (i, j, k, ℓ), $i \neq k$, $j \neq \ell$.

Consider the continuous relaxation of RLT1 obtained by removing the $\boldsymbol{x}$ binary restrictions, and call this problem $\overline{\text{RLT1}}$. Every feasible solution to $\overline{\text{RLT1}}$ with $\boldsymbol{x}$ binary is readily shown to be an extreme point of the feasible region but, as mentioned earlier, there can exist extreme points with fractional components of $\boldsymbol{x}$. The task is to identify special cases of Problem QAP for which $\overline{\text{RLT1}}$ will be optimal at some extreme point having $\boldsymbol{x}$ binary. Since $\overline{\text{RLT1}}$ is polynomial in the size of the input data, such cases will then be polynomially solvable instances of QAP.

We note here that the size of Problem RLT1 can be reduced without affecting the equivalence between Problems QAP and RLT1, nor the strength of the relaxation $\overline{\text{RLT1}}$. This reduction can be accomplished in two ways. First, and as observed in [2], we can use (2.4) to eliminate all variables $y_{ijk\ell}$ having $i > k$ and $j \neq \ell$ from the problem, and then remove (2.4) and the nonnegativity restrictions on the associated $y_{ijk\ell}$. This substitution lessens the problem size by each of $\frac{n^2(n-1)^2}{2}$ variables and equality constraints, and by the same number of nonnegativity restrictions. Second, since X is an assignment set, any single equation can be removed *prior* to applying the reformulation-linearization-technique without affecting the relaxation strength of the resulting form. The savings from this second reduction is a single constraint in X and $n(n-1)$ constraints within either (2.2) or (2.3). Temporarily ignoring nonnegativity on $\boldsymbol{x}$ and $\boldsymbol{y}$, the combined effect of the two reductions is to transform Problem RLT1 from having n^2 variables $\boldsymbol{x}$ and $n^2(n-1)^2$ variables $\boldsymbol{y}$ in $2n^2(n-1) + \frac{n^2(n-1)^2}{2} + 2n$ equality constraints to having n^2 variables $\boldsymbol{x}$ and $\frac{n^2(n-1)^2}{2}$ variables $\boldsymbol{y}$ in $(n^2 - n + 1)(2n - 1)$ equality constraints, for a savings of $\frac{n^2(n-1)^2}{2}$ variables and $\frac{n^2(n-1)^2}{2} + n(n-1) + 1$ equality constraints. Relative to variable nonnegativity, the number of restrictions on $\boldsymbol{x}$ is constant at n^2 while, as mentioned above, the number of restrictions on $\boldsymbol{y}$ is reduced from $n^2(n-1)^2$ to $\frac{n^2(n-1)^2}{2}$, for a savings of $\frac{n^2(n-1)^2}{2}$. For ease of presentation, we choose to retain all variables and constraints, keeping in mind that the problem can be so reduced. (These reductions will allow us in Section 2.3.3 to simplify a published set of conditions, mentioned earlier as our third overall contribution.)

We write the dual to $\overline{\text{RLT1}}$, and refer to it as Problem $\overline{\text{DRLT1}}$. Let $\boldsymbol{u} = u_{ik\ell}$ for all $(i,k,\ell), i \neq k$, $\boldsymbol{v} = v_{jk\ell}$ for all $(j,k,\ell), j \neq \ell$, $\boldsymbol{w} = w_{ijk\ell}$ for all $(i,j,k,\ell), i < k, j \neq \ell$, and $\boldsymbol{\lambda} = \lambda_{ijk\ell}$ for all $(i,j,k,\ell), i \neq k, j \neq \ell$, denote multipliers to (2.2), (2.3), (2.4), and (2.5) respectively. Further let $\boldsymbol{\pi}^1 = \pi_i^1$ for all i and $\boldsymbol{\pi}^2 = \pi_j^2$ for all j denote multipliers to the two families of equality constraints defining X in (2.1), and let $\boldsymbol{\pi}^3 = \pi_{ij}^3$ for all (i,j) denote multipliers to the nonnegativity restrictions in (2.1). Then the dual is as follows.

$$
\begin{aligned}
\overline{\text{DRLT1}}: \text{ maximize } & \sum_i \pi_i^1 + \sum_j \pi_j^2 \\
\text{subject to } & -\sum_{i \neq k} u_{ik\ell} - \sum_{j \neq \ell} v_{jk\ell} + \pi_k^1 + \pi_\ell^2 + \pi_{k\ell}^3 = c_{k\ell} \quad \forall\, (k,\ell) \\
& u_{ik\ell} + v_{jk\ell} + w_{ijk\ell} + \lambda_{ijk\ell} = C_{ijk\ell} \;\; \forall\, (i,j,k,\ell),\; i < k,\; j \neq \ell \\
& u_{ik\ell} + v_{jk\ell} - w_{k\ell ij} + \lambda_{ijk\ell} = C_{ijk\ell} \;\; \forall\, (i,j,k,\ell),\; i > k,\; j \neq \ell \\
& \boldsymbol{\pi}^3 \geq \mathbf{0},\; \boldsymbol{\lambda} \geq \mathbf{0}
\end{aligned}
$$

We find it useful to define a set S consisting of those values $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\lambda})$ having $\boldsymbol{\lambda} \geq \mathbf{0}$ which are feasible to the last $(n-1)^2 n^2$ equations of $\overline{\text{DRLT1}}$; these equations are the dual constraints corresponding to the primal variables $y_{ijk\ell}$. That is,

$$
S \equiv \left\{ \begin{array}{l} (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\lambda}) : \boldsymbol{\lambda} \geq \mathbf{0}, \\ \qquad u_{ik\ell} + v_{jk\ell} + w_{ijk\ell} + \lambda_{ijk\ell} = C_{ijk\ell} \;\; \forall\, (i,j,k,\ell),\; i < k,\; j \neq \ell, \\ \qquad u_{ik\ell} + v_{jk\ell} - w_{k\ell ij} + \lambda_{ijk\ell} = C_{ijk\ell} \;\; \forall\, (i,j,k,\ell),\; i > k,\; j \neq \ell \end{array} \right\}. \qquad (2.6)
$$

Now consider the below result which gives, in terms of S, a necessary and sufficient condition for a binary point $\hat{\boldsymbol{x}}$ to be part of an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to $\overline{\text{RLT1}}$, so that $\hat{\boldsymbol{x}}$ is optimal to QAP.

Theorem 2.1

Given an instance of Problem QAP, a binary point $\hat{\boldsymbol{x}}$ is part of an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to $\overline{\text{RLT1}}$, so that $\hat{\boldsymbol{x}}$ is optimal to QAP, if and only if there exists a $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S$ such that $\hat{\boldsymbol{x}}$ is optimal to

$$\min\left\{\sum_k \sum_\ell \left(c_{k\ell} + \sum_{i \neq k} \hat{u}_{ik\ell} + \sum_{j \neq \ell} \hat{v}_{jk\ell}\right) x_{k\ell} : \boldsymbol{x} \in X\right\}, \tag{2.7}$$

with $\hat{\lambda}_{ijk\ell}\hat{x}_{ij}\hat{x}_{k\ell} = 0$ for all $(i, j, k, \ell), i \neq k, j \neq \ell$.

Proof

Suppose a binary point $\hat{\boldsymbol{x}}$ is part of an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to $\overline{\text{RLT1}}$. The KKT necessary conditions state that there exists a $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\pi}}^1, \hat{\boldsymbol{\pi}}^2, \hat{\boldsymbol{\pi}}^3)$ feasible to $\overline{\text{DRLT1}}$ such that $\hat{\pi}^3_{k\ell}\hat{x}_{k\ell} = 0$ for all (k, ℓ) and $\hat{\lambda}_{ijk\ell}\hat{y}_{ijk\ell} = 0$ for all (i, j, k, ℓ), $i \neq k$, $j \neq \ell$. Then $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S$. Moreover, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\pi}}^1, \hat{\boldsymbol{\pi}}^2, \hat{\boldsymbol{\pi}}^3)$ satisfies the KKT conditions to (2.7) so that $\hat{\boldsymbol{x}}$ is optimal to (2.7). Finally, $\hat{\boldsymbol{y}}$ is defined in terms of $\hat{\boldsymbol{x}}$ as $\hat{y}_{ijk\ell} = \hat{x}_{ij}\hat{x}_{k\ell}$ for all (i, j, k, ℓ), $i \neq k$, $j \neq \ell$, so that $\hat{\lambda}_{ijk\ell}\hat{x}_{ij}\hat{x}_{k\ell} = 0$, as desired.

Now, given a binary point $\hat{\boldsymbol{x}}$, suppose there exists a $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S$ so that $\hat{\boldsymbol{x}}$ is optimal to (2.7) and $\hat{\lambda}_{ijk\ell}\hat{x}_{ij}\hat{x}_{k\ell} = 0$ for all $(i, j, k, \ell), i \neq k, j \neq \ell$. The point $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ with $\hat{y}_{ijk\ell} = \hat{x}_{ij}\hat{x}_{k\ell}$ for all (i, j, k, ℓ), $i \neq k$, $j \neq \ell$, and the point $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\pi}}^1, \hat{\boldsymbol{\pi}}^2, \hat{\boldsymbol{\pi}}^3)$ with $(\hat{\boldsymbol{\pi}}^1, \hat{\boldsymbol{\pi}}^2, \hat{\boldsymbol{\pi}}^3)$ an optimal set of duals to the $\boldsymbol{x} \in X$ constraints of (2.7) together satisfy the KKT sufficiency conditions of $\overline{\text{RLT1}}$, making $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ optimal to $\overline{\text{RLT1}}$. □

Theorem 2.1 gives, in terms of (2.7) and the set S of (2.6), necessary and sufficient conditions for $\overline{\text{RLT1}}$ to have an optimal binary solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$. Relationships also exist between the objective values to $\overline{\text{RLT1}}$ and (2.7). Given any $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ satisfying (2.2)–(2.4) and any $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S$, we have that

$$\begin{aligned}
&\sum_k \sum_\ell \left(c_{k\ell} + \sum_{i \neq k} \hat{u}_{ik\ell} + \sum_{j \neq \ell} \hat{v}_{jk\ell} \right) \hat{x}_{k\ell} + \sum_{i \neq k} \sum_{j \neq \ell} \sum_k \sum_\ell \hat{\lambda}_{ijk\ell} \hat{y}_{ijk\ell} \\
&\quad = \sum_k \sum_\ell c_{k\ell} \hat{x}_{k\ell} + \sum_{i \neq k} \sum_{j \neq \ell} \sum_k \sum_\ell C_{ijk\ell} \hat{y}_{ijk\ell} + \sum_{i \neq k} \sum_k \sum_\ell \hat{u}_{ik\ell} \left(\hat{x}_{k\ell} - \sum_{j \neq \ell} \hat{y}_{ijk\ell} \right) \\
&\qquad + \sum_{j \neq \ell} \sum_k \sum_\ell \hat{v}_{jk\ell} \left(\hat{x}_{k\ell} - \sum_{i \neq k} \hat{y}_{ijk\ell} \right) + \sum_{i < k} \sum_{j \neq \ell} \sum_k \sum_\ell \hat{w}_{ijk\ell} \left(\hat{y}_{k\ell ij} - \hat{y}_{ijk\ell} \right) \\
&\quad = \sum_k \sum_\ell c_{k\ell} \hat{x}_{k\ell} + \sum_{i \neq k} \sum_{j \neq \ell} \sum_k \sum_\ell C_{ijk\ell} \hat{y}_{ijk\ell}.
\end{aligned} \tag{2.8}$$

The first equation follows from the expression of $C_{ijk\ell}$ in S of (2.6) and the second is due to $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ satisfying (2.2)–(2.4). These equations of (2.8) lead to two consequences, presented as the first two remarks below.

Remark 2.1

Given any $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ satisfying (2.2)–(2.4) and any $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S$ such that $\hat{\lambda}_{ijk\ell} \hat{x}_{ij} \hat{x}_{k\ell} = \hat{\lambda}_{ijk\ell} \hat{y}_{ijk\ell} = 0$ for all $(i, j, k, \ell), i \neq k, j \neq \ell$, the following three objective values equal.

1. Problem $\overline{\text{RLT1}}$ evaluated at $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$,

2. Problem (2.7) evaluated at $\hat{\boldsymbol{x}}$,

3. Problem QAP evaluated at $\hat{\boldsymbol{x}}$.

To elaborate on Remark 2.1, the objective values to $\overline{\text{RLT1}}$ and (2.7) equal by the first and last expressions of (2.8) because $\hat{\lambda}_{ijk\ell} \hat{y}_{ijk\ell} = 0$ for all $(i, j, k, \ell), i \neq k, j \neq \ell$. The objective values to (2.7) and QAP also equal, because (2.8) holds true when $\hat{y}_{ijk\ell} = \hat{x}_{ij} \hat{x}_{k\ell}$ for all $(i, j, k, \ell), i \neq k, j \neq \ell$, and $\hat{\lambda}_{ijk\ell} \hat{x}_{ij} \hat{x}_{k\ell} = 0$ for all $(i, j, k, \ell), i \neq k, j \neq \ell$.

Remark 2.2

A modification of the "if" direction of Theorem 2.1 provides a lower bound on the optimal objective value to $\overline{\text{RLT1}}$ as well as an upper bound on the optimal objective value to RLT1.

Suppose there exists a $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S$ so that a binary $\hat{\boldsymbol{x}}$ is optimal to (2.7), but the condition that $\hat{\lambda}_{ijk\ell}\hat{x}_{ij}\hat{x}_{k\ell} = 0$ for all $(i, j, k, \ell), i \neq k, j \neq \ell$, is relaxed. By decreasing within $\overline{\text{RLT1}}$ each original objective coefficient $C_{ijk\ell}$ having $\hat{x}_{ij}\hat{x}_{k\ell} = 1$ by the quantity $\hat{\lambda}_{ijk\ell}$, the conditions of Theorem 2.1 would be satisfied for this revised program. Thus, $\hat{\boldsymbol{x}}$ would be part of an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to this revised version of $\overline{\text{RLT1}}$, providing a lower bound on the optimal objective value to $\overline{\text{RLT1}}$ itself. But this same $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is feasible to RLT1, and thus gives an upper bound on the optimal objective value to this problem. In summary, the first expression of (2.8) provides an upper bound on RLT1 while this same expression, less $\sum_{i \neq k} \sum_{j \neq \ell} \sum_{k} \sum_{\ell} \hat{\lambda}_{ijk\ell}\hat{y}_{ijk\ell}$, provides a lower bound on $\overline{\text{RLT1}}$. When the condition on $\hat{\boldsymbol{\lambda}}$ is not relaxed, the upper bound to RLT1 equals the lower bound to $\overline{\text{RLT1}}$, establishing optimality to both problems.

The first three special cases of Problem QAP considered in the next section focus on subsets of S having $\boldsymbol{\lambda} = \boldsymbol{0}$. For convenience, we adopt the notation that $S^0 \equiv \{(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\lambda}) \in S, \boldsymbol{\lambda} = \boldsymbol{0}\}$. Recalling that S was defined in terms of the dual space to $\overline{\text{RLT1}}$ so that $\boldsymbol{\lambda}$ represents the multipliers on inequalities (2.5), restricting $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\lambda}) \in S^0$ effectively directs attention to that relaxed version of $\overline{\text{RLT1}}$, call it $\overline{\text{RLT1}'}$, which is obtained by deleting the $\boldsymbol{y} \geq \boldsymbol{0}$ restrictions. As shown in Theorem 2.2, Problem $\overline{\text{RLT1}'}$ provides a characterization of those instances of QAP for which $S^0 \neq \emptyset$ and, as a result, encompasses the optimality conditions for all three cases.

We restate Remark 2.1 as Remark 2.3 below to accommodate the set S^0 and the linear program $\overline{\text{RLT1}'}$. Remark 2.3 is simpler than Remark 2.1 because $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$ has $\hat{\boldsymbol{\lambda}} = \boldsymbol{0}$. Also, we are able to replace $\overline{\text{RLT1}}$ in point 1 of Remark 2.1 with $\overline{\text{RLT1}'}$ in point 1 of Remark 2.3 because the objective functions to $\overline{\text{RLT1}}$ and $\overline{\text{RLT1}'}$ are identical.

Remark 2.3

Given any $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ satisfying (2.2)–(2.4) and any $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$, the following three objective values equal.

1. Problem $\overline{\text{RLT1}'}$ evaluated at $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$,

2. Problem (2.7) evaluated at $\hat{\boldsymbol{x}}$,

3. Problem QAP evaluated at $\hat{\boldsymbol{x}}$.

Now consider Theorem 2.2 below that focuses on Problem $\overline{\text{RLT1}'}$ and the set S^0. Here, unlike the conditions of Theorem 2.1, a point $\hat{\boldsymbol{x}}$ does not depend on the chosen $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$ in order to satisfy $\hat{\lambda}_{ijk\ell}\hat{x}_{ij}\hat{x}_{k\ell} = 0$ for all $(i, j, k, \ell), i \neq k, j \neq \ell$. This lack of dependency permits a stronger relationship between an optimal solution $\hat{\boldsymbol{x}}$ to $\overline{\text{RLT1}'}$, (2.7), and QAP than afforded by Theorem 2.1 for an optimal solution $\hat{\boldsymbol{x}}$ to $\overline{\text{RLT1}}$, (2.7), and QAP.

Theorem 2.2

Given an instance of Problem QAP, Problem $\overline{\text{RLT1}'}$ has a finite optimal solution if and only if $S^0 \neq \emptyset$. When $S^0 \neq \emptyset$, the following statements are equivalent for binary $\hat{\boldsymbol{x}} \in X$.

1. $\hat{\boldsymbol{x}}$ is part of an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to $\overline{\text{RLT1}'}$,

2. $\hat{\boldsymbol{x}}$ is optimal to (2.7) at any $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$,

3. $\hat{\boldsymbol{x}}$ is optimal to QAP.

Proof

We begin with the first statement, and then establish the equivalence of points 1 through 3 when $S^0 \neq \emptyset$.

Relative to the first statement, suppose that $\overline{\text{RLT1}'}$ has a finite optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$. Following the logic of the "only if" direction of the proof of Theorem 2.1, the KKT necessary conditions state that there exists a $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\pi}}^1, \hat{\boldsymbol{\pi}}^2, \hat{\boldsymbol{\pi}}^3)$ feasible to $\overline{\text{DRLT1}}$ with $\hat{\boldsymbol{\lambda}} = \boldsymbol{0}$ (since (2.5) is not present in $\overline{\text{RLT1}'}$). Then $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$. Conversely, suppose that $S^0 \neq \emptyset$. Since $\overline{\text{RLT1}'}$ is feasible, and every solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to $\overline{\text{RLT1}'}$ satisfies (2.2)–(2.4) and $\hat{\boldsymbol{x}} \in X$, we infer from points 1 and 3 of Remark 2.3 that $\overline{\text{RLT1}}'$ is bounded, as Problem QAP is bounded.

Relative to points 1 through 3, Remark 2.3 gives us that the optimal objective value to $\overline{\text{RLT1}}'$ is greater than or equal to that of both (2.7) and QAP because every $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ feasible to $\overline{\text{RLT1}}'$ has $\hat{\boldsymbol{x}}$ feasible to both (2.7) and QAP, with the same objective value. To complete the proof, it is sufficient to show that for every $\hat{\boldsymbol{x}} \in X$, there exists a $\hat{\boldsymbol{y}}$ having $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ feasible to $\overline{\text{RLT1}}'$. Define $\hat{\boldsymbol{y}}$ by $\hat{y}_{ijk\ell} = \hat{x}_{ij}\hat{x}_{k\ell}$ for all (i, j, k, ℓ), $i \neq k$, $j \neq \ell$. This completes the proof. □

Recalling from Section 2.1 that an instance of Problem QAP is *linearizable* if there exists a size n linear assignment problem having the same objective function value at all feasible binary solutions, Theorem 2.2 and Remark 2.3 combine to give us that a sufficient condition for Problem QAP to be linearizable is that $\overline{\text{RLT1}}'$ be bounded (have a finite optimal solution) for the given objective function. Given that $\overline{\text{RLT1}}'$ is bounded, Theorem 2.2 assures that there exists a vector $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$, so that Remark 2.3 has (2.7) taking the same objective function value as Problem QAP at all feasible, binary points.

Observe that, while the conditions of Theorem 2.1 identify a more general family of solvable instances of Problem QAP than do the conditions of Theorem 2.2, this generality comes at a price. Theorem 2.1 is more general because, as noted above, every bounded instance of $\overline{\text{RLT1}}'$ ensures the existence of a vector $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$ by Theorem 2.2, so that every binary optimal solution $\hat{\boldsymbol{x}}$ to the assignment problem (2.7), together with $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}})$, satisfies the conditions of Theorem 2.1. Determining whether the conditions of Theorem 2.2 can be satisfied, that is, determining whether $S^0 = \emptyset$, is accomplished in polynomial time. However, checking whether the conditions of Theorem 2.1 can be satisfied is not such a simple task. A consequence of Theorem 2.2 is that, if an optimal solution exists to $\overline{\text{RLT1}}'$, then an optimal solution exists to $\overline{\text{RLT1}}'$ with $\hat{\boldsymbol{x}}$ binary. In contrast, there can exist an optimal solution to $\overline{\text{RLT1}}$ with no optimal solution having $\hat{\boldsymbol{x}}$ binary. The task of determining whether the conditions of Theorem 2.1 can be satisfied is equivalent to checking whether such an alternative optimal solution exists. In general, checking the existence of an alternative optimal binary solution to the continuous relaxation of a 0-1 linear program is NP-complete, as it is equivalent to the 0-1 feasibility problem, which is NP-complete (see, for example, [43, Proposition 6.6 on p. 133] or [48, Corollary 18.1b on p. 248]). While $\overline{\text{RLT1}}$ has special structure, we do not believe this structure is sufficient to afford a polynomial-time check. Notably, there exist special cases of Theorem 2.1, not enveloped by Theorem 2.2, which can be solved in polynomial time. Upcoming Corollary 2.4 of Subsection 2.3.4 poses such a case.

The first, second, and fourth special cases of the next section address the Koopmans-Beckmann form of Problem QAP. These three cases each reduce QAP to an assignment problem of the form (2.7), but under differing conditions. Theorem 2.3 below exploits the structure of $\overline{\text{RLT1}'}$ to make an observation relative to those instances of QAP that are solvable by Theorem 2.1 or 2.2. Theorem 2.3 allows us, in the next section, to reexpress the conditions of the first and fourth cases in a more general setting. Of importance in the proof is that equations (2.8) do not require $\hat{\boldsymbol{\lambda}} \geq \mathbf{0}$ in order to be valid; these equations simply require that the chosen $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ and $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}})$ have $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ satisfying (2.2)–(2.4) and $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}})$ satisfying the equality restrictions of S. This theorem is a slight generalization of a result found in [18, pp. 113-114], where $f^* = d^*$.

Theorem 2.3

Given any scalars f^* and d^*, every Koopmans-Beckmann instance of Problem QAP that is identified, by either Theorem 2.1 or Theorem 2.2, as being solvable by an assignment problem of the form (2.7) is also solvable by (2.7) when f^* is subtracted from each flow f_{ik} and when d^* is subtracted from each distance $d_{j\ell}$.

Proof

Given any scalars f^* and d^*, it is sufficient to show that every $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ feasible to $\overline{\text{RLT1}}'$ has

$$\kappa + \sum_{i \neq k} \sum_{j \neq \ell} \sum_{k} \sum_{\ell} (f_{ik} - f^*)(d_{j\ell} - d^*)\hat{y}_{ijk\ell} = \sum_{i \neq k} \sum_{j \neq \ell} \sum_{k} \sum_{\ell} f_{ik} d_{j\ell} \hat{y}_{ijk\ell}, \tag{2.9}$$

where

$$\kappa = n(1-n)d^* f^* + d^* \left(\sum_{k} \sum_{i \neq k} f_{ik} \right) + f^* \left(\sum_{\ell} \sum_{j \neq \ell} d_{j\ell} \right).$$

Then the objective value to $\overline{\text{RLT1}}'$ decreases by the same scalar κ at every feasible $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$, so that the optimal solution set is unchanged. The optimal solution set to Problem $\overline{\text{RLT1}}$ is also unchanged because the feasible region to $\overline{\text{RLT1}}$ is contained within that of $\overline{\text{RLT1}}'$, and because the objective functions to $\overline{\text{RLT1}}'$ and $\overline{\text{RLT1}}$ are identical.

We begin by defining a $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}})$ that satisfies the equality restrictions of the set S of (2.6) so that we can later invoke (2.8). Let $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}})$ have $\hat{\boldsymbol{w}} = \boldsymbol{0}$, and have

$$\hat{u}_{ik\ell} = d^* \left(f_{ik} - \frac{f^*}{2} \right) \;\; \forall \, (i,k,\ell), i \neq k,$$

$$\hat{v}_{jk\ell} = f^* \left(d_{j\ell} - \frac{d^*}{2} \right) \;\; \forall \, (j,k,\ell), j \neq \ell,$$

and

$$\hat{\lambda}_{ijk\ell} = (f_{ik} - f^*)(d_{j\ell} - d^*) \;\; \forall \, (i,j,k,\ell), i \neq k, j \neq \ell.$$

Then

$$\hat{u}_{ik\ell} + \hat{v}_{jk\ell} + \hat{\lambda}_{ijk\ell} = f_{ik} d_{j\ell} = C_{ijk\ell} \;\; \forall \, (i,j,k,\ell), i \neq k, j \neq \ell,$$

as desired. Now, every $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ feasible to $\overline{\text{RLT1}'}$ has

$$\begin{aligned}
&\kappa + \sum_{i\neq k}\sum_{j\neq \ell}\sum_{k}\sum_{\ell}(f_{ik}-f^*)(d_{j\ell}-d^*)\hat{y}_{ijk\ell} \\
&\quad = n(1-n)d^*f^* + d^*\left(\sum_{k}\sum_{i\neq k} f_{ik}\right) + f^*\left(\sum_{\ell}\sum_{j\neq \ell} d_{j\ell}\right) + \sum_{i\neq k}\sum_{j\neq \ell}\sum_{k}\sum_{\ell}\hat{\lambda}_{ijk\ell}\hat{y}_{ijk\ell} \\
&\quad = d^*\sum_{k}\left[\sum_{i\neq k}\left(f_{ik}-\frac{f^*}{2}\right)\right] + f^*\sum_{\ell}\left[\sum_{j\neq \ell}\left(d_{j\ell}-\frac{d^*}{2}\right)\right] + \sum_{i\neq k}\sum_{j\neq \ell}\sum_{k}\sum_{\ell}\hat{\lambda}_{ijk\ell}\hat{y}_{ijk\ell} \\
&\quad = d^*\sum_{k}\left[\sum_{\ell}\sum_{i\neq k}\left(f_{ik}-\frac{f^*}{2}\right)\hat{x}_{k\ell}\right] + f^*\sum_{\ell}\left[\sum_{k}\left(\sum_{j\neq \ell} d_{j\ell}-\frac{d^*}{2}\right)\hat{x}_{k\ell}\right] \\
&\qquad + \sum_{i\neq k}\sum_{j\neq \ell}\sum_{k}\sum_{\ell}\hat{\lambda}_{ijk\ell}\hat{y}_{ijk\ell} \\
&\quad = \sum_{k}\sum_{\ell}\left[\sum_{i\neq k} d^*\left(f_{ik}-\frac{f^*}{2}\right) + \sum_{j\neq \ell} f^*\left(d_{j\ell}-\frac{d^*}{2}\right)\right]\hat{x}_{k\ell} + \sum_{i\neq k}\sum_{j\neq \ell}\sum_{k}\sum_{\ell}\hat{\lambda}_{ijk\ell}\hat{y}_{ijk\ell} \\
&\quad = \sum_{i\neq k}\sum_{j\neq \ell}\sum_{k}\sum_{\ell} f_{ik}d_{j\ell}\hat{y}_{ijk\ell},
\end{aligned}$$

establishing (2.9). Here, the first equation is by the definitions of κ and $\hat{\boldsymbol{\lambda}}$, the second and fourth equations are algebra, the third equation holds true for all $\hat{\boldsymbol{x}} \in X$, and the last equation follows from the first and third expressions of (2.8) without the $c_{k\ell}\hat{x}_{k\ell}$ terms, upon substituting $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ as defined. □

2.3 Readily Solvable Cases in Terms of RLT1

Four readily solvable cases of Problem QAP available in the literature are shown in this section to restrict the objective coefficients $C_{ijk\ell}$ in such a manner that the conditions of Theorem 2.1 and/or Theorem 2.2 are satisfied. The first three cases are presented as corollaries to Theorem 2.2, so that $\overline{\text{RLT1}'}$ (and consequently $\overline{\text{RLT1}}$) solves QAP. The fourth case is posed as a corollary to Theorem 2.1, so that $\overline{\text{RLT1}}$ solves QAP. The following four subsections examine the four cases, one case per subsection.

2.3.1 Symmetric Flows and Skew Symmetric Distances

The works of [13, 18] consider instances of the Koopmans-Beckmann form [37] of QAP described in Section 2.1, where the objective coefficients on the quadratic expressions have $C_{ijk\ell} = f_{ik}d_{j\ell}$ for all $(i, j, k, \ell), i \neq k, j \neq \ell$. For the instances of concern, the flows are symmetric, so that $f_{ik} = f_{ki}$ for all $(i, k), i < k$, and the distances are skew symmetric, so that $d_{j\ell} = -d_{\ell j}$ for all $(j, \ell), j < \ell$. (An analogous argument holds for the case where the flows are skew symmetric and the distances are symmetric.) These works show that, for such cases, Problem QAP is reducible to the assignment problem

$$\min \left\{ \sum_k \sum_\ell c_{kl}x_{kl} : \boldsymbol{x} \in X \right\}. \tag{2.10}$$

Corollary 2.1 and its proof give this result as a special case of Theorem 2.2.

Corollary 2.1

Consider a Koopmans-Beckmann instance of Problem QAP having $f_{ik} = f_{ki}$ for all $(i, k), i < k$, and $d_{j\ell} = -d_{\ell j}$ for all $(j, \ell), j < \ell$. Then the following statements are equivalent for binary $\hat{\boldsymbol{x}} \in X$.

1. $\hat{\boldsymbol{x}}$ is part of an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to $\overline{\text{RLT1}'}$,
2. $\hat{\boldsymbol{x}}$ is optimal to (2.10),
3. $\hat{\boldsymbol{x}}$ is optimal to QAP.

Proof

By Theorem 2.2, the proof is to define a $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$ so that (2.7) takes the form (2.10). Define $\hat{\boldsymbol{u}} = \boldsymbol{0}$, $\hat{\boldsymbol{v}} = \boldsymbol{0}$, $\hat{\boldsymbol{w}}$ by $\hat{w}_{ijk\ell} = f_{ik}d_{j\ell}$ for all $(i, j, k, \ell), i < k, j \neq \ell$, and $\hat{\boldsymbol{\lambda}} = \boldsymbol{0}$. Then (2.7) takes the form (2.10). Moreover, $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$ since

$$\hat{u}_{ik\ell} + \hat{v}_{jk\ell} + \hat{w}_{ijk\ell} + \hat{\lambda}_{ijk\ell} = 0 + 0 + f_{ik}d_{j\ell} + 0 = C_{ijk\ell} \;\; \forall \, (i, j, k, \ell), i < k, j \neq \ell,$$

and

$$\hat{u}_{ik\ell} + \hat{v}_{jk\ell} - \hat{w}_{k\ell ij} + \hat{\lambda}_{ijk\ell} = 0 + 0 - f_{ki}d_{\ell j} + 0 = C_{ijk\ell} \;\; \forall \; (i,j,k,\ell), i > k, j \neq \ell.$$

This completes the proof. □

Two comments are warranted. First, Theorem 2.3 allows us to generalize the distance structure of Corollary 2.1 to satisfy $d_{j\ell} - d^* = -(d_{\ell j} - d^*)$ for all $(j,\ell), j < \ell$, for any scalar d^*. Then $d_{j\ell} + d_{\ell j} = 2d^*$ for all $(j,\ell), j < \ell$, for the chosen d^*. This generalization is also obtainable from [18, pp. 113-114], since decreasing all flows f_{ik} by d^* preserves symmetry. Second, since the proof sets $\hat{\boldsymbol{u}} = \mathbf{0}$ and $\hat{\boldsymbol{v}} = \mathbf{0}$, that relaxed version of $\overline{\text{RLT1}'}$ obtained by removing equations (2.2) and (2.3) will provide an optimal $\hat{\boldsymbol{x}}$ to QAP when the specified conditions are satisfied.

2.3.2 Decomposition of the Flow or Distance

A second solvable instance of the Koopmans-Beckmann form of QAP found in [13, 18] has the $n(n-1)$ scalars f_{ik} expressed in terms of two n-vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ so that $f_{ik} = \alpha_i + \beta_k$, giving $C_{ijk\ell} = (\alpha_i + \beta_k)\, d_{j\ell}$ for all $(i,j,k,\ell), i \neq k, j \neq \ell$. (An analogous argument holds for the case where the scalars $d_{j\ell}$ can be instead expressed in terms of two such vectors.) The manuscripts [13, 18] show that, for such cases, Problem QAP is reducible to the assignment problem

$$\min \left\{ \sum_k \sum_\ell \left(c_{k\ell} + \sum_{j \neq \ell} (\beta_k d_{j\ell} + \alpha_k d_{\ell j}) \right) x_{k\ell} : \boldsymbol{x} \in X \right\}. \tag{2.11}$$

Corollary 2.2 and its proof establish this result as a special case of Theorem 2.2.

Corollary 2.2

Consider a Koopmans-Beckmann instance of Problem QAP having $f_{ik} = \alpha_i + \beta_k$ for all

$(i,k), i \neq k$. Then the following statements are equivalent for binary $\hat{\boldsymbol{x}} \in X$.

1. $\hat{\boldsymbol{x}}$ is part of an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to $\overline{\text{RLT1}'}$,

2. $\hat{\boldsymbol{x}}$ is optimal to (2.11),

3. $\hat{\boldsymbol{x}}$ is optimal to QAP.

Proof

By Theorem 2.2, the proof is to define a $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$ so that (2.7) takes the form (2.11). Define $\hat{\boldsymbol{u}} = \mathbf{0}$, $\hat{\boldsymbol{v}}$ by $\hat{v}_{jk\ell} = \beta_k d_{j\ell} + \alpha_k d_{\ell j}$ for all $(j,k,\ell), j \neq \ell$, $\hat{\boldsymbol{w}}$ by $\hat{w}_{ijk\ell} = -\alpha_k d_{\ell j} + \alpha_i d_{j\ell}$ for all $(i,j,k,\ell), i < k, j \neq \ell$, and $\hat{\boldsymbol{\lambda}} = \mathbf{0}$. Then (2.7) takes the form (2.11). Moreover, $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$ since

$$\begin{aligned}
&\hat{u}_{ik\ell} + \hat{v}_{jk\ell} + \hat{w}_{ijk\ell} + \hat{\lambda}_{ijk\ell} \\
&\qquad = 0 + (\beta_k d_{j\ell} + \alpha_k d_{\ell j}) + (-\alpha_k d_{\ell j} + \alpha_i d_{j\ell}) + 0 \\
&\qquad = \beta_k d_{j\ell} + \alpha_i d_{j\ell} = C_{ijk\ell} \qquad\qquad \forall\ (i,j,k,\ell), i < k, j \neq \ell,
\end{aligned}$$

and

$$\begin{aligned}
&\hat{u}_{ik\ell} + \hat{v}_{jk\ell} - \hat{w}_{k\ell ij} + \hat{\lambda}_{ijk\ell} \\
&\qquad = 0 + (\beta_k d_{j\ell} + \alpha_k d_{\ell j}) - (-\alpha_i d_{j\ell} + \alpha_k d_{\ell j}) + 0 \\
&\qquad = \beta_k d_{j\ell} + \alpha_i d_{j\ell} = C_{ijk\ell} \qquad\qquad \forall\ (i,j,k,\ell), i > k, j \neq \ell.
\end{aligned}$$

This completes the proof. □

Since the proof sets $\hat{\boldsymbol{u}} = \mathbf{0}$, that relaxed version of $\overline{\text{RLT1}'}$ obtained by removing equations (2.2) will provide an optimal $\hat{\boldsymbol{x}}$ to QAP when the conditions of Corollary 2.2 are satisfied. Theorem 2.3 does not generalize the conditions of Corollary 2.2 because there are no restrictions on the distances $d_{j\ell}$, and because decreasing all flows f_{ik} by some scalar f^* is equivalent to decreasing all scalars α_i by ρ and all scalars β_k by $f^* - \rho$, for any chosen scalar ρ.

2.3.3 Vector Decomposition of the Quadratic Cost Coefficients

The paper [28] considers instances of Problem QAP where the quadratic objective coefficients $C_{ijk\ell}$ allow a decomposition in terms of vectors $\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \ldots, \boldsymbol{\gamma}^{15}$ having $\boldsymbol{\gamma}^1, \ldots, \boldsymbol{\gamma}^4 \in \mathbb{R}^{n^3}, \boldsymbol{\gamma}^5, \ldots, \boldsymbol{\gamma}^{10} \in \mathbb{R}^{n^2}, \boldsymbol{\gamma}^{11}, \ldots, \boldsymbol{\gamma}^{14} \in \mathbb{R}^n$, and $\boldsymbol{\gamma}^{15} \in \mathbb{R}$ so that

$$\begin{aligned} C_{ijk\ell} =& \gamma^1_{ijk} + \gamma^2_{ij\ell} + \gamma^3_{ik\ell} + \gamma^4_{jk\ell} + \gamma^5_{ij} + \gamma^6_{ik} + \gamma^7_{i\ell} + \gamma^8_{jk} + \gamma^9_{j\ell} + \gamma^{10}_{k\ell} \\ &+ \gamma^{11}_i + \gamma^{12}_j + \gamma^{13}_k + \gamma^{14}_\ell + \gamma^{15} \\ &\forall\ (i,j,k,\ell),\ (i \neq k, j \neq \ell) \text{ or } (i = k, j = \ell). \end{aligned} \tag{2.12}$$

The subscripts on each vector $\boldsymbol{\gamma}^p$, $p \in \{1, \ldots, 14\}$, are used to denote the individual elements of the corresponding vector, and the coefficients $c_{k\ell}$ are represented by $C_{k\ell k\ell}$. The result of [28] is that, when a decomposition of the form (2.12) is possible, Problem QAP can be solved as the assignment problem

$$K + \min\left\{\sum_k \sum_\ell \psi_{k\ell} x_{k\ell} : \boldsymbol{x} \in X\right\}, \tag{2.13}$$

where

$$\psi_{k\ell} = \sum_i \left(\gamma^1_{k\ell i} + \gamma^2_{k\ell i} + \gamma^3_{ik\ell} + \gamma^4_{ik\ell}\right) + n(\gamma^5_{kl} + \gamma^{10}_{kl}), \tag{2.14}$$

and

$$K = \sum_i \sum_j \left(\gamma^6_{ij} + \gamma^7_{ij} + \gamma^8_{ij} + \gamma^9_{ij}\right) + n \sum_i \left(\gamma^{11}_i + \gamma^{12}_i + \gamma^{13}_i + \gamma^{14}_i\right) + n^2 \gamma^{15}. \tag{2.15}$$

Here, the optimal $\hat{\boldsymbol{x}}$ with objective value $\hat{z}$ solves Problem QAP, also with objective value $\hat{z}$.

Corollary 2.3 and its proof show this same result as a special case of Theorem 2.2.

Corollary 2.3

Consider an instance of Problem QAP having objective coefficients $c_{k\ell}$ and $C_{ijk\ell}$ such that

$C_{ijk\ell}$ can be expressed as in (2.12) for all (i,j,k,ℓ), $i \neq k$, $j \neq \ell$. Then the following statements are equivalent for binary $\hat{\boldsymbol{x}} \in X$.

1. $\hat{\boldsymbol{x}}$ is part of an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to $\overline{\text{RLT1}'}$,

2. $\hat{\boldsymbol{x}}$ is optimal to (2.13),

3. $\hat{\boldsymbol{x}}$ is optimal to QAP.

Proof

By Theorem 2.2, the proof is to define a $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$ so that (2.7) has the same optimal solution set as (2.13). Define $\hat{\boldsymbol{u}}$ by

$$\hat{u}_{ik\ell} = \gamma^1_{k\ell i} + \gamma^3_{ik\ell} + \gamma^5_{k\ell} + \gamma^6_{ik} + \gamma^7_{i\ell} + \gamma^{10}_{k\ell} + \gamma^{11}_i + \gamma^{13}_k + \gamma^{14}_\ell + \gamma^{15} \quad \forall\, (i,k,\ell), i \neq k,$$

define $\hat{\boldsymbol{v}}$ by

$$\hat{v}_{jk\ell} = \gamma^2_{k\ell j} + \gamma^4_{jk\ell} + \gamma^8_{jk} + \gamma^9_{j\ell} + \gamma^{12}_j \quad \forall\, (j,k,\ell), j \neq \ell,$$

define $\hat{\boldsymbol{w}}$ by

$$\hat{w}_{ijk\ell} = (\gamma^1_{ijk} - \gamma^1_{k\ell i}) + (\gamma^2_{ij\ell} - \gamma^2_{k\ell j}) + (\gamma^5_{ij} - \gamma^5_{k\ell}) \quad \forall\, (i,j,k,\ell), i < k, j \neq \ell,$$

and set $\hat{\boldsymbol{\lambda}} = \boldsymbol{0}$. Then (2.7) becomes

$$\min \left\{ \sum_k \sum_\ell \Big(c_{k\ell} + \psi_{k\ell} - \delta_{k\ell} + \phi_k + \theta_\ell \Big) x_{k\ell} : \boldsymbol{x} \in X \right\}, \tag{2.16}$$

where

$$\begin{aligned} \delta_{k\ell} =& \gamma^1_{k\ell k} + \gamma^2_{k\ell\ell} + \gamma^3_{kk\ell} + \gamma^4_{\ell k\ell} + \gamma^5_{k\ell} + \gamma^6_{kk} + \gamma^7_{k\ell} + \gamma^8_{\ell k} + \gamma^9_{\ell\ell} + \gamma^{10}_{k\ell} \\ & + \gamma^{11}_k + \gamma^{12}_\ell + \gamma^{13}_k + \gamma^{14}_\ell + \gamma^{15} \quad \forall\, (k,\ell), \end{aligned} \tag{2.17}$$

$$\phi_k = \sum_i \left(\gamma^6_{ik} + \gamma^8_{ik} + \gamma^{11}_i + \gamma^{12}_i \right) + n(\gamma^{13}_k + \gamma^{15}) \quad \forall\, k, \tag{2.18}$$

$$\theta_\ell = \sum_i \left(\gamma^7_{i\ell} + \gamma^9_{i\ell} \right) + n\gamma^{14}_\ell \quad \forall\, \ell, \tag{2.19}$$

and where $\psi_{k\ell}$ is as defined in (2.14). The assignment structure of X allows us to remove ϕ_k and θ_ℓ from within the optimization problem of (2.16), so that (2.7) has the same optimal solution set as

$$\sum_k \phi_k + \sum_\ell \theta_\ell + \min\left\{\sum_k \sum_\ell \Big(c_{k\ell} + \psi_{k\ell} - \delta_{k\ell}\Big) x_{k\ell} : \boldsymbol{x} \in X\right\}. \tag{2.20}$$

But (2.20) has the same optimal solution set as (2.13) because (2.18) and (2.19) have $\sum_k \phi_k + \sum_\ell \theta_\ell = K$, where K is as defined in (2.15), and because (2.12) and (2.17) have $C_{k\ell k\ell} = \delta_{k\ell}$ for all (k,ℓ), where the coefficients $c_{k\ell}$ are represented by $C_{k\ell k\ell}$. Thus, (2.7) has the same optimal solution set as (2.13).

To finish the proof, we must show that $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$. We have

$$\begin{aligned}
\hat{u}_{ik\ell} &+ \hat{v}_{jk\ell} + \hat{w}_{ijk\ell} + \hat{\lambda}_{ijk\ell} \\
&= \big(\gamma^1_{k\ell i} + \gamma^3_{ik\ell} + \gamma^5_{k\ell} + \gamma^6_{ik} + \gamma^7_{i\ell} + \gamma^{10}_{k\ell} + \gamma^{11}_i + \gamma^{13}_k + \gamma^{14}_\ell + \gamma^{15}\big) \\
&\quad + \big(\gamma^2_{k\ell j} + \gamma^4_{jk\ell} + \gamma^8_{jk} + \gamma^9_{j\ell} + \gamma^{12}_j\big) + \big(\big(\gamma^1_{ijk} - \gamma^1_{k\ell i}\big) + \big(\gamma^2_{ij\ell} - \gamma^2_{k\ell j}\big) + \big(\gamma^5_{ij} - \gamma^5_{k\ell}\big)\big) + 0 \\
&= \gamma^1_{ijk} + \gamma^2_{ij\ell} + \gamma^3_{ik\ell} + \gamma^4_{jk\ell} + \gamma^5_{ij} + \gamma^6_{ik} + \gamma^7_{i\ell} + \gamma^8_{jk} + \gamma^9_{j\ell} + \gamma^{10}_{k\ell} \\
&\quad + \gamma^{11}_i + \gamma^{12}_j + \gamma^{13}_k + \gamma^{14}_\ell + \gamma^{15} = C_{ijk\ell} \qquad \forall\,(i,j,k,\ell), i < k, j \neq \ell,
\end{aligned}$$

and

$$\begin{aligned}
\hat{u}_{ik\ell} &+ \hat{v}_{jk\ell} - \hat{w}_{k\ell ij} + \hat{\lambda}_{ijk\ell} \\
&= \big(\gamma^1_{k\ell i} + \gamma^3_{ik\ell} + \gamma^5_{k\ell} + \gamma^6_{ik} + \gamma^7_{i\ell} + \gamma^{10}_{k\ell} + \gamma^{11}_i + \gamma^{13}_k + \gamma^{14}_\ell + \gamma^{15}\big) \\
&\quad + \big(\gamma^2_{k\ell j} + \gamma^4_{jk\ell} + \gamma^8_{jk} + \gamma^9_{j\ell} + \gamma^{12}_j\big) - \big(\big(\gamma^1_{k\ell i} - \gamma^1_{ijk}\big) + \big(\gamma^2_{k\ell j} - \gamma^2_{ij\ell}\big) + \big(\gamma^5_{k\ell} - \gamma^5_{ij}\big)\big) + 0 \\
&= \gamma^1_{ijk} + \gamma^2_{ij\ell} + \gamma^3_{ik\ell} + \gamma^4_{jk\ell} + \gamma^5_{ij} + \gamma^6_{ik} + \gamma^7_{i\ell} + \gamma^8_{jk} + \gamma^9_{j\ell} + \gamma^{10}_{k\ell} \\
&\quad + \gamma^{11}_i + \gamma^{12}_j + \gamma^{13}_k + \gamma^{14}_\ell + \gamma^{15} = C_{ijk\ell} \qquad \forall\,(i,j,k,\ell), i > k, j \neq \ell,
\end{aligned}$$

where the first equality in each set is due to substitution, the second is algebraic, and the third follows from (2.12). This completes the proof. □

The conditions of Theorem 2.2 have two advantages over those of (2.12). First, in terms of size, it is much simpler to check whether there exists a $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\lambda}) \in S^0$ than to check feasibility to (2.12). Conditions (2.12) comprise a linear system of $n^2(n^2 - 2n + 2)$ equations in $4n^3+6n^2+4n+1$ variables. In contrast, the set S^0 is a linear system of $n^2(n-1)^2$ equations in $2n^2(n-1) + \frac{n^2(n-1)^2}{2}$ variables, with these two values representing the number of variables $y_{ijk\ell}$ present within RLT1 and the number of constraints in (2.2)–(2.4), respectively. However, the observation of Section 2.2 that reduces, within RLT1, the number of variables $\boldsymbol{y}$ to $\frac{n^2(n-1)^2}{2}$ and the number of equality constraints involving $\boldsymbol{y}$ to $n(n-1)(2n-1)$ allows us to reduce the associated set S^0 to $\frac{n^2(n-1)^2}{2}$ equations in $n(n-1)(2n-1)$ variables, which is significantly smaller than (2.12) in terms of both equations and variables. The second advantage of Theorem 2.2 over conditions (2.12) is that the former, in comparison to the latter, identifies a richer family of instances of Problem QAP that can be solved as an assignment problem. To illustrate this second advantage, consider the example below.

Example 1

Consider an instance of Problem QAP having $n = 4$, and having $C_{1234} = C_{3214} = 1$ and $C_{1432} = C_{3412} = -1$, and all other objective coefficients equal to 0. The reader can verify that there exists no solution to (2.12). Yet, define $\hat{\boldsymbol{u}} = \boldsymbol{0}$, $\hat{\boldsymbol{v}} = \boldsymbol{0}$, $\hat{\boldsymbol{\lambda}} = \boldsymbol{0}$, and $\hat{\boldsymbol{w}} = \boldsymbol{0}$, with the exceptions that $\hat{w}_{1234} = 1$ and $\hat{w}_{1432} = -1$, to obtain that $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S^0$ so that the conditions of Theorem 2.2 are satisfied. Here, every binary $\hat{\boldsymbol{x}} \in X$ is optimal to Problem QAP and part of an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to Problem $\overline{\text{RLT1}'}$, with objective value 0.

Interestingly, conditions (2.12) can be interpreted in terms of the dual space to a suitably defined linear program, and this interpretation leads to an alternate reduction based on the removal of superfluous variables and constraints. Unlike the richer conditions of Theorem 2.2, however, the reduced system identifies the same set of solvable instances of Problem QAP as does (2.12), because consistency of either system implies consistency to the other. The first reduction is to discard all variables $\boldsymbol{\gamma}^j$ for $j \in \{5, \ldots, 15\}$. The second reduction is to remove all equations with (i, j, k, ℓ) having $i = k$, $j = \ell$. To elaborate, in a similar manner to the way in which the set S of (2.6) was motivated in terms of the dual space to $\overline{\text{RLT1}}$, conditions (2.12) can be motivated so that each $\boldsymbol{\gamma}^j$ corresponds to a designated family of constraints, and so that each constraint corresponds to a primal variable $y_{ijk\ell}$. The constraints associated with $\boldsymbol{\gamma}^j$ for $j \in \{5, \ldots, 15\}$ turn out to be redundant, making the associated variables unnecessary. The equations associated with $y_{ijk\ell}, i = k, j = \ell$, are also unnecessary, as each such $y_{k\ell k\ell}$ simplifies to $x_{k\ell}$. (The details of the motivating linear program and the identification of redundant constraints are described in the appendix.) Upon removing the redundant restrictions and unnecessary variables $y_{ijk\ell}$ from this larger program, the resulting form is the following representation of Problem QAP due to Frieze and Yadegar [29], as presented below.

$$
\begin{aligned}
\text{FY:} \quad & \text{minimize} \quad \sum_{k} \sum_{\ell} c_{k\ell} x_{k\ell} + \sum_{i \neq k} \sum_{j \neq \ell} \sum_{k} \sum_{\ell} C_{ijk\ell} y_{ijk\ell} \\
& \text{subject to} \quad \sum_{\ell \neq j} y_{ijk\ell} = x_{ij} \qquad \forall\, (i, j, k),\ i \neq k \qquad (2.21) \\
& \qquad \sum_{k \neq i} y_{ijk\ell} = x_{ij} \qquad \forall\, (i, j, \ell),\ j \neq \ell \qquad (2.22) \\
& \qquad \sum_{j \neq \ell} y_{ijk\ell} = x_{k\ell} \qquad \forall\, (i, k, \ell),\ i \neq k \\
& \qquad \sum_{i \neq k} y_{ijk\ell} = x_{k\ell} \qquad \forall\, (j, k, \ell),\ j \neq \ell \\
& \qquad y_{ijk\ell} \geq 0 \qquad \forall\, (i, j, k, \ell),\ i \neq k,\ j \neq \ell \\
& \qquad \boldsymbol{x} \in X,\ \boldsymbol{x} \text{ binary}
\end{aligned}
$$

Let $\overline{\text{FY}}$ denote the continuous relaxation of Problem FY obtained by removing the $\boldsymbol{x}$ binary restrictions, and let $\overline{\text{FY}'}$ denote the relaxation of $\overline{\text{FY}}$ obtained by removing the nonnegativity restrictions on $\boldsymbol{y}$ in $\overline{\text{FY}}$. The appendix establishes that the dual to $\overline{\text{FY}'}$ has a solution if and only if (2.12) has a solution. Then, since $\overline{\text{FY}'}$ has $4n^2(n-1)$ equations in the $n^2(n-1)^2$ variables $\boldsymbol{y}$, system (2.12) can be thereby reduced to $n^2(n-1)^2$ equations in $4n^2(n-1)$ variables, as the dual constraints associated with the variables $\boldsymbol{x}$ do not affect dual feasibility. Of significance here is that the system remains larger than the earlier-described reduced version of S^0, which has $\frac{n^2(n-1)^2}{2}$ equations in $n(n-1)(2n-1)$ variables.

Notably, $\overline{\text{RLT1}'}$ identifies a richer family of solvable instances of QAP than $\overline{\text{FY}'}$, and $\overline{\text{RLT1}}$ identifies a richer family of solvable instances of QAP than $\overline{\text{FY}}$. This dominance follows from a result of [2] which shows that each of the constraints found within (2.21) and (2.22) is implied by (2.2)–(2.4). Specifically, given any $(i,j,k), i \neq k$, summing the equation $\sum_{\ell \neq j} y_{k\ell ij} = x_{ij}$ of (2.2) with the $(n-1)$ equations $y_{ijk\ell} - y_{k\ell ij} = 0$, $\ell \neq j$, of (2.4) gives the equation $\sum_{\ell \neq j} y_{ijk\ell} = x_{ij}$ of (2.21). Similarly, given any $(i,j,\ell), j \neq \ell$, summing the equation $\sum_{k \neq i} y_{k\ell ij} = x_{ij}$ of (2.3) with the $(n-1)$ equations $y_{ijk\ell} - y_{k\ell ij} = 0, k \neq i$, of (2.4) gives the equation $\sum_{k \neq i} y_{ijk\ell} = x_{ij}$ of (2.22). Relative to strict dominance, Example 1 shows that it is possible for $\overline{\text{RLT1}'}$ to solve QAP when $\overline{\text{FY}'}$ does not, and a modified example of [2], presented below as Example 2, shows that it is possible for $\overline{\text{RLT1}}$ to solve QAP when $\overline{\text{FY}}$ does not.

Example 2

Consider an instance of Problem QAP having $n = 4$, and with objective coefficients $c_{k\ell}$ and $C_{ijk\ell}$ defined so that $C_{1233} = C_{1443} = C_{2243} = C_{3112} = C_{4312} = C_{3314} = C_{4114} = C_{3322} = C_{4122} = C_{3124} = C_{4324} = C_{1431} = C_{2231} = C_{2433} = C_{1241} = C_{2441} = C_{1422} = C_{2214} = C_{3341} = C_{4133} = C_{1224} = C_{2412} = C_{3143} = C_{4331} = 0$, and all other coefficients equal to 1. The conditions of Theorem 2.2 cannot be satisfied so that (2.12) also cannot be satisfied, making Problems $\overline{\text{RLT1}'}$ and $\overline{\text{FY}'}$ unbounded. However, an optimal $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ to $\overline{\text{FY}}$ is given by $\hat{x}_{12} = \hat{x}_{14} = \hat{x}_{22} = \hat{x}_{24} = \hat{x}_{31} = \hat{x}_{33} = \hat{x}_{41} = \hat{x}_{43} = \frac{1}{2}$, $\hat{y}_{ijk\ell} = \frac{1}{2}$ for all $(i,j,k,\ell), i \neq k, j \neq \ell$, having $C_{ijk\ell} = 0$, and all other variables $\hat{x}_{k\ell}$ and $\hat{y}_{ijk\ell}$ equal to 0. The optimal objective value is 4. For this same problem, an optimal $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ to $\overline{\text{RLT1}}$ has $\bar{x}_{14} = \bar{x}_{22} = \bar{x}_{31} = \bar{x}_{43} = 1$, all other $\bar{x}_{ij} = 0$, and $\bar{y}_{ijk\ell} = \bar{x}_{ij}\bar{x}_{k\ell}$ for all (i,j,k,ℓ), $i \neq k$, $j \neq \ell$. The optimal objective value is 8.

Before progressing to the next section, we note here that conditions (2.12) of Corollary 2.3 subsume the conditions of Corollary 2.2, but not those of Corollary 2.1. Given that the conditions of Corollary 2.2 are satisfied so that $C_{ijk\ell} = (\alpha_i+\beta_k)d_{j\ell}$ for all $(i,j,k,\ell), i \neq k$, $j \neq \ell$, we can set $\gamma^2_{ij\ell} = \alpha_i d_{j\ell}$ for all (i,j,ℓ), $j \neq \ell$, set $\gamma^4_{jk\ell} = \beta_k d_{j\ell}$ for all (j,k,ℓ), $j \neq \ell$, and set all other $\boldsymbol{\gamma}^p = \mathbf{0}$ to satisfy Corollary 2.3. However, Example 1 satisfies the conditions of Corollary 2.1 with $f_{13} = f_{31} = d_{24} = 1$, $d_{42} = -1$, and all other f_{ik} and $d_{j\ell}$ equal 0 but, as mentioned within Example 1, there exists no solution to (2.12).

2.3.4 Flow or Distance Reduction

Motivated by the *chr18b* instance found in the problem test bed of [16], the paper [27] gives a set of conditions that are sufficient for identifying an optimal solution to a Koopmans-Beckmann instance of Problem QAP. We restate these conditions as Corollary 2.4 below, and show how these conditions follow from Theorem 2.1.

Corollary 2.4

Consider a Koopmans-Beckmann instance of Problem QAP having $c_{k\ell} = 0$ for all (k,ℓ), and let $0 = \min_{(i,k), i\neq k}\{f_{ik}\}$ and $d' = \min_{(j,\ell), j\neq \ell}\{d_{j\ell}\}$ denote the minimum flow and distance, respectively. If there exists a binary $\hat{\boldsymbol{x}} \in X$ such that for each $(i,j,k,\ell), i \neq k, j \neq \ell$, with $\hat{x}_{ij} = \hat{x}_{k\ell} = 1$, either $f_{ik} = 0$ or $d_{j\ell} = d'$, then $\hat{\boldsymbol{x}}$ is optimal to QAP.

Proof

Suppose that there exists a binary point $\hat{\boldsymbol{x}}$ satisfying the given conditions. By Theorem 2.1, it is sufficient to define a $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S$ so that $\hat{\boldsymbol{x}}$ is optimal to (2.7) and so that $\hat{\lambda}_{ijk\ell} = 0$ for all $(i,j,k,\ell), i \neq k, j \neq \ell$, having $\hat{x}_{ij} = \hat{x}_{k\ell} = 1$. Define $\hat{\boldsymbol{v}} = \mathbf{0}$, $\hat{\boldsymbol{w}} = \mathbf{0}$, define $\hat{\boldsymbol{u}}$ by

$$\hat{u}_{ik\ell} = f_{ik} d' \qquad \forall\ (i,k,\ell), i \neq k,$$

and subsequently define $\hat{\boldsymbol{\lambda}}$ in terms of $\hat{\boldsymbol{u}}$ by

$$\hat{\lambda}_{ijk\ell} = C_{ijk\ell} - \hat{u}_{ik\ell} = f_{ik}(d_{j\ell} - d') \quad \forall\, (i,j,k,\ell), i \neq k, j \neq \ell,$$

so that $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}})$ satisfies the equality restrictions in S of (2.6). By definition of d', we have $\hat{\boldsymbol{\lambda}} \geq \mathbf{0}$, so that $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\lambda}}) \in S$. Also, for each $(i,j,k,\ell), i \neq k, j \neq \ell$, with $\hat{x}_{ij} = \hat{x}_{k\ell} = 1$, we have $\hat{\lambda}_{ijk\ell} = 0$ by the premise that either $f_{ik} = 0$ or $d_{j\ell} = d'$. Finally, for the given $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}})$, the linear program (2.7) becomes

$$\min \left\{ \sum_k \sum_\ell \left(\sum_{i \neq k} f_{ik} d' \right) x_{k\ell} : \boldsymbol{x} \in X \right\}. \tag{2.23}$$

The point $\hat{\boldsymbol{x}}$ is primal feasible, and the values $(\hat{\boldsymbol{\pi}}^1, \hat{\boldsymbol{\pi}}^2, \hat{\boldsymbol{\pi}}^3)$ given by $\hat{\pi}^1_k = \sum_{i \neq k} f_{ik} d'$ for all k, $\hat{\boldsymbol{\pi}}^2 = \mathbf{0}$, and $\hat{\boldsymbol{\pi}}^3 = \mathbf{0}$ form a complementary dual solution. Thus, $\hat{\boldsymbol{x}}$ is optimal to (2.7), and the proof is complete. □

From a graph perspective, [27] points out that for those cases in which a "flow graph" of the positive f_{ik} realizes a path structure, and a "distance graph" of the $d_{j\ell}$ has a special grid structure, then the conditions of Corollary 2.4 are satisfied.

Theorem 2.3 allows an interesting generalization of the flow structure of Corollary 2.4. Since Theorem 2.3 allows us to subtract any scalar f^* from all the flows f_{ik} without changing the set of optimal solutions, it is not necessary in Corollary 2.4 to require that $0 = \min_{(i,k), i \neq k} \{f_{ik}\}$; any scalar f' will suffice in place of 0. A formal statement of this generalization is presented as a variant of Corollary 2.4 below.

Corollary 2.4 (generalized)

Consider a Koopmans-Beckmann instance of Problem QAP having $c_{k\ell} = 0$ for all (k,ℓ), and let $f' = \min_{(i,k), i \neq k} \{f_{ik}\}$ and $d' = \min_{(j,\ell), j \neq \ell} \{d_{j\ell}\}$ denote the minimum flow and distance, respectively. If there exists a binary $\hat{\boldsymbol{x}} \in X$ such that for each $(i,j,k,\ell), i \neq k, j \neq \ell$, with

$\hat{x}_{ij} = \hat{x}_{k\ell} = 1$, either $f_{ik} = f'$ or $d_{j\ell} = d'$, then $\hat{\boldsymbol{x}}$ is optimal to QAP.

Observe that the earlier-referenced result [18, pp. 113-114] allowing all flows and distances to be changed by the same scalar, without altering the set of optimal solutions, can also be used to obtain this generalization, since adding the scalar f' to all distances $d_{j\ell}$ does not change the index pairs (j, ℓ) having the minimum distance.

2.4 Conclusions

While having a large number of applications in various fields of study, the quadratic assignment problem has proven itself very difficult to solve. General solution procedures are limited to approximately $n = 30$ facilities. As a result, researchers have focused attention on special objective structures that admit more readily (polynomially) solvable instances. This chapter shows how four such structures can be explained in terms of Problem RLT1, the level-1 RLT representation. Problem RLT1 is polynomial in size in terms of n and, as is to be expected since otherwise we would have $P = NP$, the continuous relaxation, referred to as $\overline{\text{RLT1}}$, has fractional extreme points, in addition to n-factorial binary such points. The theoretical foundation of this effort, found within Theorems 2.1 and 2.2, uses the KKT conditions to establish necessary and sufficient conditions for $\overline{\text{RLT1}}$, and also a relaxation $\overline{\text{RLT1}'}$, to have an optimal binary solution.

The four special cases, found within [13, 18, 27, 28], are explained in terms of Corollaries 2.1, 2.2, and 2.3 of Theorem 2.2 and Corollary 2.4 of Theorem 2.1, with one corollary devoted to each case. These corollaries collectively show that $\overline{\text{RLT1}}$ subsumes all cases in the following sense; if the conditions of any of these cases are satisfied, then $\overline{\text{RLT1}}$ will have an optimal binary solution. In fact, the first three cases found within Corollaries 2.1 through 2.3 show the stronger result that $\overline{\text{RLT1}'}$ will have an optimal binary solution. Notably, however, $\overline{\text{RLT1}}$ identifies a richer family of solvable instances than do these corollaries. Specifically, Example 2 shows that $\overline{\text{RLT1}}$ can yield an optimal binary solution to QAP when (2.12) has no solution, so that the conditions of Corollary 2.3 cannot be satisfied. In addition, the conditions of none of Corollaries 2.1, 2.2, or 2.4 can be satisfied, as each considers Koopmans-Beckmann instances. Example 2 cannot be placed in Koopmans-Beckmann form, as there exist no $f_{13}, f_{14}, d_{23}, d_{24}$ having $C_{1234} = f_{13}d_{24} = 1$, $C_{1243} = f_{14}d_{23} = 1$, and $C_{1233} = f_{13}d_{23} = 0$.

This chapter makes two other notable contributions. First, Theorem 2.3 shows that the optimal solution set for Koopmans-Beckmann instances is invariant under certain objective transformations, allowing a straightforward generalization of the conditions of Corollaries 2.1 and 2.4. Second, conditions (2.12) used in the third case are simplified.

Avenues for further research arise. Our results show that boundedness of the linear program $\overline{\text{RLT1}'}$ resulting from the level-1 RLT approach is a sufficient condition for an instance of Problem QAP to be linearizable. We conjecture that this same condition is also necessary, an idea that is explored in Chapter 3. We are also investigating the utility of the level-1 and higher-level RLT forms for establishing new, or independently verifying other known, sets of sufficiency conditions for polynomially solvable instances of Problem QAP.

2.5 Appendix

Conditions (2.12) can be motivated in terms of a mixed 0-1 linear reformulation of Problem QAP in such a manner that each $\boldsymbol{\gamma}^j$ corresponds to a distinct family of constraints to a suitable relaxation. A reduction of this formulation leads to a known mixed 0-1 linear form of QAP, as well as to a simpler version of (2.12).

Consider Problem LP1 that is defined in the same $(\boldsymbol{x}, \boldsymbol{y})$ variable space as Problem RLT1, with the following exception. Unlike RLT1, the variables $y_{ijk\ell}$ having $i = k$ and $j = \ell$ are found within LP1 so that, for each (k, ℓ), $y_{k\ell k\ell}$ denotes the product $x_{k\ell}x_{k\ell}$. In this manner, each $y_{ijk\ell}$ in LP1 corresponds to a "compatible" four-tuple (i, j, k, ℓ) as defined in [28] which, as pointed out below, in turn corresponds to an equation of (2.12). Specifically, and though not explicitly stated as such for simplicity, only those variables $y_{ijk\ell}$ having either $(i \neq k, j \neq \ell)$ or $(i = k, j = \ell)$ are assumed present in LP1. (An equivalence between Problems QAP and LP1 is observed at the end of this section.)

$$\text{LP1:} \quad \text{minimize} \quad \sum_i \sum_j \sum_k \sum_\ell C_{ijk\ell} y_{ijk\ell}$$

$$\text{subject to} \quad \sum_\ell y_{ijk\ell} = x_{ij} \qquad \forall\ (i, j, k) \tag{2.24a}$$

$$\sum_k y_{ijk\ell} = x_{ij} \qquad \forall\ (i, j, \ell) \tag{2.24b}$$

$$\sum_j y_{ijk\ell} = x_{k\ell} \qquad \forall\ (i, k, \ell) \tag{2.24c}$$

$$\sum_i y_{ijk\ell} = x_{k\ell} \qquad \forall\ (j, k, \ell) \tag{2.24d}$$

$$\sum_k \sum_\ell y_{ijk\ell} = n x_{ij} \qquad \forall\ (i, j) \tag{2.24e}$$

$$\sum_j \sum_\ell y_{ijk\ell} = 1 \qquad \forall\ (i, k) \tag{2.24f}$$

$$\sum_j \sum_k y_{ijk\ell} = 1 \qquad \forall\ (i, \ell) \tag{2.24g}$$

$$\sum_i \sum_\ell y_{ijk\ell} = 1 \qquad \forall\ (j,k) \tag{2.24h}$$

$$\sum_i \sum_k y_{ijk\ell} = 1 \qquad \forall\ (j,\ell) \tag{2.24i}$$

$$\sum_i \sum_j y_{ijk\ell} = n x_{k\ell} \qquad \forall\ (k,\ell) \tag{2.24j}$$

$$\sum_j \sum_k \sum_\ell y_{ijk\ell} = n \qquad \forall\ i \tag{2.24k}$$

$$\sum_i \sum_k \sum_\ell y_{ijk\ell} = n \qquad \forall\ j \tag{2.24l}$$

$$\sum_i \sum_j \sum_\ell y_{ijk\ell} = n \qquad \forall\ k \tag{2.24m}$$

$$\sum_i \sum_j \sum_k y_{ijk\ell} = n \qquad \forall\ \ell \tag{2.24n}$$

$$\sum_i \sum_j \sum_k \sum_\ell y_{ijk\ell} = n^2 \tag{2.24o}$$

$$y_{ijk\ell} \geq 0 \qquad \forall\ (i,j,k,\ell) \tag{2.24p}$$

$$\boldsymbol{x} \in X,\ \boldsymbol{x} \text{ binary} \tag{2.24q}$$

Analogous to the way in which Problem $\overline{\text{RLT1}}$ was constructed from RLT1 by removing the $\boldsymbol{x}$ binary restrictions, and the way in which Problem $\overline{\text{RLT1}'}$ was constructed from $\overline{\text{RLT1}}$ by removing the $\boldsymbol{y} \geq \boldsymbol{0}$ inequalities, construct Problem $\overline{\text{LP1}}$ from LP1 by removing the $\boldsymbol{x}$ binary restrictions of (2.24q), and construct Problem $\overline{\text{LP1}'}$ from $\overline{\text{LP1}}$ by removing the $\boldsymbol{y} \geq \boldsymbol{0}$ inequalities of (2.24p). Then conditions (2.12) are the dual equations corresponding to the variables $y_{ijk\ell}$ of $\overline{\text{LP1}'}$, where equations (2.24a)–(2.24o) are assigned dual multipliers $\boldsymbol{\gamma}^1 - \boldsymbol{\gamma}^{15}$, respectively. Consequently, the dual to $\overline{\text{LP1}'}$ has a solution if and only if (2.12) has a solution, as the $\boldsymbol{x} \in X$ constraints of (2.24q) allow dual feasibility with respect to the variables $\boldsymbol{x}$ regardless of the $\boldsymbol{\gamma}^1 - \boldsymbol{\gamma}^{15}$ values.

Problems LP1, $\overline{\text{LP1}}$, and $\overline{\text{LP1}'}$ contain redundant constraints and extraneous variables, allowing the formulations to be reduced in size. Relative to constraints, in the presence of the restrictions $\boldsymbol{x} \in X$ of (2.24q), constraints (2.24a)–(2.24c) imply (2.24e)–(2.24o), and so these latter eleven families of restrictions are redundant in each of these three problems. Observe that for fixed (i, j), summing (2.24a) over the n values of k gives equation (2.24e) while for fixed (k, ℓ), summing (2.24c) over the n values of i gives equation (2.24j). For fixed (i, k), summing (2.24a) over j gives (2.24f) while for fixed (j, k), summing (2.24a) over i gives (2.24h). Similarly, for fixed (i, ℓ), summing (2.24b) over j gives (2.24g) while for fixed (j, ℓ), summing (2.24b) over i gives (2.24i). Continuing, for each i, summing (2.24a) over j and k gives (2.24k) while, for each j, summing (2.24b) over i and ℓ gives (2.24l). For each k, summing (2.24a) over i and j gives (2.24m) while, for each ℓ, summing (2.24b) over i and j gives (2.24n). Finally, summing (2.24a) over all i, j, and k gives (2.24o).

Relative to extraneous variables, all $y_{k\ell k\ell}$ can be eliminated from these three problems, as every such variable appears in exactly one constraint of each of (2.24a)–(2.24d), and each of the four associated constraints individually sets $y_{k\ell k\ell} = x_{k\ell}$. Thus, we can also remove all $4n^2$ such constraints from these formulations.

The reduced versions of Problems LP1, $\overline{\text{LP1}}$, and $\overline{\text{LP1}'}$ that are obtained by removing all redundant constraints and extraneous variables are referred to in Section 2.3.3 as Problems FY, $\overline{\text{FY}}$, and $\overline{\text{FY}'}$, respectively. As the dual to $\overline{\text{LP1}'}$ clearly has a solution if and only if the dual to $\overline{\text{FY}'}$ has a solution, we can use our above observation that the dual to $\overline{\text{LP1}'}$ has a solution if and only if (2.12) has a solution to conclude that the dual to $\overline{\text{FY}'}$ has a solution if and only if (2.12) has a solution. Also of interest is that Problem FY was earlier shown [29] to be a valid representation of Problem QAP, giving us that Problem LP1 is a valid form.

Chapter 3

Characterizing Linearizable QAPs by the Level-1 RLT

3.1 Introduction

The quadratic assignment problem (qap) can be formulated as

$$\text{P: minimize}\left\{\sum_k\sum_\ell c_{k\ell}x_{k\ell}+\sum_i\sum_j\sum_{k\neq i}\sum_{\ell\neq j}C_{ijk\ell}x_{ij}x_{k\ell}:\boldsymbol{x}\in\boldsymbol{X},\ \boldsymbol{x}\text{ binary}\right\},$$

where

$$\boldsymbol{X}\equiv\left\{\boldsymbol{x}\geq\boldsymbol{0}:\sum_j x_{ij}=1\ \forall\, i,\ \sum_i x_{ij}=1\ \forall\, j\right\} \tag{3.1}$$

is the assignment set. Note that this formulation is equivalent to Problem QAP of Chapter 2 and, like in the previous chapter, we assume that all indices and summations run from 1 to n unless otherwise noted. Since it is desired to minimize a quadratic objective function over the set $\boldsymbol{X}$, the problem name results. The qap is NP-hard, and first arose in a facility location scenario [37]. For this problem, it is desired to situate n facilities on n location sites, with each site housing exactly one facility, in such a manner as to minimize the combined cost of construction and material flow. Here, $C_{ijk\ell}=f_{ik}d_{j\ell}$ to represent the product of the flow f_{ik} between pairs of facilities i and k with the distance $d_{j\ell}$ between pairs of location sites j and ℓ. Then each coefficient $C_{ijk\ell}$ represents the cost of material flow between facilities i and k, given that facility i is located on site j *and* facility k is located on site ℓ. Each coefficient $c_{k\ell}$ represents a construction cost for situating facility k on site ℓ. This version of P is the Koopmans-Beckmann form. More general forms that do not have $C_{ijk\ell} = f_{ik}d_{j\ell}$ have also been studied. The qap boasts many applications [17, 24, 26, 30, 38, 52, 53], with surveys in [14, 18, 42, 45]. The most efficient solution strategies [1, 8, 33, 34] are limited to problems having $n \leq 30$.

Due to the problem difficulty, a strategy for solving Problem P has been to seek out special objective function structures that allow optimal solutions to be obtained in polynomial time. For Koopmans-Beckmann forms, various works [13, 15, 19, 20, 22, 27, 40] exploit specific flow and distance structures for this purpose. Other contributions [21] provide variations of Problem P that can be solved in either polynomial or pseudo-polynomial time. Of particular interest in this study is the identification of objective coefficients that allow P to be transformed into an equivalent *linear* assignment problem; such problems are referred to as being *linearizable*. Formally stated, a size n instance of Problem P with objective coefficients $c_{k\ell}$ and $C_{ijk\ell}$ is defined to be *linearizable* if there exists a scalar κ and coefficients $\hat{c}_{k\ell}$ so that

$$\kappa + \sum_k \sum_\ell \hat{c}_{k\ell} x_{k\ell} = \sum_k \sum_\ell c_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k \neq i} \sum_{\ell \neq j} C_{ijk\ell} x_{ij} x_{k\ell} \text{ for all } \boldsymbol{x} \in \boldsymbol{X},\ \boldsymbol{x} \text{ binary.} \tag{3.2}$$

Clearly, every instance of P that is expressible in the form (3.2) is polynomially solvable, because each is reducible to a linear assignment problem. Although not explicitly stated in these terms, the works [13, 18, 28] essentially provide sufficient conditions for recognizing such instances of Problem P. The paper [35] extends this body of work by providing conditions that are both necessary and sufficient, and also gives an $O(n^4)$ algorithm for checking whether these conditions are satisfied. The paper [46] shows that, for Koopmans-Beckmann forms, the conditions can be checked in $O(n^2)$ time.

The primary contribution of this chapter is to give a different, polyhedral-based necessary and sufficient condition for identifying an instance of Problem P as linearizable and, in the process, to merge the notion of linearizable with the continuous relaxation of a known mixed 0-1 linear reformulation of the problem. The linear form is a relaxed version of the level-1 (reformulation-linearization-technique) RLT representation of Problem P, as introduced in [2]. The paper [7] shows that the sufficient conditions of [13, 18, 28] are subsumed by this representation, but no mention is made as to necessity. Specifically, we show that the qap is linearizable if and only if the dual to this relaxed version has a solution. Because this version is always feasible, we have that the qap is linearizable if and only if it is bounded. A consequence of this study is that the level-1 RLT identifies a richer family of solvable instances of the qap than those which are linearizable in the sense that an optimal binary solution will exist to the relaxed linear program for every linearizable instance, but the program can have a binary solution without the qap being linearizable.

This chapter is organized as follows. In the next section, we briefly summarize the level-1 RLT form, present a more compact representation that is obtained via a substitution of variables, and provide a roadmap for identifying a maximal set of linearly independent equations that is implied by the resulting constraints. Section 3.3 identifies the linearly independent equations, with the identification applying the RLT process to a basis of the assignment polytope $\boldsymbol{X}$ of (3.1), and also constructing a second set of equations having a network substructure, to blend with equations defining $\boldsymbol{X}$. Section 3.4 provides the main result, showing that an instance of the qap is linearizable if and only if a relaxed version of the level-1 RLT form is bounded. This is accomplished using two steps. The first step shows that all equations that are valid for the level-1 RLT form are implied by the linearly independent equations of Section 3.3, verifying this set as maximal. The second step then establishes the relationship to (3.2). A numeric example is also given to show that the relaxed level-1 RLT can provide a binary solution for instances of the qap that are not linearizable. A consequence of the first step of Section 3.4 is the characterization of the dimensions of the level-1 RLT and various relaxations. These dimensions are studied in

Section 3.5, and Section 3.6 provides concluding remarks.

3.2 Level-1 RLT and Overview of Structure

In this section, we review the level-1 RLT form of Problem P, and provide a roadmap for equivalently rewriting the constraints in a form that is more amenable to our study. The details of this rewrite are presented in Sections 3.3 and 3.4, with Section 3.3 identifying an implied set of linearly independent (LI) equations and Section 3.4 showing this set to be maximal.

3.2.1 Level-1 RLT Form

The RLT methodology was introduced in [4, 5, 6] to reformulate linearly constrained quadratic 0-1 optimization problems into mixed-binary linear programs that afford a tight linear programming relaxation. It was later extended (see [3, 50, 49, 51] and their references) into a broader theory for computing polyhedral outer-approximations of general discrete and nonconvex sets. Given a mixed-discrete optimization problem, the RLT constructs a hierarchy of successively tighter linear programming approximations, culminating at the highest level with a linear program whose feasible region gives an explicit algebraic description of the convex hull of feasible solutions. The levels are obtained using the two steps of *reformulation* and *linearization*. For mixed-binary programs, the reformulation step multiplies "product factors" of the problem variables with the constraints, and enforces the binary identity that $x^2 = x$ for binary x. The "linearization" step then substitutes a continuous variable for each resulting product term.

Relative to the qap, various authors [1, 33, 34] have reported success in using different RLT levels to compute bounds within branch-and-bound schemes. Our specific interest here is with the level-1 form, as detailed in [2] and given below.

$$\text{RLT1:} \quad \text{minimize} \quad \sum_{k}\sum_{\ell} c_{k\ell}x_{k\ell} + \sum_{i}\sum_{j}\sum_{k\neq i}\sum_{\ell\neq j} C_{ijk\ell}y_{ijk\ell}$$

$$\text{subject to} \quad \sum_{j\neq\ell} y_{ijk\ell} = x_{k\ell} \qquad \forall\ (i,k,\ell),\ i\neq k \tag{3.3}$$

$$\sum_{i\neq k} y_{ijk\ell} = x_{k\ell} \qquad \forall\ (j,k,\ell),\ j\neq \ell \tag{3.4}$$

$$y_{ijk\ell} = y_{k\ell ij} \qquad \forall\ (i,j,k,\ell),\ i<k,\ j\neq \ell \tag{3.5}$$

$$y_{ijk\ell} \geq 0 \qquad \forall\ (i,j,k,\ell),\ i\neq k,\ j\neq \ell \tag{3.6}$$

$$\boldsymbol{x} \text{ binary},\ \boldsymbol{x}\in\boldsymbol{X} \tag{3.7}$$

Problem RLT1 derives from Problem P as follows. The reformulation step multiplies every constraint defining $\boldsymbol{X}$ in (3.1), including the $\boldsymbol{x} \geq \boldsymbol{0}$ inequalities, by each binary variable $x_{k\ell}$. Then $x_{k\ell} = x_{k\ell}x_{k\ell}$ is substituted for each (k,ℓ). The structure of $\boldsymbol{X}$ enforces that $x_{ij}x_{k\ell} = 0$ for all (i,j,k,ℓ) with $i = k, j \neq \ell$, or with $i \neq k, j = \ell$. For each remaining $x_{ij}x_{k\ell}$ product, the linearization step then substitutes a continuous variable $y_{ijk\ell}$. Specifically, for every (k,ℓ), the $n-1$ equations found within each of (3.3) and (3.4) are computed by multiplying the restrictions $\sum_j x_{ij} = 1\ \forall\ i$ and $\sum_i x_{ij} = 1\ \forall\ j$ of $\boldsymbol{X}$ by the variable $x_{k\ell}$, respectively. Equations (3.5) recognize that $x_{ij}x_{k\ell} = x_{k\ell}x_{ij}$ for all $(i,j,k,\ell), i<k, j\neq\ell$. Inequalities (3.6) result from multiplying the $\boldsymbol{x} \geq \boldsymbol{0}$ inequalities of $\boldsymbol{X}$ by the variables $x_{k\ell}$, upon discarding the products set to 0.

The RLT theory gives us that Problems P and RLT1 are equivalent in that an optimal solution to either problem yields an optimal solution to the other. This result [2] follows because, for every $\hat{\boldsymbol{x}} \in \boldsymbol{X}$, $\hat{\boldsymbol{x}}$ binary, a point $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is feasible to constraints (3.3)–(3.6) if and only if $\hat{y}_{ijk\ell} = \hat{x}_{ij}\hat{x}_{k\ell}$ for all $(i,j,k,\ell), i\neq k, j\neq\ell$.

As suggested in [2], we can use (3.5) to remove, via substitution, all variables $y_{ijk\ell}$ having $i > k, j \neq \ell$, from Problem RLT1, and then discard constraints (3.5). The below formulation results, where $C'_{ijk\ell} = C_{ijk\ell} + C_{k\ell ij}$ for all $(i,j,k,\ell), i<k, j\neq\ell$.

$$\text{RLT1}': \quad \text{minimize} \quad \sum_{k}\sum_{\ell} c_{k\ell}x_{k\ell} + \sum_{i}\sum_{j}\sum_{k>i}\sum_{\ell\neq j} C'_{ijk\ell}y_{ijk\ell}$$

$$\text{subject to} \quad \sum_{j\neq\ell} y_{ijk\ell} = x_{k\ell} \qquad \forall\,(i,k,\ell),\ i<k \tag{3.8}$$

$$\sum_{j\neq\ell} y_{k\ell ij} = x_{k\ell} \qquad \forall\,(i,k,\ell),\ i>k \tag{3.9}$$

$$\sum_{i<k} y_{ijk\ell} + \sum_{i>k} y_{k\ell ij} = x_{k\ell} \qquad \forall\,(j,k,\ell),\ j\neq\ell \tag{3.10}$$

$$y_{ijk\ell} \geq 0 \qquad \forall\,(i,j,k,\ell),\ i<k,\ j\neq\ell \tag{3.11}$$

$$\boldsymbol{x} \text{ binary},\ \boldsymbol{x}\in\boldsymbol{X} \tag{3.12}$$

Here, equations (3.3) become (3.8) and (3.9), and equations (3.4) become (3.10). Note that this formulation is different from the Problem RLT1$'$ of Chapter 2.

Observe the difference in size between Problems RLT1 and RLT1$'$. Problem RLT1 has n^2 variables $\boldsymbol{x}$ and $n^2(n-1)^2$ variables $\boldsymbol{y}$, for a total of $n^2(n^2-2n+2)$. Relative to constraints, RLT1 has $2n^2(n-1)$ equations in (3.3) and (3.4), $\frac{n^2(n-1)^2}{2}$ equations in (3.5), and $2n$ equations in $\boldsymbol{X}$ of (3.7), for a total of $\frac{n^2(n-1)(n+3)}{2}+2n$, in addition to the nonnegativity restrictions on $\boldsymbol{x}$ and $\boldsymbol{y}$. Due to the variable substitution, Problem RLT1$'$ has $\frac{n^2(n-1)^2}{2}$ fewer variables and constraints than does RLT1, in addition to half the number of nonnegativity restrictions in (3.6) found in (3.11). Specifically,

$$\text{RLT1}' \text{ has } n^2 + \frac{n^2(n-1)^2}{2} \text{ variables and } 2n^2(n-1)+2n \text{ equations.} \tag{3.13}$$

Problem RLT1$'$ and a variant introduced in [7] that is obtained by removing the nonnegativity restrictions on $\boldsymbol{y}$ in (3.11), call this variant RLT1$''$, will be of importance throughout this study. For future reference, we let $\overline{\text{RLT1}}'$ and $\overline{\text{RLT1}}''$ be the linear programming relaxations of Problems RLT1$'$ and RLT1$''$, respectively, that are obtained by removing the binary restrictions on $\boldsymbol{x}$.

3.2.2 Roadmap for Exploiting RLT Structure

In this section, we present a roadmap for identifying a maximal set of LI equations that is implied by (3.8)–(3.10) and the equations of $\boldsymbol{X}$. This identification is the key first step for establishing a necessary and sufficient condition for an instance of the qap to be linearizable and, as a consequence, for characterizing the dimension of the the convex hull of feasible solutions to RLT1$'$, as well as that of the feasible regions to $\overline{\text{RLT1}}'$ and $\overline{\text{RLT1}}''$. The roadmap is followed in Sections 3.3 and 3.4.

By the RLT process, restrictions (3.8)–(3.10) and the equations of $\boldsymbol{X}$ form a linearly dependent (LD) set. Certain redundant constraints have been pointed out in prior works [7, 47] for RLT1, and these dependencies immediately transfer to RLT1$'$. The paper [7] eliminates $n(n-1)+1$ equations by observing that any single equation in $\boldsymbol{X}$ is redundant, so that the $n(n-1)$ equations present in either (3.3) or (3.4), which are computed by multiplying the chosen equation with each variable $x_{k\ell}$, are also redundant. In a different study, the paper [47] explains that either the first or second set of n equations of $\boldsymbol{X}$ is implied, so that the selected n equations can be removed. We show in the next two sections that RLT1$'$ contains a maximal set of $2(n-1)^3+n(n-1)$ LI equations, so that $3n^2-3n+2$ of the $2n^2(n-1)+2n$ total equations reported in (3.13) are implied. For future reference, this number of LI equations is computed by

$$\Big(2(n-1)^3+n(n-1)\Big) = \Big(2n^2(n-1)+2n\Big) - \Big(3n^2-3n+2\Big), \tag{3.14}$$

with the right side of the expression subtracting the number of LD equations from the total. In particular, Section 3.3 provides this number of LI constraints that are implied by the equations of RLT1$'$, and Section 3.4 shows the number to be maximal.

To illustrate our approach pictorially, consider the linear system of equations in matrix form found in Figure 3.1 below, which is expressed in terms of the parameters m_1, m_2, m_3 and n_1, n_2, n_3, n_4 as given, where $m_1 = n_2$ and $m_2 = n_3$, with $n_1+n_2+n_3+n_4 = n^2 + \frac{n^2(n-1)^2}{2}$ equalling the number of variables in RLT1$'$ as noted in (3.13), and with

$m_1 + m_2 + m_3 = 2(n-1)^3 + n(n-1)$, found in the left expression of (3.14). As indicated by the multiplication, the first set of n_1 variables corresponds to $\boldsymbol{x}$ while the last three sets of $n_2 + n_3 + n_4$ variables comprise a partition of $\boldsymbol{y}$ into $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, and $\boldsymbol{y}_3$. The first $m_1 + m_2$ equations are constructed to be implied by (3.8)–(3.10), and the equations of $\boldsymbol{X}$, so as to satisfy the following properties: the first m_1 equations associate the set of n_2 variables $\boldsymbol{y}_1$ with an identity matrix $\boldsymbol{I}_{m_1}$ of size m_1, and the second m_2 equations contain none of these variables $\boldsymbol{y}_1$, but associate an invertible matrix $\boldsymbol{E}$ with the set of n_3 variables $\boldsymbol{y}_2$. The final m_3 equations represent any selection of $2n-1$ equality constraints found in $\boldsymbol{X}$ of (3.1). The last set of n_4 variables $\boldsymbol{y}_3$ are those variables $\boldsymbol{y}$ not present in either $\boldsymbol{y}_1$ or $\boldsymbol{y}_2$.

$$\begin{array}{c} m_1\{ \\ m_2\{ \\ m_3\{ \end{array} \left[\begin{array}{c:c:c:c} \overbrace{\boldsymbol{A}}^{n_1} & \overbrace{\boldsymbol{I}_{m_1}}^{n_2} & \overbrace{\boldsymbol{B}}^{n_3} & \overbrace{\boldsymbol{C}}^{n_4} \\ \hdashline \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{E} & \boldsymbol{F} \\ \hdashline \boldsymbol{G} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right] \left[\begin{array}{c} \boldsymbol{x} \\ \hdashline \boldsymbol{y}_1 \\ \hdashline \boldsymbol{y}_2 \\ \hdashline \boldsymbol{y}_3 \end{array}\right] = \left[\begin{array}{c} \boldsymbol{0} \\ \hdashline \boldsymbol{0} \\ \hdashline \boldsymbol{1} \end{array}\right]$$

with

- $m_1 = 2(n-1)^3$, $m_2 = (n-1)(n-2)-1$, $m_3 = 2n-1$
- $n_1 = n^2$, $n_2 = 2(n-1)^3$, $n_3 = (n-1)(n-2)-1$, $n_4 = \frac{(n-3)(n-2)(n-1)n}{2} + 1$

Figure 3.1: Linearly independent equations implied by $\overline{\text{RLT1}}'$.

Sections 3.3 and 3.4 relate to Figure 3.1 as follows. Section 3.3 details the system formation from the constraints of RLT1$'$, emphasizing the construction of the identity $\boldsymbol{I}_{m_1}$ and the invertibility of $\boldsymbol{E}$. Since the last $m_3 = 2n-1$ equations include no variables $\boldsymbol{y}$, and since $\boldsymbol{G}$ is of full row rank, we will then have that the $m_1+m_2+m_3$ equations are LI. Section 3.4 uses this LI set to establish the desired necessary and sufficient linearizable condition. In the process, it shows that the coefficients $\boldsymbol{\beta}$ on every equation $\boldsymbol{\alpha x} + \boldsymbol{\beta y} = \kappa$ that is valid for RLT1$'$ can be computed as a linear combination of the first $m_1 + m_2$ equations of the matrix as a stepping stone to proving that the $m_1+m_2+m_3$ LI equations of Figure 3.1 are maximal for $\overline{\text{RLT1}}'$.

3.3 Identification of Linearly Independent Equations

In this section, and consistent with equation (3.14) and Figure 3.1, we identify $2(n-1)^3 + n(n-1) = m_1 + m_2 + m_3$ LI equations that are implied by the equations of RLT1$'$. The first set of m_1 equations and the second set of m_2 equations are considered in Subsections 3.3.1 and 3.3.2, respectively. In these subsections, we emphasize the partitioning of the variables $\boldsymbol{y}$ to obtain $\boldsymbol{I}_{m_1}$ and the invertible matrix $\boldsymbol{E}$. Subsection 3.3.3 combines these equations with the additional $m_3 = 2n-1$ LI equations associated with $\boldsymbol{G}$ to obtain the independence of all equations.

3.3.1 First Set of Linearly Independent Equations

To identify the first set of $m_1 = 2(n-1)^3$ LI equations, we exploit a property of the RLT process relative to elementary row operations. Specifically, given a linear system of equations, every equation that can be computed by first performing elementary row operations on the system, and then applying the RLT, can likewise be computed by first applying the RLT, and then performing elementary row operations. In light of this property, we rewrite the set $\boldsymbol{X}$ of (3.1) by eliminating any single (redundant) equation, and by expressing the remaining $2n-1$ equations so that the variables $(x_{1n}, x_{2n}, \ldots, x_{nn}, x_{n1}, x_{n2} \ldots, x_{n(n-1)})$ serve as a basis. The below set $\boldsymbol{X}'$ results.

$$\boldsymbol{X}' \equiv \left\{ \begin{array}{ll} \boldsymbol{x} \geq \boldsymbol{0}: & x_{in} = 1 - \sum_{j<n} x_{ij} \ \forall\, i < n, \\ & x_{nj} = 1 - \sum_{i<n} x_{ij} \ \forall\, j < n, \\ & x_{nn} = \sum_{i<n} \sum_{j<n} x_{ij} + (2-n) \end{array} \right\} \tag{3.15}$$

The above-stated property assures that every restriction computed by multiplying an equation of $\boldsymbol{X}'$ by a variable $x_{k\ell}$, and then performing the substitution $x_{k\ell}x_{k\ell} = x_{k\ell}$ (again enforcing $x_{ij}x_{k\ell} = 0$ for all (i, j, k, ℓ) with $i = k, j \neq \ell$, or with $i \neq k, j = \ell$), together with the linearization operation that includes removing all variables $y_{ijk\ell}$ having $i > k, j \neq \ell$, is implied by (3.8)–(3.10).

Bearing this property in mind, compute the $m_1 = 2(n-1)^3$ equations (3.16)–(3.20) below in the following manner. For each $i < n$, multiply the equation defining x_{in} by each $x_{k\ell}, k \neq i, k < n, \ell < n$, to obtain the $\frac{(n-1)^2(n-2)}{2}$ equations found in each of (3.16) and (3.17), depending on whether $i < k$ or $i > k$. For each $j < n$, multiply the equation defining the variable x_{nj} by each $x_{k\ell}, \ell \neq j, k < n, \ell < n$, to obtain the $(n-1)^2(n-2)$ equations (3.18). Multiply the equation defining x_{nn} by each $x_{k\ell}, k < n, \ell < n$, to obtain the $(n-1)^2$ equations (3.19). Finally, for each $i < n$, multiply the equation defining x_{in} by each $x_{n\ell}, \ell < n$, to obtain the $(n-1)^2$ equations (3.20).

$$y_{ink\ell} = x_{k\ell} - \sum_{j \neq \ell, n} y_{ijk\ell} \quad \forall\, (i, k, \ell),\ i < k < n, \ell < n \tag{3.16}$$

$$y_{k\ell in} = x_{k\ell} - \sum_{j \neq \ell, n} y_{k\ell ij} \quad \forall\, (i, k, \ell),\ k < i < n, \ell < n \tag{3.17}$$

$$y_{k\ell nj} = x_{k\ell} - \sum_{i<k} y_{ijk\ell} - \sum_{k<i<n} y_{k\ell ij} \quad \forall\, (j, k, \ell),\ j \neq \ell,\ j < n,\ k < n,\ \ell < n \tag{3.18}$$

$$y_{k\ell nn} = \sum_{i<k} \sum_{j \neq \ell, n} y_{ijk\ell} + \sum_{k<i<n} \sum_{j \neq \ell, n} y_{k\ell ij} + x_{k\ell}(3-n) \quad \forall\, (k, \ell),\ k < n,\ \ell < n \tag{3.19}$$

$$y_{inn\ell} = x_{n\ell} - \sum_{j \neq \ell, n} y_{ijn\ell} \quad \forall\, (i, \ell),\ i < n,\ \ell < n \tag{3.20}$$

Consider the Lemma below.

Lemma 3.1

Equations (3.16)–(3.20) form a LI set.

Proof

Every equation in (3.16)–(3.18) contains a single variable $y_{ijk\ell}$ having exactly one of the indices j, k, ℓ equal to n, and these $2(n-1)^2(n-2)$ variables are distinct. Thus, (3.16)–

(3.18) form a LI set. Every equation in (3.19) and (3.20) contains a single variable $y_{ijk\ell}$ having exactly two of the indices j, k, ℓ equal to n, and these $2(n-1)^2$ variables are distinct. Moreover, none of these variables appear within (3.16)–(3.18). Thus, (3.16)–(3.18), together with (3.19) and (3.20), form a LI set. □

Observe the relationship between (3.16)–(3.20) and the first m_1 equations of the matrix of Figure 3.1. The matrix $\boldsymbol{I}_{m_1}$ corresponds to those variables $y_{ijk\ell}$ in RLT1′ such that

$$y_{ijk\ell} \text{ has at least one index } j, k, \ell \text{ equal to } n. \tag{3.21}$$

A count of these variables is $2(n-1)^2(n-2)$ with a single such subscript and $2(n-1)^2$ with two such subscripts, for a total of m_1. The matrix $\boldsymbol{A}$ corresponds to the variables $\boldsymbol{x}$ found within (3.16)–(3.20), while the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ correspond to the variables $y_{ijk\ell}$ having no index j, k, ℓ equal to n. (These matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are further described in Subsection 3.3.2 following Theorem 3.1.)

3.3.2 Second Set of Linearly Independent Equations

Similar to the first set of m_1 equations, the second set of $m_2 = (n-1)(n-2) - 1$ equations is implied by (3.8)–(3.10) and the equations of $\boldsymbol{X}$. In addition, the $m_1 + m_2$ equations combine to form a LI set. The explanations follow.

Consider the $\frac{(n-1)(n-2)}{2}$ *linear* equations found in each of (3.22) and (3.23) below, where the notation $\{\cdot\}_L$ is used to denote the previously defined RLT linearization operation that substitutes a continuous variable $y_{ijk\ell}, i < k, j \neq \ell$, for every occurrence of the quadratic term $x_{ij}x_{k\ell}$ (or equivalently $x_{k\ell}x_{ij}$).

$$\left\{\left(1-\sum_{i<n} x_{ij}\right)\left(1-\sum_{i<n} x_{i\ell}\right)\right\}_L = 0 \ \ \forall\,(j,\ell), j<\ell<n \tag{3.22}$$

$$\left\{\left(1-\sum_{j<n} x_{ij}\right)\left(1-\sum_{j<n} x_{kj}\right)\right\}_L = 0 \ \ \forall\,(i,k), i<k<n \tag{3.23}$$

Lemma 3.2 below establishes that these equations are implied by the equality constraints of RLT1′.

Lemma 3.2

The $(n-1)(n-2)$ equations (3.22) and (3.23) are implied by (3.8)–(3.10) and the equations of $\boldsymbol{X}$.

Proof

Consider any $(j,\ell), j<\ell<n$, where the restriction $\{x_{nj}x_{n\ell}\}_L = 0$ is enforced, as in the RLT process applied to Problem P when computing (3.8)–(3.10). We have that

$$\begin{aligned}
\left\{\left(1-\sum_{i<n} x_{ij}\right)\left(1-\sum_{i<n} x_{i\ell}\right)\right\}_L &= \left\{\left(1-\sum_{i<n} x_{ij}\right)\left(1-\sum_{i} x_{i\ell}+x_{n\ell}\right)\right\}_L \\
&= \left\{\left(1-\sum_{i<n} x_{ij}\right)\left(1-\sum_{i} x_{i\ell}\right)\right\}_L + \left\{x_{n\ell}\left(1-\sum_{i} x_{ij}\right)\right\}_L \\
&= \left(1-\sum_{i} x_{i\ell}\right) - \sum_{i<n}\left\{x_{ij}\left(1-\sum_{i} x_{i\ell}\right)\right\}_L \\
&\quad + \left\{x_{n\ell}\left(1-\sum_{i} x_{ij}\right)\right\}_L
\end{aligned}$$

where the second equation follows from $\{x_{nj}x_{n\ell}\}_L = 0$. The $n+1$ equations $(1-\sum_i x_{i\ell}) = 0$, $\left\{x_{ij}\left(1-\sum_i x_{i\ell}\right)\right\}_L = 0 \ \forall\, i<n$, and $\left\{x_{n\ell}\left(1-\sum_i x_{ij}\right)\right\}_L = 0$ appear in (3.8)–(3.10) and $\boldsymbol{X}$, so the associated equation in (3.22) is implied by (3.8)–(3.10) and the equations of $\boldsymbol{X}$.

Similarly, consider any $(i,k), i<k<n$, where the restriction $\{x_{in}x_{kn}\}_L = 0$ is

enforced. We have that

$$\begin{aligned}
\left\{\left(1-\sum_{j<n} x_{ij}\right)\left(1-\sum_{j<n} x_{kj}\right)\right\}_L &= \left\{\left(1-\sum_{j<n} x_{ij}\right)\left(1-\sum_{j} x_{kj}+x_{kn}\right)\right\}_L \\
&= \left\{\left(1-\sum_{j<n} x_{ij}\right)\left(1-\sum_{j} x_{kj}\right)\right\}_L + \left\{x_{kn}\left(1-\sum_{j} x_{ij}\right)\right\}_L \\
&= \left(1-\sum_{j} x_{kj}\right) - \sum_{j<n}\left\{x_{ij}\left(1-\sum_{j} x_{kj}\right)\right\}_L \\
&\quad + \left\{x_{kn}\left(1-\sum_{j} x_{ij}\right)\right\}_L
\end{aligned}$$

where the second equation follows from $\{x_{in}x_{kn}\}_L = 0$. The $n+1$ equations $\left(1-\sum_j x_{kj}\right) = 0$, $\left\{x_{ij}\left(1-\sum_j x_{kj}\right)\right\}_L = 0 \; \forall\, j<n$, and $\left\{x_{kn}\left(1-\sum_j x_{ij}\right)\right\}_L = 0$ appear in (3.8)–(3.10) and $\boldsymbol{X}$, so the associated equation in (3.23) is implied by (3.8)–(3.10) and the equations of $\boldsymbol{X}$. This completes the proof. □

Equations (3.22) and (3.23) written in terms of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ of Problem RLT1′ result in (3.24) and (3.25) below, respectively, since $x_{ij}x_{i\ell} = 0$ for all $(i,j,\ell), i<n, j<\ell<n$, and $x_{ij}x_{kj} = 0$ for all $(i,j,k), j<n, i<k<n$.

$$1-\sum_{i<n}(x_{ij}+x_{i\ell})+\sum_{i=1}^{n-2}\sum_{k=i+1}^{n-1}(y_{ijk\ell}+y_{i\ell kj}) = 0 \;\; \forall\,(j,\ell), j<\ell<n \tag{3.24}$$

$$1-\sum_{j<n}(x_{ij}+x_{kj})+\sum_{j=1}^{n-2}\sum_{\ell=j+1}^{n-1}(y_{ijk\ell}+y_{i\ell kj}) = 0 \;\; \forall\,(i,k), i<k<n \tag{3.25}$$

Equations (3.24) and (3.25) reveal two useful structures of (3.22) and (3.23). First, each of the $\frac{(n-1)^2(n-2)^2}{2}$ variables $y_{ijk\ell}$ found within (3.24) and (3.25) appears once with coefficient 1 in (3.24) and once with coefficient 1 in (3.25). Consequently, relative to the variables $y_{ijk\ell}$, multiplying every equation in (3.25) by the scalar -1 yields a complete bipartite network consisting of arcs directed from the $\frac{(n-1)(n-2)}{2}$ nodes associated with equations

(3.24) to the $\frac{(n-1)(n-2)}{2}$ nodes associated with equations (3.25). Second, a variable $y_{ijk\ell}$ appears in an equation of (3.24) and (3.25) if and only if the variable $y_{i\ell kj}$ also appears in that same equation. Thus, within the graph, each arc is effectively duplicated. Figure 3.2 illustrates the bipartite graph, where the left and right columns of nodes are labeled with the ordered pairs $(j, \ell), j < \ell$, and $(i, k), i < k$, consistent with equations (3.24) and (3.25), respectively, and where an arc incident with nodes (j, ℓ) and (i, k) represents the variables $y_{ijk\ell}$ and $y_{i\ell kj}$. For simplicity, all arcs are drawn undirected.

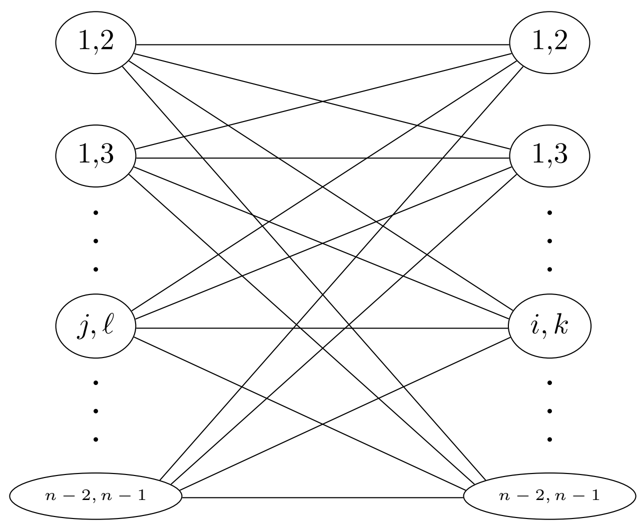

Figure 3.2: Complete, bipartite network motivated by equations (3.24) and (3.25).

Consider Lemma 3.3 below.

Lemma 3.3

Every selection of $(n-1)(n-2)-1$ equations from (3.24) and (3.25) (equivalently (3.22) and (3.23)) forms a LI set.

Proof

Since the coefficients on the variables $y_{ijk\ell}$ in equations (3.24) and (3.25) form the node-arc incidence matrix of a connected graph, the removal of any single equation ensures linear independence of the remaining $(n-1)(n-2)-1$ equations. $\square$

Now, recall that Lemma 3.1 of Subsection 3.3.1 established that the $m_1 = 2(n-1)^3$ equations (3.16)–(3.20) form a LI set, while Lemma 3.3 showed that every selection of $(n-1)(n-2)-1$ equations from (3.24) and (3.25) forms a LI set. We show in Theorem 3.1 below that (3.16)–(3.20), together with any $(n-1)(n-2)-1$ equations from (3.24) and (3.25), combine to form a LI set.

Theorem 3.1

The $m_1 = 2(n-1)^3$ equations of (3.16)–(3.20), together with any selection of $m_2 = (n-1)(n-2)-1$ equations from (3.24) and (3.25), form a LI set.

Proof

Lemma 3.1 established that the m_1 equations (3.16)–(3.20) form a LI set while Lemma 3.3 showed that any collection of m_2 equations from (3.24) and (3.25) also forms a LI set. Each of the m_1 variables $y_{ijk\ell}$ associated with the matrix $\boldsymbol{I}_{m_1}$ of Figure 3.1 was observed in (3.21) to have at least one of the indices j, k, ℓ equal to n. None of these variables appear in any of the equations (3.24) and (3.25), completing the proof. $\square$

Lemma 3.3 and Theorem 3.1 follow the roadmap of Figure 3.1 in the sense that they define the matrices $\boldsymbol{D}$, $\boldsymbol{0}$, $\boldsymbol{E}$, and $\boldsymbol{F}$ found within the second set of m_2 equations, and also characterize the second and third sets of n_3 and n_4 variables $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$, respectively. As a result, the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ found within the first set of m_1 equations can be more precisely defined. Specifically, Lemma 3.3 states that we can select any $m_2 = (n-1)(n-2)-1$ equations from (3.24) and (3.25) to form the second set of LI equations within Figure 3.1. Furthermore, from within these equations, we can define the invertible matrix $\boldsymbol{E}$ in terms of any collection of n_3 LI variables $\boldsymbol{y}_2$, and the matrices $\boldsymbol{D}$ and $\boldsymbol{F}$ to represent the coefficients on the variables $\boldsymbol{x}$ and the coefficients on the remaining n_4 variables $\boldsymbol{y}_3$, respectively. The matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are then accordingly defined to coincide with $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$. The proof of Theorem 3.1 identifies the $m_2 \times n_2$ matrix $\mathbf{0}$, so as to conclude that the first $m_1 + m_2$ equations are LI.

For future reference, relative to the selection of the n_3 variables $\boldsymbol{y}_2$ to associate with the matrix $\boldsymbol{E}$, we choose those $y_{ijk\ell}$ taking either of the two forms

$$y_{(n-2),j,(n-1)\ell} \text{ with } j < \ell < n \quad \text{or} \quad y_{i(n-2)k(n-1)} \text{ with } i < k < n. \tag{3.26}$$

Figure 3.3 graphically depicts these variables, using the same sets of nodes as Figure 3.2, and a subset of the arcs. Unlike Figure 3.2, each arc of Figure 3.3 represents that single variable $y_{ijk\ell}$ having $j < \ell$ as noted in (3.26). In this manner, the last set of variables $\boldsymbol{y}_3$ in Figure 3.1 consists of those $n_4 = \frac{(n-1)^2(n-2)^2}{2} - \Big((n-1)(n-2) - 1\Big)$ variables $y_{ijk\ell}$ found within (3.24) and (3.25), but not represented in Figure 3.3. The specific selection of m_2 equations from (3.24) and (3.25) is not important to this study, as the deleted equation will be shown implied by the $m_1 + m_2 + m_3$ remaining equations.

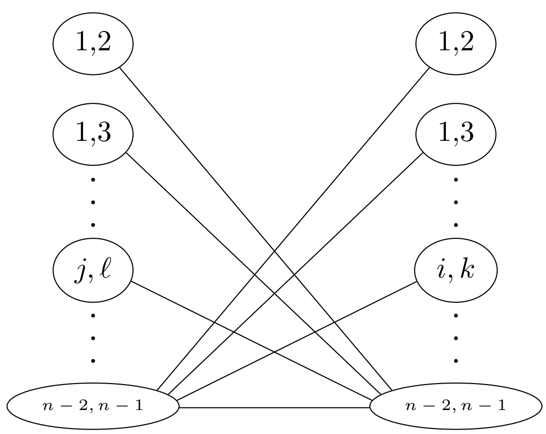

Figure 3.3: Set of n_3 basic variables from Figure 3.2 associated with matrix $\boldsymbol{E}$ of Figure 3.1.

3.3.3 Maximal Set of Linearly Independent Equations

Theorem 3.1 shows that the first $m_1 + m_2$ equations in the matrix of Figure 3.1, as identified in Subsections 3.3.1 and 3.3.2, form a LI set. It readily follows that the $2n - 1$

equations of $\boldsymbol{X}'$ in (3.15), which define the third set of m_3 equations and the matrix $\boldsymbol{G}$, continue to yield a LI set when augmented with these $m_1 + m_2$ equations. This result is stated formally in Theorem 3.2 below.

Theorem 3.2

The $m_1 = 2(n-1)^3$ equations of (3.16)–(3.20), together with any selection of $m_2 = (n-1)(n-2) - 1$ equations of (3.24) and (3.25), and the $m_3 = 2n - 1$ equations of $\boldsymbol{X}'$ in (3.15), form a LI set.

Proof

The basis $\left(x_{1n}, x_{2n}, \cdots, x_{nn}, x_{n1}, x_{n2}, \cdots, x_{n(n-1)}\right)$ of $\boldsymbol{X}'$ in (3.15) has the equations defining $\boldsymbol{X}'$ to be LI. Theorem 3.1 established the linear independence of the first $m_1 + m_2$ equations. No variables $\boldsymbol{x}$ appear in the matrices $\boldsymbol{I}_{m_1}$ or $\boldsymbol{E}$ of Figure 3.1, completing the proof. $\square$

Theorem 3.2 shows that the $m_1 + m_2 + m_3$ equations associated with Figure 3.1 form a LI set, but it does not prove that this set is maximal for $\overline{\text{RLT1}}'$; that is, Theorem 3.2 does not show that *every* equation found in (3.8)–(3.10) is implied. Section 3.4 proves the stronger result that every linear equation which is valid for the convex hull of feasible solutions to RLT1$'$ is implied by these same $m_1 + m_2 + m_3$ equations. It then uses this result to establish a condition that is both necessary and sufficient for Problem P to be linearizable.

3.4 Necessary and Sufficient Condition for Linearizable

Section 3.3 set the stage for our establishing a polyhedral-based necessary and sufficient condition for Problem P to be linearizable. In this section, we show that an instance of Problem P is linearizable if and only if the linear program $\overline{\text{RLT1}}''$ is bounded. Our argument is a two-step approach, with each of Subsections 3.4.1 and 3.4.2 devoted to a step. Following the discussion at the end of Subsection 3.2.2, the first step is to prove that, given any equation of the form $\boldsymbol{\alpha x} + \boldsymbol{\beta y} = \kappa$ that is satisfied by all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to RLT1$'$, there exists a linear combination of the $m_1 + m_2 + m_3$ equations of Figure 3.1 that yields this equation. The second step uses this result to establish a correspondence between the existence of a dual solution to $\overline{\text{RLT1}}''$ and the desired necessary and sufficient condition for Problem P to be linearizable.

3.4.1 Characterizing all Equations Valid for Problem RLT1$'$

Consistent with the LI constraints identified in Section 3.3 and depicted in Figure 3.1, we partition the variables $y_{ijk\ell}$ from Problem RLT1$'$ into two sets $\boldsymbol{y}_B$ and $\boldsymbol{y}_N$. The set $\boldsymbol{y}_B$ consists of the $2(n-1)^3 + (n-1)(n-2) - 1$ variables $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, and the set $\boldsymbol{y}_N$ consists of the $\frac{(n-3)(n-2)(n-1)n}{2} + 1$ variables $\boldsymbol{y}_3$. Accordingly, we partition the index set (i, j, k, ℓ) of the variables $y_{ijk\ell}$ appearing in RLT1$'$ in terms of $\boldsymbol{y}_B$ and $\boldsymbol{y}_N$ so that

$$I_B = \{(i, j, k, \ell) : y_{ijk\ell} \in \boldsymbol{y}_B\} \text{ and } I_N = \{(i, j, k, \ell) : y_{ijk\ell} \in \boldsymbol{y}_N\}. \tag{3.27}$$

Consider the following Theorem.

Theorem 3.3

Every equation of the form

$$\sum_k \sum_\ell \alpha_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} \beta_{ijk\ell} y_{ijk\ell} = \kappa \tag{3.28}$$

that is satisfied for all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to RLT1$'$ can be computed as a linear combination of the $m_1 + m_2 + m_3$ equations of Figure 3.1.

Proof

Consider any equation of the form (3.28) that is satisfied by all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to RLT1$'$. As

$\boldsymbol{I}_{m_1}$ and $\boldsymbol{E}$ are invertible matrices, the first $m_1 + m_2$ equations of Figure 3.1 imply a unique equation of the form

$$\sum_k \sum_\ell \hat{\alpha}_{k\ell} x_{k\ell} + \sum_{(i,j,k,\ell)\in I_B} \beta_{ijk\ell} y_{ijk\ell} + \sum_{(i,j,k,\ell)\in I_N} \hat{\beta}_{ijk\ell} y_{ijk\ell} = 0 \tag{3.29}$$

that is satisfied by all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to RLT1$'$, where the coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B$ are as given in (3.28).

The proof consists of two parts that combine to compute (3.28) as a linear combination of (3.29) and the m_3 equations $\boldsymbol{Gx} = \boldsymbol{1}$ of Figure 3.1. The first part shows that the $\beta_{ijk\ell}$ of (3.28) and $\hat{\beta}_{ijk\ell}$ of (3.29) must have $\beta_{ijk\ell} = \hat{\beta}_{ijk\ell}$ for all $(i,j,k,\ell) \in I_N$. Then (3.29) will simplify to

$$\sum_k \sum_\ell \hat{\alpha}_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} \beta_{ijk\ell} y_{ijk\ell} = 0. \tag{3.30}$$

This simplification is accomplished by showing that, within any equation of the form (3.28) that is satisfied by all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to RLT1$'$, the coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_N$ are uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B$. The second part of the proof shows that there exist multiples $\boldsymbol{\lambda} \in \mathbb{R}^{m_3}$ of the m_3 equations $\boldsymbol{Gx} = \boldsymbol{1}$ of Figure 3.1 that give the equation

$$\sum_k \sum_\ell (\alpha_{k\ell} - \hat{\alpha}_{k\ell}) x_{k\ell} = \kappa. \tag{3.31}$$

The sum of (3.30) and (3.31) is (3.28), so that (3.28) can be computed as a linear combination of the $m_1 + m_2 + m_3$ equations of Figure 3.1.

The two parts of the proof follow.

1. As the constraints of RLT1$'$ enforce $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, $y_{ijk\ell} = x_{ij}x_{k\ell}$ for all $(i,j,k,\ell), i < k, j \neq \ell$, the first part of the proof is to show that, given an equation of the form

$$\sum_k \sum_\ell \alpha_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} \beta_{ijk\ell} x_{ij} x_{k\ell} = \kappa \tag{3.32}$$

that is satisfied for all $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, the coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_N$ are uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B$. As a first step, observe that each of the $n!$ points satisfying $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, gives rise to an equation of the form (3.32) in the coefficients $\alpha_{k\ell}$ and $\beta_{ijk\ell}$, and scalar κ. Moreover, given any (p,q), the $(n-1)!$ solutions to $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, having $x_{pq} = 1$ similarly give rise to $(n-1)!$ equations of the form (3.32), and these equations are precisely those that result from a size $n-1$ instance of Problem P in the variables $x_{k\ell}, k \neq p, \ell \neq q$, with the $\alpha_{k\ell}$ and κ suitably adjusted. Consistent with (3.21) and (3.26), define the index sets I_B^{pq} and I_N^{pq} in the spirit of (3.27), but this time in terms of the reduced variable set. When (p,q) has $p \leq n-3, q \leq n-3$, then

$$I_B^{pq} = \{(i,j,k,\ell), i,k \neq p \text{ and } j,\ell \neq q : y_{ijk\ell} \in \boldsymbol{y}_B\} \tag{3.33}$$

and

$$I_N^{pq} = \{(i,j,k,\ell), i,k \neq p \text{ and } j,\ell \neq q : y_{ijk\ell} \in \boldsymbol{y}_N\} \tag{3.34}$$

because, for such cases, we have that $I_B^{pq} \subset I_B$ and $I_N^{pq} \subset I_N$. Using (3.33) and (3.34), the proof of the first part is by induction on the size n of RLT1$'$, with the base cases having $n = 3$, $n = 4$, and $n = 5$ found in the Appendix. For the inductive step, consider an instance of RLT1$'$ having $n \geq 6$, and suppose that the result holds true for all size $n-1$ instances.

Arbitrarily select a coefficient β_{rstu} with $(r,s,t,u) \in I_N$, and note that there exists a variable x_{pq} having $p \leq n-3, p \neq r, p \neq t$, and $q \leq n-3, q \neq s, q \neq u$. By the inductive hypothesis, the coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_N^{pq}$ are uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B^{pq}$. Therefore, since $(r,s,t,u) \in I_N^{pq}$ and $I_B^{pq} \subset I_B$, we have that β_{rstu} is uniquely defined in terms of those coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B$. This completes the inductive argument.

2. Since all extreme points of the assignment polytope $\boldsymbol{X}$ are binary, equation (3.31) holds true for all $\boldsymbol{x} \in \boldsymbol{X}$. In particular, since $\tilde{\boldsymbol{x}}$ defined by $\tilde{x}_{ij} = \frac{1}{n}$ for all (i,j) is in $\boldsymbol{X}$, every $\boldsymbol{d} \in \mathbb{R}^{n^2}$ satisfying $\boldsymbol{Gd} = \boldsymbol{0}$ must have $\sum_k \sum_\ell (\alpha_{k\ell} - \hat{\alpha}_{k\ell}) d_{k\ell} = 0$ because, for sufficiently small $\epsilon > 0$, $\tilde{\boldsymbol{x}} + \epsilon \boldsymbol{d} \in \boldsymbol{X}$ and must therefore satisfy (3.31). Then no solution exists to $\boldsymbol{Gd} = \boldsymbol{0}$ having $\sum_k \sum_\ell (\alpha_{k\ell} - \hat{\alpha}_{k\ell}) d_{k\ell} > 0$, so that Farkas' Lemma (see Bazaraa et al. [10, Section 5.3, pp. 234-237], for example) establishes the existence of multipliers $\boldsymbol{\lambda} \in \mathbb{R}^{n^2}$ of $\boldsymbol{Gx} = \boldsymbol{1}$ yielding (3.31).

This concludes the proof. □

Note that any equation of the form (3.28) that is satisfied by all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to $\overline{\text{RLT1}}'$ is necessarily satisfied by all $(\boldsymbol{x}, \boldsymbol{y})$ feasible to RLT1′. Therefore, a direct consequence of Theorem 3.3 is that any equation valid for $\overline{\text{RLT1}}'$ can be computed as a linear combination of the $m_1+m_2+m_3$ equations of Figure 3.1, establishing these equations as a maximal LI set for $\overline{\text{RLT1}}'$. There are two immediate consequences. First, the results found in the papers [7, 47] that identify redundant equality constraints within Problem RLT1′ are extended by Theorem 3.3, as Theorem 3.3 identifies a maximal set of implied constraints. Second, the lower bound to Problem P due to $\overline{\text{RLT1}}'$ is equal to that of the much smaller linear program obtained by replacing (3.8)–(3.10) and the equations of $\boldsymbol{X}$ in (3.12) of $\overline{\text{RLT1}}'$ with the $m_1 + m_2 + m_3$ equations of Figure 3.1. Since $\overline{\text{RLT1}}'$ has been shown to yield strong bounds [2, 32], the smaller form may prove useful computationally.

3.4.2 The Necessary and Sufficient Linearizable Condition

In order to motivate our necessary and sufficient condition for an instance of Problem P to be linearizable, let us write the dual region corresponding to the variables $y_{ijk\ell}$ of $\overline{\text{RLT1}}''$. Specifically, let $\boldsymbol{\phi} = \phi_{ik\ell}$ for all $(i,k,\ell), i < k$, $\boldsymbol{\gamma} = \gamma_{ik\ell}$ for all $(i,k,\ell), i > k$, and $\boldsymbol{\pi} = \pi_{jk\ell}$ for all $(j,k,\ell), j \neq \ell$, be the dual variables corresponding to constraints (3.8)–

(3.10), respectively. Then the region is given by

$$T \equiv \left\{ (\boldsymbol{\phi}, \boldsymbol{\gamma}, \boldsymbol{\pi}) : \phi_{ik\ell} + \gamma_{kij} + \pi_{jk\ell} + \pi_{\ell ij} = C'_{ijk\ell} \ \ \forall \, (i,j,k,\ell), \ i<k, \ j \neq \ell \right\}.$$

For clarity, we recall from the definition of RLT1$'$ that $C'_{ijk\ell} = C_{ijk\ell} + C_{k\ell ij}$ for all $(i,j,k,\ell), i < k, j \neq \ell$.

Lemma 3.4

An instance of Problem P is linearizable if and only if $T \neq \emptyset$.

Proof

(if)

Suppose that $T \neq \emptyset$. Compute a linear combination of the equations (3.8)–(3.10) using any $(\boldsymbol{\phi}, \boldsymbol{\gamma}, \boldsymbol{\pi}) \in T$ to obtain

$$\sum_k \sum_\ell \hat{c}_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} C'_{ijk\ell} y_{ijk\ell} = \kappa \ \ \forall \, (\boldsymbol{x}, \boldsymbol{y}) \text{ feasible to RLT1}',$$

so that

$$\sum_k \sum_\ell \hat{c}_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} C'_{ijk\ell} x_{ij} x_{k\ell} = \kappa \ \ \forall \, \boldsymbol{x} \text{ binary}, \, \boldsymbol{x} \in X.$$

Rewriting this last equation, we have

$$\sum_k \sum_\ell c_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} C'_{ijk\ell} x_{ij} x_{k\ell} = \kappa + \sum_k \sum_\ell (c_{k\ell} - \hat{c}_{k\ell}) x_{k\ell} \ \ \forall \, \boldsymbol{x} \text{ binary}, \, \boldsymbol{x} \in X,$$

meaning that Problem P is linearizable.

(only if)

Now suppose that Problem P is linearizable, so that there exist coefficients $\hat{c}_{k\ell}$ for all (k, ℓ)

and a scalar κ satisfying (3.2). It follows that the equation

$$\sum_{k}\sum_{\ell}(c_{k\ell}-\hat{c}_{k\ell})x_{k\ell}+\sum_{i}\sum_{j}\sum_{k>i}\sum_{\ell\neq j}C'_{ijk\ell}y_{ijk\ell}=\kappa \tag{3.35}$$

is valid for all $(\boldsymbol{x},\boldsymbol{y})$ feasible to RLT1$'$. From Theorem 3.3, we know that (3.35) can be computed as a linear combination of the $m_1+m_2+m_3$ equations of Figure 3.1, implying that $T\neq\emptyset$. $\square$

Lemma 3.5

$T\neq\emptyset$ if and only if the linear program $\overline{\text{RLT1}}''$ is bounded.

Proof

Since $\mathbf{0}\leq\boldsymbol{x}\leq\mathbf{1}$ in $\overline{\text{RLT1}}''$, this linear program is bounded if and only if there does not exist an attractive direction $\boldsymbol{d}$ in the variables $y_{ijk\ell}, i<k, j\neq\ell$, so that the set

$$D\equiv\left\{\begin{array}{ll}\boldsymbol{d}\in\mathbb{R}^{\frac{n^2(n-1)^2}{2}}: & \displaystyle\sum_{i}\sum_{j}\sum_{k>i}\sum_{\ell\neq j}C'_{ijk\ell}d_{ijk\ell}<0\\ & \displaystyle\sum_{j\neq\ell}d_{ijk\ell}=0\ \forall\ (i,k,\ell),\ i<k\\ & \displaystyle\sum_{j\neq\ell}d_{k\ell ij}=0\ \forall\ (i,k,\ell),\ i>k\\ & \displaystyle\sum_{i<k}d_{ijk\ell}+\sum_{i>k}d_{k\ell ij}=0\ \forall\ (j,k,\ell),\ j\neq\ell\end{array}\right\}$$

has $D=\emptyset$. Farkas' Lemma has $D=\emptyset$ if and only if $T\neq\emptyset$. $\square$

Combining Lemmas 3.4 and 3.5 gives Theorem 3.4, as desired.

Theorem 3.4

An instance of Problem P is linearizable if and only if the linear program $\overline{\text{RLT1}}''$ is bounded.

Proof

Follows directly from Lemmas 3.4 and 3.5. □

Theorem 3.4 characterizes all linearizable instances of Problem P in terms of $\overline{\text{RLT1}}''$. This characterization leads to two notable observations. First, since the feasible region to $\overline{\text{RLT1}}'$ is contained within that of $\overline{\text{RLT1}}''$, it is reasonable to expect that the former problem can have an optimal binary solution while the latter does not, thereby identifying more solvable instances of Problem P. This is indeed the case, as can be verified with [7, Example 2]. Thus, Problem $\overline{\text{RLT1}}'$ can identify a richer family of polynomially solvable instances of Problem P than those that are linearizable. The second observation deals with a computed optimal solution to Problem $\overline{\text{RLT1}}'$. As noted in [7], if this solution is not binary, the task of checking whether there exists an optimal solution which *is* binary may not be simple. This concern does not arise in using $\overline{\text{RLT1}}''$ to determine linearizable instances, as Theorem 3.4 does not invoke any extreme point structure.

3.5 Dimensions of the Level-1 RLT and Relaxations

In this section, we use the results of Theorems 3.2 and 3.3 to obtain the dimension of the level-1 RLT form, as well as that of several relaxations. To begin, define the polyhedral set P_1 by

$$P_1 = \{(\boldsymbol{x}, \boldsymbol{y}) : (\boldsymbol{x}, \boldsymbol{y}) \text{ is feasible to the } m_1 + m_2 + m_3 \text{ equations of Figure 3.1}\}.$$

Furthermore, define the polyhedral sets P_2, P_3, P_4 and P_5 relative to the feasible regions of Problems $\overline{\text{RLT1}}''$, $\overline{\text{RLT1}}'$, and RLT1$'$ by

$$P_2 = \{(\boldsymbol{x}, \boldsymbol{y}) : (\boldsymbol{x}, \boldsymbol{y}) \text{ is feasible to the equality constraints of } \overline{\text{RLT1}}''\},$$

$$P_3 = \{(\boldsymbol{x}, \boldsymbol{y}) : (\boldsymbol{x}, \boldsymbol{y}) \text{ is feasible to } \overline{\text{RLT1}}''\},$$

$P_4 = \{(\boldsymbol{x}, \boldsymbol{y}) : (\boldsymbol{x}, \boldsymbol{y}) \text{ is feasible to } \overline{\text{RLT1}}'\}, \text{ and}$

$P_5 = \{(\boldsymbol{x}, \boldsymbol{y}) : (\boldsymbol{x}, \boldsymbol{y}) \text{ is feasible to the convex hull of solutions to RLT1}'\}.$

Theorem 3.5

Given the polyhedral sets P_1, P_2, P_3, P_4 and P_5 defined above,

$$n + \frac{(n-1)^2(n-2)^2}{2} = \dim(P_1) = \dim(P_2) = \dim(P_3) = \dim(P_4) = \dim(P_5),$$

where $\dim(\bullet)$ denotes the dimension of the set $\bullet$.

Proof

Observe that

$$P_1 \supseteq P_2 \supseteq P_3 \supseteq P_4 \supseteq P_5,$$

with the four containments, from left to right, following because all of the equations defining P_1 are implied by the equality constraints of $\overline{\text{RLT1}}''$, because $P_3 = \{(\boldsymbol{x}, \boldsymbol{y}) \in P_2 : \boldsymbol{x} \geq \boldsymbol{0}\}$, because $P_4 = \{(\boldsymbol{x}, \boldsymbol{y}) \in P_3 : \boldsymbol{y} \geq \boldsymbol{0}\}$, and because $\overline{\text{RLT1}}'$ is a polyhedral relaxation of RLT1$'$. Then

$$\dim(P_1) \geq \dim(P_2) \geq \dim(P_3) \geq \dim(P_4) \geq \dim(P_5).$$

Note that there are n^2 variables $\boldsymbol{x}$ and $\frac{n^2(n-1)^2}{2}$ variables $\boldsymbol{y}$ within the $m_1+m_2+m_3 = 2(n-1)^3 + n(n-1)$ equations defining P_1. Since Theorem 3.2 shows that these equations are LI, we have

$$\dim(P_1) = \left(n^2 + \frac{n^2(n-1)^2}{2}\right) - \left(2(n-1)^3 + n(n-1)\right) = n + \frac{(n-1)^2(n-2)^2}{2}.$$

If $\dim(P_5)$ was strictly less than $\dim(P_1)$, then there would be some equation valid for the convex hull of solutions to RLT1$'$ that is not implied by the equations defining P_1. But by Theorem 3.3, any equation valid for all $(\boldsymbol{x}, \boldsymbol{y}) \in P_5$ is implied by the equations defining P_1, meaning that $\dim(P_5) = \dim(P_1)$. This completes the proof. □

3.6 Conclusions

The quadratic assignment problem is an NP-hard discrete, nonlinear optimization problem that arises in many diverse applications. The most successful exact solution methods use linear programming reformulations in higher variable spaces to compute tight lower bounds that efficiently prune the binary search tree. In particular, the celebrated RLT procedure has proven to be an extremely important theoretical tool for developing algorithms that can solve large (up to size $n = 30$) instances of the qap to optimality. In recent years, due to the difficulty associated with computing exact solutions, a research trend has been to identify problem instances that are solvable in polynomial time. One such class of readily solvable qap instances consists of those that are defined to be *linearizable*, meaning that they can be equivalently rewritten as linear assignment problems.

This chapter bridges the gap between these two seemingly unrelated avenues of research by completely characterizing linearizable instances of the qap in terms of the level-1 RLT form. Specifically, we show in Theorem 3.4 that an instance of the qap is linearizable if and only if the linear program $\overline{\text{RLT1}}''$ is bounded. Furthermore, we explain how $\overline{\text{RLT1}}'$ can identify a larger family of solvable qaps than just those that are linearizable.

The theoretical backbone of this study is a thorough examination of the polyhedral structure of RLT1$'$ and the explicit identification of a maximal set of LI equality constraints, depicted pictorially in Figure 3.1. Theorem 3.3 establishes that any equation valid for the convex hull of feasible solutions to RLT1$'$ can be computed as a linear combination of these LI equations, which leads to two noteworthy contributions. First, it provides the theory that is required to prove the necessary and sufficient linearizable condition of Theorem 3.4. Second, it allows for the specification of the dimensions (given in closed form in Theorem 3.5) of the feasible regions associated with RLT1$'$ and several relaxations.

As this polyhedral approach to studying readily solvable qap instances is new, there are numerous avenues for further research. For example, one could consider using the level-1 and/or higher-level RLT forms to characterize other known conditions that permit the qap to be solved in polynomial time, such as those mentioned in [13, 15, 19, 20, 22, 40]. Another interesting study would be to determine whether variations of this approach can be used to characterize readily solvable instances of related combinatorial optimization problems, such as the quadratic semi-assignment problem, the three-dimensional assignment problem, the traveling salesman problem, and the set partitioning problem. Also, as suggested in Subsection 3.4.1, it would be interesting to explore whether the smaller representation of $\overline{\text{RLT1}}'$ afforded by the LI equations of Figure 3.1 could be exploited to develop efficient algorithms to compute RLT bounds.

3.7 Appendix

Theorem 3.3, Part 1 - Base Cases

Consider a size $n = 3$, $n = 4$, or $n = 5$ instance of Problem RLT1′. Given an equation of the form

$$\sum_k \sum_\ell \alpha_{k\ell} x_{k\ell} + \sum_i \sum_j \sum_{k>i} \sum_{\ell \neq j} \beta_{ijk\ell} x_{ij} x_{k\ell} = \kappa \tag{3.36}$$

that is satisfied for all $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{x}$ binary, the coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_N$ are uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B$.

Proof

Distinguish each of the $n!$ binary $\boldsymbol{x} \in \boldsymbol{X}$ with a row vector $\boldsymbol{\theta} \in \mathbb{R}^n$ consisting of a permutation of the numbers $1, \cdots, n$, so that $\boldsymbol{x}_{\boldsymbol{\theta}}$ denotes that binary solution having $x_{i\theta(i)} = 1$ for all i and $x_{ij} = 0$ otherwise. Then, for example, $\boldsymbol{x}_{(3,1,2)}$ with $n = 3$ has $x_{13} = x_{21} = x_{32} = 1$, and $x_{ij} = 0$ otherwise. Let $E(\boldsymbol{x}_{\boldsymbol{\theta}})$ denote the linear equation obtained by setting $\boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{\theta}}$ within (3.36), and let $f(\boldsymbol{x}_{\boldsymbol{\theta}})$ denote the value of the function $f(\boldsymbol{x}) \equiv \sum_{(i,j,k,\ell) \in I_B} \beta_{ijk\ell} x_{ij} x_{k\ell}$ evaluated at $\boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{\theta}}$.

Base Case: $\boldsymbol{n = 3}$

When $n = 3$, there are $3! = 6$ binary points $\boldsymbol{x} \in \boldsymbol{X}$, and $I_N = \{(1221)\}$. The 6 points, and the corresponding equations implied by (3.36), are listed below.

(1) $E(\boldsymbol{x}_{(1,2,3)}):\ \alpha_{11} + \alpha_{22} + \alpha_{33} + f(\boldsymbol{x}_{(1,2,3)}) = \kappa$

(2) $E(\boldsymbol{x}_{(1,3,2)}):\ \alpha_{11} + \alpha_{23} + \alpha_{32} + f(\boldsymbol{x}_{(1,3,2)}) = \kappa$

(3) $E(\boldsymbol{x}_{(2,1,3)}):\ \alpha_{12} + \alpha_{21} + \alpha_{33} + f(\boldsymbol{x}_{(2,1,3)}) + \beta_{1221} = \kappa$

(4) $E(\boldsymbol{x}_{(2,3,1)}):\ \alpha_{12} + \alpha_{23} + \alpha_{31} + f(\boldsymbol{x}_{(2,3,1)}) = \kappa$

(5) $E(\boldsymbol{x}_{(3,1,2)}):\ \alpha_{13} + \alpha_{21} + \alpha_{32} + f(\boldsymbol{x}_{(3,1,2)}) = \kappa$

(6) $E(\boldsymbol{x}_{(3,2,1)}):\ \alpha_{13} + \alpha_{22} + \alpha_{31} + f(\boldsymbol{x}_{(3,2,1)}) = \kappa$

Compute the linear combination $(1)-(2)-(3)+(4)+(5)-(6)$ of these equations to obtain that $\beta_{1221} = f(\boldsymbol{x}_{(1,2,3)}) - f(\boldsymbol{x}_{(1,3,2)}) - f(\boldsymbol{x}_{(2,1,3)}) + f(\boldsymbol{x}_{(2,3,1)}) + f(\boldsymbol{x}_{(3,1,2)}) - f(\boldsymbol{x}_{(3,2,1)})$. This completes the $n = 3$ base case.

Base Case: $\boldsymbol{n = 4}$

When $n = 4$, there are $4! = 24$ binary points $\boldsymbol{x} \in \boldsymbol{X}$. The 24 points, and the corresponding equations implied by (3.36), are listed below. Here, $I_N = \{(1122), (1123), (1132), (1133), (1221), (1231), (1321), (1322), (1331), (1332), (2231), (2331), (2332)\}$.

(1) $E(\boldsymbol{x}_{(1,2,3,4)}) : \alpha_{11} + \alpha_{22} + \alpha_{33} + \alpha_{44} + \beta_{1122} + \beta_{1133} + f(\boldsymbol{x}_{(1,2,3,4)}) = \kappa$

(2) $E(\boldsymbol{x}_{(1,2,4,3)}) : \alpha_{11} + \alpha_{22} + \alpha_{34} + \alpha_{43} + \beta_{1122} + f(\boldsymbol{x}_{(1,2,4,3)}) = \kappa$

(3) $E(\boldsymbol{x}_{(1,3,2,4)}) : \alpha_{11} + \alpha_{23} + \alpha_{32} + \alpha_{44} + \beta_{1123} + \beta_{1132} + \beta_{2332} + f(\boldsymbol{x}_{(1,3,2,4)}) = \kappa$

(4) $E(\boldsymbol{x}_{(1,3,4,2)}) : \alpha_{11} + \alpha_{23} + \alpha_{34} + \alpha_{42} + \beta_{1123} + f(\boldsymbol{x}_{(1,3,4,2)}) = \kappa$

(5) $E(\boldsymbol{x}_{(1,4,2,3)}) : \alpha_{11} + \alpha_{24} + \alpha_{32} + \alpha_{43} + \beta_{1132} + f(\boldsymbol{x}_{(1,4,2,3)}) = \kappa$

(6) $E(\boldsymbol{x}_{(1,4,3,2)}) : \alpha_{11} + \alpha_{24} + \alpha_{33} + \alpha_{42} + \beta_{1133} + f(\boldsymbol{x}_{(1,4,3,2)}) = \kappa$

(7) $E(\boldsymbol{x}_{(2,1,3,4)}) : \alpha_{12} + \alpha_{21} + \alpha_{33} + \alpha_{44} + \beta_{1221} + f(\boldsymbol{x}_{(2,1,3,4)}) = \kappa$

(8) $E(\boldsymbol{x}_{(2,1,4,3)}) : \alpha_{12} + \alpha_{21} + \alpha_{34} + \alpha_{43} + \beta_{1221} + f(\boldsymbol{x}_{(2,1,4,3)}) = \kappa$

(9) $E(\boldsymbol{x}_{(2,3,1,4)}) : \alpha_{12} + \alpha_{23} + \alpha_{31} + \alpha_{44} + \beta_{1231} + \beta_{2331} + f(\boldsymbol{x}_{(2,3,1,4)}) = \kappa$

(10) $E(\boldsymbol{x}_{(2,3,4,1)}) : \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} + f(\boldsymbol{x}_{(2,3,4,1)}) = \kappa$

(11) $E(\boldsymbol{x}_{(2,4,1,3)}) : \alpha_{12} + \alpha_{24} + \alpha_{31} + \alpha_{43} + \beta_{1231} + f(\boldsymbol{x}_{(2,4,1,3)}) = \kappa$

(12) $E(\boldsymbol{x}_{(2,4,3,1)}) : \alpha_{12} + \alpha_{24} + \alpha_{33} + \alpha_{41} + f(\boldsymbol{x}_{(2,4,3,1)}) = \kappa$

(13) $E(\boldsymbol{x}_{(3,1,2,4)}) : \alpha_{13} + \alpha_{21} + \alpha_{32} + \alpha_{44} + \beta_{1321} + \beta_{1332} + f(\boldsymbol{x}_{(3,1,2,4)}) = \kappa$

(14) $E(\boldsymbol{x}_{(3,1,4,2)}) : \alpha_{13} + \alpha_{21} + \alpha_{34} + \alpha_{42} + \beta_{1321} + f(\boldsymbol{x}_{(3,1,4,2)}) = \kappa$

(15) $E(\boldsymbol{x}_{(3,2,1,4)}) : \alpha_{13} + \alpha_{22} + \alpha_{31} + \alpha_{44} + \beta_{1322} + \beta_{1331} + \beta_{2231} + f(\boldsymbol{x}_{(3,2,1,4)}) = \kappa$

(16) $E(\boldsymbol{x}_{(3,2,4,1)}) : \alpha_{13} + \alpha_{22} + \alpha_{34} + \alpha_{41} + \beta_{1322} + f(\boldsymbol{x}_{(3,2,4,1)}) = \kappa$

(17) $E(\boldsymbol{x}_{(3,4,1,2)}) : \alpha_{13} + \alpha_{24} + \alpha_{31} + \alpha_{42} + \beta_{1331} + f(\boldsymbol{x}_{(3,4,1,2)}) = \kappa$

(18) $E(\boldsymbol{x}_{(3,4,2,1)}) : \alpha_{13} + \alpha_{24} + \alpha_{32} + \alpha_{41} + \beta_{1332} + f(\boldsymbol{x}_{(3,4,2,1)}) = \kappa$

(19) $E(\boldsymbol{x}_{(4,1,2,3)}) : \alpha_{14} + \alpha_{21} + \alpha_{32} + \alpha_{43} + f(\boldsymbol{x}_{(4,1,2,3)}) = \kappa$

(20) $E(\boldsymbol{x}_{(4,1,3,2)}) : \alpha_{14} + \alpha_{21} + \alpha_{33} + \alpha_{42} + f(\boldsymbol{x}_{(4,1,3,2)}) = \kappa$

(21) $E(\boldsymbol{x}_{(4,2,1,3)}) : \alpha_{14} + \alpha_{22} + \alpha_{31} + \alpha_{43} + \beta_{2231} + f(\boldsymbol{x}_{(4,2,1,3)}) = \kappa$

(22) $E(\boldsymbol{x}_{(4,2,3,1)}) : \alpha_{14} + \alpha_{22} + \alpha_{33} + \alpha_{41} + f(\boldsymbol{x}_{(4,2,3,1)}) = \kappa$

(23) $E(\boldsymbol{x}_{(4,3,1,2)}) : \alpha_{14} + \alpha_{23} + \alpha_{31} + \alpha_{42} + \beta_{2331} + f(\boldsymbol{x}_{(4,3,1,2)}) = \kappa$

(24) $E(\boldsymbol{x}_{(4,3,2,1)}) : \alpha_{14} + \alpha_{23} + \alpha_{32} + \alpha_{41} + \beta_{2332} + f(\boldsymbol{x}_{(4,3,2,1)}) = \kappa$

The 13 linear combinations of (1)–(24) listed below use the functions $f(\boldsymbol{x}_{(i,j,k,\ell)})$ to recursively express the $|I_N| = 13$ coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_N$ in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B$.

- (1) − (2) − (3) + (4) + (5) − (6):

  $\beta_{2332} = f(\boldsymbol{x}_{(1,2,3,4)}) - f(\boldsymbol{x}_{(1,2,4,3)}) - f(\boldsymbol{x}_{(1,3,2,4)}) + f(\boldsymbol{x}_{(1,3,4,2)}) + f(\boldsymbol{x}_{(1,4,2,3)}) - f(\boldsymbol{x}_{(1,4,3,2)})$

- (7) − (8) − (9) + (10) + (11) − (12):

  $\beta_{2331} = f(\boldsymbol{x}_{(2,1,3,4)}) - f(\boldsymbol{x}_{(2,1,4,3)}) - f(\boldsymbol{x}_{(2,3,1,4)}) + f(\boldsymbol{x}_{(2,3,4,1)}) + f(\boldsymbol{x}_{(2,4,1,3)}) - f(\boldsymbol{x}_{(2,4,3,1)})$

- (13) − (14) − (15) + (16) + (17) − (18):

  $\beta_{2231} = f(\boldsymbol{x}_{(3,1,2,4)}) - f(\boldsymbol{x}_{(3,1,4,2)}) - f(\boldsymbol{x}_{(3,2,1,4)}) + f(\boldsymbol{x}_{(3,2,4,1)}) + f(\boldsymbol{x}_{(3,4,1,2)}) - f(\boldsymbol{x}_{(3,4,2,1)})$

- −(11) + (12) + (21) − (22):

  $\beta_{1231} = -f(\boldsymbol{x}_{(2,4,1,3)}) + f(\boldsymbol{x}_{(2,4,3,1)}) + f(\boldsymbol{x}_{(4,2,1,3)}) - f(\boldsymbol{x}_{(4,2,3,1)}) + \beta_{2231}$

- −(8) + (10) + (19) − (24):

  $\beta_{1221} = -f(\boldsymbol{x}_{(2,1,4,3)}) + f(\boldsymbol{x}_{(2,3,4,1)}) + f(\boldsymbol{x}_{(4,1,2,3)}) - f(\boldsymbol{x}_{(4,3,2,1)}) - \beta_{2332}$

- −(7) + (12) − (13) + (18):

  $\beta_{1321} = -f(\boldsymbol{x}_{(2,1,3,4)}) + f(\boldsymbol{x}_{(2,4,3,1)}) - f(\boldsymbol{x}_{(3,1,2,4)}) + f(\boldsymbol{x}_{(3,4,2,1)}) - \beta_{1221}$

- (14) − (16) + (20) − (22):

  $\beta_{1322} = f(\boldsymbol{x}_{(3,1,4,2)}) - f(\boldsymbol{x}_{(3,2,4,1)}) + f(\boldsymbol{x}_{(4,1,3,2)}) - f(\boldsymbol{x}_{(4,2,3,1)}) + \beta_{1321}$

- $-(9) + (10) - (15) + (16)$:

  $\beta_{1331} = -f(\boldsymbol{x}_{(2,3,1,4)}) + f(\boldsymbol{x}_{(2,3,4,1)}) - f(\boldsymbol{x}_{(3,2,1,4)}) + f(\boldsymbol{x}_{(3,2,4,1)}) - \beta_{1231} - \beta_{2331} - \beta_{2231}$

- $(17) - (18) + (23) - (24)$:

  $\beta_{1332} = f(\boldsymbol{x}_{(3,4,1,2)}) - f(\boldsymbol{x}_{(3,4,2,1)}) + f(\boldsymbol{x}_{(4,3,1,2)}) - f(\boldsymbol{x}_{(4,3,2,1)}) + \beta_{1331} + \beta_{2331} - \beta_{2332}$

- $-(1) + (2) - (7) + (8)$:

  $\beta_{1133} = -f(\boldsymbol{x}_{(1,2,3,4)}) + f(\boldsymbol{x}_{(1,2,4,3)}) - f(\boldsymbol{x}_{(2,1,3,4)}) + f(\boldsymbol{x}_{(2,1,4,3)})$

- $-(3) + (4) - (13) + (14)$:

  $\beta_{1132} = -f(\boldsymbol{x}_{(1,3,2,4)}) + f(\boldsymbol{x}_{(1,3,4,2)}) - f(\boldsymbol{x}_{(3,1,2,4)}) + f(\boldsymbol{x}_{(3,1,4,2)}) - \beta_{2332} - \beta_{1332}$

- $-(3) + (5) - (9) + (11)$:

  $\beta_{1123} = -f(\boldsymbol{x}_{(1,3,2,4)}) + f(\boldsymbol{x}_{(1,4,2,3)}) - f(\boldsymbol{x}_{(2,3,1,4)}) + f(\boldsymbol{x}_{(2,4,1,3)}) - \beta_{2332} - \beta_{2331}$

- $-(1) + (6) - (15) + (17)$:

  $\beta_{1122} = -f(\boldsymbol{x}_{(1,2,3,4)}) + f(\boldsymbol{x}_{(1,4,3,2)}) - f(\boldsymbol{x}_{(3,2,1,4)}) + f(\boldsymbol{x}_{(3,4,1,2)}) - \beta_{1322} - \beta_{2231}$

This completes the $n = 4$ base case.

Base Case: $n = 5$

When $n = 5$, there are $5! = 120$ binary points $\boldsymbol{x} \in \boldsymbol{X}$. Instead of enumerating the associated equations, we employ an inductive argument on the $n = 4$ base case. Toward this end, arbitrarily select a coefficient β_{rstu} with $(r, s, t, u) \in I_N$, for which $(r, t) \neq (1, 2), (s, u) \neq (1, 2)$, and $(s, u) \neq (2, 1)$. This accounts for every variable $\beta_{ijk\ell}$ with $(i, j, k, \ell) \in I_N$ except for those in the set

$$R = \{\beta_{1122}, \beta_{1123}, \beta_{1124}, \beta_{1132}, \beta_{1142}, \beta_{1221}, \beta_{1223}, \beta_{1224}, \beta_{1231}, \beta_{1241}, \beta_{1321}, \beta_{1322}, \beta_{1421}, \beta_{1422}, \beta_{1423}, \beta_{2132}, \beta_{2142}, \beta_{2231}, \beta_{2241}, \beta_{3241}\}.$$

There exists a variable x_{pq} having $p \leq 2, p \neq r, p \neq t$, and $q \leq 2, q \neq s, q \neq u$. By the $n = 4$ base case, the coefficients $\beta_{ijk\ell}$ having $(i, j, k, \ell) \in I_N^{pq}$ are uniquely defined in terms of the

$\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B^{pq}$, with I_B^{pq} and I_N^{pq} defined as in (3.33) and (3.34), respectively. Therefore, since $(r,s,t,u) \in I_N^{pq}$ and $I_B^{pq} \subset I_B$, we have that β_{rstu} is uniquely defined in terms of the coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B$.

Now consider an arbitrary coefficient from the set $R_1 = \{\beta_{2132}, \beta_{2142}, \beta_{2231}, \beta_{2241}, \beta_{3241}\}$. Selecting $(p,q) = (1,3)$, note that $I_B^{pq} \subset I_B \cup \{(2,2,3,4),(2,2,4,4)\}$, and that β_{2234} and β_{2244} were already shown to be uniquely defined in terms of the coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B$. By the $n = 4$ base case, the coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_N^{pq}$ are uniquely defined in terms of the $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B^{pq}$. Therefore, since $R_1 \subset I_N^{pq}$, our arbitrary $\beta_{ijk\ell} \in R_1$ is uniquely defined in terms of the coefficients $\beta_{ijk\ell}$ having $(i,j,k,\ell) \in I_B$.

The argument above can be recursively used to establish the result for the remaining coefficients $\beta_{ijk\ell} \in R$ as follows.

1. Selecting $(p,q) = (2,3)$, and noting that $I_B^{pq} \subset I_B \cup \{(1,2,3,4),(1,2,4,4)\}$, establishes the result for an arbitrary coefficient from the set $R_2 = \{\beta_{1132}, \beta_{1142}, \beta_{1231}, \beta_{1241}\}$.

2. Selecting $(p,q) = (3,1)$, and noting that $I_B^{pq} \subset I_B \cup \{(2,2,4,3),(2,2,4,4)\}$, establishes the result for an arbitrary coefficient from the set $R_3 = \{\beta_{1223}, \beta_{1224}, \beta_{1322}, \beta_{1422}, \beta_{1423}\}$.

3. Selecting $(p,q) = (3,2)$, and noting that $I_B^{pq} \subset I_B \cup \{(2,1,4,3),(2,1,4,4)\}$, establishes the result for an arbitrary coefficient from the set $R_4 = \{\beta_{1123}, \beta_{1124}, \beta_{1321}, \beta_{1421}\}$.

4. Selecting $(p,q) = (3,3)$, and noting that $I_B^{pq} \subset I_B \cup \{(1,2,2,4), (1,2,4,4), (2,2,4,4), (2,1,4,2), (2,1,4,4)\}$, establishes the result for an arbitrary coefficient from the set $R_5 = \{\beta_{1122}, \beta_{1221}\}$.

Since $R = \cup_{i=1}^{5} R_i$, this completes the $n = 5$ base case. $\square$

Bibliography

[1] W. Adams, M. Guignard, P. Hahn, W. Hightower, A level-2 reformulation-linearization technique bound for the quadratic assignment problem, *European Journal of Operational Research*, Vol. 180, No. 3, pp. 983-996, 2007.

[2] W. Adams, T. Johnson, Improved linear programming-based lower bounds for the quadratic assignment problem, *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, Vol. 16, pp. 43-75, 1994.

[3] W. Adams, H. Sherali, A hierarchy of relaxations leading to the convex hull representation for general discrete optimization problems, *Annals of Operations Research*, Vol. 140, No. 1, pp. 21-47, 2005.

[4] W. Adams, H. Sherali, A tight linearization and an algorithm for zero-one quadratic programming problems, *Management Science*, Vol. 32, No. 10, pp. 1274-1290, 1986.

[5] W. Adams, H. Sherali, Linearization strategies for a class of zero-one mixed integer programming problems, *Operations Research*, Vol. 38, No. 2, pp. 217-226, 1990.

[6] W. Adams, H. Sherali, Mixed integer bilinear programming problems, *Mathematical Programming*, Vol. 59, No. 3, pp. 279-305, 1993.

[7] W. Adams, L. Waddell, Linear programming insights into solvable cases of the quadratic assignment problem, *Discrete Optimization*, Vol. 14, pp. 46-60, 2014.

[8] K. Anstreicher, N. Brixius, J. Goux, J. Linderoth, Solving large quadratic assignment problems on computational grids, *Mathematical Programming*, Vol. 91, No. 3, pp. 563-588, 2002.

[9] A. Assad, W. Xu, On lower bounds for a class of quadratic 0-1 programs, *Operations Research Letters*, Vol. 4, No. 4, pp. 175-180, 1985.

[10] M. Bazaraa, J. Jarvis, H. Sherali, *Linear Programming and Network Flows*, John Wiley & Sons, Inc., 2010.

[11] M. Bazaraa, H. Sherali, Benders' partitioning scheme applied to a new formulation of the quadratic assignment problem, *Naval Research Logistics Quarterly*, Vol. 27, No. 1, pp. 29-41, 1980.

[12] R.E. Burkard, E. Çela, V. Demidenko, N. Metelski, G. Woeginger, A unified approach to simple special cases of extremal permutation problems, *Optimization*, Vol. 44, pp. 123-138, 1998.

[13] R.E. Burkard, E. Çela, V. Demidenko, N. Metelski, G. Woeginger, Perspectives of easy and hard cases of the quadratic assignment problem, SFB Report 104, Institute of Mathematics, Technical University, Graz, Austria, 1997.

[14] R.E. Burkard, E. Çela, P. Pardalos, L. Pitsoulis, The quadratic assignment problem, *Handbook of Combinatorial Optimization*, Vol. 3, pp. 241-338, 1998.

[15] R.E. Burkard, E. Çela, G. Rote, G.J. Woeginger, The quadratic assignment problem with a monotone anti-Monge matrix and a symmetric Toeplitz matrix: easy and hard cases, *Mathematical Programming*, Vol. 82, No. 1-2, pp. 125-158, 1998.

[16] R.E. Burkard, S. Karisch, F. Rendl, QAPLIB - a quadratic assignment program library, *Journal of Global Optimization*, Vol. 10, No. 4, pp. 391-403, 1997, QAPLIB is found on the web at < http://www.seas.upenn.edu/qaplib/ > .

[17] R.E. Burkard, J. Offermann, Entwurf von Schreibmaschinentastaturen mittels quadratischer Zuordnungsprobleme, *Zeitschrift für Operations Research*, Vol. 21, pp. B121-B132, 1977.

[18] E. Çela, *The Quadratic Assignment Problem: Theory and Algorithms*, Kluwer Academic Publishers, Dordrecht, 1998.

[19] E. Çela, V.G. Deineko, G. Woeginger, Another well-solvable case of the QAP: Maximizing the job completion time variance, *Operations Research Letters*, Vol. 40, No. 6, pp. 356-359, 2012.

[20] E. Çela, V.G. Deineko, G. Woeginger, Well-solvable cases of the QAP with block-structured matrices, *Discrete Applied Mathematics*, Vol. 186, pp. 56-65, 2015.

[21] E. Çela, N.S. Schmuck, S. Wimer, G.J. Woeginger, The Wiener maximum quadratic assignment problem, *Discrete Optimization*, Vol. 8, No. 3, pp. 411-416, 2011.

[22] V.G. Deineko, G.J. Woeginger, A solvable case of the quadratic assignment problem, *Operations Research Letters*, Vol. 22, No. 1, pp. 13-17, 1998.

[23] M. Dell'Amico, J. Carlos Diaz, M. Iori, R. Montanari, The single-finger keyboard layout problem, University of Modena and Reggio Emilia, Italy, 2008.

[24] J. Dickey, J. Hopkins, Campus building arrangement using TOPAZ, *Transportation Research*, Vol. 6, No. 1, pp. 59-68, 1972.

[25] H.A. Eiselt, G. Laporte, A combinatorial optimization problem arising in dartboard design, *The Journal of the Operational Research Society*, Vol. 42, pp. 113-118, 1991.

[26] A. Elshafei, Hospital layout as a quadratic assignment problem, *Operations Research Quarterly*, Vol. 28, No. 1, pp. 167-179, 1977.

[27] G. Erdoğan, B. Tansel, A note on a polynomial time solvable case of the quadratic assignment problem, *Discrete Optimization*, Vol. 3, No. 4, pp. 382-384, 2006.

[28] G. Erdoğan, B. Tansel, Two classes of quadratic assignment problems that are solvable as linear assignment problems, *Discrete Optimization*, Vol. 8, No. 3, pp. 446-451, 2011.

[29] A. Frieze, J. Yadegar, On the quadratic assignment problem, *Discrete Applied Mathematics*, Vol. 5, No. 1, pp. 89-98, 1983.

[30] A. Geoffrion, G. Graves, Scheduling parallel production lines with changeover costs: practical applications of a quadratic assignment/LP approach, *Operations Research*, Vol. 24, No. 4, pp. 595-610, 1976.

[31] A. Haghani, M.-C. Chen, Optimizing gate assignments at airport terminals, *Transportation Research Part A: Policy and Practice*, Vol. 32, No. 6, pp. 437-454, 1998.

[32] P.M. Hahn, T. Grant, N. Hall, A branch-and-bound algorithm for the quadratic assignment problem based on the Hungarian method, *European Journal of Operational Research*, Vol. 108, No. 3, pp. 629-640, 1998.

[33] P.M. Hahn, J. Krarup, A hospital facility problem finally solved, *The Journal of Intelligent Manufacturing*, Vol. 12, No. 5-6, pp. 487-496, 2001.

[34] P.M. Hahn, Y.-R. Zhu, M. Guignard, W.L. Hightower, M.J. Saltzman, A level-3 reformulation-linearization technique bound for the quadratic assignment problem, *INFORMS Journal on Computing*, Vol. 24, No. 2, pp. 202-209, 2012.

[35] S.N. Kabadi, A.P. Punnen, An $O(n^4)$ algorithm for the QAP linearization problem, *Mathematics of Operations Research*, Vol. 36, No. 4, pp. 754-761, 2011.

[36] L. Kaufman, F. Broeckx, An algorithm for the quadratic assignment problem using Benders' decomposition, *European Journal of Operational Research*, Vol. 2, No. 3, pp. 207-211, 1978.

[37] T. Koopmans, M. Beckmann, Assignment problems and the location of economic activities, *Econometrica*, Vol. 25, No. 1, pp. 53-76, 1957.

[38] J. Krarup, P. Pruzan, Computer-aided layout design, *Mathematical Programming Study*, Vol. 9, M.L. Balinski and C. Lemarechal (eds.), North Holland Publishing Company, Amsterdam, pp. 75-94, 1978.

[39] G. Laporte, H. Mercure, Balancing hydraulic turbine runners: a quadratic assignment problem, *European Journal of Operational Research*, Vol. 35, pp. 378-382, 1988.

[40] M. Laurent, M. Seminaroti, The quadratic assignment problem is easy for Robinsonian matrices with Toeplitz structure, *Operations Research Letters*, Vol. 43, No. 1, pp. 103-109, 2015.

[41] E. Lawler, The quadratic assignment problem, *Management Science*, Vol. 19, No. 4, pp. 586-599, 1963.

[42] E.M. Loiola, N.M. Maia de Abreu, P.O. Boaventura-Netto, P. Hahn, T. Querido, A survey for the quadratic assignment problem, *European Journal of Operational Research*, Vol. 176, No. 2, pp. 657-690, 2007.

[43] G.M. Nemhauser, L.A. Wolsey, *Integer and Combinatorial Optimization*, John Wiley & Sons, Inc., 1988.

[44] C. Nugent, T. Vollmann, J. Ruml, An experimental comparison of techniques for the assignment of facilities to locations, *Operations Research*, Vol. 16, No. 1, pp. 150-173, 1968.

[45] P.M. Pardalos, F. Rendl, H. Wolkowicz, The quadratic assignment problem: a survey and recent developments, *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, Vol. 16, pp. 1-42, 1994.

[46] A.P. Punnen, S.N. Kabadi, A linear time algorithm for the Koopmans-Beckmann QAP linearization and related problems, *Discrete Optimization*, Vol. 10, No. 3, pp. 200-209, 2013.

[47] B. Rostami, F. Malucelli, A revised reformulation-linearization technique for the quadratic assignment problem, *Discrete Optimization*, Vol. 14, pp. 97-103, 2014.

[48] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, Inc., 1986.

[49] H. Sherali, W. Adams, A hierarchy of relaxations and convex hull characterizations for mixed-integer zero-one programming problems, *Discrete Applied Mathematics*, Vol. 52, No. 1, pp. 83-106, 1994.

[50] H. Sherali, W. Adams, A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems, *SIAM Journal on Discrete Mathematics*, Vol. 3, No. 3, pp. 411-430, 1990.

[51] H. Sherali, W. Adams, *A Reformulation-Linearization Technique for Solving Discrete and Continuous Nonconvex Problems*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1999.

[52] L. Steinberg, The backboard wiring problem: a placement algorithm, *SIAM Review*, Vol. 3, No. 1, pp. 37-50, 1961.

[53] I. Ugi, J. Bauer, J. Friedrich, J. Gasteiger, C. Jochum, W. Schubert, Neue anwendungsgebiete für computer in der chemie, *Angewandte Chemie*, Vol. 91, pp. 99-111, 1979.

[54] L. Wei, W. Qi, D. Chen, P. Liu, E. Yuan, Optimal sequencing of a set of positive numbers with the variance of the sequence's partial sums maximized, *Optimization Letters*, Vol. 7, No. 6, pp. 1071-1086, 2013.