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Variations on Graph Products and Vertex Partitions

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical Sciences

by
Jobby Jacob
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Accepted by:
Dr. Renu Laskar, Committee Co-Chair
Dr. Wayne Goddard, Committee Co-Chair
Dr. Hiren Maharaj
Dr. Gretchen Matthews

Abstract

In this thesis we investigate two graph products called double vertex graphs and complete double vertex graphs, and two vertex partitions called dominator partitions and rankings.

We introduce a new graph product called the complete double vertex graph and study its properties. The complete double vertex graph is a natural extension of the Cartesian product and a generalization of the double vertex graph. The *complete double vertex graph* of G, denoted by $CU_2(G)$, is the graph whose vertex set consists of all $\binom{n+1}{2}$ unordered pairs of elements of V (duplicates allowed). That is, the vertex set consists of all 2-element multisets of the form $\{a, a\}$ and unordered pairs of the form {a,b}, where $a \neq b$. Two vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \bigcap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G.

We establish many properties of complete double vertex graphs, including results involving the chromatic number of $CU_2(G)$ and the characterization of planar $CU_2(G)$. We also investigate the important problem of reconstructing the factors of double vertex graphs and complete double vertex graphs. We reconstruct G from $U_2(G)$ and $CU_2(G)$ for different classes of graphs, including cubic graphs.

Next, we look at the properties of dominator partitions of graphs. A *dominator partition* of a graph G is a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ such that every vertex $v \in V(G)$ is a dominator of at least one class $V_j \in \Pi$; that is, v is adjacent to every vertex in V_j. A dominator partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ is *minimal* if any partition Π' obtained from Π by forming the union of any two classes into one class, $V_i \cup V_j,\ i \neq j$, is no longer a dominator partition. The *dominator partition number* of a graph G is the minimum order of a dominator partition of G and the *upper dominator partition number* of a graph G is the maximum order of a minimal dominator partition of G.

We characterize minimal dominator partitions of a graph G. This helps us to study the properties of the upper dominator partition number and establish bounds on the upper dominator partition number of different families of graphs, including trees. We also calculate the upper dominator partition number of certain classes of graphs, including paths and cycles, which is surprisingly difficult to calculate.

Properties of rankings are studied in this thesis as well. A function $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ is a *k-ranking* of G if for $u, v \in V(G)$, $f(u) = f(v)$ implies that every $u - v$ path contains a vertex w such that $f(w) > f(u)$. By definition, every ranking is a proper coloring. The *rank number* of G, denoted $\chi_r(G)$, is the minimum value of k such that G has a k-ranking. A k-ranking f is a *minimal k-ranking* of G if for all $x \in V(G)$ with $f(x) > 1$, the function g defined on $V(G)$ by $g(z) = f(z)$ for $z \neq x$ and $1 \leq g(x) < f(x)$ is not a ranking. The *arank number*, denoted $\psi_r(G)$, is defined to be the maximum value of k for which G has a minimal k-ranking.

We establish more properties of minimal rankings, including results related to permuting the labels of minimal χ_r-rankings and minimal ψ_r-rankings. In addition, we investigate rankings of the Cartesian product of two complete graphs, also known as the rook's graph. We establish bounds on the rank number of a rook's graph and calculate its arank number using multiple results we obtain on minimal rankings of a rook's graph.

Dedication

To my wife Bonnie, to my parents and to everyone in my extended family...

Acknowledgments

I would like thank my wonderful advisors, Dr. Renu Laskar and Dr. Wayne Goddard for all the help and guidance they provided me throughout my graduate career. Without their encouragement and support I would not have been able to produce this thesis and more importantly I would not be the person that I am right now. They had a major role in helping me to develop both professionally and personally.

I would also like to thank my committee members, Dr. Hiren Maharaj and Dr. Gretchen Matthews, for their suggestions and help in completing this thesis. Also, I would like to thank all the teachers I had for guiding my career, and would especially like to mention Dr. Herman Senter. I also thank the Department of Mathematical Sciences at Clemson for providing me with all the necessary facilities during my graduate studies.

My inadequate words would not be enough praise for the help and encouragement given to me by my wife Bonnie. Ever since we met she had supported me in every step of my career and continues to do so, and I am really lucky to have her by my side.

Nobody can thank their parents enough for their support and help, and I am no exception. My parents made meeting their children's needs their first priority in good times and in bad times, and I am proud of my father Jacob, my mother Valsa, my sister Pibby and my brother Sobby.

I could not thank my extended family enough for the support they gave me; they are the best that anyone could ever have. Everyone in my extended family has treated me as their son and brother and encouraged me in every phase of my life. I am grateful to God that I have such a wonderful family.

There is one more special person I would like to mention here. His name is Blackie and he is our cat. I know this is silly, but I cannot help it. I used to dislike cats and he literally changed my life. He is a great pet and a wonderful playmate to help my mind relax, which, as anybody can guess, was really helpful during my graduate studies.

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Chapter 1

Introduction

How many colors are needed to color a political map of the world? This is a very simple question, and yet it took more than a century to settle.

Let us look at the small example shown in Figure 1.1. Suppose we need to know the minimum number of colors needed to color the regions such that two regions who share a boundary receive different colors. Since it is a small map, we can easily see that we require at least 4 colors, and in fact only 4 colors are required.

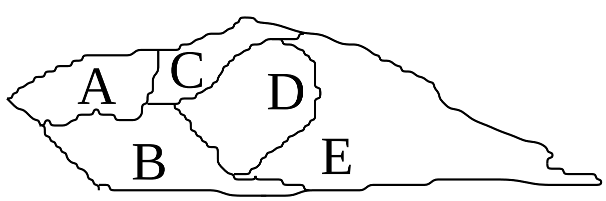


Figure 1.1: A political map

We can think about this problem using a graph theoretical model as shown in Figure 1.2. (A formal definition of a graph and other terminologies are given in Section 1.1.) The vertices of the model represent the regions, and two vertices are adjacent if and only if the corresponding regions share a common boundary. Now our problem is as follows: what is the minimum number of colors required to color the vertices of this graph such that two adjacent vertices receive different colors? As before, since the graph is small, we can see that the answer is 4. (Note that vertices B, C, D and E need to each receive a distinct color in order to satisfy our requirement.)

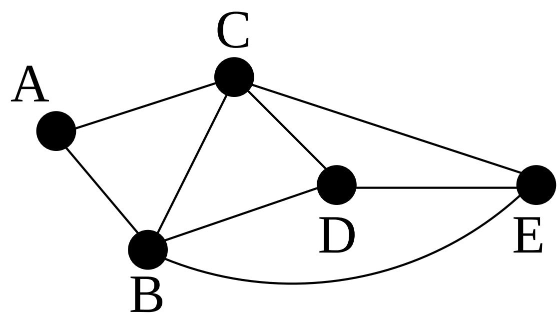


Figure 1.2: Graph theoretical model for the coloring problem

Now, how about the minimum number of colors required for the political map of the world? In 1852, a student named Francis Guthrie at University College London made the curious observation that he could color the counties of a map of England with 4 colors and discussed it with his brother, Frederick. Frederick Guthrie then asked his professor Augustus De Morgan, and thus came the Four Color Conjecture.

For more than a century, this conjecture remained one of the simplest unsolved problems in mathematics in that the problem can be explained to anyone. Many people worked on the Four Color Conjecture, to the point that it was even called the "Four Color Disease". The Four Color Conjecture became the Four Color Theorem in 1976 after Kenneth Appel and Wolfgang Haken produced a proof using the help of computers.

The coloring problem is a kind of vertex partitioning problem in which the vertices of a graph are partitioned into different sets such that each set or each vertex satisfies some property $\mathcal{P}$. In the case of the coloring problem, each set needs to satisfy the property that no two vertices in the same set are adjacent. There are different variations on colorings that have been studied in the literature. There are multiple books on graph colorings, for example [28, 30]. For a brief survey on different coloring problems, the reader is referred to the survey by Laskar, Jacob and Lyle [31].

Depending on the definition of property $\mathcal{P}$, one can investigate the minimum number of sets in a partition, and the maximum number of sets provided two sets cannot be merged. Similarly, for some property $\mathcal{P}$ the invariants that are relevant would be the maximum number of sets in a partition and the minimum number of sets provided that none of the sets can be split.

For example, for the coloring problem, the chromatic number is the minimum number of colors that are needed to color the vertices of a graph. That is, what is the minimum number of sets allowed in a partition such that no two vertices in the same set are adjacent? The achromatic number denotes the maximum number of sets that are allowed in a partition such that no two vertices in any set are adjacent and no two sets can be merged while satisfying the first requirement.

On the other hand, there are some problems which require partitioning the vertices into two sets and counting the minimum or maximum number of vertices in one set having some property $\mathcal{P}$. Examples include dominating sets and independent sets. (Formal definitions are in Chapter 3 and Section 1.1.) In the case of independent sets, the vertices are required to be partitioned into two sets such that one set has the property that no two vertices in the set are adjacent. The independence number is the size of the largest independent set. Also, one can ask what the size of the smallest independent set is such that if any vertex is added to the set then the resultant set is no longer independent.

There are many applications that involve studying vertex partitions, or to be more precise, calculating the minimum or maximum number of sets in a partition where each set has some prescribed property $\mathcal{P}$. However, it might be difficult to calculate certain graph parameters related to vertex partitions for some large graphs. Graph products play a significant role in finding these parameters for large and complicated graphs. Some popular graph products include the Cartesian product and the strong product among others.

The main appeal of studying graph products is that they allow us to consider large and complicated graphs as combinations of smaller graphs under the product, and to study the larger graph by studying its factors. For example, in the case of coloring, it is known that the chromatic number of a Cartesian product of two graphs is the maximum of the chromatic numbers of its factors.

Suppose we need to find the chromatic number of a large, complicated graph. If we can recognize the larger graph as the Cartesian product of smaller graphs, then we can find the chromatic number of the factors and calculate the chromatic number of the complicated graph. Note that there are algorithms to find factors of Cartesian products.

However, for some parameters such as the domination number, it is still unknown how the domination numbers of the factors affect the domination number of the Cartesian product in general. Note that for some classes of graphs like trees, this problem is solved.

Thus, other graph products are also being studied in the hope that they will provide a way to calculate some invariants of complicated graphs by studying those of the factors.

1.1 Definitions

In this section, we give definitions of the concepts and terminologies used in this thesis.

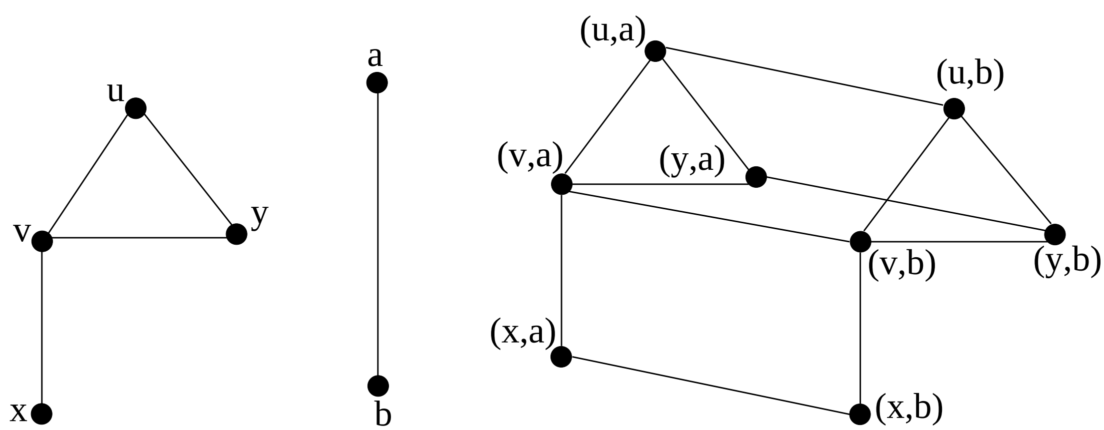
A *graph* $G = (V, E)$ consists of two sets of objects, $V = \{v_1, v_2, \ldots\}$ called vertices and $E = \{e_1, e_2, \ldots\}$ called edges such that every edge e_k is associated with two vertices v_i and v_j called its end-vertices. Note that this definition permits the end-vertices of an edge to be the same (not distinct). In that case we call the edge a *loop*. However, throughout this thesis, the graphs we consider are simple finite graphs, which means that there are no loops or multiple edges between any two vertices and the graph has finite number of vertices.

Two vertices u and v are *adjacent* if they are the end-vertices of an edge. In this case we say u and v are neighbors. The *degree* of a vertex v in G, denoted $\deg_G(v)$, is the number of edges of which v is an end-vertex. A graph is *regular* if the degrees of all vertices are the same. A *complete graph* on n vertices, denoted by K_n, is a graph on n vertices such that every pair of vertices is adjacent.

A function $f : V(G) \to \{1, 2, \ldots, k\}$ is a *k-coloring* of a graph G if for any two vertices u and v, $f(u) = f(v)$ implies that u and v are not adjacent. In other words, adjacent vertices receive different colors under f. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum k such that G has a k-coloring. Thus a k-coloring of a graph G is a partition $\Pi = \{V_i, V_2, \ldots, V_k\}$ of the vertices of G such that for any V_i, no two vertices in V_i are adjacent. A k-coloring is *complete* if for every $1 \leq i < j \leq k$, there is a vertex $u \in V_i$ and a vertex $v \in V_j$ such that u and v are adjacent. The *achromatic number*, denoted by $\psi(G)$, is the maximum k such that G has a complete coloring. In other words, the achromatic number is the maximum k such that the vertices can be partitioned into k color classes provided no two classes can be merged together and still have a coloring.

For a graph G, $S \subseteq V(G)$ is an *independent set* if no two vertices of S are adjacent. The size of the largest independent set in G is called the *independence number* of G and is denoted by $\beta_0(G)$. Thus, we can view a coloring of a graph as the partitioning of the vertices of the graph into sets such that each set is an independent set.

Let G and H be two graphs. The *Cartesian product* of G and H, denoted by $G \Box H$, is defined as follows. The vertex set of $G \Box H$ is the Cartesian product of sets $V(G)$ and $V(H)$, that is, $V(G) \times V(H)$. Two vertices (u, x) and (v, y) are adjacent in $G \Box H$ if and only if either $u = v$ and x and y are adjacent in H, or $x = y$ and u and v are adjacent in G. An example of a Cartesian product is shown in Figure 1.3.



(a) Graph G (b) Graph H (c) Graph $G \Box H$

Figure 1.3: Example of a Cartesian product

1.2 Overview

In this section, we give an overview of this thesis. In Chapter 2 we introduce a new graph product called the complete double vertex graph, and investigate the properties of the new product. The complete double vertex graph is a natural extension of the Cartesian product and is a generalization of the double vertex graph. In the same chapter we also study the important problem of reconstructing the graph G from the double vertex graph and the complete double vertex graph of G.

In Chapters 3 and 4 we look at two different vertex partitioning problems. In Chapter 3 we study the dominator partitions of graphs, which partition the vertex set of a graph based on some domination properties. In particular, we look at the properties of the upper dominator partition number, and calculate the upper dominator partition number of certain graphs, including paths and cycles.

In Chapter 4 we study a vertex partitioning problem called ranking. This is a variation of the vertex coloring problem. We establish further properties of minimal rankings, and study rankings of the Cartesian product of two complete graphs, $K_n \square K_n$, also known as the rook's graph. We establish bounds for the rank number of $K_n \square K_n$ and calculate the arank number of $K_n \square K_n$.

Chapter 5 contains some open problems related to graph products and vertex partitions.

Chapter 2

Double Vertex Graphs and Complete Double Vertex Graphs

There are many graph functions with which one can construct a new graph from a given graph or set of graphs, such as the Cartesian product and the line graph. One such graph function is called the *double vertex graph*. This was introduced by Alavi et al. in [1], and studied in [2, 3] inter alia. For a survey, see [6].

Let $G = (V, E)$ be a graph with order $n \geq 2$. The *double vertex graph* of G, denoted by $U_2(G)$, is the graph whose vertex set consists of all $\binom{n}{2}$ unordered pairs of V such that two vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \bigcap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G. An example of a double vertex graph is given in Figure 2.1.

The motivation for graph products includes the advantage of studying properties of a given larger graph by studying those properties on a set of smaller graphs, which are the factors of the larger graph under some graph product. Graph products can also be viewed as a tool to systematically produce new graphs from a graph or a set of graphs.

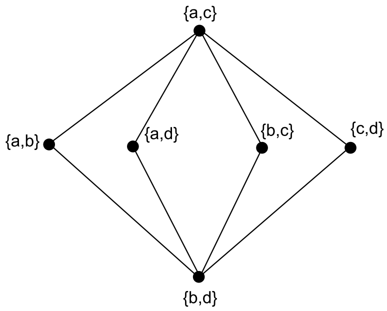


Figure 2.1: The double vertex graph of a 4-cycle, *abcda*

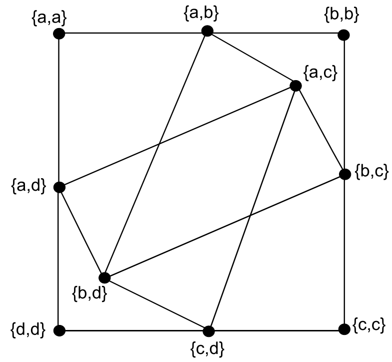


Figure 2.2: The complete double vertex graph of a 4-cycle, *abcda*

If we consider the Cartesian product as a unary operation, that is, the Cartesian product of G with itself, the vertex set of the Cartesian product consists of ordered pairs of vertices of G. What happens if we allow the vertex set of the product to be unordered pairs of vertices of G? In [26] we introduced this concept and called the new product the *complete double vertex graph.* This product was implicitly introduced by Chartrand et al. in [10], and used in [20]. The *complete double vertex graph* of G, denoted by $CU_2(G)$, is the graph whose vertex set consists of all $\binom{n+1}{2}$ unordered pairs of elements of V (duplicates allowed). That is, it contains all the vertices of $U_2(G)$ and all 2-element multisets of the form $\{a, a\}$. Again, two vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \bigcap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G. Figure 2.2 gives an example of a complete double vertex graph.

The complete double vertex graph is a natural extension of the Cartesian product and also a generalization of the double vertex graph. We study properties of double vertex graphs and complete double vertex graphs, and also investigate the important problem of reproducing the original graph from these two graph functions.

Most of the results in this chapter are obtained by Jacob, Goddard and Laskar in [26].

2.1 Basics of Double Vertex Graphs

In this section we look at some basic properties of double vertex graphs, which will help us study the problem of reconstructing G from $U_2(G)$.

Observation 2.1.1. [1] *If G has n vertices and m edges, then the double vertex graph of G has $n(n-1)/2$ vertices and $m(n-2)$ edges.*

Indeed for each edge of G there are $n-2$ edges of $U_2(G)$.

Observation 2.1.2. [1] *The degree of the vertex $\{x, y\}$ in $U_2(G)$ is:*
(i) $\deg_G(x) + \deg_G(y)$, *if* $xy \notin E(G)$,
(ii) $\deg_G(x) + \deg_G(y) - 2$, *otherwise.*

Corollary 2.1.3. [1] *If G is a connected graph, then $U_2(G)$ is regular if and only if G is either a complete graph or $K(1, 3)$.*

The following results have been proved by the respective authors.

Theorem 2.1.4. [1, 6] *a) $U_2(G)$ is a tree if and only if $G = K_2$ or $G = P_3$.*
b) $U_2(G)$ is a cycle if and only if $G = K_3$ or $G = K(1, 3)$.
c) If G is connected, $U_2(G)$ is Eulerian if and only if the degrees of all vertices in G have the same parity.
d) $U_2(K_n)$ is the line graph of K_n.

Theorem 2.1.5. [1] *G is connected if and only if $U_2(G)$ is connected. Indeed, if G has k components each of order at least two, then $U_2(G)$ has $k(k+1)/2$ components.*

Theorem 2.1.6. [6, 41] *If G is a k-connected graph with $k \geq 3$, then $U_2(G)$ is $(2k-2)$-connected.*

Theorem 2.1.7. *For any graph G, $\chi(U_2(G)) \leq \chi(G)$.*

Proof. Let f be a chromatic coloring of G. For any vertex $\{a, b\}$ of $U_2(G)$, define

$$g(\{a, b\}) = f(a) + f(b) \mod \chi(G).$$

Clearly g uses at most $\chi(G)$ labels.

Let $\{a, b\} \in V(U_2(G))$. By the definition of $U_2(G)$, any vertex adjacent to $\{a, b\}$ is of the form $\{a, c\}$ with $bc \in E(G)$ or $\{d, b\}$ with $ad \in E(G)$. Wlog, assume $\{a, b\}$ is adjacent to $\{a, c\}$. We will show that $g(\{a, b\}) \neq g(\{a, c\})$.

$\{a, b\}$ is adjacent to $\{a, c\}$ implies, since f is a $\chi(G)$-coloring of G, $f(b) \neq f(c)$ and $\max\{f(b), f(c)\} \leq \chi(G)$.

$$\begin{aligned} g(\{a, b\}) &= f(a) + f(b) \mod \chi(G) \\ &\neq f(a) + f(c) \mod \chi(G) \\ &= g(\{a, c\}) \end{aligned}$$

Thus g is a proper coloring of $U_2(G)$. Since g uses at most $\chi(G)$ labels, we have $\chi(U_2(G)) \leq \chi(G)$. □

This bound is sharp as the following observation shows.

Observation 2.1.8. $\chi(U_2(C_4)) = \chi(C_4) = 2$.

However, there are graphs where $\chi(U_2(G)) \neq \chi(G)$.

Observation 2.1.9. $\chi(U_2(K_4)) = 3 < \chi(K_4) = 4$.

Proof. $U_2(K_4)$ is the line graph of K_4, shown in Figure 2.3, and thus $\chi(U_2(K_4)) = 3$. □

Theorem 2.1.10. *If G contains k triangles, then $U_2(G)$ contains $k(n-2)$ triangles, where $n = |V(G)|$.*

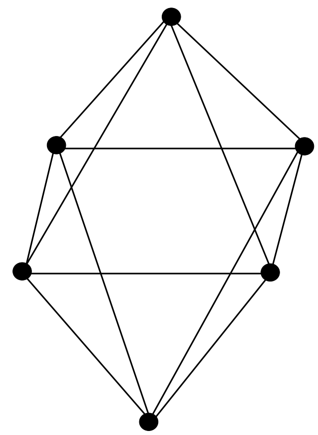


Figure 2.3: $U_2(K_4)$

Proof. For any triangle abc in G, the vertices $\{a, b\}$, $\{b, c\}$ and $\{a, c\}$ form a triangle in $U_2(G)$. Also for any $d \neq a, b, c$, the vertices $\{d, a\}$, $\{d, b\}$ and $\{d, c\}$ form a triangle in $U_2(G)$. Since there are $n - 3$ choices for d, for each triangle in G, $U_2(G)$ has at least $1 + (n - 3) = n - 2$ triangles. Thus if G has k triangles, then $U_2(G)$ contains at least $k(n - 2)$ triangles.

On the other hand, consider any triangle T' in $U_2(G)$. Let two of its vertices be $\{a, b\}$ and $\{b, c\}$. Then the third vertex is either $\{a, c\}$ or $\{b, e\}$ for some e and there are $n - 3$ choices for e. It follows that T' has one of the above forms which implies $U_2(G)$ has at most $k(n - 2)$ triangles. $\square$

In particular, $U_2(G)$ has a triangle if and only if G has one.

For more results on double vertex graphs, see [1, 2, 3, 6].

2.2 Properties of Complete Double Vertex Graphs

We now explore some properties of complete double vertex graphs, $CU_2(G)$.

Observation 2.2.1. *If G has n vertices and m edges, then $CU_2(G)$ has $n(n + 1)/2$*

vertices and nm edges.

As in the case of double vertex graphs, if G is empty then so is $CU_2(G)$.

Observation 2.2.2. *Let x and y be vertices of a graph G. Then the degree of the vertex $\{x, y\}$ of the complete double vertex graph of G is:*
(i) $\deg_G(x)$, if $x = y$, and
(ii) $\deg_G(x) + \deg_G(y)$, otherwise.

Proof. The vertex $\{x, x\}$ in $CU_2(G)$ is adjacent all vertices $\{x, a\}$ where a is adjacent to x in G. Thus degree of the vertex $\{x, x\}$ is $\deg_G x$.

The vertex $\{x, y\}$ with $x \neq y$ in $CU_2(G)$ is adjacent to all vertices $\{x, a\}$ where a is adjacent to y in G and to all vertices $\{y, b\}$ where b is adjacent to x in G. Thus degree of the vertex $\{x, y\}$ is $\deg_G x + \deg_G y$. □

Corollary 2.2.3. *If the graph $CU_2(G)$ is regular, then it is empty.*

Proof. Assume $CU_2(G)$ is regular. Let x and y be distinct vertices in G. Then the vertices $\{x, x\}$, $\{y, y\}$ and $\{x, y\}$ all have the same degree in $CU_2(G)$. By Observation 2.2.2, this can occur only if $\deg x = \deg y = 0$. □

For example, $CU_2(G)$ is never a cycle.

Theorem 2.2.4. *The graph $U_2(G)$ is an induced subgraph of $CU_2(G)$ and the graph G is an induced subgraph of $CU_2(G)$. Indeed, the edges of $CU_2(G)$ can be partitioned into n sets such that each set induces a copy of G.*

Proof. Suppose $V(G) = \{1, 2, \ldots, n\}$. Consider $S = \{\,\{i, i\} \mid 1 \leq i \leq n\,\}$. By the definition of $U_2(G)$ and $CU_2(G)$, $CU_2(G) - S = U_2(G)$ and hence $U_2(G)$ is an induced subgraph of $CU_2(G)$.

Let $S_i = \{ \{i, j\} \mid 1 \le j \le n \}$ for $1 \le i \le n$. By the definition of $CU_2(G)$ and S_i, any edge in $CU_2(G)$ is contained in a unique $\langle S_i \rangle$, where $\langle S_i \rangle$ is the graph induced by S_i. Also, each $\langle S_i \rangle$ will be a copy of G. □

Corollary 2.2.5. *If G contains a cycle of length r, then $CU_2(G)$ contains a cycle of length r.*

Theorem 2.2.6. *The chromatic number of $CU_2(G)$ is the same as the chromatic number of G.*

Proof. By Theorem 2.2.4, $CU_2(G)$ contains a copy of G. Thus

$$\chi(CU_2(G)) \ge \chi(G). \tag{2.1}$$

Let f be a chromatic coloring of G. For any vertex $\{a, b\}$ of $CU_2(G)$, define

$$g(\{a, b\}) = f(a) + f(b) \mod \chi(G).$$

Clearly, g uses at most $\chi(G)$ labels. As in the case of double vertex graphs, we can show that g is a coloring of $CU_2(G)$ and hence

$$\chi(CU_2(G)) \le \chi(G). \tag{2.2}$$

From inequalities 2.1 and 2.2 we get $\chi(CU_2(G)) = \chi(G)$. □

Corollary 2.2.7. *$CU_2(G)$ is bipartite if and only if G is bipartite.*

Theorem 2.2.8. *G is connected if and only if $CU_2(G)$ is connected. Indeed, if G has k components, then $CU_2(G)$ has $k(k+1)/2$ components.*

Proof. We will show, using induction on k, that if G is a graph with k components then $CU_2(G)$ has $k(k+1)/2$ components.

Suppose $k = 1$. This means G is connected. Let $\{a, b\}$ and $\{x, y\}$ be two vertices of $CU_2(G)$. We will show that there is a path between $\{a, b\}$ and $\{x, y\}$ in $CU_2(G)$.

Since G is connected, let $au_1u_2 \dots u_ix$ be an ax path in G. Also let $bv_1v_2 \dots v_jy$ be a by path in G. Then by the definition of $CU_2(G)$,

$$\{a, b\}\{a, v_1\}\{a, v_2\} \dots \{a, v_j\}\{a, y\}\{u_1, y\}\{u_2, y\} \dots \{u_i, y\}\{x, y\}$$

is a path in $CU_2(G)$. Thus $CU_2(G)$ is connected.

Assume that if G is a graph with $k - 1$ connected components, then $CU_2(G)$ has $(k - 1)k/2$ components.

Let G be a graph with k components, say $G_1, G_2, \dots, G_k$. Consider $G' = G - G_k$. By the induction hypothesis, $CU_2(G')$ has $(k - 1)k/2$ connected components. Let $S_i = \{\, \{a, u\} \mid a \in V(G_k), u \in V(G_i)\,\}$. Using similar arguments as in the case when $k = 1$, one can show that $\langle S_i \rangle$ is connected. Also, by the definition of $CU_2(G)$ and S_i, if $\{x, y\} \in S_i$ is adjacent to any vertex $\{p, q\}$ then $\{p, q\} \in S_i$. Thus, $CU_2(G) = CU_2(G') \cup \langle S_1 \rangle \cup \langle S_2 \rangle \cup \dots \cup \langle S_k \rangle$. Therefore, $CU_2(G)$ has $k(k + 1)/2$ connected components. This implies that if G is disconnected, then $CU_2(G)$ is disconnected and hence the proof. $\square$

Observation 2.2.9. *If $G = \cup_{i=1}^{k} G_i$ then $CU_2(G) = \cup_{i=i}^{k} CU_2(G_i) \cup_{i,j=1,i\neq j}^{k} G_{ij}$ where G_{ij} is isomorphic to $G_i \square G_j$, the Cartesian product of G_i and G_j.*

Corollary 2.2.10. *Let G be a connected graph. The graph $CU_2(G)$ is Eulerian if and only if $\deg_G(v)$ is even for every $v \in V(G)$.*

Theorem 2.2.11. [20] *If G is k-connected, then so is $CU_2(G)$.*

Theorem 2.2.12. *The complete double vertex graph of G is a tree if and only if $G = K_1$ or K_2.*

Proof. The graph $CU_2(P_3)$ contains a cycle and thus, if G contains a P_3 as subgraph, then $CU_2(G)$ is not a tree. Also, by Theorem 2.2.8, if G is disconnected then $CU_2(G)$ is disconnected. Hence, if $CU_2(G)$ is a tree then G must be a K_1 or K_2. On the other hand, $CU_2(K_1) = K_1$ and $CU_2(K_2) = P_3$. Hence the proof. $\square$

2.3 Planarity

Definition 2.3.1. *A graph G is a planar graph if G can be drawn on a plane such that no two edges of G crosses each other.*

Alavi et al. determined for which graph G the double vertex graph is planar.

Theorem 2.3.1. [3] *Let G be a connected graph. The graph $U_2(G)$ is planar if and only if G is either a path or a connected subgraph of any of the six graphs shown in Figure 2.4.*

A similar result holds for complete double vertex graphs:

Theorem 2.3.2. *Let G be a connected graph. The graph $CU_2(G)$ is planar if and only if either G is a path or a connected subgraph of any of the six graphs shown in Figure 2.5.*

Proof. ($\Rightarrow$) If G is one of the five graphs in Figure 2.5 or a path, then by construction, $CU_2(G)$ is planar.

($\Leftarrow$) Now assume that $CU_2(G)$ is planar. We know that $U_2(G)$ is an induced subgraph of $CU_2(G)$ and hence $U_2(G)$ is planar. Thus by Theorem 2.3.1, G is a path or a subgraph of one of the six graphs shown in Figure 2.4. If G is a path, then we are done. If G is not a path, then one can check that the maximal subgraphs of the graphs in Figure 2.4 whose complete double vertex graphs are planar are listed in Figure 2.5. $\square$

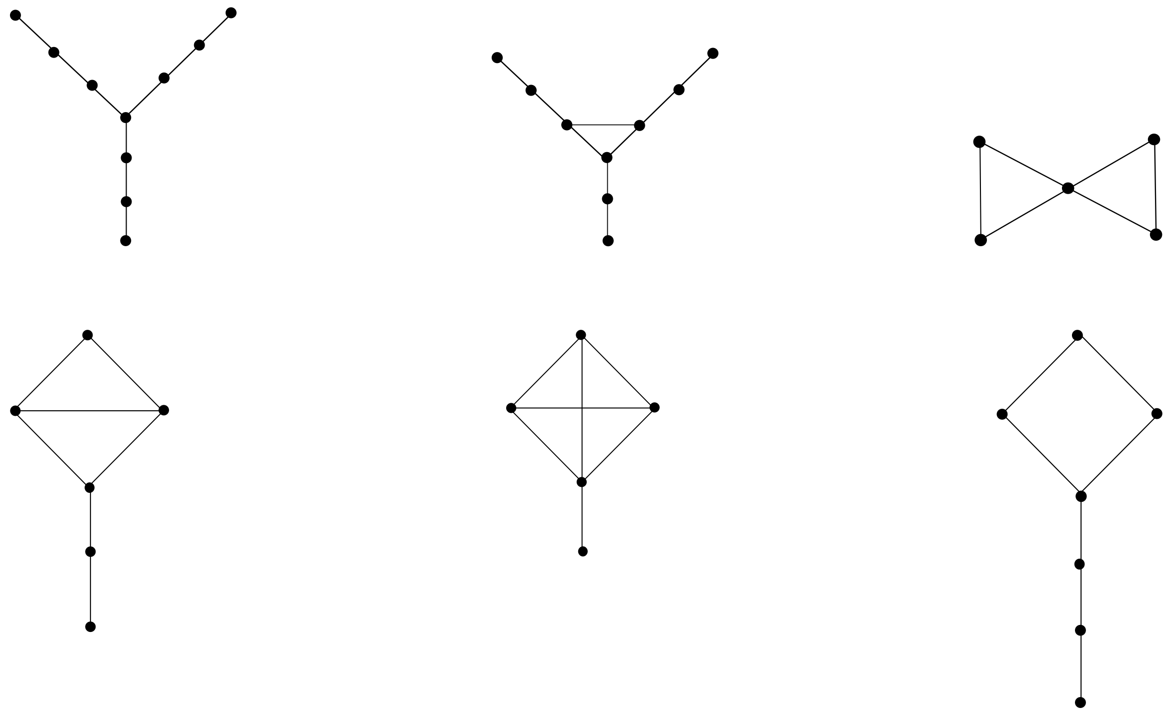


Figure 2.4: Graphs whose double vertex graphs are planar

2.4 Hamiltonian Properties

Definition 2.4.1. *A Hamiltonian graph is a graph with a spanning cycle, also called a Hamiltonian cycle.*

The following results about Hamiltonicity have been obtained.

Theorem 2.4.1. [4] *For $n = 4$ or $n \geq 6$, $U_2(C_n)$ is not Hamiltonian.*

A cycle with an odd chord is a graph obtained by adding the edge $1k$ to C_n, where k is odd, assuming that the cycle has vertices $1, 2, \ldots, n$.

Theorem 2.4.2. [5, 6] *Let G be a Hamiltonian graph of order $n \geq 4$. Then $U_2(G)$ is Hamiltonian if and only if some Hamiltonian cycle of G has an odd chord or if $n = 5$.*

We believe similar results hold for the complete double vertex graph. We provide a result for a specific chord in Theorem 2.4.5.

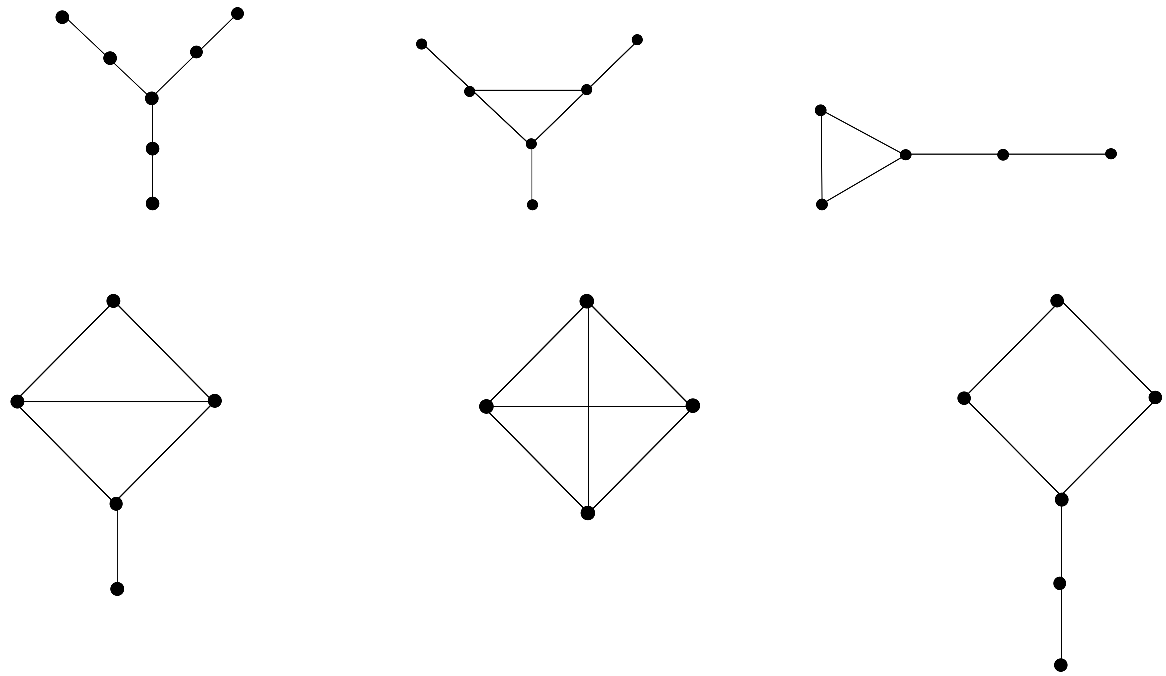


Figure 2.5: Graphs whose complete double vertex graphs are planar

Definition 2.4.2. [11] *A graph G is t-tough if for any vertex cut S the number of components of $G-S$ is at most $|S|/t$.*

Theorem 2.4.3. [11] *Every Hamilton graph is 1-tough.*

Theorem 2.4.4. *For $n \geq 4$, $CU_2(C_n)$ is not Hamiltonian. In fact it is not 1-tough.*

Proof. Let the vertices of C_n be labeled $1, 2, \ldots, n$ and let $S = \{\{1,2\}, \{2,3\}, \ldots, \{n,1\}\}$. Note that for the example shown in Figure 2.6, the vertices in S are the vertices adjacent to the outer (degree 2) vertices.

Thus, the graph $CU_2(C_n) - S$ has at least n isolated vertices, since the neighbors of the vertices $\{i,i\}$, $1 \leq i \leq n$, in $CU_2(C_n)$ are contained in S. Thus $CU_2(C_n) - S$ has at least $n+1$ components, if $n \geq 4$. However, $|S| = n$ and so $CU_2(C_n)$ is not 1-tough for $n \geq 4$. Thus, $CU_2(C_n)$ is not Hamiltonian for $n \geq 4$. $\square$

For a smaller example to follow the proof of Theorem 2.4.5, see Figure 2.2.

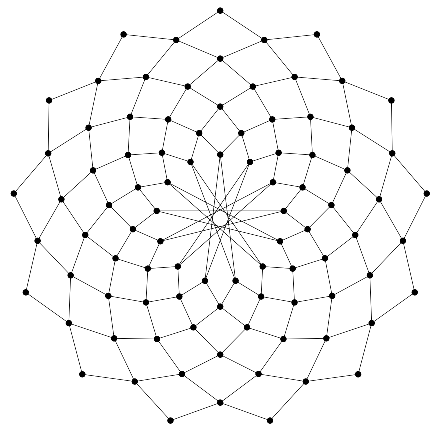


Figure 2.6: $CU_2(C_{13})$

Theorem 2.4.5. *Let G be a cycle on n vertices. Let G' be obtained from G by adding a chord between two vertices of G having distance two between them. Then $CU_2(G')$ is Hamiltonian.*

Proof. __**Case 1:**__ n is odd.

The idea is that $CU_2(C_n)$ has a spanning 2-factor when n is odd, and edges that correspond to the chord will serve as bridges between the factors.

Assume that the vertices of G are numbered 0 to $n-1$ and let the chord added to get G' be 02. Let $H = CU_2(G)$. For any vertex $\{i,j\}$ in H, the distance $d(i,j)$ between i and j in G is in the range $0 \leq d \leq (n-1)/2$. Construct a spanning 2-factor for H as follows:

For $0 \leq i \leq \lfloor \frac{n-3}{4} \rfloor$, let S_i be the graph induced by vertices of the form $\{u,v\}$ such that $d(u,v) \in \{2i, 2i+1\}$. Each S_i is a cycle on $2n$ vertices, and if $n \equiv 3 \mod 4$, then $S = \{S_i \mid 0 \leq i \leq \lfloor \frac{n-3}{4} \rfloor\}$ forms a spanning 2-factor. If $n \equiv 1 \mod 4$, then let $j = \lfloor \frac{n-3}{4} \rfloor + 1$ and let S_j be the cycle induced by the n vertices of the form $\{u,v\}$ such that $d(u,v) = 2j$. Thus, when $n \equiv 1 \mod 4$, $S \cup S_j$ forms a spanning 2-factor of H.

Now, let $H' = CU_2(G')$. Adding the chord to G adds n edges to H; call each such edge a *bridge-edge*. In each cycle S_i except for $i \geq (n-3)/4$ there are two consecutive vertices $\{0, (n-2i) \mod n\}$ and $\{0, n-2i-1\}$, and two bridge-edges joining these to $\{2, (n-2i) \mod n\}$ and $\{2, n-2i-1\}$, which are consecutive in cycle S_{i+1}. We use these bridge-edges to go between cycles to form a Hamiltonian cycle of H'.

A Hamiltonian cycle obtained using this idea for C_{11} with the chord 02 is given in Figure 2.7.

Case 2: n is even.

Let G be a cycle of the form $\{0, n-1, 1, 2, \ldots, n-2\}$ and G' be obtained by adding the chord 01. Consider $C = G' - \{n-1\}$. Now C is a cycle on $n-1$ vertices where n is even, and hence as in Case 1 we can find a spanning 2-factor S for $CU_2(C)$. Also, the subgraph of $CU_2(G')$ induced by $S' = \{\{0, n-1\}, \{1, n-1\}, \ldots \{n-1, n-1\}\}$ is a cycle on n vertices and $S \cup S'$ is a spanning 2-factor for $H' = CU_2(G')$.

For each $S_i \in S$, the consecutive vertices $\{0, 2i\}$ and $\{0, 2i+1\}$ are adjacent to the consecutive vertices $\{2i, n-1\}$ and $\{2i+1, n-1\}$ respectively in S'. Hence we can form a Hamiltonian cycle in $H' = CU_2(G')$. $\square$

2.5 Reconstruction of G from $CU_2(G)$ and $U_2(G)$

One of the major challenges in the study of graph products is to reproduce the original graph from the graph product. In this section we examine some more properties of these graph products which help us to reconstruct some classes of graphs from $CU_2(G)$ and $U_2(G)$.

Note that reconstructing the graph G from the Cartesian product $G \square G$ is solved. But the techniques do not seem to be applicable here. For more details on reconstructing a graph from its Cartesian product see [25].

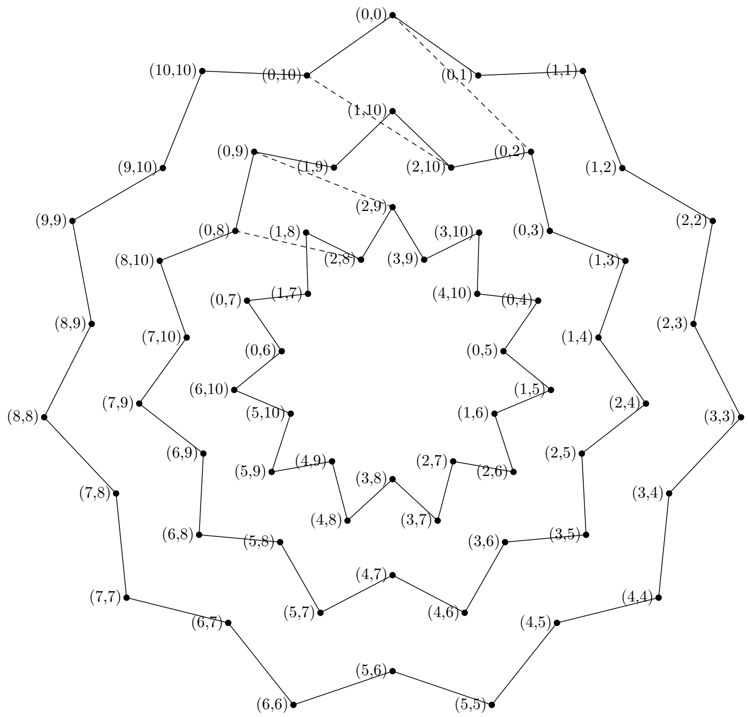


Figure 2.7: A Hamiltonian cycle in $CU_2(G)$ where $G = C_{11}$ with the chord 02.

2.5.1 Reconstructing G from $CU_2(G)$

We start with the complete double vertex graph case. We call the vertices of the form $\{x, x\}$ *twin-pairs*.

Theorem 2.5.1. *Let G be a graph. Then $xy \in E(G)$ if and only if the twin-pairs $\{x, x\}$ and $\{y, y\}$ have a common neighbor in $CU_2(G)$.*

Proof. The only possible vertex that could be a common neighbor of the pairs $\{x, x\}$ and $\{y, y\}$ in $CU_2(G)$ is the pair $\{x, y\}$. This is a common neighbor if and only if $xy \in E(G)$. $\square$

Corollary 2.5.2. *If one can identify the twin-pairs of $CU_2(G)$, then one can construct the line-graph of G, and hence one can reconstruct the graph G.*

Note that the line graphs of K_3 and $K_{1,3}$ are isomorphic. However, one can distinguish $CU_2(K_3)$ from $CU_2(K_{1,3})$ by the number of vertices.

Corollary 2.5.3. *If G is either regular or the degree of every vertex in G is odd, then one can reconstruct G from $CU_2(G)$.*

Proof. If all vertices of G are of degree r, then the degree of the twin-pairs is r and that of the non-twin-pairs is $2r$. So one can identify the twin-pairs and reconstruct G.

If the vertices of G are of odd degree, then the twin-pairs of $CU_2(G)$ have odd degree, while any other vertex of $CU_2(G)$ has even degree. So one can identify the twin-pairs and hence reconstruct G. $\square$

2.5.2 Reconstructing G from $U_2(G)$

We next consider the double vertex graph case. We call $\{a, b\} \in V(U_2(G))$ a *line-pair* if and only if $ab \in E(G)$. Hence each vertex of a double vertex graph is either a line-pair or a non-line-pair.

Theorem 2.5.4. *Two line-pairs in $U_2(G)$ are adjacent if and only if the corresponding edges lie in a triangle.*

Proof. Assume line-pairs $\{a, b\}$ and $\{a, c\}$ are adjacent in $U_2(G)$. By the definition of the double vertex graph, $bc \in E(G)$, and hence ab, ac, bc forms a K_3.

If ab, ac are edges in a K_3 in G, then by the definition of $U_2(G)$, the pairs $\{a, b\}$ and $\{a, c\}$ will be adjacent. $\square$

Theorem 2.5.5. *Two line-pairs in $U_2(G)$ have a common neighbor in $U_2(G)$ if and only if the corresponding edges are either adjacent in G or lie in a 4-cycle of G.*

Proof. ($\Leftarrow$) If edges ab and bc are adjacent, then $\{a, c\}$ is a common neighbor to $\{a, b\}$

and $\{b, c\}$ in $U_2(G)$. If edges ab and cd lie in a 4-cycle of G, say $abcda$, then $\{a, b\}$ and $\{c, d\}$ have common neighbors $\{a, c\}$ and $\{b, d\}$.

($\Rightarrow$) Suppose two line-pairs $\{a, b\}$ and $\{x, y\}$ have a common neighbor in $U_2(G)$. If the two line-pairs overlap, say $a = x$, then clearly the corresponding edges are adjacent. If the line-pairs don't overlap, then the common neighbor has one element from $\{a, b\}$ and one element from $\{x, y\}$. Say the common neighbor is $\{a, x\}$. Then, $abxya$ forms a 4-cycle in G. □

Corollary 2.5.6. *Suppose G has no 4-cycle. If one can identify the line-pairs in $U_2(G)$, then one can construct the line graph of G and hence one can reconstruct G.*

Note that the line graphs of K_3 and $K_{1,3}$ are isomorphic. However, as in the case of $CU_2(G)$, one can distinguish $U_2(K_3)$ from $U_2(K_{1,3})$ by the number of vertices.

Corollary 2.5.7. *If G is regular and has no 4-cycle, then one can reconstruct G from $U_2(G)$.*

Proof. If G is regular, then the line-pairs of $U_2(G)$ have degree 2 less than the non-line-pairs, by Observation 2.1.2. So one can recognize them, and hence by Corollary 2.5.6 one can reconstruct G. □

2.5.3 Reconstructing Cubic Graphs

As one can see from Theorem 2.5.5, the presence of 4-cycles in G seems to make the reconstruction a little harder. To overcome this, we restrict our attention to 3-regular graphs, also called cubic graphs.

Theorem 2.5.8. *Let G be a cubic graph. The corresponding edges of two line-pairs lie in an induced 4-cycle in G if and only if the line-pairs lie in an induced $K_{2,4}$ in*

$U_2(G)$ with the 4 line-pairs as one partite set and the 2 non-line-pairs as the other partite set.

Proof. ($\Rightarrow$) By the definition of $U_2(C_4)$, as shown in Figure 2.1.

($\Leftarrow$) Consider an induced $H = K_{2,4}$ in $U_2(G)$ as in the hypothesis. Let a be any vertex in a line-pair of H. Then a cannot occur in all 4 line-pairs, since G is cubic.

Suppose a occurs in 3 line-pairs; say $\{a, b\}, \{a, c\}, \{a, d\}$. Then a must occur in both of the non-line-pairs of H; say $\{a, x\}$ and $\{a, y\}$. Then the fourth line-pair is $\{x, y\}$. This implies that x is adjacent to all of b, c, d and y in G, which is a contradiction. Thus, it follows that a occurs in at most 2 line-pairs. Let $\{a, b\}$ be such a line-pair. Then, by the definition of $U_2(G)$, one of the non-line-pairs must be either $\{a, x\}$ or $\{b, x\}$ for some $x \in V(G)$.

Consider a non-line-pair of H, say the pair $\{a, x\}$. Then x is in at most two line-pairs by the previous paragraph. It follows that vertices a and x each lie in exactly two line-pairs of H, because all four line-pairs of H are adjacent to $\{a, x\}$. Also, if the other non-line-pair contains a or x, then it contains the other one too, a contradiction. Thus, the two non-line-pairs do not overlap. Wlog, say the other non-line pair is $\{b, y\}$.

Then the line-pairs are $\{a, b\}$, $\{a, y\}$, $\{b, x\}$ and $\{x, y\}$. These four line-pairs induce a 4-cycle in G. Hence the proof. $\square$

Hence, in a cubic graph one can identify the line-pairs, and hence one can identify the induced 4-cycles as well as the non-induced 4-cycles. The idea is to construct the line graph, except that at this point in some cases one can only identify the 4 vertices which form a cycle, without knowing the order of the vertices.

Theorem 2.5.9. *Let G be a cubic graph. Suppose two line-pairs lie in $K_{2,4}$ representing an induced C_4 in G, but not in a second $K_{2,4}$. Then the two line-pairs are* *adjacent in G if and only if in $U_2(G)$ there exists a line-pair $\{p, q\}$ which does not lie in the same $K_{2,4}$ as the two line-pairs, and $\{p, q\}$ is at a distance 2 from at least one of the line-pairs, but has a path of length 2 from the other line-pair.*

Proof. ($\Rightarrow$) Suppose ab and ac lie in an induced 4-cycle in G. Since G is a cubic graph, there exists $d \in V(G)$ such that $ad \in E(G)$. However, since we assumed that neither ab nor ac lie in a second $K_{2,4}$, we have cannot have both $bd \in E(G)$ and $cd \in E(G)$. Let $cd \notin E(G)$. Therefore, in $U_2(G)$, the line-pair $\{a, d\}$ is at a distance 2 from $\{a, c\}$ and has a path of length 2 from $\{a, b\}$ through the vertex $\{b, d\}$.

($\Leftarrow$) Suppose that two distinct line-pairs $\{a, b\}$ and $\{x, y\}$ have a line-pair at distance 2 from at least one of them and a path of length 2 from the other. If the elements of the line-pair are distinct from the line-pairs $\{a, b\}$ and $\{x, y\}$, say $\{c, d\}$, then there is a 4-cycle in G containing either ab or xy, and cd. This implies that either $\{a, b\}$ or $\{x, y\}$ will lie on a second $K_{2,4}$ in $U_2(G)$. In any case, we get a contradiction, as we assumed that the line-pairs $\{a, b\}$ and $\{x, y\}$ do not lie in a second $K_{2,4}$. Also, the line-pair cannot overlap with both line-pairs $\{a, b\}$ and $\{x, y\}$, because if it does then it will be in the same $K_{2,4}$ as $\{a, b\}$ and $\{x, y\}$. If the line-pair overlaps with one line-pair, say $\{a, b\}$, then $\{x, y\}$ will be part of a second $K_{2,4}$ in $U_2(G)$.

Hence if we assume that $\{a, b\}$ and $\{x, y\}$ do not overlap, then we get a contradiction, and hence the proof. □

One can therefore determine the order of the edges in the case of 4-cycles. If two 4-cycles overlap, then one can identify the overlapping $K_{2,4}$ and hence determine the overlapping edges.

Corollary 2.5.10. *Given $U_2(G)$, one can reconstruct G if any of the following is true.*

(i) G is a cubic graph

(ii) G is regular and contains no 4-cycles

(iii) G contains no 4-cycles and one can identify the line-pairs in $U_2(G)$

2.5.4 Reconstruction of Trees

Trees are one of the popular classes of graphs studied when some properties or techniques are difficult to apply to arbitrary graphs. In [1], Alavi, Behzad, Erdős and Lick made the following observation and stated Theorem 2.5.12.

Observation 2.5.11. [1] *If a connected graph H whose order is at least three is the double vertex graph of some graph G of order n, then G has a vertex of degree one if and only if H contains $n-2$ independent edges whose removal from $H = U_2(G)$ results in exactly two components, one of which is $G - v$ and the other, $U_2(G - v)$.*

Theorem 2.5.12. [1] *Let T and T' be trees. Then $U_2(T) \cong U_2(T')$ if and only if $T \cong T'$.*

However, [1] does not provide full proofs for these results. In regards to complete double vertex graphs, we are able to prove one direction of the equivalent result.

Theorem 2.5.13. *Let $H = CU_2(G)$ be connected. Then there exist $(n-1)\delta(G)$ edges whose deletion will result in a graph with more than one component, one of which is isomorphic to G.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Note that for any vertex v_i of G, the subgraph of H induced by $S = \{\{v_i, v_1\}, \{v_i, v_2\}, \ldots \{v_i, v_n\}\}$, denoted by $H[S]$, is isomorphic to G and $H - S$ is isomorphic to $CU_2(G - v)$.

Let $\deg_G(v_1) = \delta = \delta(G)$. Let $S = \{\, \{v_1, v_i\} \mid 1 \leq i \leq n \,\}$. The vertex $\{v_1, v_i\}$ of $CU_2(G)$, $2 \leq i \leq n$, is adjacent to exactly δ vertices not in S. Let M be the set of edges joining the vertices in S to the vertices not in S. Clearly, $|M| = (n-1)\delta$ and

$H - M$ has more than one component, one of which is the graph induced by S which is isomorphic to G. $\square$

We are unable to show that one can reconstruct a tree from its complete double vertex graph. However, we have verified by computer up to order 10 that different trees give different complete double vertex graphs. Thus we conjecture:

Conjecture 2.5.14. *Let T and T' be trees. Then $CU_2(T) \cong CU_2(T')$ if and only if $T \cong T'$.*

Chapter 3

Dominator Partitions of Graphs

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \dots v_n\}$. Partitioning the vertices of G into disjoint sets such that each set satisfies some property $\mathcal{P}$ is a commonly studied and interesting problem in graph theory. One well-studied vertex partitioning problem is the vertex coloring. In the case of coloring, the property that needs to be satisfied is that every set must be an independent set. Dominator partitions of graphs, which require the sets to satisfy some property related to domination, are introduced by Hedetniemi et al. in [23].

A set $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set S in G, in which case S is called a γ-set. A vertex $v \in V$ *dominates* a set $S \subseteq V$, denoted by $v \succ S$, if it is adjacent to every vertex $w \in S$, in which case we say that v is a *dominator* of S. Domination and its many variations are well-studied topics in graph theory, for example see [22, 24].

The study of dominator partitions is part of a larger study of vertex partitions $\Pi = \{V_1, V_2, \dots, V_k\}$ where each class V_i or each vertex $v \in V$ has some domination property. The first well-studied domination partition concepts were that of the *do-*

matic number d(G) and *idomatic number id(G)*, introduced in 1977 by Cockayne and Hedetniemi [12].

Another kind of partitioning of the vertex set is introduced by Hedetniemi et al. in [23]. A *dominator partition* of a graph G is a partition $\Pi = \{V_1, V_2, \dots, V_k\}$ of $V(G)$ such that every vertex $v \in V(G)$ is a dominator of at least one class $V_j \in \Pi$; that is, for every $v \in V(G)$ there exists a j, $1 \le j \le k$, such that $v \succ V_j$. Note that a vertex $v \in V_i$ is a dominator of its own class if $V_i \subseteq N[v]$, where $N[v] = \{u \mid uv \in E(G)\} \cup \{v\}$. Since every vertex dominates itself, it follows that the trivial partition $\Pi = \{\{v_1\}, \{v_2\}, \dots, \{v_n\}\}$ is a dominator partition. Thus, every graph has a dominator partition.

A dominator partition $\Pi = \{V_1, V_2, \dots, V_k\}$ is *minimal* if any partition Π' obtained from Π by forming the union of any two classes into one class, that is $V_i \cup V_j$, $i \neq j$, is no longer a dominator partition. The *dominator partition number* of a graph G, denoted $\pi_d(G)$, is the minimum order of a dominator partition of G. The *upper dominator partition number* of a graph G, denoted $\Pi_d(G)$, is the maximum order of a minimal dominator partition of G.

In this chapter, we will discuss some new results we obtained in [27], which lead to finding the upper dominator partition number of paths and cycles. In [27], we also calculated bounds for the upper dominator partition number of a tree.

3.1 Bounds for $\pi_d(G)$ and $\Pi_d(G)$

The following observations are made by Hedetniemi et al. in [23].

Observation 3.1.1. [23] *For any graph G of order n,*

1. $1 \le \pi_d(G) \le \Pi_d(G) \le n$.

2. $\pi_d(G) = 1 = \Pi_d(G) \Leftrightarrow G = K_n$.

3. $\pi_d(G) = n = \Pi_d(G) \Leftrightarrow G = \overline{K_n}$.

A very tight bound for $\pi_d(G)$ is proved in [23].

Theorem 3.1.2. [23] *For any graph G, $\gamma(G) \leq \pi_d(G) \leq \gamma(G) + 1$.*

In regards to $\Pi_d(G)$, the only bound known for $\Pi_d(G)$ is in terms of the minimum and maximum degree of the graph. The following inequality is proved in [23].

Theorem 3.1.3. [23] *For any graph G of order n,*

$$\frac{n}{1 + \Delta(G)} \leq \pi_d(G) \leq \Pi_d(G) \leq n - \delta(G).$$

Example 3.1.1. $\Pi_d(K_{m,n}) = max\ \{m, n\} = (n + m) - \delta(K_{m,n})$. *In particular, if G is a star on n vertices, $K_{1,n-1}$, then $\Pi_d(G) = n - \delta(G) = n - 1$.*

However, for a general graph, especially when G is a tree, this upper bound may not be a tight bound. We will see that $n - \delta(G)$ is not a tight bound for a path or a cycle by calculating $\Pi_d(G)$ when G is a path or a cycle.

3.2 Private Dominator Classes and Π_d of Trees

Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a dominator partition of a graph G. A class $V_i \in \Pi$ is said to be a *private dominator class*, abbreviated as PDC, if there exists $v \in V(G)$ such that v is a private neighbor of V_i; that is, $v \succ V_i$ and v does not dominate any other class of Π.

Example 3.2.1. $\Pi = \{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_6\}\}$, shown in Figure 3.1, is a dominator partition of P_6, and every class of Π is a PDC.

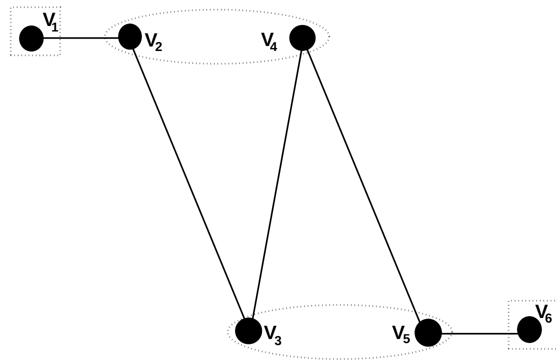


Figure 3.1: A dominator partition of P_6 where all classes are PDCs.

Example 3.2.2. Consider $P_4 = \{v_1, v_2, v_3, v_4\}$ where $v_i v_{i+1} \in E(P_4)$ for $i = 1, 2, 3$, and a partition of P_4, $\Pi = \{\{v_1\}, \{v_2, v_3\}, \{v_4\}\}$. Here the class $\{v_2, v_3\}$ is not a PDC because there is no vertex in P_4 which dominates only $\{v_2, v_3\}$.

In [27], we obtained a characterization for a dominator partition to be minimal.

Theorem 3.2.1. *Let* Π *be a dominator partition of a graph* G. Π *is a minimal dominator partition if and only if there is at most one class in* Π *which is not a PDC.*

Proof. ($\Rightarrow$) Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a minimal dominator partition of a graph G. Assume that there exist $V_i, V_j \in \Pi$ such that V_i and V_j are not PDCs. Consider the partition $\widehat{\Pi}$ obtained from Π by merging V_i and V_j. Let $v \in V(G)$. Since Π is a dominator partition, $v \succ V_m$ for some m, $1 \leq m \leq k$ in Π. If $m \neq i, j$ then $v \succ V_m$ in $\widehat{\Pi}$ also. Wlog, if we assume that $m = i$, then since V_i is not a PDC, v should dominate at least one more class, say $v \succ V_r$. If $r \neq j$ then $v \succ V_r$ in $\widehat{\Pi}$ also. If $r = j$, then $v \succ V_i \cup V_j$ in $\widehat{\Pi}$. So in any case, $\widehat{\Pi}$ is a dominator partition, which contradicts the minimality of Π.

($\Leftarrow$) Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a dominator partition of a graph G such that there is at most one class which is not a PDC. Let V_i, $V_j \in \Pi$. Since Π has at most one class which is not a PDC, either V_i or V_j or both must be PDCs. Wlog assume that V_i is a PDC. So there exists $v \in V(G)$ such that v dominates only V_i in Π and in particular v does not dominate V_j. Thus in a partition $\widehat{\Pi}$ obtained by merging V_i and V_j, v does not dominate any classes, and so $\widehat{\Pi}$ is not a dominator partition. Hence Π is a minimal dominator partition. $\square$

Theorem 3.2.2. *For any graph G, $\Pi_d(G) \geq \beta_0(G)$, where $\beta_0(G)$ is the independence number of G.*

Proof. We will construct a minimal dominator partition of order at least β_0 for G. Let S be an independent set of vertices of G such that $|S| = \beta_0(G)$. Construct Π by taking all the vertices in S as singleton classes, and $V - S$ as one class. Since S is a maximal independent set, any vertex not in S is adjacent to a vertex in S, and hence Π is a dominator partition. If all the singleton classes formed using the vertices in S are PDCs, then Π is a minimal dominator partition by Theorem 3.2.1 and $|\Pi| = \beta_0(G) + 1$.

So, for some $x \in S$, suppose $\{x\}$ is not a PDC. Then, since S is a set of independent vertices, we must have $x \succ V - S$. Now, form a partition Π' from Π by merging x into $V - S$. Since S is a set of independent vertices and $x \in S$, no vertex of $S - \{x\}$ dominates $(V - S) \cup \{x\}$, and hence all the classes of Π' are PDCs. So, by Theorem 3.2.1, Π' is a minimal dominator partition and $|\Pi'| = \beta_0$. Thus, we have $\Pi_d(G) \geq \beta_0(G)$. $\square$

For a bipartite graph G, $\beta_0(G) \geq n/2$. Thus we have the following corollary.

Corollary 3.2.3. *If G is a bipartite graph on n vertices, then $\Pi_d(G) \geq n/2$.*

We saw in Example 3.1.1 that the complete bipartite graph $K_{m,m}$ has $\Pi_d = m$. Thus the lower bound is sharp for bipartite graphs. However, we show now that it is not quite sharp for trees. We will need the following lemma.

Lemma 3.2.4. *Let T be a tree and $v \in V(T)$. Consider a tree T^* formed by adding two vertices x and y such that x is adjacent to y and y is adjacent to v. Then $\Pi_d(T^*) \geq \Pi_d(T) + 1$.*

Proof. Let Π be a minimal dominator partition of T such that $|\Pi| = \Pi_d(T)$. Consider $\Pi' = \Pi \cup \{x, y\}$. Since x and y dominate the class $\{x, y\}$, Π' is a dominator partition of T^*. Further, since x is adjacent only to y, x dominates only $\{x, y\}$ in Π' and hence $\{x, y\}$ is a PDC. Also, since no vertex other than x and y dominates $\{x, y\}$, any PDC in Π is a PDC in Π' as well. So Π' has at most one class which is not a PDC. Hence by Theorem 3.2.1, Π' is a minimal dominator partition of T^* and $|\Pi'| = |\Pi| + 1$. Thus, $\Pi_d(T^*) \geq \Pi_d(T) + 1$. □

Theorem 3.2.5. *For any tree T on n vertices, $n \geq 4$, $\Pi_d(T) \geq \lceil \frac{n+1}{2} \rceil$.*

Proof. By Corollary 3.2.3, we know for any tree T, $\Pi_d(T) \geq \frac{n}{2}$. We will show using induction on n that for any tree T with more than three vertices, $\Pi_d(T) > \frac{n}{2}$. When $n = 4$, $\Pi_d(T) > \frac{n}{2}$, because $\Pi_d(K_{1,3}) = 3$ and $\Pi_d(P_4) = 3$. Assume that for any tree with k vertices where $3 < k < n$, $\Pi_d(T) > \frac{k}{2}$.

Suppose T^* is a tree on n vertices such that $\Pi_d(T^*) = \frac{n}{2}$. By Theorem 3.2.2 and the fact that the independence number of a bipartite graph on n vertices is at least $n/2$, we get $\beta_0(T^*) = \frac{n}{2}$. Thus it follows that T^* has a perfect matching. Now consider a diametral path in T^*. Since T^* has a perfect matching, the diametral path must end in a leaf x which is adjacent to a vertex y such that $deg_{T^*}(y) = 2$. Now $T = T^* - \{x, y\}$ is a tree in $n - 2$ vertices and by the induction hypothesis, we have $\Pi_d(T) > \frac{n-2}{2}$. Thus, by the Lemma 3.2.4,

$$\begin{aligned}\Pi_d(T^*) &\geq \Pi_d(T) + 1 \\ &> \frac{n-2}{2} + 1\end{aligned}$$

$$= \frac{n}{2}.$$

This contradicts our assumption that $\Pi_d(T^*) = \frac{n}{2}$, and hence the theorem. $\square$

This bound is sharp as the following observation shows.

Observation 3.2.6. *Consider the octopus O_b which is constructed as follows: take a star with b edges, $b \geq 1$, and subdivide every edge exactly once. The octopus has order $n = 2b + 1$. Then $\Pi_d(O_b) = \frac{n+1}{2}$.*

Proof. We will use induction on b to prove show that $\Pi_d(O_b) \leq \frac{n+1}{2}$. When $b = 1$, O_b is a P_3 and $\Pi_d(P_3) = 2$. Assume that $\Pi_d(O_{b-1}) \leq b = \frac{n-1}{2}$.

Let Π be a minimal dominator partition of O_b. If all the leaves of O_b are singleton classes, then $|\Pi| = \frac{n+1}{2}$. Suppose there exists a leaf v of O_b such that $V_i = \{u, v\}$ forms a class in Π, where u is adjacent to v. In this case, consider $\Pi' = \Pi - V_i$. No vertex of O_b except u and v dominates V_i, and hence Π' is a dominator partition of $O_b - \{u, v\} = O_{b-1}$. Also, in Π the class V_i is a PDC because v dominates only V_i.

Let V_j such that $j \neq i$ be a PDC in Π, and w be a private neighbor of V_j. Since u dominates V_i and v does not dominate any class other than V_i, we get $w \notin \{u, v\}$. Thus $w \in V(O_{b-1})$ and w is a private neighbor of V_j; hence V_j is a PDC of Π'. Therefore every PDC in Π, except V_i, is a PDC in Π' too. Since Π has at most one class which is not a PDC, Π' has at most one class which is not a PDC, and hence Π' is a minimal dominator partition of O_{b-1}, by Theorem 3.2.1. By the induction hypothesis,

$$\begin{aligned} |\Pi| &= |\Pi'| + 1 \\ &\leq \frac{n-1}{2} + 1 \\ &= \frac{n+1}{2}. \end{aligned}$$

On the other hand, suppose none of classes of Π are of the form $\{u, v\}$ where v is a leaf of O_b and $uv \in E(O_b)$. As discussed earlier, if all leaves form singleton classes, then we are done. So, let class V_i contain a leaf, say v, and let $|V_i| > 1$. Then the neighbor of v, say u, must form a singleton class. This implies, since $|V_i| > 1$ and $v \in V_i$, that the class V_i is not a PDC in Π. Since Π is a minimal dominator partition of O_b, by Theorem 3.2.1, every class V_j with $j \neq i$ is a PDC in Π.

In this case we will show that the root r of O_b is contained in V_i, where the root of O_b is the vertex that is not adjacent to a leaf. Suppose $r \in V_j$, where $j \neq i$. Since V_j is a PDC, there exists a vertex w such that w dominates only V_j. However, r dominates the class $\{u\}$, and thus $w \neq r$. Moreover, since $r \in V_j$ and $r \neq w$, we have $rw \in E(O_b)$. Since w dominates only V_j and $r \in V_j$, the leaf, say x, adjacent to w must form a singleton class. However, this would imply that w dominates $\{x\} \neq V_j$ also. This is a contradiction to the assumption that w dominates only V_j. Therefore $r \in V_i$.

Also, since we assumed that there are no classes in Π of the form $\{p, q\}$, where p is a leaf and p is adjacent to q, for any leaf p we have either $p \in V_i$ or $q \in V_i$ where p is adjacent to q because Π has at most one class which is not a PDC and V_i is not a PDC. In other words, we have $|V_i| > 2$.

Now consider $O_{b-1} = O_b - \{u, v\}$. Also consider $\Pi' = \Pi - (\{u\} \cup V_i) \cup (V_i - \{v\})$. We will show that Π' is a minimal dominator partition of O_{b-1}. Consider any vertex x in O_{b-1}. If x dominates V_j in Π where $V_j \neq \{u\}$, then x dominates $V_j \in \Pi'$. If x dominates only $\{u\}$ in Π, then $x = r$. In this case, all neighbors of r except u are in V_i, and no leaves other than v are in V_i. This implies r dominates $V_i - \{v\}$ in Π'. Therefore Π' is a dominator partition.

Now we will show that Π' is a minimal dominator partition. Consider $V_j \in \Pi'$. Suppose V_j is a PDC in Π and let x be a private neighbor of V_j. If x dominates

$V_i - \{v\}$ in Π', then V_j must be of the form $\{y\}$ where y is a leaf. In this case y is also a private neighbor of V_j, and hence V_j is a PDC in Π'. If x does not dominate $V_i - \{v\}$ in Π', then x dominates only V_j in Π', and hence V_j is a PDC in Π'. Therefore Π' has at most one class which is not a PDC, and hence Π' is a minimal dominator partition of O_{b-1} by Theorem 3.2.1. By the induction hypothesis we have,

$$\begin{aligned} |\Pi| &= |\Pi'| + 1 \\ &\leq \frac{n-1}{2} + 1 \\ &= \frac{n+1}{2}. \end{aligned}$$

Therefore, $\Pi_d(O_b) \leq \frac{n+1}{2}$. However, by Theorem 3.2.5, $\Pi_d(O_b) \geq \frac{n+1}{2}$ and thus $\Pi_d(O_b) = \frac{n+1}{2}$. □

3.3 Π_d for a Path and a Cycle on n Vertices

Establishing the exact value of Π_d for a path or a cycle turns out to be surprisingly hard. A lot of cases must be considered even when we try to calculate a slightly tighter upper bound than $n - \delta(G)$. In [27], however, we provided the exact values of $\Pi_d(P_n)$ and $\Pi_d(C_n)$.

Theorem 3.2.1 has an immediate corollary when the graph is a path or a cycle.

Corollary 3.3.1. *Let Π be a minimal dominator partition of P_n or C_n. Then all the classes of Π, except at most one, must be K_1, K_2, $\overline{K_2}$ or P_3.*

3.3.1 The Upper Dominator Partition Number of P_n

Observation 3.3.2. $\Pi_d(P_6) \geq 4$.

A minimal dominator partition Π of P_6 with $|\Pi| = 4$ is given in Example 3.2.1.

Theorem 3.3.3. $\Pi_d(P_n) \geq \left\lfloor \frac{2n+1}{3} \right\rfloor$.

Proof. We will prove this by constructing a minimal dominator partition Π of size $\left\lfloor \frac{2n+1}{3} \right\rfloor$ for P_n, $n \geq 6$. (If $n < 6$, then we can easily find a minimal dominator partition of the required size).

- $n \equiv 0 \mod 6$: Repeat the pattern for P_6 as in Example 3.2.1 for every six vertices.

- $n \equiv 1 \mod 6$: Repeat the pattern for P_6 as in Example 3.2.1 for every six vertices and keep the last vertex as a singleton class.

- $n \equiv 2 \mod 6$: Repeat the pattern for P_6 as in Example 3.2.1 for every six vertices and keep the last two vertices as a K_2 class.

- $n \equiv 3 \mod 6$: Repeat the pattern for P_6 as in Example 3.2.1 for every six vertices and keep the last three vertices as a K_2 class and a K_1 class.

- $n \equiv 4 \mod 6$: Repeat the pattern for P_6 as in Example 3.2.1 for every six vertices starting from the third vertex. Keep the first and the last vertices as K_1 classes, and the second and the last-but-one vertex as a $\overline{K_2}$ class.

- $n \equiv 5 \mod 6$: Repeat the pattern for P_6 as in Example 3.2.1 for every six vertices and keep the last five vertices in the pattern K_1 class, P_3 class, K_1 class.

One can verify that for each of the above cases Π has at most one class which is not a PDC and $|\Pi| = \left\lfloor \frac{2n+1}{3} \right\rfloor$. □

Note that in the above construction when all the classes of Π are PDCs, we have $|\Pi| = \left\lfloor \frac{2n+1}{3} \right\rfloor = \left\lfloor \frac{2n}{3} \right\rfloor$.

Observation 3.3.4. *Let* Π *be a minimal dominator partition of* P_n *where all the classes are PDCs. Then there will be even number of* $\overline{K_2}$ *classes and they occur in pairs in the pattern of* $K_1, \overline{K_2}, \overline{K_2}, K_1$ *forming a* P_6*.*

Proof. Let V_i be a $\overline{K_2}$ class and let $V_i = \{u, v\}$. Since V_i is a PDC, there exists a vertex x such that x dominates only V_i, in particular x is adjacent to both u and v. Let $x \in V_j$. Since V_i is a $\overline{K_2}$ class and x is adjacent to both vertices of V_i, we have $V_i \neq V_j$. Since x dominates only V_i, there exists a vertex y such that $y \in V_j$. We assumed that all classes of Π are PDC, and so V_j is a PDC. Let z be the private neighbor of V_j. However, x is adjacent to both u and v, and $deg(x) \leq 2$, so z must be either u or v and $|V_j| = 2$. Also, V_j must be a $\overline{K_2}$ class, because x is not adjacent to y. Moreover, since Π is a dominator partition, y must dominate a K_1 class and the vertex in V_i that does not dominate V_j must dominate a K_1 class. Hence the proof. $\square$

Theorem 3.3.5. *Let* Π *be a minimal dominator partition of* P_n*,* $n > 1$*, such that all the classes of* Π *are PDCs. Then* $|\Pi| \leq \left\lfloor \frac{2n}{3} \right\rfloor$*.*

Proof. We will prove this theorem using induction on n. For the base case with $n = 2$ the theorem clearly holds. Assume that for all paths with less than n vertices the theorem holds. Consider a minimal dominator partition Π of P_n such that all the classes of Π are PDCs. Since all the classes of Π are PDCs, the classes must be K_1, K_2, $\overline{K_2}$ and P_3. Moreover, since Π is minimal, not all the classes can be K_1. Let V_i be a class of Π which is not a K_1.

Suppose V_i is a $\overline{K_2}$. By Observation 3.3.4, there exists $\overline{K_2}$ class, say V_j, which pairs up with V_i. Delete the vertices of V_i and V_j from the path to obtain two paths P_{n_1} and P_{n_2} with n_1 and n_2 vertices respectively. While deleting the vertices of V_i and V_j, if the K_1 classes adjacent to V_i and V_j do not have a private neighbor other than themselves and the vertices of V_i and V_j, then delete them too. (This ensures that $n_1 \neq 1$ and $n_2 \neq 1$.) Now the classes of Π can be classified into Π_1 and Π_2 such that Π_1 contains the classes of Π with vertices of P_{n_1}, and Π_2 contains the classes with vertices of P_{n_2}. Now, Π_1 and Π_2 are minimal dominator partitions of P_{n_1} and P_{n_2} respectively and all the classes are PDCs. Thus depending on whether we deleted the K_1 classes adjacent to V_i and V_j,we have the following.

$$\begin{aligned}|\Pi| &= |\Pi_1| + |\Pi_2| + 2\\ &\leq \left\lfloor \frac{2n_1}{3} \right\rfloor + \left\lfloor \frac{2n_1}{3} \right\rfloor + 2\\ &\leq \left\lfloor \frac{2(n-4)}{3} \right\rfloor + 2\\ &\leq \left\lfloor \frac{2n}{3} \right\rfloor.\end{aligned}$$

Or,

$$\begin{aligned}|\Pi| &= |\Pi_1| + |\Pi_2| + 3\\ &\leq \left\lfloor \frac{2n_1}{3} \right\rfloor + \left\lfloor \frac{2n_1}{3} \right\rfloor + 3\\ &\leq \left\lfloor \frac{2(n-5)}{3} \right\rfloor + 3\\ &\leq \left\lfloor \frac{2n}{3} \right\rfloor.\end{aligned}$$

Or,

$$\begin{aligned}\Pi &= |\Pi_1| + |\Pi_2| + 4\\ &\leq \left\lfloor \frac{2n_1}{3} \right\rfloor + \left\lfloor \frac{2n_1}{3} \right\rfloor + 4\\ &\leq \left\lfloor \frac{2(n-6)}{3} \right\rfloor + 4\end{aligned}$$

$$\leq \left\lfloor \frac{2n}{3} \right\rfloor.$$

If V_i is a K_2 class, then delete V_i and use similar arguments as above. If V_i is a P_3 class, then delete V_i and the K_1 classes adjacent to V_i if necessary as in the case where V_i is a $\overline{K_2}$ class, and use similar arguments as above. □

Theorem 3.3.6. *Let Π be a minimal dominator partition of P_n such that Π has exactly one class V_i which is not a PDC. However, suppose there exists $w \in V(P_n)$ such that $w \succ V_i$. Then, $|\Pi| \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$.*

Proof. Let Π be a minimal dominator partition of P_n satisfying the hypothesis of the theorem. Since $w \succ V_i$, which is not a PDC, and $deg_{P_n}(w) \leq 2$, V_i must be K_1 or K_2 or $\overline{K_2}$.

Suppose V_i is a $\overline{K_2}$. Since V_i is not a PDC, $w \succ V_k,\ k \neq i$. But $deg(w) \leq 2$ and so $V_k = \{w\}$. Let $V_i = \{u, v\}$. Also, either u or v, or both are not adjacent to a K_1 class other than V_k, because Π is minimal. Delete u, w, v and any K_1 class adjacent to u or v if necessary, as in the proof of Theorem 3.3.5. This results in at most two paths, viz. P_{n_1} and P_{n_2} with n_1 and n_2 vertices respectively. As before, let Π_1 and Π_2 be the classes of Π with vertices from P_{n_1} and P_{n_2} respectively. Since the only class in Π which is not a PDC is V_i, Π_1 and Π_2 are minimal dominator partitions of P_{n_1} and P_{n_2} respectively and all the classes are PDCs. Now, by applying Theorem 3.3.5 for Π_1 and Π_2, and using similar algebra as in the proof of Theorem 3.3.5, one can show that $|\Pi| \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$.

If V_i is a K_1 class, then delete the vertex of V_i and apply Theorem 3.3.5. If V_i is a K_2 class, then delete the vertices of V_i and the necessary adjacent K_1 classes and apply Theorem 3.3.5. □

Theorem 3.3.7. *Let Π be a minimal dominator partition of P_n such that there exists a class $V_i \in \Pi$ where no vertex of P_n dominates V_i. Then $|\Pi| \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$.*

Proof. Suppose Π is a minimal dominator partition of P_n satisfying the hypothesis. Clearly, any vertex of V_i dominates either a K_1 class or a $\overline{K_2}$ class. Also note that if any vertex v of V_i dominates a $\overline{K_2}$ class, V_j, then v is the only private neighbor for V_j.

Case 1: *All vertices of V_i dominate $\overline{K_2}$ classes.*

In this case V_i contains only isolated vertices. Suppose $|V_i| = 2$ and let $V_i = \{u, v\}$. (If $|V_i| > 2$, then we can make $|V_i| = 2$ by merging all but two vertices of V_i with the $\overline{K_2}$ class each vertex dominates. The resultant partition is a minimal dominator partition of P_n and has the same number of classes as Π.) Let $v \succ V_j = \{x, y\}$, where V_j is a $\overline{K_2}$ class, and let $x \succ V_a = \{a\}$ and $y \succ V_b = \{b\}$.

Case 1a: The class V_a has a private neighbor other than x and a.

Delete v and V_j from the path to form two paths, P_{n_1} and P_{n_2}. Let Π_1 and Π_2 respectively be the classes of Π having vertices from P_{n_1} and P_{n_2}. Now merge u with the $\overline{K_2}$ class that u dominates to form a P_3. Wlog, assume that $V_b \in \Pi_2$. Π_1 and Π_2 are minimal dominator partitions of P_{n_1} and P_{n_2}, because all classes of Π_1 are PDCs and the only class which could be a non-PDC in Π_2 is V_b. However, in that case, Π_2 satisfies the hypothesis of Theorem 3.3.6. Note that a cannot be an end-vertex of P_n in this case and so $n_1 > 1$. Thus applying Theorems 3.3.5 and 3.3.6, we have

$$\begin{aligned}
|\Pi| &= |\Pi_1| + |\Pi_2| + 2 \\
&\leq \left\lfloor \frac{2n_1}{3} \right\rfloor + \left\lfloor \frac{2n_2+1}{3} \right\rfloor + 2 \\
&\leq \left\lfloor \frac{2(n-3)+1}{3} \right\rfloor + 2 \\
&\leq \left\lfloor \frac{2n+1}{3} \right\rfloor.
\end{aligned}$$

Case 1b: The class V_a does not have a private neighbor other than x and a, and the class V_b does not have a private neighbor other than y and b.

In this case consider u instead of v. Let $u \succ V_k = \{r, s\}$, where V_k is a $\overline{K_2}$ class, and let $r \succ V_c = \{c\}$ and $s \succ V_d = \{d\}$. If V_c has a private neighbor other than r and c, then we have case 1a. Suppose V_c does not have a private neighbor other than r and c and V_d does not have a private neighbor other than s and d. Delete the vertices $u, v, x, y, a, b, r, s, c$ and d from the path. This creates paths P_{n_1}, P_{n_2}, and P_{n_3}. Let Π_1, Π_2 and Π_3 be the classes in Π having vertices from P_{n_1},P_{n_2} and P_{n_3} respectively. Moreover, $n_i \neq 1$ because of the minimality of Π. Π_i is a minimal dominator partition of P_{n_i} and all classes of Π_i are PDCs. So we have,

$$\begin{aligned}
|\Pi| &= |\Pi_1| + |\Pi_2| + |\Pi_3| + 7 \\
&\leq \left\lfloor \frac{2n_1}{3} \right\rfloor + \left\lfloor \frac{2n_2}{3} \right\rfloor + \left\lfloor \frac{2n_3}{3} \right\rfloor + 7 \\
&\leq \left\lfloor \frac{2(n-10)}{3} \right\rfloor + 7 \\
&\leq \left\lfloor \frac{2n+1}{3} \right\rfloor .
\end{aligned}$$

Case 2: *All vertices of V_i dominate K_1 classes.*

Let $|V_i| = k$.

Case 2a: Every K_1 class which is dominated by vertices of V_i also has a private neighbor not in V_i.

Delete V_i from the path to form at most $k+1$ paths. (Note that in this case, any $\overline{K_2}$ class which is not V_i is a PDC and hence has the property mentioned in Observation 3.3.4.) Let Π_i be the class in Π with vertices from P_{n_i}. So for all i, all of the classes of Π_i are PDCs, and thus Π_i is a minimal dominator partition of P_{n_i}. So, we have,

$$\begin{aligned}
|\Pi| &= |\Pi_1| + |\Pi_2| + \ldots + |\Pi_{k+1}| + 1 \\
&\leq \left\lfloor \frac{2n_1+1}{3} \right\rfloor + \left\lfloor \frac{2n_2+1}{3} \right\rfloor + \ldots + \left\lfloor \frac{2n_{k+1}+1}{3} \right\rfloor + 1 \\
&\leq \left\lfloor \frac{2(n-k)+k+1}{3} \right\rfloor + 1 \\
&\leq \left\lfloor \frac{2n+1}{3} \right\rfloor, \quad \text{if } k > 2.
\end{aligned}$$

(Note that here we cannot apply Theorem 3.3.5 as some of the paths produced may be P_1.)

Since no vertex of P_n dominates V_i, $|V_i| = k > 1$. Suppose $|V_i| = k = 2$. Since no vertex of P_n dominates V_i and Π is minimal, $n_i > 1$ for some i. Let $n_1 > 1$. We have,

$$\begin{aligned}
|\Pi| &= |\Pi_1| + |\Pi_2| + |\Pi_3| + 1 \\
&\leq \left\lfloor \frac{2n_1}{3} \right\rfloor + \left\lfloor \frac{2n_2+1}{3} \right\rfloor + \left\lfloor \frac{2n_3+1}{3} \right\rfloor + 1 \\
&\leq \left\lfloor \frac{2(n-2)+2}{3} \right\rfloor + 1 \\
&\leq \left\lfloor \frac{2n+1}{3} \right\rfloor.
\end{aligned}$$

Case 2*b*: There exists a vertex $v \in V_i$ such that $v \succ V_j = \{x\}$ and v is the only private neighbor of V_j.

Merge v into V_j to form a K_2 class. If $V_i - v$ can be combined with a class $V_k \in \Pi$ to form a dominator partition (such a partition is minimal since all the classes are PDCs), then $2 \leq |V_i| \leq 3$. Wlog, assume that $|V_i| = 2$. (If $|V_i| = 3$, then there exists a vertex in V_i which can be merged into the K_1 class it dominates to make $|V_i| = 2$ maintaining the cardinality and the minimality of Π.) Delete V_i, V_j and V_k from the path. (If there is a tie for V_k, then select V_k to be the one such that the path created after the deletion is not P_1. Such a choice is always possible because of the minimality of Π.) The deletion of these classes creates at most 3 paths, $P_{n_1}, \ldots, P_{n_3}$ (note that $n_i \neq 1,\ \forall i$), and defining Π_i as above we have,

$$\begin{aligned}
|\Pi| &= |\Pi_1| + |\Pi_2| + |\Pi_3| + 3 \\
&\leq \left\lfloor \frac{2(n-4)}{3} \right\rfloor + 3 \\
&\leq \left\lfloor \frac{2n+1}{3} \right\rfloor .
\end{aligned}$$

Suppose $V_i - v$ cannot be combined with any class to form a dominator partition. If $V_i - v$ has a vertex with the same property as v then do the same procedure again. (Stop the process if V_i reduces to a K_1 or K_2 and apply Theorem 3.3.6). If there is no vertex in $V_i - v$ with the same property as v, then we have case 2a.

Case 3: There exists $v \in V_i$ such that v dominates a $\overline{K_2}$ class, say $V_p = \{x, y\}$, and there exists $u \in V_i$ such that u dominates a K_1 class.

Merge v into V_p to form a P_3. If $V_i - v$ can be combined with a class $V_k \in \Pi$ to form a dominator partition, then we know $2 \leq |V_i| \leq 3$. Again, as in case 2b assume $|V_i| = 2$. Let $x \succ V_a = \{a\}$ and $y \succ V_b = \{b\}$.

If V_a does not have a private neighbor other than x and a, and V_b does not have a private neighbor other than y and b, then delete the vertices of V_a, V_b, V_i, V_k and V_p from the path. Each path thus produced has more than one vertex. Define Π_i as before. Then all the classes of Π_i are PDCs. So we have,

$$\begin{aligned}
|\Pi| &= |\Pi_1| + |\Pi_2| + |\Pi_3| + 5 \\
&\leq \left\lfloor \frac{2(n-7)}{3} \right\rfloor + 5
\end{aligned}$$

$$\leq \left\lfloor \frac{2n+1}{3} \right\rfloor.$$

However, if V_a has a private neighbor other than x and a, and V_b does not have a private neighbor other than y or b, then delete the vertices of V_i, V_k, V_b and V_p. We then have,

$$\begin{aligned} |\Pi| &= |\Pi_1| + |\Pi_2| + |\Pi_3| + 4 \\ &\leq \left\lfloor \frac{2(n-6)}{3} \right\rfloor + 4 \\ &\leq \left\lfloor \frac{2n+1}{3} \right\rfloor. \end{aligned}$$

Finally, if V_a has private neighbors other than x and a, and V_b has private neighbors other than y and b, then delete the vertices of V_i, V_k and V_p and we have

$$\begin{aligned} |\Pi| &= |\Pi_1| + |\Pi_2| + |\Pi_3| + 3 \\ &\leq \left\lfloor \frac{2(n-5)}{3} \right\rfloor + 3 \\ &\leq \left\lfloor \frac{2n+1}{3} \right\rfloor. \end{aligned}$$

Suppose $V_i - v$ cannot be combined with any of the classes of Π. If no other vertices of V_i dominate $\overline{K}_2$ classes, then we have case 2 and we are done. If there exists $m \in V_i - v$ such that m dominates a $\overline{K}_2$ class, then repeat case 3. (Stop the process if we end with a K_1 or K_2 and apply Theorem 3.3.6.) Hence the proof. □

From Theorems 3.3.3, 3.3.5, 3.3.6 and 3.3.7 we have the following corollary.

Corollary 3.3.8. $\Pi_d(P_n) = \left\lfloor \frac{2n+1}{3} \right\rfloor$ *for* $n \geq 1$.

3.3.2 The Upper Dominator Partition Number of C_n

In [27], we calculated the upper dominator partition number of a cycle on n vertices. Note that $\Pi_d(C_3) = 1$ and $\Pi_d(C_4) = 2$.

Theorem 3.3.9. *$\Pi_d(C_n) \geq \left\lfloor \frac{2n+1}{3} \right\rfloor$, for $n > 4$.*

Proof. We will construct a minimal dominator partition of size $\left\lfloor \frac{2n+1}{3} \right\rfloor$ of C_n, $n > 4$, as follows.

- $n \equiv 0 \mod 6$. Delete an edge in C_n to form a P_n. Find a minimal dominator partition Π of P_n, using the construction in the proof of Theorem 3.3.3. Π is a minimal dominator partition of C_n with $|\Pi| = \left\lfloor \frac{2n+1}{3} \right\rfloor$.

- $n \equiv i \mod 6$, where $i = 1, 2, 5$. Delete two edges of C_n to form paths P_i and P_{n-i} with i and $n-i$ vertices respectively. Find a minimal dominator partition Π_1 for P_i, and using the construction in the proof of Theorem 3.3.3 find a minimal dominator partition Π_2 for P_{n-i}. Now $\Pi = \Pi_1 \cup \Pi_2$ is a minimal dominator partition for C_n and $|\Pi| = \left\lfloor \frac{2n+1}{3} \right\rfloor$.

- $n \equiv i \mod 6$, where $i = 3, 4$. Delete two edges from C_n to form paths P_{i+6} and $P_{n-(i+6)}$. $\Pi_1 = \{\ \{v_1\}, \{v_2, v_4\}, \{v_3, v_7\}, \{v_5\}, \{v_6, v_8\}, \{v_9\}\ \}$ and $\Pi_1 = \{\ \{v_1\}, \{v_2, v_4\}, \{v_3, v_8\}, \{v_5\}, \{v_6\}, \{v_7, v_9\}, \{v_{10}\}\ \}$ is a minimal dominator partition for P_{i+6} when $i = 3$ and $i = 4$ respectively. Find a minimal dominator partition Π_2 for $P_{n-(i+6)}$ using the construction in the proof of Theorem 3.3.3. Now $\Pi = \Pi_1 \cup \Pi_2$ is a minimal dominator partition of C_n and $|\Pi| = \left\lfloor \frac{2n+1}{3} \right\rfloor$.

So we have $\Pi_d(C_n) \geq \left\lfloor \frac{2n+1}{3} \right\rfloor$, $n > 4$. □

Theorem 3.3.10. $\Pi_d(C_n) \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$, $n \geq 3$.

Proof. Let Π be a minimal dominator partition of C_n. We produce a minimal dominator partition for a path from Π. We will consider the following 6 cases.

1. There exists a K_2 class in Π. Delete the vertices of the K_2 class from C_n to form a path and delete the K_2 class from Π to form Π'. Π' is a minimal dominator partition of P_{n-2} and so $|\Pi| = |\Pi'| + 1 \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$.

2. Two K_1 classes are adjacent. Delete the edge connecting two K_1 classes to produce P_n. Now, Π is a minimal dominator partition of P_n and hence $|\Pi| \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$.

3. There is a P_3 class in Π. In this case delete the vertices of P_3 and the vertices of the K_1 classes adjacent to P_3 if if they do not have a private neighbor other than vertices of the deleted P_3 class. In any case, using simple algebra, $|\Pi| \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$.

4. There are two $\overline{K_2}$ classes in Π that are adjacent. In this case delete the vertices of both $\overline{K_2}$ classes. Also delete the vertices of the K_1 classes adjacent to the $\overline{K_2}$ classes if the K_1 classes do not have a private neighbor other than vertices of the deleted $\overline{K_2}$ classes. In any case, using simple algebra, $|\Pi| \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$.

5. If Π does not have any of the above four situations, then there must be a class $V_i \in \Pi$ which is not a PDC. So if there is a $\overline{K_2}$ class other than V_i, say $V_j = \{p, q\}$ then V_j must have a private neighbor in V_i. If both K_1 classes that are adjacent to V_j have private neighbors other than the vertices of V_j and themselves, then delete the vertices of V_j and its private neighbor $v \in V_i$ to form P_{n-3}. Then depending on whether $V_i - v$ can be merged with a class of Π, we have

$$\begin{aligned} |\Pi| &\leq \left\lfloor \frac{2(n-3)+1}{3} \right\rfloor + 1 \text{ or } \left\lfloor \frac{2(n-3)+1}{3} \right\rfloor + 2 \\ &\leq \left\lfloor \frac{2n+1}{3} \right\rfloor . \end{aligned}$$

If a K_1 class, say $\{x\}$, adjacent to V_j does not has a private neighbor other than x and one vertex of V_j, say p, then delete the edge incident with x other than xp. Now we have P_n and Π is a minimal dominator partition of P_n. Hence $|\Pi| \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$.

6. Π has only K_1 classes and a non-PDC. In this case, since none of the K_1 classes are adjacent (because case 2 is not satisfied), the vertices of the K_1 classes form an independent set. Hence the number of K_1 classes is at most $\left\lfloor \frac{n}{2} \right\rfloor$, and thus $|\Pi| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$, $n \geq 3$.

So in any case we have $|\Pi| \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$, $n \geq 3$. □

By Theorems 3.3.9 and 3.3.10 we have the following corollary.

Corollary 3.3.11. $\Pi_d(C_n) = \left\lfloor \frac{2n+1}{3} \right\rfloor, \ n > 4.$

Chapter 4

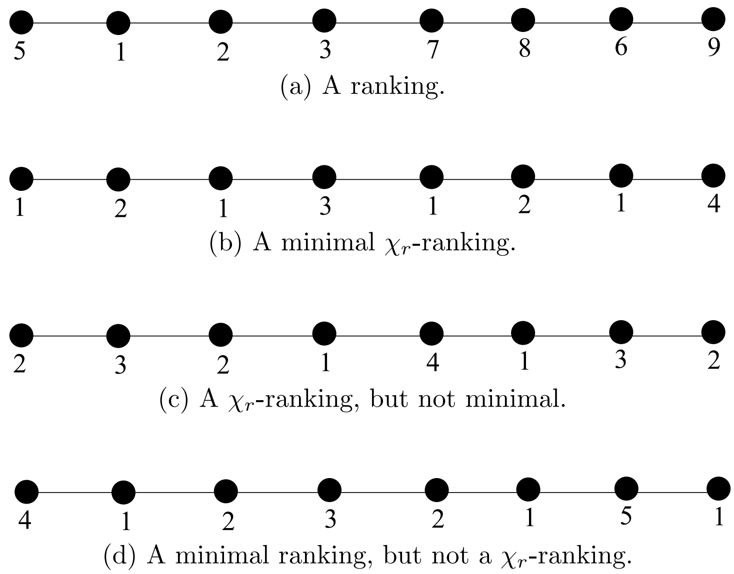
Minimal Rankings of Graphs

In the last chapter we looked at a vertex partitioning problem, the dominator partitions. As mentioned in the previous chapter, one of the well studied vertex partitioning problem is the vertex coloring problem. In this chapter we will discuss a variation of the coloring problem called ranking.

Let $G = (V, E)$ be an undirected graph. A function $f : V(G) \to \{1, 2, \ldots, k\}$ is a (vertex) *k-ranking* of G if for $u, v \in V(G)$, $f(u) = f(v)$ implies that every $u - v$ path contains a vertex w such that $f(w) > f(u)$. By definition, every ranking is a proper coloring. The *rank number* of G, denoted $\chi_r(G)$, is the minimum value of k such that G has a k-ranking. If the value of k is unimportant, then f will be referred to simply as a ranking of G.

Ghoshal, Laskar and Pillone introduced minimal rankings in [18]. A k-ranking f is a *minimal k-ranking* of G if for all $x \in V(G)$ with $f(x) > 1$, the function g defined on $V(G)$ by $g(z) = f(z)$ for $z \neq x$ and $1 \leq g(x) < f(x)$ is not a ranking.

In other words, a ranking is minimal if the reduction of any one label violates the ranking conditions. The *arank number*, denoted $\psi_r(G)$, is defined to be the maximum value of k for which G has a minimal k-ranking [18].



(a) A ranking.

(b) A minimal χ_r-ranking.

(c) A χ_r-ranking, but not minimal.

(d) A minimal ranking, but not a χ_r-ranking.

Figure 4.1: A few examples of ranking for P_8.

The problem of finding the rank number of a graph is studied because of its many applications including the design of very large scale integration layouts (VLSI), Cholesky factorization of matrices in parallel, and scheduling problems of assembly steps in manufacturing systems [13, 15, 35, 36, 39].

One early result dealing with rankings of graphs is given in [8] by Bodlaender et al. As mentioned before, Ghoshal, Laskar and Pillone introduced the concept of minimal rankings [18]. Later, Laskar and Pillone considered some complexity issues of minimal rankings as well as properties of minimal rankings [19, 32, 33].

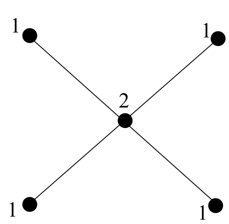


Figure 4.2: χ_r-ranking of $K_{1,4}$

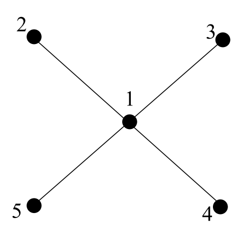


Figure 4.3: ψ_r-ranking of $K_{1,4}$

Finding the rank number and the arank number of an arbitrary graph is difficult, so attempts have been made to find the rank and the arank numbers of classes of graphs. Some examples include:

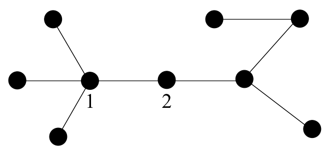
- $\chi_r(K_{1,n-1}) = 2$ and $\psi_r(K_{1,n-1}) = n$.
- $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$. [8]
- $\psi_r(P_n) = \lfloor \log_2(n+1) \rfloor + \lfloor \log_2(n+1-(2^{\lfloor \log_2 n \rfloor - 1})) \rfloor$. [29]
- $\chi_r(P_2 \square P_n) = \psi_r(P_n) + 1$. [38]

In this chapter we consider the Cartesian product of complete graphs, $K_n \square K_n$. $K_n \square K_n$ is also studied in the context of statistical design of experiments, and is a two-class association scheme first introduced by Bose [9].

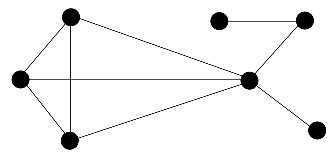
The Cartesian product $K_n \square K_n$ is also called the rook's graph, denoted $R_{n,n}$. A *rook's graph* is a graph that represents all legal moves of the rook chess piece on an $n \times n$ chessboard. Thus, vertices represent squares on the chessboard, and two vertices are adjacent if and only if a rook, when placed on one square, can reach the other square in one legal move, which is either horizontally or vertically. Recently much interest has been developed in studying various graph parameters related to the legal moves of different chess pieces on an $n \times n$ chessboard. The graph parameters studied are usually domination-related parameters and the chess pieces are queens, kings, bishops, rooks, and knights. The reader is referred to an excellent survey article in [21].

In general, a χ_r-ranking may not be a minimal ranking. Here, however, by $\chi_r(G)$-ranking we mean a minimum ranking of G which is also minimal. If f is a ranking and $x \in V(G)$, then $f(x)$ is the *label* of x. If $f(x) = f(y)$ implies $x = y$, then the label is *distinct*; otherwise it is a *repeated* label. The concept of a reduction, introduced in

[18], is as follows: given a graph $G = (V, E)$ and a subset $S \subseteq V(G)$, define a graph $G^* = (V - S, E^*)$ where for $u, v \in V - S$, $uv \in E^*$ if and only if either $uv \in E(G)$, or there exists a path $u - w_1 - w_2 - \ldots - w_m - v$ in G where $w_i \in S$ for $1 \leq i \leq m$. We say that the graph G^* is the reduction of G by S and use the notation G^*_S. An example of a graph and its reduction is given in Figure 4.4.



(a) $S = \{1, 2\}$



(b) G^*_S

Figure 4.4: Example of a reduction process

4.1 Properties of Rankings

In this section, we cite some of the already established results on rankings that will be useful for the rest of this chapter.

1. If H is a subgraph of G, then $\chi_r(H) \leq \chi_r(G)$ [14].

2. If H is an induced subgraph of G, then $\psi_r(H) \leq \psi_r(G)$ [8].

3. A minimal k-ranking is an onto function [18].

4. If G is a graph on n vertices, then $\psi_r(G) = n$ if and only if $\Delta(G) = n - 1$ [18].

5. If f is a minimal k-ranking and $S_i = \{x | f(x) = i\}$ for $1 \leq i \leq k$ then $|S_1| \geq |S_2| \geq \ldots \geq |S_k|$ [18].

6. If f is a minimal ranking, then the set R of vertices with repeated labels is a dominating set for G [18].

7. Let G be a graph and let f be a minimal χ_r-ranking of G. If $S_1 = \{x|f(x) = 1\}$, then $\chi_r(G^*_{S_1}) = \chi_r(G) - 1$ [18].

8. Let G be a graph and let f be a minimal ψ_r-ranking of G. If $S_1 = \{x|f(x) = 1\}$, then $\psi_r(G^*_{S_1}) = \psi_r(G) - 1$ [18].

9. For any graph G, $\chi_r(G) \geq 1 + \delta(G)$ and $\psi_r(G) \geq 1 + \Delta(G)$ [18].

10. A k-ranking f is minimal if and only if for all u with $f(u) = i > 1$ and for each j such that $1 \leq j < i$, one of the following is true [19].

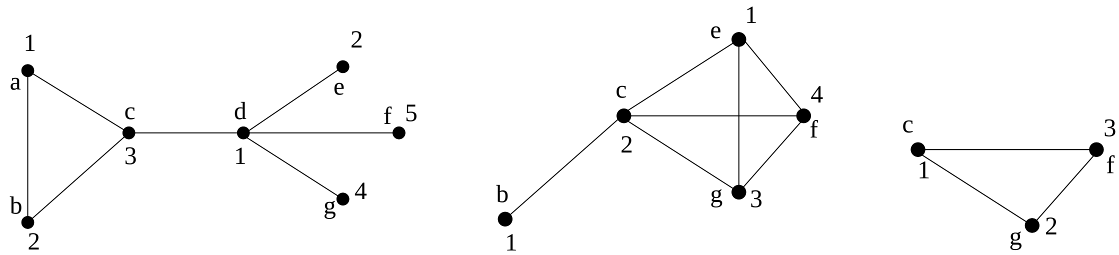
    (a) There exist vertices x and y with $f(x) = f(y) \geq j$, and u is the only vertex on some $x - y$ path such that $f(u) > f(y)$.

    (b) There exists a vertex w with $f(w) = j$, and there exists a $u - w$ path such that for every vertex x on the path $f(x) \leq f(w)$.

11. Let G be a graph and suppose f is a minimal k-ranking of G. Let $S = \{x|f(x) > j\}$ where $1 \leq j < k$ and let C be a connected component of $\langle V - S \rangle$, the induced subgraph of $V - S$. Then f_C, the restriction of f to C is a minimal ranking of C [19].

In the rest of the chapter, Property i will refer to the i^{th} property in the above list.

One of the main results in [18] is that if G is a graph and f is a ranking of G, then the function $g(x) = f(x) - 1$ is a ranking for $G^*_{S_1}$, where $S_1 = \{x|f(x) = 1\}$. In fact, if the original ranking, f, is a minimal $\chi_r(G)$-ranking then g will be a minimal $\chi_r(G^*_{S_1})$-ranking. If f is a minimal $\psi_r(G)$-ranking then g is a $\psi_r(G^*_{S_1})$-ranking. This process can be repeated, as shown in Figures 4.5a, 4.5b and 4.5c. This is the essence of Property 7 and Property 8.



(a) G and $S_1 = \{a, d\}$ (b) $G' = G^*_{S_1}$ and $S'_1 = \{b, e\}$ (c) $(G^*_{S_1})^*_{S'_1}$

Figure 4.5: An illustration of the reduction process.

Laskar, Pillone, Jacob and Eyabi established further properties of minimal rankings in [34]. The authors also considered the rook's graph, established bounds for the rank number of a rook's graph and calculated its arank number in [34].

4.2 Further Properties of Minimal Rankings

The following bounds for the rank number of an arbitrary graph G are immediate.

Lemma 4.2.1. *If G is a graph on n vertices then $\omega(G) \leq \chi(G) \leq \chi_r(G) \leq n - \beta_0(G) + 1$.*

Proof. We know that if G has a clique of size k, then any coloring of G requires at least k colors. Therefore $\omega(G) \leq \chi(G)$. By the definition of ranking, any ranking is a proper coloring and hence $\chi(G) \leq \chi_r(G)$.

We will now show that $\chi_r(G) \leq n - \beta_0(G) + 1$. Let S be an independent set of G such that $|S| = \beta_0(G)$. Construct a labeling f as follows. Label the vertices in S using the label 1 and the rest of the $n - \beta_0(G)$ vertices using labels $2, 3, \ldots, n - \beta_0(G) + 1$ such that f is of order $n - \beta_0(G) + 1$. Thus, vertices with repeated labels are in S and vertices which are not in S have labels from 2 to $n - \beta_0(G) + 1$, which are distinct labels. However, S is an independent set, and therefore any path between two vertices in S, that are labeled 1, must have at least one vertex which is not in S.

Thus f is a ranking, and hence $\chi_r(G) \leq n - \beta_0(G) + 1$. $\square$

Theorem 4.2.2. *Let f be a minimal χ_r-ranking of a graph G such that there exist two vertices u and v with distinct labels. Then the function g defined by*

$$g(x) = \begin{cases} f(x), & \text{if } x \neq u, v \\ f(u), & \text{if } x = v \\ f(v), & \text{if } x = u \end{cases}$$

is a minimal χ_r-ranking. In other words, swapping labels of two distinct label vertices does not destroy the minimality condition of ranking.

Proof. We will first show that the function g is a ranking. Suppose for two distinct vertices a and b, $g(a) = g(b)$. Since g is the same as f for vertices with repeated labels we have $f(a) = g(a) = g(b) = f(b)$. Let P_{ab} be a path from a to b in G. Since f is a ranking there exists $w \in V(P_{ab})$ such that $f(w) > f(a)$. If $u \neq w$ and $v \neq w$, then by the definition of g we have $g(w) = f(w) > f(a) = g(a)$. So, wlog, assume $u = w$. Then we have,

$$\begin{aligned} g(w) &= g(u) \\ &\geq \min\{f(u), f(v)\} \\ &> f(a), \text{ by Property 5} \\ &= g(a). \end{aligned}$$

Thus the function g is a ranking.

Now we will show that g is a minimal ranking. Let $x \in V(G)$ and let h be a labeling obtained from g by reducing $g(x)$ to a smaller label. That is, let $h(z) = g(z)$ for all $z \in V(G) - \{x\}$ and $1 \leq h(x) < g(x)$. We will show that h is not a ranking.

__**Case 1:**__ $g(x)$ is a repeated label.

In this case, by the definition of g, $g(x) = f(x)$. Consider the subgraph of G induced by H, denoted by $\langle H \rangle$, where $H = \{z \in V(G) | f(z) \leq f(x)\}$. By Property 11, f restricted to $\langle H \rangle$ is minimal, which implies $f(x)$ cannot be reduced to a smaller label. Since $f(x) = g(x)$, it follows that h is not a ranking.

__**Case 2:**__ $g(x)$ is a distinct label.

In this case we will consider two subcases.

__**Case 2a:**__ $g(x)$ is reduced to a repeated label of g. That is, $h(x)$ is a repeated label under h.

Consider the function h' such that $h'(z) = f(z)$ for all $z \in V(G) - \{x\}$ and $h'(x) = h(x)$. Since f is a minimal ranking, h' is not a ranking. This implies that for some $a, b \in V(G)$ with $h'(a) = h'(b)$ there exists a path P_{ab} such that for all $s \in V(P_{ab})$, we have $h'(s) \leq h'(a)$. Note that $h'(z) = h(z)$ for all $z \in V(G) - \{u, v\}$ and in particular, if $h'(z) \leq t$, where t is the largest repeated label in f, then $h'(z) = h(z)$. Thus $h(s) \leq h(a)$ for every $s \in V(P_{ab})$ and h is not a ranking.

__**Case 2b:**__ $g(x)$ is reduced to a label which is equal to a distinct label of g, i.e, $t < h(x) < g(x)$, where t is the largest repeated label of g. That is, $h(x) = g(y) = h(y)$ where $x \neq y$ and $h(x) > t$.

Let $\mathcal{R}_f = \{z | 1 \leq f(z) \leq t\}$. By Property 7, it follows that $\chi_r(G^*_{\mathcal{R}_f}) = \chi_r(G) - t$. Since $\chi_r(G) = n - |\mathcal{R}_f| + t$, we now have $\chi_r(G^*_{\mathcal{R}_f}) = n - |\mathcal{R}_f|$. However, $|V(G^*_{\mathcal{R}_f})| = n - |\mathcal{R}_f|$ and thus using Lemma 4.2.1 we get $\beta_0(G^*_{\mathcal{R}_f}) = 1$. This implies $G^*_{\mathcal{R}_f}$ is a complete graph on $n - |\mathcal{R}_f|$ vertices. This implies that in G for any two vertices a and b with distinct labels under f, either $ab \in E(G)$ or there exists a path P_{ab} in G such that the internal vertices of P_{ab} are in $\mathcal{R}_f$.

This implies that x and y are either adjacent in G or have a path between them with all internal vertices from $\mathcal{R}_f$. Since $h(z) = f(z)$ for any $z \in \mathcal{R}_f$, it follows that

h is not a ranking.

Thus g is a minimal ranking. Since g has $\chi_r(G)$ labels, g is a minimal χ_r-ranking as required. □

Corollary 4.2.3. *If f is a minimal χ_r-ranking, then any permutation of the distinct labels is also a minimal χ_r-ranking.*

Corollary 4.2.4. *If f is a minimal χ_r-ranking of a graph G and t is the largest repeated label, then $G^*_{\{z|f(z)\leq t\}}$ is a clique.*

With ψ_r-rankings, the situation is a little different. It is not necessarily true that the smallest distinct label can be swapped with another distinct label, because the resulting ranking might not be minimal. Figure 4.6 shows an example of a ψ_r-ranking in which swapping the smallest distinct label with another distinct label produces a ranking which is not minimal.

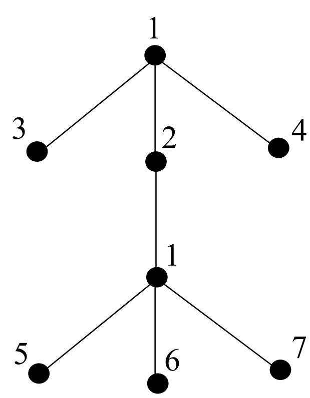


Figure 4.6: The smallest distinct label, 2, cannot be swapped with a larger label.

However, the next theorem shows that the labels of two vertices in a ψ_r-ranking can be swapped if the labels are both greater than the smallest distinct label.

Theorem 4.2.5. *Let f be a minimal ψ_r-ranking of a graph G and let t be the largest repeated label. If u and v are vertices of G such that $u \neq v$, $f(u) > t + 1$ and*

$f(v) > t+1$, then the function g defined by

$$g(x) = \begin{cases} f(x), & \text{if } x \neq u, v \\ f(u), & \text{if } x = v \\ f(v), & \text{if } x = u \end{cases}$$

is a minimal ψ_r-ranking.

Proof. As in the case of the proof of Theorem 4.2.2, the function g will be a ranking. We will show that g is a minimal ranking. Let $h(z) = g(z)$ for all $z \in V(G) - \{x\}$ and $1 \leq h(x) < g(x)$. We will show that h is not a ranking. We consider three cases as in the proof of Theorem 4.2.2. For Case 1 and Case 2a, we can use the same arguments as in the proof of Theorem 4.2.2 to show that h is not a ranking, so here we will prove only one case, Case 2b.

Case 2b: $g(x)$ is reduced to a label which is equal to a distinct label in g, i.e, $t < h(x) < g(x)$.

As before, let $\mathcal{R}_f = \{z | 1 \leq f(z) \leq t\}$. By Property 8, we have, $\psi_r(G^*_{\mathcal{R}_f}) = \psi_r(G) - t$. Since $\psi_r(G) = n - |\mathcal{R}_f| + t$, we now have $\psi_r(G^*_{\mathcal{R}_f}) = n - \mathcal{R}_f$. However, $|V(G^*_{\mathcal{R}_f})| = n - |\mathcal{R}_f|$ and this implies, by Property 4, that $G^*_{\mathcal{R}_f}$ contains a vertex y adjacent to all vertices in $G^*_{\mathcal{R}_f}$.

As explained in Figure 4.5, restricting f to $G^*_{\mathcal{R}_f}$ and subtracting t from each label will give rise to a ψ_r-ranking for $G^*_{\mathcal{R}_f}$. This implies that $g(y) = f(y) = t+1$. If $h(x) = t+1$, then y and x have the same label under h and $xy \in E(G^*_{\mathcal{R}_f})$. This implies that either $xy \in E(G)$ or there is a path between x and y in G with all internal vertices from $\mathcal{R}_f$. Since $h(z) = f(z)$ for any $z \in \mathcal{R}_f$, it follows that h is not a ranking. On the other hand if $h(x) > t+1$, then $G^*_{\{z|f(z) \leq t+1\}}$ is a complete graph, and using a similar argument to Case 2b in the proof of Theorem 4.2.2, we can show that h is not a ranking.

Thus g is a minimal ranking with ψ_r labels and hence g is a minimal ψ_r-ranking. $\square$

Corollary 4.2.6. *Let f be a minimal ψ_r-ranking of a graph G and let t be the largest repeated label. Any permutation of the distinct labels which are greater than $t+1$ is a minimal ψ_r-ranking.*

Corollary 4.2.7. *If f is a minimal ψ_r-ranking of a graph G and t is the largest repeated label, then $G^*_{\{z:f(z)\leq t+1\}}$ is a clique.*

4.3 Minimal Rankings of a Rook's Graph

As mentioned in the beginning of this chapter, a *rook's graph*, $R_{n,n}$, is a graph that represents all legal moves of the rook chess piece on an $n \times n$ chessboard. In other words, vertices of $R_{n,n}$ can be represented as ordered pairs (i, j), $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Two vertices are adjacent if and only if they have one coordinate in common.

For simplicity, for the rest of the chapter, $R_{n,n}$ will be drawn as an $n \times n$ grid.

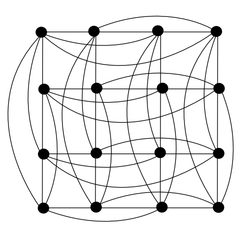


Figure 4.7: $R_{4,4}$

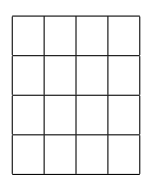


Figure 4.8: A simpler representation of $R_{4,4}$

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Theorem 4.3.1. *Let f be a minimal ranking of $R_{n,n}$. Then every row and every column of $R_{n,n}$ contains a repeated label and a distinct label under f.*

Proof. First we will show that every row of $R_{n,n}$ has a repeated label under f. On the contrary, assume $R_{n,n}$ has a row i which does not contain a repeated label. That is, for $j = 1, 2, \ldots, n$, we have $f(v_{i,j}) > t$, where t is the largest repeated label. Let $a = f(v_{i,j})$ for some $1 \leq j \leq n$. Since f is a minimal ranking, by Property 10, for every k such that $1 \leq k < a$, one of the following is true.

1. There exist vertices x and y with $f(x) = f(y) \geq k$, and $v_{i,j}$ is the only vertex on some $x - y$ path such that $f(v_{i,j}) > f(y)$.

2. There exists a vertex w with $f(w) = k$, and there exists a $v_{i,j} - w$ path such that for every vertex x on the path $f(x) \leq f(w)$.

Suppose for some $1 \leq k < a$, Condition 1 is true, and let P be such a path. Since all the vertices of row i have labels greater than t, P will not contain any vertices from row i other than $v_{i,j}$. This implies that $P' = P - \{v_{i,j}\}$ is a path from x to y such that $f(z) \leq f(x)$ for all $z \in V(P')$. This is a contradiction because f is a ranking. Thus Condition 2 must be true for all $1 \leq k < a$.

However, if $k = 1$, then $v_{i,j}$ must be adjacent to a vertex labeled 1. This implies that for every j such that $1 \leq j \leq n$, $v_{i,j}$ is adjacent to a vertex labeled 1. This is not possible because row i does not have a vertex labeled 1, and no row can have two vertices labeled 1. Thus f is not minimal, which is a contradiction. Hence every row of $R_{n,n}$ contains a repeated label under f. Using the same arguments we can show that every column of $R_{n,n}$ has a repeated label.

We will now show that every row and column of $R_{n,n}$ has a distinct label. Again, on the contrary assume row i contains only repeated labels. Let $v_{i,j}$ have the largest label in row i. Since $f(v_{i,j})$ is a repeated label, let $v_{k,l}$ be such that $f(v_{k,l}) = f(v_{i,j})$. Now, since $v_{i,j}$ has the largest repeated label in row i it follows that $f(v_{i,l}) < f(v_{i,j})$, and thus the path $v_{i,j} - v_{i,l} - v_{k,l}$ will not have any vertex labeled higher than $f(v_{i,j})$.

This is a contradiction. Hence every row must have a vertex with distinct label. Using the same arguments we can show that every column of $R_{n,n}$ contains a distinct label. $\square$

Lemma 4.3.2. *Let f be a minimal ranking of $R_{n,n}$. There exist a row i and a column j such that the total number of distinct labels in row i and column j together is at least n.*

Proof. Let the vertices v_{ij} and v_{kl} be labeled t, where t is the largest repeated label under f. That is, let $f(v_{ij}) = f(v_{kl}) = t$. Since f is a ranking, every path between v_{ij} and v_{kl} has a vertex with a higher label than t. This means that every path between v_{ij} and v_{kl} has a distinct label. Therefore the paths $v_{ij} - v_{rj} - v_{rl} - v_{kl}$ and $v_{ij} - v_{ir} - v_{kr} - v_{kl}$ must have a distinct label for all r where $1 \leq r \leq n$. This implies either column j or l has at least $\lceil \frac{n}{2} \rceil$ distinct labels and either row i or row k has at least $\lceil \frac{n}{2} \rceil$ distinct labels. Hence the lemma. $\square$

Lemma 4.3.3. *Let f be a minimal ranking of $R_{n,n}$. Also, let t be the largest repeated label in f and $S_i = \{v|f(v) = i\}$. If $t = n - 1 - k$, where $k \geq 0$, then $\sum_{i=1}^{t} |S_i| \geq 2n - (k + 2)$.*

Proof. Let $t = n-1-k$, where $k \geq 0$. We want to show that $\sum_{i=1}^{t} |S_i| \geq 2n-(k+2)$. On the contrary, assume that $\sum_{i=1}^{t} |S_i| \leq 2n - (k + 3)$. By Theorem 4.3.1, every row and every column of $R_{n,n}$ has a repeated label. Suppose there are δ_r rows with exactly one repeated label. This implies that $n - \delta_r$ rows have at least two repeated label vertices. Thus we have,

$$\begin{aligned} 2n - (k+3) \geq \sum_{i=1}^{t} |S_i| &\geq \delta_r + 2(n - \delta_r) \\ &= 2n - \delta_r. \end{aligned} \tag{4.1}$$

It follows from Equation (4.1) that $\delta_r \geq k+3$. Similarly, if δ_c is the number of columns with exactly one repeated label, then $\delta_c \geq k+3$.

Case 1: Among the δ_r rows and δ_c columns, there exists a row i and a column j such that $v_{i,j}$ is a repeated label vertex and $f(v_{i,j}) > 1$.

Note that in this case every other label in row i and column j is a distinct label.

Consider the labeling g defined as follows:

$$g(v_{k,l}) = \begin{cases} 1, & \text{if } k = i \text{ and } l = j \\ f(v_{k,l}), & \text{otherwise.} \end{cases}$$

Since f is a minimal ranking, g will not be a ranking. This means there exist $u, v \in V(G)$ such that $g(u) = g(v)$ and a path P between u and v such that $g(z) \leq g(u)$ for every $z \in V(P)$. If $v_{i,j} \notin V(P)$, then $g(z) = f(z)$ for every $z \in V(P)$. Since f is a ranking, there exists $z \in V(P)$ such that $f(z) > f(u)$, that is $g(z) = f(z) > f(u) = g(u)$. This is a contradiction, so assume $v_{i,j} \in V(P)$. Let z be the vertex adjacent to $v_{i,j}$ in P. However, z is in column j or row i which implies that z is a vertex with distinct label under f and thus $g(z) = f(z) > t \geq f(u) \geq g(u)$. This is a contradiction.

Case 2: Among the δ_r rows and δ_c columns, there does not exist a row i and a column j such that $v_{i,j}$ is a repeated label vertex and $f(v_{i,j}) > 1$.

Note that if $|S_1| \geq k+2$, then

$$\begin{aligned} \sum_{i=1}^{t} |S_i| &\geq k+2+\sum_{i=2}^{t} |S_i| \\ &\geq k+2+2(n-2-k) \\ &= k+2+2n-4-2k \\ &= 2n-(k+2), \text{ which is a contradiction.} \end{aligned}$$

Thus the number of vertices with label 1 is at most $k+1$. We know that there are $\delta_r \geq k+3$ rows and $\delta_c \geq k+3$ columns with exactly one repeated label. Thus there exist at least two rows among the δ_r rows and at least two columns among the δ_c columns with a repeated label greater than 1. Since we assumed that $\sum_{i=1}^{t} |S_i| \leq 2n-(k+3)$ and Case 1 does not hold, it follows that one of the following is true.

1. There exist a row, say row i, among the δ_r rows, and a column, say column j, such that $v_{i,j}$ is a repeated label vertex, $f(v_{i,j}) > 1$ and column j does not contain a vertex labeled 1.

2. There exist a column, say column j, among the δ_c columns, and a row, say row i, such that $v_{i,j}$ is a repeated label vertex, $f(v_{i,j}) > 1$ and row i does not contain a vertex labeled 1.

Wlog, assume condition 1 is true. Note that every vertex in row i other than $v_{i,j}$ is a distinct label vertex. That is, we have,

- $1 < f(v_{i,j}) \leq t$
- $f(v_{i,l}) > t$, if $l \neq j$
- $f(v_{k,j}) > 1$, $1 \leq k \leq n$.

Define g as follows:

$$g(v_{k,l}) = \begin{cases} 1, & \text{if } k=i \text{ and } l=j \\ f(v_{k,l}), & \text{otherwise.} \end{cases}$$

Since f is a minimal ranking, g will not be a ranking. This means there exist $u, v \in V(G)$ such that $g(u) = g(v)$ and a path P between u and v such that $g(z) \leq g(u)$ for every $z \in V(P)$. As proved in Case 1 we must have $v_{i,j} \in V(P)$.

Since row i and column j do not contain a vertex labeled 1 under f, row i and column j will not contain a vertex labeled 1 other than $v_{i,j}$ under g.

Therefore, if $u = v_{i,j}$, then P will contain at least one vertex labeled greater than 1 under g, which is a contradiction. Therefore, assume $P = uz_1z_2 \dots z_k v_{i,j} z_{k+1} \dots z_r v$. An example is shown in Figure 4.9.

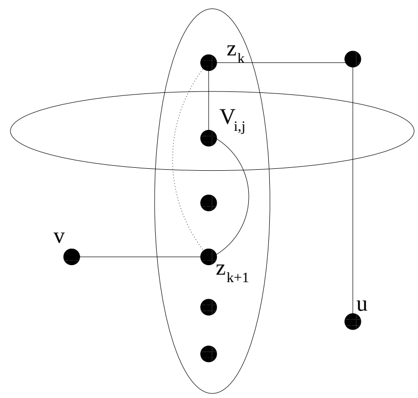

Figure 4.9: A uv path containing the vertex $v_{i,j}$.

Then z_k and z_{k+1} will be in column j (because every vertex in row i, except $v_{i,j}$, has a higher label than t and $g(z) \leq g(u) \leq t$ for every $z \in V(P)$). Thus $P' = P - \{v_{i,j}\}$ will be a path from u to v and $g(z) = f(z)$ for all $z \in V(P')$. Therefore, we have $f(z) = g(z) \leq g(u) = f(u)$ for all $z \in V(P')$, which is a contradiction because f is a ranking.

Thus, in both cases we get a contradiction, and hence $\sum_{i=1}^{t} |S_i| \geq 2n - (k+2)$. $\square$

4.4 Results on $\chi_r(R_{n,n})$ and $\psi_r(R_{n,n})$

In this section we will find bounds for $\chi_r(R_{n,n})$ and determine the exact value of $\psi_r(R_{n,n})$.

Theorem 4.4.1. *For all $n \in Z^+$, $\chi_r(R_{n,n}) \leq \frac{2n^2}{3} + \frac{n}{3}$.*

Proof. We will use induction on n to prove this theorem. When $n = 1$, the result is true. Assume that the result is true for values less than or equal to n. Consider $R_{n+1,n+1}$.

<u>Case 1:</u> $n + 1$ is even.

In this case, construct a ranking for $R_{n+1,n+1}$ using a ranking for $R_{\frac{n+1}{2},\frac{n+1}{2}}$ by dividing $R_{n+1,n+1}$ as shown in Figure 4.10.

I	II
III	IV

Figure 4.10: $R_{n+1,n+1}$

Label regions I and IV with labels 1 to $\chi_r(R_{\frac{n+1}{2},\frac{n+1}{2}})$. Label regions II and III with labels $\chi_r(R_{\frac{n+1}{2},\frac{n+1}{2}})+1$ to $\chi_r(R_{\frac{n+1}{2},\frac{n+1}{2}})+2\left(\frac{n+1}{2}\right)^2$. The resulting labeling is a ranking for $R_{n+1,n+1}$. Therefore we have,

$$\begin{aligned}
\chi_r(R_{n+1,n+1}) &\leq \chi_r(R_{\frac{n+1}{2},\frac{n+1}{2}}) + 2\left(\frac{n+1}{2}\right)^2 \\
&\leq \frac{2}{3}\left(\frac{n+1}{2}\right)^2 + \frac{1}{3}\left(\frac{n+1}{2}\right) + 2\left(\frac{n+1}{2}\right)^2 \\
&= \frac{2}{3}(n+1)^2 + \frac{n+1}{6} \\
&\leq \frac{2}{3}(n+1)^2 + \frac{n+1}{3}.
\end{aligned}$$

<u>Case 2:</u> $n + 1$ is odd.

In this case we divide $R_{n+1,n+1}$ into 4 regions as in the previous case, with the following sizes. Region I will be of size $\frac{n}{2} \times \left(\frac{n}{2}+1\right)$, region II will be of size $\frac{n}{2} \times \frac{n}{2}$, region III will be of size $\left(\frac{n}{2}+1\right) \times \left(\frac{n}{2}+1\right)$ and region IV will be of size $\left(\frac{n}{2}+1\right) \times \frac{n}{2}$.

Note that we can label regions I and IV using $\chi_r(R_{\frac{n}{2},\frac{n}{2}}) + \frac{n}{2}$ labels. Again as in the previous case, label regions II and IV using labels from $\chi_r(R_{\frac{n}{2},\frac{n}{2}}) + \frac{n}{2} + 1$ to $\chi_r(R_{\frac{n}{2},\frac{n}{2}}) + \frac{n}{2} + \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}+1\right)^2$. Thus we have,

$$\begin{aligned}
\chi_r(R_{n+1,n+1}) &\leq \chi_r(R_{\frac{n}{2},\frac{n}{2}}) + \frac{n}{2} + \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}+1\right)^2 \\
&\leq \frac{2}{3}\left(\frac{n}{2}\right)^2 + \frac{1}{3}\left(\frac{n}{2}\right) + \frac{n}{2} + \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}+1\right)^2 \\
&= \frac{2n^2}{3} + \frac{5n}{3} + 1 \\
&= \frac{2}{3}(n+1)^2 + \frac{n+1}{3}.
\end{aligned}$$

This completes the proof. $\square$

An example of a ranking constructed in Theorem 4.4.1 for $R_{5,5}$ is shown in Figure 4.11.

1	2	4	6	7
3	1	5	8	9
10	11	12	4	5
13	14	15	2	1
16	17	18	1	3

Figure 4.11: A ranking for $R_{5,5}$

Lemma 4.4.2. *$\chi_r(R_{n,n}) \geq \chi_r(R_{n-1,n-1}) + n$, for $n \geq 2$.*

Proof. Consider a χ_r-ranking f for $R_{n,n}$. By Lemma 4.3.2 there exist a row i and a column j such that the total number of distinct labels in row i and column j together is at least n.

By Corollary 4.2.3 we can permute the distinct labels, so wlog assume that row i and column j contain the largest distinct labels. Deleting row i and column j from $R_{n,n}$ we will get $R_{n-1,n-1}$. Let f_{n-1} be f restricted to $R_{n-1,n-1}$. We will show that

f_{n-1} is a ranking. Suppose P is a path in $R_{n-1,n-1}$ between vertices u and v such that $f_{n-1}(u) = f_{n-1}(v)$. Since P is in $R_{n-1,n-1}$, it follows that P does not contain vertices from row i or column j and since f is a ranking, there exists $z \in V(P) \subseteq V(R_{n-1,n-1})$ such that $f(z) > f(u)$. However, since $z, u \in V(R_{n-1,n-1})$ it follows that $f_{n-1}(z) = f(z)$ and $f_{n-1}(u) = f(u)$. Thus f_{n-1} is a ranking of $R_{n-1,n-1}$ and hence $\chi_r(R_{n-1,n-1}) \leq |f_{n-1}| \leq \chi_r(R_{n,n}) - n$. □

Theorem 4.4.3. $\chi_r(R_{n,n}) \geq \frac{n^2+n}{2}$.

Proof. We will use induction on n. When $n = 1$, the result is true. Assume the result is true for n. Consider $R_{n+1,n+1}$. By Lemma 4.4.2 we have,

$$\begin{aligned}
\chi_r(R_{n+1,n+1}) &\geq \chi_r(R_{n,n}) + n + 1 \\
&\geq \frac{n^2+n}{2} + n + 1 \\
&= \frac{(n+1)^2 + (n+1)}{2}.
\end{aligned}$$

This completes the proof. □

However, this bound is not necessarily a sharp bound when $n > 2$. When $n = 3$, $\chi_r(R_{n,n}) = 7$. Nonetheless, from Theorems 4.4.1 and 4.4.3 we have the following result.

Corollary 4.4.4. $\frac{n^2+n}{2} \leq \chi_r(R_{n,n}) \leq \frac{2n^2}{3} + \frac{n}{3}$.

We now consider the arank number of a rook's graph.

Theorem 4.4.5. $\psi_r(R_{n,n}) \geq n^2 - n + 1$.

Proof. We will construct a minimal ranking for $R_{n,n}$ of size $n^2 - n + 1$. Consider the function $f : V(R_{n,n}) \to \{1, 2, \ldots, n^2 - n + 1\}$ defined as follows:

$$f(v_{i,j}) = \begin{cases} j, & \text{if } i = 1 \\ n - (i-1), & \text{if } j = n \\ (i-1)(n-1) + j + 1, & \text{otherwise.} \end{cases}$$

The labels $1, 2, \ldots, n-1$ are repeated once, and occur once in the first row and once in the last column. Thus, any path between the repeated labels will have either the label n or a label larger than n and hence f is a ranking.

Now we will show that f is minimal. Since the first row contains the labels $1, 2 \ldots, n$, none of these labels can be reduced. The same argument holds for the last column.

Let i and j be such that $1 < i \leq n$ and $1 \leq j < n$. Let g be defined as follows.

$$g(v_{k,l}) = \begin{cases} a < f(v_{i,j}), & \text{if } k = i \text{ and } l = j \\ f(v_{k,l}), & \text{otherwise.} \end{cases}$$

To show that f is a minimal ranking, we need to show that g is not a ranking.

Case 1: $1 \leq g(v_{i,j}) = a \leq n$.

If $g(v_{i,j}) \geq g(v_{1,j})$, then consider the path $P = v_{i,j} - v_{1,j} - v_{1,k}$, where $g(v_{1,k}) = a$. However, $g(v_{1,j}) \leq a$ and so g is not a ranking. Therefore assume $g(v_{i,j}) < g(v_{1,j})$. Wlog, assume $g(v_{1,j}) \geq g(v_{i,n})$. Consider the path $P = v_{1,j} - v_{i,j} - v_{i,n} - v_{k,n}$, where $g(v_{k,n}) = g(v_{1,j})$. However, $g(v_{i,j}) = a < g(v_{1,j})$ and $g(v_{i,n}) \leq g(v_{1,j})$, which implies that g is not a ranking.

Case 2: $n + 1 \leq g(v_{i,j}) = a < n^2 - n + 1$.

Consider the path $P = v_{i,j} - v_{1,j} - v_{1,l} - v_{k,l}$, where $g(v_{k,l}) = g(v_{i,j}) = a$. $g(v_{1,j}) \leq n < a$ and $g(v_{1,l}) \leq n < a$ and hence g is not a ranking.

Thus in any case g is not a ranking. Therefore, f is a minimal ranking and $\psi_r(R_{n,n}) \geq |f| = n^2 - n + 1$. □

1	2	3	4	5
6	7	8	9	4
10	11	12	13	3
14	15	16	17	2
18	19	20	21	1

Figure 4.12: A minimal ranking of size $n^2 - n + 1$ for $R_{n,n}$ when $n = 5$.

Theorem 4.4.6. *Let f be a minimal k-ranking of $R_{n,n}$. Then $k \leq n^2 - n + 1$.*

Proof. Let t be the largest repeated label in f. Let $S_i = \{v | f(v) = i\}$.

If $t > n - 1$, then we have

$$\begin{aligned} k &= n^2 - \sum_{i=1}^{t} |S_i| + t \\ &\leq n^2 - 2t + t \\ &= n^2 - t \\ &< n^2 - (n-1). \end{aligned}$$

Suppose $t \leq n - 1$. Let $t = n - 1 - r$, where $r \geq 0$. By Lemma 4.3.3, we have $\sum_{i=1}^{t} |S_i| \geq 2n - (r + 2)$. Thus,

$$\begin{aligned} k &= n^2 - \sum_{i=1}^{t} |S_i| + t \\ &\leq n^2 - (2n - (r+2)) + n - 1 - r \\ &= n^2 - n + 1. \end{aligned}$$

This completes the proof. □

From Theorem 4.4.5 and Theorem 4.4.6, we get the following result.

Theorem 4.4.7. $\psi_r(R_{n,n}) = n^2 - n + 1.$

Chapter 5

Open Problems

In this chapter, we will discuss some of the open problems that are related to the topics discussed in this thesis.

Conjecture 1. *Let T and T' be trees. Then $CU_2(T) \cong CU_2(T')$ if and only if $T \cong T'$, where $CU_2(T)$ is the complete double vertex graph of T.*

One challenging graph product problem is to reconstruct the factors from the graph products. In the case of Cartesian products, the problem of reconstructing the factors from the Cartesian product is solved. In Chapter 2, we showed that for some classes of graphs we can reconstruct G from its double vertex graph and complete double vertex graph.

Problem 2. *For any arbitrary graph H, can we recognize whether H is a double vertex graph of some graph G?*

Problem 3. *Is it possible to develop an algorithm to reconstruct G from $U_2(G)$?*

Answering the above questions involves answering the following.

Problem 4. *Is it true that for any two graphs G and H, $G \cong H$ if and only if*

$U_2(G) \cong U_2(H)$?

The above questions are open in the case of the complete double vertex graphs as well.

Problem 5. *Is it true that for any two graphs G and H, $G \cong H$ if and only if $CU_2(G) \cong CU_2(H)$? Is it possible to recognize whether a given graph is a complete double vertex graph? How can one reconstruct G from its complete double vertex graph for all classes of graphs?*

For Cartesian products, one well-investigated problem is the domination number of the Cartesian product. Vizing [40], in 1963, conjectured that $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$. Vizing's conjecture is verified for some classes of graphs, most notably when one of the factors is a tree by Barcalkin and German [7].

Conjecture 6. *For any two arbitrary graphs G and H, $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.*

In the case of complete double vertex graphs, since $CU_2(G)$ contains a copy of G, we get $\gamma(CU_2(G)) \leq \gamma(G)|V(G)|$.

Problem 7. *What else can we say about the domination number of $CU_2(G)$ and $U_2(G)$?*

Problem 8. *How does the dominator partition number and the upper dominator partition number of the factors affect that of the graph products, especially that of Cartesian products, double vertex graphs and complete double vertex graphs?*

Problem 9.[23] *For which graphs is $\pi_d(G) = \Pi_d(G)$?*

Some examples of graphs in which the above equality holds are complete graphs, the complement of a complete graph and the cycle on 4 vertices.

The authors in [23] discuss two variations of dominator partitions. One is called the total dominator partition, in which each vertex is required to be a dominator of at least one class other than its own. It is observed in [23] that $\gamma_t(G) \leq \pi_{td} \leq \gamma_t(G)+1$, where $\pi_{td}(G)$ is the total dominator partition number of G, which is analogous to dominator partition number, and $\gamma_t(G)$ is the well-known total domination number of G.

Problem 10. *What more can we say about the total dominator partition number of G?*

Problem 11. *What about the upper total dominator partition number of any arbitrary graph? Calculating the upper total dominator partition number of an arbitrary graph seems to be difficult, as is the upper dominator partition number. However, can we find the upper total dominator partition number of some classes of graphs as in the case of upper dominator partition number?*

The other variation of the dominator partition mentioned in [23] is called the independent dominator partition, where each class is required to be independent in addition to the requirements of being a dominator partition. Independent dominator partitions are also called dominator colorings of graphs. The authors of [23] and Gera et al. [16, 17] are independently investigating dominator colorings of graphs. However, not much attention has been made to the dominator achromatic number, which is analogous to the upper dominator partition number.

Problem 12. *What can we say about the dominator achromatic number of graphs?*

In Chapter 4 we calculated the arank number of $R_{n,n} = K_n \square K_n$. However, we were unable to find the rank number of $R_{n,n}$.

Conjecture 13. $\chi_r(R_{n,n}) = \left\lfloor \frac{2n^2+n}{3} \right\rfloor$.

Problem 14. *What can we say about the rank and arank number of graph products such as the Cartesian product, the double vertex graph and the complete double vertex graph?*

Note that since $G\Box H$ contains copies of both G and H, we have $\chi_r(G\Box H) \geq \max\{\chi_r(G), \chi_r(H)\}$ and $\psi_r(G\Box H) \geq \max\{\psi_r(G),\ \psi_r(H)\}$. Similarly, since $CU_2(G)$ contains copies of G, we have $\chi_r(CU_2(G)) \geq \chi_r(G)$ and $\psi_r(CU_2(G)) \geq \psi_r(G)$.

Problem 15. *What is the rank number and the arank number of $P_n\Box P_n$? In [38], Novotny et al. showed that $\chi_r(P_2\Box P_n) = \psi_r(P_n) + 1$. However, the authors posed $\psi_r(P_2\Box P_n)$ as an open problem.*

Problem 16. *Does there exist a graph G such that for every integer k, where $\chi_r(G) \leq k \leq \psi_r(G)$, there is a minimal k-ranking of G? Not all graphs possesses this property; for example K_n does not. Narayan [37] showed that paths have this property. What other classes of graphs satisfy this property?*

Problem 17. *We know that $\chi(G) \leq \chi_r(G)$. Is there any relation between $\psi(G)$ and $\psi_r(G)$?*

Naturally, these are not the only questions one might consider from the topics discussed in this thesis. Hopefully, they will provide us with some interesting insights into the topics of graph products and vertex partitions.

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