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The Effect of Prior Experiences and Sequence of Instruction Upon Preservice Teachers' Pedagogical Beliefs and Mathematical Knowledge for Teaching

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THE EFFECT OF PRIOR EXPERIENCES AND SEQUENCE OF INSTRUCTION
UPON PRESERVICE TEACHERS' PEDAGOGICAL BELIEFS AND
MATHEMATICAL KNOWLEDGE FOR TEACHING

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Curriculum and Instruction

by
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Abstract

This study investigated the effects of sequence of instruction in a laboratory-based mathematics content course for elementary teachers on preservice teachers' (PSTs') pedagogical beliefs and Mathematical Knowledge for Teaching (MKT) through a quasi-experimental design in which the order of laboratory tasks and explanations of content was altered between sections for a given unit and between units within a given section. This study also investigated the prior experiences a group of preservice elementary teachers shared as students of mathematics and examined how these experiences related to their beliefs regarding effective pedagogy through a phenomenology. Additionally, the perspective of an experienced instructor, who was new to facilitating a sequence of instruction in which PSTs explored mathematical before a formal explanation of content was delivered, was examined through a case study.

Data were collected through mathematical autobiographies written by PSTs, pre- and post-assessments of MKT, classroom observations, exit surveys completed by PSTs, and interviews with the instructor and six PSTs. Results from these data revealed that most PSTs experienced a traditional school mathematics characterized by review, delivery of content, worked examples, and practice, and many PSTs also framed their interpretations of effective pedagogy within this traditional context. No significant differences were found in mean gains between those sections that explored content through laboratory tasks before an explanation was given and those that confirmed content through laboratories that followed an explanation of the content. Although the instructor did not initially identify differences between sequence groups, as the semester

progressed, she found the discourse richer and PSTs more independent in their problem solving in the exploratory section, and these benefits came with no drop off in MKT.

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CHAPTER ONE

Introduction

The goal of many mathematics teacher-preparation programs is not only to arm future teachers with knowledge of content and pedagogy but also to establish beliefs aligned with a large, growing body of literature supporting reform in mathematics education (Richardson, 1996). This study examines the effects of sequence of instruction in a laboratory-based mathematics content course for elementary teachers on preservice teachers' (PSTs') Mathematical Knowledge for Teaching (MKT) and pedagogical beliefs. As this particular content course included collaborative tasks through a laboratory component of the course, the focus of this study was to explore how these laboratory tasks are used. Laboratory activities can either precede a formal explanation of content as an exploratory task or follow an explanation confirming the concepts previously presented. While theory and literature from a reform-oriented research community advocates for an explore-explain sequence, still other learning theories and preservice teachers' own experiences as students of mathematics support an explain-confirm sequence. This chapter discusses the need for considering sequence of instruction in a content course for elementary teachers. The chapter continues with a presentation of the problem statement, research questions, and professional significance of these inquiries to teacher education.

Motivating Questions

While it is an ambitious task to prepare secondary teachers with the content and pedagogical knowledge they will need to teach a discipline they have chosen to pursue, it

is an even greater challenge to prepare elementary teachers to master various domains with which they may not be completely comfortable. Mathematics teacher educators are given the challenge of convincing future teachers to depart from twelve-plus years of experience learning mathematics in an oftentimes single (Ball, 1989; NCTQ, 2008) methods course while also ensuring teachers have adequate knowledge of the domain. How then does one persuade future teachers that their students truly can make sense of mathematics despite years of lived memorization, repetition, and vacuously learned processes?

Lortie (1975) offers a theory of an apprenticeship of observation in which educators teach as they were taught, not as they are told. Implications of this theory suggest that a content course could have a great effect on preservice teachers' practice. One can imagine that implementing reform practices as a new teacher without experiencing such instruction as a student of mathematics would be difficult at best. As Schifter (2005) explains,

One important way to help teachers develop new conceptions of what can happen in their classrooms is to allow them to experience *as students* classrooms that enact the new approach to teaching, classes that provide learning experiences powerful enough to challenge 16 and more years of traditional education. . . . Through mathematics lessons that challenge teachers at their own levels of mathematics competence, they can both increase their mathematical knowledge and experience a depth of learning that is, for many of them, unprecedented. (p. 88, italics in original)

While mathematics content courses are a venue for preservice teachers to develop deep conceptual understandings of the fundamental ideas they will teach, these courses could also be an avenue for future teachers to explore mathematics themselves. Perhaps then preservice teachers could become apprentices of reform-based instruction.

Both the Conference Board of the Mathematical Sciences (2002, 2007) and National Council on Teacher Quality (2008) stress the importance of mathematics content courses for teachers and suggest at least nine credit hours of mathematics specific to elementary content. Some researchers have explored the effects of content courses on preservice teachers' mathematical beliefs (Ambrose, 2004; Philipp et al., 2007; Charalambous, Panaoura & Philippou, 2009; Hart, 2002; Wilcox et al., 1990, Lubinski & Ott, 2004, Spielman & Loyd, 2004) and content knowledge (Mathews & Seaman, 2007; Lueke, 2009). However, the current research base does not focus on specific well-defined aspects of these courses, such as sequence of instruction. Instead, this research focuses on combining content courses with field experiences (Ambrose, 2004; Philipp et al., 2007) or methods courses (Hart, 2002) or makes general claims of being standards-based (Lubinski & Ott, 2004), non-traditional (Wilcox et al., 1990), or curriculum-based (Spielman & Lloyd, 2004) in nature with few details concerning how one might implement similar practices.

Problem Statement

Despite recommendations for integrating education faculty in the instruction of mathematics content (CBMS, 2002, 2007), most content courses are taught by mathematics faculty (NCTQ, 2008) who are not professionally equipped to target the

specialized content knowledge required of elementary teachers. As Ball and colleagues noted, “In our research we began to notice how rarely these mathematical demands could be addressed with mathematical knowledge learned in university mathematics courses” (Ball, Thames & Phelps, 2008, p. 398). These courses are “neither demanding in the content nor their expectations of students” (NCTQ, 2008, p.47), with a superficial delivery of content appropriate for elementary or middle school students (NCTQ, 2008).

Although mathematics courses for elementary teachers have the potential to influence both the pedagogical beliefs and MKT of teachers positively, they rarely do so:

Unfortunately, subject matter courses in teacher preparation tend to be academic in both the best and worst sense of the word, scholarly and irrelevant, either way remote from classroom teaching . . . Although there are exceptions, the overwhelming majority of subject matter courses for teachers, and teacher education courses in general, are viewed by teachers, policy makers, and society at large as having little bearing on the day-to-day realities of teaching and little effect on the improvement of teaching and learning (Ball, Thames & Phelps, 2008 p. 404).

Therefore, there is evidence that content courses for elementary teachers are not meeting the needs of future teachers.

Research Questions

To better understand the effects of a mathematics course for elementary teacher on preservice teachers’ pedagogical beliefs and MKT, this study addresses the following questions:

1. How do the lived experiences, as students of mathematics, of a group of preservice teachers influence their pedagogical beliefs?
 - a. What common experiences do these preservice teachers share as students of mathematics related to the *Principles and Standards for School Mathematics* (NCTM, 2000)? That is, are these experiences congruent or in conflict with reform-based instruction?
 - b. How do these PSTs describe the nature of mathematics, and how do their beliefs regarding mathematics relate to their pedagogical beliefs and school experiences?
 - c. What evidence of Lortie's (1975) apprenticeship of observation emerges in these preservice teachers' mathematical autobiographies? That is, do these preservice teachers plan to teach in ways similar to their own experiences as students?
2. How does experiencing exploratory and confirmatory sequences of instruction influence a group of preservice elementary teachers' pedagogical beliefs?
 - a. What pre-existing beliefs regarding sequence of instruction emerge from PSTs' descriptions of their future classrooms and example lessons?
 - b. What evidence of changes in these beliefs regarding sequence of instruction can be found in PSTs' example lessons at the end of the semester?

- c. What advantages and disadvantages in exploratory and confirmatory sequences do these preservice teachers perceive as a result of their experience in this course?
- H₀2-1 PSTs' intended instruction sequence is independent of their preferred sequence as students.
- H₀2-2 There is no significant difference in the proportion of PSTs that preferred exploratory sequences of instruction between those that experienced a five-week exploratory unit and those that experienced a 10-week exploratory unit.
3. Is there evidence of a relationship between sequence of instruction and gains in preservice teachers' MKT as measured by *Learning Mathematics for Teaching* (LMT, 2004, 2008)?
- H₀3-1: There is no significant difference in preliminary MKT scores and scores at the end of the course.
- H₀3-2: There is no evidence of an interaction between pre-MKT and section or sequence groups. That is, the group effects are the same for all ability levels.
- H₀3-3: There is no significant difference in gains in MKT scores across sections within a particular unit when controlled for by preliminary MKT.
- H₀3-4: There is no significant difference in gains in MKT scores across exploratory and confirmatory sequence groups within a particular unit when controlled for by preliminary MKT.

- H₀3-5: There is no difference in MKT gains among preservice teachers that value exploratory sequences, confirmatory sequences, or are neutral during both exploratory and confirmatory units when controlled for by preliminary MKT. That is, preservice teachers' sequence preference is not related to sequence-specific gains.
4. How do an experienced instructor's perceptions of students' learning change when implementing exploratory sequences of instruction?
 - a. Is there evidence from classroom observations that the instructor successfully alters the sequence of instruction across sections?

Methodologies

This study used a mixed-model (Creswell, 2009) methodology in which both qualitative and quantitative methods are used depending on the particular research question. This mixed-model approach contrasts with other mixed-methods approaches in that there is no mixing phase in which qualitative and quantitative data inform each other.

Research Question One explores the lived experiences with school mathematics for group of PSTs. Because this question focuses on the "essence of human experiences about a phenomenon as described by participants" (Creswell, 2009, p. 13), a phenomenology is used. The aim of this research is to understand school mathematics from the perspective of a group of future teachers. As Bondas and Eriksson (2001) noted, "Phenomenological research pursues not only the sense people make of things but what they are making sense of" (p. 826). Therein, this research examines both PSTs' recollections of school mathematics and how these PSTs understand this phenomenon.

Research Question Two explores PSTs' beliefs regarding how instruction should be sequenced. Qualitative inquiries into this question are considered through PSTs' descriptions of their future classrooms and example lessons as none of the quantitative measures review in Chapter Two met the needs of the current study.

Research Question Three explores the relationship between two variables, sequence of instruction and gains in MKT, when controlled for by initial MKT. Therefore, quantitative methods are used.

Research Question Four explores the process of implementing an exploratory sequence of instruction for an experienced instructor. This question is considered through a case study with the goal of exploring and describing the process of implementing an exploratory sequence of instruction. The subject of the inquiry (case), the instructor, is referred to as Ms. B, and her experiences and perception are situated in the context of the process of implementing an exploratory sequence of instruction. This case is "bound by time and activity" (Creswell, 2009, p. 13); therefore, in-depth inquiries drawing from multiple data sources were sustained over the course of the semester in which this instructor implemented an exploratory sequence.

Significance

With a body of research focusing on evaluating and changing preservice teachers' pedagogical beliefs through teacher preparation, this study adds to the research base by inquiring into beliefs preservice teachers establish as students of mathematics. With a better understanding of their lived experiences and beliefs, teacher educators can better meet the needs of future teachers.

This study also attempts to support a body of literature showing the valuable influence content courses can have on future teachers' beliefs and content knowledge for teaching mathematics. By looking at the sequence of instruction, as opposed to claiming general reform-based practice, this study has specific, targetable implications for content instructors. This research also has immediate implications for participants themselves. Having experienced exploring and making sense of mathematics, perhaps for the first time, preservice teachers may enter methods courses with a belief that their students can make sense of mathematics as they did.

In response to claims that content courses are not currently addressing the content knowledge for teaching mathematics (Ball, Thames & Phelps, 2008; NCTQ, 2008), this study also attempts to evaluate whether an exploratory sequence of instruction can help solve this problem. By evaluating the effects of sequence of instruction on gains in content knowledge, this study has the potential to add to a body of research supporting exploratory learning and the theories of learning on which these practices are founded.

Nonetheless, the aforementioned implications are not truly significant if instructors find this sequence of instruction unmanageable. By exploring the perspective of an experienced instructor attempting to use an exploratory sequence of instruction, perhaps for the first time, perceived barriers and benefits will be identified. Practical benefits add to the evidence for exploratory learning, and barriers can be corrected and avoided.

Definition of Key Terms

Laboratory-based instruction. For the purposes of this study, laboratory-based mathematics instruction involves students working together in small groups to complete a mathematical task that oftentimes includes the use of concrete models, manipulatives, or other technologies. Manipulatives are defined as any concrete materials students use to solve a problem or complete a mathematical task. Models are any objects, drawings, or mental images that represent a concept and through which properties of this concept can be expressed. Other technologies include virtual (electronic) manipulatives, dynamic geometry software such as Geometer's Sketchpad®, calculators, or other computational or graphical applications used to simulate mathematical actions or represent concepts. The use of these manipulatives, models, or other technologies are oftentimes referred to as hands-on activities.

Although concrete instructional aids are often used in laboratories, not all laboratories employ them; for example, a teacher may simply pose a problem for groups to consider. It is important, however, that the task targets specific mathematical content. Not included in the understanding of laboratory-based instruction presented here are games, brainteasers, riddles, or other activities that only incidentally address mathematical content, despite the fact that they may require critical thinking, reasoning, and problem solving. Also excluded from this understanding of laboratories are exercises void of context.

Exploratory sequence of instruction. With the abovementioned understanding of laboratory-based instruction in mind, two classifications of laboratory-based

instruction are offered: exploratory and confirmatory. Exploratory laboratories are those in which the task posed precedes a formal explanation of the content addressed. Although the teacher may review fundamental ideas required to tackle the task or offer meaning to the models students might use before the laboratory, students gain experiences related to the intentionally-targeted content during the laboratory. A student may obtain or construct the desired knowledge during the laboratory, or this content may not become apparent until discussions that follow. Mastery during the activity is not imperative; albeit, exploratory laboratories do require that group members attempt the task before solution strategies are presented.

Confirmatory sequence of instruction. The second sequence of laboratory-based instruction used in this study is a confirmatory sequence in which the teacher first explains the content after which group members collaboratively verify or apply mathematical ideas. Confirmatory laboratories include modeling a concept after it has been presented or applying a previously explained concept to a novel context. In confirmatory laboratories, the activity is used to solidify the ideas that the teacher presented. It should be noted that this typology of instructional sequence does not consider instruction in which content is explained concurrently with laboratory activities because this would more closely align with whole-class instruction, which does not meet the small-group criteria for laboratory-based instruction.

Reform-based instruction. Reform-based instruction, also referred to as standards-based instruction in some literature, aligns with the framework set forth by the National Council of Teachers of Mathematics (NCTM) in three reform documents: *The*

Curriculum and Evaluation Standards for School Mathematics (1989), *The Professional Standards for School Mathematics* (1991), and *The Principles and Standards for School Mathematics* (2000). Reform-based instruction emphasizes conceptual understandings of mathematics that go beyond performing procedures. Reform-based instruction differs from traditional mathematics instruction in the roles of students and teachers. In reform-based instruction, students actively make sense of the mathematics as they engage in worthwhile mathematical tasks (NCTM, 1991), and teachers facilitate students' meaning making as they mediate the classroom discourse. NCTM intentionally avoids offering prescribed methods for reform-based instruction (NCTM, 1991) as teaching is considered a complex endeavor that cannot be reduced to an instructional formula. For this study, reform-based instruction includes practices that embody the five process standards (NCTM, 2000): problem solving, reasoning and proof, communication, connections, and representations.

Problem solving is defined as "engaging in a task for which the solution method is not known in advance" (NCTM, 2000, p. 52). Students enact the reasoning and proof standard as they formulate and explore conjectures and justify their claims by expressing their analytical thinking. The communication standard stresses students expressing mathematics orally and in writing as they "organize and consolidate their mathematical thinking" (NCTM, 2000, p. 60). The connections standard includes both connections among mathematical concepts and applications to meaningful contexts. Representation includes both the process of embodying mathematics and the symbols, diagrams, pictures, graphs, and models used to stand for an idea. That is, representation includes

both "the act of capturing a mathematical concept or relationship in some form and . . . the form itself" (NCTM, 2000, p. 67).

Traditional mathematics instruction. In contrast to reform-based instruction, traditional instruction does not exemplify the process standards (NCTM, 2000) and focuses on duplicating procedures. A more detailed description of traditional mathematics instruction is reported in Chapter Four as four instructional phases—review, delivery of content, worked examples, and practice—emerged from PSTs' descriptions of a school mathematics void of process standards.

Teacher beliefs. Although the review of literature that follows in Chapter Two more completely develops the construct of beliefs, it is important to note that epistemological beliefs refer to convictions regarding how students learn. Pedagogical beliefs relate to practices that a teacher thinks promote student learning. The current study also focuses on pedagogical beliefs specific to sequence of instruction, that is how PSTs suppose activities and explanations should be ordered to optimize student learning.

Mathematical Knowledge for Teaching. Based on the work of the Learning Mathematics for Teaching research group (Ball, Thames & Phelps, 2008; Hill, Ball & Schilling, 2008), MKT is defined as "mathematical knowledge 'entailed by teaching'—in other words, mathematical knowledge needed to perform the recurrent task of teaching mathematics to students" (Ball, Thames & Phelps, 2008). This construct is developed more completely in the review of literature that follows in Chapter Two.

Researcher's Perspective of Sequence of Instruction

As with all research, the fact that the researcher chooses to explore a phenomenon introduces the influence of the researcher. Therein, the bias of the researcher influences aspects of all research. Understanding the influence of my own bias on the current study, I aim to be aboveboard in my own beliefs regarding how instruction should be sequenced.

I believe activities and explanations should follow an exploratory sequence of instruction whenever possible. This belief is grounded by the educational goal I envision for students. With the goal of preparing students to independently confront novel problems, I believe an exploratory sequence of instruction better prepares students to face such non-routine problems because of the benefits of learning through discovery noted by Bruner (1964, 1967) and the instructional implications of NCTM reviewed in detail in Chapter Two.

I find value in all of the theoretical perspectives presented in Chapter Two, including those that support confirmatory sequences of instruction, and do not ascribe to a single theory of learning. However, I believe the theories that are argued to support a confirmatory sequence of instruction in Chapter Two are based on less ambitious educational outcomes—namely, producing and predicting behaviors (or, in the context of mathematics, producing rote computational results).

With the ambitious goal of creating “as autonomous and self-propelled a thinker as we can” (Bruner, 1961, p 23) and an eclectic theoretical perspective, I believe an exploratory sequence of instruction better prepares students to call upon, use, or create

the mathematical knowledge needed to confront problems that they will face outside the classroom.

CHAPTER TWO

Review of Literature

This chapter reviews literature relevant to the current study. The chapter is divided into three sections. First, theoretical bases for confirmatory and exploratory sequences of instruction are examined. Second, research of teachers' beliefs is reviewed. Third, literature pertaining to teachers' Mathematical Knowledge for Teaching is considered.

Theoretical Bases for Sequence of Instruction

The purpose of this section is to show that sequence of instruction is a valuable instructional component supported by theories of learning and instruction that are drawn from the current research base. Therein, the current study's inquiries into instructional sequence are not haphazard but purposefully chosen and grounded in theories supported by the literature. This section examines a theoretical framework for both confirmatory and exploratory laboratory sequences. First, the theoretical bases for confirmatory laboratories are examined in Gagné's conditions of learning, Bandura's social-cognitive, and information processing theories. Then, an exploratory laboratory sequence is shown to be congruent with a constructivist epistemology. Within the context of constructivism, the theoretical contributions of Piaget, Vygotsky, and Bruner and instructional implications of the National Council of Teacher of Mathematics (NCTM) are used to support an exploratory laboratory sequence.

Confirmatory tasks. One means of implementing laboratory-based instruction is through confirmatory tasks in which the teacher first explains the content after which

students collaboratively apply the concepts learned, practice procedures, and verify mathematical ideas through laboratory activities. Confirmatory laboratories also include modeling a concept after it has been presented or applying a previously explained concept to a different or novel context. In confirmatory laboratories, the teacher uses mathematical tasks to solidify the ideas previously presented. The instructor first delivers the content and then reinforces it through the laboratory task.

Gagné's conditions of learning. From his research training pilots during World War II (Gagné, 1962), Gagné developed a hierarchical theory of learning with an accompanying instructional theory (Gagné, 1968). Gagné (1962) critiqued John Dewey's (1938, 1951) *learning by doing* and theories of conditioning, in which the performance of the behavior precedes the response of the teacher, because action leads or occurs in tandem with learning. Instead, Gagné used task analysis to identify intermediate, observable outcomes of learning, which he called *capacities* of learning. Gagné suggested that instructors evaluate the prerequisite knowledge or skills required to accomplish the desired outcome and decompose subordinate capabilities in a hierarchical fashion. Thus, Gagné viewed learning as a linear, connected, and cumulative process of systematically and sequentially acquiring knowledge and skills that build to an end goal.

For Gagné, instruction consisted of carefully sequencing and developing ancillary skills to achieve the desired educational outcome. In addition to his conditions of learning, Gagné offered a theory of instruction (1985) consisting of a sequence of nine instructional events. Of particular interest to this research are events four through six for which the instructor is to (4) present the content, (5) provide learning guidance, and (6)

elicit performance. With the presentation of content preceding performance (practice), Gagné's theory of instruction is compatible with a confirmatory laboratory sequence. Further, Gredler (1992) described the process of learning guidance, in which subordinate skills are developed through cues and clues: "The learner discovers how to combine some *previously learned rules* to generate a solution to a problem that is new to the learner" (p. 159, italics added). Gagné's theory of learning is congruent with a confirmatory laboratory sequence because the teacher must carefully guide the acquisition of sequenced principles before the desired outcome is tested in real-world or laboratory contexts.

Social-cognitive theory. Social-cognitive theory views learning in a communal context. Social-cognitive theorists believe that learners can obtain information from observing others, an idea that departed from behaviorism's focus on individual reinforcement. This theory grew from Bandura's (Bandura, Ross & Ross, 1961) study which found that children observing aggressive acts towards an inflatable doll were more likely to display aggressive acts in similar situations. Modeling is a significant component of learning for followers of social-cognitive theory: "Individuals learn new behaviors through the observation of models and through the effects of their own actions" (Gredler, 1992, p. 309). In addition to the individual reinforcement of behaviorism, Bandura believed models can vicariously reinforce learners. Social-cognitive theorists believe children can learn from observing others interact with the environment.

With an emphasis on modeling, social-cognitive theory supports confirmatory laboratory sequencing. In this context, the teacher is to model the desired behaviors. The

students perceive the teacher's success with the behavior as reinforcing. Seeing the teacher successfully complete the task gives students a sense of efficacy (Bandura, 1977) and the expectation of similar success. Gredler (1992) identified three components for social-cognitive-based instruction: providing appropriate models, creating a reinforcing expectation of success, and guiding students as they rehearse behaviors. The explanation provided by the teacher, which may include worked examples, serves as a model. The confirmatory laboratory activity then supplies a social context for rehearsal while also providing additional examples.

Information-processing theory. Information processing theory views the mind as analogous to a computer. Information is taken in through the sensory registers and passed along to short-term or working memory, which is comparable to a computer's Random Access Memory (RAM). One's short-term memory is limited by time and capacity, so information is often lost from working memory before it can be stored in long-term memory. Long-term memory, similar to a computer's hard drive, holds networks or schemata of information. The executive control acts as the operating system of the mind and activates schemata, passing information from working to long-term memory. Learning can then be understood as a process of attending to, coding, storing, and retrieving information (Atkinson & Shiffrin, 1968; Driscoll, 2005; Gredler, 1992; Lutz & Huitt, 2003).

As Gredler (1992) explained, from an information processing perspective, "a new strategy should be modeled by the teacher accompanied by explicit, detailed information about ways to implement the process and when and where to use the strategy" (p. 197).

To reduce demands on short-term memory, the teacher should first direct students' attention to trim extraneous input. Then, she should activate relevant schema, deliver manageable portions of content, and allow students to practice. Driscoll (2005) noted instructional implications for information-processing theory in which she recommended "providing organized instruction" (p 104) followed by "arranging extensive and variable practice" (p. 105). The confirmatory laboratory activity then acts as a time to practice in order to facilitate the processes of encoding.

Cognitive load. Of particular importance to our typology of laboratory-based instruction is the limited nature of working memory. Kirschner, Sweller, and Clark (2006) explained that working memory can be overburdened during exploratory tasks in their piece "Why Minimum Guidance During Instruction Does not Work: An Analysis of the Failure of Constructivist, Discovery, Problem-based, Experimental, and Inquiry-based Teaching." These scholars argued that during minimally guided instruction learners must search the problem space, discriminate and attend to relevant information, and activate pertinent schemata from long-term memory, placing great demands on the limited working memory. Therefore, the overworked short-term memory has little capacity remaining to code and store the newly acquired information, and the mathematical content may not be successfully stored in long-term memory. From a cognitive load perspective, "learning and problem solving are different and incompatible processes" (Sweller, van Merriënboer, & Paas 1998, p 271).

Therefore, an exploratory laboratory sequence would not be congruent with this theory of learning. Instead, Sweller and colleagues (1998) advocated for teacher-worked

examples: “Studying worked examples also eliminates the means-ends search, and so heavy use of worked examples as a substitute for problem solving may be also beneficial” (p 273). After students study the teacher-worked examples, these scholars recommended that students work completion problems with a well-defined problem space to “provide a bridge between the worked examples and conventional problems” (275-276). This sequence of teacher-worked examples followed by completion problems is clearly congruent with a confirmatory laboratory sequence.

Exploratory tasks. Exploratory laboratories are those in which the task posed precedes a formal explanation of the content addressed. Learning then occurs as students collaboratively grapple with the task. This is not to say that the instructor does not scaffold students’ exploration of the content, only that this content has not been formally presented through lecture. An assumption of this sequence, which is argued in more detail in the section that follows, is that more explanation, as compared to a confirmatory laboratory sequence, will come from students themselves, both during the collaborative laboratory task and the formal discussion of the content that follows the task.

Constructivism. The premise of constructivism is that students create their own understanding. It is the learner who actively creates knowledge as she interacts with the world and reflects on these experiences. For the constructivist, knowledge is not conveyed; it must be constructed. In the section that follows, the ontological assumptions of constructivism are discussed and applied to mathematics. The contributions of the theories of Piaget and Vygotsky to constructivism are examined and applied to an exploratory laboratory sequence.

Ontological assumptions of constructivism. Paramount to radical constructivism are its assumptions regarding the origins of knowledge. Radical constructivists (von Glaserfeld, 1990, 1998, 2005) view the knowledge one constructs as dependent on one's experiences; therefore, one cannot represent an independent reality because she has no access to it. Constructivists question the existence of a collective reality as each individual constructs reality based on her individual experiences. These ontological debates, however, are not new. As many researchers (e.g., Dossey, 1992; Cooney & Wiegel, 2003; von Glaserfeld, 1990) have noted, this debate dates back at least as far as Plato and Aristotle. Plato believed that knowledge was discovered from an external, universal reality, whereas Aristotle thought knowledge was created from one's perceived representation of the world. As Piaget (1970) explained,

For the empiricist point of view, a "discovery" is new for the person who makes it, but what is discovered was already in existence in external reality and there is therefore no construction of new realities. . . . By contrast, for the genetic epistemologist, knowledge results from continuous construction, since in each act of understanding, some degree of invention is involved; in development, the passage from one stage to the next is always characterized by the formation of new structures which did not exist before, either in the external world or in the subject's mind (pp 77).

For the radical constructivist, the more important issue is not whether the knowledge one constructs comes from an independent, unambiguous reality, but whether it is viable (von Glaserfeld, 1990), that is whether it is useful in accomplishing a task.

“Knowledge is not something people possess in their heads, but rather something people do together” (Gergen, 1982, in Wheatley 1991, p11). From the perspective of radical constructivism, a confirmatory laboratory sequence would not be appropriate because the instructor cannot explain a universal mathematical body of knowledge because she does not have direct access to the sovereign reality from which the body of knowledge is drawn. Additionally, constructivists do not consider mathematics to be a static body of knowledge but “the activity of constructing relationships and patterns” (Wheatley, 1991, p 11). Therein, “mathematics should be thought of as a human activity of ‘mathematizing’—not as a discipline of structures to be transmitted, discovered, or even constructed” (Fosnot, 2005, p 280).

This view of mathematics as an activity, not a complete set of concepts, is more congruent with an exploratory laboratory sequence. Students learn about mathematics by participating in mathematics: “Mathematics is effectively learned only by experimenting, questioning, reflecting, discovering, inventing, and discussing” (Ahmed, 1987 in Wheatley, 1991, p 13). These processes by which Ahmed argues mathematics is learned are also the processes by which one does mathematics, and these processes should be fostered in an exploratory laboratory sequence. With a view of mathematics as an activity, “to ‘do’ mathematics is to conjecture—to invent and extend ideas about mathematical objects—and to test, debate, and revise or replace those ideas” (Schifter, 2005, p 81).

Contributions of Piaget. Much of Piaget’s genetic epistemology (1961, 1970, 1972, 1977, 1980) is situated in an age-dependent theory of development in which

learners pass through four stages: sensory motor, preoperational, concrete operational, and formal operational stages. Of particular interest to constructivists, however, is Piaget's description of how learners move from one stage to another when experiencing disequilibrating situations in which they reflect on actions. In the discussion that follows, this theory of learning is not bound to an age-dependent stage context; learners of any age or stage of development are assumed by constructivists to learn through these processes.

Through his work with pond snails, Piaget observed that the activity of an organism drives its evolution (Piaget, 1980). New structures cause an imbalance in the genome. This perturbation of the genome causes a series of possibilities to result in the form of mutations. The organism then returns to a stable state, referred to as equilibrium, through the process of autoregulation as these new structures are either accepted or other structures are reorganized to integrate these changes.

Similarly, Piaget saw learning as cognitive evolution in which learners are perturbed and either assimilate or accommodate knowledge as they return to a state of equilibrium: "In short, every new problem provokes a disequilibrium (recognizable through types of dominant errors), the solution of which consists in a re-equilibration, which brings about a new original synthesis of two systems" (Piaget, 1961, p 281). Learners assimilate new knowledge that is congruent with their current understanding of the world. Accommodation requires a reorganization of knowledge structures, referred to as schemata, to integrate meanings. Of particular importance for the constructivist are the situations that place the learner in a state of disequilibrium and encourage the learner to autoregulate. As Piaget (1977) explained, "It is clear that one of the sources of progress

in the development of knowledge is to be found in nonbalance, as such which alone can force a subject to go beyond his present state and to seek new equilibriums” (p 12).

Piaget equated this non-balance or disequilibrium with motivation as it “produced the driving force of development” (Piaget, 1961, p 13). Disequilibrium can be considered the catalyst for learning. Learners experience disequilibrium and are perturbed to make sense of the situation (re-equilibrate) when they realize their current understanding is no longer sufficient:

[One] source of nonbalance consists of gaps which leave requirements unfulfilled and are expressed by the insufficiency of a scheme. . . . a gap becomes a disturbance when it indicates the absence of an object, the lack of conditions necessary to accomplish an action, or want of knowledge that is indispensable in solving a problem. (Piaget, 1977, p 19)

Sources of disequilibrium may then be positive, in the sense of a perceived contradiction when the learner’s current understanding is challenged, or negative, when the learner has gaps in her understanding (Piaget, 1980).

As Fosnot and Perry (2005) wrote, “Disequilibrium facilitates learning. ‘Errors’ need to be perceived as a result of learners’ conceptions, and therefore not minimized or avoided” (p 34). In regard to an exploratory laboratory sequence, maximizing learning becomes a matter of maximizing disequilibrium and the resulting re-equilibration through disturbances to students’ current understandings in the form of laboratory tasks. The task perturbs students to a state of disequilibrium in which they collaboratively make

sense of the mathematics, returning to a state of equilibrium. The whole-class explanation that follows would also assist students as they return to equilibrium.

This is not to say that, from a constructivist's perspective, one cannot learn through traditional lecture-practice sequences; certainly most learners have confronted gaps in their understanding and constructed knowledge through these traditional means. However, the claim here is that an exploratory laboratory sequence maximizes disequilibrium and encourages students “not simply to return to the former state, but to go beyond it in the direction of the best possible equilibrium” (Piaget, 1980).

When examining a confirmatory laboratory sequence from a Piagetian perspective, the purpose of the laboratory is less clear because the presentation of information is assumed to both disturb and reconcile one’s understanding. Further, the purpose of the laboratory is not to disequilibrate and re-equilibrate but to produce answers in line with the explanation of the teacher:

Students of mathematics often apply only one criterion to their evaluation of their own constructs, asking “is it in agreement with the experts?” (Or, in less constructivist terms, “Is it right?”). As a result, their knowledge of mathematics becomes isolated and formalized from the rest of their experiences, which is constructed from their action on the world in a more spontaneous and interactive fashion. Memorization and imitation of examples produce the “right answer,” the desired outcome, in a local, well-defined problem space and thereby outpace the more difficult endeavor of constructing the idea and of coordinating its

interactions with the other qualities of powerful construction (Confrey, 1990, p. 112).

Confrey did not argue that students could not construct knowledge through more passive, traditional means but that these constructions are less powerful than those that are actively constructed through authentic problem-solving tasks. Returning to the biological basis of Piaget's epistemology, recall that "The organism acts constantly upon its environment instead of merely submitting to it" (Piaget, 1980, p112).

Contributions of Vygotsky. The premise of Vygotsky's (1962) theory of learning is that children obtain knowledge by interacting with the world. Knowledge is thus the product of social interactions and experiences, and these experiences are situated within a culture and social situation. Therefore, knowledge is dependent on time and place. Vygotsky thought knowledge is constructed as the learner attempts to mediate her environment. Linked to Vygotsky's emphasis on cultural and social interactions, language and the use of symbols are also integral in making meaning of one's world. Vygotsky "draws our attention to the larger social structures in which educating is embedded" (Confrey, 1995a, p. 41), introducing a socio-cultural perspective to the learning of mathematics. Vygotsky also contributed the concept of the *zone of proximal development*, which is a range of tasks that a learner might be able to accomplish with assistance from a more capable person, perhaps a more experienced peer or an adult (Vygotsky, 1978). Scaffolding, in which the teacher controls portions of a task beyond a student's current capability in order that the learner will succeed with the task, is also a contribution of this theory.

While the zone of proximal development is a significant contribution, the influence of social interactions, culture, and language are most significant to social constructivists. Whereas the constructivist focuses on how the individual constructs her own knowledge, the social constructivist is concerned with how groups construct knowledge collaboratively. Sierpiska and Lerman (1996) described this distinction as a mere difference of perspective. Constructivists view knowledge from the point of view of the individual, whereas social constructivists take the view of an observer. Groups then construct knowledge as they interact with culture and environment. Sierpiska and Lerman (1996) also underscored differences in how these three theories view language. For the constructivist, language is an expression of knowledge. For the Vygotskian, language is a medium through which cultural knowledge is transmitted. For the social constructivist, however, language is knowledge. Language creates reality and cannot be seen as a separate object; knowledge is discourse. Therein, meaning is taken as shared (Yackel & Cobb, 1996). Ideas are taken to be true as they make sense to the community. As individuals are disequibrated so is the culture. Through language group members' ideas converge toward a culturally accepted reality.

Consequences of the social constructivists' epistemology are seen through the co-construction of sociomathematical norms (Yackel & Cobb, 1996; Yackel, Cobb, & Wood 1991). These are normative aspects of mathematical discourse constructed through whole-class or group dialogue, not predetermined or imposed by the teacher. While the teacher guides the discourse as a representative of the mathematical community, norms are constructed together. Sociomathematical norms include what an acceptable

mathematical explanation sounds like. What does it mean to be mathematically different? Elegant? Efficient?

One assumption of this study is that the discourse in an exploratory laboratory sequence will differ from a confirmatory sequence both during the laboratory tasks and the explanation. First, consider group discussions during a confirmatory laboratory sequence. Because the teacher has already presented the content as infallible, conversations shift from convincing each other of the validity of one's strategy to comparing the results to the content presented by the teacher. As Confrey (1990) noted above, students then focus on the correctness of the solution, insofar as it agrees with the teacher. Compare this to the exploratory laboratory sequence in which a solution strategy is less obvious. Conversations focus on the legitimacy of the mathematics constructed as students critique each other's ideas until a commonly agreed upon strategy emerges.

In addition, discourse during the explanation phase is assumed to differ between sequences. During the whole-class discussion that precedes the laboratory task, the teacher presents the content as certain with minimal input from students in that this information is new to them. In contrast, consider a discussion that follows an exploratory laboratory sequence. Here individual groups have already constructed strategies and come to a taken-as-shared (Yackel & Cobb, 1996) understanding. Now these groups can converge on a whole-class understanding with the teacher mediating the discussion. Further, students have experienced the mathematics being discussed and are more likely to contribute to a teacher-led discussion than when the information is presented as new and external to the students.

Bruner. While Bruner's writings have a great number of educational implications, his research in modes of representation and discovery learning are particularly pertinent to the research presented here. Bruner's theory of modes of representation is considered a theory of learning as it suggests that students come to know mathematics as they move from the concrete to the abstract. In contrast, Bruner's contributions to discovery learning are considered to be a theory of instruction. Certainly Bruner's ideas of representation can be integrated into discovery, but these ideas are not necessarily synonymous. In this section, Bruner's research on representations is described. Then, his writings on discovery are considered.

Modes of representation. Bruner made the seemingly bold claim that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (Bruner, 1960, p 33). Bruner supported his thesis by noting "Any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in a recognizable form" (Bruner, 1967, p 44). Whereas Piaget might have asked the question of whether the learner is ready for the content, Bruner was more interested in how the content can be made accessible to the student (Driscoll, 2005).

Bruner believed that subject matter could be adapted to the learner by altering its economy, power, and mode of representation (Bruner, 1967). Economy is defined as "the amount of information that must be held in the mind and processed to achieve comprehension" (Bruner, 1967, p45). For example, a formula is a more concise (economical) way to summarize a phenomenon as compared to a list of quantities in a

table. Power is the degree to which the learner makes connections between ideas in order to use the knowledge at hand. For example, memorizing an alphabetical list of connected cities would be less powerful, when attempting to develop a travel route, than a network diagram illustrating these connections. An economical structure could certainly be powerless; consider, for example, a formula that might be useless to a learner. However, it is less common for a powerful structure to be uneconomical (Bruner, 1967). As highlighted below, both economy and power vary with mode of representation.

Bruner (1964, 1967) described knowledge through three domains: enactive, iconic, and symbolic representations. In an enactive mode, learners construct knowledge through motor responses. For example, a learner may be able to act on principles of center when positioning herself on a seesaw yet cannot articulate what she has done. Another learner, using an iconic representation, may use pictures, graphics, or mental images, to summarize a concept. This learner may relate to the concept of a lever through an image in a textbook in which the pivot and fulcrum are illustrated. In a symbolic mode, learners relate to content through a symbolic system, often language. A student may use this representation to express the phenomenon of a lever orally or using mathematical equations related to Newton's laws of motion.

This theory of learning has direct instructional implications; just as children learn by progressing through these modes, instruction should progress in a similar manner. Bruner advocates for instruction to be sequenced from enactive through iconic to symbolic but notes that students with a well-developed symbolic system may forgo the first two stages (Bruner, 1964, 1967). This sequence of representation was illustrated in

research (Bruner, 1967) in which four eight-year-old children learned to factor quadratic equations through these three modes of representation by first physically arranging algebra tiles (enactive), then drawing the tiles (iconic), and finally representing these arrangements with algebraic expressions (symbolic). Similarly, Bruner taught ten nine-year-olds about group theory through a sequence of actions, images, and symbols (Bruner, 1967).

As explanations are expressed through language, a symbolic representation, Bruner's theory of representations is more congruent with an exploratory laboratory sequence. During exploratory tasks, group members first enact the mathematics, perhaps using concrete manipulatives and calling on various iconic representations, before the symbolic whole-class explanation. In contrast, a confirmatory laboratory sequence moves from a symbolic, or even iconic, mode of representation during the explanation to enacting the mathematics during the laboratory task.

Discovery. For Bruner, the goal of education is to create "as autonomous and self-propelled a thinker as we can" (Bruner, 1961, p 23). It should be noted that the aim of education for Bruner is quite different from other theories examined here that measure learning by one's ability to complete a procedure in a well-defined problem space. To achieve this ambitious objective, Bruner advocated for discovery learning, which he defined as "all forms of obtaining knowledge for oneself by the use of one's own mind . . . permitting the student to put things together for himself, to be his own discoverer" (Bruner, 1961, p.22).

Although Bruner did not go into details of the ontological nature of knowledge, the term discovery often implies the existence of an external, objective reality that can be discovered, which is not a common assumption of constructivists: “It is difficult to talk about *discovering* something, such as a pattern or a structure, if we are unwilling to regard it as ‘there,’ existing apart from the individual” (Goldin, 1990, p 45, italics added). Nevertheless, Piaget, a founding contributor to the constructivist epistemology, also used the language of discovery (1961, 1970, 1977). The constructivist may therefore interpret discovery as the invention or reinvention of knowledge.

For Bruner, discovery is not haphazard; scaffolds and models are needed (Bruner, 1971). This form of well-scaffolded, purposeful exploration, described in his later work (Bruner, 1971), is similar to guided discovery, as opposed to pure discovery. In his theory of instruction, Bruner advocated for activation, maintenance, and direction in discovery (Bruner, 1967). During activation, teachers pique and maintain students’ interest by presenting a task with some “optimal level of uncertainty” (Bruner, 1967, p. 43) or ambiguity to invoke students’ curiosity. Direction is needed to ensure students have a “sense of the goal of the task” (Bruner, 1967, p. 44). The exploration is maintained through guidance provided by the teacher to ensure that the consequences of exploring incorrect alternatives do not exceed the benefits.

Bruner gave a number of benefits for learning through discovery. Although his terminology, identified in italics, changed over the course of his writings, these benefits are more-or-less consistent. First, learning through discovery increases the *intellectual potency* (Bruner, 1961) or *information flow* (Bruner, 1971). That is, “Practice in

discovering for oneself teaches one to acquire information in a way that makes that information more readily viable in problem solving” (Bruner, 1961, p 26). Bruner also referred to the connections made in the storage of knowledge through discovery as *compatibility* (Bruner, 1971). Bruner argued that learning through discovery allows the learner to “fit [new material] into his own system of associations, subdivisions, categories, and frame of reference, in order that he can make it his own and thus be able to use the information in a fashion compatible with what he already knows” (Bruner, 1971, p 71). Because of these connections, learning through discovery aids in the *conversation of memory* (Bruner, 1961). Bruner explained that difficulties remembering are due to retrieval and that learning through discovery allows one to organize information such that it can be recovered more efficiently: “organization of information that reduces the aggregate complexity of material by imbedding it into a cognitive structure a person has constructed will make that material more accessible for retrieval” (Bruner, 1961, p. 32).

Additionally, students who learn through discovery experience a *shift from extrinsic to intrinsic rewards* (Bruner, 1961). Through discovery motives move toward “the autonomy of self-reward or, more properly by reward that is discovery itself” (Bruner, 1961, p. 26). Gratification comes from coping with the task and satisfying one’s own curiosity. Therein, the learner is *activated* (Bruner, 1971) such that she “feels rewarded for the exercise of thinking” (p. 71).

Further, learning through discovery teaches heuristics for problem solving: “It is only through the exercise of problem solving and the effort of discovery that one learns

the working heuristics of discovery, and the more likely is one to generalize what one has learned” (Bruner, 1961, p 31). It is through engaging in problem solving tasks that one learns strategies for confronting problems. *Problem solving through hypothesizing* (Bruner, 1971) gives students experiences framing conjectures and developing a plan to test these claims.

Learning through discovery also changes students’ *attitudes* (Bruner, 1971) toward their perceived capacity to solve problems. Although Bruner was not yet using the language of Bandura’s (1977) self-efficacy, these constructs are quite similar. Bruner believed that

You have got to convince students (or exemplify for them, which is a much better way of putting it) of the fact that there are implicit models in their heads which are useful. . . to have the children recognize that they can use their own heads in their own education. (Bruner, 1977, pp72-73)

Therein, Bruner saw discovery learning as a means to show students that they can, in fact, solve non-routine problems.

Lastly, students can discover concepts for which they do not yet have the language to describe, which Bruner (1971) referred to as *self-loop*. Consider the aforementioned example of a student that can position herself on a seesaw to balance her opposing totterer but does not yet have the language to explain what she has done. Integrating Bruner’s theories of representation and discovery, one might say that students in the enactive stage can learn through discovery when traditional means of instruction, which rely on symbolic representation, are not yet accessible.

Certainly discovery learning is congruent with an exploratory laboratory sequence insomuch as students put things together for themselves. Therein, each of the abovementioned benefits of learning through discovery also applies to learning through exploratory laboratory tasks.

Standards of the NCTM. Situated within a constructivist theoretical framework, the instructional perspective of the National Council of Teachers of Mathematics (NCTM) is considered in support of an exploratory laboratory sequence. As mentioned previously, constructivism is a theory of learning, not a theory of instruction. However, one can draw a number of instructional implications from this epistemology. As Stiff (2001), President of NCTM 2000-2002, pointed out, “Like unicorns, ‘constructivist math’ does not exist. There are, however, several theories about learning that are categorized as ‘constructivism,’ and they can be linked to Standards-based mathematics,” yet Stiff warned that “NCTM’s *Principles and Standards* is not synonymous with constructivism” (Stiff, 2001). Nevertheless, constructivism is arguably a major theoretical contributor to NCTM’s work with the (1990) JRME monograph, *Constructivist Views on the Teaching and Learning of Mathematics*, devoted to this theory. Just as NCTM does not commit to a single theory of learning, neither does the council subscribe to one instructional strategy: “Teaching is a complex practice and hence not reducible to recipes or prescriptions” (NCTM, 1991, p. 22).

Despite NCTM’s eclectic theoretical stance, this section argues that the sense making that takes place during an exploratory laboratory sequence is attuned to the problem solving and reasoning and proof standards for school mathematics (NCTM,

2000) and the worthwhile mathematical tasks professional standard (NCTM, 1991).

Using a task analysis guide published by NCTM (Stein et al., 2000), exploratory tasks are shown to have a higher cognitive demand than confirmatory laboratories. Further, it is argued that the teacher's role in discourse and students' role in discourse professional standards (NCTM, 1991) are embodied in an exploratory laboratory sequence.

Worthwhile mathematical tasks. In an exploratory laboratory sequence, students confront problems before the teacher has prescribed a solution strategy. Therein, the problem-solving standard is illustrated as “Problem solving means engaging in a task for which the solution method is not known in advance” (NCTM, 2000, p. 52). The problem solving that takes place during an exploratory laboratory sequence is “not only a goal of learning mathematics but also a major means of doing so” (NCTM, 2000, p. 52). Just as Bruner (1961) stated,

It is only through the exercise of problem solving and the effort of discovery that one learns the working heuristic of discovery, and the more one has practice, the more likely is one to generalize what one has learned into a style of problem solving or inquiry that serves for any kind of task one may encounter (p. 31).

That is, through the problem solving that takes place during exploratory laboratories, students, in turn, become better problem solvers.

One of the tenants of the reasoning and proof standards is that students “make and investigate mathematical conjectures” (NCTM, 2000, p 57). If the teacher has already presented the concept, as in a confirmatory laboratory sequence, there is little to speculate about during the laboratory task. An exploratory laboratory sequence, however, invites

students to express their conjectures and describe their thinking. NCTM's focus on making and testing conjectures also parallels Bruner's (1971) aforementioned discussion of hypothesizing in discovery. NCTM (2000) notes, "Doing mathematics involves discovery. Conjecture—that is, informed guessing—is a major pathway to discovery" (p. 57).

One means of implementing these standards is by posing worthwhile mathematic tasks (NCTM, 1991), which "call for problem formulation, problem solving, and mathematical reasoning" (p 25). These are tasks that "capture students' curiosity, and that invite them to speculate and pursue hunches" (p 25). Considering a confirmatory laboratory sequence, there is little left to speculate about other than whether one's result agrees with the teacher, as Confrey (1990) noted above in her discussion of constructivism. Therein, the "explain-practice" sequence is "a poor source for tasks" (Wheatley, 1991, p 16). Instead, "Favorable conditions for learning exist when a person is faced with a task for which no known procedure exists" (Wheatley, 1991, p 15), as is the case with an exploratory laboratory sequence.

Worthwhile tasks should also "represent mathematics as an ongoing human activity" (NCTM, 1991, p. 25). Looking back to the discussion of constructivism above, such tasks show mathematics not as a body of knowledge but as the activity of *doing* mathematics, or "mathematizing" (Fosnot, 2005). Further, mathematics is not static, as with the empiricist's perspective opposed by Piaget (1970), but ongoing. Once again, exploring mathematics through laboratory tasks preceding a formal explanation represents mathematics not as a static collection of concepts but as a human activity.

Stein and colleagues (2000) offered a task analysis guide with four levels of cognitive demand. From least demanding to most cognitively complex, these classifications are *memorization tasks*, *procedures without connections tasks*, *procedures with connections tasks*, and *doing mathematics tasks*. Confirmatory laboratory tasks are classified as procedures with connections because these tasks

- suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. . . .
- require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedure in order to successfully complete the task and develop understanding. (Stein et al., 2000, p. 16)

Because the instructor has already presented content related to the task, or perhaps worked a similar problem, she narrows the problem space. Therefore at least one approach to the task has been recommended. Still, these tasks may be applied in a different context such that procedures must be used thoughtfully.

Exploratory tasks, however, are more cognitively complex and are classified as doing mathematics because such tasks

- require complex and nonalgorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example).

- require students to explore and understand the nature of mathematical concepts, processes, or relationships. . . .
- require considerable cognitive effort and may involve some level of anxiety for the students due to the unpredictable nature of the solution process required.

(Stein et al., 2000, p. 16)

As an approach for confronting the problem has not yet been presented, students must generate their own strategies and justify the appropriateness their strategies to group-mates. Therefore, the cognitive demand of the task is raised because the teacher has not narrowed the problem space. Particularly pertinent to Chapter 4 of this document is the forewarning that students may be uncomfortable with such cognitively demanding tasks.

Discourse. As previously mentioned in the discussion of social constructivism, the discourse generated during these two sequences is assumed to differ. Discourse is so important to NCTM that three of their five professional standards (NCTM, 1991) relate to discourse. In examining the teacher’s and students’ roles in discourse, an exploratory laboratory sequence is shown to align more closely with these professional standards.

In regard to the teacher’s role in discourse standard, “Instead of doing virtually all talking, modeling, and explaining themselves, teachers must encourage and expect students to do so” (NCTM, 1991, p38). In an exploratory laboratory, students are more likely to explain the content to each other during the task, insofar as there is a level of uncertainty that requires justification, and students will have more experiences to contribute during the whole-class discussion that follows (Marshall, Smart, & Horton, 2009). Teachers must also decide “when and how to attach mathematical notation and

language to students' ideas" (NCTM, 1991, p.35). Notice the sequence implied here; first students generate ideas, then the teacher connects convention to these student-generated concepts. This is the goal of the explanation phase of an exploratory laboratory sequence: to formalize the concepts students have constructed.

In *Students' Role in Discourse* (NCTM, 1991), learners should:

- make conjectures and present solutions;
- explore examples and counterexamples to investigate a conjecture;
- try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers; [and]
- rely on mathematical evidence and argument to determine validity (NCTM, 1991, p. 45).

Once again, students are more likely to make, and therefore articulate, conjectures when an explanation has not yet been given. Discussions of the validity of approaches to problems are also more likely when the teacher has not yet offered an assumedly infallible solution path.

4Ex2 Instructional Model. While a number of instructional models illustrate the importance of allowing students time to explore content collaboratively before a formal explanation is offered, the 4Ex2 Instruction Model (Marshall, Smart, & Horton, 2009; Marshall & Horton, 2009) is addressed here because of its influence on the author in generating the research questions explored in this study. The 4Ex2 is a dynamic model for inquiry-based instruction with four phases—Engage, Explore, Explain, and Extend—

representing the 4E's of the 4Ex2. Teacher reflection and formative assessment are integrated into each phase, composing the *by two* of the 4Ex2.

During the Engage phase, students' curiosities are piqued, similar to Bruner's (1961) activation, while the instructor assesses students' prior knowledge and provides direction (Bruner, 1967) for the task. In the Explore phase, students collaboratively make sense of the content, as they "predict, design, test, collect, and/or reason" (Marshall, Smart, & Horton, 2009, p 508) through a worthwhile mathematical task (NCTM, 1991). During the Explain phase, students share their strategies, invoking the communication standard (NCTM, 2000) as the teacher mediates the discourse. "The disequilibrium experience caused in students during the Engage and Explore now begins to gain resolution as understanding and knowledge [is] articulated during the Explain phase" (Marshall, Smart, & Horton, 2009, p. 511). The Extend phase stretches students as they "apply, elaborate, transfer, and generalize knowledge to novel situations" (Marshall, Smart, & Horton, 2009, p. 511).

While the model is dynamic in that the four phases do not necessarily occur only once in the above-mentioned sequence, "the Model is predicated on having the Explain phase follow the Explore phase" (Marshall, Smart, & Horton, 2009, p. 509). The authors of the model note, "If explanation precedes exploration, which is typical in non-inquiry instruction, students are thrust into passive learning situations that rarely challenge them" (Marshall, Smart, & Horton, 2009, p. 509-510). Therefore, it is paramount that the Explore phase precedes the Explain phase.

Beliefs

Having established sequence of instruction as a valuable instructional component based on theories of learning and instruction drawn from the literature, the review of literature that follows turns to teachers' beliefs. As Research Questions One and Two explore PSTs' pedagogical beliefs, the purpose of this section is to show that beliefs are a valid and valuable construct worth exploring. This section begins by defining the construct of mathematical beliefs as it relates to teaching. Then, empirical research on elementary teachers' mathematical beliefs is examined. After reviewing belief measures used in the literature, research relating teachers' beliefs to instructional practices, teachers' content knowledge, and student achievement is reviewed. Research related to influencing preservice teachers' mathematical beliefs through methods courses, field experiences, and mathematics content courses is also considered.

Defining beliefs. The term “teacher beliefs” is not consistently used in the research literature (Kagan, 1992; Pajares, 1992), even less those beliefs specific to mathematics. In this section beliefs are contrasted with other similar constructs such as attitudes, emotions, values, and knowledge.

In a seminal chapter on affect—a construct encompassing beliefs, attitudes, and emotions—McLeod (1992) describes beliefs as more cognitive in nature, more stable, and held less intensely than attitudes or emotions. Whereas attitudes or emotions might change often with little thought, beliefs are established over time with more cognitive consideration. For example, the frustration felt from a challenging problem and affirmation felt when finally making sense of this problem are emotions. One's like or

dislike of the subject is considered an attitude, but the conviction that mathematics is important is a belief.

Philipp (2007) makes the following distinction between beliefs and values: “A belief *that* is a belief, but a belief *in* is about values” (p. 265, emphasis in original). For example, one may believe *that* reform-based instruction leads to a more conceptual understanding as opposed to a more traditional delivery of content. This pedagogical belief is “associated with a true/false dichotomy whereas values are associated with a desirable/undesirable dichotomy” (Philipp, 2007, p. 265). However, one may believe *in* student autonomy in the mathematics classroom. In the second example, one values student autonomy as a desirable component of mathematics learning.

Further, Thompson (1992) points out that knowledge and beliefs differ in conviction and consensus. Whereas beliefs can be held with varying degrees of conviction, one rarely passionately or flippantly knows something. Additionally, knowledge is usually thought of as taken to be true by the majority, whereas one who believes something to be true is aware that others might disagree. This assumption, however, might be challenged if one’s ontological understandings do not lead her to accept a universal truth. For the radical constructivist who challenges the existence of an external, universal reality, the difference between knowledge and beliefs is less distinct.

Taking a more inductive approach, consider now some mathematical beliefs to be examined in this study. One’s epistemological beliefs relate to how one comes to know mathematics. Pedagogical beliefs relate to instructional practices believed to be most effective. Beliefs about one’s efficacy have to do with how a teacher might interpret her

ability to change students' learning outcomes. All of these beliefs are based on years of experiences, are more stable and felt less intensely than emotions or attitudes, and vary in degree of conviction and consensus.

Measuring beliefs. Having established beliefs as a valid construct drawn from the current research base, the review of literature that follows examines means for measuring these beliefs. The purpose of this section is twofold. In reviewing these measures, one can better understand the empirical research that follows. Additionally, in understanding the purposes and shortcomings of these measures, the current study's qualitative approach to understanding beliefs is supported as the measures currently available do not target beliefs regarding sequence of instruction. Though a great number of quantitative instruments have been used to assess teachers' epistemological, pedagogical, and efficacy beliefs regarding mathematics, those instruments that were repeatedly used in the literature are examined here.

Standards Beliefs Instrument. Zollman and Mason (1992) developed the Standards' Beliefs Instrument to assess teachers' pedagogical beliefs as they relate to the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). The 16-item, four-point-Likert scale asks teachers whether they strongly agree, agree, disagree, or strongly disagree with statements that are either direct quotes or slight modifications from the *Standards* (NCTM, 1989) regarding pedagogic practices. For example, Item 1 states, "Problem solving should be a SEPARATE, DISTINCT part of the mathematics curriculum" (Zollman & Mason, 1992, p. 363 emphasis in original), which relates directly to the standard stating that "Problem solving is not a distinct topic but a process

that should permeate the entire program and provide the context in which concepts and skills can be learned” (NCTM, 1989, p. 23).

While items on this measure relate directly to the seminal reform document of the time, the items are transparent and respondents may select responses they believe are desired. Additionally, because these items are either direct quotes with positive valence or negated quotes with negative valence one could argue that the instrument is measuring awareness of the *Standards* (NCTM, 1989) as opposed to pedagogic beliefs. Although the items are measured using a Likert scale, they are phrased rather dichotomously. In the item above, for example, problem solving is either distinct or indistinct.

Schoenfeld’s questionnaire. Schoenfeld’s questionnaire (1989) was originally designed to measure high school geometry students’ mathematical perceptions, beliefs, and behaviors; however, portions of this scale have been adapted to assess teachers’ beliefs about school mathematics (Hart, 2002, 2004; Wilkins & Brand, 2004; Benbow, 1995). This 81-item questionnaire consists of 60 four-point-Likert items, 10 multiple-choice items, and 11 open-ended questions. The instrument measures beliefs related to attributions of success, perceptions of mathematics, school practices, mathematics as compared to English and social studies, geometric proof, motivation, and scholastic performance. For example, Item 21 states, “Some people are good at math and some just aren’t” (Schoenfeld, 1989, p. 352), and Item 38 reads “The best way to do well in math is to memorize all the formulas” (Schoenfeld, 1989, p. 352).

No evidence of reliability or validity of this scale was reported. The sub-scales mentioned above were not statistically confirmed using factor analysis. The length of this

survey, its intended high school audience, and its focus on geometry also make it a poor fit for this study.

Mathematics Beliefs Scales (MBS). In the research examined here, the Mathematics Beliefs Scales (MBS) (Fennema, Carpenter, & Loef, 1990) is the most commonly used measure of epistemological and pedagogical beliefs (Peterson et al., 1989; Fennema et al. 1996; Staub & Stern, 2002; Swars et al. 2007; 2009; Vacc & Bright, 1999; Capraro, 2001; 2005). Researchers have also noted use of the MBS under various names, including the Mathematics Beliefs Instrument (Swars et al., 2007; 2009) and the Cognitively Guided Instruction Beliefs Scale (Fennema et al., 1996; Vacc & Bright, 1999). Various primary references (Fennema, Carpenter, & Loef 1990; Peterson et al., 1989) have also been cited with some revisions between the two versions. The MBS is a 48-item, five-point-Likert scale related to addition and subtraction, which consists of four subscales of 12 items: *how children learn mathematics*, *how mathematics should be taught*, *relationship between concepts and procedures*, and *sequencing topics*. Items from these four scales respectively include:

- It is important for a child to discover how to solve simple word problems for him/herself. . . .
- Teachers should teach exact procedures for solving word problems. . . .
- Children will not understand addition and subtraction until they have mastered some basic number facts. . . .
- When considering the next topic to be taught, one must consider the logical organization of mathematics. (Peterson et al., 1989, p. 6-7)

Capraro (2001, 2005) proposed a parsimonious version of the 1990 version of the MBS. This researcher reduced the instrument to an 18-item version with three scales: *student learning, stages of learning, and teacher practices*. While the 48-item version had an internal consistency Cronbach's alpha reliability of 0.68 with 123 classroom teachers, the 18-item version obtained a reliability of 0.86.

Although this prolifically published scale measures teachers' pedagogical beliefs and is situated in an elementary mathematics context, the scale uses a context specific to number and operation. Further, there are discrepancies between the number of subscales between the 1989 and 1990 version with no published reliability or validity for the revised version. Capraro was contacted (personal communication, February 22, 2011) regarding the subscales of the shortened scale, but the computer holding these data is no longer operational.

Mathematics Beliefs Instrument (MBI). Hart (2002) created an instrument by adapting Zollman and Mason's (1992) scale and Schoenfeld's (1989) questionnaire and adding a few original questions. The Mathematics Belief Instrument (MBI), not to be confused with references to the MBS with the same name (Swars et al., 2007, 2009), borrows items from Zollman and Mason's (1992) scale but dichotomizes respondents' answers to either agree or disagree for the items borrowed from the Standards Beliefs Instrument. Nine 4-point-Likert items were taken from Schoenfeld's (1989) questionnaire with the addition of three items about success in math courses related to sex, ethnicity, and speed of computation and two items related to teacher efficacy.

Weaknesses in the original scales from which items were borrowed influence this instrument as well. By dichotomizing the items taken from the Standards Beliefs Instrument, desired responses are made more obvious. No evidence of reliability or validity was reported.

Integrating mathematics and pedagogy (IMAP) Web-based survey. Ambrose and colleagues (2003, 2004) pointed out that Likert scales give no information about how respondents interpret the item or the importance of the item to a respondent. Such scales also often lack context. As mentioned previously, these items are also often transparent and dichotomous. Therefore, these researchers offer a quantitative alternative to open-ended surveys in which respondents answer questions based on samples of student work or videos on students' mathematical thinking. Researchers can then use 17 rubrics to quantify the responses as showing no evidence, weak evidence, evidence, or strong evidence of seven beliefs about mathematics, pedagogy, and epistemology. For example, after considering Figure 2.1, respondents are asked, "Do you think Carlos could make sense of and explain Sarah's strategy? Why or Why not?" (Ambrose et al., 2004, p. 68).

Figure 2.1. Sample Question from IMAP Survey (Ambrose et al., 2004, p. 68)

Carlos	Sarah
<p style="text-align: center;">Written on Paper</p> $\begin{array}{r} 1\ 1 \\ 149 \\ +286 \\ \hline 435 \end{array}$	<p style="text-align: center;">149 + 286</p> <p>Sarah says, "Well, 149 is only 1 away from 150, so 150 and 200 is 350, and 80 more is 430, and 6 more is 436. Then I have to subtract the 1, so it is 435."</p>

In addition to the coding rubrics offered in the training manual (IMAP, 2003), examples of typical responses for each coding are offered.

While this survey allows for both qualitative and quantitative data, the quantifying of these data can be quite time consuming. Because of the labor-intensive coding process, this measure is designed to show individual growth in small numbers of teachers, not to compare means of larger groups. These probes are much less transparent than many of the Likert scales reviewed, but, as with any rubric, transforming qualitative data to measurable quantities introduces the influence of the researcher's interpretation.

Implications of beliefs research. As the current study explores PSTs' pedagogical beliefs, the purpose of this section is to underscore the importance of considering this construct as beliefs have been connected to teachers' pedagogy, teachers' content knowledge, and students' learning. Therein, this study's focus on evaluating and influencing PSTs' pedagogical beliefs may have implications to PSTs' future practice and student outcomes based on the literature that follows. In this section, implications of

research on teachers' pedagogical beliefs are examined as they relate to teachers' practice, content knowledge, and student achievement.

Practice. It may not be surprising that teachers' mathematical beliefs influence their instructional practice. In fact, some researchers define beliefs as mental constructs that guide behavior or filters through which decisions are made (Pajares, 1992). This has implications for practice inherent in one's understanding of the construct. Lerman (1983) explains that "one's perspective on mathematics teaching is a logical consequence of one's epistemological commitment" (1983, p. 59). In his philosophical piece, Lerman ties two ontological perspectives to epistemological beliefs and gives instructional implications of each—mathematics as a body of knowledge and mathematics through problem solving. Ernest (1989) gives three views of mathematics: "unrelated but utilitarian rules and facts. . . static but unified body of certain knowledge. . . [and] dynamic, continually expanding field of human creation and invention, a cultural product" (p. 2). From these three views of mathematics, his model explains the role of the teacher as an instructor, explainer, or facilitator.

However, Richardson (1996) argues that the relationship between beliefs and practice is more complex than a linear, causal relationship: "The perceived relationship between beliefs and actions is interactive. Beliefs are thought to drive actions; however, experience and reflection on action may lead to changes in and/or additions to beliefs" (p. 104). Leatham (2006) offers a *sensible system* as a framework for understanding the relationship between teachers' beliefs and practice. When explaining perceived contradictions between beliefs and practice, Leatham challenges the assumptions that

teachers can readily articulate their beliefs and that researchers' interpretations of what teachers express correspond to the underlying beliefs held by the teacher. Therefore, inconsistencies in beliefs and practice are attributed to problems articulating and interpreting beliefs, not the lack of sensible connection between beliefs and practice.

Beginning with qualitative research linking beliefs and practices, in the case study of two secondary mathematics teachers, Beswick (2007) found evidence supporting Ernest's (1989) model connecting ontology to practice. Nine beliefs emerged from surveys, interviews, and classroom observations that connected the nature of mathematics to beliefs regarding students' mathematics learning and, in turn, connected mathematics learning to the role of the teacher.

Archer (1999) compared the beliefs and practices of 17 elementary and 10 secondary teachers of mathematics through interviews. She found that elementary teachers tended to view mathematics as connected and applicable to students' lives. Therefore, elementary teachers used activities that represented the real world with overarching themes connecting mathematics with other disciplines. However, secondary teachers viewed mathematics as a self-contained, linear body of knowledge. These views led to more traditional instruction of teacher-worked examples followed by guided practice and individual practice with little to no context.

In a collective case study of five high school teachers, Cross (2009) found that teachers' ontological beliefs regarding the nature of mathematics connected to their beliefs about how students learn and, in turn, the teachers' pedagogy. For example, one teacher believed mathematics to be a collection of facts and rules and therefore believed

understanding mathematics to be “knowing how and when to use the formula correctly to get the right answer” (p. 337). This teacher believed her role was to expose students to formulas and show them how to use them. Therefore, this teacher stressed memorization and practice. For each of these cases, the researcher found ontological beliefs influencing pedagogical beliefs and the resulting practice.

Through a collective case study of four third-grade teachers, Bray (2011) found that teachers’ beliefs influenced their error-handling practices. From a qualitative analysis of classroom observations, interviews, and IMAP Web-based surveys (Ambrose, Phillip, Chauvot, & Clement, 2003), Bray found that the extent to which flawed solutions were focused on during whole-class discussion was related to teachers’ beliefs. These teachers avoided discussions of flawed solutions because they believed their students had academic deficits and such analysis would confuse students and bruise their self-confidence. Therefore, it was their beliefs regarding their students’ mathematical abilities, not the nature of mathematics, that influenced their error-handling practice.

In a case study of two elementary teachers, Sztajn (2003) found that beliefs beyond mathematics influence teachers’ pedagogical decisions. Through classroom observations and semi-structured interviews, Sztajn found that teachers’ concepts of students’ needs, not mathematical beliefs, influenced their practice. A teacher of students with lower socioeconomic status believed her role was to mold students into good citizens, and her teaching followed the text in structured lessons. Another teacher of students with higher socioeconomic status saw her role as preparing students for higher-order thinking and used problem-driven explorations in her instruction. In addition to the

support this research gives to Anyon's (1980) "hidden curriculum," through these contrasting cases, the researcher connected teachers' beliefs about their students' needs to practice.

Other researchers found teachers' beliefs as barriers to implementing reform-based practices. Grant, Hiebert, and Wearne (1998) worked with 12 in-service, primary teachers to create reform-based lessons. They found that teachers who viewed mathematics as a collection of skills were less likely to interpret and implement reform-based instruction as intended. Similarly, Borko and others (1997) found that unchanged beliefs regarding pedagogy resulted in inappropriately assimilating ideas presented during staff development. However, when these beliefs were challenged, developers were successful in changing beliefs and consequent practice. Bolden and Newton (2008) found that teachers believed time, aspects of accountability, and standardized tests to be major barriers to investigative teaching in their case study of three primary teachers. In the qualitative analysis of his quasi-mixed methods study of 25 preservice teachers during an early field experience, Benbow (1995) concluded that future teachers' pre-existing beliefs about mathematics influenced their planning and implementation of lessons.

Turning now to mixed-methods research, Peterson et al. (1989) used sequential explanatory mixed-methods sampling in which scores from the MBS informed the selection of seven teachers with cognitively-based perspectives and seven teachers with less cognitively-based perspectives. Classroom observations and interviews showed that cognitively based teachers emphasized counting strategies and word problems, whereas less cognitively based teachers stressed learning addition and number facts with less

attention paid to word problems until these facts were mastered. Although the “cognitively based” classifications of teachers aligns with the study’s focus, the MBS instrument used to select these groups does not measure teachers’ orientation toward instruction based on students’ development as the labels might imply.

Fennema and colleagues (1996) used data from transcribed interviews, observations, and field notes to situate 18 elementary teachers in one of four levels of instruction during a 4-year Cognitively Guided Instruction (CGI) professional development. Scores from the MBS and levels of instruction increased over the 4-year intervention. The authors note an “obvious” relationship between levels of instruction and beliefs, but do not attempt to statistically correlate these data.

In quantitative studies, Staub and Stern (2002) found teachers’ MBS scores to be significantly, positively correlated with the frequency of performance-oriented tasks presented by the teacher in 33 German Grade 3 classrooms.

Nevertheless, Hart (2004) found contrary results. She followed eight of the 14 preservice teachers from an earlier study (Hart, 2002) into their first year of teaching. While results from the Mathematics Beliefs Instrument showed these teachers maintained the reform-based belief gains from the previous study (Hart, 2002) into their first years of teaching, Hart did not find evidence that teachers were able to implement pedagogy that was consistent with these beliefs during classroom observations.

Content knowledge. Although Ernest’s (1989) model does not explain connections between teachers’ mathematical beliefs and their mathematical and pedagogical content knowledge, connections between mathematical beliefs and content

knowledge have been found. In the aforementioned study of 7 cognitively based (CB) and 7 less cognitively based (LCB) teachers, Peterson et al. (1989) found that CB teachers had greater pedagogical content knowledge than LCB teachers. Six of the seven CB teachers correctly identified distinctions in word problems, whereas only one of the seven LCB teachers could identify such differences. CB teachers could also identify more counting strategies than LCB teachers but were no more likely to predict which strategy a student would use.

Swars and others (2007, 2009) found that scores on the MBS for 103 preservice teachers in 2007 and 24 in 2009 were positively correlated with scores on a measure that used selected items from the number and operations, patterns, function, and algebra, and geometry scales of the Learning Mathematics for Teaching (LMT, 2004) instrument. Although content subscales of the LMT were not examined individually, the *learner* and *curriculum* subscales of the MBS were significantly correlated with the summative LMT score. Therefore, preservice teachers with more mathematical knowledge for teaching were more likely to believe learners can construct their own mathematical understandings and take a problem-solving approach to curriculum. Again, these subscales are not among those originally published for the MBS (Peterson et al., 1989) but are assumed to be part of a later, unpublished version of the instrument (Fennema, Carpenter, & Loef 1990).

However, in a study of 481 in-service elementary teachers, Wilkins (2008) found that content knowledge—as measured by 45 items selected from the Third International Mathematics and Science Study, the Second International Mathematics Study, and some items constructed by the author—was negatively correlated ($r=-0.15$, $p<0.001$) with

teachers' beliefs regarding the effectiveness of inquiry-based instruction, assessed with a four-point Likert scale published by Horizon Research (2000). Additionally, in an Analysis of Variance, K-2 teachers were found to score lower on measures of content knowledge than teachers of grades three-five, yet K-2 teachers believed inquiry-based instruction to be more effective than upper-elementary teachers. Still, this finding may be confounded by the sophistication of the content being taught.

Wilkins' (2008) findings connecting pedagogical beliefs to the grade level being taught are consistent with Archer's (1999) findings comparing elementary and secondary teachers. When comparing the findings of Wilkins (2008) with Peterson et al. (1989), it should be noted that Wilkins measured common content knowledge, whereas Peterson and colleagues measured pedagogical content knowledge. It should also be noted that the construct Mathematical Knowledge for Teaching in Swars et al. (2007, 2009) includes pedagogical content knowledge, specialized content knowledge, and common content knowledge; however, the measures cited only measure the latter two of the three. The differences in the types of content knowledge being measured could explain these seemingly contrary results. Inquiries into these classifications of content knowledge are examined further in the section on "Mathematical Knowledge for Teaching."

Student achievement. As previously mentioned, Peterson and colleagues (1989) used data from belief interviews and MBS score to select seven cognitively based teachers (CB) and seven less cognitively based (LCG) teachers for further study. In addition to the abovementioned difference in instructional practices and pedagogical content knowledge, these researchers also found that students with CB teachers scored

higher than the children with LCB teachers on a problem-solving assessment created by the researchers. However, there were no significant differences on a researcher-generated assessment of number facts.

Fennema and fellow researchers (1996) found that student achievement increased during the 4-year Cognitively Guided Instruction (CGI) professional development, as did teachers' scores on the MBS, for 18 in-service teachers. Although direct correlation between MBS scores and students' scores on a concepts and problem-solving test were not reported, the researchers did find a positive relationship between level of instruction, which was linked to beliefs, and scores on a concept and problem-solving assessment. Teacher beliefs, instructional practice, and student achievement all increased over the longitudinal study, but researchers did not attempt to connect beliefs and achievement statistically. Similar to the findings of Peterson et al. (1989), there were no significant differences in scores on an assessment of computational fluency.

Staub and Stern (2002) also reported links between teachers' epistemological beliefs, as measured by the MBS, and student achievement by way of instruction. Teachers with higher MBS scores more frequently presented tasks focused on conceptual understanding. The frequency of such tasks was significantly correlated with students' achievement gains on a multiplication-division word-problem assessment for students of 22 teachers in the study.

Summary: Implications of beliefs research. Ernest (1989) linked teachers' beliefs to their practice through a model that connects ontological beliefs to epistemological beliefs and, in turn, to instructional practice. Empirical research supports this model by

connecting beliefs regarding the nature of mathematics to beliefs regarding the role of the teacher (Beswick, 2007; Cross 2009) and error-handling practices (Bray, 2011). Reform-congruent pedagogical beliefs have been connected to the use of performance-oriented tasks (Staub & Stern, 2002), counting strategies, and word problems (Peterson et. al, 1989). Reform-resistant pedagogical beliefs have also been identified as barriers to implementing reform-based professional development (Grant, Hiebert & Waine; Borko et al., 1997; Bolden & Newton, 2008). In addition to connections to practice, beliefs have been associated with teachers' pedagogical content knowledge (Peterson et al., 1989) and students' achievement (Peterson et al., 1989, Fennema et al. 1996; Staub & Stern 2002). Because of the relationship among beliefs, practice, and learning, the research presented here highlights the importance of exploring PSTs' beliefs further in the current study.

Changing beliefs. Given these potential links between teachers' mathematical beliefs and instructional practice, teachers' content knowledge, and student achievement, it is not surprising that researchers study the effects of teacher education programs on preservice teachers' beliefs, as the current study also does. Specifically, the research presented here examines the effect of mathematics methods courses, early field experiences, and mathematics content courses on preservice teachers' mathematical beliefs.

Methods courses. Wilkins and Brand (2004) found significant gains on 89 preservice teachers' MBI scores after participating in a mathematics methods course that "centered around an investigative approach to teaching mathematics" (p. 226). Within the authors' brief description of this methods course, it is difficult to determine what makes

this course investigative other than the title of the text used, *Fostering Children's Mathematical Power: An Investigative Approach to K-8 Mathematics Instruction* (Baroody & Coslick, 1998). Still, preservice teachers had more reform-minded beliefs after participating in this methods course.

Vacc and Bright (1999) also found significant positive changes in 34 preservice teachers' scores on the MBS after teachers' were introduced to Cognitively Guided Instruction during a mathematics methods course. While some detail is given to the course requirements and instructional strategies used, other than the use of a CGI framework that focuses on children's thinking, it is not obvious what aspects of this course distinguish this methods course from others.

Swars et al. (2007) report significant gains in 103 preservice teachers' scores on the MBS after a two-course methods sequence. Additionally, these researchers also found that pedagogical beliefs and efficacy beliefs, as measured by the MBS and Mathematics Teaching Efficacy Beliefs Instrument (Enoch, Smith & Huinker, 2000) respectively, were unrelated before the methods course. However, beliefs about pedagogy and efficacy were positively correlated after the two-course methods sequence. That is, after the methods course, PSTs with more reform-oriented pedagogical beliefs also believed they were more capable of increasing study learning.

In a similar study of 24 pre-service teachers in a later cohort, Swars and fellow researchers (2009) found significant increases in MBS scores toward reform-congruent pedagogical beliefs during the first methods course, insignificant decreases in scores after the second methods course, and a significant drop in MBS scores after student teaching.

The authors ascribe this decline in MBS scores to “continued enculturation in existing classroom practices and the development of somewhat more realistic expectations for successful learning outcomes given the demands of teaching” (p. 62), conclusions that were based on data from follow-up interviews. Nonetheless, pre-post change in MBS scores over the entire program still showed a significant positive change. Once again, other than the use of a CGI text— *Children’s Mathematics: Cognitively Guided Instruction* (Carpenter et al. 1999)—no novel features of these methods course are described for either study.

While these studies provide evidence that methods courses can be a venue for influencing teachers’ pedagogical beliefs, they do not provide teacher educators with specifics regarding what aspects of these courses influenced beliefs. If we are to assume these studies are assessing the effects of the texts or specific CGI materials, a quasi-experimental design that tests these materials against others might be more appropriate.

Field experiences. Benbow (1995) examined the effects of an early field experience on 25 preservice teachers’ beliefs as measured by the Indiana Mathematics Belief Scales, the Elementary School Mathematics Teaching Beliefs Inventory, and Schoenfeld’s (1989) Questionnaire. However, no significant changes in pre- and postscores beliefs scales were found. Using data from interviews, Benbow concluded that teachers’ pre-existing beliefs were confirmed during this early field experience.

Lloyd (2005) studied the effects of a student-teaching experience on one secondary teacher’s beliefs related to the role of the teacher. From interviews and written assignments from a previous methods course, data revealed an initial belief that good

teachers do not lecture and that the teacher should remain “off stage” (p. 450) during student-centered investigations. After this student-teaching experience, the PST found that his previous belief oversimplified the teacher’s role and that teachers should take an active role directing discourse. Therein, it was this PST’s field experience that helped him develop more complex beliefs related to the teacher’s role.

Ambrose (2004) evaluated the effects of early field experiences combined with a mathematics content course on preservice teachers’ mathematics beliefs in the Integrating Mathematics and Pedagogy project. PSTs focused on the number and operations content standard both in the content course and sessions with elementary-aged students. Pairs of PSTs worked with a single child during these sessions. The 15 PSTs showed gradual gains over the course of the intervention as seen in data collected from a pilot version of the IMAP Web-based survey, interviews, and field notes.

Ambrose continued her work in Philipp and others’ (2007) study of 159 pre-service teachers. As part of the IMAP project, while taking mathematics content courses, 50 teachers watched videos of children solving problems and then worked with children to solve problems; another 27 PSTs only watched the videos but had no live component; 23 PSTs visited reform-based classrooms; 25 PSTs visited convenient classrooms in which teachers’ reform orientation was unknown, and 24 PSTs had no field or video experience. Teachers with combined video and live experiences showed significant changes toward reform-oriented pedagogical beliefs in four of the seven beliefs measured by the IMAP survey over those without field experiences. The video-live group also showed significant gains over the convenient-placement group on five beliefs and over

the reform-placement group on one belief. Although similar results were reported for the video-only group as compared to reform-placement, convenient-placement, and control groups, no comparisons were reported between the live-video and video-only groups.

Although Philipp et al. examined the integration of field experiences with the learning of mathematics content, few details are given about the specifics of the content course as the content course was considered a constant across all field placements. This study provides evidence that combining the learning of content and field experiences can influence pedagogical beliefs; however, it is unclear what role the content course played. The purpose of this dissertation is to explore the effects of specific aspects a content course prior to field experiences.

Mathematics content courses. Charalambous, Panaoura, and Philippou (2009) found that two mathematics content courses centering on the history of mathematics were successful in changing 94 preservice teachers' epistemological and efficacy beliefs as indicated on an instrument the authors created. These mathematics courses consisted of two, one-hour lectures and one 90-minute "'hands-on-activity' session" (p.166) each week. The authors do not report whether these activity sessions confirmed ideas presented in lecture or if they were exploratory in nature. No reliability statistics were provided, but researchers supported the validity of their instrument through qualitative data from interviews and confirmed subscales through factor analysis.

Hart (2002) found 14 preservice teachers showed significant gains on the Mathematics Beliefs Instrument after taking a combined content–methods course with an exploratory approach to content. The six credit hours of mathematics and six hours of

methods were combined into a single course taught over three semesters. This course was taught from a “constructivist philosophy” (p. 5) in which concepts were introduced and explored using “a problem situation that was designed to provoke a dilemma” (p. 6).

Wilcox and colleagues (1990) analyzed data from interviews with four students that participated in three reform-based mathematics content courses for elementary teachers. Again, these nontraditional mathematics courses are vaguely defined. Researchers explain that participants were engaged in “analyzing, abstracting, generalizing, inventing, proving, and applying” and that they were required to “communicate their understanding in multiple ways” (p. 5). Although the content course emphasized learning communities, the term learning community is equally ill defined: “Our image of community was richer than simply having groups of students work together on a problem and then report their findings. Our vision of community was a classroom where students and teacher together engaged in mathematical inquiry” (p. 6). Nevertheless, the authors found the course effective in establishing norms of collaboration including shifts in epistemological authority and a change in beliefs about community.

Lubinski and Otto (2004) studied the effect of a problem-centered mathematics content course on 16 preservice teachers’ mathematical beliefs. Although these authors report in detail about the course itself, giving examples of problem-driven lessons, they simply quote typical student responses to survey questions with no analysis of these data or implications for teacher educators. However, the researchers do claim that the goal of

changing teachers' epistemological beliefs through content courses was slowly being accomplished.

Spielman and Lloyd (2004) compared a traditional textbook section of mathematics courses for elementary teachers to a section that used two reform-oriented middle school curricula: *Mathematics in Context* and *Connected Mathematics Project*. The authors found significant changes on *classroom authority* and *learning sources* subscales on their own instrument measuring teacher beliefs, but not on the *community* subscale, for the 38 PSTs in the reform section. However, little change in beliefs was found for the 24 students in the textbook section. Although authors support the validity of their instrument by “involving multiple persons in the design of survey items, designing items in response to themes in the research literature, and basing item design on the results of pilot work” (p. 36), no reliability statistics or factor analysis of subscales are reported. By combining curricula in the experimental section, it becomes difficult to determine how each curriculum was used and which curriculum influenced PSTs' beliefs. Although this study focuses on beliefs, not content, it is problematic that content instructors used middle school curricula at the post-secondary level. This supports findings of the National Council on Teacher Quality (NCTQ, 2008) that such courses do not cover the content PSTs need but focus on content at an elementary or middle-grades level.

Summary: Changing beliefs. The research presented above shows that teacher educators have already begun to inquire into PSTs' beliefs and attempt to influence these beliefs through methods courses (Wilkins & Brand, 2004; Vacc & Bright, 1999; Swars et

al., 2007, 2009), field experiences (Benbow, 1995; Lloyd, 2005; Ambrose, 2004; Philipp et al. 2007) and content courses (Charalambous, Panaoura & Philippou, 2009; Hart, 2002; Wilcox et al., 1990; Lubinski & Otto, 2004; Spielman & Lloyd, 2004). Like other studies that explored the influence of content courses, the current study examines beliefs from the perspective of PSTs as students of mathematics. Nevertheless, the current study adds to this literature as it focuses on a specific, targetable instructional component, namely sequence of instruction.

Mathematical Knowledge for Teaching

Research Question Three explores the effects of sequence of instruction on PSTs' Mathematical Knowledge for Teaching (MKT). To better understand the specific type of knowledge considered in the current study, this section begins by developing a model for MKT. Then, empirical research on teachers' content knowledge is presented to further illustrate the importance of considering this specialized knowledge. Within this presentation of empirical research, work related to general mathematical content knowledge is reviewed. Then, research specific to this model of MKT is discussed.

Model for Mathematical Knowledge for Teaching. In the section that follows Shulman's (1986) typology of content knowledge is introduced. The work of Ball and Bass is reviewed as they apply Shulman's framework to elementary mathematics instruction. Then, the development of a more holistic model of teachers' Mathematical Knowledge for Teaching (MKT) is chronicled.

Pedagogical content knowledge. In the presidential address to the American Educational Research Association, Shulman (1986) introduced the "missing paradigm"

(p. 6) of subject-matter-specific knowledge. After reviewing teacher licensing exams of the late 19th century, which were concerned primarily with content, Shulman turned to the research-based assessments of the 1980s, which focused almost exclusively on general pedagogical knowledge. Having established such distinctions between content and pedagogy as a new phenomenon, the author challenged this assumed partition of content and pedagogy and expressed a need for a more coherent understanding of the knowledge required for teaching.

Shulman (1986) offered three categories of content knowledge: subject-matter content knowledge, pedagogical content knowledge, and curricular knowledge. This subject-matter content knowledge goes beyond recalling and demonstrating facts and procedures of the domain: “The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so” (p. 9, italics in original). In addition to a deep, conceptual understanding of the subject, teaching requires an understanding of content that is specific to teaching. This pedagogical content knowledge includes which examples best illustrate a concept, which representation or analogy works best in a particular circumstance, and knowledge of what misconceptions students might have. Shulman’s third category of content knowledge, curricular knowledge, includes knowledge of programs and materials available within the content domain. This curricular knowledge is further divided into lateral and vertical curricular knowledge. Lateral curricular knowledge includes the relationship the content has to other domains within a given grade level. Vertical curricular knowledge is one’s understanding of past

and future years' content within one's own domain and how this relates to the current curriculum.

Based on interviews, classroom observations, and examining materials of teachers as they move from teacher education programs to becoming novice teachers, Shulman (1987) later expanded his model of knowledge for teaching to include general pedagogical knowledge not limited to a particular domain. This includes knowledge of learners as well as educational contexts, ends, purposes, and values. This work also evaluated sources of teacher knowledge, which includes content-specific scholarship, general educational scholarship, educational materials, and wisdom of practice. While this work references a qualitative research base, no specific methods for data collection and analysis, subject or grade level taught by participating teachers, or the exact number of participants is given.

Wilson, Shulman, and Richert (1987) published a more comprehensive description of their research methods with another variation of this model of teacher knowledge. This work followed 12 secondary social studies, mathematics, English, and biology teachers from a teacher education program into their first year of teaching. Through task-based, semi-structured interviews, researchers developed intellectual biographies of preservice teachers' subject-matter knowledge. In other transcript-guided interviews, preservice teachers described their undergraduate experiences, highlighting courses that influenced their intellectual development. Data were also collected using free association, card sort tasks, and analyses of texts. Once participants began their first year

of teaching, researchers collected observational data as teachers planned, taught, and reflected on instruction.

From these data researchers constructed two theoretical frameworks regarding teacher knowledge. The first was a professional knowledge base similar to Shulman's (1986, 1987) work. Knowledge for teaching was divided into general pedagogical knowledge as defined by Shulman (1987), subject-matter knowledge as in Shulman's (1986, 1987) work, and pedagogical content knowledge, which included the curricular knowledge Shulman (1986, 1987) previously treated as a distinct category.

The second product of this research was a model for the construction of pedagogical content knowledge. Through cycles of preparation, instruction, and reflection, knowledge of subject matter, general pedagogy, curriculum, and learners is transformed to create pedagogical content knowledge. Teachers construct new pedagogical content knowledge as they move through the stages of comprehension, transformation, instruction, evaluation, reflection, and new comprehension. Of particular interest is the stage of transformation in which teachers critically interpret the subject matter, select appropriate representations, adapt the content for students in general, and tailor the transformed content to their particular students. One result of this model is that teachers develop pedagogical content knowledge through the practice of teaching. One troubling implication of this model is that if pedagogical content knowledge is gained only through the act of teaching, methods courses for which there is no clinical teaching experience attached and in-class professional development opportunities have no potential for developing such knowledge.

Content knowledge in elementary mathematics. Although not yet using the language of Shulman and colleagues, Ball (1988, 1990a, 1990b, and 1991) also investigated teachers' subject matter knowledge; specifically, Ball examined preservice elementary and secondary teachers' mathematical subject matter knowledge for teaching. In this line of research, Ball introduced *substantive* knowledge of mathematics defined by three criteria. Substantive knowledge of mathematics must be correct, connected, and include underlying meanings. To have substantive knowledge of dividing fractions, for example, a teacher should not only be able to correctly invert and multiply, but she must have an understanding of why this procedure works and how one might represent this concept, be able to apply a relevant context to a problem, and connect fractions to division as well as link representations and contexts for whole-number division to division of fractions.

Ball (1990a, 1991) also distinguished between knowledge *of* mathematics and knowledge *about* mathematics. Knowledge *about* mathematics includes ontological issues related to mathematics:

What counts as an “answer” in mathematics? What establishes the validity of an answer? What is involved in doing mathematics? In other words, What [sic] do mathematicians do? Mathematical knowledge is based on both convention and logic. Which ideas are arbitrary or conventional and which are logical? What is the origin of some of the mathematics we use today and how does mathematics change? (Ball, 1990a, p. 458)

Ball argued that this knowledge of the nature of the field is important for teachers and distinct from one's traditional knowledge *of* mathematics.

In 1996, Ball continued to explore the mathematical knowledge required for teaching by teaming up with H. Bass, a research mathematician (Ball 1999; Ball & Bass 2000a, 2000b, 2001, 2003a, 2003b). Ball and Bass reviewed videotapes from Ball's 1989-1990 third-grade class using a job-analysis approach to explore the mathematical knowledge required to teach elementary mathematics and the interaction between mathematics and pedagogy. It is in this line of research that Ball and Bass applied pedagogical content knowledge (Shulman 1986, 1987; Wilson, Shulman & Richert, 1987) to the teaching of elementary mathematics. Ball and Bass used the widely published (Ball 1999; Ball & Bass 2000a, 2000b, 2001) example of students' discussion of even and odd numbers to illustrate the mathematical and pedagogical content knowledge required of elementary teachers. Although this research does not necessarily advance the theory of pedagogical content knowledge, Ball and Bass situated this construct in the context of elementary mathematics and gave examples of the specialized mathematical and pedagogical content knowledge required of elementary teachers.

Mathematical Knowledge for Teaching (MKT). The Learning Mathematics for Teaching research team began in 2001 to pilot multiple-choice items to measure the mathematical knowledge required to teach elementary mathematics (Ball, Bass & Hill 2004; Hill Schilling & Ball, 2004; Ball, Hill & Bass, 2005). Researchers began writing items for two of Shulman's (1986) three domains of content knowledge: subject-matter knowledge and pedagogical content knowledge.

Researchers further divided subject-matter knowledge into *common content knowledge* and *specialized content knowledge*. Common content knowledge refers to mathematical knowledge required for teaching elementary mathematics that would also be held by other educated adults, for example, determining the decimal halfway between 1.1 and 1.11 (Ball, Bass & Hill, 2004; Hill Schilling & Ball, 2004). Specialized content knowledge, however, is mathematical knowledge required for teaching but not generally held by non-educators. Consider the problem below.

Imagine that you are working with your class on multiplying large numbers.

Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
$\begin{array}{r} 35 \\ \times 25 \\ \hline 125 \\ +75 \\ \hline 875 \end{array}$	$\begin{array}{r} 35 \\ \times 25 \\ \hline 175 \\ +700 \\ \hline 875 \end{array}$	$\begin{array}{r} 35 \\ \times 25 \\ \hline 25 \\ 150 \\ 100 \\ +600 \\ \hline 875 \end{array}$

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers? (Ball, Bass & Hill 2004; Hill Schilling & Ball, 2004; Ball, Hill & Bass, 2005; Hill et al., 2007; LMT, 2008).

Here one must first decompose student work to understand students' strategies and then determine whether these strategies will generalize. Such a task requires mathematical knowledge not generally held by individuals outside the field of education.

The domain of *knowledge of content and students*, based on Shulman's (1986) pedagogical content knowledge, requires both an understanding of mathematics and students' thinking, which includes "typical errors, reasons for those errors, developmental sequences, [and] strategies for solving problems" (Hill, Schilling & Ball, 2004, p. 17).

Take for example the problem below.

Mr. Fitzgerald has been helping his students learn how to compare decimals. He is trying to devise an assignment that shows him whether students know how to correctly put a series of decimals in order. Which of the following sets of number will best suit that purpose? (Mark ONE answer.)

- | | | | | |
|----|---|------|------|------|
| a) | .5 | 7 | .01 | 11.4 |
| b) | .60 | 2.53 | 3.14 | .45 |
| c) | .6 | 4.25 | .565 | 2.5 |
| d) | Any of these would work well for this purpose. They all require the student to read and interpret decimals. | | | |

(Hill, Schilling & Ball, 2004, p. 28; Hill, Ball & Schilling 2008, p. 400)

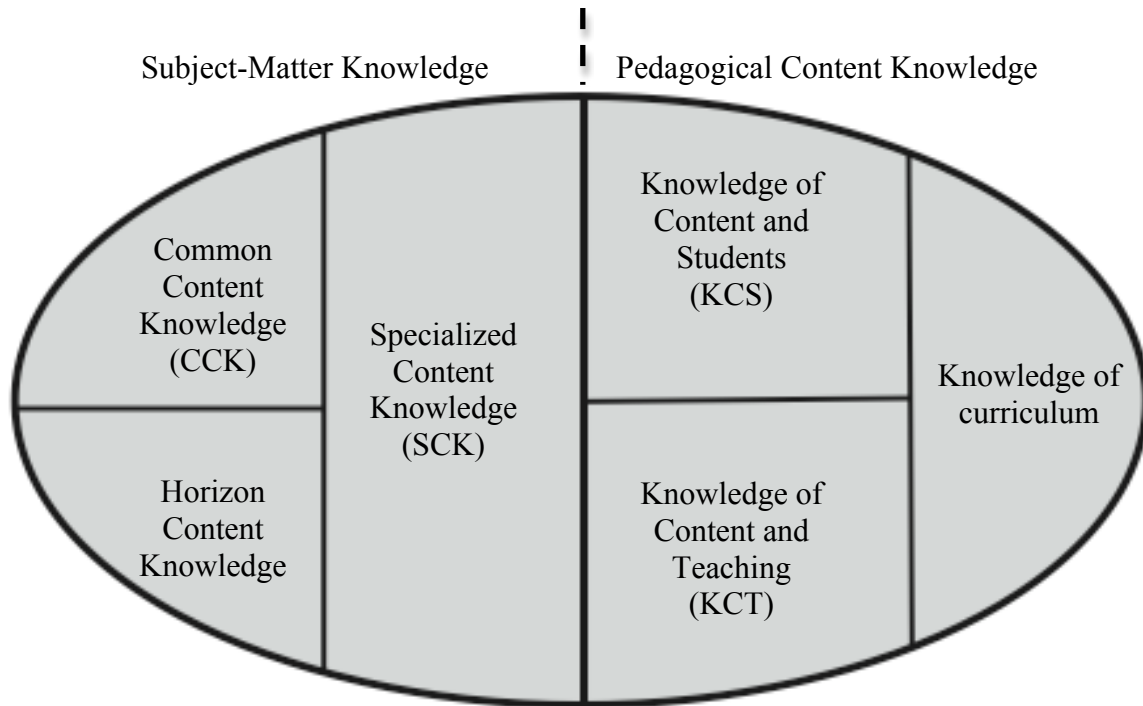
In addition to the mathematical content, one must realize that students sometimes ignore the decimal and order the numbers based on the magnitude of the digits to answer this question. Therefore, this question requires knowledge of both students and content.

Current model of MKT. As the Learning Mathematics for Teaching research group continued to refine their instruments, they once again returned to Shulman's typology of content knowledge. Ball and colleagues (Ball, Thames & Phelps, 2008; Hill, Ball & Schilling 2008) further separated pedagogical content knowledge into *knowledge*

of content and students and *knowledge of content and teaching*. Researchers came to view tasks such as selecting appropriate representations, choosing examples and counterexamples, and sequencing content as requiring knowledge of content and teaching, whereas such knowledge was previously included in knowledge of content and students. This new domain included the knowledge of mathematics and pedagogy required to make instructional decisions.

As seen in Figure 2.2, as of 2008, pedagogical content knowledge included portions of Shulman's (1986, 1987) curricular knowledge similar to the model provided by Wilson Shulman, and Richert (1987). However, Ball, Thames, and Phelps (2008) admit that distinctions between curricular knowledge and other forms of pedagogical knowledge are unclear: "We are not yet sure whether [knowledge of content and curriculum] may be a part of our category of knowledge of content and teaching or whether it may run across several categories or be a category in its own right" (p. 403).

Figure 2.2 Domain map for mathematical knowledge for teaching (Ball, Thames & Phelps, 2008; Hill, Ball & Schilling, 2008).



The construct Shulman (1986) referred to as *vertical curricular knowledge*, which includes knowledge of previous and future content and its relationship to the current curriculum, is categorized as a component of subject matter knowledge and referred to as *horizon content knowledge* (Ball, 1993), not to be confused with Shulman’s (1986) horizontal curricular knowledge.

In sum, subject-matter knowledge includes common content knowledge held by the general population, specialized content knowledge particular to educators, and horizon content knowledge that includes content from previous and future grade levels and how this relates to the current curriculum. Pedagogical content knowledge, which

addresses the interplay of content and pedagogy, includes knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum.

One could argue that these classifications are not necessarily mutually exclusive. For example, common content knowledge could be thought of as a subset of specialized content knowledge. Consider the example above of the three students' strategies for multiplying 25 and 35. While this certainly requires specialized knowledge not generally held by non-educators, it also requires common knowledge of the traditional algorithm. Similarly, in the example in which Mr. Fitzgerald must select an example for ordering decimals, knowledge of typical student errors is required, but the choice of example has instructional implications as well. It could be argued that all knowledge required to make instructional decisions also requires knowledge of students.

Empirical research on elementary teachers' content knowledge. This section begins with a review of empirical research prior to the model of Mathematical Knowledge for Teaching (Ball, Thames & Phelps, 2008; Hill, Ball & Schilling, 2008) used in this study. Then, more recent research specific to this model of MKT and the Learning Mathematics for Teaching measures used in this study is reviewed.

General content knowledge of teachers. While many of the studies that follow use mathematical contexts specific to teaching, these researchers do not make distinctions between common and specialized content knowledge. First, research focused on assessing the mathematical content knowledge of teachers through classroom observations and task-based interviews is reviewed. Then, studies in teacher education

that evaluate the effects of methods courses, content courses, and field experience are discussed.

Assessing general content knowledge. Early inquiries into elementary teachers' content knowledge used case studies of small numbers of elementary teachers. Leinhardt and Smith (1985) examined content knowledge related to fractions for eight fourth-grade mathematics teachers, four expert and four novice, using data from interviews, card-sorting tasks consisting of problems from a fourth-grade textbook, and transcriptions of videotaped lessons. Researchers found considerable differences in subject-matter knowledge and the level of conceptual presentation of content even among those teachers deemed as experts because of their students' growth in test scores over a five-year period. Some expert teachers demonstrated a conceptual understanding of fractions, whereas others relied solely on algorithms.

Ball (1990a, 1990b) used questionnaires to examine the content knowledge of 217 elementary education majors and 35 mathematics majors with task-based interviews for ten elementary and nine secondary majors. The researcher found these preservice elementary and secondary teachers' mathematical understandings of division of fractions inadequate for teaching mathematics conceptually. Tirosh and Graeber (1990) also used task-based interviews to illustrate misconceptions of 21 preservice elementary teachers who correctly calculated the quotient of a division problem with divisor less than one but agreed with a statement claiming quotients must always be smaller than the dividend.

Borko and fellow researchers (1992) used interviews and observations to evaluate content knowledge related to division of fractions for a student teacher with three years of

academic training as a mathematics major. Researchers found that this preservice middle school teacher had a weak understanding of the concepts underlying mathematical procedures. The researchers attributed this teacher's deficits to taking advanced courses in mathematics as opposed to courses specifically designed for preservice teachers that stress the meaningful learning of mathematics.

Simon (1993) assessed 33 preservice elementary teachers' knowledge of division using an open-response questionnaire requiring more than a procedural understanding of division and eight perspective elementary teachers using task-based interviews. The researcher determined that these preservice teachers' conceptual knowledge was weak and suggested refocusing mathematics courses for elementary teachers.

Ma (1999) also used task-based interviews to compare 11 experienced U.S. teachers, 12 novice U.S. teachers, and 72 Chinese teachers. The researcher found that more U.S. teachers than Chinese teachers showed only a procedural understanding of subtraction with borrowing and multi-digit multiplication. Most Chinese teachers could construct a representation for the division of fractions, whereas no U.S. teachers showed an understanding for concepts underlying the procedure. Chinese teachers approached the novel claim of a student regarding the relationship between perimeter and area by investigating the problem independently and seeking out a counterexample for the claim, whereas U.S. teachers did not. The researcher found Chinese elementary teachers' knowledge coherent and U.S. teachers' content knowledge fragmented.

This research highlights deficits in both preservice and in-service U.S. elementary teachers' mathematical content knowledge. Specifically, research focused on the number

and operations content standard. Teachers in these studies relied heavily on procedural understandings of elementary mathematics and lacked the conceptual understandings needed to teach the content.

Teacher education and general content knowledge. With this body of research underscoring elementary teachers' fragmented content knowledge, research in teacher education focused on improving preservice teachers' content knowledge through methods courses, content courses, and field experiences.

Quinn (1997) investigated the effects of two methods courses that incorporated manipulatives, technology, and cooperative learning on 28 preservice elementary teachers' and 19 preservice secondary teachers' content knowledge as measured by select items from the Essential Elements of Elementary School Mathematics Test. This multiple-choice assessment measured PSTs' common content knowledge up to a sixth-grade level. Pretests showed secondary teachers performed significantly higher than elementary teachers. Posttests revealed significant gains for elementary PSTs' content knowledge after completing the methods courses; however, secondary PSTs did not show similar gains, possibly due to a ceiling effect from their high preliminary scores. Nevertheless, no confidence interval or variance is reported for the mean gain of 1.7 questions on a 24-item assessment.

Davis and McGowen (2001) conducted a case study of one preservice elementary teacher enrolled in a content course for future teachers in which tasks were posed and then followed by video clips of children engaging in the same tasks. Through qualitative data reported for a single task, researchers found this PST "made a significant change in

her understanding of mathematics” (p. 8). Researchers report that this subject’s pre- to posttest scores “improved from not satisfactory to excellent;” however, this growth was on an unidentified content assessment and no supporting statistical evidence was provided.

Matthews and Seaman (2007) also investigated the effects of content courses on PSTs’ mathematical content knowledge. These researchers compared scores on their own assessment of content knowledge for 29 elementary PSTs that took a previous (experimental) Logic of Arithmetic course to 19 PSTs that took a (control) general mathematics course for elementary teachers. While the authors describe the content addressed in the experimental course, no novel aspects of this course are described other than extensions to other bases and weekly small-group discussion sessions. No information about the control course is given, and researchers do not report whether the instructor or the time between the content courses and data collection were held constant. All participants were enrolled in a methods course at the time of data collection, which might also influence participants’ content knowledge, given the findings of Quinn (1997). With these methodological shortcomings in mind, researchers did not find a significant difference in mean scores of the two groups. When group effect was added to a regression model that used students’ ACT score and GPA to predict content scores, a significant effect was found. However, one may assume ACT score and college GPA to be highly correlated; therefore, multicollinearity in this model is problematic.

The Integrating Mathematics and Pedagogy (Philipp et al., 2007) research team explored the effect of combining various field experiences with mathematics content

courses on PSTs' content knowledge as measured by the IMAP Content Instrument (IMAP, 2004), an assessment for teachers that focuses on place value and rational number. While enrolled in a content course, 50 PSTs watched videos of children solving problems and then interacted with children during six problem-solving sessions; 27 PSTs watched the videos but had no live component; 23 PSTs participated in apprenticeships with reform-oriented teachers; 25 PSTs were apprentices to a convenient collection of supervising teachers whose reform orientation was unknown, and 34 PSTs in a control group had no video or field experience. No pair-wise comparison among these groups showed significantly different gains in content knowledge. However, when researchers pooled the two video groups and the apprenticeship groups with the control group, significant differences in gains were found between groups that focused on children's mathematical thinking through videos and those that did not. Although it was noted that various mathematics graduate students taught four separate sections of the content course taken in tandem, no instructor variables were considered in this analysis of content knowledge. Additionally, motivational differences in groups might be argued as video and apprenticeship participants were paid up to \$600 for their time, whereas the control participants were not compensated. Further, it is not obvious why the researchers combined the control group with the apprenticeship groups after pair-wise comparisons were not significant.

Peterson and Williams (2008) investigated the effects of student-teaching experiences on PSTs' content knowledge in a case study contrasting one PST-cooperating-teacher pair with little change in content knowledge, focusing instead on

classroom management, with another pair that focused on the mathematics from a student's perspective. Through interviews, recorded conversations between PSTs and cooperating teachers, and a follow-up questionnaire, researchers found that for the pair that focused on classroom management, mathematical knowledge was something separate from teaching, and content learned in college courses, which focused on procedures, was considered sufficient for teaching. In the contrasting case, conversations focused on what students understood, and the researchers concluded that this PST came to value mathematical knowledge through preparing for and engaging in teaching. While the data provided support for a difference in beliefs about content knowledge between these cases, conclusions made about changes in content knowledge appear unsupported as no measure of content knowledge was described.

When gauging the effects of teacher preparation programs on content knowledge across these studies, it becomes difficult to synthesize findings due to methodological issues in these studies. Questions of changes in content knowledge attributed to a specific component of teacher education are more compatible with a quantitative analysis, not case study (Davis & McGowen; Peterson & Williams, 2008). In studies that do attempt to answer such questions quantitatively, interventions are poorly defined and the quantitative analysis is weak (Quinn, 1997; Mathews & Seaman, 2007). The one study with more sound methods (Philipp et al., 2007) does not control for variations in the content courses taken during data collection, and their rationale for combining control and apprenticeship groups after their initial findings were insignificant is weak.

Therefore, further research is needed to evaluate the effects of teacher-education efforts on PSTs' content knowledge.

Effects of general content knowledge. In a study of teacher trainees from two institutions in England and Wales, Goulding, Rowland, and Barber (2002) examined ordinal scores (low, medium, or high) on a 16-item measure of teachers' content knowledge required by the Institute of Education as they relate to teacher trainees' teaching performance. Teaching performance was categorized as weak, capable, or strong from observations of field placements. At one institution, differences in teachers' ability to teach numeracy were found among content knowledge groups for 154 teacher trainees. High content-knowledge teachers were more likely assessed as strong numeracy teachers. At another institution, content classification was linked to assessments of planning and teaching effectiveness, with poor-content trainees associated with weakness in planning and teaching primary mathematics for 164 teacher trainees. Although these researchers allude to rejections of null hypotheses, no test statistics or p-values are reported. Additionally, the authors do not describe the assessments of planning and teaching used. These findings might not generalize to U.S. contexts. However, they do reinforce the idea of the importance of content knowledge for effective teaching.

In a study of 72 third-grade teachers and 1,043 students in Belize, Mullens, Murane, and Willett (1996) found that teachers' ordinal scores (A, B, C, D, or E) on an eighth-grade exit exam when they were students was a significant predictor of their current students' gains on an instrument developed by the Ministry of Education ($r=3.64$, $p<.001$). Students answered, on average, 10.4 questions (SD 7.0) correctly on the pretest

and 20.5 questions (SD 11.3) on the posttest on this 69-item assessment. Therein, each letter grade higher a teacher scored during her own eighth-grade year is linked to her students gaining 3-4 items, on average.

These results should not, however, be generalized to the United States given that one in four teachers had only a primary education and only 52% completed a teacher-training program. Further, assessments of teachers' content knowledge were taken prior to their secondary or post-secondary training with a variable number of years between data collection of teachers and their students; one might assume that teachers' content knowledge changed since their eighth-grade years. Additionally, the assessment of students is suspect as no reliability or validity information is provided. With mean scores of approximately 10 (SD 7) and 20 (SD 11) questions on a 69-item assessment falling far below the 50% standard, the distribution is likely skewed. That is, the left interval two standard deviations below the mean would fall below the lower bound of answering no questions correctly. Still, this study adds to a body of literature connecting teachers' content knowledge to their students' achievement.

Harbison and Hanushek (1992) examined the relationship between 349 teachers' scores on the same criterion-referenced assessment of fourth-grade numeracy as their students ($n=1,789$) in rural northeastern Brazil. Students' mean score was 50.1% correct with a standard deviation of 23.5 percent. Surprisingly, teachers' mean score was only 87.3% with a standard deviation of 12.6 percent. In a regression model considering over 30 state, program, student, peer, school, and teacher characteristic variables, teachers' content scores were a significant predictor ($p<.0001$) of students' scores with a regression

coefficient of 0.52. Therefore, one standard deviation difference in teachers' knowledge of fourth-grade numeracy translated to roughly a quarter of a standard deviation increase in their students' scores when all other characteristic variable were held constant.

Again, these results might not generalize to the U.S. given the rural, international context. Additionally, these teachers had, on average, only seven to eight years of schooling with 20% of teachers having four or fewer years of schooling. In addition to the limited and variable educational backgrounds of teachers, comparing teachers' scores on an assessment designed and referenced for fourth graders may lack validity. Still, this research supports the importance of teachers' content knowledge.

Based on an analysis of public-use data from the National Education Longitudinal Study (NELS) of 1988, Rowan, Chiang, and Miller (1997) linked tenth-grade mathematics teachers' responses to a single item of high-school-level common content knowledge to their students' standardized scores on the NELS mathematics achievement assessment. Students with teachers that answered the question correctly performed, on average, 0.02 standard deviations higher than students with teachers that incorrectly answered the question ($p=.015$). As the authors note, using a single item as an estimate of teachers' content knowledge lacks reliability. Further, no information about the sample's representativeness or randomness is offered, and data collected in 1988 might not generalize to current educational settings. Further, the effect size of 0.02 SD, although statistically significant, was quite small.

Given the international contexts of three of these four studies and weaknesses in instruments used to measure teachers' content knowledge, further research in a U.S.

context using reliable and valid instruments is needed to connect teachers' content knowledge to their performance and students' achievement. In the section that follows, such research situated in the model of MKT described above, which uses Learning Mathematics for Teaching's content assessments for teachers, attempts to fill in these gaps in the research base.

Research specific to Mathematical Knowledge for Teaching. Before the introduction of measures specific to MKT, researchers used college coursework, degrees earned, and scores on licensure exams as estimates of content knowledge when attempting to link such teacher characteristics to students' achievement (Wayne & Young, 2003). With the introduction of quantitative measures to assess teachers' MKT (Hill, Schilling, & Ball, 2004), research shifted to connecting MKT to teacher characteristics, linking teachers' MKT to quality of instruction and student achievement, and assessing the effects of teacher development efforts on teachers' MKT.

MKT and teacher characteristics. Hill (2007) investigated links between MKT and teachers' credentials and experience for a national sample of 591 middle school teachers stratified by region and urbanicity. Participants were assessed on scales in number and operations and algebra (LMT, 2004), which assessed both common and specialized content knowledge. Hill found that teachers with high school credentials performed an average of a half a standard deviation higher than those without. Former elementary teachers and teachers without mathematics-specific credentials scored a third of a standard deviation below others in the study, on average. Regression of teachers' MKT scores on teacher-characteristic variables showed each additional mathematics

course added 0.2 SD ($p < 0.001$), on average, to teachers' MKT scores; each methods course added 0.1 SD ($p < 0.05$), and experience added 0.01 SD ($p < 0.05$) per year. Hill also found MKT to be negatively correlated with percent of students receiving free or reduced lunch ($r = -.19$, $p < 0.0001$), percent of African-American students ($r = -0.17$, $p < 0.0001$), and percent of Hispanic students ($r = -0.12$, $p < 0.01$). From these results, Hill suggested that middle school teachers be high school certified and proposed addressing the inequitable distribution of MKT through professional development, post-secondary coursework, and recruiting more qualified teachers.

This research seems to counter the claim (Ball, Thames & Phelps, 2008; NCTQ, 2008) that content courses are not meeting the needs of teachers. Interestingly, the residual effect of a single previous content course rivals the effect of 20 years of experience. This research highlights the importance of content courses for elementary teachers.

In a similar study of 438 California, K-6 teachers, Hill and Lubienski (2007) also found an inequitable distribution of MKT in number and operations (LMT, 2004). Again, teachers in schools with larger proportions of Hispanic students and students receiving free and reduced lunch scored significantly lower. However, years of experience was not a statistically significant predictor of MKT in this study, but highest grade level taught was a significant predictor ($r = 0.12$, $p < 0.001$). The effects of mathematics and methods courses were not considered in this study.

With another national representative, stratified sample of 625 elementary teachers, Hill (2010) found correlations between indicators of teachers' educational

background and MKT in number and operations to be relatively weak. The number of mathematics and methods courses was significantly correlated with MKT, but with coefficients of only 0.09 and 0.06 respectively. These effects were much smaller than in earlier work (Hill, 2007). No significant relationship was found between MKT and math-specific professional development. However, grade level taught was positively correlated with MKT scores ($r=0.30$, $p<.001$), with K-one teachers scoring half a standard deviation below the sample mean. In this study, experience was a significant predictor with a modest effect, but the categorical analysis of years of experience makes it difficult to compare to previous studies (Hill, 2007; Hill & Lubienski, 2007). Still, these experience effects ranged from 0.04 to 0.1 SD for experience classifications. Again, students' free-and-reduced-lunch status was negatively correlated with teachers' MKT scores ($r=-.09$, $p<0.05$).

In all three of these studies, the inequitable distribution of MKT among socioeconomic status and race is problematic. Findings connecting years of experience are inconsistent and difficult to compare between studies. Although the long-term effects of college coursework vary considerably between studies, there is evidence to counter other research (Ball, Thames & Phelps, 2008; NCTQ, 2008) that concludes mathematics content courses for teachers do not address the MKT needed to teaching elementary mathematics.

It should also be noted that these researchers use the term Mathematical Knowledge for Teaching, which is composed of three components of subject-matter knowledge and three divisions of pedagogical content knowledge. However, the

measures used in these studies assess only common and specialized content knowledge, two of the three aspects of subject-matter knowledge, with no distinctions made between the two. Perhaps semantic clarity is needed when measures of common and specialized content knowledge are reported as measuring MKT, a construct that also includes horizon content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum not measured by these instruments.

Effects of MKT on teaching and learning. Hill et al. (2008) explored connections between teachers' MKT and the mathematical quality of their instruction in a mixed-methods study of ten teachers in which the Mathematics Quality of Instruction (MQI) rubric (LMT, 2006) was validated through case studies of five elementary teachers. Three MQI scores were collected before a professional development institute (Hill & Ball, 2004), and six lessons were scored after the institute for each of the ten teachers. Because MQI scores did not significantly change over the intervention, mean MQI scores were calculated for each teacher. Teachers' MKT was assessed using subscales in number and operations, geometry, and algebra (LMT, 2004) among a larger group of 636 elementary teachers. MKT scores were significantly positively correlated with the *responding appropriately to students* subscale of the MQI and negatively correlated with the *total errors* and *errors in language* subscales.

To validate these findings and answer questions of how MKT influences the quality of instruction, five cases were selected, stratified by MKT and MQI scores. Through these cases, researchers concluded “a powerful relationship between what a teacher knows, how she knows it, and what she can do in the context of instruction” (Hill

et al., 2008, p. 496). Hill and colleagues also found this relationship to be mediated by teachers' beliefs regarding how mathematics should be taught:

Anna thought her students in need of liking mathematics, and adjusted her instruction to leave little mathematics for students to dislike. Rebecca has very circumscribed views of mathematics itself, and thus enacted that very circumscribed mathematics in her classroom. By contrast, the views of Lauren and Noelle were more expansive, specific, and mathematical in nature. (p. 500)

Therein, this ancillary finding may add to a body of research that connects teachers' beliefs regarding the nature of mathematics and their practice.

The quality of higher-MKT teachers' instruction was attributed to avoiding errors and providing a "denser, more rigorous mathematics instruction" (Hill et al., 2008, p. 487). Lower-MKT teachers were found to be more variable with more reliance on textbooks.

Similar to critiques of other studies using these assessments to measure MKT, the instruments used in this study assess only common and specialized content knowledge with no distinction made between the two. One might argue that the *responding appropriately to students* and *avoiding mathematical errors* could be correlated with common content knowledge just as easily as specialized content knowledge.

In a 3-year study of 699 elementary teachers and 2,963 students in a non-random yet nationally representative sample, Hill, Rowan, and Ball (2005) explored the effects of teachers' MKT on student achievement. Researchers measured MKT using an instrument that assessed common and specialized content knowledge in number concepts and

operations along with patterns, functions, and algebra. While items were piloted in Knowledge of Content and Students, these items did not meet the researchers' criteria for inclusion due to low reliability and multidimensionality in factor analysis. Students' achievement data were collected on the Terra Nova Complete Battery.

Teachers' content knowledge for teaching mathematics was a significant predictor of students' mathematics gains on the Terra Nova. Each standard deviation difference in teachers' mathematical content knowledge translated to one-half to two-thirds of a month of additional growth over an academic year. MKT scores predicted achievement better than teacher background variables or average time spent on mathematics and had an effect size comparable to students' socioeconomic status, ethnicity, and gender. Scores on a reading content assessment were not found to significantly predict students' math gains; this suggests that the effect was from content-specific knowledge, not general aptitude. Similar to findings in other studies (Hill, 2007, 2010; Hill & Lubienski, 2007), teachers' MKT scores were negatively correlated with students' minority status ($r=-0.16$, $p<0.0001$).

In a study of new mathematics teachers in New York City, Rockoff, Jacob, Kane, and Staiger (2008) found scores on an unspecified scale of the Learning Mathematics for Teaching assessment (LMT, 2004) to be significantly positively correlated ($r=0.028$, $p=0.024$) with students' mathematics achievement, as measure by an unidentified standardized test, for 337 fourth- through eighth-grade teachers. Interestingly, the links between MKT and achievement were stronger than those attributed to general cognitive ability ($r=0.016$, $p=0.17$) as measured by Raven's Matrices Standard Version. This

supports the findings of Hill, Rowan, and Ball (2005) that greater effects are contributed by content-specific knowledge, not general aptitude. MKT scores were also more significantly correlated with students' achievement than scores on the Haberman Pre-Screener, a commercial teacher-screening instrument ($r=0.023, p=0.11$).

These researchers noted limitations in generalizing their findings as this study considered only newly hired teachers, not all applicants for these teaching positions. Therefore, the current hiring process might have introduced bias. The regression coefficient for MKT should be interpreted with care as these researchers used the percent correct on the assessment (normalized to have a standard deviation of one) as opposed to Item Response Theory scores, which use a standard normal scale referenced on a large, nationally representative sample of teachers varying in years of experience, as in other studies using these measures. Therefore, MKT scores are relative only to the other 337 new mathematics teachers hired in New York. Additionally, Rockoff, Jacob, Kane, and Staiger (2008) did not report which content scales they administered or whether they constructed their own scale by pulling items from various content domains. Therein, the reliability of their assessment is unknown. Still, although small, these correlations were significant.

Looking across these three studies, there is evidence that teachers' content knowledge for teaching mathematics affects the quality of their instruction and, in turn, their students' achievement. These findings are consistent with other abovementioned studies that address only common content knowledge. Although there is evidence that

subject-specific knowledge, not general intelligence, can influence student outcomes, more research is needed that focuses on the “specialized” aspect of content knowledge.

MKT and teacher education. With a body of research connecting MKT to student learning, it is not surprising that teacher educators study the effects of professional development on teachers’ MKT. Hill & Ball (2004) evaluated the effects of California’s Mathematics Professional Institutes: Elementary Number and Operations on 398 elementary teachers’ common and specialized content knowledge in number concepts and operations. Across all 15 institutes, participants gained an average of 0.48 standard deviation units ($p < 0.0001$). Researchers also found the length of the institute, whether one, two, or three weeks, a significant predictor of MKT gains ($r = 0.33$, $p < 0.05$). These professional development institutes were not described in detail, and differences among institutes were not addressed. Nevertheless, this does suggest that professional development opportunities can have a positive effect on MKT.

Lueke (2008) examined the effects of content courses at two universities on 101 PSTs’ MKT as measured by an instrument constructed from the pool of Learning Mathematics for Teaching items (LMT, 2004). Through qualitative analysis, the author contrasted the two cases across universities. However, no significant differences in MKT were found at the beginning or end of the semester. The author reported a mean gain of about a standard deviation, but did not report standard errors or confidence intervals for these gains. In a step-wise multiple regression of institution, attitude, score rating PSTs’ perceived relevance of the course, and pretest score onto PSTs’ posttest score, pretest was the only significant predictor. This researcher did attempt to distinguish between common

and specialized content knowledge. While pretest scores did not differ between schools for either construct, there was some evidence that posttest scores differed between institutions on specialized ($p=0.065$) and common ($p=0.076$) content knowledge. However, neither means nor the magnitudes of these differences were reported. When attempting to explain these differences in specialized and common content knowledge gains between institutions, Lueke reported that interview data were inconclusive.

Summary: Review of Literature.

As Research Questions Two and Three explore the effects of sequence of instruction on PSTs' beliefs and MKT, the literature presented in this chapter shows that the two sequences of instruction examined in the current study are supported by theories of learning and instruction. Further, as Research Questions One and Two inquire into PSTs' beliefs, research is also presented here that shows teachers' beliefs to be an important construct worth exploring. Additionally, Research Question Three focuses on PSTs' MKT, and the literature in this chapter also illustrates the importance of considering this specialized type of knowledge. In Chapter Three, the methods used to investigate these research questions are described.

CHAPTER THREE

Methods

This chapter describes the research methods used in the current study. The setting, participants, and mathematics content courses are described. In addition, the instruments for measuring Mathematical Knowledge for Teaching (MKT) and protocols for qualitative data collection are reviewed. This chapter concludes with methods for analyzing these data.

Setting

This study took place at a public, coeducational, land-grant, research university in the southeastern United States in the fall of 2010. The Carnegie Foundation (2011) classified this university as *full time* in that at least 80% of undergraduate students are enrolled full time. The university was also classified as *more selective* in that its admission's test scores for entering freshmen place these students in the top fifth of applicants. This university was also classified as a *high transfer-in* institution in that at least 20% of first-year undergraduates transfer from another institution. Additionally, this institution was classified as a *large four-year, primarily residential* university with at least 10,000 degree-seeking undergraduate of which 25-49% live on campus.

Participants

The convenience sample of preservice teachers for this study was drawn from three sections of Mathematics for Elementary School Teachers II (MEST II) taught by a single instructor in the fall of 2010. The sections consisted of 29, 33, and 35 preservice elementary and early-childhood teachers. Participants were recruited the first day of the

semester. The researcher explained the study to each section and distributed informed consent letters. Those students that agreed to participate in the study returned the signed consent form to the instructor. Of the 97 PSTs in these three sections, 96 agreed to participate in the study. Of these 96 participants, 94 were female and two were male. Demographic data were not collected.

The female instructor of these sections, referred to as Ms. B in this study, had previously taught these courses for roughly 12 years and had previously taught secondary mathematics in both public and private settings. She holds a bachelor of arts in secondary mathematics education and a master of education in secondary mathematics.

Content Course

Mathematics for Elementary School Teachers II is a content course for elementary and early-childhood preservice teachers that addresses the following content identified in the university catalog:

Simple probability and descriptive statistics are reviewed. Two- and three-dimensional geometry including polygons, polyhedra and their properties; congruence, similarity, and construction; coordinate system; standard measurement, area, surface area, volume; and motion geometry are explored.

Content, according to State standards, is taught with appropriate methodology for teaching K-6. (University, 2010)

MEST II is a four-credit-hour course that meets 4 days each week. This course includes a laboratory component in which PSTs engaged in 13 collaborative tasks, roughly one per week excluding exam weeks, over the course of the semester. Although PSTs registered

for a separate laboratory course, Ms. B had the flexibility to work on laboratory activities during any of the four class meetings.

This course is intended for first-semester sophomores and requires Mathematics for Elementary School Teachers I (MEST I), or an equivalent course, as a prerequisite. MEST I covers the algebra and number and operations standards and has a prerequisite of Essential Mathematics for the Informed Society, or an equivalent course, in which probability and statistics (one of the content standards also studied in MEST II) is also addressed. This course precedes methods courses for both majors.

Both MEST I and MEST II used the text *A Problem Solving Approach to Mathematics for Elementary School Teachers, 10th Edition* (Billstein, Libeskind & Lott, 2010). In the National Council on Teacher Quality's (2010) report on the preparation of elementary teachers, this book was the most commonly used of the 127 content courses surveyed. This text was ranked positively across all content standards, with the highest score in algebra for all texts reviewed. NCTQ recognized this text as "far better than average" (NCTQ, 2008, p. 80).

Sequence of instruction. MEST II addressed the data analysis and probability standard and the geometry content standard. For the purposes of this study, the course was divided into two units. First, PSTs participated in a roughly 5-week unit that focused on the data analysis and probability content standard. A second approximately 10-week unit addressed the geometry content standard.

Section One (N=34) and Section Two (N=33) were combined to form one sequence group (N=67) because a few students' schedules required them to alternate

between these two sections. The second sequence group consisted of Section Three (N=29). During the first 5-week, data analysis and probability unit, Ms. B used an exploratory sequence of instruction with Sections One and Two, in which three of the five laboratory tasks were posed before a formal explanation of the content was given. Taking into account the two laboratories that did not use an exploratory sequence, the first laboratory of the semester used a confirmatory sequence in all sections and is discussed further in Chapter Four as Ms. B initially had a different understanding of what it meant to explore mathematics. Another laboratory in this unit focused on displaying data and required knowledge of mathematical conventions related to box plots. Ms. B did not find it productive to learn about these conventions through an exploratory sequence.

Section Three experienced a confirmatory laboratory sequence during the first unit in which content was first explained through lecture and worked examples before each of the five laboratory tasks. For the second unit, Sections One and Two experienced a confirmatory laboratory sequence in which Ms. B first explained the content before each of the eight collaborative tasks was posed. PSTs enrolled in Section Three experienced an exploratory laboratory sequence for seven of the eight laboratory tasks posed in the geometry unit. Table 3.1 illustrates the sequences of instruction used in these two units.

Table 3.1

Sequences of Instruction for Content Units

	Data Analysis & Prob. (5 weeks)	Geometry (10 weeks)
Sections 1 & 2 (N=67)	Explore 3 of 5 labs	Confirm 8 of 8 labs
Section 3 (N=29)	Confirm 5 of 5 labs	Explore 7 of 8 labs

Laboratory tasks. Twenty laboratory tasks were piloted (Sloop & Che, 2011) the previous semester, seven for data analysis and probability and 13 for geometry. Of these seven data analysis and probability laboratories, six were written for an exploratory sequence. The one laboratory written with a confirmatory sequence related to displaying data and required prerequisite knowledge of mathematical conventions related to box plots. Of the 13 laboratories created for the geometry unit, all 13 were written for an exploratory sequence with nine using Geometer's Sketchpad® dynamic geometry software. All of these materials were made available to Ms. B at the end of Spring semester of 2010. Ms. B used many of the laboratories as written, shortened a few longer laboratories, and incorporated some laboratory tasks of her own.

Data Collection

As data were collected, PSTs' names were removed from their responses and replaced with identification codes. These codes included a section identifier and a randomly assigned participant number, which allowed responses to remain anonymous while identifying section groups. For example, PST 3-17 corresponds to Student 17 in

Section Three. These identification codes also appear in Chapter Four where PSTs' responses are reported.

Pre- and post-quantitative data were collected on PSTs' Mathematical Knowledge for Teaching (MKT) in probability, data, and statistics (LMT, 2008a) and geometry (LMT, 2004) in each of the three MEST II sections. These assessments were taken in class. Preliminary assessments were taken in both content areas the second week of the semester. Post-MKT data were collected after the completion of the first 5-week unit for probability, data, and statistics. Post-MKT data were collected the last week of the semester for geometry.

Preliminary qualitative data were collected within the first two weeks through mathematics autobiographies. Throughout the semester qualitative data were collected through classroom observations and interviews with Ms. B. During the last week of the semester, qualitative data were also collected through exit surveys for all participants and through interviews for two PSTs from each section stratified by preliminary MKT scores.

Mathematical Knowledge for Teaching (MKT). Learning Mathematics for Teaching (LMT), a research group focused on assessing the specialized knowledge required to teach elementary mathematics, created multiple-choice measures of MKT for elementary teachers with scales specific to geometry; number concepts and operations; patterns, function and algebra; and place value. Scales were also available for MKT for middle school teachers in geometry; number concepts and operations; and patterns, function and algebra. Additionally, there were scale for grades four through eight in geometry; probability, data, and statistics; proportional reasoning; and rational number.

The elementary geometry (LMT, 2004) and grades-four-through-eight probability, data, and statistics (LMT, 2008) scales were used in the current study. The researcher participated in an LMT training session in the spring of 2009.

Although some of LMT's more recent scales offer data related to the specific domain of MKT (see Figure 2.2 for the domain map of MKT) measured by each item, all items on the geometry scale loaded under a single factor (LMT, 2004), subject-matter knowledge. This scale was assumed to measure both common and specialized content knowledge; however, item-specific distinctions were not made. Although the geometry scale does not claim to measure pedagogical content knowledge, to be consistent with the literature, the construct measured is referred to as Mathematical Knowledge for Teaching. The more recent probability, data, and statistics scale (LMT, 2008) does identify the specific domain each item assesses and covers common and specialized content knowledge as well as knowledge of content and students and knowledge of content and teaching.

Reliability. The grades-four-through-eight geometry scale was considered but not used. While this scale had higher item reliabilities, between 0.87 and 0.94, it discriminates best among above-average teachers, with item difficulties of 11 of the 73 items one standard deviation above the mean (LMT, 2008b). This scale was also significantly longer than the 2004 elementary version. The 2004 elementary version discriminates best slightly below the mean with item reliabilities in the "mid .8s or higher" for alternative forms (LMT, 2004, p. 2). There was only one form of the

probability, data, and statistics scale available. The test-retest reliability of this single form was 0.91.

Validity. For both scales, construct validity was established by using a “job analysis” approach in creating the model of MKT, as described in the previously mentioned literature. Further, construct validity was ensured through use of research literature, personal experience, and field notes (LMT, 2004, 2008a). Criterion validity was established by associating MKT scores with the quality of instruction for in-service teachers measured through cognitive interviews and videotapes of mathematics instruction (Hill et al., 2008). Further evidence of criterion validity was established by linking LMT scores, consisting of items from all content domains, with student achievement on three of McGraw-Hill’s Terra Nova assessments for kindergarten, first-grade, and third- and fourth-grade students (Hill, Rowan & Ball, 2005).

Mathematical autobiographies. As a requirement of the MEST II course, all students submitted mathematical autobiographies within the first two weeks of the semester. Ms. B granted the researcher access to the autobiographies of students that chose to participate in the study. In a brief essay introducing her- or himself to the instructor, PSTs described what mathematics is to them. PSTs also described a typical day in mathematics class in elementary, middle, high school, and college as they remembered it. They wrote about strategies they found particularly helpful and those that did not benefit them. Preservice teachers were also asked to write about what a typical day in their future mathematics class might look like. Then, they described how they

might teach their class about the area of a rectangle or elementary probability. The prompt and grading rubric given to PSTs are in Appendices A and B.

Classroom observations. All sections of MEST II were observed at least twice for a total of ten observations over the semester. The purposes of these observations were to ensure the fidelity of the sequence of instruction and to gain insight into differences between the two sequences. Field notes were taken and audiotapes of classroom discourse were recorded. Although all dialogue from classroom observations was not transcribed, these recordings were used to capture specific dialogue identified in field notes.

Instructor interviews. Ms. B was interviewed four times over the semester. Interviews were audio taped and transcribed. These interviews were based on classroom observations and field notes. Appendices C - F contain the protocols for these interviews. During each interview Ms. B was also asked about any differences she noticed between sequences. Results of the MKT assessments were also shared and discussed during these interviews. These interviews lasted between 10 and 20 minutes.

Student interview protocol. A stratified sample of MEST II participants was selected for interviews based on initial MKT scores. For each section, PSTs' preliminary scores for both content measures were ranked and divided into thirds to form low-, middle-, and high-scoring groups. Two participants were selected from each section such that one low-scoring, middle-scoring, and high-scoring PST was chosen for probability, data, and statistics and geometry. Table 3.2 shows the stratification scheme for selection

of participant for interviews. Interviews lasted between 5 and 15 minutes and were audio recorded and transcribed. A protocol for these interviews can be found in Appendix G.

Table 3.2

Stratification of Interview Selection based in Preliminary MKT Scores

	Low MKT		Mid MKT		High MKT	
	Geo	Prob	Geo	Prob	Geo	Prob
Section 1	1			1		
Section 2			1			1
Section 3		1			1	

Notes. 1=one PST selected based on preliminary MKT score.

Exit surveys. At the end of the semester preservice teachers were given open-ended surveys in which they were asked (1) which sequence they preferred and why as well as (2) how they planned to sequence activities and explanations in their own class and why. PSTs were again asked (3) how they might teach a lesson on the area of a rectangle or elementary probability. These surveys were completed in class and are found in Appendix H.

Data Analyses

Research Question One. PSTs' mathematical autobiographies were used to answer Research Question One: How do the lived experiences, as students of mathematics, of a group of preservice teachers influence their pedagogical beliefs? These data were analyzed using a phenomenological reduction (Moustakas, 1994) of data in which significant statements were extracted from which meaning units were constructed. Meaning units were collapsed into meaning clusters to form themes related to prospective teachers' experiences and pedagogical beliefs.

Research Question Two. Data from PSTs' mathematical autobiographies, interviews, and exit surveys were used to answer Research Question Two: How does experiencing exploratory and confirmatory sequences of instruction influence a group of preservice elementary teachers' pedagogical beliefs? To uncover PSTs' pre-existing beliefs about how instruction should be sequenced, each phase of PSTs' descriptions of their future instruction and example lessons was labeled using open codes. Next, example lessons from PSTs' mathematical autobiographies and exit surveys were coded again related to the order in which phases occurred. To identify possible changes in beliefs, a similar analysis was performed on example lessons from exit surveys administered at the end of the semester.

To test null hypotheses H_{02-1} and H_{02-2} , which related to PSTs' preferred sequence, contingency table were constructed for student preferences versus intended sequence as teachers and for student preferences versus sequence group. A Pearson's chi-squared statistic and Fischer's exact test statistic were calculated using JMP, a statistical software produced by SAS Inc. However, before these tests could be performed, the following assumptions of a chi-squared analysis were considered (Hinkle, Wiersma & Jurs, 2003):

1. The sample was randomly selected from the population.
2. The sample size is large enough that the expected value of each cell is at least five in 75% of the cells.

Fischer's exact test was used for the two-by-two comparison of sequence group and sequence preference due to the sample size. However, for the three-by-three comparison

of student preference and intended instructional sequence, a chi-squared analysis was used despite violations of assumption two above, as Fischer's exact test can only be applied to two-by-two comparisons.

Research Question Three. An Analysis of Covariance (ANCOVA) was used to test Research Question Three: Is there evidence of a relationship between sequence of instruction and gains in preservice teachers' MKT? To test the related null hypotheses, differences in means scores were examined for sections, sequence groups, and sequence preference groups in an ANCOVA with preliminary MKT scores used as a covariate. However, before these analyses could be performed the following assumptions were considered (Hinkle, Wiersma & Jurs, 2003):

1. The samples are independent and random.
2. Scores on the dependent measure are normally distributed.
3. The variances for each group are equal.
4. The relationship between the covariate and dependent measure is linear.
5. The regression lines for each group are parallel. That is, the covariate affects all groups similarly, or there is no interaction between the covariate and treatment.

Therein, null hypothesis H_{03-2} —group effects are the same for all ability levels—is also assumption 5 for the ANCOVA used to test the other null hypotheses.

Research Question Four. Data from observations, field notes, interviews, and laboratory worksheets were used in the case study analysis (Stake, 1995) of Ms. B, in which multiple data sources informed the development of emerging themes to answer

Research Question Four: What benefits and barriers does an experienced instructor perceive when implementing exploratory sequences of instruction? Results of these analyses are reported in Chapter Four.

CHAPTER FOUR

Results

This chapter begins with themes generated from a phenomenological reduction of preservice teachers' mathematical autobiographies. Within this phenomenology, lived experiences with school mathematics for these preservice teachers (PSTs) are reported as they relate to specific process standards (NCTM, 2000). Next, PSTs' beliefs regarding the nature of mathematics are explored through a reduction of PSTs' definitions of mathematics. Connections between PSTs' experiences and pedagogical beliefs are examined through data in which PSTs describe pedagogy they found helpful as students of mathematics and practices from which they believe they did not benefit.

After examining PSTs' previous experience with school mathematics, the nature of mathematics, and interpretations of pedagogy as students, PSTs' beliefs regarding sequence of instruction are explored through descriptions of their future classrooms, sample lessons, exit surveys and interviews. Possible changes in future teachers' beliefs regarding sequence of instruction are examined. Then, PSTs' beliefs regarding exploratory and confirmatory sequences of instruction are considered from their perspective as students of mathematics and as future teachers.

Next, quantitative data for PSTs' Mathematical Knowledge for Teaching (MKT) are reported and relationships to sequence of instruction for content units and preferences regarding sequence of instruction are considered. Then, the case of an instructor attempting to implement an exploratory sequence of instruction is described through analysis of data from observations, field notes, interviews, and planning sessions. The

chapter ends by summarizing these results as they answer the four research questions presented in Chapter One.

Experiences with School Mathematics

The National Council of Teacher of Mathematics (NCTM, 2000) envisions classrooms similar to the following:

Students confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. . . . Orally and in writing, students communicate their ideas and results effectively. They value mathematics and engage actively in learning it. (p. 3)

That is, mathematics should be connected; students should use mathematical reasoning when engaging in problem solving and communicate their findings using a variety of representations. Compare this vision for school mathematics to the class in which students “sit down, be quiet, take notes, (wonder when we would ever need this in real life) and then take tests” (PST 3-01), in which there is little evidence of the problem solving, reasoning and proof, communication, connections, or representations process standards (NCTM, 2000).

In the section that follows, common themes related to the process standards among these preservice teachers' mathematics experiences are illustrated with evidence from their mathematical autobiographies. Then, less common experiences related to specific process standards are highlighted.

Traditional school mathematics. Twenty-four of the 89 (27%) mathematical autobiographies described traditional educational experiences at all grade levels, characterized by review of previous content, lecture, and guided practice. In these preservice teachers' (PSTs') autobiographies, no evidence of any of the five process standards is found. For example, one PST wrote, "A typical day consisted of going over homework from the night before, learning the new lesson, [and] then doing practice problems afterwards" (PST 1-12). This sequence of review, lecture, and practice was common as seen in this PST's response: "Once the concept of the day has been presented, the students work on their class work individually or begin on their homework" (PST 2-33). Still another PST wrote, "The teacher went over homework from the night before. Then the teacher would give notes on the current day's lesson and usually end with a set of examples" (PST 3-24). Yet another PST wrote, "We would take notes, which usually involved a list of rules we had to memorize then class ended with working on problems from the book" (PST 1-26). In this PST's response, as with many others', teacher-worked examples dominated PSTs' experiences with school mathematics: "I only remember sitting in the desks watching the teacher at the projector as they droned on and on doing problem after problem" (PST 3-08). In these traditional classrooms, there were clear roles

for both the teacher and the student: “Teachers taught the lesson, [and] students took notes” (PST 3-15).

Although no evidence of the process standards was apparent in these traditional experiences, this is not to say preservice teachers’ experiences with traditional mathematics were necessarily negative. For example, this PST wrote,

A typical day in any of my math classes was going over the homework from the night before and then continuing on to the next lesson. . . . It was always very helpful when the teacher taught math with the blackboard. I have to see things written down in order to understand them. The chalkboard was always very helpful for me. (PST 2-14)

Additionally, another future elementary teacher explained,

A typical day in this class [consisted] of checking homework and moving onto the lesson. My teacher taught at a very fast pace, and we had to learn at this pace as well. Taking this class was probably the most stressful and challenging thing I have ever done, but it was also the most rewarding.

These teachers found traditional practice in which the teacher reviewed homework and delivered content beneficial and rewarding. Although the PSTs described above experienced only traditional school mathematics, other PSTs explained that they used concrete representations in elementary school but experienced a transition to traditional mathematics in secondary grades.

Transition from concrete representations to traditional school mathematics.

Through the middle grades, children's mathematical representations usually are about objects and actions from their direct experience. Primary school students might use objects to represent the number of wheels on four bicycles or the number of fireflies in a story. They may represent larger numbers of objects using place-value mats or base-ten blocks. . . . Representations make mathematical ideas more concrete and available for reflection. Students can represent ideas with objects that can be moved and rearranged. Such concrete representations lay the foundation for the later use of symbols. (NCTM, 2000, p. 68, 137)

Thirty-five of the 89 (39%) autobiographies showed evidence of the representation standard in the elementary grades followed by a transition to traditional explain-practice instruction similar to those described above in secondary grades. For example, one PST wrote of an elementary teacher who “used diagrams on the board for visual learners; she had physical blocks for us to use to help those with more of motor senses and she also spent time on it verbally.” However, “Math in middle school was more of a lecture style classroom. Depending on the teacher, we would either take notes on what she was saying or we would be following an overhead projector which had notes for us” (PST 1-17). Teachers continued to deliver content through lecture in high school as this PST explained: “We would enter the class, sit down and the teacher would begin lecturing and writing notes on the board” (PST 1-17). This PST experienced concrete representations in elementary school and a transition to traditional education in middle and high school.

The preservice teachers who experienced a transition to traditional education often described their elementary experience as hands-on, illustrating the representation standard, with a shift to lecture in later years. These shifts toward lecture were often accompanied by changes in PSTs' perceptions of the subject. For example, one PST wrote,

We also did a lot of hands on activities in the earlier grades. In middle school, I remember math being hard. This is when everything started being confusing and frustrating. There was not as much hands on activities. The math class was mainly taught through lecture. In high school, Math began to [steer] towards taking notes.
(PST 2-03)

For this PST mathematics became difficult with the absence of concrete representations. Similarly, another PST described a shift in middle school from concrete representations to worked examples with a focus on textbooks. For example,

In elementary school, math was fun. We used a lot of blocks and counting toys to learn. However, once I got to middle school we mainly did book work, which was where I started to not understand math. Once I got to high school and math was purely lecture notes and homework out of a textbook, I became a very poor math student. (PST 2-24)

Not only did the instructional practices change, but there was also a change in PSTs' perceptions of mathematics and their own mathematical prowess as illustrated in the responses above.

Less common evidence of process standards. Although evidence was found for the representation standard only in early years for the 35 preservice teachers described above, one future teacher described a rich elementary experience. This PST recalled an elementary classroom that was “related to your life,” in which “the teacher usually relates the lesson/problem to [the] real world and relatable issues to make it easier for the students.” In this classroom, “the teacher doesn’t talk that much.” Instead, students “usually work collectively as a class to solve problems” (PST 1-03). For this PST, elementary mathematics was connected to everyday life, and students, as opposed to the teacher, communicated problem-solving strategies. However, this PST also experienced a transition to lecture-driven mathematics instruction. In middle school, “the teacher lectures more,” and in high school, “the teacher lectures almost the entire lesson, without stopping much for questions. . . . Once the teacher is done explaining, the students usually practice problems on their own.” (PST 1-03). This PST described a less common elementary experience that included multiple process standards, not just the representation standard; however, this PST also experienced a transition to traditional mathematics in middle school.

Although much less prevalent, there were a few instances of concrete representations in students’ secondary experiences. One future teacher wrote about high school geometry and algebra classes that “had more hands on things mixed in” (PST 1-05). Similarly, another PST identified an uncommon experience with concrete representations in high school: “In high school, the teachers mainly sat at the projector doing problems too, except in my geometry class we did a lot of hands-on things such as

making shapes out of paper.” Evidence of similar secondary experiences was much less common, with only three PSTs alluding to instances of the representation standard in high school, and these experiences were isolated to a single class or teacher.

Even less common were experiences engaging in problem solving. In addition to the one preservice teacher’s elementary experience collectively solving problems described above, a second PST told of a single high school course that required non-routine mathematical thinking:

I may have had an interesting teacher occasionally, but for the most part it was easy and almost boring. The teacher taught a lesson, and I pretty much understood it right away. Until calculus, that is. AP Calculus provided me with new concepts and ideas that were more complex than simple algebra and graphs. It challenged me and made me think outside the box, and I liked that. (PST 3-01)

This PST's experience thinking "outside the box" was, however, isolated to a single course.

Evidence of preservice teachers communicating their mathematical thinking was also in short supply. In fact, one PST wrote of a high school experience void of student-to-student communication: “We came into the classroom, sat in our seats, and listened to the teacher talk and then followed along with examples. We were . . . not permitted to talk to the people around us” (PST 1-33). Only one other preservice teacher, in addition to the aforementioned rich elementary experience, described an elementary experience in which students communicated their mathematical thinking. This PST recalled that the teacher “split us into small groups so that we could collaborate and work to understand

whichever math lesson was being taught. . . . Each group would be able to explain one of the problems to the class” (PST 2-04). In this uncommon example, students communicated mathematics in small groups; then, groups consolidated their thinking to develop an explanation for the entire class.

No preservice teacher wrote of connections among mathematical ideas or topics. However, four PSTs wrote of connections to real-world applications of mathematics, whereas two PSTs wrote of an isolated school mathematics void of connections to out-of-class experiences. Still, these connected experiences were only present in a single course or grade level. For example, one PST described an elementary experience in which there “were always fun ways to relate the math to my life whether it was using candy to count or having 3-D blocks to count in hundreds” (PST 2-21). This PST considered enacting (Bruner, 1964) mathematics with physical representations as connected to life. Another future teacher described a middle school experience with connections to the real world:

I also enjoyed learning how to take the score of a baseball game, and applying this skill to real life when we visited the local baseball stadium to watch a game and practice keeping score. . . . I learned that I had a passion for math and loved applying it to life situations such as finding the volume of the Sydney Opera House. (PST 3-12).

This PST enjoyed applying connections to experiences outside the classroom.

However, other PSTs did not have similar experiences; one future teacher became frustrated with school mathematics “because I knew I would never use this in my future.” (PST 2-08). Similarly, another PST wrote, “A typical day in my math classes throughout

my student career always involved me learning something very confusing that I thought, ‘I will never use this in real life’ ” (PST 3-27). These two PSTs did not see the mathematics they were learning as *viable* (von Glaserfeld, 1990).

No PST provided evidence of the reasoning and proof standard in their descriptions of school mathematics.

Summary: School experiences. These preservice teachers' interpretations of their school experiences were analyzed in search of evidence of the five process standards (NCTM, 2000). Thirty-five of the 89 PSTs described school experiences that incorporated hands-on activities, providing evidence of the representation standard, in elementary grades and a transition to traditional review, lecture, and guided practice in secondary grades. Twenty-four PSTs described strictly traditional experiences with no evidence of any process standard at any grade level. Evidence of the process standards was uncommon and confined to a specific instance, grade level, or teacher. Only five PSTs provided evidence of the representation standard in secondary grades. Three PSTs wrote of isolated incidences of connections to the experiences outside of school, whereas two PSTs wrote of a lack of connections. One PST provided evidence of the communication standard, and one PST wrote of an experience void of student communication. Additionally, only one student wrote of non-routine problem solving. No evidence of the reasoning and proof standard was found. Only one PST provided evidence of more than one process standard, and this rich mathematics experience was isolated to elementary grades.

Nature of Mathematics

Teachers' beliefs regarding the nature of mathematics have been linked to pedagogical beliefs and instructional practice (Archer, 1999; Beswick, 2007; Cross, 2009; Ernest, 1989; Lerman, 1983). This research explores such beliefs further and attempts to connect beliefs regarding the nature of mathematics to PSTs' experiences with school mathematics as well as pedagogical beliefs. Two major themes emerged in the analysis of PSTs' definitions of mathematics: mathematics as a body of knowledge and mathematics as an activity. Table 4.1 illustrates the meaning clusters, associated meaning units, and number of significant statements supporting each theme.

Table 4.1

Phenomenological Reduction of PSTs' Definitions of Mathematics

Meaning Cluster (no. sig. state.)	Meaning Unit	No. Sig. Statements
Mathematics as a Body of Knowledge (55)		
Study of (41)	Study of numbers and other domains	12
	Study of numbers	7
	Study of relationships	5
	Study of properties	3
	Study of change	3
	Science of numbers	2
	Study of patterns	2
	Study of structures	1
	Study of algorithms	1
	Study of calculations	1
	Study of equations	1
	Science of reasoning	1
	Study of formulas	1
	Subject dealing with numbers	1
Symbolic system (8)	System	3
	Symbols	3
	Language	2
List of content (6)	List of school courses	2
	List of operations	2
	List of content	2
Mathematics as an activity (24)		
Performing procedures (14)	Using numbers to calculate or compute	5
	Manipulating numbers and symbols	4
	Finding numeric answers	2
	Configuring numbers	1
	Using numbers to determine an unknown	1
	Process of performing operations	1
Solving Problems (10)	Using numbers for problem solving	3
	Using numbers to understand	3
	Process of thinking	1
	Process of assigning meaning	1
	Process of coming to conclusions	1
	Thinking	1
Unassociated meaning units (3)	Exploring problem solving	1
	Mathematics as strategies	2
	Mathematics as a relationship	1

Note: no. sig. state. = the number of significant statements

Mathematics as a body of knowledge. Fifty-five of the 83 (66%) significant statements supported mathematics as a body of knowledge. This theme consists of three meaning clusters: mathematics as the study of various topics, mathematics as a list of content, and mathematics as a symbolic system.

Mathematics as the "study of." Forty-one PSTs (49%) described mathematics as the study of various topics. For example, one PST wrote “Mathematics is the study of quantity and patterns” (PST 1-06). Other typical definitions included “the study of numbers, shapes, and logic” (PST 1-09), “the study of all things related to numbers” (PST 1-13), “the study of structure, space, and change” (PST 2-03), and “the study of numbers and formulas” (PST 3-29). For PSTs who believed mathematics to be the study of various topics, there was no evidence that students were doing the studying. That is, PSTs were not actively investigating or analyzing these domains. Instead, references to the study of mathematics were more closely aligned with the titles of an academic subject.

Mathematics as a list of content. Six PSTs (7%) listed content included in their understanding of mathematics without defining this content as a study. The PST’s definition that follows illustrates one such response: “Mathematics is math, arithmetic, algebra, geometry, calculus, and statistics” (PST 1-07). Still another PST described mathematics as “a subject taught in school's [sic] focusing on different concepts involving numbers, shapes, and logic” (PST 1-19). As illustrated by these responses, such understandings of mathematics are confined to a school context.

Mathematics as a symbolic system. Eight PSTs (10%) described mathematics as a symbolic system. For example, one PST wrote, “Mathematics is the system of numbers and formulas used to solve problems and equations” (PST 1-12). Similarly, another PST explained, “Mathematics is [a] system used to calculate” (PST 3-26). These symbols were described as "their own language" (PST 1-15) and for some as "a foreign language that I may or may not understand on any given day" (PST 3-10). Although this symbolic system can be used for solving and calculating, mathematics was described as a set of external principles, not the activity of using this system. To use one PST’s analogy of mathematics as a language, mathematics is comparable to the rules of grammatical convention, not the act of communicating.

Mathematics as an activity. Twenty-four preservice teachers’ (29%) described mathematics as an activity. Although these definitions portrayed mathematics as an action, the processes that PSTs identified were not all congruent with the understanding of mathematics as “the activity of constructing relationships and patterns” (Wheatley, 1991, p 11) held by constructivists. Two types of actions were identified in these 24 PSTs' descriptions of mathematics: Fourteen PSTs (17%) described mathematics as the act of performing procedures, and 10 PSTs (12).described it as the act of solving problems.

Performing procedures. Some PSTs described mathematics as the process of "configuring" (PST 1-04) or “manipulating numbers to solve equations” (PST 1-16). The actions these PSTs used to describe mathematics related to procedures. Further, another PST wrote that "Mathematics is the manipulation of numbers and variables” (PST 1-29).

For these PSTs mathematics was the process of arranging and moving mathematical symbols.

Similarly, mathematics was also understood as the action of following an algorithm to determine the one correct answer. The following PST's definition illustrates an understanding of mathematics as performing rigid procedures:

Mathematics is finding an answer to a problem through a list of constant steps. In mathematics there is always one correct answer and an infinite amount of incorrect answers, you simply need to know the steps to find the one correct answer. (PST 2-33).

Although these 14 teachers viewed mathematics as an activity, the actions by which one does mathematics were prescribed.

Solving problems. A number of preservice teachers described mathematics as the process of solving problems. For example, one such PST described mathematics as “using numbers to solve a problem” (PST 1-27). These PSTs believed “Mathematics is a way of using theories, equations, numbers and/or figures to solve or explain problems” (PST 1-03). Although these PSTs reference problem solving, it was unclear whether these problems were non-routine in the NCTM sense of problem solving (NCTM, 2000). Similarly, other PSTs believed mathematics to be the “process of coming to some numerical conclusion” (PST 1-30) or the process of “finding answers” (PST 3-09). Although these definitions of mathematics do not explicitly reference rote activities, it was not clear whether these conclusions and answers were a product of prescribed algorithms or non-routine thinking.

Only three PSTs (4%) provided explicit evidence of a standards-based perspective of problem solving. One such response follows:

Mathematics is more than just adding or subtracting and there being an exact, definite answer. Sure, in your actual class, obtaining the correct answer is definitely important but I believe that the thought process is more important than the actual answer. As a teacher, I would regard the mathematical process of obtaining an answer as much more important. (PST 2-22)

This PST believed mathematics is the process of thinking, which goes beyond traditional algorithms. Similarly, another PST described mathematics as “logical thinking that helps shape the world” (PST 3-28). For this PST mathematics was logical; that is, mathematics required reasoning. Another PST wrote, “Mathematics, to me, is exploring different ways to solve equations and problems” (PST 2-19). By exploring multiple avenues, mathematics was conveyed as non-routine. Nevertheless, such responses were rare.

Summary: Nature of mathematics. Three meaning clusters—math as a study, list of content, and symbolic system—were associated with mathematics as an external body of knowledge. Further, many of the typical responses showed mathematics to be confined to a school context or specific courses. Such codified, static understandings of mathematics are not congruent with a constructivist epistemology that views mathematics as an activity (Wheatley, 1991; Fosnot, 2005; Schifter, 2005).

Although some PSTs viewed mathematics as a human activity, the processes by which these PSTs viewed mathematics were not congruent with a problem-solving or “mathematizing” (Fosnot, 2005) perspective described in constructivist literature

(Wheatley, 1991; Fosnot, 2005; Schifter, 2005). When attempting to connect preservice teachers' lived experiences with school mathematics to their understanding of the subject, analysis becomes difficult because there is such little variation in school experiences. Overwhelmingly, these PSTs either experienced traditional school mathematics or transitioned to review-explain-practice pedagogy by middle grades. Additionally, understandings of mathematics as the activity of problem solving were rare. In a secondary comparison of those PSTs' school experiences with views of mathematics as the activity of problem solving, no clear differences were identified.

Interpretations of Pedagogy as Students

One means of connecting PSTs' experiences as students of mathematics to their pedagogical beliefs was to examine pedagogical practices they interpreted as particularly helpful and those they did not find beneficial as students of mathematics. In the analysis of these data, 127 significant statements were extracted from PSTs' mathematics autobiographies related to beneficial practices and 49 related to practices they did not find helpful. These 176 significant statements were associated with 52 meaning units.

From these 52 meaning units, eight meaning clusters emerged: review, delivery of content, worked examples, practice, learning styles, group work, process standards, and assessment. Four of these eight meaning clusters were associated with a theme of traditional educational experiences characterized by review, delivery of content, worked examples, and practice. The remaining meaning clusters—learning styles, group work, process standards, and assessment—were not associated with any particular theme. The

13 meaning units not associated with any particular cluster and the number of significant statements associated with each are identified in Appendix I.

This section first examines the theme of traditional education through its four components of review, delivery of content, worked examples, and practice. Then, the unassociated meaning clusters (learning styles, group work, and process standards) are considered.

Traditional pedagogy. Given that these preservice teachers' experiences with school mathematics were overwhelmingly traditional, it is not surprising that their interpretations of effective pedagogy would also be situated within a traditional framework. Eighty-four of the 176 significant statements (approximately 48%) were associated with tradition education characterized by review, delivery of content, worked examples, and practice. Table 4.2 shows the meaning clusters, associated meaning units, and number of significant statements supporting the theme of traditional mathematics experiences.

Table 4.2

Phenomenological Reduction of PSTs' Interpretations of Pedagogy: Traditional Mathematics Experiences

Meaning Cluster (no. sig. state.)	Meaning Unit	No. of Sig. State.	
		Beneficial	Unhelpful
Traditional mathematics experiences			
Review (5)	Review of homework	3	
	Review	2	
Delivery of content (32)	Lecture alone ^a		12
	Notes	6	4
	Lecture	2	2
	Lecture followed by practice ^b	3	
	Boring lectures		1
	Notes from text		1
	Explicit algorithm given	1	
Examples (27)	Examples	8	
	Step-by-step examples	7	
	Teacher-worked examples	6	
	Example with explanation	2	
	Teacher-worked examples followed by guided practice ^b	2	
	Single example		1
	Algorithm without example		1
Practice (28)	Practice	12	
	Repetition	7	
	Group work: practice ^b	3	
	Lecture followed by practice ^b	3	
	Teacher-worked examples followed by guided practice ^b	2	
	Too much practice		1

Note. No. of Sig. State. = the number of significant statements. ^aThe meaning unit "Lecture alone" was assigned to significant statements that reference lecture the entire period or specific references to lecture without support from examples or guided practice. ^bThis meaning unit was associated with more than one meaning cluster.

It should be noted that each PST's response did not produce the same number of significant statements. Therefore, there is not a one-to-one correspondence between significant statements supporting a meaning unit and the number of PSTs in the analyses

that follow. Although Tables 4.2 and 4.3 may provide information regarding how common certain responses were, in this section attempts are not made to quantify the exact number of PSTs that held a particular belief.

Review. Just as many PSTs described a typical day that began with a review of homework, significant statements extracted from these PSTs' interpretations of effective pedagogy also supported beliefs that review is advantageous. One such significant statement follows: "I find it beneficial when a teacher goes over the homework assigned so I am sure I am on the right track and I can correct my mistakes when the work is still fresh in my mind" (PST 1-15). Similarly, other significant statements described review as a time to identify misconceptions after students have practiced the skill at hand similar to this PST's response:

Overall, homework, given at all age levels, is very helpful. It gives students a chance to do the work independently and test their knowledge. Then, the opportunity will be given to review the homework and ask any questions pertaining to the ones that they may have missed. (PST 1-30)

Although these PSTs were not using the language of schema theory, there was evidence that they viewed such reviews as activating prior knowledge. For example, one PST wrote, "Teachers would review [sic] old material to refresh your memory before starting new material" (PST 2-18). Among this group of PSTs, review of previous content was consistently seen as a component of effective pedagogy.

Delivery of content. Significant statements supported the delivery of content through lecture and the presentation of notes to be a valuable pedagogical practice as

illustrated by this response: “I found the lecture classes to be the most beneficial to me, because I learn best by being thoroughly explained the material step by step and by writing it down to have to study later” (PST 1-18). Similarly, another significant statement follows: “I think that the teachers who lectured while working either at the board or at an overhead and did many examples works well for me” (PST 1-34).

Many significant statements supported lecture as something separate from illustrating teacher-worked examples as evident in their critiques of a lecture void of such examples. For instance, one such response follows: “Lecturing in math did not benefit me at all. I would rather see something be done then [sic] to be told how to do it” (PST 2-03). This significant statement shows that lecture to be a time during which procedures were explained separately from when these procedures were demonstrated. Still other significant statements supported lectures to be effective only when delivered with enthusiasm:

Lecturing does not seem to benefit me as much because I find myself “day dreaming” throughout class. If my teacher that is giving the lecture is enthusiastic about what they are teaching then it makes me feel the same way and I find that I am more interested in the information and I want to learn it. (PST 3-23)

From these responses, a belief in the value of lecture is illustrated; however, a belief that lecture should occur in tandem with worked examples and be delivered with zeal was identified.

Another aspect of the delivery of content was the distribution of notes. PSTs found giving notes to be a valuable part of the process of teaching as illustrated by this response:

Strategies that I find particularly helpful are very detailed notes that simplify math. I need things to be explained very thoroughly instead of just being shown how to do a problem. When teachers just show you how to do a problem without explaining the ins and outs, it's hard for me to keep up. (PST 2-01)

While PSTs found merit in the note-taking process, significant statements support that note taking can be uninteresting: “I do believe writing down notes is helpful. It is easy to refer to when it comes to preparing for the test. However, I do not like always taking notes. I become unmotivated and bored” (PST 2-29). Interestingly, this significant statement, as did others, showed the end goal of effective pedagogy to be a test, and notes were helpful because they aided in the preparation for tests.

Similar to PSTs' critiques of lecture without examples, PSTs found note taking without guided practice to be problematic. For example, one PST wrote, “I found classes where I only took notes on the idea not beneficial because I think it is important to practice the concept when the teacher is there so you can get help if needed” (PST 1-28). Thus, significant statements support the delivery of content through lecture and giving notes to be beneficial strategies for instruction but warn that such practices should also be accompanied by other components of traditional instruction.

Worked examples. As mentioned above, these PSTs interpreted worked examples as something separate from lecture, which was a time when teachers explained an

algorithm and gave notes, as evident in this response: “When a problem was simply explained in written steps & not demonstrated to the class, I found it a lot harder” (PST 2-30). However, when lectures were supported with examples, PSTs found teacher-worked examples beneficial. One such significant statement follows: “When a certain problem was worked through several times by the teacher in front of the class, the idea usually stuck better” (PST 2-30). Further, another response is below:

I like a lot of examples so I can look at how to solve certain problems. I do not like just being told how to solve a problem and given one example. I really need to work several examples to fully understand how to solve the problem. (PST 1-29)

This response shows that multiple examples were needed and did not find a single instance sufficient.

These preservice teachers viewed worked examples as an instance of a precise set of steps that learners should imitate. To cite an example, “The most helpful strategies for me have been giving good examples that clearly show me step by step how to work through the problem” (PST 2-17). Further, another significant statement follows: “Something that is helpful to me is the teacher going through the problem step by step and then going through a few examples so I am able to see how to do different types of problems” (PST 3-11). Worked examples were interpreted as an illustration of a particular algorithm that learners can reference when this procedure is needed, as this significant statements illustrates, “It was always helpful for me to be able to go back and look at how I did a problem in class step by step” (PST 1-17).

Significant statements support a belief that multiple examples of a procedure, accompanied by a step-by-step explanation of the algorithm, were an important component of instruction. Within their explanations of the importance of worked examples, a traditional understanding of mathematics is illustrated in which a precise set of steps can be reproduced. PSTs' view of mathematics as confined to a well-defined problem space is illustrated in their belief that teacher-worked examples can be reproduced when needed.

Practice. Once a procedure had been explained and illustrated, many PSTs found it beneficial to practice this procedure in class with teacher guidance. For example, the following significant statement supports this belief: “The practice helped me to get a good feel of how the problem was to be done before I had to solve similar problems for homework later on” (PST 2-10). Significant statement revealed a belief that guided practice should be followed by individual practice as well. This is illustrated in the following response:

In mathematics, I think that just practicing a lot helped me more. After going over a new concept, stopping to do a problem or two on my own benefitted me because sometimes you may think you understand it, but when you actually try and work it out on your own, you realize that there's more to it than you think. (PST 2-22)

These significant statements supported practice as valuable and stressed the volume of practice needed, as illustrated in this typical response: “The best way for me to understand how to do something is for me to practice it so the more practice I received the better” (PST 2-31).

Further, some PSTs believed that practice should be repetitive. For example, the significant statement that follows supports this belief: “I learn better by hand writing math problems and practicing them over and over again” (PST 2-11). Additionally, another PST wrote, “I personally find that working problems over and over until they are mastered pretty well is the most helpful way to learn math problems” (PST 1-32).

Once again, teachers’ descriptions of sound pedagogy parallel their experiences with a traditional sequence of review, delivery of content, examples, and practice. A number of PSTs stressed the volume of practice and believed repetition to be an important part of learning mathematics. Similar to PSTs’ beliefs regarding the importance of examples, repeating procedures also illustrated an understanding of mathematics confined to a rigid problem space in which routines can be replicated.

Unassociated meaning clusters. Although a majority of preservice teachers’ interpretations of pedagogy were situated within a context similar to their experiences with traditional school mathematics, four meaning clusters emerged that were not defined by traditional instruction. Table 4.3 shows the meaning clusters, associated meaning units, and number of significant statements supporting meaning clusters that did not support a common theme. In the following section, three of these four unassociated meaning clusters are reported. Once again, the number of significant statements is not necessarily synonymous with the number of PSTs.

Table 4.3

Phenomenological Reduction of PSTs' Interpretations of Pedagogy: Unassociated Meaning Clusters

Meaning Cluster (no. sig. state.)	Meaning Unit	No. of Sig. State.	
		Beneficial	Unhelpful
Unassociated meaning clusters			
Learning styles (21)	Visual learner	7	
	Hands-on as visual ^b	7	
	Only auditory style		4
Group work (21)	Hands-on as kinesthetic ^b	3	
	Group work: no rationale	4	4
	Group work: comfort	2	1
	Group work: communication ^b	3	
	Group work: practice	3	
	Group work: perspective	2	
	Group work: left out		1
	Group work: teacher's role		1
Process standards (31)	Connection to experience	6	
	Connection to other subjects	1	
	Group work: communication ^b	3	
	Hands-on activities	11	
	Hands-on as visual ^b	7	
Assessments (7)	Hands-on as kinesthetic ^b	3	
	Graded homework		2
	Quizzes		1
	Timed tests		1
	Projects		1

Note. No. of Sig. State. = the number of significant statements. ^bThis meaning unit was associated with more than one meaning cluster.

Learning styles. In 18 of the 89 (20%) preservice teachers' examples of beneficial pedagogy, evidence of three preferences for sensory input was found: visual, auditory, and tactile (kinesthetic). For example, one PST explained,

“My 4th grade math teacher left a mark on me in her class through her tactics. If we were ever adding, multiplying, or subtracting, she would always have us use objects to show us a physical example of the numbers being represented. I really

liked this strategy because it gave you a visual, hands on, and verbal explanation of how it was done.” (PST 3-27)

Preservice teachers believed they learn best when instruction aligns with their learning style and that instruction should vary to accommodate all learners’ preferences: “In my mathematics class, I want to incorporate all types of learning: visual, auditory and kinetic [sic]” (PST 3-19).

Visual learners. Seven PSTs described themselves as visual learners and preferred visual means of instruction. For example, one PST wrote, “I’m a visual learner so anything that I could see I would understand” (PST 1-20). Similarly, another PST explained, “The strategies that I find helpful are using models and the white board/overhead to learn lessons. I’m more of a hands-on and visual person” (PST 2-07).

PSTs that described themselves as visual learners often explained that they do not benefit from instruction delivered strictly through auditory means, as this PST illustrated:

Personally I am more of a visual learner so it helps me a lot to have actual concepts written on the board or using some hands-on activities such as using a spinner to learn probability. It helps me to visualize the problem so using pictures or graphs helps me to grasp the question and find a solution. Alternatively, just listening to a teacher describe concepts or doing worksheets doesn't help me as much as it doesn't stick in my brain as much. (PST 1-24)

Similarly, another PST wrote, “I’m a visual learner so anything that I could see I would understand. I’m not an audio learner so listening with nothing for me to see or touch then I would not benefit from the lesson” (PST 1-20). Among these responses, no PST

identified her- or himself as a strictly auditory learner; auditory means of sensory input were referenced only in contrast to their visual preferences.

Seven preservice teachers referenced the use of hands-on activities as beneficial to them due to their visual preferences. For example, this PST explained, “The hands-on activities helped me because I am a visual learner and it helps me to literally see what is happening” (PST 3-01). Similarly, another PST wrote, “I also like hands on activities that help me visualize the work I am trying to learn.” (PST 1-15). This connection between hands-on activities and visual learning was further illustrated by this PST’s response: “I found the use of hands on activities helpful in learning math, especially when the activity was edible. I’m a visual person, so being able to see difficult concepts laid out before me was beneficial.” (PST 3-05). Interestingly, these PSTs attributed value to hands-on activities not because they allow students to enact the mathematics as they move from concrete to abstract representation (Bruner, 1964, 1967) but because they make the mathematics visible.

Kinesthetic learners. While many PSTs valued hand-on activities because of their visual sensory input preference, three PSTs attributed value to such activities due to their tactile learning preference. For instance, one PST explained, “Hands on strategies particularly helped me in math because I am a tactic [sic] learner” (PST 2-03). Similarly, another PST wrote, “Anything hands-on I found particularly helpful because I am very kinesthetic and like to be able to touch and feel and be able to move around what I am learning” (PST 3-08). Although not using the language of learning styles, another PST wrote: “I really liked more hands on teaching because that’s how I learn. It was easier for

me to be able to connect math with something I could touch" (PST 1-09). PSTs who preferred learning through kinesthetic inputs were less consistent in labeling their preference. Some attributed their preference to touching objects (tactile), whereas others associated their preference with moving objects (kinesthetic).

Moreover, 21 of the 176 (12%) significant statements regarding effective pedagogy, representing 18 of the 89 (20%) of PSTs, referred to matching mode of instruction with learners' learning styles, a theory not supported by research on teaching and learning (Pashler, McDaniel, Rohrer, & Bjork, 2009). In analyzing PSTs' responses, no evidence was found that explained how this theory penetrated PSTs' pedagogical beliefs.

Group work. Preservice teachers' beliefs regarding group work were the most varied among the beliefs examined in this research. Fourteen PSTs (16%) wrote of the benefits of group work, whereas seven PSTs specifically mentioned group work as a unfavorable instructional practice.

Some PSTs referenced group work as a valuable time to collaboratively practice the mathematics delivered by the teacher. One such typical response follows: "Working in groups on problems in class was always helpful to me. Sometimes it is hard to remember exactly how to solve problems even right after a lesson, and it is very helpful to have peers to work together to solve the problem" (PST 3-04). Again, a routine understanding of mathematics was portrayed as group work aided in remembering how to solve a problem.

Other PSTs valued group work because it provided a comfortable setting to ask questions, which was less intimidating than asking the teacher in front of the entire class.

For example, one PST wrote,

Group work is actually fairly helpful to me in most situations. When I don't understand the material as well, it is easier for me to then work in a group and get help and understanding rather than speak up and ask the professor in front of the whole class, which I am very uncomfortable doing most of the time. (PST 1-13)

Similarly, another PST found peers more approachable than the teacher:

I think that it is helpful when teachers encourage students to use the biggest resource in the classroom: the students. It is so helpful to ask people around you for help without struggling in the seat alone or feeling like the teacher is not approachable. (PST 1-33).

For these future teachers, discussing mathematics with one's peers is less intimidating than addressing the teacher. However, not all PSTs found group settings to be more comfortable. For example, this PST found group settings uncomfortable:

Some of my teachers used to split us up randomly to work together in math and I feel as though that never benefited me because I was less comfortable with my partner I was working with since I didn't know them, or their level of math intelligence. (PST 3-14)

While inquiries into PSTs' beliefs regarding group work showed that future teachers value comfortable learning environments, group work was not consistently identified as such.

Examining PSTs' beliefs regarding group work also revealed strong beliefs regarding the role of the teacher. For instance, one PST wrote, "The method I despise the most is peer teaching. I truly believe this is a teacher's attempt to get out of work. You have the college degree not your student. You need to teach!" (PST 3-28). This PST strongly believed that teachers should deliver the content and viewed group work as an escape from one's pedagogical duties. Another PST found group work unbeneficial because those students that understood the content took an active role, whereas students struggling with the content were passive: "Something not so helpful is group work only because if you don't really know the concept the person in the group who does know it spends all the time doing the work and your [sic] watching but not really learning much." (PST 1-25)

Still a few PSTs found group work to be a useful time to communicate their mathematical thinking. One such response follows: "Working in groups and being able to discuss problems and explain things to classmates helps me to fully grasp the idea or concept" (PST 2-17). Similarly, another PST wrote, "I also find group work on an activity to be more beneficial to me, opposed to individual work, because group members can bounce ideas off each other and help each other understand the material" (PST 3-17).

Not only did group work allow the student communicating the mathematics an opportunity to organize her or his thinking, but it also gave those listening to the explanation a perspective different from the teacher's. Some PSTs, therefore, valued group work because it offered mathematics from a student's perspective: "Sometimes peers understand where the mistakes are made because we all make the same ones, so

they can easily instruct us to the right path and answer. I love working with groups to get others' opinions on how to approach problems as well" (PST 3-12). Similarly, another PST found group work advantageous "because sometimes it takes someone other than a teacher to be able to explain things to make them simpler to understand" (PST 3-21).

Preservice teachers' experiences with collaborative learning in mathematics courses have shaped varied beliefs regarding group work. Responses varied in both the value of group work and reasons they found group work particularly beneficial or obstructive. However, through examining PSTs' beliefs about group work, beliefs regarding the value of comfortable learning environments, the role of the teacher, students' perspectives, and communication were highlighted.

Process standards. In addition to the abovementioned evidence of the communication standard (NCTM, 2000) in group work, six significant statements provided evidence that at least some PSTs believed mathematics instruction should be connected. PSTs wrote of the benefits of real-world examples as such applications made the mathematics relevant:

When the mathematics lessons were applied to a real life situation, I feel like I appreciated math more and I wasn't just another high school kid forced to be in class wondering, "Why are we learning this?" It kept the math relevant to today, even though some techniques were conceived hundreds of years ago. (PST 3-01).

Similarly, another PST wrote, "I really like when teachers make lessons relevant to us. It makes me want to pay attention because I feel like I will actually use what we are learning in a real-life situation and can find it helpful to my life in some way" (PST 1-

03). Additionally, one PST found the laboratories in the content course examined in this study to be beneficial because these laboratories "are able to draw a connection from the concepts I have been taught to the real world, and learning how the lesson applies to my life always helps me understand the material better" (PST 3-10). Similar to PSTs' experiences with school mathematics, there were a few instances of mathematics as connected to or applied to the real world, but no references to connections among mathematical ideas were found.

In examining the influence of PSTs' experiences with school mathematics on these beliefs, two of the six PSTs that valued a connected mathematics were also among the six PSTs that wrote of connected school experiences. Although no causal claims are made, instances of connected school experiences were rare as was evidence of pedagogical beliefs related to connections. Still, there was some overlap among these uncommon experiences and beliefs.

As previously mentioned, ten significant statements supported two meaning units relating hands-on activities to PSTs' learning styles. The use of concrete objects was interpreted as visual, tactile, and kinesthetic representations of mathematics. Eleven additional significant statements were identified that support the use of hands-on activities. However, these references did not reveal how concrete representations might benefit learners. For example, one PST wrote, "I always really enjoyed hands on activities. They are much easier to wrap your head around" (PST 2-25). Combined, these statements showed that some PSTs believed such representations are important, but did not offer insight into how such models aid in student learning. In the next section, further

inquiries into PSTs' beliefs regarding hands-on activities are explored through prospective teachers' descriptions of their anticipated future instructional practices.

Summary: Interpretation of pedagogy as students. Similar to their experiences with school mathematics, PSTs' beliefs regarding effective pedagogy were situated in a traditional framework of review, delivery of content, worked examples, and practice. Additionally, a belief that instruction should match students' preferred learning styles emerged. Beliefs regarding group work varied both in the value attributed to collaborative tasks and the reasons that such tasks are beneficial. Similar to their experiences with school mathematics, a few less frequent instances of the communication, connection, and representations standards were identified. However, no evidence of the reasoning and proof or problem-solving standards was identified in PSTs' interpretations of effective pedagogy.

Beliefs Regarding Sequence of Instruction

Data pertaining to PSTs' beliefs regarding sequence of instruction was collected at the beginning of the semester for 89 PSTs through their mathematical autobiographies. In these autobiographies, PSTs were asked to describe a typical day in their future classroom and to explain how they might teach a lesson on either the area of a rectangle or elementary probability. At the end of the semester, data were collected using hand-written exit surveys for 83 PSTs in which they were again asked how they planned to teach a lesson on either the area of a rectangle or elementary probability. The content domains chosen for these pre- and post-example lessons can be found in Appendix L. Table 4.4 summarizes these data. Although it appears that more PSTs chose geometry for

their post-example lessons in Sections One and Three, no clear pattern emerged between sequence groups or in changes between content domains for the pre- and post-example lessons.

Table 4.4

Number of PSTs who Addressed Geometry or Probability in Example Lessons

Sequence Group 1								Sequ. Group 2			
Section 1				Section 2				Section 3			
Pre		Post		Pre		Post		Pre		Post	
Geo	Prob	Geo	Prob	Geo	Prob	Geo	Prob	Geo	Prob	Geo	Prob
15	16	22	8	15	15	14	12	15	13	20	7

Note. Sequ.= sequence; Geo = Geometry; Prob = Probability.

It should be noted that not every response offered insight into their intentions regarding sequence of instruction. Of the 261 responses analyzed, only 97 (36%) offered explicit evidence of an intended sequence of instruction. From these data four sequences of instruction were identified: traditional sequences, confirmatory sequences, exploratory sequences, and no formal explanation. Table 4.5 illustrates variations of these four sequences and the number of instances of each sequence identified in PSTs' descriptions of a typical day in their class, example lesson at the beginning of the semester, and example lesson at the end of the semester.

Table 4.5

Number of Instances of PSTs' Intended Sequences of Instruction

Sequence	No. of instances of Sequ. in		
	Typical Day	Pre-Example Lesson	Post-Example Lesson
Traditional Sequence without Activity (44)			
Review; Explain; Example	1		
Review; Explain; Practice	3	1	1
Review; Explain			1
Review; Example	1		
Explain; Example; Practice	2	3	1
Explain; Example	2	1	4
Explain; Practice	2	6	4
Explain only			7
Example; Practice	2		
Example only	1		
Practice only	1		
Confirmatory Sequence (28)			
Review; Explain; Activity; Practice	3		
Review; Explain; Examples; Activity	1		
Explain; Activity; Practice		2	1
Explain; Example; Activity; Practice	1	1	1
Explain; Example; Activity	1		
Explain; Activity; Practice	2		
Explain; Example; Practice; Activity	1		
Explain; Activity		8	3
Example; Practice; Activity	1		
Example; Activity; Practice	1		
Example; Activity		1	
Exploratory Sequence (7)			
Activity; Explain; Practice	1	1	
Activity; Explain		5	
No Formal Explanation (18)			
Review; Activity; Practice	1		
Activity only		12	3

No. of instance of Sequ. in = the number of instances of a particular sequence in; è = followed by.

Traditional sequence. Traditional sequences of instruction were characterized by review, delivery of content, worked examples, and practice. In these 44 instances of

traditional sequences (45%), no collaborative tasks were posed, and concrete models were not used by the students. Although each phase of a traditional sequence was not always present, those phases present always occurred in the same order. Take, for example, one PST's preliminary description of a typical day in her or his future classroom:

First we get introduced to the topic and the teacher usually gives definitions of certain words. Then we learn the basic steps. After the basic steps we work examples. After the examples we are usually assigned some kind of homework for practice. (PST 2-12)

This PST planned to explain content, give examples, and assign practice. Here, the teacher delivers the mathematics while the students passively observe and practice these procedures.

Evidence of traditional sequences was also found in PSTs' example lessons. A typical preliminary response associated with a traditional sequence follows:

If I were teaching my math class the area of the rectangle, then I would begin by drawing a rectangle on the board to show the exact shape we are discussing, write the formulas for the perimeter and the area to show the difference, and explain the difference between the two formulas. After the small lesson, I would write certain numbers for both the length and width, re-write the formula for the area, and explain which number represents the length and the width of the rectangle. Then, I would explain how to find the answer and show a few other examples. After

showing examples, I would give the students a few problems for them to try, and then check their answers when they are finished. (PST 2-02)

This PST planned to begin by explaining an abstract formula and showing students how to use this formula with examples and practice exercises. This example follows a sequence of explanation, examples, and practice without group activities or concrete models.

In some example lessons identified as traditional, PSTs did incorporate concrete representations. However, students did not engage with these objects; the teacher demonstrated the mathematics as students observed. Take, for example, this preliminary example lesson:

I would begin teaching my class a lesson on probability by reviewing fractions, at first, and explaining the idea of a part over a whole. Once I had reviewed the concept, I would take out a bag that could not be seen through and place 3 red, 4 blue, and 1 yellow marbles in the bag. And explain that since there were 8 marbles in the bag, we had a 3 out of 8 chance pulling out a red marble, etc. I would then go around to different students in the class and ask what color marble we had the most of in the bag (blue) and then explain that because we had more blue marbles, we had a bigger chance of pulling out one of those rather than pulling out the one yellow marble in the bag. I would use this again to talk about fractions and how $\frac{1}{2}$ of the bag of marbles contained blue ones, so we were most likely to pull out one of those. I would then ask different students in the class

what the probability would be for pulling out a red marble, and ask the same question for the yellow one. (PST 1-02)

Although this PST planned to incorporate concrete objects into the lesson, students would not use the objects to explore or confirm the mathematics; the teacher demonstrates the mathematics instead.

Confirmatory sequence. Twenty-eight of the 97 (29%) examples of instructional sequence were classified as confirmatory. Confirmatory sequences included group activities and often incorporated students engaging with concrete representations. However, these collaborative tasks were posed after an explanation of content or worked examples. One such typical preliminary response follows:

Hopefully a typical day in my math class will include a few opening review questions, teaching a short lesson, hands on activities to go with the lesson or group work to understand the concept, then homework/ individual work to make sure each student understands the concept. (PST 1-16)

A typical day in the PST's classroom would follow a review-explain-activity-practice sequence in which the group task follows the explanation of content.

Evidence of confirmatory sequences of instruction was also found in PSTs' preliminary example lessons. For example, one PST wrote,

If I were to teach a lesson on probability, I would have a PowerPoint prepared with a definition of what probability is and the basic concepts of it. I would then show examples related to probability and how to solve them. Next, would be the hands-on activity. Since I want to teach small children, my activity would

probably consist of the students having some type of multi-colored candy (such as, Skittles) in a plastic bag and they have to find the probabilities of pulling out certain colored candies. . . . After the activity was over, the students would complete a worksheet. (PST 2-19)

This PST's example lesson illustrates an explain-example-activity-practice sequence in which the content delivered in the PowerPoint was reinforced during the group activity.

Exploratory sequence. Only seven of the 97 (7%) examples of instructional sequence provided evidence of an exploratory sequence in which mathematics tasks are posed to students before a formal explanation of content or worked examples. All seven instances were collected at the beginning of the course. For example, one PST wrote, "A typical day in my math class will be starting the lesson with a fun activity to get the attention of my class, teaching the material, doing practice problems together, trying a few on their own, and then giving a few for homework" (PST 2-15). For this PST, the activity precedes an explanation. However, it is not clear whether the purpose of the task is solely to engage students in the mathematics or whether students will explore the content through this task.

Evidence of an exploratory sequence was also found in PSTs' example lessons. One such lesson is report below:

On a lesson of probability, I would probably give my kids a bag with M&M's and have them count the number of M&M's in the bag. Then I would have them pull out a bunch of M&M's (one at a time, and returning them to the bag before another one is drawn) and write down their results. Then I would ask them

questions like do you think it is likely for me to pull out a yellow one, and so on.

After this experiment I would explain that this is probability, explain the

definition of probability, and go into my lecture. (PST 1-31)

This PST planned to have students first run simulations to get a sense of which outcome is more probable before content was formally explained or examples were worked.

No formal explanation. In sixteen of the 97 (16%) examples of instructional sequence, PSTs presented example lessons in which activities were described without explicit evidence of an explanation from either the students or the teacher. For example, one PST initially wrote,

Whenever I am teaching a math class in the future, I plan to start the day with a little review question from the information we learned the day before just so that it can be fresh in my students' minds. Then I plan to get into small groups and start working on the specific hands-on activity that addresses the necessary information that needs to be covered in a fun and simple way. I like to use colorful things, arts and crafts, building blocks, everyday materials, etc. to help get the point across. Then, at the end of class, I plan to assign work to do at home sometimes with parents or siblings such as games so that the topics covered in class can be reiterated. (PST 3-02)

This PST does not clearly describe a time when content is explained. One might infer that students learn the content from the hands-on activity posed. All other instances of activities with no formal explanation came through PSTs' description of example lessons.

In these example lessons, PSTs only described the task they planned to pose. For example, one PST initially explained,

To teach a lesson on probability, I would use M&Ms and a bowl. Have the class work in pairs. Give each pair an even number of red, blue, green, yellow, and brown M&Ms. The students will first find the theoretical probability of drawing each color out of the bowl. Then they will actually draw a piece of candy out of the bowl and record. After repeating 20 times, the students will record the number of times they drew each color and find the experimental probabilities. After the experiment, the students can eat the candy! (PST 2-21)

In this example lesson, similar to others in which no explanation phases is described, it is not clear how students would know what theoretical probability is or how one would find it. The task posed does not make the content self-evident and requires prior knowledge. Similarly, the other example lessons identified as lacking formal explanations do not provide clear evidence that students are learning the mathematics from the task. Further inquiries are needed to determine whether PSTs envisioned students learning mathematics from the activity alone or whether information is missing from their instructional plan.

Changes in intended instructional sequences. Over the course of the semester, all PSTs experienced both exploratory and confirmatory sequences of instruction as students of a content course for elementary teachers. To assess the effects of this course on PSTs' beliefs regarding sequence of instruction, changes between PSTs' initial intended sequences and post-sequence preferences were examined. Because of the

aforementioned uncertainties in the instructional plan of the activity-only sequence, these examples were not included in this analysis. Twenty-nine preliminary examples and 23 post-example lessons remained in this analysis with pre- and post-data for 11 PSTs.

Seven of these 11 PSTs showed no change in their intended sequence, indicating a traditional sequence both at the beginning and end of the semester. For example, one of these unchanged PSTs planned to "first explain that probability is like the chance of something happening." Then this PST "would give them examples." This PST continued to elaborate on a particular example and did not include any collaborative tasks. At the end of the semester this PST explained that the lesson on the area of a rectangle would "start with defining a rectangle and other quadrilaterals." Then, this PST would "talk about perimeter and what that means" and "introduce area with examples." The lesson would end with "the formula for area of a rectangle (lw)" (PST 3-14). This PST's instructional sequence, similar to the other six PSTs that showed no change, included an explanation and examples without posing any tasks.

Three PSTs that initially showed intentions of using exploratory sequences preferred traditional sequences after participating in this content course. For example, this PST initially planned to teach probability as follows:

I would just take Skittles, or another candy they enjoy, and put them in a bag and ask them to randomly pull one without looking. Then we would write everything on the board and I would explain to them that they were finding the "what if" of Skittles. (PST 2-25).

After experiencing both exploratory and confirmatory sequences the same PST explained, "I would show them the formula, explain what each part of it means and where it came from, and simplify it, if it needs to be, by examples such as the tile on the floor" (PST 2-25).

One PST's example lesson illustrated an exploratory sequence at the beginning of the semester: "I would first do an exercise using colored candy (such as M&M's) and have students draw 20 times and determine the probability. Then I would give the students notes and practice problems" (PST 1-12). At the end of the semester this same PST's example lesson illustrated a confirmatory sequence: "I will first teach the lesson & explain how you find area. Then I will have them do an activity that elaborates on this, such as using geoboards" (PST 1-12). This PST showed intentions of an exploratory sequence at the beginning of the semester yet preferred a confirmatory sequence at the end of the semester.

From examining these 11 PSTs' initial and final example lessons, it appears that these PSTs' beliefs either remained traditional or shifted towards a more traditional sequence. Additionally, Table 4.6 illustrates the number of PSTs' example lessons with evidence of particular sequences of instruction at the beginning and end of the semester for all PSTs, not only those 11 PSTs with both pre- and post-evidence of sequential preference.

Table 4.6

Instance of Sequences of Instruction in Example Lesson: Pre and Post

Sequence of Instruction	No. of Ex. Lessons	
	Pre	Post
Traditional	11	18
Confirmatory	12	5
Exploratory	6	0
Total	29	23

These data may also provide evidence of a shift toward a more traditional sequence of instruction as no PST described an example lesson that incorporates an exploratory sequence at the end of the semester. Hypothesis tests for the difference between the pre- and post- proportions for each sequence were not performed because the samples were not random or independent and each sample does not contain at least 10 successes and 10 failures.

Additionally, interview data do not support a change in these PSTs' beliefs regarding sequence of instruction resulting from experiencing both sequences. When asked how this course has influenced how they plan to teach, none of the teachers interviewed mentioned sequence of instruction. Instead, most PSTs mentioned growing their repertoire of models and strategies as well as deepening their understandings of mathematics, not instructional sequence. For example one PST wrote,

It really opened my eyes because when I was in elementary school I had certain ways that I did things, but then I got to this class and found my way and I found more ways. So, it actually helped me understand what I did when I was younger and I feel like learning all the different ways is helpful because learning the way I

did it works for me, but it might not work for other people. So, learning all the different ways that we learn is helpful. I know one of the things we learned was Lattice Multiplication or division or something. I had never done that, but now it's really helpful, so I know learning all the different ways is going to be good for me when I am teaching because if a student doesn't get one thing I can try a different approach. (PST 3-11)

This PST, similar to others interviewed, found the course helpful in expanding her or his collection of mathematical strategies, but not because of the various sequences of instruction used.

Exploratory versus confirmatory sequences. After experiencing both exploratory and confirmatory sequences of instruction, 83 PSTs reported their preferred sequence as students of mathematics as either confirmatory, exploratory, or both. Similarly, PSTs were asked which of these sequences they planned to use in their own classroom. PSTs' responses are reported in Table 4.7 below.

Table 4.7

Preferred Sequence of Instruction as Students and Intended Sequence as Teachers

		Intended Seq. as Teachers			
		Both	Explore	Confirm	Total
Preferred Seq. as Students	Both	1	0	0	1
	Explore	5	1	1	7
	Confirm	25	0	50	75
	Total	31	1	51	83

Note. seq.=sequence

Learning preferences. From their perspectives as students, 75 of the 83 PSTs preferred a confirmatory sequence in this course, whereas only seven preferred an exploratory sequence and one preferred both equally.

Exploratory sequence. Considering the seven PSTs (8%) that preferred an exploratory sequence as students of mathematics, one PST wrote, "I liked being able to try and figure out on our own versus being told how to do everything" (PST 1-03). Another PST also found it "interesting to figure stuff out on my own" (PST 3-01). Still another PST reported that "the meaning was clearer when the instructor explained the purpose of the lab after because I could link it to previous knowledge" (PST 3-04). Similarly, another PST wrote, "I remember things better when I have to work them out for myself—instead of just being told how to do it" (PST 3-13).

Confirmatory sequence. Of the 75 PSTs (90%) that preferred a confirmatory sequence, ten preservice teachers viewed the laboratory tasks as a time to practice what they had learned. For example, one PST wrote, "this way the material we just learned could be enforced and practiced to help understand and apply the concepts" (PST 1-17). Additionally, four more PSTs preferred to apply the concepts from the lecture in the laboratory tasks. One instance of this follows: "If I had no clue how to complete the lab activity, I didn't learn anything anyways, so it was a waste. I was able to understand and apply much more easily when the explanation was first" (PST 1-02). Still four additional PSTs thought the purpose of laboratories was to reinforce the skills taught during lecture. For example, one PST wrote, "I felt like the information was being reinforced and I was not blindly going into it." Among these 18 PSTs, laboratories were seen as a time to confirm the mathematics from the lecture.

Five PSTs found the exploratory laboratories confusing and therefore frustrating. One such typical response follows: "I often confused myself when I didn't learn the correct material, and also became frustrated." Similarly, another PST preferred a confirmatory sequence because "I didn't feel as overwhelmed and confused when trying to complete it" (PST 1-15). Still other PSTs found exploratory laboratories "very discouraging" (PST 2-26). Similarly, another PST preferred a confirmatory laboratory sequence because "I felt much more confident when I did the lab. When the lab was first, it was easy to become frustrated when the material was unknown" (PST 2-04). This group of eight PSTs felt uncomfortable with the uncertainty of exploratory tasks and became confused and frustrated.

Other PSTs felt that exploratory tasks lacked direction. For instance, one PST preferred a confirmatory sequence because "you can understand the lab questions and have a specific goal of what you will be learning" (PST 3-15). Similarly, another PST explained, "The lab is pointless when you have absolutely no idea what you are looking for" (3-10). Additionally, interview data also illustrated that PSTs felt exploratory laboratories lacked direction:

Going through the lab was good, but I didn't really know [pause]. We went through it pretty quickly, and I couldn't really separate the things in my mind. Like, this is this kind of triangle and this type of circle, and this is where these intersect. I think we did the notes afterwards, and as she was going through the notes I was like separating them out and tried to connect it to the lab. But, that got really jumbled in my head. (PST 3-05)

Similarly, another PST explained, "If we knew why we were doing this or why it's important, we could understand it more" (PST 1-15).

Another PST explained a lack of direction when attempting computer-based laboratory prior to an explanation:

I felt like with the labs on the computer we were just doing it to like use the program, so it was weird because it was like we were learning the program, not what we were supposed to learn through the program. So, it was more like learning about Sketchpad instead of learning about geometry. . . . [In another laboratory that followed an explanation,] stuff made sense because we had talked about it and we knew what we were doing. We knew where the lines are going to

be and all about it. It was a lot easier and a lot better to understand it more because we knew why we were doing it. (PST 1-05)

Without a lecture directing the PST toward the purpose of the task, this PST focused more on completing the laboratory and using the software than on the underlying mathematics.

Although a few PSTs preferred to attempt to make sense of the mathematics before a formal explanation was given, the majority of PSTs preferred a confirmatory sequence because it reinforced that mathematics they learned, exploratory sequence were confusing and therefore frustrating, and an exploratory sequence lacked direction.

Student Preferences and sequence group. Table 4.8 illustrates the number of PSTs that preferred exploratory, confirmatory, or both sequences for both sequence groups. For the group that experienced a 10-week exploratory unit, six of the 25 PSTs (24%) preferred an exploratory sequence of instruction, whereas only one of the 58 PSTs (2%) that experienced a five-week exploratory unit preferred an exploratory sequence of instruction. Although more than 20% of the cells in Table 4.7 have an expected value less than five, a Pearson's chi-squared test was performed; however, the results should be taken cautiously. This test resulted in a chi-square test statistic of $\chi^2=11.53$ ($p=0.003$). With the limitations of the sample size in mind, there may be evidence that a greater proportion of PSTs in the longer exploratory unit preferred an exploratory sequence of instruction.

Table 4.8

Preferred Sequence of Instruction by Sequence Group

		Preferred Seq. as Students			
		Both	Explore	Confirm	Total
Sequence Group	1 (5-wk. explore)	1	1	56	58
	2 (10-wk. explore)	0	6	19	25
	Total	1	7	75	83

Note. seq.=sequence

Intended teaching sequence. When asked which sequence they plan to use in their own instruction, 50 PSTs planned to use a confirmatory sequence; 31 PSTs planned to use both sequences, and only 2 PSTs planned to use an exploratory sequence. To test the null hypothesis that PSTs' intended instructional sequence as teachers is independent of the preferred sequence as students, a chi-squared likelihood ratio was calculated resulting in $\chi^2=12.95$ ($p<0.01$). Therein, the evidence supports the conjecture that preferred learning sequence is related to intended teaching sequence. However, it should be noted that these results are suspect as the expected value of each cell in Table 4.8 is not greater than five.

Confirmatory Sequence. Of the 51 that intended to use confirmatory sequences, 50 also preferred confirmatory sequences as students with only one preferring an exploratory sequence. Qualitative analysis of PSTs' explanations of their intended instructional sequence support quantitative connections between student preferences and

teaching intention in that the same justifications for a confirmatory sequence also emerged from PSTs' rationale for their intended instructional sequence. In fact, three PSTs explicitly stated they planned to use a certain sequence, as one stated "because that was what worked best for me" (PST 2-22).

When examining PSTs' reasons for using a confirmatory sequence, again a view of laboratory tasks as a time to practice, apply, and reinforce emerged. Seven PSTs referenced laboratories as a time to practice similar to the following: "Practicing what you've learned is a good way to further understand the concepts" (PST 1-18). Three PSTs wrote of how confirmatory laboratories allow students to apply what they have learned: "The children can then apply what they've learned in the lab so I can be sure they understand" (PST 3-05). Additionally, four PSTs explained that confirmatory laboratories reinforce lecture similar to the following: "I feel giving the lab afterwards reinforces what has been taught" (PST 1-11). Therein, these 14 PSTs viewed laboratory activities to be a time for their future students to confirm the mathematics previously presented.

Still other PSTs found an exploratory sequence caused confusion, whereas a confirmatory laboratory "eliminates confusion and helps them understand what they are doing" (PST 3-12). By avoiding confusion, students would be less likely to "become frustrated and give up" (PST 1-31). One PST explained,

It will probably frustrate the children if I give them a worksheet and tell them to add if they have never done it before. The lab first then content was frustrating to me, and I was taught some of these math techniques in elementary school and middle school. I was just rusty." (PST 1-07)

Additionally, by eliminating confusion and resulting frustrations, "students feel confident about their work and are able to fully understand what they are doing" (PST 1-28). These seven PSTs planned to use confirmatory tasks because they believe their future students will be more comfortable with this sequence of instruction.

Both Sequences. Of the 31 PSTs that intended to use both confirmatory and exploratory sequences, 25 preferred a confirmatory sequence as students, five preferred an exploratory sequence, and only one preferred both sequences equally. There was much more variation in student preferences for PSTs that intend to use both sequences as compared to those PSTs that intend to use confirmatory sequences. Similarly, there was also a variety of rationales for using both sequences.

The most common reason for incorporating both sequences in their future classes was that students learn differently; therefore, teachers should differentiate their instruction. Nine PSTs intend to use both sequences for reasons similar to the following: "I plan to use both strategies because different students in my classroom may learn different ways, so I want to try to let them learn in the best way possible" (PST 2-19). These PSTs believed that all students do not necessarily learn as they do. For example, one PST planned to vary the sequence of instruction "because each student learns differently. Each student doesn't learn the same way I do." (PST 2-03)

Still other PSTs planned to vary their sequence of instruction depending on the content. For example one PST wrote, "I think that if a subject is building on something that has just been taught, then exploration through a lab is appropriate. If the content is brand new, content explanation should come first" (PST 1-30). Whereas some PSTs

planned to use different sequences depending on the cumulative nature of the content, other PSTs planned to alter the sequence depending on the difficulty of content. For example, one PST explained that a "more complicated lesson might need to be explained to the class first" (PST 3-01). These six PSTs planned to match their sequence of instruction to the specific content.

Another group of five PSTs planned to alter their instructional sequence to incorporate problem solving into their mathematics instruction. For example, one such PST explained, "I will use both in my classroom—it is important for students to develop problem solving skills" (PST 1-33). Similarly, another PST planned to alter the sequence of instruction to promote autonomy in problem solving: "I would plan to use explanation followed by labs most of the time, but it is also good to do labs first so kids can have a sense of independence and practice doing tasks without previous instruction" (PST 2-16).

Intended teaching sequence and sequence group. Table 4.9 illustrates the number of PSTs that planned to use an exploratory sequence, confirmatory sequence, or both sequences in their future classrooms for both sequence groups. Compared to distributions of preferences as students in Table 4.7, differences between sequence groups are less obvious for the intended instructional sequence distributions. Although more than 20% of the cells in Table 4.8 have an expected value less than five, Pearson's chi-squared test was performed; however, these results may be suspect due to the small sample size. This test resulted in a chi-squared test statistics of $\chi^2=3.26$ ($p=0.20$). Therefore, there is insufficient evidence to conclude that the distributions of intended sequence as teachers

differ between those PSTs that experienced a five-week exploratory unit and those that experienced a 10-week exploratory unit.

Table 4.9

Intended Instructional Sequence as Teachers by Sequence Group

		Intended Seq. as Teachers			
		Both	Explore	Confirm	Total
Sequence Group	1 (5-wk. explore)	20	0	38	58
	2 (10-wk. explore)	11	1	13	25
	Total	31	1	51	83

Note. seq.=sequence

Summary: Beliefs regarding sequence of instruction. In analyzing PSTs' intended instructional sequence from their descriptions of their future instruction, a traditional sequence of review, delivery of content, worked examples, and practice dominated. This sequence parallels PSTs' own experiences with traditional school mathematics and the framework for effective pedagogy described above. If PSTs did intend to include laboratory tasks, these tasks usually followed an explanation of content. Evidence of exploratory sequences was rare at the beginning of the semester and absent at the end of the semester. Beliefs either remained traditional or shifted towards a traditional sequence. Implications of these results are discussed further in Chapter Five.

When comparing confirmatory and exploratory sequences at the end of the semester, most PSTs preferred confirmatory sequences as students and intended to use

confirmatory sequences in their future instruction. Although there was limited variation in sequential preferences there was evidence that sequential preferences as students was associated with intended sequence as teachers. Additionally, there is evidence that a greater proportion of PSTs from the 10-week exploratory unit, as opposed to the 5-week exploratory unit, preferred an exploratory sequence of instruction as students. When examining PSTs' rationales for a confirmatory sequence both as students of mathematics and as future teachers, PSTs believed collaborative tasks should give students a time to practice, apply, and reinforce mathematics and should avoid confusion or frustration by first explaining the content. PSTs who planned to alter their sequence planned to do so in an attempt to differentiate their instruction because they believed that all students may not learn as they did.

Mathematical Knowledge of Teaching

Preservice teachers' Mathematical Knowledge for Teaching (MKT) was evaluated using Learning Mathematics for Teaching's measures in probability, data, and statistics for grades 4-8 teachers (LMT, 2008) and geometry for grades K-6 teachers (LMT, 2004) at the beginning of the semester. After completing a roughly five-week unit on data analysis and probability, PSTs were evaluated again on the same form of the probability, data, and statistics MKT assessment. At the end of the semester, after completing a roughly ten-week geometry unit, PSTs took a parallel form of the geometry content measure. Pre- and post-LMT means and standard deviations are reported by section for both content measures in Table 4.10 below.

Table 4.10

MKT Means by Section: Pre-, Post-, and Gains for Prob., data, and Stat. and Geometry

	Prob., data, and stat. means (SD)			Geometry means (SD)		
	Pre-	Post-	Gains	Pre-	Post-	Gains
Section 1	-0.10 (0.65) N=34	0.39 (0.58) N=32	0.45 (0.46) N=32	-0.49 (0.52) N=34	0.46 (0.60) N=29	0.90 (0.54) N=29
Section 2	-0.18 (0.57) N=32	0.04 (0.52) N=32	0.25 (0.51) N=31	-0.35 (0.49) N=32	0.16 (0.43) N=29	0.56 (0.46) N=29
Section 3	-0.13 (0.53) N=29	0.28 (0.52) N=27	0.39 (0.49) N=27	-0.23 (0.70) N=29	0.50 (0.69) N=26	0.76 (0.60) N=26
Overall	-0.13 (0.58) N=95	0.24 (0.56) N=91	0.36 (0.49) N=90	-0.36 (0.57) N=95	0.23 (0.55) N=91	0.73 (0.54) N=84

Notes. Item Response Theory mean scores (norm referenced on in-service elementary teachers) are reported with a mean of zero and standard deviation of one.

Further, confidence intervals for the means reported in Table 4.10 are reported in Table 4.11.

Table 4.11

95% Confidence Intervals by Section: Pre-, Post-, and Gains

	Prob., data, and stat. 95% CI						Geometry means 95% CI					
	Pre-		Post-		Gains		Pre-		Post-		Gains	
	lower	upper	lower	upper	lower	upper	lower	upper	lower	upper	lower	upper
Sec. 1	-.32	0.12	0.18	0.60	0.28	0.62	-.68	-.31	0.22	0.68	0.69	1.11
Sec. 2	-.39	0.03	-.14	0.23	0.07	0.43	-.53	-.18	0.00	0.33	0.38	0.73
Sec. 3	-.32	0.07	0.08	0.49	0.19	0.59	-.49	0.04	0.22	0.78	0.51	1.00
All	-.25	-.01	0.12	0.35	0.26	0.47	-.48	-.25	0.12	0.35	0.62	0.86

Notes. Sec.=section

Considering the 90 PSTs that took both the pre- and post-assessments in probability, data, and statistics, PSTs gained on average 0.36 SD units with a 95% confidence interval from 0.26 to 0.47. Because this 95% confidence interval does not

contain zero the null hypothesis that this course had no effect on PSTs' MKT in probability, data, and statistics was rejected. Therein, there is evidence that this course had a significant effect on PSTs' MKT in probability, data, and statistics. The publishers of this assessment (LMT, 2008) note effects of 0.25 to be noteworthy, 0.3 to be significant, 0.5 to be moderate, and 0.75 to be substantial. Therefore, these gains fall in the noteworthy to significant range.

Similarly, for the 84 PSTs that took both the pre- and post-assessments in geometry, PSTs gained an average of 0.73 standard deviations with a 95% confidence interval from 0.62 to 0.86 standard deviations. Again, this interval does not contain zero, so the null hypothesis of no course effect on MKT in geometry was rejected. The effect of this course PSTs' MKT in geometry was substantial.

Difference in the effect sizes of these two content areas should be noted and could be attributed to the amount of instructional time allocated to each content standard. The mean gains were roughly twice as large in the 10-week geometry unit as compared to the 5-week data analysis and probability unit.

Analysis of Covariance: Section. To test whether these MKT gains differed significantly between sections, an Analysis of Covariance (ANCOVA) was performed on MKT gains by section with initial MKT scores as a covariate. In addition to the assumption of an Analysis of Variance (ANOVA)—(1) independent, random samples, (2) normally distributed dependent variable, and (3) homogeneity of variance—an ANCOVA also assumes (4) a linear relationship between the covariate and dependent variable and (5) regression lines for each group are parallel (Hinke, Wiersma, & Jurs,

2003). Although assumption (1) was violated given the convenience samples used, assumption (2) was met given that Item Response Theory (IRT) scores are normal with a mean of zero and a standard deviation of 1. Levene's test for unequal variances produced a p-value of 0.67, so there was no evidence that assumption (3) was violated. Assumption (4) was tested by regressing preliminary MKT scores on MKT gains producing a t-ratio of -4.83 ($p < 0.0001$). To test assumption (5) a section by pre-MKT interaction term was included in the preliminary ANCOVA model. These interaction terms produced p-values of $p = 0.57$ and $p = 0.80$. Therefore, there was no evidence that the slopes differed by section.

After considering these assumptions, an ANCOVA was performed without the interaction terms using indicator variables for Section 1 and Section 2, which compared these sections to Section 3. With a p-value of 0.07, the mean gain in Section 1 was marginally different from Section 3 with an effect of 0.12 SD when controlled for by initial MKT score. With a p-value of 0.02, mean gains in Section 2 were significantly different from Section 3 with an effect of -0.15 SD when controlled for by initial MKT score. Therefore, there is evidence that gains in MKT in probability, data, and statistics differed by section.

Similarly, assumptions for the ANCOVA were considered for MKT in geometry. Levene's test for unequal variance produced a p-value of 0.67, so there was insufficient evidence to conclude that the assumption of equal variance was violated. Regressing preliminary geometry score on gains produced a t-ratio of -4.19 ($p < 0.0001$), so the assumption of linearity between the covariate and dependent variable was not violated.

Section by preliminary MKT score interaction terms produced p-values of 0.63 and 0.38, so there was no evidence of differences in the slopes among sections.

After these assumptions were examined, an ANCOVA was performed without these interaction terms. Again, dummy variables were introduced to compare differences in mean gains for Section 1 and Section 2 to that of Section 3. With a p-value of 0.09 and an effect of 0.13 SD, there was marginal evidence that mean gains for Section 1 differed from Section 3 when controlled for by initial MKT in geometry. With a p-value of 0.01 and an effect size of -0.19 SD, there was significant evidence that mean gains in Section 2 differ from Section 3 when controlled for initial MKT in geometry. Again, there is evidence that gains in MKT differ by section.

Analysis of Covariance: Sequence. To determine if these differences among sections could be attributed to differences in sequence of instruction, Sections 1 and 2 were combined as these sections experienced an exploratory sequence during the data analysis and probability unit and a confirmatory sequence during the geometry unit, whereas Section 3 experienced a confirmatory sequence in the first unit and an exploratory sequence in the second. Means, standards deviations, and confidence intervals for MKT scores are reported in Table 4.12 for sequence groups.

Table 4.12

Mean MKT Gains by Sequence: Prob., data, and Stat. and Geometry

	Prob., data, and stat. means (SD)			Geometry means (SD)		
	Mean (SD)	Lower 95%	Upper 95%	Mean (SD)	Lower 95%	Upper 95%
Sequence 1	0.35 (0.49) N=63	Explore 0.22	0.47	0.73 (0.53) N=58	Confirm 0.59	0.87
Sequence 2	0.39 (0.49) N=27	0.19	Confirm 0.59	0.76 (0.60) N=26	0.51	Explore 1.00

Notes.

With overlapping confidence intervals for the two sequence groups for both content units, there is no significant difference in gains by sequence group. When looking within a sequence group across units, one should note that both groups gained almost twice as much MKT in the geometry unit. One possible reason, although others are considered in Chapter Five, for the difference in these gains is that the geometry unit was roughly twice as long (10 weeks) as the probability unit (5 weeks). Because of the differences in instructional time allocated to these units, differences within a given sequence group across content units was not statistically tested.

To determine if there were differences in mean gains within a given content unit across sequence groups when controlled for by initial MKT score, an ANCOVA was performed. As with the section analysis, assumptions were first examined for MKT in probability, data, and statistics. Levene's test for unequal variances ($p=0.62$) did not show evidence that the assumption of equal variance was violated. As previously examined in the ANCOVA for section, preliminary MKT scores and MKT gains are assumed to be

linear. With a p-value of 0.63, there is insufficient evidence to conclude that sequence-group slopes are not parallel.

With these assumptions in mind, an ANCOVA was performed on MKT gains in probability, data, and statistics with preliminary MKT score as a covariate for the exploratory and confirmatory sequence groups. With a p-value of 0.62, there is insufficient evidence to conclude that mean gains in MKT scores differ between sequence groups when controlled for by initial MKT score.

Similarly, an ANCOVA was performed on differences in gains in MKT in geometry across sequence groups when controlled for by initial MKT score after the assumptions for an ANCOVA were considered. As before, Levene's test showed no evidence that the assumption of equal variances was violated ($p=0.56$). As in the analysis of ANCOVA for section, the relationship between initial MKT and gains in MKT for geometry are assumed to be linear. The t-ratio for sequence group by initial MKT interaction term ($p=0.61$) did not provide evidence that the assumption of equal sloped for sequence groups was violated.

After considering these assumptions, an ANCOVA was performed without this interaction term. Effects of sequence group on mean gains in MKT for geometry did not prove to be significant ($p=0.42$) when controlled for by initial MKT score.

Sequence preference and MKT. Next, differences in mean gains were explored for sequence preference groups by sequence. Because only one PST preferred both sequences equally, this PST was not considered in analysis. The distribution of PSTs' sequence preferences over the two sequence groups is reported in Table 4.8 above.

Differences in mean MKT gains were examined between preference groups for both content units using an ANCOVA with initial MKT score as a covariate. For the data analysis and probability unit in which the Sequence One group explored content and the Sequence Two group confirmed through laboratory activities, the assumptions of equal variance and parallel slopes were considered in addition to the assumptions previously examined in the ANCOVA for section. It should be noted that preference groups were constructed post hoc based on PSTs' interpretations of their experiences with both sequences. Therefore, these groups are neither random nor independent. Levene's test for unequal variances did not show evidence that the assumption of equal variances was violated ($p=0.89$). Additionally, the preference by preliminary MKT in probability, data, and statistics interaction was not significant ($p=0.26$), so there was insufficient evidence to conclude the assumption of parallel sloped was violated.

After examining these assumptions, ANCOVAs were performed without this interaction term for each sequence group. There was insufficient evidence to conclude that the mean gain for the six PSTs that preferred an exploratory sequence differed from the 18 PSTs that preferred a confirmatory sequence ($p=0.54$), when controlled for by initial MKT, during a confirmatory unit in data analysis and probability. Statistical tests were not performed on the Sequence One group because only one PST preferred an exploratory sequence. With a group size of one, there was no within-group variation, so the resulting ANCOVA is suspect.

Similarly, an ANCOVA was performed for mean gains in MKT in geometry with initial MKT score as a covariate for the two preference groups in both sequence groups.

Again, preference groups are not independent or random. In addition to the previously examined assumptions, there was insufficient evidence to conclude that the assumption of equal variance was violated ($p=0.80$). There was also inadequate evidence to conclude that the slopes differed between preference groups ($p=0.68$).

After considering these assumptions, an ANCOVA was performed without this interaction term. There was insufficient evidence ($p=0.88$) to conclude that the mean gains in MKT for the six PSTs that preferred an exploratory sequence differed significantly from the 20 PSTs that preferred a confirmatory sequence during a geometry unit that used an exploratory sequence of instruction. Again, statistical tests were not performed for Sequence One group due to limited sample size.

Summary: MKT. Mean gains in MKT were noteworthy to significant in probability, data, and statistics over the five-week data analysis and probability unit. Mean gains in MKT were substantial in geometry over the 10-week geometry unit. These mean gains differed among sections for both content units; however, section differences were not statistically attributed to differences in sequence of instruction. There was evidence that a greater proportion of students preferred an exploratory sequence of instruction that participated in the 10-week exploratory unit as compared to the 5-week exploratory unit. Nevertheless, PSTs' sequence preferences were not associated with differences in MKT gains during units that used various sequences of instruction.

Instructor's Perspective of Sequence of Instruction

In Chapter Two, theoretical bases for confirmatory and exploratory sequences of instruction are presented. The purpose of exploring the case of Ms. B, however, is to gain

insight into the practical advantages and disadvantages of these sequences of instruction from the perspective of an experienced instructor. Therein, this practical perspective may add viability to these theories or perhaps show instructional implications of these theories of learning to be realistically infeasible.

Rather than using a rigid protocol to examine specific beliefs related to sequence of instruction, data from classroom observations, field notes, interviews, and laboratory worksheets were used to explore the case of this experienced teacher implementing two sequences of instruction. Themes developed as data were collected and could then be examined further as data collection and analysis occurred simultaneously.

First, Ms. B's developing understanding of exploratory sequence of instruction is described. Then, assumptions related to differences in discourse are considered. The chapter ends with Ms. B's evolving perspective of sequence of instruction.

What does it mean to explore mathematics? I met with Ms. B several times before the study began to explain the purpose of the study and request Ms. B's participation. Although these preliminary meetings were not recorded, I explained exploratory and confirmatory sequences of instruction in a manner similar to the description given in Chapters One and Two. That is, in an exploratory sequence students explore the mathematical content through laboratory tasks before a formal explanation is given; in a confirmatory sequence, content is first delivered and then verified through laboratory tasks.

During the first laboratory of the semester, observational data revealed that students in the exploratory section were referring to their notes to answer various

questions, which was not congruent with my conception of an exploratory sequence.

During an interview that followed, inquiries were made into Ms. B's understanding of an exploratory sequence of instruction.

Ms. B explained that in all three sections she reviewed terminology related to probability and "talked about mutually exclusive and independent events, and then we started looking at two-step experiments in rolling two dice or flipping two coins." Ms. B went on to explain that she worked an example of a three-stage probability and illustrated this with a tree diagram. Because this is the content students were to "explore" in the lab, it became clear that Ms. B's understanding of exploring mathematics was different from mine.

When describing the laboratory activity for the exploratory group, Ms. B said, "I didn't tell them anything up front. I didn't remind them how we did the two dice. . . . I just let them get started." However, for the confirmatory sequence, she reviewed the content previously discussed, explained the laboratory, and worked a few examples from the laboratory showing students how to organize their data in a table.

For Ms. B, to explore mathematics was to apply previously learned concepts in a novel context. Because she had already worked examples related to two-stage probability, students were not generating strategies to complete the task; they were using what they had been shown in the previous lesson in a different context. Additionally, Ms. B understood the explanation to be specific to laboratory context. For example, she believed that she "didn't tell them anything up front" because she did not explain the specifics of this new context.

Because the fidelity of other aspects of this study depended on mutual understanding of an exploratory sequence, an exploratory sequence was explained again in the context of the current lesson. That is, students would generate their own strategies for solving these problems related to two-stage probabilities before the instructor ever discussed the content or worked an example. Because this concept of exploring mathematics was so foreign to Ms. B, she requested assistance in planning her next laboratory task.

In a separate planning meeting, I discussed how one might implement an exploratory sequence in the context of a laboratory in which students develop the idea of permutations while building trains from Cuisenaire rods. I demonstrated possible strategies for solving the problem without extensive prior knowledge of permutations and illustrated how the explanation that follows might connect to the task, therein drawing from students' experiences. After this planning meeting, Ms. B implemented an exploratory sequence of instruction aligned with the goals of the study.

When reflecting on this change in her understanding of exploring mathematics, Ms. B explained, "I think that it is just getting the feel for things. . . . There is a learning curve for me too." Before she could successfully implement an exploratory sequence of instruction, Ms. B had to first reconceptualize what it means to explore mathematics.

Discourse. One assumption made in Chapter Two is that the discourse generated in an exploratory laboratory sequence would be significantly different from that of a confirmatory sequence. It was assumed that during the exploratory tasks, focus would shift from reproducing the mathematics presented by the teacher to constructing

strategies to confront the task via rich mathematical discourse. Additionally, during whole-class discussions with an exploratory sequence, more of the explanation would come from the students as they now would have experiences to connect to the mathematics.

However, initial observational data do not support this assumption. Students of both sequences remained mostly passive during explanations of content, only asking questions of clarification during lectures. The discourse during exploratory laboratories was not significantly different either. During exploratory and confirmatory laboratories, numerous instances of PSTs raising their hands and waiting idly for the teacher to tell them how to confront a problem were identified in field notes. Even during exploratory laboratories, conversations centered on finding the "right" answer. When Ms. B was asked about differences in the discourse between sequences, she replied,

I haven't noticed a difference because usually they are pretty interactive anyway. I don't know, but I'd have to think about that—if I could tell that much of a difference between [sequences]. But, nothing jumps out at me. Like I said, I think the ones that are going to be more verbal and all that would do it both ways. I'll have to think about that one.

Here, Ms. B's perspective supported observational data that indicated similar discourse patterns between sequence groups in the first few weeks of the course.

Gradual change. During each of the four interviews, Ms. B was asked about any differences she noticed between the two sequence groups. During the first interview

during the second week of the semester, she did not see clear differences in students' learning across sequences. When asked if she noticed any differences, Ms. B replied,

Not a whole lot. . . . I think a couple of people in the second [exploratory] class were a little more frustrated. They weren't sure what to do. They did seem all very confused about that table and how to fill it out. They are pretty good about helping each other too. You know watching the other group. Not a huge difference.

To probe further into these differences, I brought up an instance I observed during a confirmatory laboratory during which a group wrote out the entire sample space, as modeled by the teacher, for a problem that seemed, to me, to be intuitive. Still, Ms. B did not see differences between sections and attributed these students' solution strategy to a lack of confidence:

Some of them did the table, but they knew [the answer] would be two-thirds of the time. They just needed to see that. They aren't confident. I think it was probably about half and half with the third [confirmatory] section too. Some knew it right off the bat. I didn't see a huge difference."

Ms. B viewed students' reliance on teacher-modeled procedures as an issue of confidence that was not confined to a particular sequence.

During our second interview approximately six weeks into the semester, Ms. B maintained her view that there was little difference in student learning between sequences. Ms. B gave the following example of how students in neither exploratory nor confirmatory sequence groups made connections between measures of center in a

distribution-center problem (found in Appendix J) for which students attempted to find an optimal location of a distribution center that would minimize the total distance between a number of stores:

They could come up with 16 being the best place to put the distribution center.

Then, when it came time to consider the mean, median, and mode, they did not all make the connection: Oh, that's the median. . . . It was probably half and half that came up with that. And, I didn't see too much difference in the third

[confirmatory] class despite the fact that we had talked about mean, median, and mode. I would still say about the same number of them made the connection. That surprised me that they were about the same as the first two sections.

Ms. B did not see a clear difference between students that were applying what they had learned about measures of center to a new context and those that were exploring properties of the median through this task.

By our third meeting 13 weeks into the semester and well into the geometry unit, Ms. B began to notice differences between students' learning across sequence groups, which she attributed to differences in learners' personalities:

A lot of this—it is personalities. There are certain kids that are more, um, prone to ask questions. You know. I think they would do it whether they were in the explore- or the explain-first group. I think certain personalities lean one way or the other. Um, it makes a difference because I have known these kids before. I had most of them last semester. . . . That one girl that fussed, she would be that way if she had been in the explain-first group. She is just that way period. She is

just very, you know, particular. Everything has to be spelled out: Here is how you do it.

At this point, Ms. B remained sequence neutral and attributed students' sequence preferences to their personalities.

However, she did begin to see differences in students' level of comfort with exploratory tasks. As she explained, "I probably see a little more—not in a bad way—but just a little more frustration with the explore first. They are a little—you know—but they don't get upset." Additionally, Ms. B noted differences in students' performance, as compared to previous semesters, but did not attribute these differences to sequence of instruction: "I think for the first time ever everybody is passing, which I don't know if I have ever had that. . . . I've always had one or two in [MEST II] that just can't handle it. I think maybe it was doing more labs or maybe it was the Sketchpad." Ms. B went on to explain how Geometer's Sketchpad®, not sequence of instruction, has helped improve student learning:

I loved it. . . . I do think it made an improvement as far as their understanding. . . . When you look at how they answer questions, it seemed to be a little bit better than I had gotten on labs in the past, as far as how they answer the why's. . . . I think they had a little better concept of some of the stuff, especially the constructions. I thought the constructions went a little bit better than doing it all with compass and straight edge, although we did both.

Although Ms. B noted differences in students' performance, she did not attribute these differences to changes in sequence of instruction. Instead, she noted how incorporating dynamic geometry software for the first time helped improve student learning.

By our final meeting at the end of the semester, Ms. B began to identify differences in the two sequences. However, she maintained her belief that some personalities were better suited for a particular sequence of instruction:

I think they have gotten a little more independent than I have seen in the past, some of them. A lot of it has to do with personalities. Some of them have gotten a lot better about thinking through it a little bit, where they have not done that so much in the past. But they got to that point by the end, especially with the geometry because they had it a longer period of time to be doing it on their own. Interestingly, Ms. B noted the time required for students to adjust to this new sequence. Aforementioned quantitative findings related to differences in the proportion of students that preferred an exploratory sequence in the 10-week unit support Ms. B's observation. Implications of these results are discussed further in Chapter Five.

Ms. B also noted that the instructional time devoted to explaining content was reduced in an exploratory sequence:

I think it has definitely cut down the explanation time if they do the lab first. Of course, I tend to be kind of long winded, but I have not done it as much. I don't feel like I have had to go into as much of an explanation. A lot of that is Sketchpad. You know, the labs themselves were in depth.

Here, Ms. B noted a practical advantage of an exploratory sequence. With an in-depth exploration of content, formal explanations could be shortened, saving valuable instructional time. This idea is also discussed further in Chapter Five.

When asked about her interpretation of students' perspectives of an exploratory laboratory sequence, Ms. B noted resistance from some students but went on to explain, I think they get more out of that than they realize. They think that they got frustrated and say "I didn't know what I was doing" or "I didn't understand where I was going." But I think they get more out of it than they would like to admit. I think it forces them to think about what is happening instead of me telling them. Ms. B gave the following example of an exploratory laboratory (found in Appendix K) in which students used Geometer's Sketchpad® to create tessellations from various transformations:

I think it forces them to try to process why that happened or why this is 60 degrees. Even in just doing that one [laboratory], they had to stop and think how many degrees does it make sense to rotate that. Then, when they had to rotate the whole figure, they had to really stop and think about it because I didn't tell them anything about how they would have to do that. . . . I think it was good they didn't have me saying you are going to do this and this and this. And, the lab didn't tell them. I think it forces them to be a little more thoughtful about it.

Here, Ms. B recognized that an exploratory laboratory sequence forced students to be more thoughtful as opposed to following procedures they had been given.

In addition to noting that students in the exploratory sequence had become more thoughtful, Ms. B also began to notice differences in the discourse between sequences. As mentioned above, Ms. B did not initially notice differences between the discourse generated through various sequences of instruction. Observational data also supported Ms. B's conclusion. By the end of the semester, however, Ms. B began to notice differences in student-to-student interactions and less of a reliance on the teacher for guidance. Ms. B explained,

The other thing I have seen more is more interaction between the groups. They have been two [students] here and three here but end up being all five of them, which is fine with me because they end up interacting with each other more instead of calling me over there, which is good. That is more in the explore section because they are kind of desperate, so they will start relying on each other and start listening and seeing what the group behind them is doing. That's part of it. It's kind of like "Well, if she is not going to tell us, then we will go to each other." I think they have done great. I've been really pleased with it. It has been interesting.

Although not evident at the beginning of the semester, Ms. B began to recognize differences in student discourse by the end of the semester. Perhaps, this could be attributed to changes in sociomathematical norms (Yackel & Cobb, 1996), which took time to construct. Implications for professional development are discussed further in Chapter Five.

Summary: Case of Ms. B. Initially, Ms. B had a different interpretation of what it meant to explore mathematics. At the beginning of the semester, she believed that students explored mathematics by applying previously learned concepts to a novel context. After I made my understanding of an exploratory sequence clear through planning sessions with Ms. B, she was able to implement an exploratory sequence of instruction consistently throughout the semester.

Ms. B's perspective on instructional sequence slowly changed over the course of the semester. At the beginning of the semester, she noted few differences between sequences. As the semester progressed, she identified differences in students' comfort levels and frustration with an exploratory sequence, but attributed these differences to students' personalities. She noted improvements in student learning, but did not attribute these changes to sequence of instruction. By the end of the semester, however, Ms. B recognized students in an exploratory sequence became more independent and thoughtful in their problem solving and student-to-student discourse began to change.

Revisiting Research Questions

Recall Research Question One: How do the lived experiences, as students of mathematics, of a group of preservice teachers influence their pedagogical beliefs? Almost all PSTs experienced either completely traditional mathematics or hands-on elementary years with a transition to traditional education in middle grades. Evidence of process standards in PSTs' recollections of school mathematics was scarce. Most PSTs also understood mathematics to be a body of knowledge or the act of performing rote procedures. Problem-solving perspectives of mathematics were rare. Additionally, most

PSTs interpreted effective pedagogy within the same traditional framework of review, delivery of content, worked examples, and practice. Beliefs regarding learning styles, group work, and process standards were also identified. Evidence is presented that these teachers intend to teach traditionally, as they were taught.

Research Question Two follows: How does experiencing exploratory and confirmatory sequences of instruction influence a group of preservice elementary teachers' pedagogical beliefs? In examining pre-existing beliefs regarding sequence of instruction, most PSTs did not plan to use collaborative tasks. Of those that did intend to incorporate tasks, a confirmatory sequence dominated. It appears that PSTs' beliefs regarding sequence of instruction either remained traditional or shifted towards a traditional sequence after experiencing both exploratory and confirmatory sequences. PSTs' intended instructional sequence as teachers was found to depend on their preferred sequence as students. Moreover, a greater proportion of PSTs that experienced a longer exploratory unit with seven exploratory laboratories also preferred exploratory sequences, as compared to the shorter exploratory unit with only three exploratory labs.

Recall Research Question Three: Is there evidence of a relationship between sequence of instruction and gains in preservice teachers' Mathematical Knowledge for Teaching as measured by *Learning Mathematics for Teaching* (LMT, 2004, 2008)? Although all PSTs made significant gains in MKT, these mean gains did not differ between sequence groups. However, differences were found between sections both within a single sequence group and across sequence groups. No section- or sequence-by-MKT

interaction effects were found. No sequence-dependent differences were found in MKT gains between PSTs that preferred an exploratory sequence and those that did not.

Research Question Four follows: What benefits and barriers does an experienced instructor perceive when implementing exploratory sequences of instruction? Initially, Ms. B had a different understanding of what it meant to explore mathematics than that of the researcher. At first, Ms. B saw few differences between sequence groups. As the semester continued, she began to notice differences in PSTs' frustration levels but attributed these to differences in personalities. By the end of the semester, Ms. B noted that explanation time could be shortened, student-to-student interactions improved, and PSTs were more thoughtful and independent in the exploratory sequence group. No barriers for implementing exploratory sequences were noted. These results, and others, are discussed further in Chapter Five.

CHAPTER FIVE

Conclusions

This study examined the influence of prior experiences and sequence of instruction on preservice teachers' (PSTs') pedagogical beliefs and Mathematical Knowledge for Teaching (MKT). In this chapter, results from the study are discussed as they relate to the four research questions presented in Chapter One. Then, limitations of this study and avenues for future research are considered. The chapter concludes with a discussion of how the current study contributes to the research base.

Research Question One

The discussion that follows is guided by the following research question: How do the lived experiences, as students of mathematics, of a group of preservice teachers influence their pedagogical beliefs?

Experiences with school mathematics. Two themes emerged when PSTs' recollections of their school experiences with mathematics were analyzed: traditional mathematics and a shift from concrete representations in elementary grades to traditional instruction in secondary grades.

Traditional school experiences, in which no evidence of the process standards was identified, were characterized by review, delivery of content, worked examples, and practice. This is not to say that these PSTs never encountered the process standards, but if PSTs did use these process standards, this reform-based instruction did not influence PSTs' interpretations of school mathematics as strongly as did traditional instruction. Therein, these findings are not a definitive evaluation of the state of the reform

movement, but these findings do illuminate aspects of PSTs' school experiences that influenced PSTs' perceptions of school mathematics.

In Chapter One, content courses for elementary teachers are established as a valuable place for future teachers to experience reform-based instruction as students of mathematics. One assumption made is that PSTs' experiences with school mathematics were mostly traditional. Further, Lortie's (1975) theory of an apprenticeship of observation maintains that teachers teach as they were taught, not as instructed. Therefore, a content course for elementary teachers can be a space where PSTs are apprentices of reform-based instruction. The finding that PSTs experienced mostly traditional instruction confirms this assumption and underscores the importance of these content courses as venues for introducing reform-congruent experiences learning mathematics.

Additionally, this finding marks a formidable challenge for teacher educators when preparing PSTs to depart from these traditional experiences in their own practice. With long-standing calls for reform for over three decades (NCTM, 1980, 1989, 1991, 2000), some teacher educators may assume that PSTs are familiar with reform-based instruction. However, this research presents evidence that such practices are far removed from PSTs' own experiences. Therefore, teacher educators should approach methods courses for elementary teachers with an understanding that exploring mathematics might be foreign to these future teachers.

The second theme that surfaced when examining these PSTs' experiences with school mathematics was a shift from concrete representations in elementary grades to

lecture in middle grades. This shift was accompanied by changes in PSTs' perceptions of the subject and their own mathematical abilities. This shift to abstract instruction is aligned with Piaget's age-dependent theory of development (1961, 1970, 1972, 1977, 1980) in which adolescents move from concrete to formal operational stages. In contrast, Bruner (1964, 1967) believed that learners of any age gain knowledge as they move from enactive to iconic and finally to symbolic representations.

From a Brunerian perspective, many PSTs experienced frustrations with secondary mathematics because the content was not made accessible through concrete representations. If one takes these PSTs' voiced frustrations as indicative of the state of secondary mathematics, then one can infer that concrete models should be used throughout PSTs' secondary experiences as well as their elementary years. The *Principles and Standards* (NCTM, 2000) support the Brunerian perspective as concrete representations are supported in grades nine through 12 as well as three through eight. From a Piagetian perspective, these learners had not yet reached a formal operational stage and changes to the middle-grades and high school curricula should be considered. Regardless of one's theoretical perspective, the fact that these shifts in representations were accompanied by shifts in PSTs' perspectives of the subject and their own mathematical prowess is problematic. This finding highlights the importance of integrating the representation standard at all grade levels with implications for secondary mathematics educators and teacher educators as well.

Nature of mathematics. A majority of PSTs viewed mathematics as a codified body of knowledge confined to a school context. This isolated, external view of

mathematics is not congruent with a constructivist epistemology (Wheatley, 1991; Fosnot, 2005; Shiftner, 2005) that views mathematics as an activity. Even among those few PSTs that saw mathematics as an activity, most viewed mathematics as the act of performing rote procedures, not the process of making, testing, and refining conjectures through non-routine problem solving. Research (Ernest, 1989; Beswick, 2007; Cross, 2009) has connected a similar static understanding of mathematics to a traditional role of the teacher as an explainer.

Findings of the current study support connections in the literature between the nature of mathematics and the role of the teacher. Overwhelmingly, PSTs viewed mathematics as an external collection of procedures, and most PSTs' interpretations of effective pedagogy were situated in a traditional explain-demonstrate-practice framework. Additionally, most PSTs' intended sequences of instruction were classified as traditional or confirmatory in which the teacher explains the mathematics.

When considering possible connections between experiences with school mathematics and PSTs' views of the nature of mathematics, recall that only three of 89 PSTs' descriptions of mathematics provided explicit evidence of a reform-congruent problem-solving perspective of mathematics. Similarly, evidence of process standards in PSTs' descriptions of school mathematics was rare. With such little variation in PSTs' experiences and understandings of the nature of mathematics, one can say that most PSTs recalled traditional experiences with school mathematics, and most also viewed mathematics as a body of knowledge consisting of procedures bound to a school context.

Perhaps the lack of connections to the world outside the classroom could be attributed to an understanding of mathematics confined to a school context.

Understanding preservice teachers' views of the nature of mathematics is important to teacher educators. Teacher educators should be aware of how rare problem-solving perspectives of mathematics are among future teachers. With a better understanding of the perspectives of their PSTs, teacher educators can prepare to challenge such static, procedural views of mathematics. Further, content courses for PSTs can allow PSTs to experience mathematics as the process of constructing, investigating, and refining conjectures through problem solving. Perhaps then a more comprehensive understanding of mathematics can be constructed.

Interpretations of effective pedagogy. One means of connecting PSTs' experiences to their pedagogical beliefs was to examine instructional practices they found particularly helpful and those practices they believed they did not benefit from. Just as their school experiences were defined by review, delivery of content, worked examples, and practice, their interpretations of effective pedagogy were situated in a similar traditional framework. It may not be surprising that a group of PSTs that experienced traditional school mathematics also interpreted effective pedagogy within this traditional framework. However, it is significant that these PSTs did not find traditional practices to be problematic.

For example, their critiques of teacher-worked examples were not that these instances of a procedure lacked context and did not generalize when confronting non-routine problems. Instead, they believed that worked examples should be accompanied by

a step-by-step explanation and followed by guided practice. The point to be made here is that even PSTs who struggled through school mathematics did not find this traditional pedagogy to be problematic. Of course, one would not expect a PST that had never experienced inquiry-based instruction, for example, to identify this as effective pedagogy. However, one might predict that PSTs who found school mathematics painfully difficult would be critical of these traditional experiences through which they struggled.

Consequently, it is significant that this group of PSTs has not yet begun to question traditional practices. It appears that traditional pedagogy has been normalized to the point that PSTs do not question its effectiveness. The challenge then for teacher educators is to show PSTs an alternative to traditional mathematics and encourage them to consider the effectiveness, or lack thereof, of traditional review, lecture, worked examples, and practice.

As mentioned above, despite numerous calls for reform in mathematics education (NCTM, 1980, 1989, 1991, 2000), these data do not show strong evidence that reform ideals have influenced PSTs' pedagogical beliefs. Interestingly, however, it does appear that theories regarding learning styles have managed to penetrate at least some PSTs' pedagogical beliefs. Specifically, some PSTs in this study believed that learning is optimized when instructional modes match students' sensory preferences; however, this hypothesis is not supported by empirical research (Pashler, McDaniel, Rohrer & Bjork, 2009). How then has this unsupported instructional theory permeated PSTs' pedagogical beliefs, whereas evidence of the widely supported process standards is rare?

In addition to beliefs regarding learning styles, strong yet varied beliefs were identified related to group work. Some PSTs believed they benefitted from collaborative tasks, whereas others strongly opposed group work. Beliefs regarding the role of the teacher were associated with both PSTs' support for and resistance to group work. That is, those who believed that group work hindered their learning believed that it is the teacher's job to explain, not that of peers; thus, group work is an excuse to escape from one's instructional duties. These beliefs are in direct opposition to those held by social constructivists (Cobb & Yackel, 1996) who believe learners co-construct meaning. Further, the communication standard (NCTM, 2000) and professional standards related to students' role in discourse (NCTM, 1991) support student-to-student interactions. Therefore, teacher educators should be aware that PSTs may hold reform-resistant beliefs and be prepared to challenge such beliefs in content and methods courses.

Research Question Two

The discussion below is guided by the following research question: How does experiencing exploratory and confirmatory sequences of instruction influence a group of preservice elementary teachers' pedagogical beliefs?

Traditional sequence. Another purpose of this research was to examine the influence of incorporating two sequences of group activities and explanations on a group of preservice teachers. It was hypothesized that experiences making sense of mathematics through exploratory tasks in this course might influence how these PSTs plan to sequence their own instruction. This focus on how tasks would be used assumed, perhaps naively, that PSTs would incorporate collaborative tasks similar to those they experienced in this

course. One unexpected finding was that almost half of PSTs' descriptions of their future instruction did not include collaborative tasks. Therein, this research's focus on how tasks might be sequenced is stifled if future teachers do not plan to incorporate such tasks.

This finding also highlights intentions of traditional practice consistent with PSTs' experiences with school mathematics and their interpretation of effective pedagogy. This may add to a body of research that supports an apprenticeship of observation (Lortie, 1975). As this study preceded methods courses for these PSTs, one cannot claim that PSTs ignored the guidance of teacher educators and imitated the instruction they observed as Lortie claimed. However, it is important for teacher educators to be aware that their PSTs may enter methods courses with intentions to replicate the traditional instructional practices they observed as students.

This finding also adds to evidence of the abovementioned resistance to group work identified in PSTs' interpretations of effective pedagogy. Therefore, the challenge for teacher educators is daunting. Before PSTs are convinced that their students can collaboratively make sense of mathematics through rich mathematical discourse, they must first be persuaded that their students should collaborate.

Confirmatory sequence. Of those preservice teachers that did plan to incorporate group tasks, most planned to pose the task after she or he delivered an explanation of content. Many of the example lessons that illustrated a confirmatory sequence began with symbolic representations such as formulas and ended with enacting mathematics through hands-on activities, a sequence inconsistent with Bruner's (1964, 1967) theory, which encouraged moving from the concrete to the abstract. PSTs preferred confirmatory

sequences, both as students and as future teachers, because they believed (1) an exploratory sequence lacked direction; (2) an exploratory sequence was uncomfortable; and (3) the purpose of group tasks was to practice, apply, and reinforce the mathematics.

Regarding the first, both Bruner (1967) and the 4E×2 Instructional Model (Marshall, Smart, & Horton, 2009; Marshall & Horton, 2009) encourage a sense of direction during student explorations. Bruner claimed that learners need a “sense of the goal of the task” (p. 44). Additionally, the 4E×2 Instructional Model includes an Engage phase during which direction is established. PSTs' critiques of exploratory sequences of instruction support this research and may provide evidence that more structure was needed during an exploratory sequence. That is, without the specific goal in mind these explorations may have been less structured and tended toward free discovery as opposed to guided discovery or inquiry-based instruction.

Second, PSTs also advocated for confirmatory sequences because they found exploratory sequences confusing and therefore frustrating. In these PSTs' critiques of an exploratory sequence, a belief emerged that mathematics instruction should be comfortable and confusion should be avoided. This belief may be in conflict with a constructivist epistemology that argues “Disequilibrium facilitates learning. ‘Errors’ need to be perceived as a result of learners’ conceptions, and therefore not minimized or avoided” (Fosnot & Perry, 2005, p 34). Many of these PSTs preferred confirmatory sequences because they found states of disequilibrium uncomfortable.

Moreover, PSTs' frustrations with exploratory sequences stress the importance of Bruner's (1971) concept of maintenance during discovery. Bruner suggested that teachers

provide guidance during discovery such that an “optimal level of uncertainty” (1971, p. 43) is maintained such that the consequences of exploring seemingly incorrect alternatives do not surpass the benefits of learning through discovery. It is unclear, however, whose "optimal level" of ambiguity to maintain, the student's or the teacher's. That is, were PSTs actually stretched beyond their capacity, or was sense making so new to them that their perceived threshold for uncertainty was minuscule?

Third, PSTs that advocated for confirmatory sequences believed that the purpose of collaborative tasks is to practice, apply, and reinforce concepts presented during lecture. One possible explanation for this is that PSTs were attempting to force laboratory tasks into their traditional framework of review, delivery of content, worked example, and practice. Therefore, individual practice was replaced by collaborative tasks where students practice, apply, and reinforce skills with their peers. Once again, a need for teacher educators to encourage PSTs to critically consider this traditional framework is highlighted.

Sequence preferences. After experiencing confirmatory and exploratory sequences of instruction, data were collected related to PSTs' sequence preference. A significantly larger proportion of PSTs preferred an exploratory sequence within a group that experienced a 10-week exploratory unit consisting of seven exploratory tasks as compared to a group that experienced a five-week exploratory unit consisting of three exploratory tasks. It should also be noted that sequence preferences in the current study differed significantly from the pilot (Sloop & Che, 2011) in which a single section, guided by a different instructor, experienced an exploratory sequence throughout both

units. In the pilot, 13 of the 31 PSTs (42%) preferred exploratory sequences, whereas only 1 of the 57 (2%) in the 5-week exploratory unit and six of the 26 (23%) in the 10-week unit preferred exploratory sequences.

One possible explanation for the differences in these proportions is that PSTs in the longer unit had more experiences exploring mathematics, and beliefs related to the value of exploring mathematics took time to develop. Perhaps PSTs in the shorter unit did not have enough time to establish the sociomathematical norms (Yackel & Cobb, 1996) required for rich explorations to take place. Results from the pilot study also support this hypothesis as these PSTs experienced an exploratory sequence the entire semester.

Moreover, this shorter unit occurred at the beginning of the semester when an exploratory sequence was new to Ms. B as well. Results from the case of Ms. B also support a "learning curve" for facilitating an exploratory sequence. Therein, the quality of these explorations may have increased over the semester as Ms. B also became more familiar with facilitating student explorations.

Additionally, both quantitative and qualitative analyses of data showed connections between sequence preferences as students and intended instructional sequence as teachers. These results again support a theory of an apprenticeship of observation and call attention to the value of content courses as a space for PSTs to be apprentices of reform-based instruction. Almost all PSTs that planned to use confirmatory (or exploratory) sequences in their own classrooms also preferred this

sequence as students. However, there was more variation in the student preferences of a group of PSTs that planned to incorporate both sequences.

Among PSTs that intended to use both sequences, a belief was identified that students' needs vary and instruction should be differentiated accordingly. Despite many of these future teachers' own confirmatory preferences, this subset of PSTs did not discount the value of an exploratory sequence and held a belief that all students might not learn as they believed they did. Although a minority belief, this is important because these PSTs illustrated pedagogical beliefs not bound by their interpretation of their own learning. This finding is significant for teacher educators who introduce reform-based pedagogy to PSTs in methods courses that is quite different from PSTs' experiences with school mathematics and interpretations of their own learning.

In addition to connections between sequence preferences and instructional intentions, possible changes in intended instructional sequence were considered. From pre- and post-example lessons, it appeared that PSTs' instructional intentions either remained traditional or shifted towards a traditional sequence of instruction. After experiencing both sequences, intentions of exploratory sequences were not found in any post-example lessons. This result might seem to contradict the finding discussed in the paragraph above; however, recall that preliminary data explicitly comparing the two sequences were not collected, as these teachers had not yet experienced such instruction. Insights into PSTs' initial beliefs regarding sequence of instruction were extracted from example lessons, and all example lessons did not uncover beliefs related to instructional sequence. This finding is consistent with suspicions that experiencing frustrations with

exploratory sequences might confirm reform-resistant beliefs, which surfaced during the pilot study (Sloop & Che, 2011).

Perhaps this shift was due to PSTs' abovementioned belief that mathematics instruction should be comfortable and a hypothesized low threshold of uncertainty for PSTs who were new to exploratory sequences. Therein, PSTs became frustrated with exploratory sequences during the content course and avoided such sequences in their post-example lessons. Or, perhaps some PSTs truly valued both sequences, as other forms of data show, but these PSTs lacked the pedagogical knowledge to design an example lesson that followed an exploratory sequence. Moreover, differences in modes of data collection for these pre- and post-example lessons should also be considered. Preliminary data were collected through mathematical autobiographies written outside of class with a week to consider the lesson. Post-example lessons were described in class through hand-written surveys. Perhaps, PSTs had more time to construct their preliminary example lessons; therefore, these preliminary descriptions may have been more thoroughly thought out.

Research Question Three

The discussion below is guided by the following research question: Is there evidence of a relationship between sequence of instruction and gains in preservice teachers' Mathematical Knowledge for Teaching as measured by *Learning Mathematics for Teaching* (LMT, 2004, 2008)?

In Chapter One, research is presented that claims that content courses for elementary teachers are not meeting the needs of future teachers (NCTQ, 2008) and that

MKT is rarely developed in such courses (Ball, Thames & Phelps, 2008). This study provides strong evidence to counter these claims. Even during a five-week unit, significant gains in MKT in probability, data, and statistics (LMT, 2008) were made. In the ten-week unit, a substantial effect size on MKT in geometry (LMT, 2004) was found.

Recall that the mean gains in the ten-week geometry unit were roughly twice as large as the gains in the five-week probability and statistics unit. One possible explanation for the differences in these gains is that more instructional time was devoted to the geometry unit, as compared to the probability and statistics unit. Nevertheless, there might also be a ceiling effect as initial mean geometry scores were almost a quarter of a standard deviation lower than mean pre-test scores in probability and statistics. One might further speculate that the higher initial probability and statistics scores might be attributed to content covered in a prerequisite course, Essential Mathematics for the Informed Society, which also addresses probability. Still, these differences in content gains could also be due to the specific content addressed or the means through which PSTs engaged in laboratory tasks. Perhaps, these large gains in geometry could be attributed to the use of dynamic geometry software in the laboratories.

When considering the effects of sequence of instruction on MKT gains, although differences in section means were significant, these differences could not be attributed to sequence of instruction. That is, means differed significantly between sections even within the same sequence group despite the fact that the means between sequence groups were not significantly different. Although the current study did not find that a particular sequence fostered greater gains in MKT, the fact that section means were different

provides evidence that further inquiries are needed. Because differences in the means were found within two sections of a single sequence group, these differences would most likely be attributed to the convenience sample used, as the instructor attempted to keep the instruction constant. If these convenience samples were, in fact, different, perhaps these sample differences masked the effect of sequence of instruction. Therefore, these section differences may provide evidence that further inquiries, with random assignments to sequence groups, are needed.

Previous research (Sloop & Che, 2011) found differences in mean gains in MKT during a similar course that used an exploratory sequence of instruction between those who preferred exploratory sequences and those that did not. Students who preferred an exploratory sequence had greater gains in MKT, on average, during an exploratory course. However, because these PSTs experienced only an exploratory sequence of instruction, comparisons could not be made regarding whether PSTs who were not satisfied with the exploratory sequence used in the course would have actually fared better when sequence of instruction aligned with their sequence preference. The current study, however, attempted to look further into this question by alternating sequence of instruction between units within a given sequence group. Still, in the current study, mean gains were not significantly different between those who preferred exploratory sequences and those who preferred confirmatory sequences in either unit. It should be noted, however, that the exploratory preference group was much smaller in the current study as compared to the pilot.

Research Question Four

This discussion of the case of Ms. B is guided by the following research question: What benefits and barriers does an experienced instructor perceive when implementing exploratory sequences of instruction?

Exploring mathematics. Initially, Ms. B's understanding of exploring mathematics was to apply previously learned content to a novel context. It is important to note this initial understanding of exploring mathematics because other teacher educators may also mistakenly assume a common understanding of language associated with mathematics education. For example, teachers might associate inquiry-based instruction with the use of group activities, or teachers might think that their students are engaging in problem solving as they work traditional practice "problems" in which students repeat previously modeled procedures in a well-defined problem space. As mathematics teacher educators, researchers, or members of any community with domain-specific language, it is important to clarify such concepts that may seem commonplace, even among other members of the same community. I falsely assumed that another mathematics educator would have the same understanding of exploring mathematics as I had.

Discourse. As stated in Chapters Two and Four, I also assumed that the discourse generated during an exploratory sequence would be richer than the discourse during a confirmatory sequence. However, differences in discourse between sequences were not a given. Therefore, even experienced teachers may benefit from professional development that focuses on the teacher's role in generating discourse. Further, professional

development specific to this new approach might be needed to help teachers generate meaningful discourse through exploratory laboratories.

Changes in discourse. By the end of the semester, Ms. B began to notice differences in the discourse between sequences. However, it took time to establish these sociomathematical norms (Yackel & Cobb, 1996) in the exploratory group. Further, Ms. B also noted a "learning curve" for implementing this new sequence. Because these norms related to exploring mathematics took time to construct, it might have been a bit ambitious to expect to find a sequence effect during a five-week exploratory unit. Other researchers should also consider the time needed to effectively implement a new instructional practice when designing a study. Additionally, educators and policy makers might also learn from Ms. B's experience as they evaluate the effectiveness of changes to instruction.

Explanation time. Although time was needed to establish exploratory norms, Ms. B noted that she saved valuable instructional time during her explanations if PSTs first explored the mathematics. As students were armed with in-depth experiences with the mathematics, Ms. B could shorten her explanations. This is significant as Bolden and Newton (2008) found that teachers believed limited time to be a barrier in implementing investigative teaching. Perhaps Ms. B's experience could help ease others' concerns as they might avoid student explorations because of time constraints.

Confidence in sense making. Another interesting finding was that Ms. B attributed PSTs' replications of teacher-modeled procedures to a lack of confidence. Ms. B believed that PSTs had an intuitive understanding of the mathematics and were able to

generate their own strategies but chose to reproduce the instructor's solution strategy because they lacked confidence. If Ms. B's conclusion is correct, it is important for these future teachers to become confident in their mathematical intuition because teaching mathematics conceptually is, in essence, making seemingly obtuse concepts appear intuitive (Fischbein, 1987).

Particular personalities. When Ms. B began to notice differences in PSTs' learning between sequences, she initially attributed these differences to PSTs' personalities. Ms. B described some PSTs as inquisitive and some as "very, you know, particular." Despite observations that students were performing better than in previous years and were more independent, she held a belief in a sort of sense-making orientation: Inquisitive personalities learn well through an exploratory sequence, whereas particular personalities need step-by-step instruction. However, by the end of the study, Ms. B believed that these particular personalities also benefitted from exploratory sequences (despite their opposition) because it forced them to think.

It is important to note the slow, gradual nature of Ms. B's change in beliefs regarding sequence of instruction. First, she saw no differences between sequences. Then, she found differences dependent on one's sense-making orientation. Finally, she began to believe that even particular personalities benefitted from exploring mathematics. It is worth considering whether Ms. B would have continued to use an exploratory sequence of instruction, despite resistance from particular personalities, long enough to see its benefits had she not committed to do so for this study. This emphasizes a need for

professional-development support to be sustained while reform-congruent beliefs are being developed.

It is also problematic that these particular personalities never saw the benefits of learning through exploratory sequences that Ms. B came to realize. Perhaps, it takes more than a single unit of exploring mathematics to compete with the 12-plus years of traditional mathematics. Therein, consistency is needed between the pedagogy PSTs experience in content courses and the instructional methods encouraged by mathematics teacher educators. Perhaps then, over several courses, change can begin to take root.

Limitations

One limitation of the current study was its quasi-experimental design. Without random assignment to sequence groups, differences in dependent variables, or the lack thereof, may be attributed to the convenience sample, not sequence group.

Another limitation was the study's two possibly contradictory focuses on both Ms. B's change and an intervention that required consistent implementation. Although both questions are important, perhaps the "learning curve" Ms. B experienced should have been avoided for the fidelity of the quasi-experimental portion of the study. As noted previously, it is possible (and probable) that the quality of the explorations improved as the semester progressed.

An additional limitation of this study was the difference in instructional time and number of exploratory laboratories for each unit. Originally, I planned to consider differences in mean gains in MKT across exploratory and confirmatory units within a given sequence group. However, it was noted that gains in the longer unit were roughly

twice as large for all sections. As mentioned earlier, it might have been a bit ambitious to attempt to find sequence effects in a brief five-week unit.

Further, considering the differences in the quantity, and perhaps the quality, of the exploratory tasks PSTs experienced between the two sequence groups, the means by which qualitative data were analyzed may also be limiting. PSTs' beliefs regarding sequence of instruction were analyzed collectively, as all PSTs experienced both exploratory and confirmatory sequences of instruction. However, because some participants experienced a shorter exploratory unit when Ms. B was new to this sequence, considering sequence group in the analysis might be beneficial. Perhaps the beliefs of those PSTs with more opportunities to explore mathematics differed from those with fewer experiences with an exploratory sequence.

Additionally, some post-hoc speculations that were made related to the differences in instructional time between these two units may also be limiting. For example, differences in the proportion of PSTs' sequence preferences between units was attributed to the time spent on each unit. However, these differences could have been due to differences in initial MKT, the particular content explored, or the dynamic geometry software used in the longer unit.

As noted in Chapter Three, both of Learning Mathematics for Teaching's assessments (LMT, 2004, 2008) claim to measure MKT despite the fact that the geometry scale does not draw from domains within pedagogical content knowledge, whereas the probability, data, and statistic scale does. Further, the geometry scale used does not make item-specific distinctions within the subject-matter domain. Therefore, these scales are

limited as to whether the two content assessments were actually measuring the same construct. Further, radical constructivists' ontological assumptions related to the existence of an external body of knowledge known as mathematics might be in conflict with an assessment that claims to measure one's knowledge of this domain. That is, if mathematics is an activity, not a singular, unified body of knowledge, how can a multiple-choice assessment claim to measure this, and whose mathematics does it measure?

Additionally, one of the major advantages of learning by discovery is that by engaging in non-routine problems one becomes a better problem solver. That is, one becomes more prepared to confront other novel, non-routine problems. However, it is doubtful that this multiple-choice assessment can measure, nor does it claim to measure, one's problem-solving abilities. Therefore, there may be an inconsistency in the hypothesized advantages of learning through an exploratory sequence and the measures on which these advantages were gauged. Still, with no loss in MKT in an exploratory sequence, the advantages identified by Ms. B may be worth the effort.

As with any qualitative research (and debatably quantitative research as well), the perceptions of the researcher influence the interpretation of data. Further, the fact that phenomena are being observed affects outcomes as well. Whether or not this is a limitation of the study or a given for any research (whether quantitative or qualitative) is left to the reader.

In the discussion of these data, comparisons were made to results of a pilot study (Sloop & Che, 2011). One limitation of conclusions drawn from comparing these two

studies was that different instructors facilitated the explorations in each study. For example, one speculation made was that a larger proportion of PSTs preferred exploratory sequences in the pilot because of the additional time spent exploring. However, these differences may have been due to differences in instructors. In fact, the instructor for the pilot had considerable experience with an exploratory sequence and inquiry-based professional development. This contrasts markedly with the "learning curve" that Ms. B experienced.

In considering PSTs' mathematical autobiographies (or perhaps any self-reported data), these were PSTs' interpretations of their recollections of experiences, not an objective account (if one believes in such a thing) of reality. This was both a limitation of these data and an advantage as these subjective interpretations of experiences can shed light on PSTs' perceptions of their lived experiences.

When attempting to identify connections between PSTs' lived experiences, beliefs regarding the nature of mathematics, and pedagogical beliefs, the lack of variation in these data was a limitation. A vast majority of PSTs described traditional experiences, understandings of mathematics, and pedagogical beliefs. Therein, the lack of variation in these data limited the analysis.

In considering the theory of an apprenticeship of observation (Lortie, 1970), this study was also limited in that one can only speculate on PSTs' instructional practices based on their pedagogical intentions. Further, this study took place before methods courses, so one cannot conclude that PSTs will ignore the guidance of teacher educators in methods courses to follow their experiences as students, as Lortie proposed. This study

can only conclude that there was evidence that PSTs intended to teach as they were taught before they entered methods courses.

Data used to consider changes in PSTs' beliefs regarding sequence of instruction were also limited. Intended instructional sequence was extracted from example lessons. However, all example lessons did not offer evidence of an intended sequence. Further, aforementioned differences in the modes of data collection, whether in-class or out-of-class, may have also influenced these data. Consequently, pre- and post-data were available for only 11 PSTs, with fatigue factors most likely affecting the quality of post-example lessons. Therefore, conclusions made regarding possible changes in PSTs' beliefs were also limited by these data.

Future Research

Speculations of instructional practice were made in the current study based on intended instructional sequence. Further inquiries are needed to determine whether intended instructional sequence is actually enacted in PSTs' future classrooms. Therefore, future research should examine the relationship between sequence preference, intended instruction sequence, and practice for in-service teachers.

The current study found differences in MKT gains between sections within an instructional sequence. It was hypothesized that these section differences provided evidence that further research is needed to determine if these differences were due to the convenience sample used. Therefore, future research is needed to examine the effects of sequence of instruction of MKT with random assignment to sequence group.

The effects on MKT attributed to this laboratory-based content course add to the literature that attempts to improve teachers' MKT through teacher preparation programs (Lueke, 2008). These findings might also have significant implications for professional development for in-service teachers, as Schifter (2005) has called for teachers to experience making sense of mathematics as students. Perhaps in-service teachers can also gain MKT through task-based professional development. Further research is needed to examine the effects of similar task-based professional development opportunities on in-service teachers.

This study found considerable gains in MKT for PSTs during this content course. The long-term effects of content courses on in-service teachers' MKT vary considerably in the literature (Hill, 2007, 2010). Further longitudinal research is needed to determine the long-term effects of gains in MKT made in content courses for elementary teachers.

An unexpected and interesting finding of this study was that unsupported theories of learning styles have permeated at least some PSTs' pedagogical beliefs, whereas reform-congruent beliefs supported by research have only marginally influenced beliefs. Further inquiries are needed to evaluate the origins of these learning-style beliefs.

This study did not consider how participants' gender might influence their beliefs regarding sequence of instruction or MKT. Future research is needed to examine gender-specific differences in preservice elementary teachers' prior experiences with mathematics, pedagogical beliefs, and MKT.

In Chapter Two, shortcomings for quantitative measures of pedagogical beliefs are noted. This study attempted to assess beliefs regarding sequence of instruction

through example lessons. However, limitations of this method are noted above. If inquiries into beliefs regarding sequence of instruction are to continue, more consistent methods for evaluating these beliefs are needed. Therefore, future methodological research should explore other means for assessing beliefs regarding sequence of instruction.

Relationship to Previous Research

Much of the previous research regarding teachers' beliefs either focused on in-service teachers or PSTs involved in field experiences or methods courses under the guidance of mathematics teacher educators. Therein, these studies examined teachers' pedagogical beliefs as current or apprentice teachers. The current study, however, examined PSTs' beliefs from their perspectives as students of mathematics. By looking back at PSTs' experiences with school mathematics and their resulting interpretations of pedagogy, this study adds to the research base as it explored the beliefs PSTs hold prior to the influence of mathematics teacher education.

As noted in Chapter Two, prior research on mathematics content courses for PSTs did not focus on targetable instructional components. The current study, however, examined two well-defined sequences of instruction that are both grounded by theories of learning and instruction. This study has more explicit instructional implications than previous research and, therein, adds a practical perspective to this research base.

MKT has been established as a valuable construct in the literature as it has been connected to teachers' practice and students' achievement. However, research that explores how and where this knowledge is constructed was lacking. Although there was

evidence that previous coursework might be linked to MKT, the effect sizes varied greatly between studies. The current study, however, examined the effect of a content course for PSTs directly for specific content domains. Additionally, this study provided evidence to refute claims in the literature that content courses for elementary teachers are not providing the specialized mathematical knowledge teachers need.

This study also adds to the literature by offering the practical perspective of an experienced instructor who is new to facilitating student explorations. Although previous research identified beliefs that stifled the effects of teacher education efforts, this study found evidence for optimism for reform as Ms. B's practice changed, and she began to see the benefits of allowing her students to explore mathematics.

Final Thoughts

This study supports NCTM's (2000) claim that teaching truly is a complex endeavor. Although sequence of instruction is a targetable instructional component, simply altering one's instructional sequence does not automatically guarantee dramatic changes in student learning. Time is needed as the instructor and students construct new norms for this new sequence. This study shows that optimism in the power of reform is warranted. Among PSTs in an exploratory sequence group, Ms. B identified improvements in student-to-student discourse and noted these PSTs were more thoughtful and independent. These benefits came with no drop off in MKT gains. These gradual changes in Ms. B's perception are also promising. Therein, this study has proven beneficial to the participants themselves, and its findings are also valuable to mathematics teacher educators. This study shows that mathematics content courses can

be a valuable space for PSTs to gain MKT while also experiencing as student what it means to explore mathematics.

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APPENDICES

Appendix A

Mathematical Autobiography Prompt Given to all PSTs

MATHEMATICS AUTOBIOGRAPHY

Please answer the following questions thoughtfully and send your autobiographies to me electronically ([email address removed]) by Friday, August 20.

- What is mathematics?
- Describe a typical day in math class in elementary, middle, high school and college as you remember it.
- What strategies did you find particularly helpful and which did not benefit you?
- Describe one positive and one negative experience with mathematics and comment on your mathematical prowess.
- What will a typical day in your mathematics class look like?
- Describe how you might teach your class about the area of a rectangle or a lesson on probability.

Appendix B

Grading Rubric for Mathematical Autobiographies given to all PSTs

	0	1	2	3	Score
Definition	A definition of math was not addressed or was taken from another source with no reference provided.	Mathematics was ill-defined or defined recursively (i.e. what we learn in math class)	Mathematics was defined, but the definition lacked enough detail to help the reader discriminate the content area from others.	Mathematics was defined completely using precise language such that a reader could differentiate math from other sciences.	
Typical Day	Typical mathematics experiences were not addressed.	Autobiography addressed only 1 of the 3 school experiences.	Autobiography only addressed 2 of the 3 school experiences.	A typical day in elementary, middle, and high school was described, or it was noted that the author could not recall these events.	
Strategies	Neither helpful nor unbeneficial strategies were addressed.	Autobiography addressed only 1 aspect of learning strategies.	Autobiography addressed both helpful and unbeneficial strategies, but said strategies were ill-defined or vaguely explained.	Autobiography addressed both helpful and unbeneficial strategies for learning mathematics with sufficient detail to reproduce such strategies.	
Experiences	Neither positive nor negative experiences were addressed.	Autobiography only addressed a positive or negative experience, not both.	Autobiography described a positive and negative experience, but it was unclear why these experiences were interpreted as such.	Autobiography explained both a positive and negative experience with sufficient detail.	
Future Class	A typical day was not addressed.	A typical day in math class was only superficially addressed with no detail given to what students and/or teacher are doing.	A typical day was described, but vague language or lack of detail makes it difficult to distinguish the class described.	A typical day in one's future class was described with enough detail that the reader could identify the class being described.	

Example Lesson	An example lesson was not mentioned in the autobiography.	The example lesson described had no identifiable characteristics.	An example lesson was described, but vague language or lack of detail makes it difficult to identify this lesson among others on the same topic.	An example lesson was described with enough detail to distinguish this lesson from others on the same topic.	
On time	Autobiography was late.			Autobiography was submitted by August 20.	

Appendix C

Instructor Interview Protocol: Interview One

August 23, 2010

1. Can you tell me about what you did the two days before today?
2. Was that for all three sections?
3. Have all the students seen intersection before?
4. Can you tell me a little bit about the explore-first section?
5. Did you notice a difference in how students were performing or interacting?
6. For the first problem that is kind of intuitive because one die is all threes, one could look at Die A and see that four of the six times it would win. I noticed, with the group in front of me, that students were drawing out the entire sample space. Did you see that in the first two sections in which you did not explain the content first?
7. I recall you telling them to think about shortcuts. One group realized that the last problem would take three charts, and they wanted to know that shortcut. Can you tell me how you envision the explanation coming in those first two sections?
8. When I taught this lesson with an exploratory sequence of instruction, I noticed that I would get a lot of different strategies, whereas if I had explained it first, they might just do it the way I did. Did you see a difference?

Appendix D

Instructor Interview Protocol: Interview Two

September 28, 2010

1. Can you tell me about the labs you have done in the first unit? Which ones you were able to make exploratory for the first two sections?
2. Have you noticed any differences between sequence groups?
3. When I asked my class why, some got there and others didn't, especially with the even number of distribution centers. How did your students do with this?
4. How about time? Do you find that it takes more time?
5. Do you think you get that time back with being able to shorten your explanation?
6. Have you noticed any differences in the classroom discourse among sections?

Appendix E

Instructor Interview Protocol: Interview Three

November 17, 2010

1. Can you tell me about the labs you have done thus far in this unit and which labs you were able to alter the sequence?
2. For the sketchpad labs, what do think about it? Is Sketchpad something you would continue to use?
3. What do you think students think about it?
4. Have you noticed any differences between sequence groups?
5. Can you give me an example of this?

Appendix F

Instructor Interview Protocol: Interview Four

December 1, 2010

1. Do you have a rough guess how many labs you did in the geometry unit and how many you were able to alter the sequence?
2. Have you seen any difference in the two sequences?
3. Have you seen a difference in the discourse, the interaction between you and the students, from when they have had the labs first or the labs second?
4. Right now, which sequence would you lean toward?
5. Can you expand a little bit on how you think they are getting more out of it?
6. What is it about doing the lab first?

Appendix G

PST Interview Protocol

1. Tell me about your experience as a student of math.
2. If you had to explain to one of your students what math is, what would you tell her?
3. Do you think this course has influenced how you plan to teach mathematics? If so, how?
4. Do you remember doing activities similar to these labs as a student? If so, did these labs usually come before or after the teacher explained the content?
5. In this course, did you notice any differences in how your teacher sequenced lab activities and explanations between the probability and geometry units?
6. Which sequence did you prefer, and why? Can you give me an example of a lab that illustrates this?
7. How do you think you will use lab activities in your own classroom?

Appendix H

Exit Survey Given to All PSTs

Your instructor sometimes explained the content first; then, the lab on this content came after the explanation. Other times, you worked on the lab first, and the explanation came afterward. Which sequence of lab activity and explanation did you prefer? Explain.

In your own classroom, which sequence of activity and explanation do you plan to use. Explain.

Briefly describe how you might teach your class a lesson on either the area of a rectangle or probability.

Appendix I

Phenomenological Reduction of PSTs' Interpretations of Effective Pedagogy

Meaning Cluster (no. sig. state.)	Meaning Unit	No. of Sig. State.	
		Beneficial	Unhelpful
Traditional mathematics experiences			
Review (5)	Review	2	
	Review of homework	3	
Delivery of content (32)	Lecture	2	2
	Lecture alone ^a		12
	Lecture followed by practice ^b	3	
	Boring lectures		1
	Notes	6	4
	Notes from text		1
	Explicit algorithm given	1	
Examples (27)	Teacher-worked examples	6	
	Examples	8	
	Step-by-step examples	7	
	Example with explanation	2	
	Single example		1
	Algorithm without example		1
	Teacher-worked examples followed by guided practice ^b	2	
Practice (28)	Teacher-worked examples followed by guided practice ^b	2	
	Practice	12	
	Group work: practice ^b	3	
	Too much practice		1
	Repetition	7	
	Lecture followed by practice ^b	3	

Unassociated meaning clusters			
Learning styles (20)	Visual learner	7	
	Hands-on as visual ^b	7	
	Hands-on as kinesthetic ^b	3	
Group work (21)	Only auditory style		4
	Group work: no rationale	4	4
	Group work: comfort	2	1
	Group work: teacher's role		1
	Group work: communication ^b	3	
	Group work: practice ^b	3	
	Group work: left out		1
Process standards (31)	Group work: perspective	2	
	Connection to experience	6	
	Connection to other subjects	1	
	Group work: communication ^b	3	
	Hands-on as visual ^b	7	
Assessments (7)	Hands-on as kinesthetic ^b	3	
	Hands-on activities	11	
	Graded homework		2
	Quizzes		1
	Timed tests		1
Unassociated meaning units	Projects		1
	Formal proofs		1
	Same structure		1
	Edibles	1	
	Individual instruction	1	
	Memorization		1
	Games	2	
	Multiple strategies	4	1
	Tricks or shortcuts	3	
	Active participation	2	
	Activities	2	1
	Formulas		1
	Reading textbook		4
	Graphic organizers	2	

Note. No. of Sig. State. = the number of significant statements. ^aThe meaning unit "Lecture alone" was assigned to significant statements that reference lecture the entire period or specific references to lecture without support from examples or guided practice. ^bThis meaning unit was associated with more than one meaning cluster.

Appendix J

Distribution Center Lab

Engage

A fast food chain sets up five restaurants along a highway at mile markers 2, 4, 16, 28, and 50. The owner of the chain needs a distribution center to service the restaurants. She will have five trucks, one for each of the restaurants. Of course, she wants to save money, so she wants the total number of miles the trucks will travel to be as small as possible. Before beginning any computations, estimate a reasonable location for the 5th store, and devise a plan to systematically determine the best location of the distribution center.

Should you consider that round-trip distance traveled by the trucks, or only the distance of each restaurant from the distribution center? Why?

Explore

Where should she locate the distribution center? Show your calculations below.

Organize your data and represent it below using either a table or graph.

Explain

How does your answer relate to measures of center: mean, median, and modes?

Choose 5 new restaurant locations, and determine if the measure of center identified above also minimizes the total distance between the locations.

Explain why the measure of center above minimizes the total distance between the stores.

Extend

Suppose a 6th store was added at mile marker 80. Where should the distribution center be located now?

Explain why this is the case.

Suppose, instead, the 6th restaurant was located at mile marker 525. Where should the distribution center be located? What conclusion can you draw from this result regarding this measure of center and extreme values in the data set?

Appendix K
Rotation and Reflection Lab

Adapted from Sketchpad's Learning Center

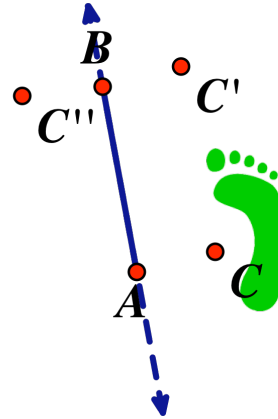
1. Open a new sketch, and save this as Lab-18.gsp. Choose **File | Document Options** and name this page *Hop*.
2. Choose **Help | Picture Gallery**. Click the **Transformation** link and find the picture of a footprint. Drag and drop (or copy and paste) the footprint picture into your sketch.
3. Construct a vertical segment to the left of the footprint. (Hold the Shift key as you construct your segment to make it vertical.)
4. Label the endpoint A and B with Point A being below Point B.
5. Select A and B in order and choose **Transform | Mark Vector**. A brief animation should appear.
6. Select the picture and choose **Transform | Translate**. In the dialogue box, click **Translate**.
7. Leave the footprint selected and translate it again. Repeat this a few times to make a series of footprints.
8. Drag each endpoint of Line AB and observe how the footprints behave.
What two things are needed to define a translation?
Suppose you were at the beach. How could you create this pattern in the sand?
9. Choose the **Line** tool by pressing and holding the **Segment** tool until the straightedge menu appears, move the cursor to the line icon, and release.
10. Click points A and B to construct Segment AB.
Leave the line selected and choose **Display | Line Style | Dashed**.
11. Double-click the line to mark it as a mirror.
12. Select the pictures (click each one or use a selection rectangle) and choose **Transform | Reflection**.



13. Drag the mirror line and observe the behavior of the reflected footprints.
 What is needed to define a reflection?
 Suppose you were at the beach. How might you use create this pattern in the sand?

14. Choose **File | Document Options**. Then, choose **Add Page | Duplicate | Hop**. Name this page *Walk* and click OK.

15. On the *Walk* page, select all pictures except the original picture and choose **Edit | Clear Pictures**.



16. Construct a point to the left of Segment AB near the footprint. Label it C.

17. Select point C, and translate it as in Step 5 and 6. Label this point C'.

18. Reflect point C' across Segment AB as in Steps 11 and 12. Label this point C''.

19. Deselect all objects. Select only points C and C'' and choose **Transform | Define Custom Transform**. Name your new transformation *Glide Reflect* and click OK.

20. Select points C, C', and C'' and choose **Display | Hide Points**.

21. Select the picture and choose **Transform | Glide Reflect**.

22. Leave the new picture selected and glide-reflect it again. Repeat many times (or use the keyboard shortcut shown next to *Glide Reflect* in the Transform menu) to make a series of footprints.



23. Drag points A and B and AB and observe how the footprints behave

What is needed to define a glide reflection?

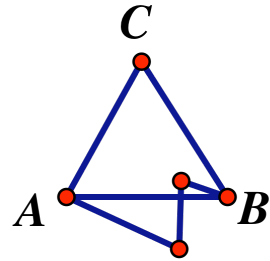
Suppose you were at the beach. How could you create this pattern?

24. Choose **File | Document Options**. Add a blank page and name it *Tessellation*.

25. Construct an equilateral triangle and label it ABC. (If you can't remember how we construct an equilateral triangle, see Lab 10 and review the linked video on the page *Equilateral Triangle*.)

26. Construct two or three connected segments from A to B.

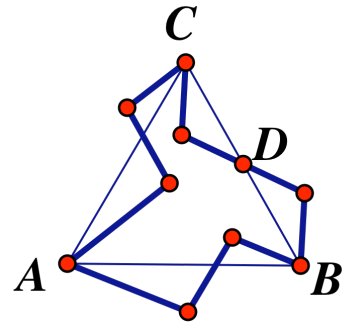
27. Mark point A as a center by double-clicking it (an animation should appear). Then, select all the segments and points you just constructed, and rotate them 60° by choosing **Transform | Rotate**.



28. Construct the midpoint of Line CB and label it D.

29. Construct two connected segments from B to D.

30. Mark D as the center of rotation, and rotate the points and segments you constructed in Step 29 180° .



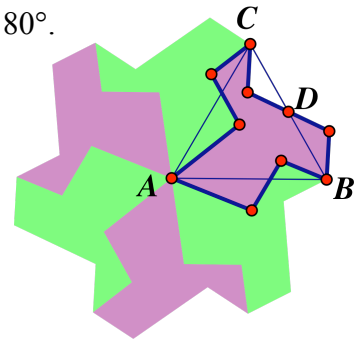
31. Drag the points to see how they behave. Make sure none of the irregular edges intersect.

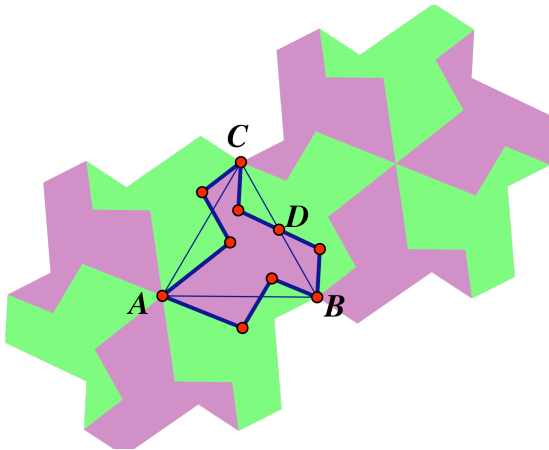
32. Construct the polygon's interior with the vertices along the irregular edges.

33. To begin tessellating, mark point A as a center and rotate the polygon's interior 5 times by the appropriate number of degrees to surround point A with 6 non-overlapping tiles. Change the color of alternating tiles.

34. Mark D as the center of rotation, and rotate all 6 tiles 180° .

35. Change the color of the new tiles to keep a clear pattern.





Look at the tiles surrounding point A. What kind of rotation would the completed tessellation have about this point?

Look at the tiles surrounding point D. What kind of rotation would the completed tessellation have about this point?

Look at the tiles surrounding point B and C. What kind of rotation would the completed tessellation have about these points?

36. Use an appropriate rotation to fill in the tiles around B and C. Then, adjust the colors of these tiles accordingly.
37. Drag the vertices of the original tile and observe the changes in you tessellation.

Appendix L

Content of PSTs Pre- and Post-Example Lessons

Sequence Group 1						Sequence Group Two		
Section 1			Section 2			Section 3		
Content of Lesson			Content of Lesson			Content of Lesson		
PST ID	Pre	Post	PST ID	Pre	Post	PST-ID	Pre	Post
1-01	--	Geo	2-01	Prob	Prob	3-01	Geo	Geo
1-02	Prob	Geo	2-02	Geo	Geo	3-02	Prob	--
1-03	Prob	Prob	2-03	Geo	--	3-03	Geo	Geo
1-04	Geo	Prob	2-04	Prob	Prob	3-04	Geo	Geo
1-05	Geo	Geo	2-05	Prob	Prob	3-05	Prob	Geo
1-06	Prob	--	2-06	Prob	Prob	3-06	--	Geo
1-07	Geo	Geo	2-07	--	--	3-07	Geo	Geo
1-08	--	Geo	2-08	Geo	Geo	3-08	Geo	--
1-09	Geo	Prob	2-09	--	Prob	3-09	Geo	Geo
1-10	Geo	Prob	2-10	Prob	Prob	3-10	Prob	Prob
1-11	Geo	Geo	2-11	Prob	Prob	3-11	Prob	Geo
1-12	Prob	Geo	2-12	Geo	Geo	3-12	Geo	Geo
1-13	Prob	Geo	2-13	Prob	Geo	3-13	Prob	Geo
1-14	--	Geo	2-14	Geo	Prob	3-14	Prob	Geo
1-15	Prob	Geo	2-15	Geo	Geo	3-15	Geo	Geo
1-16	Prob	Geo	2-16	Geo	Prob	3-16	Geo	Prob
1-17	Prob	--	2-17	Geo	--	3-17	Prob	Prob
1-18	Geo	Geo	2-18	Prob	--	3-18	Geo	Geo
1-19	Geo	Geo	2-19	Prob	Prob	3-19	Geo	Geo
1-20	Geo	--	2-20	Prob	Geo	3-20	Prob	Geo
1-21	Prob	Geo	2-21	Prob	--	3-21	Prob	Geo
1-22	--	--	2-22	Prob	Geo	3-22	Geo	Geo
1-23	--	Prob	2-23	Geo	Geo	3-23	Prob	Prob
1-24	Prob	Prob	2-24	Geo	Geo	3-24	Prob	Geo
1-25	Prob	Prob	2-25	Prob	Geo	3-25	--	Prob
1-26	Geo	Geo	2-26	Geo	Geo	3-26	Geo	Geo
1-27	Geo	Geo	2-27	Geo	Geo	3-27	Prob	Prob
1-28	Geo	Geo	2-28	--	Prob	3-28	Prob	Geo
1-29	Prob	Geo	2-29	Prob	--	3-29	Geo	Prob
1-30	Geo	Geo	2-30	Prob	Geo			
1-31	Prob	Geo	2-31	Geo	--			
1-32	Prob	--	2-32	--	Prob			
1-33	Prob	Geo	2-33	Geo	Geo			
1-34	Geo	Geo						
1-35	Prob	--						

Notes. ID=identification number; Geo=Geometry; Prob=Probability

