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Intersections and Representations of Graphs

John Light
Clemson University, bolight@alum.mit.edu

Intersections and Representations of Graphs

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J. Bowman Light
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Accepted by:
Dr. Neil J. Calkin, Committee Chair
Dr. Robert E. Jamison, Committee Chair
Dr. Hiren Maharaj
Dr. Gretchen L. Matthews

Abstract

Given two graphs G and H sharing the same vertex set, the *edge-intersection spectrum* of G and H is the set of possible sizes of the intersection of the edge sets of both graphs. For example, the spectrum of two copies of the cycle C_5 is $\{0, 2, 3, 5\}$, and the spectrum of two copies of the star $K_{1,r}$ is $\{1, r\}$. The intersection spectrum was initially studied for designs by Lindner and Fu and others and was originally extended to graphs by Eric Mendelsohn. Several examples are studied, both when G and H are isomorphic and when they are not isomorphic. It will also be shown any set S of positive integers is the edge-intersection spectrum of some pair of connected graphs.

The other two chapters cover the area of *conflict-tolerance graph representations.* These representations consist of rules for measuring the *rank* and *tolerance* of each vertex, and for determining if two vertices are in conflict, by combining and comparing the ranks and tolerances of the vertices. The edge set of the graph is then the pairs of vertices which are in conflict.

In the *odd-intersection interval model*, each vertex is represented by a subpath of a host path P, and two vertices are in conflict if and only if their corresponding subpaths intersect in an odd number of nodes. This model is not universal; in particular, the complete 4-partite graph $K_{3,3,3,3}$ is a minimal forbidden subgraph. The parity of the subpaths affects the representation; graphs in which the subpaths are of a fixed odd order are strongly chordal (in fact, these are precisely the unit interval graphs), and graphs in which the subpaths are all of even order are bipartite. The converse of the latter statement is not true, as it will be shown that the number of bipartite graphs is asymptotically larger than the number of possible representations.

A *cross-comparison graph model* is one in which each vertex v is assigned a rank r_v and a tolerance t_v, and two vertices u and v are in conflict if $r_v \geq t_u$ and $r_u \geq t_v$. Jamison showed that the cross-comparison model is universal, using n-dimensional vectors and coordinatewise comparison to

represent a graph on n vertices.

The *inefficiency* of a vector representation is the smallest number of dimensions d for which we can represent a graph G using d-dimensional vectors. The set of all graphs which can be represented using one-dimensional vectors (*efficient* cross-comparison graphs) is precisely the set of graphs which are the complement of a threshold tolerance graph (known as *co-TT graphs*; these were defined by Monma, Reed, and Trotter in [31]). The efficient cross-comparison graphs are characterized as those graphs which are chordal, and contain no *strongly asteroidal triple* — a set of three vertices, such that there is a path between any two of these vertices which contains no neighbor of the third, and which does not contain two consecutive vertices adjacent to every neighbor of the third. In addition, a graph with a d-dimensional vector representation is the intersection of d efficient graphs. The inefficiency of G is bounded below by its chordality, and above by its boxicity; both of these bounds are tight. In addition, the graph $K_{n(2)}$ has chordality and boxicity (and thus inefficiency) equal to n, and this is known to be the upper bound for boxicity, which shows that the efficiency of a graph is at most half its order, and that this bound is tight.

Dedication

Dedicated to Laney Sims Light, my motivation and inspiration.

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Introduction

1.1 Definitions

A *graph* G is an ordered pair (V, E), where $V(G)$ is the set of *vertices* (also called *nodes*) and $E(G)$ is the set of *edges*, which are unordered pairs of vertices. The *order* of G is the cardinality of the vertex set, and the *size* of G is the cardinality of the edge set. If the edge $e = uv \in E(G)$, then vertices u and v are said to be *adjacent*, and e is *incident* with u and v. Edges are adjacent if they are incident with a common vertex. The *open neighborhood* of a vertex v, denoted $N(v)$, is the set of vertices adjacent to v; the *closed neighborhood* of v, denoted $N[v]$, also includes v.

The *degree* of a vertex v in G, denoted $deg_G(v)$, is the number of edges in G that contain v as an endpoint. If $deg_G(v) = 0$, v is an *isolated vertex.* If $deg_G(v) = 1$, v is a *pendant vertex.* If $deg_G(v) = n - 1$ (where $n = |V(G)|$), then v is a *dominating vertex.* The *maximum degree* of G, denoted $\Delta(G)$, is defined as $\Delta(G) := \max\{deg_G(v) : v \in V(G)\}$. The *degree set* of G, denoted $\mathcal{D}(G)$, is the set of unique degrees of vertices in $V(G)$; i.e.

$$\mathcal{D}(G) = \{k : deg_G(v) = k \text{forsome} v \in V(G)\}.$$

The *complement* of G, denoted $\overline{G}$, has $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, then we say G *contains* H, and write $H \subseteq G$. A *vertex-induced* subgraph of G is a subset of the vertices of

G together with any edges whose endpoints are both in this subset; i.e.,

$$V(H) \subseteq V(G) \text{and} E(H) = \{uv : u, v \in V(H) \textit{ and } uv \in E(G)\}.$$

The *complete graph* on n vertices, denoted K_n, contains every possible edge. If K_n is a vertex-induced subgraph of G, then G contains a *clique* of order n. An *independent set*, denoted $\overline{K}_n$, contains an empty edge set. The *star* $K_{1,r}$ has a central vertex adjacent to the other r vertices (which are not adjacent to each other). A *path* in a graph is a sequence of vertices $a_1 - a_2 - \cdots - a_n$ such that the vertex a_i is adjacent to a_{i+1} for $1 \leq i \leq n-1$, and $a_i = a_j$ if and only if $i = j$. If a_i and a_j are adjacent if and only if $|i - j| = 1$, then the path is *irreducible.* A graph which is an irreducible path of order n is denoted P_n. A graph is *connected* if there is a path between any two vertices. A graph which is not connected is *disconnected*, and each maximal connected subgraph of this graph is a *component.*

The path gives rise to a distance metric on graphs: the *length* of a path is the number of edges in that path. The *distance* between two vertices in a graph is the length of the shortest path between these vertices.

A *cycle* is a path in which the end vertices are also adjacent. A cycle may be called *even* or *odd*; this refers to the number of vertices in the cycle. A *forest* is any graph which contains no cycles; a forest with only one component is a *tree.* A *caterpillar* is a tree which contains a path so that every vertex is either on this path, or adjacent to a vertex on this path. The *Petersen Graph* is the graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets. The *n-dimensional hypercube*, denoted Q_n, is a graph whose vertex set is the set of all n-tuples with entries in $\{0, 1\}$ and whose edges are the pairs of n-tuples which differ in exactly one position. The *Möbius ladder* M_{2n} is a cycle of length $2n$ in which each vertex is also adjacent to the vertex opposite it in the cycle; i.e., $V(M_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$ where v_i is adjacent to v_j if and only if $j = i \pm 1 \pmod{2n}$ or $j = i + n \pmod{2n}$.

A vertex whose neighborhood is a clique is called a *simplicial* vertex; if the vertices of $N(v)$ can also be ordered $u_1, u_2, \ldots, u_k$, such that $N[u_i] \subseteq N[u_j]$ if $i \leq j$, then v is a *simple* vertex. Simplicial and simple vertices are useful when studying chordal graphs. A *chord* in a cycle of a graph is an edge between two vertices which are not adjacent in the cycle. A graph is *chordal* if every cycle of order at least 4 contains a chord, and *strongly chordal* if it is chordal, and every even cycle of order at least 6 contains an *odd chord* (a chord between two vertices an odd distance apart).

Chordal and strongly chordal graphs can be characterized by *elimination orderings*. A *perfect elimination ordering* on G is an labeling of the vertex set $V(G)$ with the values $1, 2, \dots n = |V(G)|$, such that, for the vertex labeled i, the neighbors of this vertex with a label larger than i form a clique. In other words, if we remove the vertices $1, 2, \dots i-1$ from G, and examine the vertex-induced subgraph G_i that results, the vertex i is simplicial. A *strong elimination ordering* is a labeling of $V(G)$ with the values $1, 2, \dots, n = |V(G)|$, such that the vertex i is simple in G_i.

Proposition 1.1.1 (Fulkerson and Gross [16]) *A graph is chordal if and only if it has a perfect elimination ordering.*

Proposition 1.1.2 (Farber [14]) *A graph is strongly chordal if and only if it has a strong elimination ordering.*

A graph is *bipartite* if its vertex set can be broken into two independent subsets (or *parts*); this is equivalent to having no cycles of odd length. The *complete bipartite graph* $K_{r,s}$ has parts V_1 of order r and V_2 of order s, such that each vertex in V_1 is adjacent to every vertex in V_2. Similarly, a k-partite graph is a graph whose vertex set can be divided into k parts, and a complete k-partite graph such that two vertices are adjacent if and only if they are not in the same part. If each part in a complete k-partite graph has order r, then this graph is denoted $K_{k(r)}$. A graph is *chordal bipartite* if it is bipartite, and every cycle of length at least 6 has a chord.

A graph G is the *intersection* of a set of graphs $G_1, G_2, \dots, G_k$ if $V(G) = V(G_1) = \cdots = V(G_k)$ and $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_k)$. The *direct product* of G and H, denoted $G \times H$, has vertex set $V(G) \times V(H)$, where (u, v) is adjacent to (u', v') if and only if $uu' \in E(G)$ and $vv' \in E(H)$.

1.2 Intersection Graphs

1.2.1 Overview

In intersection graph theory, the vertices of a graph are usually represented by the members of some family $\mathcal{F}$ of sets (often, the set $[k] = \{1, 2, \dots, k\}$, for some finite k) and two vertices are adjacent if the intersection of their corresponding sets satisfies some specified condition. The set of rules used to define the vertex and edge sets is known as a *model.* In an intersection model, the choice of sets to represent the vertices of a graph pre-determines the edges; the specific sets corresponding to each vertex are a *representation* of the graph. A graph G is *representable* with respect to a given model if there is some representation of G under that model; i.e., there is some choice of sets for $V(G)$ that produce precisely the edges of $E(G)$. If every graph is representable in a given model, that model is *universal.*

One of the most interesting questions surrounding a given model is that of how to represent a given graph G. For models which are not universal, this problem is how to determine whether G is representable; in particular, it is interesting whether a polynomial-time algorithm exists to determine a representation (or lack thereof). For many models, this problem is NP-complete. However, if a model can be characterized by *minimal forbidden subgraphs* (a collection of graphs for which the representable graphs are precisely those graphs which contain none of the collection as a vertex-induced subgraph), a polynomial-time algorithm is guaranteed.

For models which are universal, the problem is one of the "best" representation. This could be the smallest number of sets in the family $\mathcal{F}$, or, in the case where $\mathcal{F}$ are subsets of some host set, the smallest cardinality of these subsets.

1.2.2 Applications

A basic motivating example of intersection graph models is the room scheduling problem.

Suppose that several departments at a company have meetings scheduled on a given day. If one department has a meeting scheduled from 9:30-11:00, and another department has a meeting scheduled from 10:30-11:30, then at least two conference rooms are needed to accomodate both departments. The scheduling problem can be represented by a graph G, where each vertex is represented by the interval of time required for each meeting, and two vertices are adjacent if two meetings are scheduled for the same time. The number of rooms required to host all the meetings is then the *chromatic number* of the corresponding graph, which is the smallest number k such that G is k-partite, but not $(k-1)$-partite.

A key problem in genome research is that of finding genes that exhibit similar expression patterns. DNA microarray technology has made it possible to obtain large amounts of gene expression data, but the problem of efficiently and effectively analyzing the data remains. Tanay, Sharan, and Shamir searched for significant *biclusters* (subsets of genes with similar expressions across a subset of experiments) in [36] by representing the data as a weighted bipartite graph (where the vertices in one part represent genes, and the vertices in the other part represent experiments), and then searching for complete bipartite subgraphs. Another potential representation of this problem assigns each vertex v a k-tuple $(v_1, v_2, \ldots, v_\ell)$ representing the expression level of each gene in ℓ experiments, and assigns an edge between u and v if a predetermined number of elements in their k-tuples are close in value (indicating similar expression levels across a significant number of experiments); significant biclusters would then be represented by cliques in the resulting graph. This model would require two (or more) threshold parameters: one parameter specifying the number of elements required for adjacency, and one parameter (which may be different for each experiment) for determining whether two gene expression values are "close".

1.2.3 Graph Models

The following are some of the intersection graph models that have been studied. In some of these models, the family $\mathcal{F}$ is the set of subgraphs of some host graph H. As a matter of notation in these models, the term "node" will refer to the an element of $V(H)$, and a "vertex" will be an element of $V(G)$, where G is the graph being represented under the model.

1.2.3.1 Set-Intersection Graphs

The most basic intersection graph is the *set-intersection graph*, in which each vertex is represented by a subset of some set S, and two vertices are adjacent if their corresponding sets have a nonempty intersection.

Proposition 1.2.1 (Marczewski [35]) *The set-intersection graph model is universal.*

Proof. For a graph of size m, label the edges $1, 2, \ldots, m$, and assign each vertex $v \in V(G)$ the set $S_v \subseteq [m]$, where $i \in S_v$ if and only if v is incident with the edge labeled i. Then, for vertices u and v, $S_u \cap S_v$ is nonempty if and only if u and v are incident with the same edge; in other words, if and only if u and v are adjacent. $\square$

A host set with as many *labels* (elements) as edges is not likely to be the smallest possible. As the size of the graph approaches its maximum ($\binom{n}{2}$ for a graph of order n), fewer labels will be needed; in fact, K_n requires only one label, regardless of n. If we allow vertices to be represented by the empty set, fewer labels are needed for *sparse* graphs (graphs with very few edges, relative to their order) as well; in this case, $\overline{K}_n$ requires no labels at all! This illustrates a key point in intersection graph theory: the problem in intersection representations is not how to represent cliques (in which the closed neighborhoods of every vertex are the same), nor independent sets (in which the open neighborhoods of every vertex are the same), but graphs which have a large variation in the neighborhoods their vertices.

1.2.3.2 Tree Intersection Graphs

In the *tree-intersection model*, each vertex is represented by a subtree of a host tree T (a *subtree* being a connected subgraph of a tree). This model is also assigned a tolerance t, and two vertices are adjacent if their corresponding subtrees intersect in at least t nodes. For $t = 1$, Gavril showed in [17] that the graphs representable by this model are precisely the chordal graphs.

1.2.3.3 Odd-Intersection Graphs

An *odd-intersection graph* is one in which two vertices are adjacent if the intersection of their corresponding sets contains an odd number of elements. In the case where $\mathcal{F}$ is the power set of the set $[n]$ (where $n = V(G)$), this model is universal; Eaton demontrated an algorithm in [9] which assigns each vertex a subset of cardinality at most $n - 1$. In chapter 3, we will study a combination of this model and the tree-intersection model, in which $\mathcal{F}$ are subpaths of a host path H.

1.2.3.4 Interval Graphs and Box Graphs

In an *interval graph*, each vertex is represented by an interval on the real line. The interval graph model is not universal; Lekkerkerker and Boland characterized the interval graphs in [26], both by a specific unrepresentable condition, and by a collection of minimal forbidden subgraphs.

Similar to the interval graph is the *d-box graph*, in which each vertex is represented by a box in $\mathbb{R}^d$. For an unbounded number of dimensions, the box model is universal; a graph G has *boxicity* d if d is the minimum number of dimensions for which G has a d-box representation. Because a d-dimensional box is the intersection of d intervals, a graph with boxicity d are the intersection of d interval graphs. Several results concerning box graphs and interval graphs will be examined in chapter 4.

1.2.3.5 Threshold Graphs

In the *threshold graph* model, each vertex v is assigned a weight w_v, and vertices u and v are adjacent if and only if $w_u + w_v \leq 1$. Threshold graphs were originally studied by Chvátal and Hammer [7], who showed that if G is a threshold graph, then $\overline{G}$ is also a threshold graph; and, that a new threshold graph can be obtained by adding either an isolated vertex or a dominating vertex to $V(G)$.

Related to threshold graphs are *threshold tolerance graphs*, or TT graphs, first defined by Monma, Reed, and Trotter in [31]. In this model, each vertex v is assigned a weight w_v and tolerance t_v, and $uv \in E(G)$ if and only if $w_u + w_v \geq \min\{t_u, t_v\}$. If every vertex is assigned the same tolerance, then G is a threshold graph. If the inequality is reversed, the graphs represented are the complements of the threshold tolerance graphs, known as *co-TT graphs.* These graphs were also studied in [31], and were characterized using a total ordering of the vertices which avoided certain directed configurations. We will examine co-TT graphs more closely in chapter 4

1.2.3.6 Conflict-tolerance Graphs

Consider the following extension of the interval graph model: in addition to an interval on the real line, each vertex v is assigned a tolerance t_v, and two vertices are adjacent, not if the intersection of their intervals is nonempty, but if the length of the intersection exceeds the minimum of the corresponding tolerances. This model, introduced in [19] was the first *conflict-tolerance model* to be studied.

The conflict-tolerance models were defined by Golumbic and Jamison in [18], and extended in [23]. In these broadly-defined models, each vertex is assigned a rank and tolerance, and edges are defined using a “conflict rule” which compares the weights and tolerances of two vertices; vertices which are adjacent according to this rule are said to be “in conflict”.

The TT and co-TT graphs are very specifically conflict-tolerance models, but all of the above models can be considered to fall under the conflict-tolerance umbrella. For example, to obtain a set-intersection graph, the weight of each vertex is still some subset of a host set (exactly as in the original model), but each vertex is also assigned a constant tolerance 1, and two vertices are in conflict if the intersection of their weights is at least the tolerance. This simple modification allows us to see how the set-intersection model could be extended to one in which the tolerances can vary, and the “conflict rule” can be modified so that the intersection of the sets must exceed the minimum or maximum tolerance of the vertices. These variations might have a great effect on the minimum number of labels required for the host set. Models in which the tolerance is a constant k have already been studied [10, 11, 12].

While many conflict-tolerance models use real-valued ranks and tolerances, combine these using specific binary operations (sums, products, minima and maxima), and directly compare combined ranks and tolerances, more abstract rules for conflict are possible. Chapters 3 and 4 will study two such models.

Chapter 2

Edge-Intersection Spectra

2.1 Introduction

For two graphs G and H of order n, we define the *edge-intersection spectrum* of G and H to be:

$$\mathrm{Spec}(G, H) = \{k : k = |E(G') \cap E(H')| \text{ where} G' \cong G, H' \cong H, \text{ and } V(G') = V(H')\}.$$

In other words, if we place copies of G and H onto the same vertex set, then $\mathrm{Spec}(G, H)$ is the possible number of edges that will be used in both graphs.

Eric Mendelsohn initially posed the problem of finding the edge-intersection spectrum of two cycles [30]. Suppose $G = H = C_5$. If we label the vertices of our 5-cycles $\{a, b, c, d, e\}$, and let a-b-c-d-e-a be the order in which G travels these vertices, then the representations of H in table 2.1 give us intersections of 0, 2, 3, and 5 edges.

However, there is no way to share exactly four edges between two copies of C_5; if the cycles

H	k
$a-c-e-b-d-a$	0
$a-b-d-e-c-a$	2
$a-b-c-e-d-a$	3
$a-b-c-d-e-a$	5

Table 2.1: The edge-intersection spectrum of two copies of C_5.

share four edges, then they must share the fifth. Also, there is no way to share exactly one edge; because C_5 is self-complementary, if G and H were two copies of C_5 which share exactly one edge, then G and $\overline{H}$ would share four edges. Thus, we say that $\mathrm{Spec}(C_5, C_5) = \{0, 2, 3, 5\}$.

Note that while G and H may be isomorphic, this is not necessary. We will assume, however, that G and H are both connected; specifically, G and H have no isolated vertices.

2.2 Examples when G and H are isomorphic

To gain a better understanding of the edge-intersection spectrum, we will look at some examples, starting with the case where $G = H$.

2.2.1 Cycles

We have already seen the spectrum of 5-cycles. For longer cycles, we can state a stronger result:

Proposition 2.2.1 *For $n \geq 6$, $Spec(C_n, C_n) = \{0, 1, \ldots, n-2\} \cup \{n\}$.*

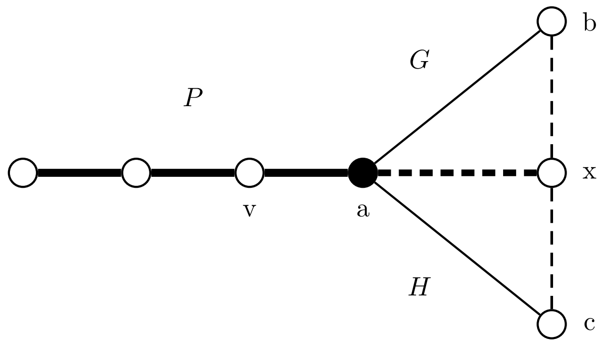
Proof. We note that n edges can always be shared, by using the same edge set for both graphs. Also, 0 edges can always be shared, as K_n contains at least two distinct hamiltonian cycles for $n \geq 5$ [6, 275-277]. However, we can never share $n-1$ edges; at this point, the final edge of the cycle is 'fixed,' and must also be shared.

For $0 < k \leq n-2$, if $k-1 \in \mathrm{Spec}(C_{n-1}, C_{n-1})$, we can show that $k \in \mathrm{Spec}(C_n, C_n)$ using the following construction:

Let G and H be two copies of C_{n-1} in K_{n-1} sharing exactly $k-1$ edges. Since $k-1 < n-1$, the components of $G \cap H$ are paths. For the moment assume $1 \leq k-1$ as well, so at least one component path P is nontrivial. Let a be an end point of P and let v be its unique neighbor in P. Then $va \in G \cap H$. Let b [resp., c] be the other neighbor of a in the cycle G [resp., H].

Now add a new vertex x to K_{n-1} to get K_n. Remove edge ab [resp., ac] from G [resp., H] and replace it by edges ax and xb [resp., ax and xc] to get an n-cycle G' [resp., H']. In other words, $G' = G - ab + ax, xb$, and $H' = H - ac + ax, xc$. A similar construction works if the components of $G \cap H$ are all isolated vertices — that is, if $k = 0$. Using this construction and

Figure 2.1: Extension of two cycles.



Spec(C_5, C_5), we can share 0, 1, 3, 4, or 6 edges between two copies of C_6. If we label the vertices of K_6 with $a - f$, representing G by a-b-c-d-e-f-a and H by a-b-d-f-e-c-a shares two edges, giving us Spec$(C_6, C_6) = 0, 1, 2, 3, 4, 6$. We can then use the construction iteratively to show the result for larger values of n. □

2.2.2 Paths

Two copies of a path can intersect in any number of edges, up to the size of the path:

Proposition 2.2.2 *For $n \geq 4$, Spec$(P_n, P_n) = \{0, 1, \ldots, n-1\}$.*

Proof. This follows directly from the result for cycles: If $k \in Sp(C_n, C_n)$, then we can delete an unshared edge (if $k \neq n$) from each cycle to get two paths that share k edges, or a shared edge (if $k \neq 0$) to get two paths sharing $k - 1$ edges. □

2.2.3 Stars

The star gives us the first example of a graph with serious restrictions on the edge - intersection spectrum:

Proposition 2.2.3 *Spec$(K_{1,n-1}, K_{1,n-1}) = \{1, n-1\}$.*

Proof. Let u and v be the central vertices of G and H, respectively. If $u = v$, then obviously G and H share $n-1$ edges. If $u \neq v$, then the only edge that G and H can (and do) share is uv. □

2.2.4 Double Stars

The double star has a complicated spectrum with large gaps. In this case, $DS(r, s)$ refers to a tree with two adjacent vertices (u and v), with r additional vertices adjacent to u and s additional vertices adjacent to v.

Proposition 2.2.4 *For $r \geq s$, $Spec(DS(r, s), DS(r, s)) =$*

$$\{0, 1, \ldots, s + 1\} \cup \{r - s + 1, \ldots, r + 1\} \cup \{2k + 1 : 0 \leq k \leq s\} \cup \{r - s + 2k + 1 : 0 \leq k \leq s\}$$

Proof. Given isomorphic double stars G, H, let uv and ab be the central edge of G and H, respectively, where $\deg_G u = \deg_H a = r + 1$ and $\deg_G v = \deg_H b = s + 1$. There are five cases to consider:

Case I: $u = a$ **and** $v = b$. For $0 \leq k \leq s$, if there are k peripheral vertices that are adjacent to v in both G and H, then there are $s - k$ such vertices adjacent to v in G but not in H, as well as $s - k$ vertices adjacent to v in H but not in G. This leaves $r - s + k$ peripheral vertices adjacent to u in both G and H. Then G and H share $r - s + 2k + 1$ edges (k edges connecting v to peripheral vertices, $r - s + k$ edges connecting u to peripheral vertices, and uv).

Case II: $u = b$ **and** $v = a$. For $0 \leq k \leq s$, if there are k peripheral vertices that are adjacent to v in both G and H, then there are $s - k$ vertices that are adjacent to v in G but not in H, and $r - k$ vertices that are adjacent to v in H but not in G. This leaves k peripheral vertices adjacent to u in both graphs. Then G and H share $2k + 1$ edges.

Case III: $u = a, v \neq b$. There are $r + 2$ edges that could possibly be shared; the $r + 1$ edges incident with u in G, plus the edge vb. There are s vertices (besides a) adjacent to b in H; if k of these vertices are adjacent to u in G, then there are $r - k$ vertices besides v that must be adjacent to u in both G and H, so at least this many edges are shared. However, if $k = s$, then ab cannot be a shared edge. In this case, either va or vb must be shared, so the fewest edges that can be shared is $r - s + 1$. Also, we cannot share both va and vb, so at most $r + 1$ edges can be shared.

Case IV: $v = a, u \neq b$. There are $s + 2$ edges that could possibly be shared; the $s + 1$ edges incident with v in G, plus the edge ub. It is possible to share any number of the edges incident with v in G, or none of those edges if $ab \notin E(G)$. However, if ab is not a shared edge, then either ua or ub must be shared, so at least one edge must be shared. Also, ua and ub cannot both be edges of H, so it is not possible to share more than $s + 1$ edges.

Because this case depends only on the degree of v, this is identical to the case where $v = b$ and $u \neq a$. Also, because G and H are isomorphic, this is also identical to the case where $u = b$ and $v \neq a$.

Case V: a, b, u, and v are all distinct. There are two subcases here. First, if a and b are both adjacent to the same vertex in G, then without loss of generality, we may assume that they are adjacent to u. Then exactly one edge (either ua or ub) must be shared. If, on the other hand, a and b are adjacent to different vertices in G, say ua and $vb \in E(G)$, then only these two edges can be shared, and we can share no edges by allowing ub and $va \in E(H)$. It is also possible to share only one of these edges, unless $r = s = 1$. □

It is possible to minimize the gaps in this spectrum; in fact, they can be eliminated entirely in one case.

Corollary 2.2.5 *For $r \geq s$, Spec$(DS(r,s), DS(r,s)) = \{0, 1, \ldots, r+s+1\}$ if and only if $r = s = 1$ or $r = s+1$.*

Proof. Note that $DS(1,1) = P_4$, so proposition 3.2 gives us our result in that case. If $r > 1$, it is impossible to share $r+s$ edges unless $r = s+1$. If this equality is satisfied, then note that $r+s+1 = 2s+2$, which is an even number. Then from the theorem above, $0 \in \mathrm{Spec}(DS(r,s), DS(r,s))$, the set $\{2k+1 : 0 \leq k \leq s\}$ contains all odd numbers from 1 to $2s+1$, and the set $\{r-s+2k+1 : 0 \leq k \leq s\} = \{2k+2 : 0 \leq k \leq s\}$ contains all even numbers from 2 to $2s+2$. □

Note that it is possible (likely, even) for many values to be repeated in this formula, especially in the case where $r = s$. For this case the set notation can be simplified.

Corollary 2.2.6 *The spectrum of $DS(r,r)$ with itself is*

$$\{0, 1, \ldots, r+1\} \cup \left\{ 2r+1-2k : 0 \leq k \leq \left\lfloor \frac{r-1}{2} \right\rfloor \right\}$$

Table 2.2 shows the edge-intersection spectra for several values of r and s. While many of the gaps in the spectra are of size one, it is possible to have large gaps when r is very large in relation to s.

r	s	**Spec**$(DS(r,s), DS(r,s))$
6	1	$\{0,1,2,3\} \cup \{6,7,8\}$
6	2	$\{0,1,2,3\} \cup \{5,7,9\}$
6	3	$\{0,\ldots,8\} \cup \{10\}$
6	4	$\{0,\ldots,7\} \cup \{9,11\}$
6	5	$\{0,\ldots,12\}$
6	6	$\{0,\ldots,7\} \cup \{9,11,13\}$
7	2	$\{0,1,2,3\} \cup \{5,6,7,8\} \cup \{10\}$
8	2	$\{0,1,2,3\} \cup \{5,7,8,9,11\}$
20	2	$\{0,1,2,3\} \cup \{5,19,20,21,23\}$
20	5	$\{0,\ldots,6\} \cup \{7,9,11\} \cup \{16,\ldots,22\} \cup \{24,26\}$

Table 2.2: Edge-intersection spectra for selected double stars.

2.3 Examples when G and H are NOT isomorphic

2.3.1 $G = K_{1,n-1}$

A simple example involving nonisomorphic graphs is the example of the path and the star. We see immediately that if one graph is the star, the edge-intersection spectrum becomes very specific.

Proposition 2.3.1 *Spec*$(K_{1,n-1}, P_n) = \{1,2\}$ *when* $n \geq 3$.

In fact, the star forces a specific spectrum on any graph G.

Theorem 2.3.2 *If G is a graph of order n with degree set $\mathcal{D}(G)$, then*
Spec$(K_{1,n-1}, G) = \mathcal{D}(G)$.

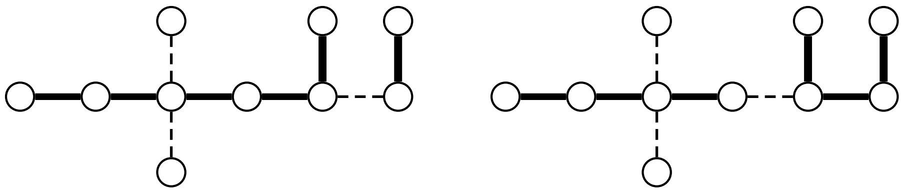
Proof. if we choose a vertex v to be the central vertex of the star, then precisely the edges incident with v in $E(G)$ are shared, so $deg_G(v) \in \text{Spec}(K_{1,n-1}, G)$. □

This leads us to the interesting result that an arbitrary set of positive integers is always the edge-intersection spectrum of some pair of graphs:

Theorem 2.3.3 *If $S = \{a_1, a_2, \ldots, a_k\}$ is a set of positive integers with $a_1 < a_2 < \cdots < a_k$, then there exist graphs G and H such that Spec$(G,H) = S$. Specifically, S is the intersection spectrum of $K_{1,n-1}$ and a graph of order $a_k + 1$ with degree set S.*

Proof. This follows from the previous theorem and a result of Kapoor, Polimeni, and Wall stating that a graph of order $a_k + 1$ exists with degree set S [25]. □

Figure 2.2: Two maximum linear subforests.



Note that S above specifically does not include zero. A spectrum including zero can be obtained by adding an isolated vertex to the non-star graph, but that violates our assumption (in the introduction) that both graphs are connected. It remains an open question whether an arbitrary set of nonnegative integers is always the edge-intersection spectrum of two connected graphs.

2.3.2 $G = P_n$

We now examine the spectrum of the path and a tree. We begin with a definition.

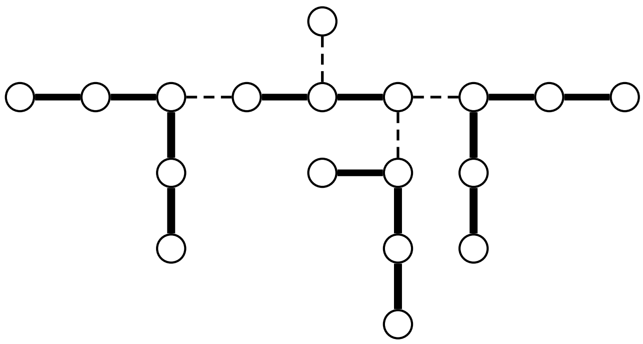
Definition 2.3.4 *The* pathic number *of a graph G, denoted $\tau(G)$ is the maximum number of edges in a* linear subgraph *of G: a vertex-induced subgraph $H \subseteq G$ such that every component of H is a path or an isolated vertex. Note that the linear subgraph of maximum size is not unique in many cases; see Figure 2.2.*

While most components of a maximum linear subforest of a tree will contain an end-vertex of the tree, it is possible for components to contain no end-vertices. It is also possible (and in many cases likely) that a maximum linear subforest will contain isolated vertices; see Figure 2.3. Every maximum linear subgraph of a tree must contain a pendant edge; we will need the following lemma in the next theorem.

Lemma 2.3.5 *Let T be a tree of order n, and F a linear subgraph of T of order n and maximum size. Then F has at least one component of nonzero size that contains an end-vertex of T.*

Proof. The proof is by induction on n. If $n = 2$, then $F = T = K_2$. For $n > 2$, assume that the lemma is satisfied for trees of all orders less than n. Assume to the contrary that all end-vertices of T are isolated in F, and consider the tree T' obtained by removing all the end-vertices of T. The corresponding vertex-induced subgraph of F is a maximum linear subgraph of T', and by the

Figure 2.3: Note the isolated vertex, and the component containing no end-vertices.



inductive hypothesis has a nontrivial component that contains an end-vertex of T'; call this vertex v. Because v could not be an end-vertex of T, it must be adjacent to an end-vertex of T; call this vertex u. But then appending the edge uv to F gives a linear subgraph of larger size, which contradicts our assumption that F is maximum. □

We claim that the edge-intersection spectrum of a path and a tree of the same order includes every integer from zero to the pathic number of the tree. To show this, we will need one further lemma.

Lemma 2.3.6 *Let F be a forest of order n. If $\Delta(F) < n-2$, then $\overline{F}$ is hamiltonian.*

Proof. The proof is by induction on m, the size of F. If $m = 0$, then $\overline{F} = K_n$, and note that $n \geq 3$, so $\overline{F}$ is hamiltonian.

For $m > 0$, assume that any forest of order n and size $m-1$ with maximum degree less than $n-2$ has a hamiltonian complement. Let uv be an edge in $E(F)$ such that $deg_F(u) = 1$. Then $deg_{\overline{F}}(u) + deg_{\overline{F}}(u) \geq n$, so $\overline{F}$ is hamiltonian if and only if $\overline{F} + uv$ is [5, 111-136]. But $\overline{F} + uv$ is the complement of $F - uv$, which is a forest of size $m-1$. Thus $\overline{F}$ is hamiltonian. □

Theorem 2.3.7 *For T a non-star tree, Spec$(P_n, T) = \{0, 1, \ldots, \tau(T)\}$.*

Proof. Note that $\text{Spec}(P_n, T) \subseteq \{0, 1, \ldots, \tau(T)\}$ as P_n and T cannot share more than the maximum number of edges in a linear subgraph of T. Thus, we must show that $k \in \text{Spec}(P_n, T)$ for $0 \leq k \leq \tau(T)$.

If $\Delta(T) < n-2$, then $\overline{T}$ is hamiltonian by the above lemma. If $\Delta(T) = n-2$, then $\overline{T}$ contains a hamiltonian path: if u is the vertex of degree $n-2$, v is the vertex not adjacent to u, w is the vertex adjacent to both u and v, and $a_1, \ldots, a_{n-3}$ are the remaining vertices, then $u, v, a_1, \ldots, a_{n-3}, w$ is a hamiltonian path. In either case, $0 \in \mathrm{Spec}(P_n, T)$.

Suppose T' is a linear subgraph of T of maximum size with $V(T') = V(T)$ and path components $a_1, \ldots, b_1; a_2, \ldots, b_2; \ldots; a_r, \ldots b_r$ of sizes $t_1, t_2, \ldots, t_r$, such that a_1 is an end vertex of T and $t_1 > 0$. We note the following:

1. $\sum_{i=1}^{r} t_i = \tau(T)$
2. It is possible that T' contains isolated vertices; in other words, for one or more values of $i, a_i = b_i$ and $t_i = 0$.
3. for $1 \leq i \leq r-1$, the edge $b_i a_{i+1}$ cannot be in $E(T)$; otherwise, T' would not be the linear subgraph of maximum size. Similarly, the edge $b_r a_1 \notin E(T)$.

Then if we let P be the path of order n formed by T' plus the edges $b_i a_{i+1}$ for $1 \leq i \leq r-1$, then P intersects T in exactly $\tau(T)$ edges. Thus, $\tau(T) \in \mathrm{Spec}(P_n, T)$.

For $0 < k < \tau(T)$, there is a number $s < r$ such that $\sum_{i=1}^{s} t_i < k$ and $\sum_{i=1}^{s+1} t_i \geq k$; if it happens that $t_1 \geq k$, then we consider s to be zero. We construct a path P that intersects T in exactly k edges as follows:

We begin with P' consisting of exactly k edges of T'. If $s = 0$, then P' is the subpath $a_1, \ldots, c$, where the vertex c is chosen such that the size of P' is k. If $s > 0, P'$ consists of the paths $a_1, \ldots, b_1; \ldots; a_s, \ldots, b_s$ and the subpath $a_{s+1}, \ldots, c$ plus the edges $b_i a_{i+1}$ for $1 \leq i \leq s$. In this case, c is chosen so the size of P' is $k + s$; note that in either case, it is possible that $c = b_{s+1}$. Consider the vertex-induced subgraph of T formed by the vertex c and all vertices of T not in $V(P')$, which is a forest of order $x = n - (k + s)$; call this graph F, and note that by our construction of T', a_1 cannot be adjacent to any vertex of F, except possibly v. There are several cases.

If $\Delta(F) < x - 2$, then $\overline{F}$ contains a hamiltonian cycle v-u_1-u_2-$\cdots$-u_{x-1}-v. Then by appending the edges $vu_1, u_1u_2, \ldots, u_{x-2}u_{x-1}$ to P', we obtain the desired path P.

If $\Delta(F) = x - 2$, there are three subcases to consider:

- If v is a vertex of degree $x-2$ in F (note that there could be two such vertices if and only if F is a copy of P_4), w is the lone vertex of F not adjacent to v, and $u_1, u_2 \ldots, u_{x-2}$ are the vertices adjacent to v, then we can complete the path P by appending the edges $a_1w, wu_1, u_1u_2, \ldots, u_{x-3}u_{x-2}$ to P'.

- If v is adjacent to the unique vertex w of degree $x-2$ in F, u is the lone vertex not adjacent to w, and $u_1, u_2, \ldots, u_{x-3}$ are the remaining vertices, then we can complete P by appending the edges $a_1w, wu, uu_1, u_1u_2, \ldots, u_{x-4}u_{x-3}$ to P'.

- If w is a vertex of degree $x-2$ in F that is not adjacent to v (this includes the case where F is a copy of P_4 and $deg_F(v) = 1$), and $u_1, u_2, \ldots, u_{x-2}$ are the vertices adjacent to w, then we can complete P by appending the edges $a_1u_1, u_1u_2, \ldots, u_{x-3}u_{x-2}$ and the edge vw to P'.

If $\Delta(F) = x-1$ (in other words, F is a star), there are two subcases to consider:

- If v is the central vertex, and $u_1, u_2, \ldots, u_{x-1}$ the peripheral vertices, then we complete P by appending the edges $a_1u_1, u_1u_2, \ldots, u_{x-2}u_{x-1}$ to P'. Note that in this case, a_1 cannot be adjacent to v because T is not a star.

- If $w \neq v$ is the central vertex, and $u_1, u_2, \ldots, u_{x-2}$ are the other peripheral vertices, then we append the edges $vu_1, u_1u_2, \ldots, u_{x-3}u_{x-2}$ and the edge wa_1 to P'.

□

2.4 Open Problems

As stated earlier, it is an open question whether an arbitrary set of nonnegative integers can always be the edge-intersection spectrum of two graphs.

The edge-intersection spectrum of two non-star trees is also an interesting problem. We know that the spectrum always includes zero [13], and suspect it includes one, but further results are unknown.

Chapter 3

Odd-Intersection Interval Graphs

3.1 Introduction

An *odd-intersection interval model* is a special case of a conflict-tolerance set model in which the host H is a path and the allowable sets are subpaths of H, and two sets are in conflict if their intersection contains an odd number of nodes. A graph G is an *odd-intersection interval graph* if it has a representation under this model; i.e., we can assign each vertex v to a subpath such that two vertices are adjacent if and only if their corresponding subpaths have an odd intersection. Figure 3.1 shows two odd-intersection interval representations of C_6. Note that there is no simple permutation of the intervals of one representation into the other (by re-ordering the intervals, deleting vertices, etc.); thus there is no canonical representation of this cycle. In fact, many graphs have multiple representations under this model.

To simplify descriptions, we will always consider the intervals to be oriented horizontally, as can be noted from the figures. While disjoint intervals can certainly be placed on the same "line" to save space in a representation, in general we will consider the intervals to be arranged vertically in such a way that no two intervals are collinear. This allows us to use directional terms to describe the relative locations of intervals. It must be noted, however, that these terms will refer only to a specific representation of a graph, as the relative positions of intervals are not fixed between representations (as a simple example, re-arranging the vertical positions of intervals without shifting any horizontally gives a new representation of the same graph).

Note that there are three ways for intervals to interact:

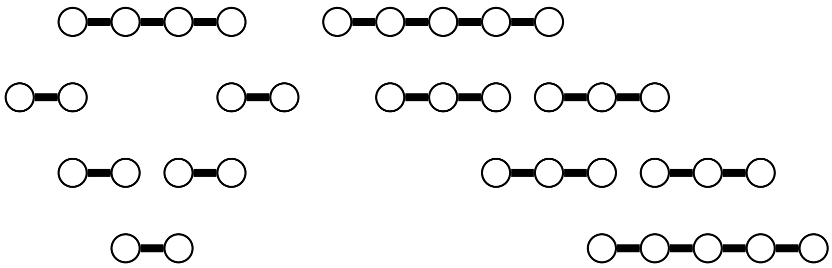


Figure 3.1: Two representations of C_6.

- The intervals could overlap, with neither contained in the other.
- One interval could be completely contained in the other, in which case the intervals represent adjacent vertices if and only if the contained interval is of odd length.
- The intervals could be completely disjoint, in which case they must always represent non-adjacent vertices.

Appendix A contains odd-intersection representations of selected graphs.

When attempting to represent a graph, the parity of interval lengths is often important. We will call an interval even (odd) if it has an even (odd) number of nodes, and refer to two intervals as (non)adjacent if they represent (non)adjacent vertices.

Lemma 3.1.1 *Given three pairwise intersecting intervals,*

1. *If the intervals are odd, they cannot represent an independent set.*
2. *If the intervals are even, they cannot represent a three-cycle.*

Proof. For both cases, we cannot have any of the three intervals contained in another, as it would immediately cause two odd intervals to be adjacent, and two even intervals to be nonadjacent. Thus the intervals must have intersections as shown in Figure 3.1 (possibly after re-arranging the order of the intervals).

To represent an independent set, the intersections marked a, b, and c must all have even length. But the length of interval v_2 is $a + b + c$, so this must be an even interval. Similarly, if we

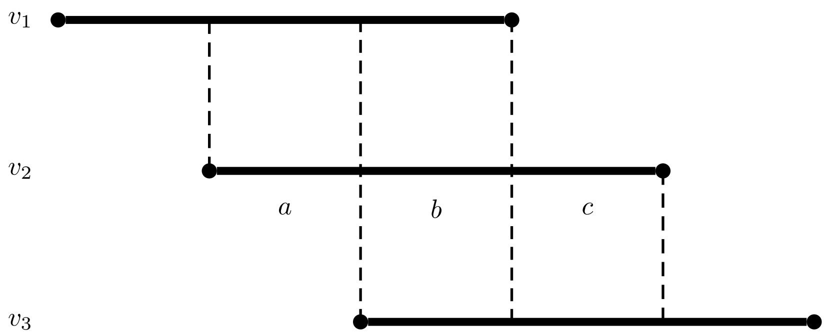


Figure 3.2: A generalized view of three pairwise-intersecting intervals.

wish to represent a three-cycle, b must have odd length, while a and c must have even length, so v_2 must be odd. □

3.2 Representability

This brings us to the question of what graphs are representable as odd-intersection interval graphs. We will refer to a graph G as *representable* if an odd-intersection interval representation exists. Otherwise, G is *non-representable.*

Theorem 3.2.1 *All rooted trees are representable. In particular, every rooted tree has a representation in which each interval is of even length, and the unique leftmost interval represents the root vertex.*

Proof. The proof is by strong induction on the order of the tree.

There is only one tree of order 1, and it can be represented by a single interval with two nodes.

If T is a rooted tree of order $n > 1$, then assume inductively that any rooted tree of order $k < n$ can be represented as above. Let r be the root vertex, and $v_1, v_2, \ldots, v_m$ be the vertices adjacent to r; note that deleting the vertex r leaves m trees, and that we can choose $v_1, v_2, \ldots, v_m$ as the roots of these trees. We represent T as follows:

- r is represented by a single interval with two nodes.

- The trees with $v_1, v_2, \ldots, v_m$ as roots have order less than n, and therefore have representations by the inductive hypothesis. The representation of the subtree rooted by v_1 is placed below the interval representing r, with each interval shifted one to the right, so that v_1 overlaps r by one node (and no other vertices of the subtree overlap r).

- The process is iterated for the subtrees rooted by $v_2, \ldots v_m$ by representing the root vertex v_i with an even interval that overlaps r by one node and extends at least one node past the rightmost interval in the previous construction. The remainder of the representation of the subtree is then placed below this interval. All intervals above v_i besides r are contained in v_i; because all of the intervals are even, non of them are adjacent to v_i. The remaining intervals in the subtree do not overlap any other interval in the previous construction, and so are not adjacent to any interval in the previous construction.

□

Corollary 3.2.2 *All trees are representable.*

Several other classes of graphs are known to be representable.

- Clearly, all complete graphs are representable, as we can simply repeat the same odd interval for each vertex.

- By repeating the same even interval and/or using disjoint intervals, we can represent an independent set.

- We have already seen that all cycles are representable.

- $K_{a,b}$ is representable. We represent the vertices of one part with a repeated even interval, and the vertices of the second part with a second repeated even interval which has an odd overlap with the first.

- $K_{a,b,c}$ is representable. In this case, the vertices of the third part are represented by disjoint odd intervals. The even intervals used to represent the other two parts must then be of sufficient length that their intersection completely contains these odd intervals.

In fact, we can extend complete multipartite graphs indefinitely, so long as each part beyond the third contains at most two vertices (Figure 3.2 is a representation of $K_{3,3,3,2}$). However, if there are four parts with more than two vertices, a representation is impossible:

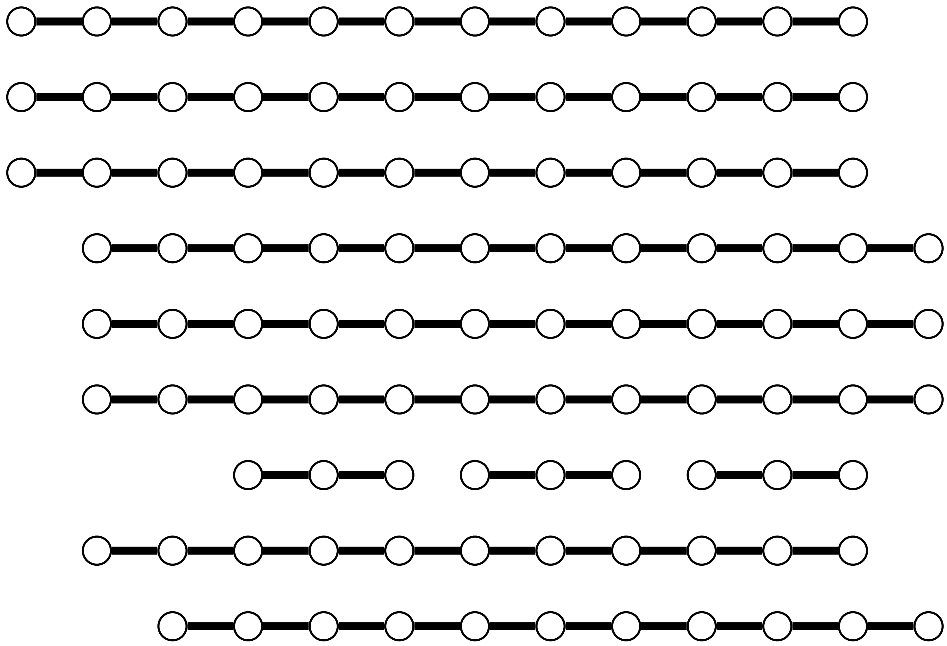


Figure 3.3: A representation of $K_{3,3,3,2}$.

Theorem 3.2.3 *$K_{4(3)}$ is not representable.*

Proof. If we choose a vertex from each of three partitions, those three vertices must form a three-cycle. By Lemma 3.1.1, at least one of these vertices must be odd. Then at most two parts (call these a and b) may have any vertices represented by even intervals. The other two parts must be represented by all odd intervals. One of these parts (part c) may be represented using at least two disjoint intervals. This leaves one part (part d), whose vertices must all be represented by odd intervals, all of which must have an odd overlap with the intervals of part c. Because there are two disjoint intervals in part c, the space between these two intervals (horizontally) must be completely contained in each interval of part d. Then part d is represented by three pairwise intersecting odd intervals. But this is a contradiction, because by Lemma 3.1.1, three pairwise intersecting odd intervals cannot represent an independent set. □

Now that we know that not all graphs can be represented, we have new questions to answer:

- Are there necessary and/or sufficient conditions for a graph to be representable?
- What is the "smallest" non-representable graph?

In general, it is difficult to prove that a graph G is not representable. Certainly, if any vertex-induced subgraph of G is not representable, then G is not representable. (Note that the term "vertex-induced" is important - adding any edge to $K_{4(3)}$ gives a representable graph.)

3.3 Parity in Intervals

In some cases, the parity of the intervals is irrelevant. For example, Figure 3.1 shows that C_6 can be represented by either all even intervals or odd intervals. These representations can be extended to any cycle; for example, the representation on the left can be extended to C_7 by adding one node to the right side of the longest interval, and adding a new interval of order two that overlaps both the longest interval and the current rightmost interval. The study of the representation of cycles gives the following result:

Proposition 3.3.1 *Under the odd-intersection interval model,*

1. *Any cycle may be represented using only odd intervals.*

2. Any even cycle may be represented using only even intervals.

To further investigate the properties of intervals of a specific parity, let us consider a "number-line" representation of the intervals; in other words, the vertices of the host path H are labeled $0, 1, 2, \ldots$, and a subpath of order m can be thought of as a set of m consecutive integers $\{a, a+1, \ldots, a+m-1 = b\}$; we will refer to this interval using the notation $[a, b]$. The intersection of two intervals is then either the empty set, or itself a set of consecutive integers $[a', b']$; the intervals are adjacent if the cardinality of this set is odd; in other words, if $b' - a' \equiv 0 \mod 2$ (NOT $1 \mod 2$, because the interval contains $b' - a' + 1$ nodes).

In the case of even intervals, we can state a few generalizations. If $I_i = [a_i, b_i]$, then $b_i - a_i \equiv 1 \mod 2$. If I_i and I_j are adjacent, then the intervals must overlap (one cannot be contained in the other); without loss of generality, assume that $a_i < a_j \leq b_i < b_j$ (note that $a_j = b_i$ gives an overlap of exactly one vertex). Then $[a_j, b_i]$ has odd cardinality, so $b_i - a_j \equiv 0 \mod 2$. But then

$$a_j - a_i = (b_i - a_i) - (b_i - a_j) \equiv 1 \mod 2.$$

The fact that the left endpoints of adjacent even intervals must have an odd difference leads to an important discovery:

Theorem 3.3.2 *If a graph G has an odd-intersection interval representation using only even intervals, such a representation is a 2-coloring of G.*

Proof. Suppose $V(G) = \{I_1, I_2, \ldots, I_n\}$, where $I_i = [a_i, b_i]$. Then let $V_1 = \{I_i : a_i - a_1 \equiv 0 (\mod 2)\}$, and $V_2 = \{I_i : a_i - a_1 \equiv 1 (\mod 2)\}$. Because the left endpoints of even intervals in conflict must have an odd difference, both V_1 and V_2 must represent independent sets, and thus G has a 2-coloring. □

The following corollaries are an immediate consequence of this theorem.

Corollary 3.3.3 *If G has an odd-intersection interval representation using only even intervals, then G is bipartite.*

Corollary 3.3.4 *If G contains an odd cycle, then any odd-intersection interval representation of G must contain at least one odd interval.*

The converse of Corollary 3.3.3 (that any bipartite graph is representable using only even intervals) is not true, because the number of bipartite graphs grows more quickly than the number of possible representations.

In a similar vein, the left endpoints of adjacent odd intervals must have an even difference. This leads us to another interesting result, using intervals of a fixed odd order.

Theorem 3.3.5 *If G has an odd-intersection interval representation using only odd intervals of the same order, then G is chordal.*

Proof. Consider the leftmost interval in the representation (if more than one interval shares the leftmost endpoint, any one can be chosen). Call this interval v; without loss of generality, let its endpoints be $[0, 2k]$. If two intervals u and w are adjacent to v, then they cannot be disjoint (as they are the same length, and v is the leftmost interval). Further, the left endpoints of u and v must have an even difference, as must the left endpoints of v and w. But this means the left endpoints of u and w have an even difference, and because they overlap, u and w are adjacent. Thus, the vertices adjacent to v form a clique.

Now, suppose u is left of w (i.e., the left endpoint of u has a smaller index than the left endpoint of w). Any vertex to the right of u that is adjacent to u must also be adjacent to w, using a similar argument to the above. Further, as v is the leftmost interval, and v is adjacent to w, any vertex to the left of u that is adjacent to u is also adjacent to w. Then, by ordering the vertices in $N(v)$ by their left endpoints, we also order the neighborhoods of those vertices by inclusion. We can then eliminate v, and repeat this process on $G - v$; iterating, we obtain a strong elimination ordering for G. $\square$

Finally, using intervals of a fixed even order gives a result we might guess from the previous two theorems:

Theorem 3.3.6 *If G has an odd-intersection interval representation using only even intervals of the same order, then G is chordal bipartite.*

Proof. By Corollary 3.3.3, G is bipartite.

Suppose that $v_1, v_2, \ldots, v_2k$ form a cycle of length at least 6 in G, where v_i is adjacent to v_{i+1} and v_{i-1}. Assume, without loss of generality, that v_1 is the leftmost interval in this cycle, and consider v_2 and v_2k. If the endpoints of these intervals are the same, then these vertices are (false) twins, and therefore v_3 must be adjacent to v_2k. If the endpoints are not the same, assume (again, without loss of generality) that the left endpoint of v_2 comes before that of v_2k. Because the intervals are of fixed length, and because v_1 is the leftmost interval, v_3 must then overlap v_2k. Further, because the left endpoints of v_2 and v_3 have an odd difference, and the left endpoints of v_2 and v_2k have an even difference, the left endpoints of v_3 and v_2k must have an odd difference, so v_3 is adjacent to v_2k. Thus, every cycle of length at least 6 in G contains a chord, and G is chordal bipartite. □

We can also establish a link between odd-intersection interval graphs and interval graphs (in which each vertex is represented by an interval on the real line, and two vertices are adjacent if the corresponding intervals have nonempty intersection) and unit interval graphs (a subclass of interval graphs in which each interval is of fixed length).

Theorem 3.3.7 ***a.*** *If G is an interval graph, then G has an odd-intersection interval representation using only odd intervals.*

b. *A graph G is a unit interval graph if and only if G has an odd-intersection interval representation using only intervals of fixed odd order.*

Proof. **a.** Create a path H' with twice as many nodes as there are intervals of G); number the nodes of H' from left to right. The endpoints of the intervals of G form an ordered set left-to-right (note that G has a representation in which no interval shares an endpoint); each interval I then leads to a subpath S' whose endpoints correspond to the ordinality of the endpoints of I. To guarantee that overlapping intervals have an odd intersection, create the host path H and subpaths $\{S\}$ by subdividing every edge of H' and the subpaths $\{S'\}$.

b. If G is a unit interval graph, G is also equivalent to an $\tilde{n}$-graph, in which vertices are represented by sets of n consecutive integers [34]. This leads to a path H' and collection of subpaths $\{S'\}$ whose nodes have the integer coordinates corresponding to these sets. To guarantee that overlapping intervals have an odd intersection, create the host path H and subpaths $\{S\}$ by subdividing every edge of H' and the subpaths $\{S'\}$.

Now, suppose G has a representation using only intervals of fixed odd order. If v has endpoints $[2i, 2k]$, then if u has left endpoint $2i < 2j + 1 < 2k$, then the intersection of u and v contains $2(k - j)$ endpoints, so u and v are not adjacent. If we number the vertices of the host path H, the set of vertices whose left endpoints have even index and the set of vertices whose left endpoints have odd index will have no edges between them. Then we can obtain a representation of the same graph by "shifting" all of the vertices in one of these sets so that they have no intersection whatsoever with any vertex in the other set. Thus, we may assume that the left endpoint of every vertex has an even index. Then, by replacing each subpath with a real interval with the same endpoints, we obtain an interval graph where each interval has the same length, which can be scaled so that each interval has unit length. □

Note that the converse of part (a.) is not true, due to the possibility of containment with intervals of different orders. In particular, recall that chordless cycles are representable using only odd intervals.

3.4 Counting Representations

3.4.1 Regular and Canonical Representations

We are interested in a method for determining the number of possible (distinct) arrangements of n intervals in the odd-intersection interval model. To begin, we will examine the question of whether there exists a canonical form for a set of similar arrangements of intervals.

Given a host path H and a set of intervals $S = I_1, I_2, \ldots, I_n$, if we subdivide any edge of H twice (similarly subdividing each interval that shares this edge), then the arrangement of the intervals in S remains essentially the same: Any interval which did not already contain this edge is unaffected, and if two intervals share this edge, then the number of nodes in their intersection increases by two, so neither the parity of the intersection, nor that of the intervals themselves, is changed. In addition, if two intervals are disjoint, and this edge is part of the "space" between those intervals, the parity of the space is likewise unchanged.

Similarly, if H contains three consecutive nodes such that every interval contains either all three nodes or none of them, then we can perform two edge contractions without changing the parity of any interval, any intersection of two intervals, or the distance between any two disjoint intervals; and thus without changing the arrangement of the intervals.

Also, we can extend any interval I_k by appending two nodes to one endpoint (we will assume that H is sufficiently long that I_k remains a subpath of H), provided that neither of the appended nodes is the endpoint of any other interval (in several cases, the nodes can be the endpoint of another interval without changing the graph, or even the basic form of the arrangement of intervals, but this condition allows us to avoid unnecessary case analysis).

Theorem 3.4.1 *If a graph G is representable, then G has a representation such that the endpoints of each interval are unique.*

Proof. We may assume a number-line representation, as in Section 3.3, with the interval $I = [a, b]$ identified by its endpoints. Begin by ordering the intervals $I_1, I_2, \ldots, I_n$, such that, for $1 \leq i \leq n$ and $I_i = [a_i, b_i]$:

1. $a_i \leq a_{i+1}$
2. If $a_i = a_{i+1}$, then $b_i \leq b_{i+1}$

Then the following algorithm produces a representation in which all of the endpoints are distinct; i.e., $a_i = a_j$ if and only if $i = j$, $b_i = b_j$ if and only if $i = j$, and $a_i \neq b_j$ for any i, j.

1. Beginning with $i = 1$, if $a_i = a_{i+1}$, or if $a_i = b_j$ for $j < i$, then:
   - subtract 2 from a_i.
   - For $j < i$, subtract 2 from a_j
   - For $j < i$, subtract 2 from b_j if $b_j < a_i$.

   This is equivalent to twice subdividing the edge of the host path H directly before a_i (as well as any interval containing that edge), then appedning two vertices to the left end of I_i. Repeat this process for $i = 2, \ldots, n$. After this step, $a_1 < a_2 < \cdots < a_n$, and $a_i \neq b_j$ for any i, j.

2. starting now with $i = n$, if $b_i = b_j$ for any $j < i$, then:
   - add 2 to b_i.
   - For $k > i$, add 2 to a_k if $b_i < a_k$. Note that if $k \leq i$, $a_k < b_i$ necessarily.
   - For $\ell \neq i$, add 2 to b_ℓ if $b_i < b_\ell$.

   This is equivalent to twice subdividing the edge of the host path H directly after b_i (as well as any interval containing that edge), then appedning two vertices to the left end of I_i. Repeat this process for $i = n - 1, \ldots, 1$. After this step, each right endpoint will be unique, giving us the desired representation.

□

With this knowledge, given any set S of subpaths of H, we can construct a set S' of subpaths of a host H' that represents the same graph, such that each node of H' is the endpoint of at most one interval. We will call the representation (H', S') a *regular* representation.

Remark 3.4.2 *The algorithm in Theorem 3.4.1 produces a representation which first orders the intervals by their left endpoints, and then, given identical right endpoints, chooses to "shift" the endpoint of the interval with the rightmost left endpoint. In this way, the algorithm minimizes the number of intersections in which one interval is contained in another. While containment in a representation is often unavoidable, having intervals overlap instead is usually more desirable when attempting to find a representation of a specific graph, as it allows more flexibility in the possible neighborhoods of the intervals involved.*

We can simplify a representation by creating a "code" that allows us to view a representation as a line of data, rather than a large number of drawn intervals. One possible implementation of such a code is as follows:

- Number the intervals in order of their left endpoints; the interval with the leftmost left endpoint is interval 1, then next left endpoint is interval 2, and so on.
- Traversing H from left to right, whenever an endpoint is reached, append the number of that interval to the code. Because the code will be read from left to right, there is no need to specify whether it is the left or right endpoint of the interval. The first entry in the code will always be 1, and each number will appear twice in the code.
- If the number of nodes between consecutive endpoints is odd, place an "o" between the corresponding numbers in the code; if it is even, place an "e" there.

For example, the code 1e2o3o1e3o2 corresponds to the representation in Figure 3.4.1. The graph represented is P_3. Note that a representation with n intervals will have a code of length $4n-1$.

We can now create an equivalence relation between arrangements of intervals. We can say two regular representations (S, H) and (S', H') are *equivalent* if their representation code is the same. Furthermore, we can create a *canonical representation* for each equivalence class of regular representations, using the code. Traversing the code from left to right:

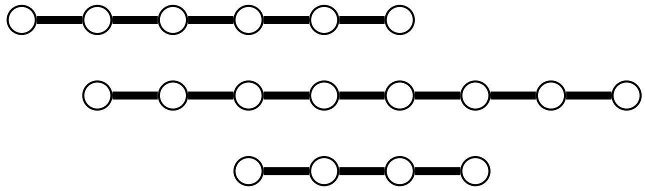


Figure 3.4: A regular representation of P_3.

- When you reach a number for the first time, create a new interval, and add a node to the host path, and to any intervals which have not yet reached their right endpoint.
- When you reach a number for the second time, append the right endpoint to the corresponding interval, and append a node to the host path, and to all intervals which have not yet reached their right endpoint.
- When you reach an "o," append a node to the host path, and to all intervals which have not yet reached their right endpoint.
- When you reach an "e," do nothing.

The creation of a canonical representation leads to our first result about the maximum order of the host path H:

Theorem 3.4.3 *If G is a graph of order n, and G is representable, then there is a representation (H, S) of G in which the order of H is at most $4n - 1$.*

Proof. Given any canonical representation of G, the length of the corresponding representation code is $4n-1$. If every non-endpoint entry is an "o," then every single character in the code requires the creation of exactly one node in H. □

Note that while a canonical representation is unique in its equivalence class, that equivalence class may not be the unique representatives of a graph. For example, P_3 is also represented by the code 1e2o3o1e2o3. Note also that the actual upper bound on the order of H is certainly less than $4n - 1$; for example, if the code begins "1o2," then removing the two leftmost nodes of interval 1 will lead to a representation of the same graph with a host path of length at most $4n - 3$, simply by allowing two intervals to share an endpoint. In fact, P_3 can be represented using a host path of order 2.

3.4.2 Counting Canonical Representations

We can use canonical representations to count the number of unique representations on n vertices:

Theorem 3.4.4 *The number of canonical representations on n vertices is* $2^{n-1}\frac{(2n)!}{n!}$.

Proof. The proof is inductive on n. There are two canonical representations for one vertex: an even interval and an odd interval. For $n > 1$, we can obtain any canonical representation on n vertices by adding a leftmost interval to a canonical representation on $n-1$ vertices. The left endpoint of this interval comes before the previous first interval; the distance between these two endpoints can be chosen to be even or odd. The right endpoint of this interval can now be placed after any endpoint (left or right), including the newly placed left endpoint, for a total of $2n-1$ places to place the right endpoint. Finally, the difference between the right endpoint and the point before it may be chosen to be even or odd, subdividing intervals if necessary — note that if this point is placed between two points in the previous representation, this creates two new sub-intervals, and choosing the parity of one will force the parity of the other. Then there are $(2)(2n-1)(2) = 2\frac{(2n)(2n-1)}{n}$ possible ways to add the new interval. By the inductive hypothesis, the number of representations on $n-1$ vertices is $2^{n-2}\frac{(2n-2)!}{(n-1)!}$, so the number of representations on n vertices is $2^{n-1}\frac{(2n)!}{n!}$, proving the induction. $\square$

If, in the previous proof, we require that all of the intervals are of a specific parity, either even or odd, then setting the parity of the distance between the left endpoint of the new interval and the first interval in the previous representation will force the parity of any subintervals created by placing the right endpoint. This divides the number of possible new representations by 2 at each step, which leads to the following count of canonical representations of specific parity:

Theorem 3.4.5 *The total number of canonical representations on n vertices, in which all intervals are even or all intervals are odd is* $\frac{(2n)!}{2(n!)}$.

3.4.3 Counting Bipartite Graphs

Our goal in this section is to show that not all bipartite graphs are representable as odd-intersection interval graphs using only even intervals. To do this, we will show that the number of bipartite graphs grows more quickly than the number of possible representations. We first define $\mathscr{G}(n, m, p)$ to be the model which generates a bipartite graph with parts of order n and m, by including each possible edge with probability p.

Lemma 3.4.6 *Almost all bipartite graphs are connected, with diameter 3. In particular, almost all balanced bipartite graphs are connected.*

Proof.

Let $G \in \mathscr{G}(n, m, \frac{1}{2})$ have parts U (order n) and V (order m), and suppose $u, v \in V(G)$.

If u and v are in the same part, WLOG U, then for each vertex w in V, the probability that w is adjacent to both u and v is $\frac{1}{4}$. The probability that no vertex in V is adjacent to both u and v is then $\left(1 - \frac{1}{4}\right)^m = \left(\frac{3}{4}\right)^m$, which goes to 0 as $m \to \infty$.

If $u \in U$ and $v \in V$, there is a $u - v$ path of length 3 if u is adjacent to some vertex w in V which is distance 2 from v. For each $w \neq v$ in V, then, the probability that there is no $w - v$ path of length 2 is $\left(\frac{3}{4}\right)^{n-1}$ ($n - 1$ because we will assume that this path does not travel through u). The probability that there is no path of the form $u - w - x - v$ (where x is some vertex in U) is then the probability that there is no $uw \notin E(G)$, plus the probability that $uw \in G$, but there is no $w - v$ path of length 2. This probability is given by the expression $\left[\frac{1}{2} + \frac{1}{2}\left(\frac{3}{4}\right)^{n-1}\right]^{m-1}$, which goes to 0 as m and n increase to ∞. We note that $\frac{1}{2} + \frac{1}{2}\left(\frac{3}{4}\right)^{n-1} < 1$, and thus this probability tends to 0 as $m \to \infty$ for any fixed n.

Finally, note that all of the above statements hold if m is fixed equal to n. □

The next result relies on a a modification of a result by Bollobás [4]. As preliminaries, denote the normal density function by $\varphi(x)$, and the normal distribution function by $\Phi(x)$. Note that, as $x \to \infty$,

$$1 - \Phi(x) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2} \tag{3.4.1}$$

For a given n, choose $G \in \mathscr{G}(n, n, 1/2)$. We are interested primarily in the degree sequence $d_1^U \geq d_2^U \geq \cdots \geq d_m^U$ of U, and the analogous degree sequence in V. Note that, for any vertex $v \in G$,

$deg(v) \sim bin(n, p)$. Let

$$b(n;k) = \binom{n}{k} p^k q^{n-k}$$

$$S(n;K,L) = \sum_{k=K}^{L} b(n;k)$$

(if no L is given, assume $L = n$). Note that, if $x = o(n^{1/6})$, then for $K = pn + x(pqn)^{1/2} = pn + o(n^{2/3})$,

$$S(n;K) \approx 1 - \Phi(x). \tag{3.4.2}$$

Finally, let X_K^U (resp. X_K^V) be the number of vertices in U (resp. V) of degree at least K. The next fifteen statements are adapted from [4]. If the proof is not supplied, it may be assumed that the proof from the original paper is essentially identical. Unless otherwise stated, the results will be for vertices in partition U; in each case, there is an analogous result for vertices in V.

Lemma 3.4.7 *Let $1 \leq K' \leq K'' \leq n$. Let $\mu' = E(X_{K'}^U), \sigma'^2 = E((X_{K'}^U - \mu'))^2$, and define μ'' and σ''^2 analogously. If a is an integer satisfying $\mu'' < a \leq \mu'$ then*

$$P(d_a \geq K'') \leq \min\left\{\frac{\mu''}{a}, \frac{\sigma''^2}{(a-\mu'')^2}\right\}$$

and

$$P(d_a < K') \leq \frac{\sigma'^2}{(\mu' - a + 1)^2}.$$

Lemma 3.4.8 *For every integer $K, 0 \leq K \leq n$, we have $E(X_K^U) = nS(n;K)$.*

Proof. The probability that any vertex in U has degree k is $\binom{n}{k} p^k q^{n-k}$. The probability that said vertex has degree at least K is then $S(n;K)$, and there are n vertices in U. □

Lemma 3.4.9 *Let K be an integer, $1 \leq K \leq n$. Let $\mu = E(X_K^U)$ and $\sigma^2 = E((X_K^U - \mu)^2)$. Then $\sigma^2 = \mu(1 - S(n;K))$.*

Proof. Let $Y_K^U = \binom{X_K^U}{2}$. Then $E((X_K^U)^2) = E(X_K^U) + 2E(Y_K^U)$. Note that, because the degrees of each vertex in U are independent and identically distributed, the probability that any two vertices have degree at least K is $(S(n;K))^2$, so $E(Y_K^U) = \binom{n}{2}(S(n;K))^2$. Then:

$$\begin{aligned}\sigma^2 &= E((X_K^U)^2) - \mu^2 \\ &= \left(\mu + (n^2-n)[S(n;K)]^2\right) - n^2[S(n;K)]^2 \\ &= \mu - n[S(n;K)]^2 \\ &= \mu\,(1 - S(n;K))\end{aligned}$$

□

Lemma 3.4.10 *Let $K = pn + x(pqn)^{1/2} = pn + o(n^{2/3})$. Let $\mu = E(X_K^U)$ and $\sigma^2 = E((X_K^U - \mu)^2)$. Then $\sigma^2 \leq \mu$.*

Proof. This follows immediately from the fact that $S(n;K)$ is a probability, namely the probability that a given vertex in U has degree at least K. □

Theorem 3.4.11 *Let $K' > K''$ be such that $K' = pn + o(n^{2/3})$ and $K'' = pn + o(n^{2/3})$. Suppose $\mu'' = E(X_{K''}^U) < a < \mu' = E(X_{K'}^U$. Then*

$$P(d_a \geq K'') \leq \min\left\{\frac{\mu''}{2}, \frac{\mu''}{(a-\mu'')^2}\right\}$$

and

$$P(d_a < K') \leq \frac{\mu'}{(\mu' - a)^2}.$$

Theorem 3.4.12 *Let a be a fixed natural number and let f_1, f_2 be fixed real numbers satisfying $e^{f_1} < a < e^{f_2}$. Then*

$$P(d_a \geq K'') \leq \min\left\{\frac{e^{f_1}}{a}, \frac{e^{f_1}}{(a-e^{f_1})^2}\right\} + o(1)$$

and

$$P(d_a < K') \leq \frac{e^{f_2}}{(e^{f_2} - a)^2} + o(1).$$

Corollary 3.4.13 *If a is fixed and $C(n) \to \infty$ arbitrarily slowly, then almost every graph in $\mathscr{G}(n, n, 1/2)$ satisfies*

$$K^* + C(n)\left(\frac{n}{\log n}\right)^{1/2} \geq d_1^U \geq d_2^U \geq \cdots \geq d_a^U \geq K^* - C(n)\left(\frac{n}{\log n}\right)^{1/2},$$

where

$$K^* = pn + (2pqn\log n)^{1/2} - \left(\frac{pqn}{8\log n}\right)^{1/2} \log\log n.$$

Theorem 3.4.14 *Suppose $a \to \infty$ and $K' < k < K''$ are such that with $\mu = E(X_k^U)$, $\mu' = E(X_{K'}^U)$, and $\mu'' = E(X_{K''}^U)$ we have*

$$|\mu - a| \le a^{1/2} \tag{3.4.3}$$

$$C(n)\mu^{1/2} \le \mu' - \mu \le \mu \tag{3.4.4}$$

and

$$C(n)\mu^{1/2} \le \mu - \mu'' \le \mu \tag{3.4.5}$$

where $C(n) \to \infty$ arbitrarily slowly. Then almost every bipartite graph satisfies $K' \le d_a^U \le K''$.

Lemma 3.4.15 *Let $c > 2$ be a constant and suppose $K = pn + x(pqn)^{1/2} = pn + o(n^{2/3})$ is such that $\mu = E(X_K^U) \to \infty$. Suppose furthermore that $C(n) \le \mu^{1/2}$ and $C(n) \to \infty$. Then the inequalities 3.4.4 and 3.4.5 are satisfied for $K' = K - \epsilon(pqn)^{1/2}$ and $K'' = K + \epsilon(pqn)^{1/2}$, where ϵ is chosen as follows:*

- *If $|x| \ge c$, then $\epsilon = C(n)/(x\mu^{1/2})$.*
- *If $|x| < c$, then $\epsilon = C(n)n^{-1/2}$.*

Theorem 3.4.16 *Let $a \le n/2$ be a natural number and let $C(n) \to \infty$ arbitrarily slowly. Define x by $1 - \Phi(x) = a/n$. Then almost every graph in $\mathscr{G}(n, n, 1/2)$ satisfies*

$$\left| d_a^U - pn - x(pqn)^{1/2} \right| \le C(n) \left(\frac{n}{a \log(n/a)} \right).$$

Corollary 3.4.17 *Let $C(n) \to \infty$ arbitrarily slowly. Then*

1. *If* $a = O\left(\log n/(\log\log n)^4\right)$ *then almost every graph in* $\mathscr{G}(n,n,1/2)$ *satisfies*

$$\left| d_a^U - pn - (2pqn\log n)^{1/2} + (\log\log n)\left(\frac{pqn}{8\log n}\right)^{1/2} + \log(2\pi^{1/2}a)\left(\frac{pqn}{2\log n}\right)^{1/2} \right| \le C(n)\left(\frac{n}{a\log n}\right)^{1/2}.$$

2. *If* $a = O((\log n)^2)$, *then define* x *by* $\frac{\varphi(x)}{x} = \frac{a}{n}$. *Then almost every balanced bipartite graph satisfies*

$$\left| d_a^U - pn - x(pqn)^{1/2} \right| \le C(n)\left(\frac{n}{a\log n}\right)^{1/2}.$$

3. *If* $a - m/2 = O(n^{1/2}$, *then almost every graph in* $\mathscr{G}(n,n,1/2)$ *satisfies*

$$\left| d_a^U - pn \right| \le C(n).$$

Theorem 3.4.18 *Suppose* $a \to \infty$ *and* $a = o(n)$. *Set*

$$\begin{aligned} K &= K(a,n) \\ &= pn + (2pqn\log(n/a))^{1/2} - \\ &\quad (\log\log(n/a) + log4\pi)\left(\frac{pqn}{8\log(n/a)}\right)^{1/2}. \end{aligned}$$

Then almost every graph in $\mathscr{G}(n,n,1/2)$ *satisfies* $\left| d_a^U - K \right| = O\left(\frac{n}{\log(n/a)}\right)^{1/2}$.

Theorem 3.4.19 *Let* $a = o(n)$ *and let* $c(n)$ *be a positive function tending to 0 arbitrarily slowly. Then almost every graph in* $\mathscr{G}(n,n,1/2)$ *satisfies*

$$d_i^U - d_{i+1}^U \ge \frac{c(n)}{a^2}\left(\frac{n}{\log(n/a)}\right)^{1/2}$$

for every $i < a$.

Theorem 3.4.20 *Suppose* $a \to \infty$ *and* $a = o(n)$. *Let* $C(n) \to \infty$ *arbitrarily slowly. Then almost every graph in* $\mathscr{G}(n,n,1/2)$ *satisfies*

$$d_i^U - d_{i+1}^U \le \frac{C(n)}{a^2}\left(\frac{n}{\log(n/a)}\right)^{1/2}$$

for some $i < a$.

Corollary 3.4.21 *If $a = o(n^{1/4})/(\log n)^{1/4}$ then almost every graph in $\mathscr{G}(n,n,1/2)$ is such that $d_1^U > d_2^U > \cdots > d_a^U$. If $a \neq o(n^{1/4})/(\log n)^{1/4}$ then almost every graph in $\mathscr{G}(n,n,1/2)$ is such that $d_i^U = d_{i+1}^U$ for some $i < a$.*

Now that we have the number of terms which are strictly decreasing, we can use this information to create a canonical labeling of the vertices in each part (this method was used in [1] to show that almost every graph admits only the trivial automorphism.)

Theorem 3.4.22 *Almost every graph in $\mathscr{G}(n,n,1/2)$ has trivial automorphism group.*

Proof. Let $G \in \mathscr{G}(n,n,1/2)$ have parts U and V as above. Order the vertices of U in descending order of degree, i.e., $d(u_1) \geq d(u_2) \geq \cdots \geq d(u_n)$. Order the vertices of V in the same manner.

Set $k = 3log_2 n$, and note that $k = o(n^{1/4})/(\log n)^{1/4}$.

For each vertex $u_i \in U$, define a function $g_U : U \leftarrow \mathbb{N}$ by

$$g_U(u_i) = \sum_{j=1}^{k} a(i,j)2^j$$

where $a(i,j)$ is 1 if u_i is adjacent to v_j and 0 otherwise. In other words, f assigns a binary string to u_i based on its adjacencies to the k highest-degree vertices in V. Similarly, $g_V : V \leftarrow \mathbb{N}$ is defined by

$$g_V(v_i) = \sum_{j=1}^{k} b(i,j)2^j$$

where $b(i,j)$ is 1 if v_i is adjacent to u_j and 0 otherwise.

We first check that there is no automorphism which maps u_i to u_j or v_i to v_j for some $i \neq j$. This may be possible if:

1. $d(u_i) = d(u_{i+1})$ or $d(v_i) = d(v_{i+1})$ for $1 \leq i \leq k$.

2. $g_U(u_i) = g_U(u_j)$ or $g_V(v_i) = g_V(v_j)$ for any $i \neq j$.

By Corollary 3.4.21, the k highest degrees in U and the k highest degrees in V will almost surely be unique, so the probability of the first condition occurring approaches 0 as $n \to \infty$. For the second condition to occur, both u_i and u_j must be adjacent to the same vertices in the set $\{v_1, \ldots, v_k\}$

or v_i and v_j must be adjacent to the same vertices in the set $\{u_1, \dots, u_k\}$. The probability of either case is $(1/2)^k = n^{-3}$, so the probability over all i and j is at most $\binom{n}{2}n^{-3} = O(n^{-1})$, which is also approaching 0 as $n \to \infty$.

So, there is almost surely no automorphism which re-arranges the vertices within either part. By Lemma 3.4.6, G is connected, so the partitioning is fixed; in other words, if there is an automorphism φ such that $\varphi(u_i) = v_j$ for any i or j, then $\varphi(U) = V$ (and, conversely, $\varphi(V) = U$).

Suppose, then, that G does not satisfy either of the above conditions, but an automorphism φ exists with $\varphi(U) = V$. Consider $\varphi^2(u_i) = \varphi(\varphi(u_i)) = u_j$. If $i \neq j$, then φ^2 is an automorphism that re-arranges the verices of U, which we have assumed is not the case. Then $\varphi^2(u_i) = u_i$ and $\varphi(v_i) = v_i$ for $1 \leq i \leq n$, so φ is an involution.

If $\varphi(u_i) = v_j$, then $d(u_i) = d(v_j)$. So, for $1 \leq i \leq k$, it must be the case that $i = j$. But if this is the case, $g_U(u_i)$ must be equal to $g_V(v_i)$ for each of these vertices as well; specifically, $g_U(u_1) = g_V(v_1)$. Thus, the probability that there is an automorphism is at most $P\left(g_U(u_1) = g_V(v_1)\right)$. Because these strings are always equal in the first position, this probability is $(1/2)^{k-1} = O(n^{-3})$, which approaches 0 as $n \to \infty$. Therefore, G almost surely has no non-trivial automorphisms. □

This, at last, leads to the promised result of this section:

Theorem 3.4.23 *Not all bipartite graphs are representable.*

Proof. It will suffice to show that the number of balanced bipartite graphs grows asymptotically more quickly than the number of representations. To simplify the arithmetic somewhat, we will consider graphs on $2n$ vertices.

Let $B_u(2n)$ be the number of unlabeled balanced bipartite graphs of order $2n$, and let $B_\ell(2n)$ be the number of labeled balanced bipartite graphs of order n. Because a balanced bipartite graph almost surely has no non-trivial automorphism, $B_u(2n) \sim \frac{1}{(2n)!}B_\ell(2n)$ asymptotically [3]. We are then interested in the asymptotic growth rate of $B_\ell(2n)$. We can create a labelled balanced bipartite graph as follows:

Let vertex 1 be in part U, and choose n of the remaining vertices to be in part V. There are then n^2 possible edges, and thus 2^{n^2} possible ways for the edges to be arranged between the two parts.

This method produces $\binom{2n-1}{n}2^{n^2}$ labelled bipartite graphs. However, any graph with multiple components has isomorphic graphs, in which the parts of one or more components are switched.

But, because Lemma 3.4.6 states that almost all balanced bipartite graphs are connected, we conclude that $B_\ell(2n)$ grows asymptotically like $\binom{2n-1}{n}2^{n^2}$.

The asymptotic growth rate of $B_u(2n)$ is then $\frac{2^{n^2}}{2(n!)(n!)}$. The number of possible representations on $2n$ vertices is $R(2n) = 2^{2n-2}\frac{(4n)!}{(2n)!}$. The ratio $B_u(2n)/R(2n) = \frac{2^{(n-1)^2}}{(4n)!}\binom{2n}{n}$. Using Stirling's approximation, the ratio $\frac{2^{(n-1)^2}}{(4n)!}$ has a growth rate of $O\left(\frac{2^{n^2}}{(4n)^n}\right)$. This can be simplified to $\left(\frac{2^n}{4n}\right)^n$, and because 2^n is growing faster than $4n$, the ratio is tending towards infinity. The binomial is also growing to infinity, so $B_u(2n)$ is growing faster than $R(2n)$. Then there is a number N such that $B_u(2n) > R(2n)$ for all $n > N$. In other words, for $n > N$, there are more balanced bipartite graphs, and therefore more bipartite graphs, then there are possible odd-intersection interval representations (regardless of the parity of the intervals). □

A study of the sequence of bipartite graphs on n nodes (a partial sequence is available at http://www.research.att.com/˜njas/sequences/A033995) indicates that the number of bipartite graphs exceeds the number of even-interval representations sometime after $n = 20$. However, it is likely that there is an unrepresentable bipartite graph on fewer nodes. In fact, we have a specific candidate in mind:

Conjecture 3.4.24 *The hypercube Q_4 is not representable using even intervals.*

3.5 Open Problems

Only one minimal forbidden subgraph is known; others must exist, particularly in light of the result of Theorem 3.4.23. In particular, it is not known whether $K_{4(3)}$ is the smallest non-representable graph. Possible candidates for this graph include the Petersen graph, and the Möbius ladder M_8.

At present, no algorithm exists for determining whether a graph is representable.

In addition, while Theorems 3.4.4 and 3.4.5 give formulae to count the number of representations of order n, many of these representations have an empty edge set. Further combinatorial study of canonical representations is warranted.

Chapter 4

Cross Comparison Graphs

4.1 Introduction

A *cross comparison graph model* is a model in which each vertex v is assigned a rank r_v and a tolerance t_v, and two vertices u and v are in conflict if $r_v \geq t_u$ and $r_u \geq t_v$. A *cross comparison representation* of a graph G is a representation in which each vertex $v \in V(G)$ is assigned a rank and a tolerance, such that the vertices u and v are in conflict if and only if $uv \in E(G)$.

A vertex v is referred to as *bounded* if $t_v \leq r_v$. A vertex v is a *retro* vertex if $r_v \leq t_v$ (we do not refer to these vertices as *unbounded*, because of the possibility that r_v and t_v are incomparable).

Jamison showed in [24] that the cross comparison representation is universal. This was done using vector representations, in which the rank and tolerance of each vertex v are vectors ($\rho(v)$ and $\varphi(v)$, respectively) in $\mathbb{R}^d$, and the comparison rule between vectors is that $\mathbf{a} \geq \mathbf{b}$ if and only if $a_i \geq b_i$ for $1 \leq i \leq d$, where a_i and b_i are the i^{th} coordinates of the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively.

Because all graphs can be represented using cross comparison vectors, it is of some interest to find the lowest value of d for which a graph G can be represented using vectors in $\mathbb{R}^d$. We will call this value the *inefficiency* of G, and denote it d_G. A graph with a cross comparison representation on $\mathbb{R}^d$ will be referred to as *d-representable*. A one-dimensional cross comparison representation is referred to as an *efficient* representation, and G is an *efficient graph* if $d_G = 1$.

4.2 Vector Representations of Cross Comparison Graphs

We can quickly get the order of G as an upper bound on d_G, by labelling the vertices of G v_1 through v_n. For vertex v, the tolerance vector $\varphi(v)$ has a 1 in position i if $v = v_i$ and 0 in that position otherwise; and the rank vector $\rho(v)$ has a 1 in position i if v is adjacent to v_i and 0 in that position otherwise. This was, in fact, the representation used by Jamison in [24] to show that cross comparison graphs are universal. However, by making use of the fact that the vector coordinates do not have to be limited to 0 and 1, we can improve the efficiency of our representation.

4.2.1 A "Canonical" Representation

It is sometimes difficult to keep track of comparisons when the ranks and tolerances of several different vertices are equal. It is useful, then, to look at graphs in which each rank and tolerance has a distinct value, so that we can focus on strict inequality in our comparisons. Fortunately, this requirement does not restrict the number or efficiency of representable graphs.

Theorem 4.2.1 *If G has a cross comparison representation using vectors in $\mathbb{R}^d$, then G has a representation using vectors in $\mathbb{R}^d$ such that the i^{th} coordinate of every vector (rank and tolerance) has a distinct value, for $1 \leq i \leq d$. Further, if G has order n, the coordinate values are integers between 1 and $2n$.*

Proof. The following algorithm may be used to re-label the i^{th} coordinate of every vector. The process may be repeated for each coordinate, re-labeling the vertices as necessary. Label the vertices $v_1, v_2, \ldots, v_n$, such that:

1. If $j < \ell$, then $t_i(v_j) \leq t_i(v_\ell)$.

2. If $j < \ell$ and $t_i(v_j) = t_i(v_\ell)$, then $r_i(v_j) \leq r_i(v_\ell)$.

3. If two vertices have equal rank and tolerance in the i^{th} coordinate, their ordering is arbitrary.

Assign each coordinate the labels $1, 2, \ldots, 2n$, such that:

1. The labels are assigned in ascending order of the original values of the coordinates; i.e., if $r_i(v_j) < t_i(v_\ell)$, then $r_i(v_j)$ is assigned a lower label than $t_i(v_\ell)$, regardless of j and ℓ.

2. If $t_i(v_j) = r_i(v_\ell)$, then $t_i(v_j)$ is assigned the lower label, regardless of j and ℓ.

3. If $j < \ell$ and $t_i(v_j) = t_i(v_\ell)$, then $t_i(v_j)$ is assigned the lower label.

4. If $j < \ell$ and $r_i(v_j) = r_i(v_\ell)$, then $r_i(v_j)$ is assigned the lower label.

The first condition on this labeling ensures that if $r_i(v_j) < t_i(v_\ell)$ in the original representation, this inequality still holds under the new labeling. The second condition ensures that if $t_i(v_\ell) \leq r_i(v_j)$ in the original labeling, that now $t_i(v_\ell) < r_i(v_j)$. Therefore, the new labeling preserves all adjacencies in the original representation, so the new labeling is still a representation of G. □

This theorem can easily be extended to make every value unique, not just the values within a specific coordinate.

Corollary 4.2.2 *If G has a cross comparison representation using vectors in $\mathbb{R}^d$, then G has a representation using vectors in $\mathbb{R}^d$ such that every coordinate of every vector (rank and tolerance) has a distinct value. Further, if G has order n, the coordinate values are integers between 1 and $2dn$.*

Proof. After applying the algorithm of Theorem 4.2.1 to obtain unique values within each coordinate, it is sufficient to add the value $(2n)(i-1)$ to each value in the i^{th} coordinate of every vector. Thus, the i^{th} coordinate contains the values $(2n)(i-1)+1$ through $2ni$. □

Remark 4.2.3 *The algorithm of Theorem 4.2.1 produces a representation which attempts to minimize the number of situations in which $t_i(u) < t_i(v) < r_i(v) < r_i(u)$ for some pair of vertices u and v. The algorithm is similar to that of Theorem 3.4.1 in the previous chapter. The reasoning behind the algorithm is also similar, and will become clearer in the next section.*

4.3 Efficiency

While we now know a representation with all coordinate values distinct is possible, it will often be convenient to allow some coordinate values to be equal in our representations. A path, for example, can be represented by assigning a set of n vertices the ranks $1, 2, \ldots, n$, and respective tolerances $0, 1, \ldots, n-1$. Then every path has an efficient representation. To see what other graphs might have efficient representations, we will reprint two lemmas from [24]:

Lemma 4.3.1 *Every retro vertex in a cross comparison graph is simplicial.*

Lemma 4.3.2 *If P is linearly ordered, then every bounded P-cross comparison graph is an interval graph. Conversely, every interval graph is a bounded P-cross comparison graph for some linear order P.*

For $d > 1$, $\mathbb{R}^d$ is not linearly ordered, but we can still extend Lemma 4.3.2 for $\mathbb{R}^d$.

Lemma 4.3.3 *Every bounded cross comparison graph on $\mathbb{R}^d$ is a d-box graph. Conversely, every d-box graph is a bounded cross comparison graph on $\mathbb{R}^d$.*

Proof. Suppose G is a bounded cross comparison graph on $\mathbb{R}^d$. For each vertex v with rank $\rho(v) = (r_1, r_2, \ldots, r_d)$ and tolerance $\varphi(v) = (t_1, t_2, \ldots, t_d)$, then these vectors map to a d-box in which the i^{th} coordinate of each corner is either $\rho_i(v)$ or $\varphi_i(v)$. Conversely, if G is a d-box graph, the box that represents vertex v has an "upper corner" $A = (a_1, a_2, \ldots, a_d)$ and a "lower corner" $B = (b_1, b_2, \ldots, b_d)$, such that $a_i \geq b_i$ for $1 \leq i \leq d$. Then by letting $\rho(v) = A$ and $\varphi(v) = B$, we create a bounded cross comparison representation for G on $\mathbb{R}^d$. $\square$

Because, in $\mathbb{R}$, every vertex is either bounded or retro (or both, if $r_v = t_v$), we can begin to characterize graphs with efficient representation.

Theorem 4.3.4 *Suppose G is a graph with $d_G = 1$. Then G is chordal.*

Proof. We show this by finding a perfect elimination ordering for G. Let S be the set of retro vertices in the representation of G. Each of these vertices is simplicial by Lemma 4.3.1, and so these vertices can be eliminated first, in any order we like. The remaining vertices are bounded, and, by Lemma 4.3.2, form an interval graph. This subgraph is chordal, and thus has a perfect elimination ordering of its own. This gives us a perfect elimination ordering for G. $\square$

The converse of this theorem is not true; a computer algorithm can find chordal graphs with no efficient representation. The 3-sun, a graph of order 6, is the smallest order chordal graph with no efficient representation. The "bad aster," with 6 edges, is the chordal graph of smallest size with no efficient representation.

The *depth* of a rooted tree T is the number of levels in T, where the root counts as level 0. The *depth* $\delta(x)$ of a particular vertex x is the level number of that vertex; i.e., the distance to the root.

Proposition 4.3.5 *Let T be a tree of maximum degree Δ. T has a vector comparison representation for $d = 2$.*

Proof. Select a vertex r to be the root of T. We can assume that the children of any node are ordered in a *birthorder*. Let $\beta(v)$ denote the birthorder of v; then $\beta(v)$ then takes on integer values between 1 and Δ. Assume that $b(r) = 1$.

Our representation will depend on vertices with the same depth having different tolerance vectors. Define $(v)^k$ to be the k^{th} direct ancestor of v, where $0 \leq k \leq \delta(v)$; so, $(v)^0 = v$, $(v)^1$ is the parent of v, $(v)^2$ is the parent's parent, and so on, until we reach $(v)^{\delta(v)} = r$. Then, we define a tolerance function $\mathscr{B}(v)$ by

$$\mathscr{B}(v) = \sum_{k=0}^{\delta(v)} \frac{\beta((v)^k)}{(\Delta+1)^{\delta(v)-k}}.$$

Lemma 4.3.6 *If u is any descendant of v, then $\mathscr{B}(u) > \mathscr{B}(v)$. If $\delta(u) = \delta(v)$ and $\mathscr{B}(u) < \mathscr{B}(v)$, then if w is any descendant of u, $\mathscr{B}(w) < \mathscr{B}(v)$.*

Proof. For the first statement, it is sufficient to show $\mathscr{B}(u) > \mathscr{B}((u)^1)$, and for this, it is sufficient to note that $\mathscr{B}(u) = \frac{b(u)}{(\Delta+1)^{\delta(u)}} + \mathscr{B}((u)^1)$.

For the second statement, because u and v are on the same level, $\mathscr{B}(v) - \mathscr{B}(u) \geq \frac{1}{(\Delta+1)^{\delta(u)}}$. If w is any descendant of u, then

$$\begin{aligned}
\mathscr{B}(w) - \mathscr{B}(u) &\leq \frac{\Delta}{(\Delta+1)^{\delta(u)+1}} + \frac{\Delta}{(\Delta+1)^{\delta(u)+2}} + \cdots + \frac{\Delta}{(\Delta+1)^{\delta(w)}} \\
&< \frac{1}{(\Delta+1)^{\delta(u)}} \\
\Rightarrow \mathscr{B}(w) - \mathscr{B}(u) &< \mathscr{B}(v) - \mathscr{B}(u) \\
\Rightarrow \mathscr{B}(w) &< \mathscr{B}(v)
\end{aligned}$$

□

Lemma 4.3.7 *The function $\mathscr{B}(v)$ assigns a unique label between 1 and 2 to each vertex v.*

Proof. By Lemma 4.3.6, $\mathscr{B}(v) \geq \mathscr{B}(r) = 1$. The upper bound comes from the fact that $\mathscr{B}(v) \leq 1 + \sum_{k=1}^{\delta(v)} \frac{\Delta}{(\Delta+1)^{\delta(v)-k}} \leq 2$.

Uniqueness can be shown inductively on $\delta(v)$. If $\delta(v) = 0$, then $v = r$, the only vertex of depth 0. For larger values of $d(v)$, let $\mathscr{B}(u) = \mathscr{B}(v)$; note that Lemma 4.3.6 implies that $\delta(u) = \delta(v)$. Then

$$(\Delta+1)^{\delta(v)-1}\mathscr{B}(v) = (\Delta+1)^{\delta(v)-1}\mathscr{B}(u) = N + \tfrac{b}{(\Delta+1)^{\delta(v)}},$$

where N is an integer and $1 \leq b \leq \Delta$. Because birthorder is also between 1 and Δ, it follows that $b = b(v) = b(u)$. Then $\mathscr{B}(v) - \frac{b(v)}{(\Delta+1)^{\delta(v)}} = \mathscr{B}(u) - \frac{b(u)}{(\Delta+1)^{\delta(u)}}$. But this means $\mathscr{B}((v)^1) = \mathscr{B}((u)^1)$. By the inductive hypothesis, u and v have the same parent. But, u and v also have the same birthorder, so $u = v$. □

We are now ready to create our vector representation. Define $\varphi(v)$ as follows:

$$\varphi(v) = \begin{bmatrix} \mathscr{B}(v) \\ \delta(v) \end{bmatrix}$$

Define $\rho(v)$ as follows:

$$\rho(v) = \begin{bmatrix} \mathscr{B}(v) + \frac{\Delta}{(\Delta+1)^{\delta(v)+1}} \\ \delta(v) + 1 \end{bmatrix}$$

To show that this representation gives us the tree T, select vertices $u, v \in V(T)$, and examine their vectors. If $|\delta(u) - \delta(v)| > 1$, then either $\rho_2(u) < \varphi_2(v)$ or $\rho_2(v) < \varphi_2(u)$, so we do not have $\varphi(v) \leq \rho(u)$, and thus there is no edge between u and v.

If $\delta(u) = \delta(v)$, then $|\mathscr{B}(v) - \mathscr{B}(u)| \geq \frac{1}{(\Delta+1)^{\delta(v)-1}}$, so either $\rho_1(u) < \varphi_1(v)$ or $\rho_1(v) < \varphi_1(u)$, and again there is no edge between u and v.

We are left with the case where $|\delta(u) - \delta(v)| = 1$; in this case, $\rho_2(v) = \varphi_2(u)$ and $\rho_2(u) = \varphi_2(v)$. Without loss of generality, let $\delta(u) = \delta(v) + 1$. If v is the parent of u, then $\mathscr{B}(v) < \mathscr{B}(u) \leq \mathscr{B}(v) + \frac{\Delta}{(\Delta+1)^{\delta(v)}}$, so $\rho_1(v) > \varphi_1(u)$ and $\rho_1(u) > \varphi_1(v)$. Suppose, then, that v is not the parent of u. If $\mathscr{B}((u)^1) < \mathscr{B}(v)$, we must have $\rho_1(u) < \varphi_1(v)$. If $\mathscr{B}((u)^1) > \mathscr{B}(v)$, then $\rho_1(v) < \mathscr{B}((u)^1) < \mathscr{B}(u)$, so $\rho_1(v) < \varphi_1(u)$. In either case, there is no edge between u and v. Thus we get the tree T. □

Corollary 4.3.8 *Let T be a tree. If T is a caterpillar, then $d_T = 1$. Otherwise, $d_T = 2$.*

Proof. If T is a caterpillar, then T has an interval representation, and $d_T = 1$ by Lemma 4.3.2. If T is not a caterpillar, then T contains a copy of the bad aster, and has no one-dimensional representation, but $d_T = 2$ by Theorem 4.3.5. □

It is worth examining, then, how retro vertices interact with bounded vertices in a cross comparison model, particularly one with one-dimensional vectors. If u is a retro vertex and v a bounded vertex, then u is in conflict with v if and only if u is "contained" in v; that is, if

$t_v \leq r_u \leq t_u \leq r_v$. We could then think of an efficient representation as an interval graph, in which the intervals corresponding to retro vertices are only adjacent to regular intervals, and only those which completely contain them.

Retro vertices cannot be adjacent to each other, regardless of dimension, as the next lemma shows.

Lemma 4.3.9 *The set of retro vertices which are not also bounded vertices in a cross comparison graph form an independent set.*

Proof. Let u and v be retro vertices, and assume, without loss of generality, that $r_v \geq r_u$. Then $r_u \leq r_v \leq t_u$, so $r_u \geq t_v$ if and only if $r_u = r_v = t_v$. Also, $r_u \leq t_u$, so u and v are adjacent if and only if $r_u = t_u = r_v = t_v$. However, in this case, both u and v are also bounded vertices. □

The next theorem shows that, if we can characterize the graphs with efficient representations, we can characterize the graphs with d-dimensional representation for any d. Suppose G is a cross comparison graph, and for $v \in V(G)$, $\rho(v)$ and $\varphi(v)$ are the respective (d-dimensional) rank and tolerance vectors of v. We define the graph G_i by letting $V(G_i) = V(G)$, and giving each vertex v rank $r(v) = \rho_i(v)$ and tolerance $t(v) = \varphi_i(v)$. By definition, G_i is an efficient cross comparison graph.

Lemma 4.3.10 *The graph G_i contains G as an edge-induced subgraph.*

Proof. If $uv \in E(G)$, then $\rho(u) \geq \varphi(v)$ and $\rho(v) \geq \varphi(u)$. By our comparison rule, this means, that $\rho_i(u) \geq \varphi_i(v)$ and $\rho_i(v) \geq \varphi_i(u)$, so $uv \in E(G_i)$. □

Theorem 4.3.11 *Let G be a cross comparison graph with d-dimensional ranks and tolerances. Then G is the intersection of d graphs with efficient cross comparison representation.*

Proof. For $1 \leq i \leq d$, let G_i be the efficient cross comparison graph defined as above. The edges of G must be in each graph G_i, by Lemma 4.3.10. Conversely, if any edge uv is in all d of these graphs, then $\rho_i(u) \geq \varphi_i(v)$ and $\rho_i(v) \geq \varphi_i(u)$ for $1 \leq i \leq d$, so $\rho(u) \geq \varphi(v)$ and $\rho(v) \geq \varphi(u)$, and therefore $uv \in E(G)$. Thus, G is precisely the intersection of these efficient cross-comparision graphs. □

The *boxicity* of a graph G is the minimum number k for which G is a k-box graph. The *chordality* of G is the minimum number of chordal graphs whose intersection is G. It has been shown that the upper bound on both chordality and boxicity of a graph of order n is $\lfloor n/2 \rfloor$, and that the graph formed by removing $\lfloor n/2 \rfloor$ non-adjacent edges from the complete graph K_n has been shown to have have boxicity and chordality both equal to $\lfloor n/2 \rfloor$ [33, 29]. Because all interval graphs are efficient, and all efficient graphs are chordal, Theorem 4.3.11 gives us bounds on d_G:

Corollary 4.3.12 *Let* $\mathrm{C}(G)$ *be the chordality of* G*, and* $\mathrm{Box}(G)$ *be the boxicity of* G*. Then*

1. $\mathrm{C}(G) \leq d_G \leq \mathrm{Box}(G)$.
2. $d_G \leq \lfloor n/2 \rfloor$
3. *The above bounds are tight.*

4.4 Characterization of efficient cross comparison graphs

We are still left with the question of how to characterize the efficient cross comparison graphs. While a characterization based on minimal forbidden subgraphs might still be out of reach, a more-studied class of graphs from [31] offers a partial solution.

Definition 4.4.1 *A* threshold tolerance graph*, or* TT graph*, is a graph* G *in which each vertex* v *is assigned a weight* w_v *and tolerance* t_v*, and* $uv \in E(G)$ *if and only if* $w_u + w_v \geq \min\{t_u, t_v\}$*.*

The complements of TT graphs, in which u and v are adjacent whenever $w_u + w_v \leq \min\{t_u, t_v\}$, are called *co-TT* graphs. However, Monma, Reed, and Trotter use a slightly different definition for these graphs in [31]:

Definition 4.4.2 *A graph* $G = (V, E)$ *is a* co-TT graph *if, for each* $v \in V$*, there are real numbers* a_v *and* b_v *such that*

$$xy \in E \Leftrightarrow a_x \leq b_y \text{ and } a_y \leq b_x.$$

To see that the two definitions are equivalent, set $a_v = w_v$ and $b_v = t_v - w_v$. The second definition, however, is exactly the definition of an efficient cross-comparision graph! This gives us an immediate characterization of graphs with $d_G = 1$.

Theorem 4.4.3 *A graph* G *is an efficient cross comparison graph if and only if* G *is a co-TT graph.*

A polynomial-time algorithm for recognition of these graphs was given in [31]. The characterization of co-TT graphs by minimal forbidden subgraphs was left as an open problem. However,

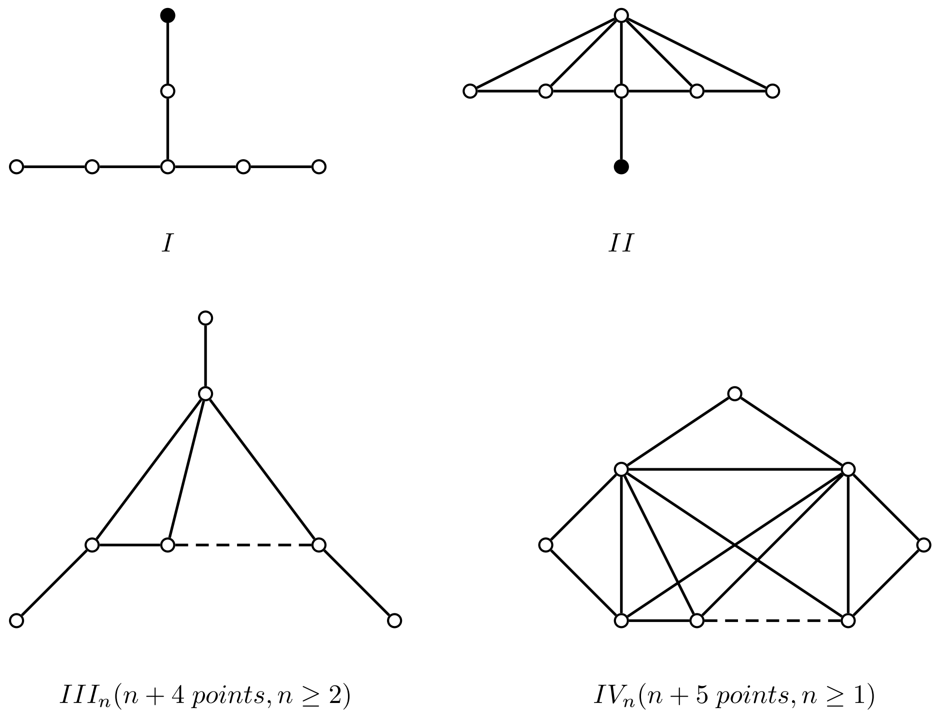


Figure 4.1: The minimal asteroidal graphs. An interval graph is chordal and contains none of the graphs above.

the equivalency of co-TT graphs and efficient cross comparison graphs will allow us to characterize these graphs.

A graph G is *asteroidal* if it contains three vertices a_1, a_2, a_3, and paths W_1, W_2, W_3, such that each path W_i connects the vertices $a_j, j \neq i$, and contains no neighbor of a_i. In this case, the vertices a_1, a_2, and a_3 are an *asteroidal triple.* Lekkerkerker and Boland characterized the interval graphs in [26] as all chordal graphs which do not contain an asteroidal triple. They also characterized the interval graphs by minimal forbidden subgraph.

Theorem 4.4.4 *A graph G is an interval graph if and only if G is chordal and contains none of the graphs in Figure 4.1 as a vertex-induced subgraph.*

Recall that every interval graph is also an efficient cross comparison graph, and that every efficient cross comparison graph is chordal. To begin working on a characterization of efficient cross comparison graphs, recall that each efficient cross comparison graph is also a co-TT graph. The following theorem from [31] strengthens the condition of chordality:

Theorem 4.4.5 *Every co-TT graph is strongly chordal.*

Farber gives a characterization of strongly chordal graphs in [14]:

Definition 4.4.6 *A* complete sun*, or* k-sun *is a G on $2k$ vertices for some $k \geq 3$, such the vertex set can be partitioned into sets $U = \{u_1, \ldots, u_k\}$ and $W = \{w_1, \ldots, w_k\}$ such that U is a clique, W is independent, and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j + 1(\mod n)$.*

If the vertex-induced subgraph $G(U)$ is chordal, but not a clique, G is an incomplete sun.

Lemma 4.4.7 *Every incomplete sun contains a complete sun.*

Theorem 4.4.8 *A graph is strongly chordal if and only if it is chordal and sun-free.*

This gives us a set of forbidden subgraphs for the efficient cross comparison graphs:

Corollary 4.4.9 *If G contains a sun, G does not have an efficient cross comparison representation.*

We may, without loss of generality, assume that all ranks and tolerances in a cross comparison representation have distinct values, and thus all inequalities will be strict. All proofs in the remainder of this section will make this assumption.

Lemma 4.4.10 *The representation of the interior vertices of an irreducible path is unique, up to reversing the order of the vertices.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of a path, ordered left to right, with ranks $r_1, r_2, \ldots, r_n$ and tolerances $t_1, t_2, \ldots, t_n$. For $1 \leq j < i \leq n$, if $i - j = 1$, we must have $r_i \geq t_j$ and $r_j \geq t_i$, but for $i - j > 2$, we must have $t_i > r_j$. This leads to the condition that

$$t_2 < r_1 < t_3 < r_2 < \cdots < t_{n-1} < r_{n-2} < t_n < r_{n-1}$$

Additionally, $t_1 < r_2$ and $r_n > t_{n-1}$, but there is some freedom in the ordinality of these two values. The ordering of the remaining $n - 2$ values relative to each other is fixed. $\square$

Theorem 4.4.11 *Suppose G is an efficient cross comparison graph, and $v_1, v_2, \ldots, v_5$ are the consecutive vertices of an irreducible subpath of G.*

1. If x is adjacent to v_3, but not v_2 or v_4, then every vertex adjacent to x is also adjacent to v_3.

2. If x is adjacent to all five vertices in this path, then every vertex adjacent to v_3 is also adjacent to x.

Proof. Let the vertices in the path have ranks and tolerances as in Lemma 4.4.10, and let the rank and tolerance of x be, respectively, r_x and t_x.

1. As a consequence of Lemma 4.4.10, $t_2 < t_3 < r_x$ and $t_x < r_3 < r_4$. We must then have $r_2 < t_x$ and $r_x < t_4$ to avoid this adjacency, so $t_3 < r_2 < t_x$ and $r_x < t_4 < r_3$. Suppose vertex y, with rank and tolerance (resp.) r_y and t_y, is adjacent to x. Then $r_y > t_x > t_3$ and $t_y < r_x < r_3$, so y is also adjacent to v_3.

2. As a consequence of Lemma 4.4.10, $t_x < r_1 < t_3$ and $r_3 < t_5 < r_x$. Suppose vertex y, with rank and tolerance (resp.) r_y and t_y, is adjacent to v_3. Then $t_y < r_3 < r_x$ and $t_x < t_3 < r_y$, so y is also adjacent to x.

□

Corollary 4.4.12 *Graphs I and II in Figure 4.1 are not efficient cross-comparision graphs.*

Proof. The solid vertex in graph I violates condition 1 of Theorem 4.4.11, and the solid vertex in graph II violates condition 2 of the same theorem. □

From Corollary 4.4.9 above, the k-suns are forbidden subgraphs. We now show that they are minimal.

Theorem 4.4.13 *Let G be a k-sun, and v be any vertex of G. Then $G - v$ has an efficient cross comparison representation.*

Proof. Removing any vertex from the 3-sun leaves an interval graph, so we may assume $k \geq 4$. By symmetry, it suffices to consider the cases $v = w_k$ and $v = u_k$ from Definition 4.4.6 above. Note also that if we remove the vertex u_k, we are left with w_k and w_{k-1} as pendant vertices; for $k \geq 4$, if we also remove these two vertices, the remaining graph is isomorphic to a $(k-1)$-sun with the vertex w_{k-1} removed. Thus, we need only consider the case $v = u_k$.

We can obtain a representation for this case as follows:

- The vertex u_i, $1 \leq i \leq k-1$, is assigned tolerance $t(u_i) = 2i - 1$ and rank $r(u_i) = 2(k + i)$.

- The vertex w_j, $1 \leq j \leq k$, is assigned tolerance $t(w_j) = 2(k+j) - 3$ and rank $r(w_j) = 2j$.

The vertices in part U all have tolerance at most $2k-3$, and rank at least $2k+2$, so u_i and u_j are adjacent for $1 \leq i, j \leq k-1$. The vertices in part W are all retro, and are thus independent. If u_i and w_j are adjacent, then we must have $2(k+j) - 3 < 2(k+i)$ and $2i - 1 < 2j$. The first inequality implies $j \leq i+1$, and the second implies that $i \leq j$, so w_j and u_i are adjacent if and only if $j = i$ or $j = i+1$. This gives the desired representation. $\square$

The graphs from families III_n and IV_n in Figure 4.1 (except for graph IV_1) have efficient cross comparison representations, but since they have no interval representations, each must include at least one retro vertex. Further, because retro vertices are simplicial, only the three shaded vertices in these subgraphs can be retro. Note that these are precisely the vertices that form the asteroidal triple that prevents each graph from having an interval representation. The remaining forbidden configurations are supergraphs of these families.

Given an asteroidal triple a_1, a_2, a_3 in a simply representable graph, a_2 is called the *middle vertex* if $r(a_1) < r(a_2) < r(a_3)$ and $t(a_1) < t(a_2) < t(a_3)$. The following lemma shows that, in each representation of a graph, each asteroidal triple has a middle vertex, and that the middle vertex is retro.

Theorem 4.4.14 *Suppose G is an efficient cross comparison graph, and that G has an asteroidal triple a_1, a_2, a_3, where $t(a_1) < t(a_2) < t(a_3)$. Then $r(a_1) < r(a_2) < r(a_3)$. Further, a_2 is retro.*

Proof. If $r(a_2) < r(a_1)$, then any vertex adjacent to a_2 must also be adjacent to a_1. Then there would be no path between a_2 and a_3 that does not contain a neighbor of a_1, so this is not an asteroidal triple. Similarly, if $r(a_3) < r(a_2)$, then any neighbor of a_3 would be adjacent to a_2, so it must be that $r(a_1) < r(a_2) < r(a_3)$.

Now, consider an irreducible path from a_1 to a_3; by Lemma 4.4.10, the interior vertices of this path are bounded. If a_2 is bounded, then for at least one vertex v on this path, $r(v) \geq t(a_2)$ and $r(a_2) \geq t(v)$, which again violates the definition of an asteroidal triple. Therefore $r(a_2) < t(a_2)$. $\square$

Note that if a_1 and a_3 are bounded, then $r(a_1) < r(a_2) < t(a_2) < t(a_3)$. The presence of an asteroidal triple in an efficient graph does not necessarily mean a specific vertex must be a middle vertex. In particular, any of the asteroidal vertices the complement of the 3-sun (graph III_2) may be a middle vertex. However, other asteroidal graphs do have specific middle vertices.

The relationship between the neighborhood of the middle vertex and the paths between the other vertices in the asteroidal triple is the first step in characterizing efficient graphs. We will refer to a vertex as *v-heavy* if it is adjacent to every neighbor of v, and define a *v-heavy path* as one which contains at least two consecutive v-heavy vertices. A vertex or path which is not v-heavy is *v-light*.

Lemma 4.4.15 *Suppose G is an efficient cross-comparison graph, and that G has an asteroidal triple a_1, a_2, a_3, in which a_2 is the middle vertex. Let W be any path from a_1 to a_3 which contains no neighbor of a_2. Then W is a_2-heavy.*

Proof. Because a_1 and a_3 are necessarily a_2-light, we must show that the interior of W contains at least two consecutive a_2-heavy vertices. Assume, without loss of generality, that $t(a_1) < t(a_2) < t(a_3)$, and (by Theorem 4.4.14) $r(a_1) < r(a_2) < r(a_3)$. If W has only one interior vertex x, then $r(x) > t(a_3) > t(a_2)$ and $t(x) < r(a_1) < r(a_2)$, so x would be adjacent to a_2 as well. Thus, W must have at least two interior vertices; let the interior of W be denoted c_1-c_2-$\cdots$-c_k. The remainder of the proof is inductive on the number of retro vertices in W.

Suppose all of the interior vertices of W are bounded. For each interior vertex w, if $r(w) > t(a_2)$, then $t(w) > r(a_2)$, or w is adjacent to a_2. There must be an interior vertex $x = c_i$ such that $t(x) < r(a_2) < r(x) < t(a_2)$, and a vertex $y = c_j$ such that $r(y) > t(a_2) > t(y) > r(a_2)$. If more than one vertex on W satisfies these inequalities, we may choose c_i to have the highest possible index and c_j to have the lowest possible index such that $j > i$. If v is any neighbor of a_2, then $r(v) > t(a_2) > t(x) > t(y)$, and $r(x) > r(y) > r(a_2) > t(v)$, so v is adjacent to x and y, and to each vertex between x and y on W; because x and y cannot be the same vertex, this means that there are at least two consecutive a_2-heavy vertices.

Suppose now that not all of the interior vertices are bounded, and choose a retro vertex u from the interior of W. Because u is retro, all of its neighbors must be bounded (and adjacent to each other).

Case I: $r(u) < r(a_2) < t(a_2) < t(u)$. In this case, every neighbor of u would also be adjacent to a_2; thus, W would contain neighbors of a_2. Therefore, this case is impossible.

Case II: $r(a_2) < r(u) < t(u) < t(a_2)$. In this case, every neighbor of a_2 is adjacent to u, and to every neighbor of u, so u and either of its neighbors in W give us our consecutive vertices.

Case III: $r(u) < r(a_2) < t(u) < t(a_2)$ or $r(u) < t(u) < r(a_2) < t(a_2)$. Note that some subset of the vertices of W form an irreducible path P. All of the interior vertices of P are bounded, and so by the inductive hypothesis must contain vertices x and y as above. Because u is a vertex of W, there is a subpath of W from u to a_3 which contains no neighbor of a_2; call this subpath W'. In addition, there is an subpath of W from a_1 to u. This subpath contains an irreducible path; the interior vertices of this path cannot be adjacent to a_3 (or they would also be adjacent to a_2). Thus, there is a path from u to a_1, and therefore to a_2, which contains no neighbor of a_3. Finally, because u cannot be adjacent to y, nor any vertex right of y in P, there is a path from a_2 to a_3 which contains no neighbor of u. Then u, a_2, and a_3 form an asteroidal triple, in which a_2 is the middle vertex, and W' is a path with fewer interior retro vertices than W. By the inductive hypothesis, W', and therefore W, are a_2-heavy.

By symmetry, the cases where $r(a_2) < r(u) < t(a_2) < t(u)$ and $r(a_2) < t(a_2) < r(u) < t(u)$ are the same. □

Lemma 4.4.15 gives a very specific condition under which a vertex may be a middle vertex. We must note that this condition only applies to one asteroidal triple at a time; in other words, if a vertex is in more than one asteroidal triple, it may be a middle vertex in one triple, even if it is prevented from being so in another. If a_1, a_2, and a_3 are an asteroidal triple in which none of the vertices may be the middle vertex, then a_1, a_2 and a_3 are unrepresentable as an efficient cross comparison graphs. We are now ready to characterize efficient cross comparison graphs, beginning by defining the unrepresentable condition.

Definition 4.4.16 *A graph G is* strongly asteroidal *if it contains three distinct points a_1, a_2, a_3, such that each pair of these points is connected by a path which contains no neighbor of the third vertex, and which does not contain two consecutive vertices adjacent to every neighbor of the third vertex. The three points are referred to as a* strongly asteroidal triple.

The paths in this definition are not necessarily irreducible. In fact, it is often the case that a v-light path is not irreducible. For example, given a k-sun, with vertices labeled as in Definition 4.4.6, the path u_1-w_1-u_2 is u_3-light, and the path u_2-w_2-u_3 is u_1-light. If $k = 3$, then u_3-w_3-u_1 is u_2-light. For $k \geq 4$, the only u_2-light path between u_1 and u_3 is u_3-w_3-u_4-$\cdots$-u_k-w_k-u_1. In any case, every k-sun contains a strongly asteroidal triple.

Theorem 4.4.17 *Let G be a chordal graph. Then G has an efficient cross-comparison representation if and only if G contains no strongly asteroidal triple.*

Proof. The "only if" portion of the theorem follows directly from Lemma 4.4.15 and Theorem 4.4.14; a strongly asteroidal triple has no middle vertex, and is therefore not representable.

Let G be a chordal graph on n vertices without a strongly asteroidal triple; we may assume that G is connected. Also, because the k-suns contain strongly asteroidal triples, we may assume that G is not just chordal, but strongly chordal. The proof that G has an efficient cross-comparison representation is inductive on n. For $n = 1$, the representation is trivial.

For $n > 1$, if G has no asteroidal triple, then G is an interval graph, and thus has a representation in which every vertex is bounded. Suppose, then, that G contains an asteroidal triple a_1, a_2, a_3. Because this triple is not strongly asteroidal, one of these vertices could be a middle vertex. Let us assume, without loss of generality, that vertex is a_1; then every path between a_2 and a_3 either contains a neighbor of a_1, or is a_1-heavy. Because G is strongly chordal, this also implies that a_1 is simplicial. Let W be an irreducible a_2-a_3 path which contains no neighbor of a_1. By our inductive hypothesis, $H = G - a_1$ has an efficient cross-comparison representation. We claim that H has a representation which satisfies the following two conditions:

1. Every vertex of $N(a_1)$ is bounded.

2. Let $I = \bigcap_{u \in N(a_1)} [t(u), r(u)] = [T, R]$. If v is a bounded vertex such that $I \subseteq [t(v), r(v)]$, then $v \in N(a_1)$.

To show (1), let b be a neighbor of a_1 which is retro. Because W is irreducible, its interior vertices have a unique representation by Lemma 4.4.10. Because W must contain at least two consecutive a_1-heavy vertices, b must be adjacent to exactly two vertices of W. Specifically, for some vertices w_1, w_2, we must have $t(w_1) < t(w_2) < r(b) < t(b) < r(w_1) < r(w_2)$. Suppose there is a bounded vertex c not adjacent to a_2, such that $t(c) < r(b) < r(c) < t(b)$, $r(b) < t(c) < t(b) < r(b)$, or $r(b) < t(c) < r(c) < t(b)$; or that there is a retro vertex c such that $r(b) < r(c) < t(c) < t(b)$. In any of these four cases, c is adjacent to both w_1 and w_2, and so W can be extended to an a_1-light path, contradicting the assumption that a_1 may be a middle vertex. However, if no such vertex exists, then b can be given a bounded representation (simply by switching $r(b)$ and $t(b)$) without affecting any adjacencies.

To show (2), recall that W must have a subpath W' of consecutive a_1-heavy vertices. Let x and y be the end-vertices of this subpath, so that $t(x) < T < r(x)$, and $t(y) < R < r(y)$. We may assume that W' has minimum length; i.e., no other irreducible a_2-a_3 path has a shorter subpath of a_1-heavy vertices. Let v be a vertex not adjacent to a_1. We wish to show that it is impossible for $t(v) < T$ and $r(v) > R$. Note that if this is the case, then v is adjacent to both x and y (or is one of these vertices), so we may replace all of the interior vertices of W' with v. If W' contains more than three vertices, then this contradicts the assumption that W' has minimum length. Therefore, we may assume that W' has at most three vertices.

In order for it to be necessary that $t(v) < T$ and $r(v) > R$, v must be adjacent to a vertex u_1 with $r(u_1) < T$ and a vertex u_2 with $t(u_2) > R$. In addition, both u_1 and u_2 must have adjacencies that force them to be on opposite sides of I. There are three cases to consider.

Case I: u_1 is adjacent to a neighbor of a_1 that is not adjacent to u_2, and u_2 is adjacent to a neighbor of a_1 that is not adjacent to u_1. Then these four vertices, along with a_1 and v, form a 3-sun, which contradicts the assumption that G is strongly chordal.

Case II: u_1 is adjacent to x and u_2 is adjacent to y. If v is neither x nor y, then x-u_1-v-u_2-y is part of an a_1-light path. Suppose, then, that v is one of these vertices; without loss of generality, suppose $v = x$. Then v-u_2-y is part of an a_1-light path.

Case III: At least one of u_1 and u_2 — without loss of generality, u_1 — is retro, and not adjacent to x (note that $v \neq x$ in this case), but is adjacent to another vertex b_1. Then b_1 is bounded, and adjacent to x, v, and I. If $r(b_1) > t(y)$, then u_1 can be moved so that $T < r(u_1) < R < t(u_1)$, so we may assume b_1 and y are not adjacent; this implies that b_1 and u_2 are not adjacent, and that x and y are also not adjacent. Either u_2 is adjacent to y, or is similarly adjacent to a vertex b_2, which we may assume is adjacent to neither x nor u_1. Finally, there is a vertex c adjacent to a_1 which is adjacent to neither u_1 nor u_2 (if not, then we have case I). Then u_1-v-u_2 contains only one a_1-heavy vertex, a_1-c-b_1-u_1 does not contain consecutive u_2-heavy vertices, and a_1-c-y-u_2 (or a_1-c-y-b_2-u_2) does not contain consecutive u_1-heavy vertices. Then u_1, u_2, and a_2 form a strongly asteroidal triple, which contradicts the assumption that G is not strongly asteroidal.

Because none of these cases is possible, it is not necessary that $I \subseteq [t(v), r(v)]$. Therefore, there is a representation where both (1) and (2) hold. Then, by letting $T < r(a_2) < t(a_2) < R$, we have a representation of G. $\square$

We are now left with the problem of determining the minimal subgraphs which contain an unrepresentable triple. Certainly, these graphs must all contain an asteroidal triple, so we look at the graphs in Figure 4.1. Graphs I and II contain a strongly asteroidal triple, as do the k-suns. For each pair of vertices in any asteroidal triple in these graphs, there is a path which does not contain two consecutive vertices adjacent to every neighbor of the third. The remaining graphs contain a triple which is asteroidal, but not strongly asteroidal. We can obtain the remaining minimal strongly asteroidal graphs by adding vertices to these graphs in such a way as to create a strongly asteroidal triple.

4.5 Open Problems

The first step in the future study of cross comparison graphs is a characterization of efficient graphs by minimal forbidden subgraphs. There are at least 52 specific forbidden graphs, plus four infinite families (including chordless cycles and suns). Appendix B contains a partial characterization.

Much like odd-intersection interval graphs, cross comparison models have "canonical" representations, but many of these representations give the same graph (in particular, the graph $\overline{K}_n$ has more than $\frac{n!}{2^n}\binom{2n}{n}$ representations). Probabilistic study of random representations to determine such attributes as expected edge densities or the probability of representing a specific graph could yield interesting results.

Studies of d-representable graphs for values of d greater than 1 is another area with a number of open problems. Trees, cycles, and cubes are all 2-representable, but little else is known. Is there a bound on d_G if G is bipartite? What if G is chordal? It is also worth investigating whether a polynomial-time algorithm exists to determine whether a graph is d-representable, or to determine d_G.

Appendices

Appendix A

Some Odd-intersection Interval Representations

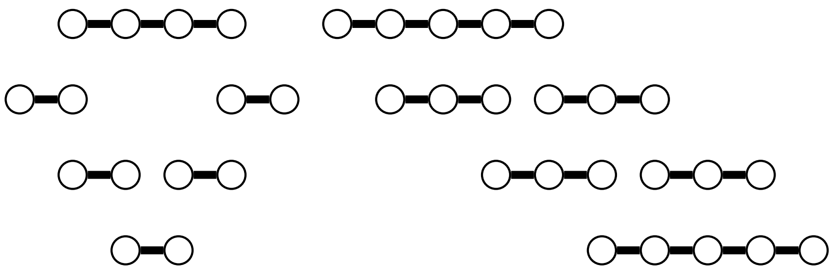


Figure A.1: Two representations of C_6.

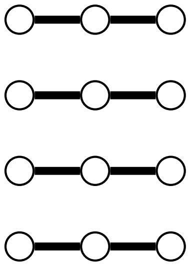


Figure A.2: K_4

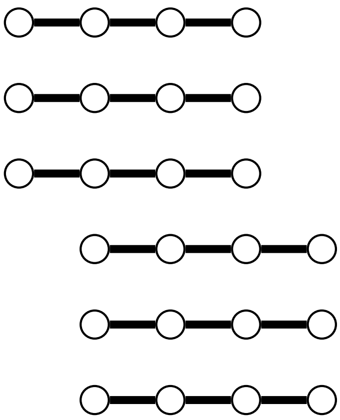


Figure A.3: $K_{3,3}$

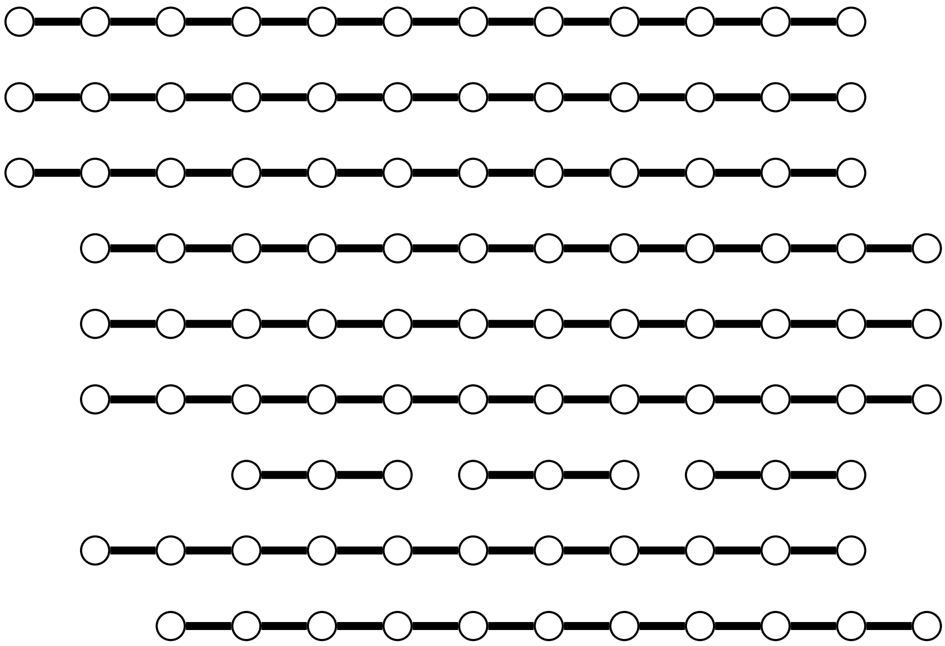


Figure A.4: A representation of $K_{3,3,3,2}$.

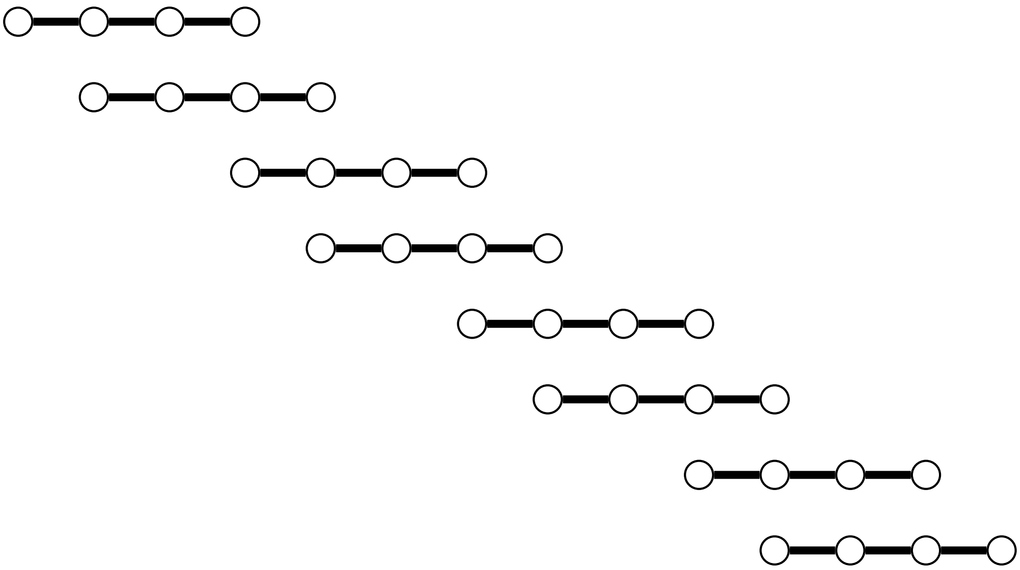


Figure A.5: $P_4 \times K_2$

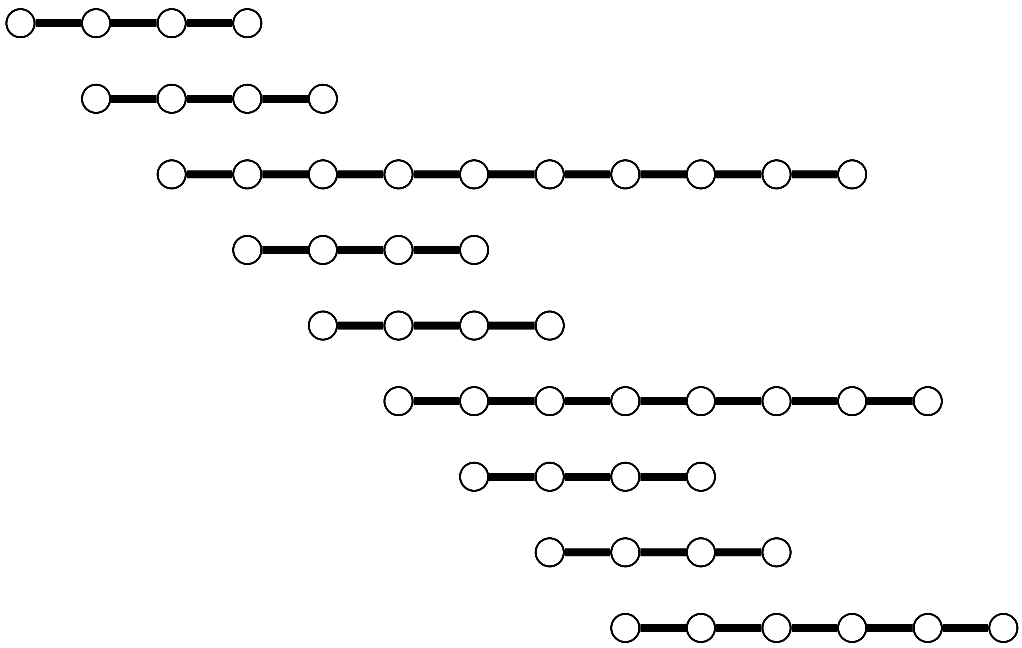


Figure A.6: $P_3 \times P_3$

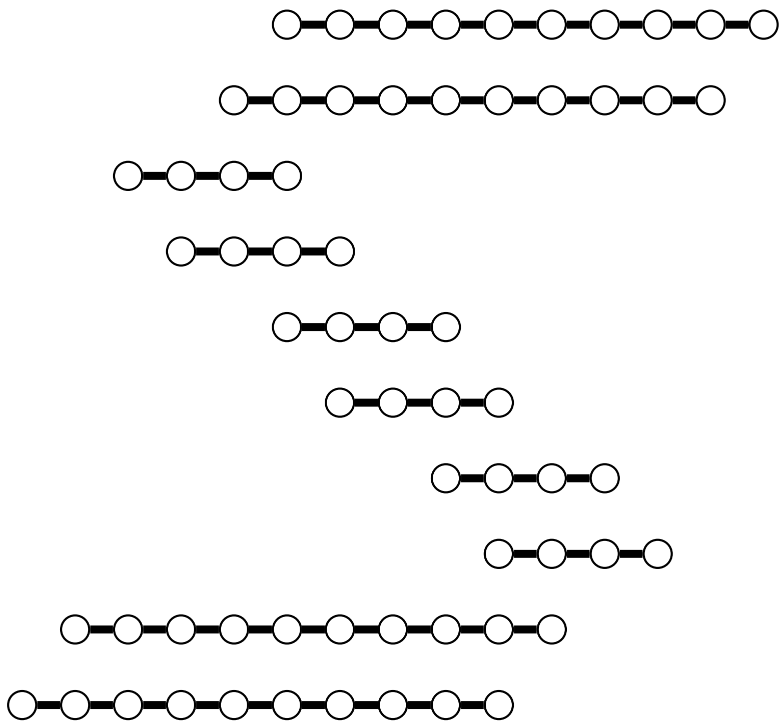


Figure A.7: The Möbius ladder M_{10}.

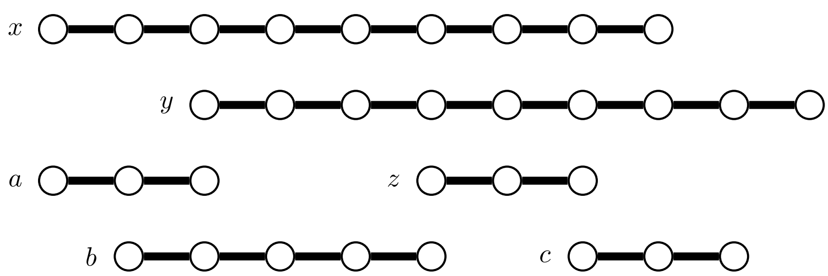


Figure A.8: The 3-sun.

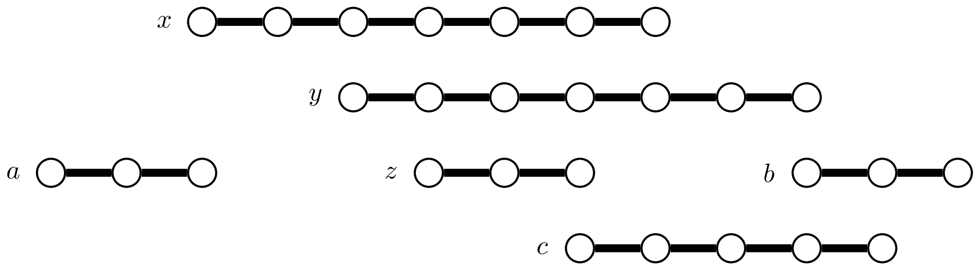
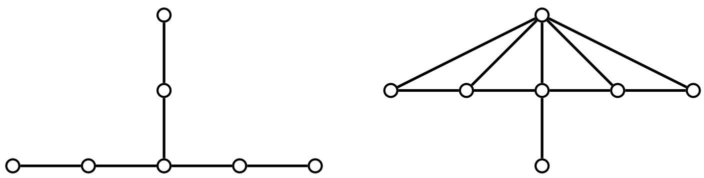


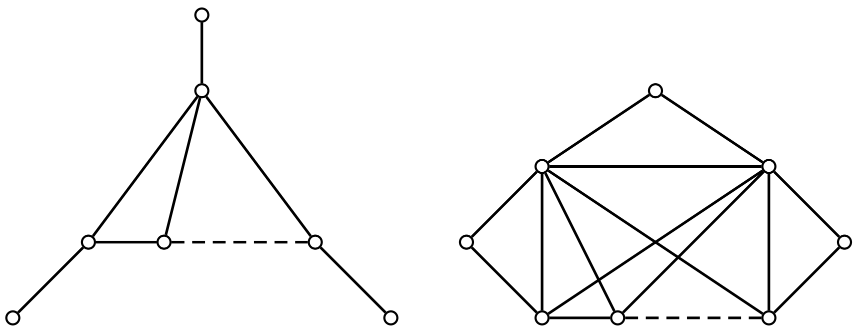
Figure A.9: The complement of the 3-sun.

Appendix B

Forbidden Subgraphs

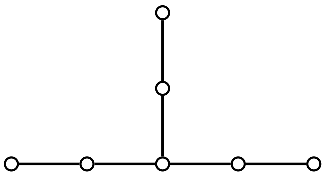


I II



III_n $(n+4\ points, n \geq 2)$ IV_n $(n+5\ points, n \geq 1)$

Figure B.1: The minimal asteroidal graphs. An interval graph is chordal and contains none of the graphs above as a vertex-induced subgraph.



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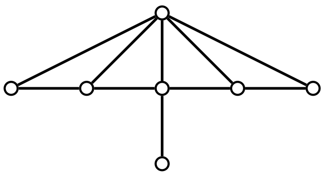

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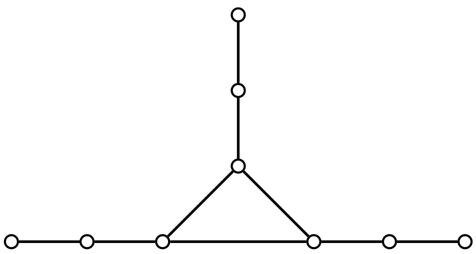
Figure B.2: Graphs which are neither interval graphs nor efficient cross comparison graphs.



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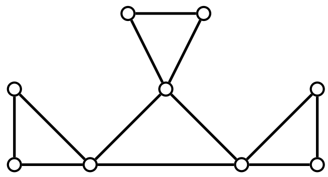

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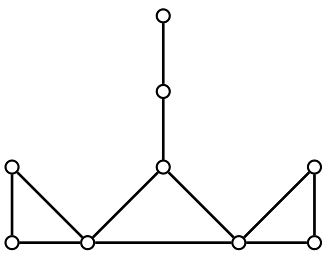

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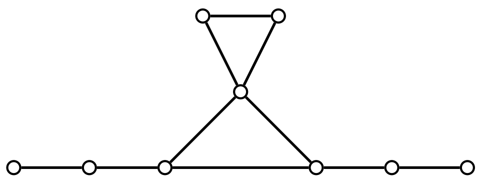

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Figure B.3: Minimal strongly asteroidal graphs formed from graph III_2.

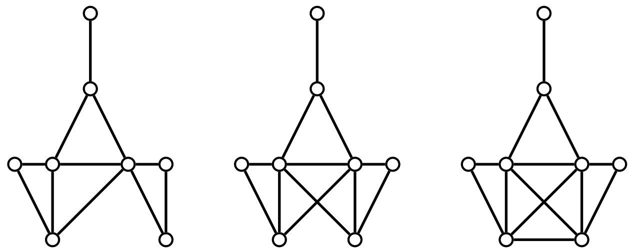

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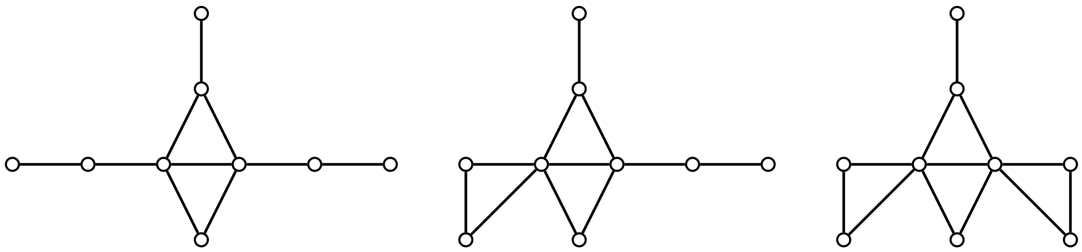
Figure B.4: Minimal strongly asteroidal graphs formed from graph III_2.



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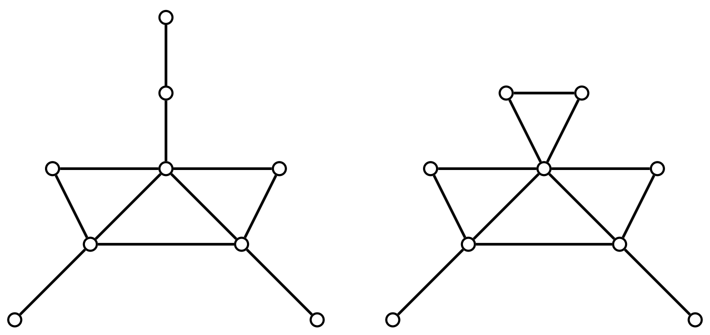

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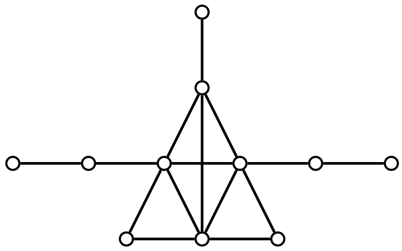
Figure B.5: Minimal strongly asteroidal graphs formed from graph III_2.



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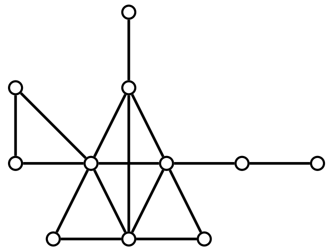

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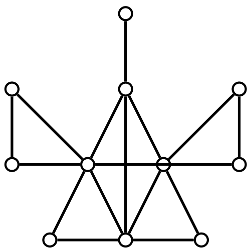

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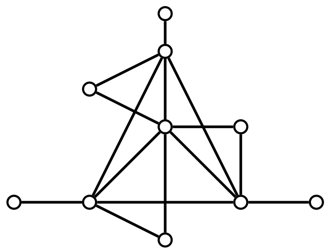

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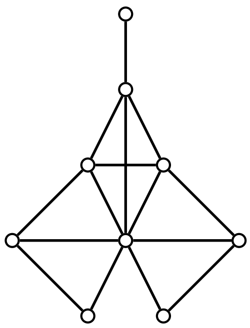
Figure B.6: Minimal strongly asteroidal graphs formed from graph III_2.



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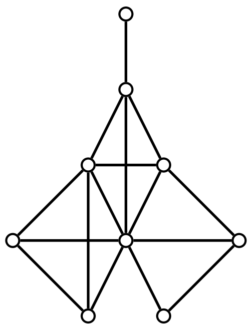

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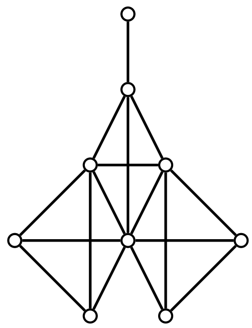

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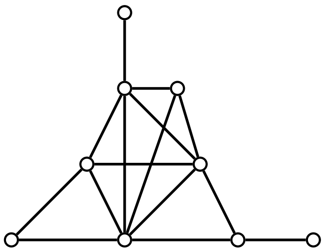
Figure B.7: Minimal strongly asteroidal graphs formed from graph III_2.



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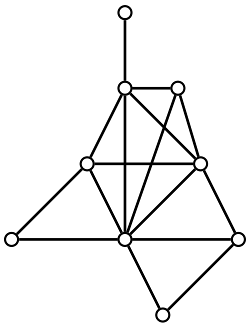

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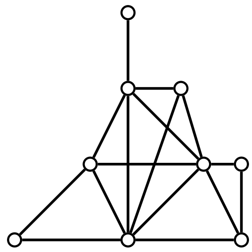
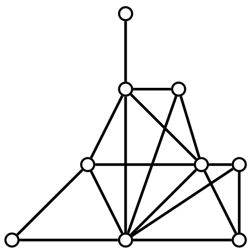
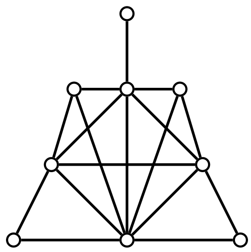


24

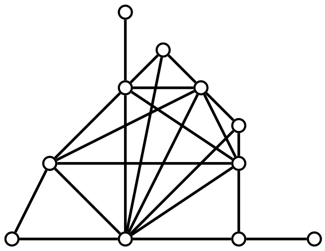


25

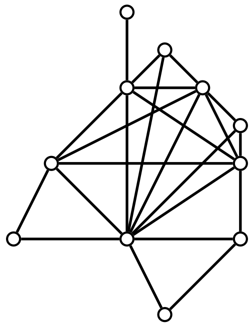


26

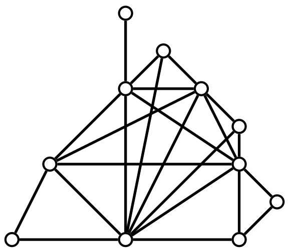
Figure B.8: Minimal strongly asteroidal graphs formed from graph III_2.



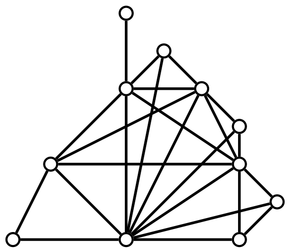
31



32

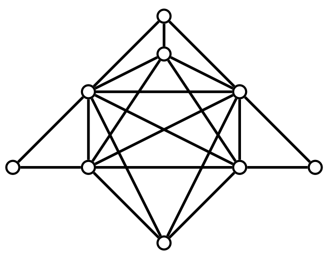


33

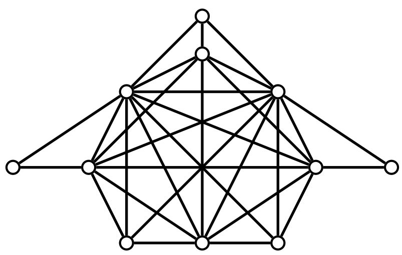


34

Figure B.9: Minimal strongly asteroidal graphs formed from graph III_2.

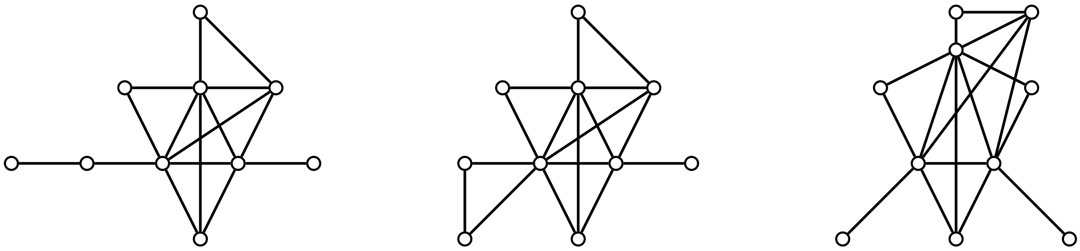


35



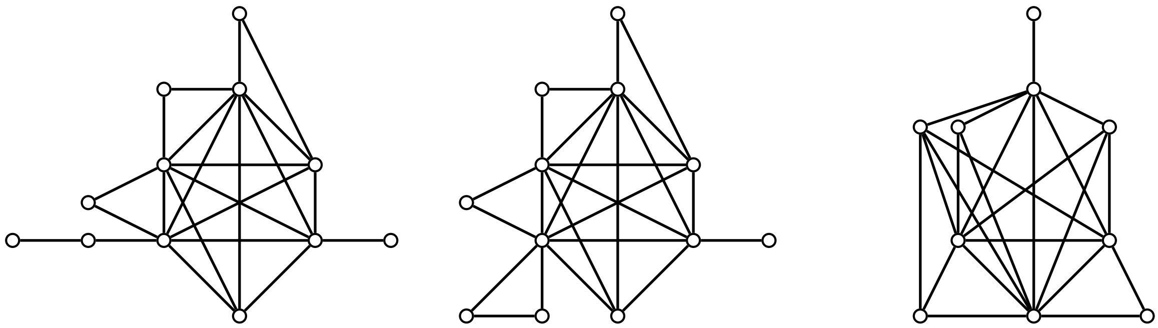
36

Figure B.10: Minimal strongly asteroidal graphs formed from graph III_2.



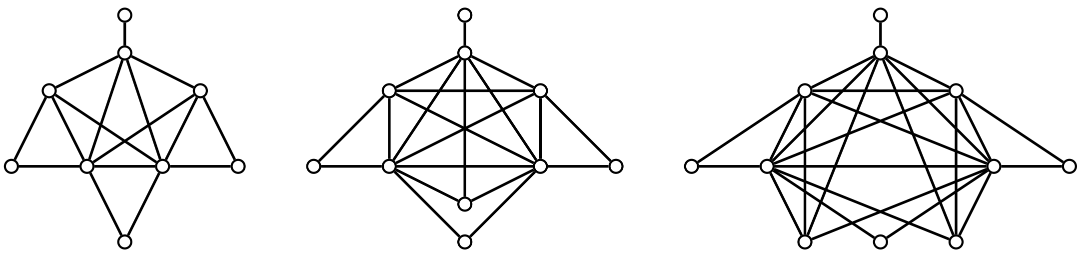
37 38 39

Figure B.11: Minimal strongly asteroidal graphs formed from graph III_2.



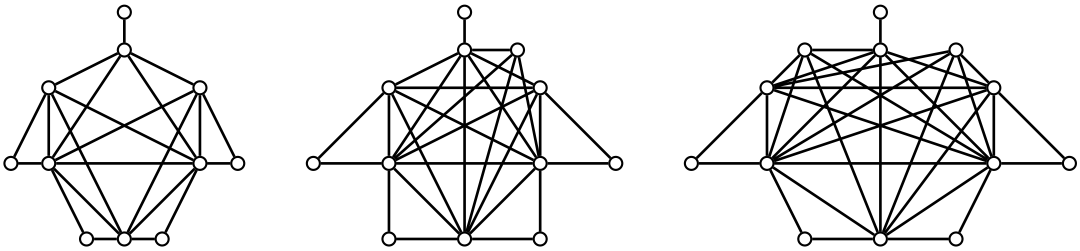
40 41 42

Figure B.12: Minimal strongly asteroidal graphs formed from graph III_2.



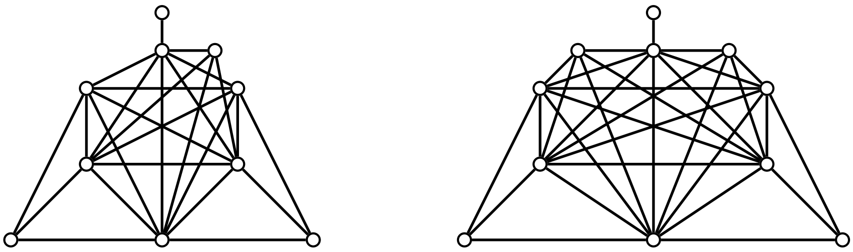
43 44 45

Figure B.13: Minimal strongly asteroidal graphs formed from graph III_2.



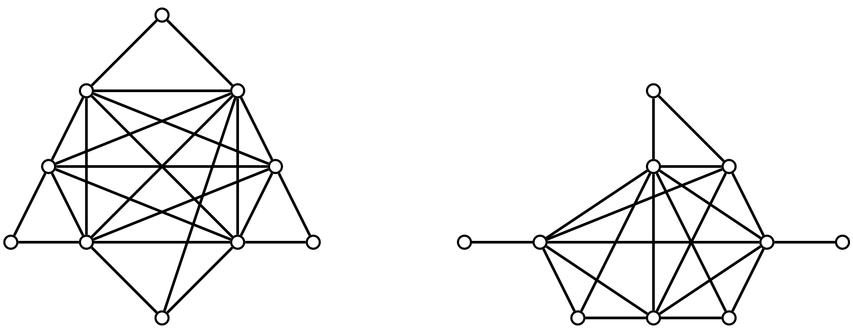
46 47 48

Figure B.14: Minimal strongly asteroidal graphs formed from graph III_2.



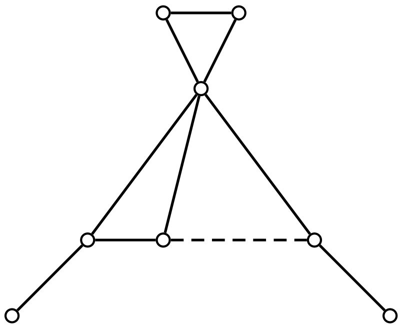
49 50

Figure B.15: Minimal strongly asteroidal graphs formed from graph III_2.

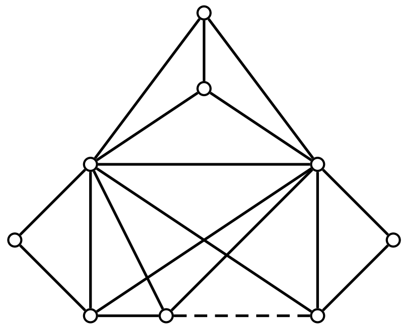


51 52

Figure B.16: Minimal strongly asteroidal graphs formed from graph III_2.



53_n $(n + 5\ points, n \geq 3)$



54_n $(n + 6\ points, n \geq 2)$

Figure B.17: Minimal strongly asteroidal graphs formed from graphs III_n $(n > 2)$ or IV_n $(n \geq 2)$.

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