8-2013

Count Time Series and Discrete Renewal Processes

James Livsey
Clemson University, jlivsey@clemson.edu

Count Time Series and Discrete Renewal Processes

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical Science's - Statistics

---

by
James Andrew Livsey
August 2013

Accepted by:
Dr. Robert Lund, Committee Chair
Dr. Brian Fralix
Dr. Colin Gallagher
Dr. Peter Kiessler

Abstract

Most data collected over time has some degree of periodicity (i.e. seasonally varying traits). Climate, stock prices, football season, energy consumption, wildlife sightings, and holiday sales all have cyclical patterns. It should come as no surprise that models that incorporate periodicity are paramount in the study of time series.

The following work devises time series models for counts (integer-values) that are periodic and stationary. Foundational work is first done in constructing a stationary periodic discrete renewal process (SPDRP). The dynamics of the SPDRP are mathematically interesting and have many modeling applications, expositions largely unexplored here. This work develops a SPDRP as a generation mechanism to produce a stationary count time series models with many desirable characteristics, including periodicity, negative autocovariances and long-memory.

After development of the SPDRP univariate count models are generalized into multiple dimensions. A multivariate renewal process has many interrelated stochastic processes. The resulting multivariate model has all the desirable properties of its univariate kin, but can also have negative autocovariances between marginal components of the series. To our knowledge, this trait is seldom achieved in current multivariate count methods in tandem with long-memory and periodicity.

Dedication

This work is dedicated to my wife Alison.

Her love and support during this entire work $\gg$ her mathematical assistance.

Love you.

Acknowledgments

My first thank you goes to Robert Lund. His tutelage reaches much farther than the classroom (as I'm sure anyone who knows him would verify). Our teamwork has produced numerous interesting mathematical results, research papers, a golf tournament victory, and a successful road trip to Omaha, NB. He has made a long-lasting impact on myself and my family. Only a devoted advisor would fly across the country, from JSM in San Diego, CA, to attend my wedding ceremony in Buffalo, NY. Hopefully, more success together lies in our future.

Secondly, a sincere thanks is in order for Brian Fralix. His zealous approach to research gave the flavor of what it takes to be successful in our discipline. Brian was the first to work with me through line-by-line pesky research issues, constantly being positive and motivating along the way. Our work led to my first published paper, accepted quickly and efficiently, a route not seen too often by my peers.

I am also indebted to all other statistics and stochastic faculty members at Clemson. All the professors have been accessible and willing to do anything possible to help me learn. Specifically, a special acknowledgments and thanks go to:

Pete Kiessler - For countless advice for job hunting in academia, how to prioritize in graduate school, and ways to think and work on research problems. Of course, there is also the many Clemson baseball game tickets and unmatched tailgating food. Without him, the College World Series trip wouldn't have been possible.

Colin Gallagher - For more insight than I am sure he'd ever know. This ranges from simple topics, like what to expect when a research paper is sent to a referee, to more important decisions like when in a graduate career is commensurate to having a wedding. And his graduate classes are responsible for all my linear models and introductory time series knowledge.

Graduate school would not have been a tenth as enjoyable if is was not for the amazing students I was fortunate enough to be around at this university. From the nine of us that banded together in the first summer boot camp, to all others that helped make intramural sports and weekend festivities possible. Especially:
James Wilson and Jeremy Brown - Without “ya’ll” (I will say it once before I leave SC), I don’t think the first two years classes would have been bearable. Fifteen hour days are only tolerable when you enjoy the company. Thanks for everything and good luck in all your future endeavors.

Finally, an acknowledgment section wouldn’t be complete without mentioning my family. The comforts provided throughout by Alison (and Boozie) can’t be chronicled - so I just leave it at saying thank you. The support and encouragement I received from my entire family, including my Mom, Dad, Brother and Stepfather, was an incredible aid in my corner. Thank you and love you all.

On to the math...

Table of Contents

List of Figures

Chapter 1

Introduction

This work is aimed at advancing modeling and forecasting of count time series. Along the way, many connections and work on discrete point processes will be discussed.

1.1 Time Series Overview

A time series is simply a sequence of random variables measured (typically) in a time-ordered fashion. Usually, the time between observations is equally spaced. One example would be the daily high temperatures in Clemson, SC. Time series assume a large variety of forms and patterns. Some common types of time series are shown in figure 1.1. The values of a time series can be periodically varying, trending, noisy, integer-valued, or any mix of these patterns.

Time series are ubiquitous and are used for communication, visualization, decision making, and description. Time series describe, for the most part, physical concepts and thus have real-world interpretations and implications. It is said that time series plots are the most frequently used form of graphic design. Examining any modern newspaper or media publication would seemingly verify this claim.

The amount of work done on modeling and forecasting of time series is vast; see [16], [6] and [7] and the references therein.

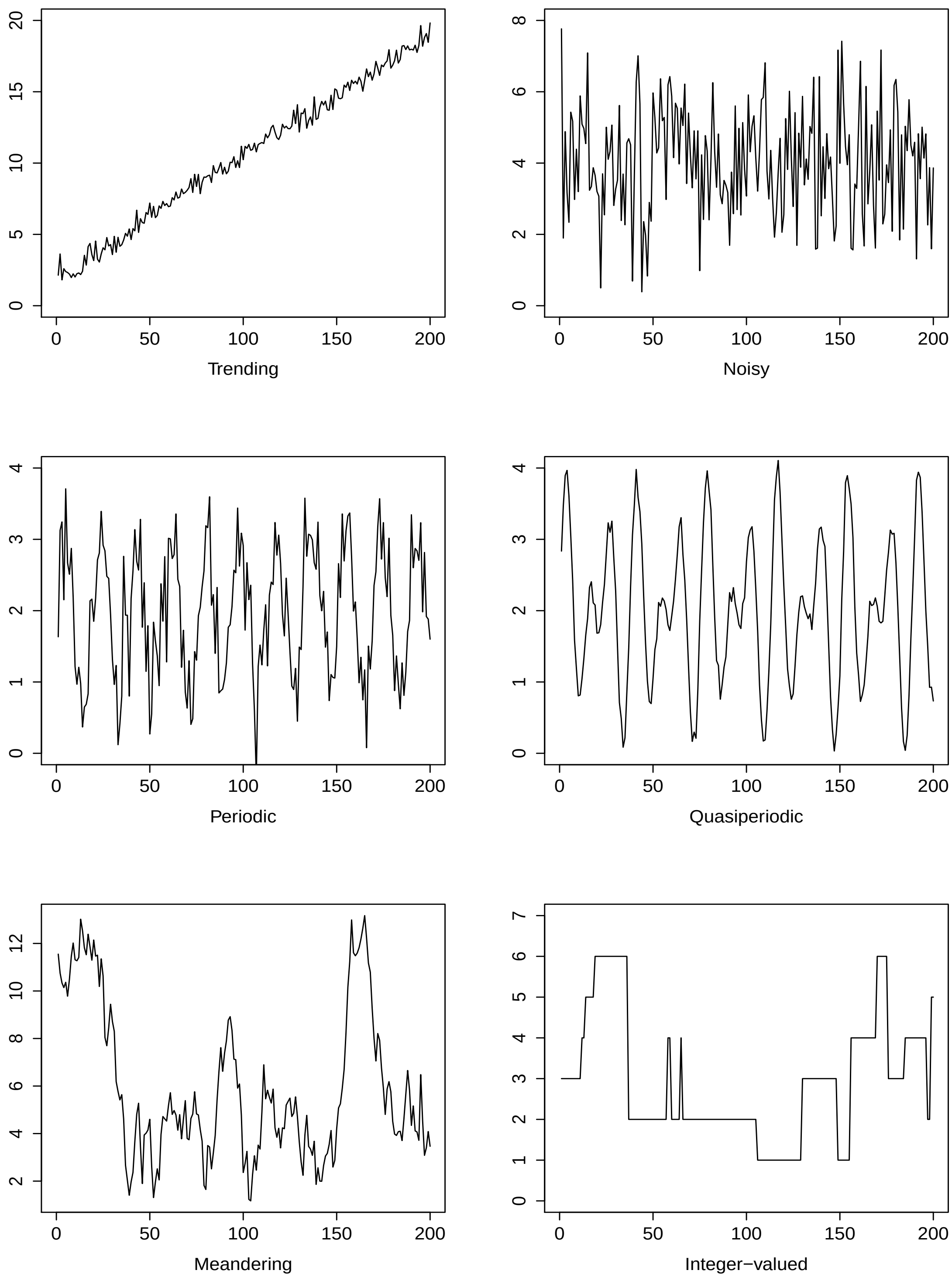

Figure 1.1: Different types of time series

Let $X_1, X_2, \ldots, X_n$ be a given time series of length n. Since many time series are correlated in time, an important quantity is the autocovariance function,

$$\gamma(t, s) := \mathrm{Cov}(X_t, X_s). \tag{1.1}$$

For example, the first-order autoregressive model (AR(1)) model (which will be discussed further in section 1.1.1 and is obtained by taking $p = 1$ and $q = 0$ in 1.5) has autocovariance function

$$\gamma(t, s) = \frac{\sigma^2 \phi_1^{|t-s|}}{1 - \phi_1^2} \tag{1.2}$$

for some $\sigma^2 > 0$ and $\phi_1 \in (-1, 1)$.

A time series $\{X_t\}$ is said to be weakly stationary (also termed stationary) if $E[X_t]$ does not depend on t and $\gamma(t, t+h)$ does not depend on t for each h. Similarly, a time series $\{X_t\}$ is strictly stationary if $(X_1, X_2, \ldots, X_n)'$ and $(X_{1+h}, X_{2+h}, \ldots, X_{n+h})'$ have the same joint distributions for all integers h and $n > 0$. Clearly, any strictly stationary series is also weakly stationary.

It should be noted here that this work considers stationary time series models. In this setting,

$$\gamma(t + h, t) = \gamma(h, 0), \tag{1.3}$$

and we will henceforth use a single argument in all autocovariance functions: $\gamma(h) := \gamma(t + h, t)$.

It is sometimes advantageous to look at correlations in lieu of covariance. The autocorrelation function of the series is defined as

$$\rho(h) := \frac{\gamma(h)}{\gamma(0)}. \tag{1.4}$$

Other properties of time series that will be of interest in our work are long-memory and periodicity. If $\sum_{h=0}^{\infty} |\gamma(h)| = \infty$, a time series is said to have long-memory. Long-memory indicates a slow rate of decay of statistical dependence between series values. Periodicity in time series is defined as a repetitive and predictable movement in mean or covariance. Periodic time series exhibit cyclic variation that occurs in a regular or semi-regular pattern; this is seen in the periodic/cyclic and quasi-periodic plots in figure 1.1.

1.1.1 ARMA Models

The most widely-used class of stationary time series are the autoregressive moving average (ARMA) class. An ARMA(p, q) series obeys the linear difference equation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q} + Z_t, \qquad (1.5)$$

where p and q are non-negative integers. ARMA(p, q) methods use the last p data values and uncorrelated innovations $\{Z_t\}_{t=0}^{\infty}$ (noises) to describe the series at time $t+1$. ARMA models are flexible and extensively used in economics, finance, and the natural sciences; they are popular because of their forecasting ability. When $p = 0$ in an ARMA(p, q) series, it is referred to as simply a moving average of order q (MA(q)); when $q = 0$ the model is called an auto regressive of order p (AR(p)).

1.2 Count Time Series

Autoregressive moving-average models work well when data is Gaussian; that is, the joint distribution of $(X_1, X_2, \ldots, X_n)'$ is Gaussian for any $n \geq 1$. This is not always the practical case. A simple illustration is in the bottom right plot in Figure 1.1. At each time, the observed series value is an integer, which is clearly not normally distributed.

A count time series is a time series where the observed data is integer-valued. Other examples include the yearly number of rare disease occurrences, the daily number of car accidents, and the hourly number of people treated in a hospital emergency room. Discrete counts may not be simply approximated by continuous variables, especially when the counts are relatively small [8]. Many attempts to model discrete counts have been made; an overview is given in [35].

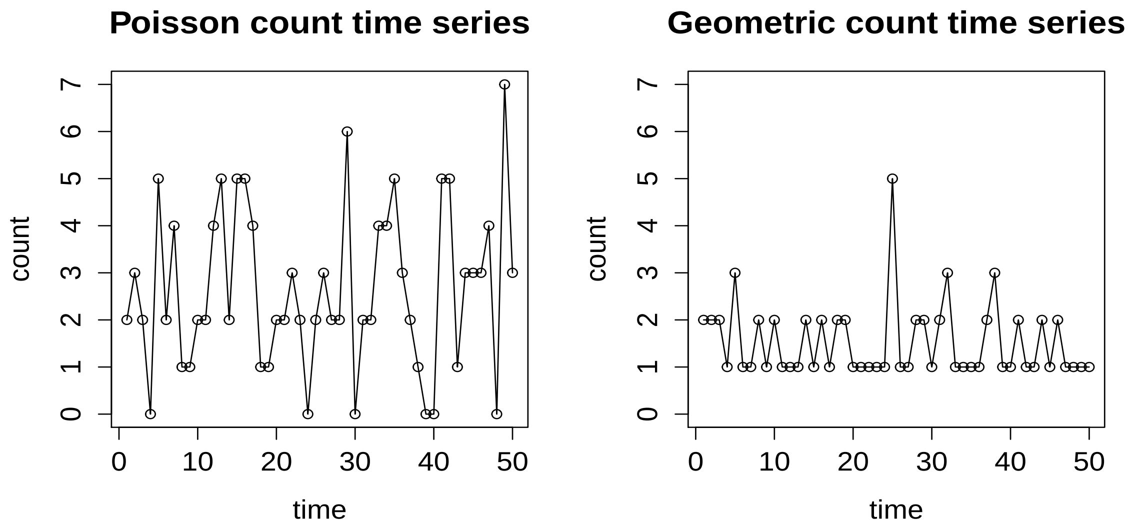

Figure 1.2: Sample Poisson and Geometric count time series

Our work often assumes a fixed marginal distribution at a given time point that does not change with time. This is particularly important in count settings since the number of possible observed values is finite. Consider Figure 1.2. The left-hand plot is a sample count time series where, at each time, the series has a marginal Poisson distribution with mean 3. The right-hand graph has a marginal distribution that is Geometrically distributed with success probability 3/4. Figure 1.2 gives the flavor of the importance of incorporating the knowledge of a marginal distribution into any model for count series.

1.2.1 DARMA Models

The first attempt to model stationary integer-valued time series was introduced in the 1970's by Jacobs and Lewis ([20], [21] and [22]). Their model, the discrete autoregressive moving-average (DARMA) model, used mixing techniques to generate any marginal distribution desired. Let

$$P(X_t \in A) = \pi(A), \tag{1.6}$$

denote the marginal distribution for any Borel set A. DARMA tactics are perhaps best illustrated in the DAR(1) recursion

$$X_t = V_t X_{t-1} + (1 - V_t) A_t, \qquad t = 1, 2, \ldots, \tag{1.7}$$

where $\{A_t\}_{t=1}^{\infty}$ is an independent and identically distributed (IID) sequence of random variables with distribution π and $\{V_t\}_{t=1}^{\infty}$ is an IID sequence of Bernoulli random variables with success probability p. If X_0 has distribution π, then induction shows X_t has marginal distribution π for each $t \geq 1$ as desired. The scheme is also stationary. The autocorrelation function of the DAR(1) model mimics the AR(1) model:

$$\rho(h) = p^h \in (0, 1) \qquad h = 0, 1, 2, \ldots \tag{1.8}$$

To obtain a model with a high lag one autocorrelation, p must be large. Inspection of equation (1.7) reveals a large probability of repeated time series values: $P(X_{t+1} = X_t) = p$. Such a trait is seldom seem in real-world data. Moreover, observe that DAR(1) techniques cannot produce negatively correlated series or series with long-memory. The same is true for the DAR(1) extensions: DAR(p) and DARMA(p, q) series.

1.2.2 INARMA Models

Integer-valued autoregressive moving-average (INARMA) models were first introduced in the 1980's ([33] and [1]). INARMA models use Bernoulli trials coupled with uncorrelated integer-valued random innovations to generate counts. Again, the INARMA tactics are simply appreciated in the INAR(1) setup. Define a thinning operator $\circ$ via

$$\alpha \circ X := \sum_{i=1}^{X} B_i(\alpha), \tag{1.9}$$

where $\{B_i(\alpha)\}_{i=1}^{\infty}$ are IID Bernoulli random variables with success probability α, and X is an integer-valued non-negative random variable. Since the resultant $\alpha \circ X \sim \text{Bin}(X, \alpha)$, this is known as binomial thinning [42]. The INAR(1) scheme is written as

$$X_t = \alpha \circ X_{t-1} + Z_t, \qquad (1.10)$$

where $\alpha \in (0,1)$ and $\{Z_t\}_{t=1}^{\infty}$ are independent and identically distributed non-negative integer-valued random variables with Z_t independent of $X_{t-1}, X_{t-2}, \ldots$. Notice in equation (1.10) that the thinning operator $\circ$ mimics the scalar multiplication of the AR(1) to maintain count values. Different distributional choices for Z_t give rise to different marginal count distributions. For example, picking Z_t with a Poisson marginal distribution with mean λ gives a count time series that is Poisson distributed with mean $\lambda(\alpha+1)$. Poisson, Geometric and Negative Binomial marginal distributional structures, among others, can be produced with INARMA methods ([33] and [34]).

INARMA methods still have deficiencies; in particular, they are unable to produce count series with negative autocovariances. This is because all thinning probabilities must be between 0 and 1.

1.3 Renewal Processes

A fundamental building block of this work is the discrete renewal process. A discrete renewal process is a stochastic model for "events" that occur in discrete time. These events will henceforth be referred to as renewals, indicating the regenerative nature of the overall process. A simple interpretation of a renewal process visualizes a device in service until it eventually fails. When it fails, a brand new device is immediately installed, that is independent of all previous devices. This process continues indefinitely. All device lifelengths are assumed independent and identically distributed. Each time a new device is installed, it is said that a renewal occurs. The length of time each device lasts is called a interarrival time or lifetime. This basic idea gives rise to many interrelated stochastic processes, which will be more rigorously defined as needed throughout.

Let L denote a non-negative integer-valued random variable representing a generic

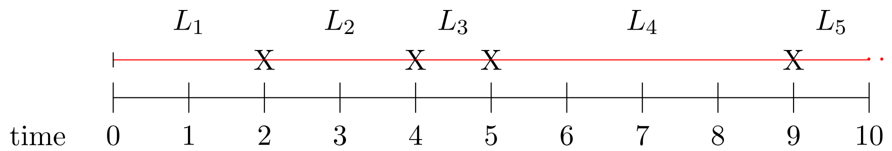

Figure 1.3: Sample discrete renewal process

lifetime. Then each lifetime L_n, $n = 1, 2, \ldots$ is equal in distribution to L. A simple example is given in Figure 1.3, where renewals occur at times $2, 4, 5, 9, \ldots$. In chapter 2, we develop a periodic version of discrete renewal processes that allows us to model count series that have periodic dynamics - say, for example, the number of rainy days in each of the 52 calendar weeks of the year.

1.4 Research Questions

Negatively correlated stationary time series are difficult to devise. In fact, generating a simple bivariate vector $(X, Y)'$, with X and Y each marginally Poisson distributed with the same mean λ, and where $\text{Cov}(X, Y) < 0$, is not trivial.

Is it useful to produce a series with negative autocorrelations? If yes, then there should be real-world data exhibiting negative covariances. It turns out that such data is not difficult to find. For example, Canadian Lynx sightings are negatively correlated from year to year. If in a given year Canadian Lynx sightings are high, the lynx over hunt the hare in the area, depleting the food supply. Thus, the following year's sightings of Canadian Lynx are low (and vice versa). Another example of a negatively correlated data is the major hurricane counts in the Atlantic Ocean and Pacific Ocean Basins shown in Figure 1.4. We select 1970 as a starting year because satellite reconnaissance was in full operation then, making it unlikely that a storm of such severity formed over open waters and went undocumented (this issue arises in early Atlantic Basin records). A Saffir-Simpson Category 3 or higher storm has wind speeds of 111 mph or more at some time during the storms life. Notice that when the Pacific Basin count is high, the Atlantic Basin count is low, and vice

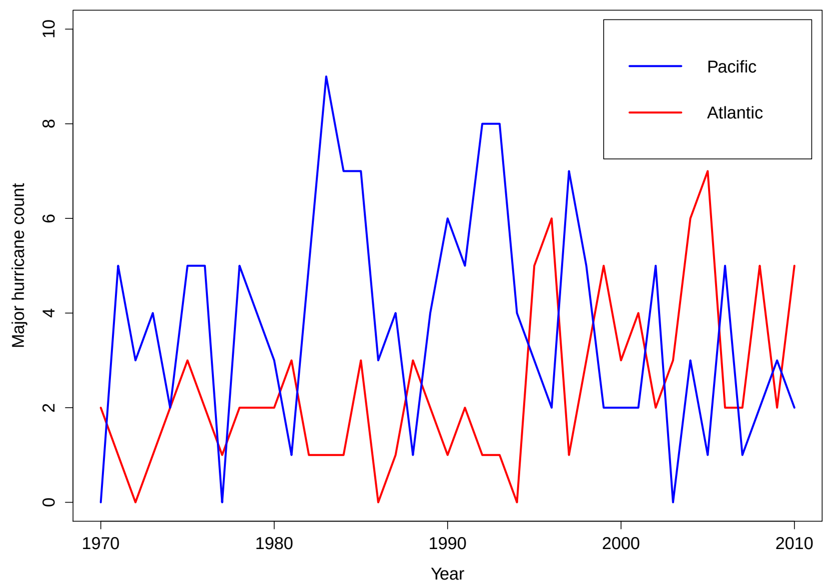

Figure 1.4: Major hurricane counts by basin of origin

versa. The sample correlation between the components is -0.43539!

There does not currently exist a current class of multivariate time series models that can easily achieve periodicity, long-memory, and negative autocovariances in tandem. In what follows, we will construct models that capture such structure. In Chapter 2, the work of Fralix, Livsey and Lund [14] is presented. Chapter 3 develops a multivariate count time series model with flexible autocorrelation structure and returns to the above hurricane data with some preliminary results.

Chapter 2

Renewal Sequences with Periodic Dynamics

Discrete-time renewal sequences play a fundamental role in the theory of stochastic processes. This paper considers periodic versions of such processes; specifically, the length of an interrenewal is allowed to probabilistically depend on the season at which it began. Using only elementary renewal and Markov chain techniques, computational and limiting aspects of periodic renewal sequences are investigated. We use these results to construct a time series model for a periodically stationary sequence of integer counts.

2.1 Background and Notation

Discrete renewal processes are ubiquitous in stochastic phenomenon and are extensively used in stochastic analyses (see Smith [41], Feller [13], Karlin and Taylor [25], Resnick [39] and Ross [40] for renewal background, history, and applications). This paper develops a periodic version of classical discrete-time renewal sequences. The period of the process, denoted by T, is assumed known and fixed throughout. The fundamental difference from an ordinary renewal process is that when a renewal occurs at time t, the time until the next renewal is allowed to depend on the season corresponding to time t. In this manner, one can build processes where renewals are more likely (or unlikely) during some seasons.

By superimposing or mixing versions of periodic renewal processes, one can construct models for periodic sequences of counts. This is useful in climatology, for example, where many phenomena have a definitive season of occurrence. Thunderstorms in the Southern United States can take place at any time in the year, but are most likely during the summer. Hurricanes, tornados, and snowstorms are other meteorological count processes with periodic features. Rare disease occurrences, accidental deaths, and animal sightings are non-meteorological examples of count series with periodic structure. Section 6 shows how periodic renewal processes can be used to describe periodically stationary count sequences. Such processes have periodic means and autocovariances (the counts are not independent).

Processes with periodic dynamics have been previously studied. General time series with periodic dynamics are overviewed in Hurd and Miamee [19] and Gardner, Napolitano, and Paura [15] from a second-order point of view. Lund and Basawa [30] study periodic time series via periodic autoregressive moving-average models. Markov chains with periodic dynamics have been considered in queueing contexts (Harrison and Lemoine [17], Lemoine [27], Heyman and Whitt [18], Asmussen and Thorisson [3]), branching processes (Jagers and Nerman [23]), storage models (Phatarfod [38], Lund [28]) and general regenerative processes (Çinlar [10], Thorisson [43]). While the analysis below is in discrete time and is somewhat pedestrian compared to say [43], its utility lies with having results carefully stated with explicit formulae supporting the theorems. Our arguments are elementary, developed fully, and our end pursuit, the generation of a time series of counts with periodic features, departs from the stochastic analysis slant of some of the above works.

2.1.1 Preliminaries

Our periodic discrete renewal process is described as follows. There are T possible seasons, which are indexed in the order $1, 2, \ldots, T$. There are T lifetimes $\{L_k\}_{k=1}^{T}$ that are used as follows: if a renewal occurs in season $\nu \in \{1, 2, \ldots, T\}$, then the time until the next renewal has the same distribution as L_ν and does not depend on past lifetimes in any way.

Let $s(t) = t - T\lfloor (t-1)/T \rfloor$ denote the season of time t so that $s(0) = T$, $s(1) = 1$, $\ldots$, $s(T-1) = T-1$, $s(T) = T$, $s(T+1) = 1$, etc..

Our periodic renewal process has renewals at the times $R_0 < R_1 < R_2 < \ldots$ and the kth lifetime is I_k, where $I_0 = R_0$ and $I_k = R_k - R_{k-1}$ for $k \geq 1$. Conditional on R_{n-1}, I_n has the same distribution as $L_{s(R_{n-1})}$. The interrenewal lifetimes are governed by T independent and identically distributed (iid) sequences $\{L_{1,k}\}_{k\geq 1}$, $\{L_{2,k}\}_{k\geq 1}$, $\ldots$, $\{L_{T,k}\}_{k\geq 1}$, with $L_{\nu,k} \stackrel{d}{=} L_\nu$ for each $\nu \in \{1, 2, \ldots, T\}$ and $k \geq 1$. The initial lifetime R_0, which may not be equal in distribution to L_ν for any season ν, is assumed independent of all other lifetimes. For each $n \geq 1$, one has

$$R_n = R_{n-1} + L_{s(R_{n-1}),n}.$$

In contrast to classical renewal theory, the interrenewal times are no longer iid. We call the process pure if $R_0 = 0$; otherwise, the process is termed delayed. Let u_n be the renewal probability at time n, i.e.

$$u_n = \sum_{k=0}^{\infty} \Pr(R_k = n).$$

The sequence $\{u_n\}_{n\geq 0}$ is called the renewal probability sequence. One goal of this paper is to find an initial distribution of R_0 that makes the renewal probabilities periodically constant: $u_{nT+\nu} = u_\nu$, for all $n \geq 0$. The utility of this will be seen in Section 6. Renewal processes satisfying this condition are called periodically stationary. When $T = 1$, our model reduces to the classical time-homogeneous renewal model.

The age chain $\{A_n\}_{n=0}^{\infty}$ is defined as $A_n = n - \sup\{R_k : R_k \leq n\}$; that is, A_n is the elapsed time since the most recent renewal previous to time n. If a renewal occurs at time n, then $A_n = 0$. If the last renewal occurring at or before time n occurred at time $k < n$, then $A_n = n - k$. Later, the derived limiting behavior of $\{A_n\}_{n=0}^{\infty}$ will show what distribution of the delay R_0 is needed to generate a periodically stationary renewal sequence.

2.2 Computation of Renewal Probabilities

As a ground zero issue, one would like to be able to compute $\{u_n\}_{n=1}^{\infty}$. For this, let $u_n^{(\nu)}$ be the probability of a renewal at time $n \geq 1$ when R_0 has the same distribution as L_ν:

$$u_n^{(\nu)} = \Pr(\text{A renewal occurs at time } n \text{ when } R_0 \text{ is a } L_\nu \text{ lifetime}).$$

While a non-delayed setup involves the season $\nu = T$ renewal probabilities $\{u_n^{(T)}\}_{n=1}^{\infty}$, it will be convenient to calculate $\{u_n^{(\nu)}\}_{n=1}^{\infty}$ simultaneously for all seasons $\nu \in \{1, 2, \dots T\}$.

The renewal probabilities at time one are simply $u_1^{(\nu)} = \Pr(L_\nu = 1)$ for each season ν. Conditioning on the time of the first renewal gives the recursion

$$u_n^{(\nu)} = \Pr(L_\nu = n) + \sum_{k=1}^{n-1} \Pr(L_\nu = k) u_{n-k}^{s(\nu+k)}. \tag{2.1}$$

From equation (2.1), it is a simple matter to compute the renewal probabilities in the order $u_1^{(1)}, \dots, u_1^{(T)}; u_2^{(1)}, \dots, u_2^{(T)}; \dots$.

As an example of the above, consider the case where L_ν has a geometric distribution with success probability p_ν. For concreteness, suppose p_ν has the sinusoidal structure

$$p_\nu = A + B \cos\left(\frac{2\pi(\nu - \tau)}{T}\right), \tag{2.2}$$

where A, B, and τ are parameters rendering $p_\nu \in (0, 1)$ for all seasons ν. Selecting $T = 3$, $A = 1/3$, $B = 3/8$, and $\tau = 3.2$ gives $p_1 = 0.2941$, $p_2 = 0.02995$, and $p_3 = 0.6759$ to four significant digits. The season 3 renewal probabilities are plotted in Figure 2.1. Notice the rapid convergence of the renewal probabilities to a periodic limit. In the limit, the season three renewal probabilities are the largest and the season two probabilities are the smallest.

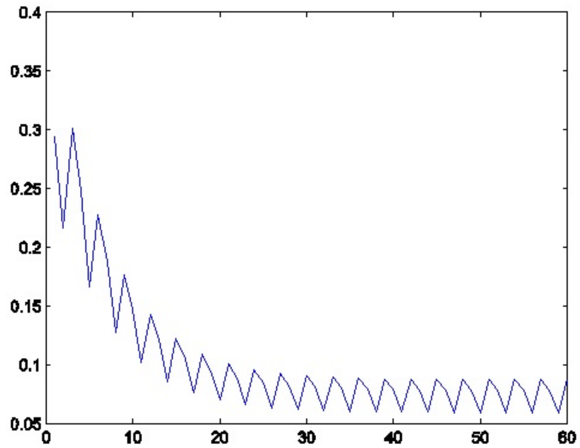

Figure 2.1: Renewal probabilities starting with a season three lifetime

2.3 Limiting Properties

This section establishes a periodic limiting distribution for $\{A_n\}_{n=0}^{\infty}$ and a deterministic periodic limit for $\{u_n\}_{n=1}^{\infty}$. Since the interest here is only in limiting behavior and the limits will not depend on the initial delay, we assume in this section that $R_0 = 0$, making $A_0 = 0$.

Our first observation shows that $\{A_n\}_{n=0}^{\infty}$ is a Markov chain with periodic transition probabilities.

Proposition 2.3.1 *$\{A_n\}_{n=0}^{\infty}$ is a Markov chain with periodic transition probabilities. The only non-zero transition probabilities are*

$$\begin{aligned} \Pr(A_{n+1} = i+1 \mid A_n = i) &= \Pr(L_{s(n-i)} > i+1 \mid L_{s(n-i)} > i) \\ \Pr(A_{n+1} = 0 \mid A_n = i) &= \Pr(L_{s(n-i)} = i+1 \mid L_{s(n-i)} > i). \end{aligned}$$

Proof Observe that if $A_n = i$, the only possible values for A_{n+1} are $i+1$ or 0. Also, if $A_n = i$, then the last renewal previous to time n must have occurred at time $n-i$ and this item is known to have lasted more than i time units. Hence, given $A_n = i$, the item in use at time n has a distribution equivalent to that of $L_{s(n-i)}$. Finally, conditional on $A_n = i$, $A_{n+1} = i + 1$ if and only if the item in use at time n lasts at least $i + 1$ time units. This verifies the first equation in the proposition; similar reasoning (or complementation) verifies the second equation. $\diamondsuit$

Unfortunately, the one-step-ahead transition probabilities $\Pr(A_{n+1} = j \mid A_n = i)$ change with n in a periodic manner. Thus, while the age process is a Markov chain, it is not a time-homogeneous Markov chain. This observation leads us to the following proposition, which shows that subsequences of the age process on a periodic lattice are time-homogeneous chains.

Proposition 2.3.2 *The sequence* $\{A_{nT+\nu}\}_{n=0}^{\infty}$ *is a time-homogeneous Markov chain for each fixed season* ν*.*

Proof As any subsequence of a Markov chain retains the Markov property, we need only verify the claimed time-homogeneity of transition probabilities. Given that $A_{nT+\nu} = i$, we know that the item in use at time $nT + \nu$ is drawn from a season $s(nT + \nu - i)$ distribution and is i units old at time $nT + \nu$. Should this item fail within the next T time units, it will be replaced by new item(s) whose distribution only depends on the season of failures(s) and not on the cycle index n. Should such replacement(s) result in an age of j at time $(n+1)T + \nu$ (note that $A_{(n+1)T+\nu}$ can be no larger that $T + i$), then this same replacement sequence would have produced an age of j at time $(n + 2)T + \nu$ should $A_{(n+1)T+\nu} = i$. Invoking periodic IID renewals proves the time-homogeneity of transition probabilities. $\diamondsuit$

We now derive conditions that ensure that $\{A_{nT+\nu}\}_{n=0}^{\infty}$ is irreducible, aperiodic, and positive recurrent. It will be helpful to introduce a Markov chain $\{S_n\}_{n=0}^{\infty}$ that keeps track of the season which the nth item was generated from. Specifically, for $\nu \in \{1, 2 \ldots, T\}$, let $C_\nu = \{\nu, T + \nu, 2T + \nu, \ldots\}$ be the set of all times at which season ν takes place and for

$\nu, \nu' \in \{1, 2, \ldots, T\}$, set

$$p_{\nu,\nu'} = \Pr(S_{n+1} = \nu' \mid S_n = \nu) = \Pr(L_\nu \in C_{s(\nu'-\nu)}).$$

Proposition 2.3.3 *If $\{S_n\}_{n=0}^\infty$ is irreducible, aperiodic, and positive recurrent and $E[L_\nu] < \infty$ for every season ν, $\{A_{nT+\nu}\}_{n=0}^\infty$ is irreducible, aperiodic and positive recurrent for each season ν and hence has a unique limiting distribution.*

Proof It suffices to show that $E_\nu[\tau_\nu] < \infty$ for each season ν, where τ_ν denotes the time spent waiting for a future renewal to take place in season ν and E_ν signifies that a renewal has currently taken place at a time whose season is ν.

We first consider the case where $p_* := \min_{i,j} p_{i,j} > 0$. Under this assumption, there is some positive probability that each and every new renewal will take place in the season ν lattice set C_ν. A geometric trials argument now shows that τ_ν is stochastically bounded by $\tau_\nu \stackrel{st}{\leq} \sum_{j=1}^N M_j^*$, where $\{M_j^*\}_{j\geq 1}$ is an iid sequence, with $M_1^* \stackrel{d}{=} \max(L_1, \ldots, L_T)$, and N is a geometric random variable with parameter p. Hence, $E_\nu[\tau_\nu] \leq E[M_1^*]E[N] < \infty$ follows from $E[M_1^*] \leq \sum_{j=1}^T E[L_j] < \infty$.

Now suppose that $\min_{i,j} p_{i,j} = 0$. Since $\{S_n\}_{n=0}^\infty$ is a finite state-space chain whose transition matrix is irreducible, aperiodic, and positive recurrent, there exists an integer $k > 1$ such that the k-step-ahead transitions of $\{S_n\}_{n=0}^\infty$ are uniformly positive for all i and j: $\min_{i,j} p_{i,j}^{(k)} > 0$ (Billingsley [5]). Given that we are currently experiencing a season ν renewal, let $\tau_\nu^{(k)}$ be the first time until ℓ more renewals occur where, (1) ℓ is a whole multiple of k, and (2) the renewal puts the chain in season ν. Applying the above argument shows that $E[\tau_\nu^{(k)}] < \infty$. But since $\tau_\nu \leq \tau_\nu^{(k)}$, $E[\tau_\nu] < E[\tau_\nu^{(k)}] < \infty$ and our work is done. $\diamondsuit$

For notation, let $\pi_k(\nu) = \lim_{n\to\infty} \Pr(A_{nT+\nu} = k)$ denote the stationary distribution of $\{A_{nT+\nu}\}_{n=0}^\infty$. Our next goal is to compute this distribution for each season ν. The following result reduces this issue to the case where $k = 0$.

Proposition 2.3.4 *For each $k \geq 0$ and each season ν,*

$$\pi_k(\nu) = \Pr(L_{s(\nu-k)} > k)\pi_0(s(\nu - k)). \tag{2.3}$$

Proof Simply use that

$$\begin{aligned}
\pi_k(\nu) &= \lim_{n\to\infty} \Pr(A_{nT+\nu} = k) \\
&= \lim_{n\to\infty} \Pr(A_{nT+\nu} = k, A_{nT+\nu-k} = 0) \\
&= \lim_{n\to\infty} \Pr(A_{nT+\nu} = k \mid A_{nT+\nu-k} = 0)\Pr(A_{nT+\nu-k} = 0) \\
&= \Pr(L_{s(\nu-k)} > k)\pi_0(s(\nu - k)).
\end{aligned}$$

$\diamondsuit$

When the lifetimes $\{L_\nu\}_{\nu=1}^{T}$ are all identically distributed (seasonally non-varying), the age process is time-homogeneous and there is only one stationary distribution to compute. In this case, (2.3) gives

$$1 = \sum_{k=0}^{\infty} \pi_k(1) = \sum_{k=0}^{\infty} \Pr(L_1 > k)\pi_0(1) = \pi_0(1)E(L_1),$$

which yields the classical result $\pi_0(1) = E[L_1]^{-1}$.

An analogous approach will be used in our periodic setting, but will require slightly more work. For each season ν, $\{\pi_k(\nu)\}_{k=0}^{\infty}$ is a probability measure. Using this and Proposition 4.4 provides, for each season ν,

$$1 = \sum_{k=0}^{\infty} \pi_k(\nu) = \sum_{j=1}^{\nu} \pi_0(j) \sum_{n=0}^{\infty} \Pr(L_j > nT+\nu-j) + \sum_{j=\nu+1}^{T} \pi_0(j) \sum_{n=0}^{\infty} \Pr(L_j > (n+1)T+\nu-j). \tag{2.4}$$

Hence, $\vec{\pi}_0 := (\pi_0(1), \pi_0(2), \ldots, \pi_0(T))$ is a solution to a linear system of T equations.

Our immediate goal is to solve (2.4) for $\pi_0(1), \ldots, \pi_0(T)$. To help solve this, we offer the following lemma. Here, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Lemma 2.3.1 *Let X be a positive integer-valued random variable. Then for each season ν*

$$\sum_{n=0}^{\infty} \Pr(X > nT + \nu) = E\left[\lfloor X/T \rfloor\right] + \Pr\left(X \in \cup_{\ell=\nu+1}^{T-1} C_\ell\right). \tag{2.5}$$

Proof Since $\Pr(X > nT+\nu) = \Pr(X \geq (n+1)T) + \Pr(X \in \{nT+\nu+1, \ldots, nT+(T-1)\})$,

$$\sum_{n=0}^{\infty} \Pr(X > nT + \nu) = \sum_{n=0}^{\infty} \Pr(X \geq (n+1)T) + \Pr(X \in B_\nu),$$

where $B_\nu = \cup_{\ell=\nu+1}^{T-1} C_\ell$. Using this and

$$\sum_{n=0}^{\infty} \Pr(X \geq (n+1)T) = \sum_{n=1}^{\infty} \Pr\left(\lfloor X/T \rfloor \geq n\right) = E[\lfloor X/T \rfloor]$$

finishes our work. $\diamondsuit$

Applying Lemma 2.3.1 to (2.4) for each season ν yields

$$1 = \sum_{j=1}^{\nu} \pi_0(j)\left[E(\lfloor L_j/T \rfloor) + \Pr(L_j \in B_{\nu-j+1})\right] + \sum_{j=\nu+1}^{T} \pi_0(j)\left[E(\lfloor L_j/T \rfloor) + \Pr(L_j \in B_{T+\nu-j+1})\right]. \tag{2.6}$$

Equation (2.6) shows that $\vec{\pi}_0$ lies in $\{x \in \mathbb{R}^T : Ax = \vec{1}\}$, where $\vec{1}$ is a T-dimensional column vector containing all ones and $A := [a_{i,j}]_{1 \leq i,j \leq T}$ is a matrix with entries

$$a_{i,j} = \begin{cases} E(\lfloor L_j/T \rfloor) + \Pr(L_j \in B_{i-j+1}), & i \geq j; \\ E(\lfloor L_j/T \rfloor) + \Pr(L_j \in B_{T+i-j+1}), & i < j \end{cases},$$

where the convention $B_0 = B_T$ is used. We now show that $\vec{\pi}_0$ can be expressed purely in terms of the stationary distribution of the seasonal chain and the first moment of the T lifetime distributions.

Readers should note that the expressions in Theorem 4.1 for $\vec{\pi}_0$ are akin to those found in Theorem 3 of [43], who studies periodic renewal processes in continuous time when the interrenewal distributions have some common absolutely continuous component over all seasons.

Theorem 2.3.1 *For each season ν,*

$$\pi_0(\nu) = \lim_{n\to\infty} \Pr(A(nT+\nu)=0) = \frac{T\gamma_\nu}{\sum_{\ell=1}^{T} E(L_\ell)\gamma_\ell}$$

where $\vec{\gamma} = \{\gamma_\nu\}_{\nu=1}^{T}$ represents the stationary distribution of $\{S_n\}_{n=0}^{\infty}$.

Proof Consider the linear system given by (2.6). Subtracting the second equation from the first equation, the third equation from the second equation, the fourth from the third, and so on, we observe that $\vec{\pi}_0$ must also satisfy a linear system of T equations, the first $T-1$ being of form

$$\pi_0(\nu) \sum_{\nu' \neq \nu} p_{\nu,\nu'} = \sum_{\nu' \neq \nu} \pi_0(\nu^{'}) p_{\nu',\nu}, \qquad 2 \leq \nu \leq T,$$

where $p_{\nu',\nu}$ is the (ν',ν)th entry in the transition matrix of $\{S_n\}_{n=0}^{\infty}$. The last equation is

$$1 = \sum_{\nu=1}^{T} \pi_0(\nu) \left\{ E(\lfloor L_\nu/T \rfloor) + \Pr(L_\nu \in B_{T-\nu+1}) \right\}.$$

Adding $\pi_0(\nu)p_{\nu,\nu}$ to both sides of the first $T-1$ equations shows that $\vec{\pi}_0$ also satisfies

$$\pi_0(\nu) = \sum_{\ell=1}^{T} \pi_0(\ell) p_{\ell,\nu}, \qquad 2 \leq \nu \leq T,$$

$$1 = \sum_{\nu=1}^{T} \pi_0(\nu) \left\{ E(\lfloor L_\nu/T \rfloor) + \Pr(L_\nu \in B_{T-\nu+1}) \right\}.$$

This new system is extremely elegant; in particular, the first $T-1$ equations are the stationary balance equations of $\{S_n\}_{n=0}^{\infty}$. From these first $T-1$ equations, one can show that $\vec{\pi}_0$ must satisfy all balance equations for the stationary distribution of $\{S_n\}_{n=0}^{\infty}$. Since $\{S_n\}_{n=0}^{\infty}$ is irreducible and positive recurrent, all solutions of the balance equations are scalar multiples of the stationary distribution $\vec{\gamma}$ of the seasonal chain (for details see e.g. [39]). Thus, there exists a constant $c > 0$ satisfying $\vec{\pi}_0 = c\vec{\gamma}$.

We finish by determining the unknown constant c. Summing all equations in (2.4) and applying $\vec{\pi}_0 = c\vec{\gamma}$ gives $T = c\sum_{\ell=1}^{T} E[L_\ell]\gamma_\ell$. Hence,

$$c = \frac{T}{\sum_{\ell=1}^{T} E[L_\ell]\gamma_\ell}$$

and our derivation is complete. $\diamondsuit$

Our results are expressed in terms of the stationary distribution $\vec{\gamma}$ of $\{S_n\}_{n=0}^{\infty}$, which depends on terms of the form $\Pr(L \in C_\nu)$. Such probabilities are tractable for many classical types of integer-valued random variables. For example, if L is geometric with parameter p, then for each season ν,

$$\Pr(L \in C_\nu) = \frac{p(1-p)^{\nu-1}}{1-(1-p)^T}, \qquad \nu = 1, 2, \ldots, T. \tag{2.7}$$

It is also possible to compute such probabilities when L is a negative binomial random variable with parameters $m \geq 2$ and $p \in (0, 1)$. One efficient way of doing this is to recognize that L is now an iid sum of m geometric random variables with parameter p. Setting G to be a matrix with $(i, j)th$ element

$$g_{i,j} = \frac{p(1-p)^{s(j-i)-1}}{1-(1-p)^T},$$

we see that $\Pr(L \in C_\nu)$ is simply the νth component of $\vec{e}_1 G^m$, where $\vec{e}_1$ is the first basis vector $(1, 0, 0, \ldots, 0)$.

These types of probabilities can also be computed explicitly when L is a Poisson random variable. Interested readers will find that $P(L \in C_\nu)$ can, in this case, be expressed rather elegantly in terms of the T roots of unity (possibly complex-valued solutions to $z^T = 1$).

Example 4.2 Suppose that $T = 2$. Then the transition matrix of $\{S_n\}_{n=0}^{\infty}$ has $(i, j)^{th}$

element $\Pr(L_i \in C_{s(j-i)})$ for $i, j \in \{1, 2\}$. The stationary distribution of $\{S_n\}_{n=0}^{\infty}$ hence has form

$$\vec{\gamma} = \left(\frac{\Pr(L_2 \in C_1)}{\Pr(L_1 \in C_1) + \Pr(L_2 \in C_1)}, \frac{\Pr(L_1 \in C_1)}{\Pr(L_1 \in C_1) + \Pr(L_2 \in C_1)} \right).$$

Hence,

$$\pi_0(1) = \lim_{n \to \infty} u_{nT+1} = \frac{2\Pr(L_2 \in C_1)}{E[L_1]\Pr(L_2 \in C_1) + E[L_2]\Pr(L_1 \in C_1)}$$

and

$$\pi_0(2) = \lim_{n \to \infty} u_{nT+2} = \frac{2\Pr(L_1 \in C_1)}{E[L_1]\Pr(L_2 \in C_1) + E[L_2]\Pr(L_1 \in C_1)}.$$

Arguing with (2.3) gives

$$\pi_k(1) = \Pr(L_2 > k)\pi_0(2)1_{\{k \text{ odd}\}} + \Pr(L_1 > k)\pi_0(1)1_{\{k \text{ even}\}}, \quad k \geq 1$$

and

$$\pi_k(2) = \Pr(L_1 > k)\pi_0(1)1_{\{k \text{ odd}\}} + \Pr(L_2 > k)\pi_0(2)1_{\{k \text{ even}\}}, \quad k \geq 1.$$

Example 4.3 Consider a setting where a renewal always occurs during season one; that is, there is always an annual replacement. If the item fails before season one, the item is replaced at that time. Such a scenario occurs when the support of L_ν is $\{1, 2, \ldots, T-\nu+1\}$ for each season ν. We start with a season T lifetime at time zero so that $S_0 = T$.

The renewal probabilities can be obtained from the renewal equations of Section 3: set $u_0 = 1$, and for each season ν,

$$u_\nu = \sum_{k=0}^{\nu-1} u_k P(L_k = \nu - k).$$

One can easily verify that for each season ν,

$$\pi_0(\nu) = \frac{T u_\nu}{\sum_{\ell=1}^{T} E[L_\ell] u_\ell}$$

satisfy the limiting equations in Theorem 4.1.

2.4 A Periodically Stationary Initial Lifetime

What should the law of the initial delay R_0 be in order for our process to be exactly periodically stationary from the onset? Specifically, we seek to identify the distribution of R_0 so that the renewal probabilities are exactly periodic: $u_{nT+\nu} \equiv \pi_0(\nu)$ for all $n \geq 0$. When $T = 1$, it is well-known that the law of R_0 is simply

$$\Pr(R_0 = n) = \frac{\Pr(L_1 \geq n+1)}{E[L_1]}, \qquad n \geq 0.$$

This is often referred to as the equilibrium distribution, or the first-derived distribution of L_1.

As our next result shows, if the law of A_0 is set to $\vec{\pi}(T)$, then the age chain is put into a periodic state (exactly). Our notation uses P_ν for the one-step-ahead transition probability matrix whose (i, j)th element is $\Pr(A_{\nu+1} = j \mid A_\nu = i)$.

Proposition 2.4.1 *Interpreted periodically with period* T, $\vec{\pi}(\nu+1) = \vec{\pi}(\nu)P_\nu$, *where* $\vec{\pi}(\nu) = \{\pi_k(\nu)\}_{k=0}^{\infty}$.

Proof The result follows from the fact that the stationary distribution of the subsequence $\{A_{nT+\nu}\}_{n=0}^{\infty}$ must be unique for each fixed season ν. In fact, the one-step-ahead transition matrix of $\{A_{nT+\nu}\}_{n=0}^{\infty}$ is $\mathbb{Q}_\nu := P_\nu P_{\nu+1} \cdots P_T P_1 \cdots P_{\nu-1}$. Since $\vec{\pi}(\nu) = \vec{\pi}(\nu)\mathbb{Q}_\nu$, we have

$$\vec{\pi}(\nu)P_\nu = \vec{\pi}(\nu)\mathbb{Q}_\nu P_\nu = \vec{\pi}(\nu)\left(P_\nu P_{\nu+1} \cdots P_T P_1 \cdots P_{\nu-1}\right) P_\nu. \tag{2.8}$$

Hence, $\vec{\pi}(\nu)P_\nu = \vec{\pi}(\nu)P_\nu \mathbb{Q}_{\nu+1}$. But since $\vec{\pi}(\nu+1) = \vec{\pi}(\nu+1)\mathbb{Q}_{\nu+1}$, uniqueness of stationary distributions shows that $\vec{\pi}(\nu)P_\nu = \vec{\pi}(\nu+1)$. $\diamondsuit$

Deriving an expression for the probability mass function of R_0 is now a simple task. Suppose that the law of A_0 is $\pi(T)$ so that the age chain is periodically stationary (exactly) from the onset. Let $\tilde{f}_k = \Pr(R_0 = k)$ be the mass function of the initial delay. Conditioning on R_0 gives the recursion

$$u_n = \tilde{f}_n + \sum_{k=0}^{n-1} u_k \Pr(L_{s(k)} = n - k).$$

Solving this for $\tilde{f}_n$ gives

$$\tilde{f}_n = \pi_0(s(n)) - \sum_{k=0}^{n-1} \pi_0(s(k)) \Pr(L_{s(k)} = n - k) \tag{2.9}$$

when the fact that R_0 was chosen to induce a periodically stationary renewal process is applied ($u_{nT+\nu} = \pi_0(\nu)$ for all n and seasons ν).

2.5 A Count Time Series Model with Periodic Dynamics

Count time series models with periodic properties can be devised with the above methods. For example, suppose one is interested in modelling the number of precipitation days in a week at a fixed locality (a day is called a precipitation day if 0.1 inches or more of rain or its snow water equivalent is recorded). Here, a binomial marginal distribution with 7 trials plausibly describes the counts in any week. However, because adjacent weeks experience similar weather, weekly counts are likely to exhibit positive correlation. Also, some localities should display periodic features. For example, rain rarely occurs in California during the summer, but is common during the winter. Figure 2.2 plots the number of precipitation days observed in 728 successive weeks at Coldfoot, Alaska spanning the 14-year period January 1, 1996 — December 31, 2009. Coldfoot is noteworthy as it claims North America's lowest observed temperature of -82°F. While this record is not officially recognized due to gauge deficiencies, Coldfoot, lying near the Brooks Range, has a seasonal but ephemeral climate. Leap year day (Feb 29) precipitations and December 31 precipitations have been neglected to induce an "exact period" of $T = 52$ weeks in the counts. This should not influence end inferences greatly. In fact, precipitation was observed on December 31 in only two of the 14 years in the record, for example.

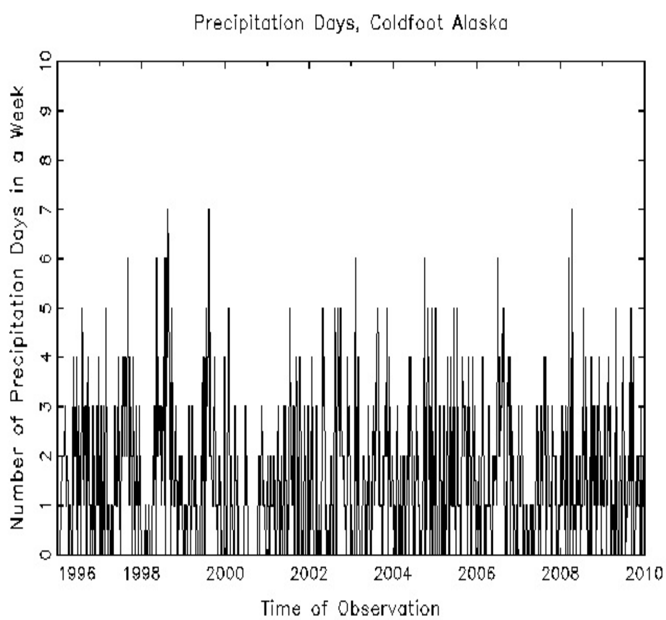

Figure 2.2: Weekly precipitation days at Coldfoot, Alaska

A model for series with the above properties is easily devised from our work. Let $\{R_t\}$ be a renewal process that is periodically stationary (exactly); that is, R_t is unity if a renewal occurs at time t and zero otherwise. Then $\Pr(R_{nT+\nu} = 1) = u_\nu$ is the probability of a renewal at time $nT + \nu$. Now suppose that $\{R_{t,1}\}_{t=1}^{\infty}, \{R_{t,2}\}_{t=1}^{\infty}, \ldots$ are independent copies of $\{R_t\}$. To model the precipitation count N_t of week t, set

$$N_t = \sum_{\ell=1}^{7} R_{t,\ell}. \tag{2.10}$$

It is easy to see that $\{N_t\}$ has marginal binomial distributions with 7 trials. Also, $\{N_t\}$ has a periodic mean and covariance structure. To see this, note that $E[R_{nT+\nu}] = u_\nu$, and hence, $E[N_{nT+\nu}] = 7\mu_\nu$. By a periodic covariance structure, (frequently termed cyclostationary, periodically stationary, or periodicially correlated in the literature), we mean that $\mathrm{Cov}(N_{n+T}, N_{m+T}) = \mathrm{Cov}(N_n, N_m)$ for all integers n and m. This is seen by noting that for $t < s$,

$$\begin{aligned}
\mathrm{Cov}(N_t, N_s) &= \sum_{i=1}^{7} \mathrm{Cov}(R_{t,i}, R_{s,i}) \\
&= 7u_t \left[\Pr(R_s = 1 | R_t = 1) - u_s\right] \\
&= 7\pi_0(t)[\Pr(R_s = 1 | R_t = 1) - \pi_0(s)],
\end{aligned}$$

and applying $\Pr(R_{s+T} = 1 | R_{t+T} = 1) = \Pr(R_s = 1 | R_t = 1)$ and $u_{t+T} = u_t = \pi_0(t)$.

As an aside, observe that marginal distributions other than Binomial can be devised with the above methods. For example, should one desire periodic Poisson marginals, then (2.10) is modified to $N_t = \sum_{\ell=1}^{M_t} R_{t,\ell}$, where $\{M_t\}_{t=1}^{\infty}$ is a periodic random sequence having Poisson marginal distributions, say $E[M_t] = \lambda_t$ with $\lambda_{t+T} = \lambda_t$. Cui and Lund [11] show how to generate geometric and other classical count structures from the above tactics. The fundamental building block to all constructions is the renewal process, which is simply a correlated sequence of zeros and ones.

Returning to the binomial problem at hand, we briefly attempt to fit a rudimentary statistical model to the Coldfoot series. For the renewal process, we posit that the season ν lifetime L_ν is geometrically distributed with success probability $p_\nu > 0$; moreover, since p_ν is periodic, we explore the first-order Fourier parametrization in (2.2), where $A > 0$, $B \in [0, 1-A]$ are unknown amplitude parameters and $\tau \in [0, T]$ is an unknown phase parameter. We fit this model by minimizing a sum of squared prediction residuals. Specifically, arguing as in [11], one can show that the weekly precipitation counts $\{X_t\}$ obey a periodic Markov structure and that

$$E[N_{t+1}|N_t] = E[N_{t+1}] + \frac{\text{Cov}(N_{t+1}, N_t)}{\text{Var}(N_t)}(N_t - E[N_t]).$$

Using $E[N_t] = 7\pi_0(t)$, $\text{Var}(N_t) = 7\pi_0(t)(1 - \pi_0(t))$, $\Pr[L_{s(t)} = 1] = p_{s(t)}$, and

$$\text{Cov}(N_{t+1}, N_t) = 7\pi_0(t)[\Pr(L_{s(t)} = 1) - \pi_0(t+1)] = 7\pi_0(t)[p_{s(t)} - \pi_0(t+1)],$$

a one-step-ahead prediction of the form

$$\begin{aligned}
\hat{N}_{t+1} &= E[N_{t+1}|N_1, \dots, N_t] \\
&= E[N_{t+1}|N_t] \\
&= 7\pi_0(t+1) + \frac{p_{s(t)} - \pi_0(t+1)}{1 - \pi_0(t)}(N_t - 7\pi_0(t))
\end{aligned}$$

is obtained.

A reasonable objective function for selecting A, B, and τ simply minimizes

$$\sum_{\ell=0}^{d-1} \sum_{\nu=1}^{T} (N_{\ell T+\nu} - \hat{N}_{\ell T+\nu})^2 = \sum_{t=1}^{n} (N_t - \hat{N}_t)^2 \tag{2.11}$$

over feasible values of A, B, and τ. In (2.11), $n = 728$ is the total number of weeks and $d = n/T = 14$ is the number of years of observed data. Observe that $\hat{N}_t$ is a function of A, B, and τ only and that $\Pr[L_{s(t)} = 1] = p_{s(t)}$.

A numerical minimization routine was used to find parameter values that minimize

(2.11): $\hat{A} = 0.2225$, $\hat{B} = 0.08333$, and $\hat{\tau} = 29.2770$. The minimum sum of squares was 1588.9517. Elaborating, Theorem 4.1 was used to compute $\vec{\pi}_0$ for each feasible triple of A, B, and τ. The values of $\{\gamma_\nu\}_{\nu=1}^{52}$ needed were obtained numerically after explicit expressions for the one-step-ahead transitions of the seasonal chain $\{S_n\}_{n=0}^{\infty}$ were computed. Specifically, L_ν having a geometric distribution with parameter p_ν implies that

$$\Pr(S_{n+1} = j | S_n = i) = \begin{cases} \frac{p_i(1-p_i)^{j-i-1}}{1-(1-p_i)^T}, & j > i \\ \frac{p_i(1-p_i)^{T+j-i-1}}{1-(1-p_i)^T}, & j \leq i \end{cases}.$$

Numerical estimates of the Hessian of this fit indicate that B is significantly positive, implying that periodic features are needed in the model. As an example of what the model fit does, Figure 2.3 plots the weekly precipitation count segment during 2001 along with one-step-ahead predictions. The one-step-ahead predictions track the observed counts reasonably well. The estimated long run probabilities of a precipitation day peak during week 29 (summer) at 0.2950 and are minimal during week three (winter) at 0.1485.

While we leave issues of whether the fit is good, etc., to a statistical inference paper, one appreciates that periodic renewal processes have utility in a variety of applications.

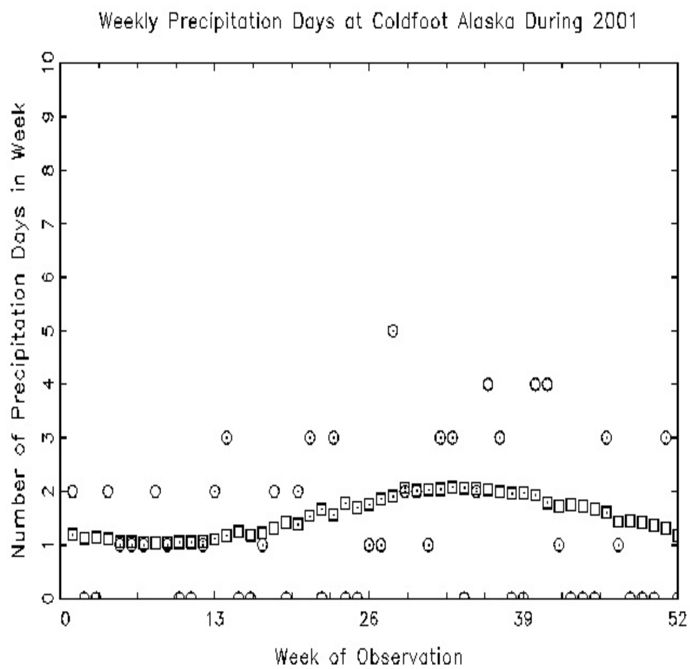

Figure 2.3: Weekly precipitation days during 2001 (Circles) and their one-step-ahead predictions (Squares)

Chapter 3

Multivariate Count Models

3.1 Preliminaries

Although much research considers the modeling of multivariate count time series, no one class of models has emerged as the most flexible, parsimonious, and widely used. This chapters develops a multivariate count time series model by superimposing stationary renewal processes. This method, though radically different from current multivariate count time series tactics, achieves a large range of autocorrelations and cross-correlations. Difficulties in modeling multivariate count time series arise in concurrently dealing with both auto and cross-correlations. We use a multivariate renewal process to help resolve issues.

Improved computing power has led to a surge in literature on generating IID, and correlated random variables ([9],[4]), especially multivariate Poisson [44]. These methods, coupled with multivariate renewal process results, will be used to increase both our modeling possibilities and our range of feasible correlations. Since multivariate time series appear in a variety of applications, flexible multivariate count models are needed [35]. Currently, the majority of multivariate count series models follow multivariate integer-valued autoregressive moving-average (MINARMA) recursions that generalize the univariate thinning operator [2]. MINARMA models have been extensively researched in the past few years. The bivariate INAR(1) (BINAR(1)) was defined in [36] and then generalized in [37]. This method uses Bernoulli trials to replace the scalar multiplication of Gaussian multivariate ARMA models. The MINAR(1) model obeys

$$\mathbf{X}_t = A \circ \mathbf{X}_{t-1} + \mathbf{Z}_t, \tag{3.1}$$

where A is an $n \times n$ matrix of independent autocorrelation parameters and $A\circ$ acts as the usual matrix multiplication, where each scalar multiplication is a binomial thinning as defined in Section 1.2.2. Here, $\{\mathbf{Z}_t\}$ are IID integer-valued innovations where $\mathbf{Z}_t$ is independent of $X_{t-1}, X_{t-2}, \ldots$. The ith marginal element of the vector $\mathbf{X}_t$ is then

$$X_{i,t} = \sum_{j=1}^{n} \alpha_{i,j} \circ X_{j,t-1} + Z_{i,t}, \tag{3.2}$$

where $\alpha_{i,j}$ is the (i,j)th element of A. In the multivariate setting, these methods induce complicated estimation problems. Moreover, the autocorrelation function of each component of (3.1) must be non-negative, a drawback that we will resolve with renewal methods.

The generalized integer-valued autoregressive (GINAR) model was generalized to the multivariate setting (MGINAR(p)) in [26]. This work, analogous to the extension from INAR to GINAR, created a stationary class of multivariate integer-valued autoregressive series with autocovariance functions identical to Gaussian VAR(p) models. Instead of defining a single thinning probability, MGINAR(p) series have p mutually independent operators $A_1, A_2, \ldots, A_p$.

$$\mathbf{X}_t = \sum_{j=1}^{p} A_j \circ \mathbf{X}_{t-j} + \boldsymbol{\epsilon}_j.$$

However, this paper later goes on to show the autocorrelation structure of the MGINAR(1) is exactly that of the AR(1) model. Thus, producing a long-memory series in this paradigm is unobtainable.

Recently, negative correlation modeling issues have received attention in the count literature. [24] attempts to introduce negative autocorrelation structure by rounding ordinary ARMA models (as opposed to the thinning operator of the aforementioned MINAR model of 3.1). This rounded integer (RINAR) model,

$$X_t = \left\langle \sum_{j=1}^{p} \alpha_j X_{t-j} + \lambda \right\rangle + \epsilon_t$$

where $< \cdot >$ is the standard rounding operator, can produce a series with negative autocorrelations. Unfortunately, this method, due to the rounding, cannot produce a stationary series with arbitrary marginal distribution. The ability to specify a marginal distribution is one benefit of the renewal model. Our multivariate renewal count model will be able to produce a negatively correlated series with long or short memory with ease.

In previous work, renewal chains have been superimposed to generate count time series in univariate settings [11]. Stationary as well as periodic point process [14] have been used to derive count models with desirable characteristics i.e., stationarity, negative lag one correlations, seasonal properties, and both long and short memory. Count time series play an important role in the modeling of small-integer valued natural phenomenon such as rare disease occurrences, animal sightings, natural deaths, etc. Also, many meteorological phenomena such as hurricanes, tornadoes, and severe snowstorms are small integers. Extending our previous work to a multivariate setting affords greater flexibility in modeling and prediction. For example, suppose one is interested in a count time series of severe snowstorms. The ability to easily generate multivariate count time series with negative autocorrelations allows snowstorm occurrences to be grouped by type, say lake-effect snowstorms and overrunning snow storms. Lake effect snow occurs when a mass of sufficiently cold air moves over a body of warmer water, creating an unstable temperature profile in the atmosphere. As a result, clouds build over the lake and eventually develop into snow showers and squalls as they move downwind. Overrunning snowstorms occur when moist, warmer air is directed up and over a mass of colder air at the surface of the earth. The warm air cools as it rises, and its moisture condenses into precipitation-producing clouds. Disjoint formation conditions make these two types of snowstorm counts negatively correlated. If a body of water is still warm, we would expect lake effect snow storms - vice-versa after the lake has frozen over. Our multivariate count time series models permit negative correlations.

The rest of the chapter proceeds as follows. In section 3.2, we introduce a motivating data set of hurricane counts. In section 3.3, we propose the renewal-based multivariate count time series model, including notation and relevant stochastic processes background. Section 3.4 handles issues of estimation and statistical inference in fitting the renewal count model.

3.2 Hurricane Counts Data

Figure 3.1 shows the number of major hurricanes (Saffir-Simpson Category 3 and above) recorded in the North Atlantic and North Pacific Basins since 1970. We select 1970 as a starting year because satellite reconnaissance was in full operation then, making it unlikely that a storm of such severity formed over open waters and went undocumented (this issue arises in early Atlantic Basin records). A Saffir-Simpson Category 3 or higher storm has wind speeds of 111 mph or more at some time during the storms life. Marginally, the two component series pass most Poissonian diagnostic tests (there is a very slight amount of overdispersion). What is perhaps unexpected is a negative sample correlation between the components: -0.43539. Active North Atlantic seasons are typically accompanied by inactive North Pacific seasons and vice versa.

Negatively correlated count series models are difficult to devise. Existing generalized linear models and count time series methods cannot handle negatively correlated data, long-memory autocovariance aspects, or periodic features. Below, we show how each of these features can be made; one can even have any subset of the features in tandem.

3.3 Renewal-based Model

3.3.1 Notation

The primary building block of our count time series model will be a stationary multivariate renewal process. Let $\mathbf{L} = (L^{(1)}, L^{(2)})'$ be a bivariate random variable taking values in $\{1, 2, 3, \ldots\} \times \{1, 2, 3, \ldots\}$. We assume knowledge of a joint probability mass

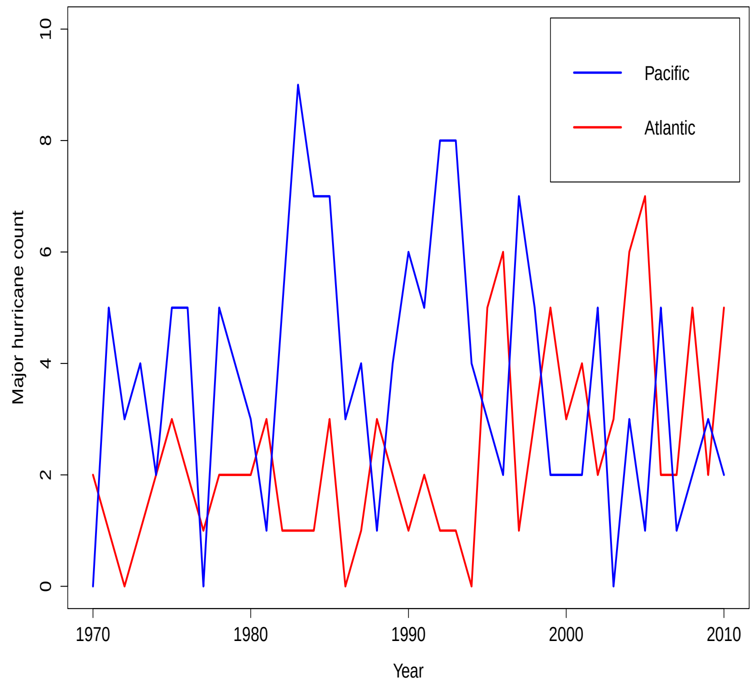

Figure 3.1: Annual number of Saffir-Simpson category 3 and stronger hurricanes in the North Pacific and North Atlantic Basins.

function $f_{n,m} = P(L^{(1)} = n \cap L^{(2)} = m)$. Now let $\mathbf{L_1} = (L_1^{(1)}, L_1^{(2)})'$, $\mathbf{L_2} = (L_2^{(1)}, L_2^{(2)})', \ldots$ be IID with distribution equal to $\mathbf{L}$. The former will henceforth be referred to as the lifetimes of the two-dimensional renewal process. A renewal is said to have taken place at time $\mathbf{t} = (t_1, t_2)$, $t_i \in \{0, 1, 2, \ldots\}$ if and only if $\mathbf{S}_n = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \mathbf{t}$ for some $n \in \{0, 1, 2, \ldots\}$. Thus, $\{\mathbf{S}_n\}_{n=0}^{\infty}$ forms the points of a bivariate renewal process. An important observation is that this also defines two separate univariate renewal process times $\{S_n^{(i)}\}_{n=0}^{\infty}$, where $S_n^{(i)} = L_1^{(i)} + L_2^{(i)} + \cdots + L_n^{(i)}$, $i = 1, 2$. Univariate renewal processes have been extensively studied ([41], [13], [25], [39], [40]). Define renewal probabilities $w_{n,m} = P(\mathbf{S}_\ell = (n, m)')$ for some $\ell \in \{0, 1, 2, \ldots\}$. Assuming $\mathbf{S}_0 = (0, 0)'$ and conditioning on the value of $\mathbf{L}_1$ gives the computational formula

$$w_{n,m} = f_{n,m} + \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} w_{i,j} f_{n-i,m-j}. \tag{3.3}$$

Throughout, we define the multivariate renewal process $\{\mathbf{R}_n\}_{n=0}^{\infty} = (R_n^{(1)}, R_n^{(2)})'$, where $R_t^{(i)} = 1$ if and only if a renewal occurs in the ith component at time t of the bivariate renewal process; $R_t^{(i)} = 0$ otherwise. Let $u_{n,m}^{(1,2)} = P(R_n^{(1)} = 1 \cap R_m^{(2)} = 1)$. Note that $\mathbf{S}_\ell = (n, m)'$ for some ℓ implies that $R_n^{(1)} = 1$ and $R_m^{(2)} = 1$, but the converse is not necessarily true. The renewal probabilities in one dimension are $u_n^{(i)} = P(R_n^{(i)} = 1)$ for $i = 1, 2$. The $u_n^{(i)}$ for $i = 1$ or 2 can be calculated with the recursive relationship

$$u_n^{(i)} = \sum_{k=0}^{n-1} u_k^{(i)} P(L^{(i)} = n - k), \tag{3.4}$$

where the convention $u_0^{(i)} = 1$ is used.

3.3.2 A Bivariate Count Time-series Model

Assume that the bivariate renewal process is stationary in the sense that $E[\mathbf{R}_n]$ is independent of n and $\text{Cov}(\mathbf{R}_n, \mathbf{R}_{n+h})$ only depends on h (how to do this with an appropriate initial lifetime will not be delved into here). Let $\{\mathbf{R}_{1,t}\}_{t=0}^{\infty}, \{\mathbf{R}_{2,t}\}_{t=0}^{\infty}, \{\mathbf{R}_{3,t}\}_{t=0}^{\infty}, \ldots$ be IID copies of the stationary bivariate renewal process. Then a bivariate count time series can be defined as

$$\begin{pmatrix} X_n^{(1)} \\ X_n^{(2)} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N_t^{(1)}} R_{i,n}^{(1)} \\ \sum_{i=1}^{N_t^{(2)}} R_{i,n}^{(2)} \end{pmatrix}, \tag{3.5}$$

where $\{\mathbf{N}_t\} = \{(N_t^{(1)}, N_t^{(2)})'\}$ is stationary multivariate count sequence with mean $\lambda = (\lambda_1, \lambda_2)'$ and autocorrelation function $\Lambda(\cdot)$.

For example, if $\mathbf{N}_t$ has a Poisson distribution, then each component $X_n^{(i)}$ is marginally Poisson with mean $\lambda_i E[L^{(i)}]$. Moreover, since the summands depend on the components in $\{\mathbf{N}_t\}$, non-zero cross-correlation is obtained. In fact, the lag h cross-correlation is

$$\text{Cov}(X_t^{(1)}, X_{t+h}^{(2)}) = \frac{\Lambda_{12}(h)}{\mu^{(1)}\mu^{(2)}} + E[\min(N_t^{(1)}, N_{t+h}^{(2)})] \left(\Delta_0 - \frac{1}{\mu^{(1)}\mu^{(2)}} \right), \tag{3.6}$$

where $\Delta_h = \lim_{n\to\infty} u_{n,n+h}^{(1,2)}$ and $\Lambda_{ij}(\cdot)$ denotes the (i,j)th element of $\Gamma(\cdot)$. It should be noted that the goal is to generate a stationary multivariate count sequence. With this in mind, the multivariate renewal sequence $\{\mathbf{R}_n\}_{n=0}^{\infty}$ is needed to be stationary. For any choice of joint lifetime distribution with $\mathrm{Cov}(L^{(1)}, L^{(2)}) \neq \pm 1$, we conjecture that

$$\Delta_h = P(R_n^{(1)} \cap R_{n+h}^{(2)}) \; \to \; \frac{1}{(E[L^{(1)}]E[L^{(2)}])} \qquad \text{as } n \to \infty. \tag{3.7}$$

Equations (3.6) and (3.7), if true, show that no flexibility in covariance is gained by assuming dependence between $L^{(1)}$ and $L^{(2)}$; henceforth, we assume independence between component lifetimes.

3.3.3 Renewal Count Model Covariance Structure

The covariance $\mathrm{Cov}(\mathbf{X}_t, \mathbf{X}_{t+h})$ depends on $\{\mathbf{N}_t\}$ via (3.6). Since our goal is to have a flexible correlation structure, choosing $\{\mathbf{N}_t\}$ with a flexible correlation structure is paramount. Henceforth, we take $\mathbf{N}_t = (N_t, N_{t-1})'$, where $\{N_t\}_{t=1}^{\infty}$ is the univariate renewal count process given by [11] with autocorrelation function $\gamma_N(\cdot)$:

$$N_t = \sum_{i=0}^{M_t} Q_{i,t}, \tag{3.8}$$

where $\{M_t\}$ is an IID sequence of Poisson random variables with mean λ_M and $\{Q_{i,t}\}_{t=1}^{\infty}$ are identically distributed stationary renewal sequences generated by the renewal lifetime $L^{(Q)}$. Independence between distinct copies of the renewal process are assumed. Thus, $(\lambda_1, \lambda_2)' = (\lambda_N, \lambda_N)'$, where $\lambda_N = E[N_t] = \lambda_M / E[L^{(Q)}]$ and

$$\gamma_N(h) = \frac{E[\min(M_t, M_{t+h})]}{E[L^{(Q)}]} \left(u_h^{(Q)} - \frac{1}{E[L^{(Q)}]} \right), \tag{3.9}$$

where $u_t^{(Q)} = P(Q_{1,t} = 1)$ and is generated with a recursive relationship analogous to (3.4). This choice allows both positive and negative covariances at all lags and gives the ability to produce long or short memory series. Now assuming $E[L^{(1)}] = \mu_1$, $E[L^{(2)}] = \mu_2$,

$$\gamma_{\mathbf{X}}(h) = Cov(\mathbf{X}_t, \mathbf{X}_{t+h}) = \begin{bmatrix} C(h)\mu_1^{-1}(u_h^{(1)} - \mu_1^{-1}) & \gamma_N(h+1)\mu_1^{-1}\mu_2^{-1} \\ \gamma_N(h-1)\mu_1^{-1}\mu_2^{-1} & C(h)\mu_2^{-1}(u_h^{(2)} - \mu_2^{-1}). \end{bmatrix} \tag{3.10}$$

Here, $C(h) = E[\min(N_t, N_{t+h})]$. One can derive a closed-form expression for $C(h)$. This is discussed later in Section 3.4. These choices give

$$E\left[\begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix}\right] = \begin{pmatrix} \lambda_N/\mu_1 \\ \lambda_N/\mu_2 \end{pmatrix}. \tag{3.11}$$

By selecting a lifetime $L^{(i)}$ such that $u_1^{(i)} = P(L^{(i)} = 1) < 1/\mu_i$ for $i = 1, 2$, the resulting marginal component series will have negative lag one autocorrelation. In fact, let $\epsilon > 0$ be a small probability and consider the first component series lifetime to take values 1, 2, or 3 with probabilities $P(L^{(1)} = 1) = P(L^{(1)} = 3) = \epsilon$ and $P(L^{(1)} = 2) = 1 - 2\epsilon$. Then $E[L^{(1)}] = 2$, $u_1^{(1)} = \epsilon$, and from (3.10),

$$Corr(X_t^{(1)}, X_{t+1}^{(1)}) = 2\epsilon - 1 \overset{\epsilon \downarrow 0}{\longrightarrow} -1$$

(arbitrarily close to -1). This is but one choice for the lifetime. Other choices yield different correlation structures. Picking $L^{(i)}$ such that $\text{Var}(L^{(i)}) = \infty$ yields a long-memory series, following the work of [11]. To see the effectiveness of this modeling technique, consider the first component of a sample path of (3.5) with $\{N_t^{(1)}\}$ IID Poisson($\lambda = 10$) and renewal lifetimes Poisson(α)+1, where $\alpha = 5$. The notation +1 is used to emphasize that $P(L = k) = e^{\alpha}\alpha^{(k-1)}/(k-1)!$ for $k = 1, 2, \ldots$ to avoid a zero lifetime. The sample path is shown in Figure 3.2 and its sample ACF and PACF are shown in Figure 3.3. The inference is that we have produced a sequence with negative lag one autocorrelation.

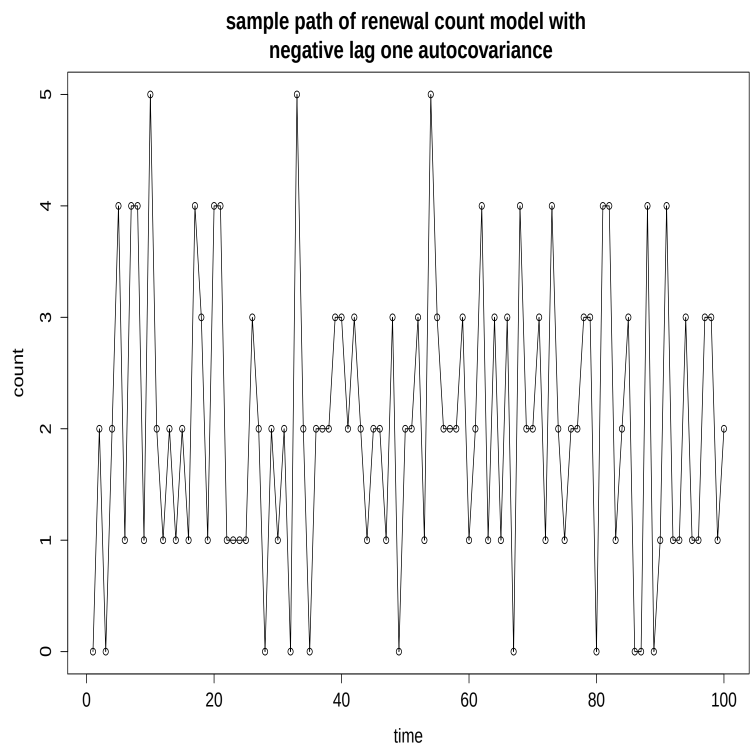

Figure 3.2: Univariate sample path of a Poisson count series

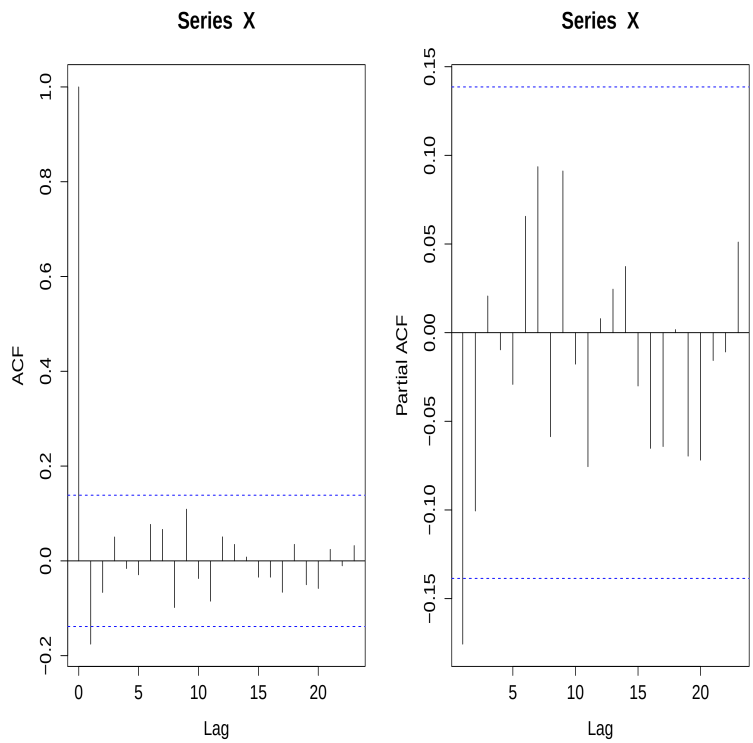

Figure 3.3: ACF and PACF of Figure 3.2 sample path

3.4 Estimation and Inference

Fitting the multivariate count model in (3.5) requires selecting a lifetime distribution for both components of $\mathbf{L}$. It is speculated that major hurricane counts are influenced by slowly varying natural phenomena. Thus, choosing a lifetime that can produce long-memory in either Atlantic or Pacific marginal counts seems fruitful. Recall that a series $\{X_t\}$ has long-memory if $\sum_{h=0}^{\infty} |\text{Cov}(X_t, X_{t+h})| = \infty$. Hence, for each component series, a discrete Pareto lifetime defined as

$$P(L^{(i)} = k) = \frac{A(\alpha^{(i)})}{k^{\alpha^{(i)}}} \qquad i = 1, 2 \tag{3.12}$$

where $\alpha^{(i)} > 2$ and $A(\alpha^{(i)})$ is a scaling constant that ensures $\sum_{k=1}^{\infty} P(L^{(i)} = k) = 1$ will be used. Note that $2 < \alpha^{(i)} < 3$ gives a long-memory covariance structure as $\text{Var}(L^{(i)}) = \infty$ but $E[L^{(i)}] < \infty$.

With lifetimes selected for each marginal component series, it only remains to specify a distribution for $L^{(Q)}$. For this, we use the zero-modified Poisson distribution

$$P(L^{(Q)} = k) = \frac{e^{-(\lambda_Q - 1)}(\lambda_Q - 1)^{(k-1)}}{(k-1)!} \qquad k = 1, 2, \ldots \tag{3.13}$$

where $\lambda_Q > 1$. The only parameter governing this lifetime is $E[L_Q] = \lambda^{(Q)}$.

We fit this model by minimizing a sum of squared prediction residuals. A reasonable objective function for selecting $\alpha^{(1)}, \alpha^{(2)}, \lambda_M$, and λ_Q simply minimizes

$$\sum_{t=1}^{n} (\mathbf{X}_t - \hat{\mathbf{X}}_t)' V_t^{-1} (\mathbf{X}_t - \hat{\mathbf{X}}_t), \tag{3.14}$$

where V_t is the prediction error covariance matrix

$$V_t = E\left[(\mathbf{X}_t - \hat{\mathbf{X}}_t)(\mathbf{X}_t - \hat{\mathbf{X}}_t)'\right]. \tag{3.15}$$

$\hat{\mathbf{X}}_t$ is calculated using best linear prediction (BLP) techniques; a multivariate version of the

Innovations Algorithm ([7], Proposition 11.4.2) is employed for the BLP. Elaborating, for a given set of parameters $\alpha^{(1)}, \alpha^{(2)}, \lambda_M$ and λ_Q (3.10) is used to explicitly compute $\gamma_{\mathbf{X}}(\cdot)$, which requires the computation of $C(h) = E[\min(N_t, N_{t+h})]$. Calculating this expected value requires the realization

$$N_{t+h}|N_t \overset{\mathcal{D}}{=} B_1 + B_2 + B_3, \tag{3.16}$$

where $B_1 \sim \text{Bin}(N_t, u_h^{(Q)}), B_2 \sim \text{Bin}(M_t - N_t, \frac{1-u_h^{(Q)}}{\lambda_Q})$, and $B_3 \sim \text{Bin}(M_{t+h} - M_t, \lambda_Q^{-1})$. In the case where $M_t = M_{t+h}$, the convention that $B_3 = 0$ with probability 1 is used. If $M_t > M_{t+h}$, a hyper-geometric conditioning argument is required to attain the number of renewals that have occurred in the the first M_{t+h} trials at time t. These identities and a sleepless afternoon allow us to evaluate $C(h)$, and hence, the covariance structure of the series.

A numerical minimization routine is utilized to find an optimal set of parameters $\alpha^{(1)}, \alpha^{(2)}, \lambda_M$ and λ_Q. While no rigorous parameter optimization is claimed, a preliminary gradient step-and-search algorithm indicates parameter estimates of $\alpha^{(1)} = 2.074712$, $\alpha^{(2)} = 2.3928$, $\lambda_M = 11.0221$, and $\lambda_Q = 1.4649$. Even if these values end up being perturbed slightly, they offer good insight. For example, the lifetimes for $L^{(1)}$ and $L^{(2)}$ are discrete Pareto, given by (3.12). Hence, these $\alpha^{(i)}$ values give $\mu_1 = E[L^{(1)}] = 2.8568$ and $\mu_2 = E[L^{(2)}] = 1.97380$. They yield, $\lambda_N = \lambda_M/\lambda_Q = 7.5241$. Marginal component means, via (3.5), are

$$E\left[\begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix}\right] = \begin{pmatrix} \lambda_N/E[L^{(1)}] \\ \lambda_N/E[L^{(2)}] \end{pmatrix} = \begin{pmatrix} 2.63374 \\ 3.81199 \end{pmatrix}. \tag{3.17}$$

For comparison, the sample means of the marginal Atlantic and Pacific ocean basin major hurricane counts are 2.4146 and 3.7317 respectively. Moreover, the fact $2 < \alpha^{(i)} < 3$ for both $i = 1, 2$ is an indicator that a long-memory count time series model is appropriate for this data. What we have seemingly found is a count series with long-memory marginal covariances and negative component correlations!

Recall, in section 3.2 the sample correlation between series was -0.43539. Taking the above set of parameter estimates as law, the lag zero cross-correlation structure of our model, via (3.10)

$$\mathrm{Corr}(X_t^{(1)}, X_t^{(2)}) = \frac{C_M(1)}{\lambda_Q \mu_1 \mu_2}(u_1^{(Q)} - \lambda_Q^{-1}), \tag{3.18}$$

where $C_M(h) = E[\min(M_t, M_{t+h})]$ can be calculated in terms of Bessel functions.

$$C_M(1) = \lambda_M \left(1 - e^{-2\lambda_M}(I_0(2\lambda_M) + I_1(2\lambda_M))\right), \tag{3.19}$$

where I_0 and I_1 denote Bessel functions of the first and second orders, respectively. For $\lambda_M = 11.0221$, equation (3.19) gives $C_M(1) = 9.1597$. Lastly, for the given $\lambda_Q = 1.4649$, $u_1^{(Q)}$ is calculated via (3.13) as

$$u_1^{(Q)} = e^{-(\lambda_Q - 1)} = e^{-4.649} \approx 0.62819. \tag{3.20}$$

These parameters produce a renewal count model with negative lag zero autocorrelation of -0.0604. We are addressing whether this is significantly negative in a statistical sense.

Chapter 4

Conclusions and Discussion

4.1 Answering the Research Questions

In the first paragraph of section 1.4, a challenge was issued to find a simple bivariate vector $(X, Y)'$, with X and Y each marginally Poisson distributed with the same mean λ, and where $\text{Cov}(X, Y) < 0$. Our calculations show how to generate random pair $(X, Y) = (N_t, N_{t-1})$ via (3.8), each having a Poisson marginal distribution with the same mean $\lambda = \lambda_N$, but with negative correlation out to $-E[\min(M_t, M_{t+h})]/\lambda$. This turns out to be very close to the most negative correlation possible. Figure 4.1 shows that we come close to the theoretical minimum correlation for differing λ values. The theoretical minimum is derived/listed in [44] as

$$\underline{\text{Corr}}(X, Y) = \text{Corr}(F_X^{-1}(U), F_Y^{-1}(1 - U)) \tag{4.1}$$

where U is a uniform zero one random variable and the inverse CDF, $F^{-1}(p)$ is interpreted as the smallest integer x such that $P(X \leq x) \geq p$. The fact that F^{-1} is discontinuous (in fact, it is right continuous) accounts for the jaggedness of the theoretical minimum.

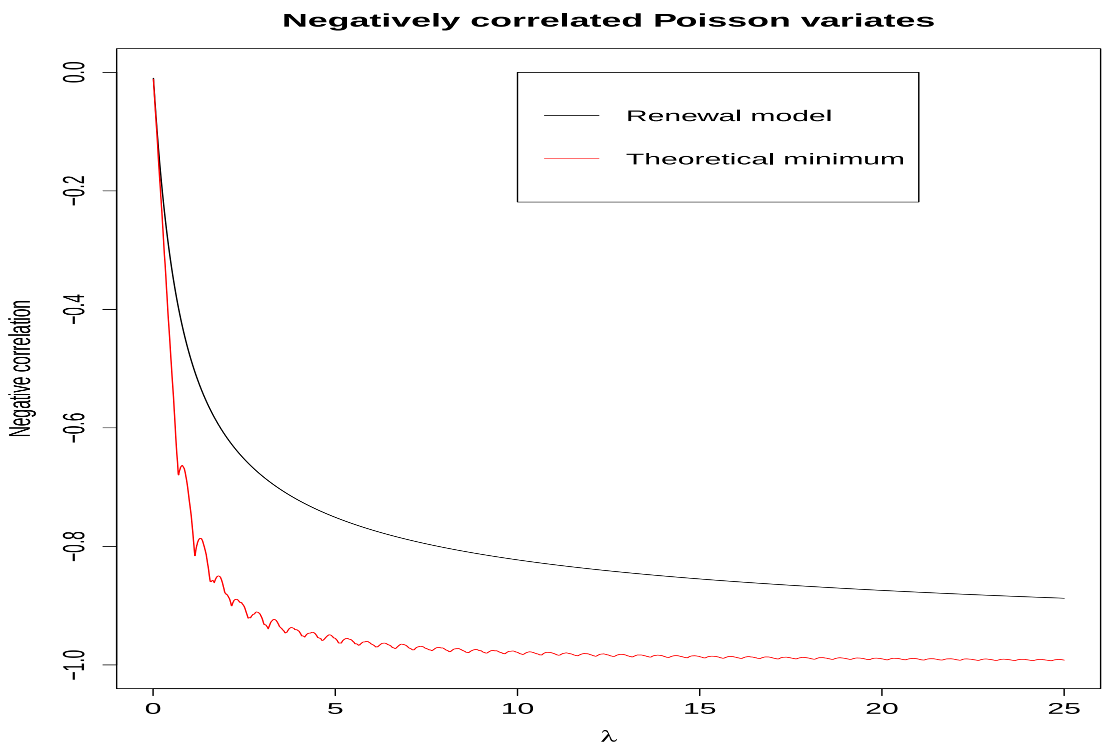

Figure 4.1: Comparison between the negative correlation achieved in renewal model and a theoretically possible minimum.

4.2 Future Research

4.2.1 Spectral Theory

The constructions of chapter 2 and 3 connect count time series with renewal processes. Consider the simple case of

$$X_t = 1_{\{R_t\}}, \tag{4.2}$$

where R_t is the event that a renewal occurs at time t and $1_{\{\cdot\}}$ is the indicator function. The covariance function of the series $\{X_t\}$ is related to the renewal probabilities $\{u_n\}_{n=0}^{\infty}$ of a non-delayed renewal process (non-delayed refers to a process where $L_0 = 0$) via

$$\gamma_N(h) = \mathrm{Cov}(N_t, N_{t+h}) = \mu^{-1}(u_h - \mu^{-1}). \tag{4.3}$$

There are immediate implication to (4.3): everything known about stationary time series can now be applied to renewal theory (and vice versa). As one example, stationary series have a well-developed spectral theory. From (4.3), a spectral theorem for the renewal probabilities $\{u_n\}_{n=0}^{\infty}$ follows with no work; specifically, the representation

$$u_h = \frac{1}{\mu} + \mu \int_{(-\pi,\pi]} e^{ih\lambda} dF(\lambda) \tag{4.4}$$

holds for some nondecreasing right-continuous function $F(\cdot)$ over $(-\pi, \pi]$. Here, F may not be a proper cumulative distribution function, but rather has total mass $\mu^{-1}(1-\mu^{-1})$. While a renewal spectral theorem is known from [12] via other methods, it follows with no work here. What can be learned via this link could be vast. For example, the spectral theory of stationary time series is well developed. Established bounds for eigenvalues of covariance matrices, Whittle-based spectral likelihoods, and transfer function techniques can now be used in renewal settings. In the other direction, rates of decay for the renewal function ([31] and [32]) quantify the memory structure of the count series. Construction of an analogous result in the multivariate case should prove extremely useful; multivariate renewal theory is notoriously difficult and underdeveloped. Much of this future work was proposed in [29].

Bibliography

[1] Al-Osh, M. and A. A. Alzaid. First-order integer-valued autoregressive (INAR(1)) process. *Journal of Time Series Analysis*, 8:261–275, 1987.

[2] Alzaid, A. A. and M. Al-Osh. An integer-valued pth-order autoregressive structure (INAR(p)) process. *J. Appl. Probab.*, 27:314–324, 1990.

[3] Asmussen, S. and H. Thorisson. A markov chain approach to periodic queues. *Journal of Applied Probability*, 24:215–225, 1987.

[4] Avramidis, A. N., Channouf, N. and P. L'Ecuyer. Efficient correlation matching for fitting discrete multivariate distributions with arbitrary marginals and normal-copula dependence. *INFORMS J. Comput.*, 21:88–106, 2009.

[5] Billingsley, P. *Probability and Measure*. Wiley, New York, third edition, 1995.

[6] Box, G., Jenkins, G. and C. Reinsel. *Time Series Analysis: Forecasting and Control*. Holden-Day, San Francisco, 1970.

[7] Brockwell, P. and R. Davis. *Time series: Theory and methods*. Springer-Verlag, New York, 1991.

[8] Brockwell, P. and R. Davis. *Introduction to Time Series and Forecasting*. Springer Verlag, 2002.

[9] Cario, M. C. and B. L. Nelson. Modeling and generating random vectors with arbitrary marginal distributions and correlation matrix. Technical report, Department of Industrial Engineering and Management Sciences, Northwestern University, Evanston, IL, 1997.

[10] Çinlar, E. Periodicity in markov renewal theory. *Advances in Applied Probability*, 6:61–78, 1974.

[11] Cui, Y. and R. Lund. A new look at time series of counts. *Biometrika*, 96:781–792, 2009.

[12] Daley, D. J. and D. Vere-Jones. *An Introduction to the Theory of Point Processes: Volume II: General Theory and Structure*. Springer, New York, 2007.

[13] Feller, W. *An Introduction to Probability Theory and its Applications*, volume I. Wiley, New York, third edition, 1968.

[14] Fralix, B., Livsey, J. and R. Lund. Renewal sequences with periodic dynamics. *Probab. Engrg. Inform. Sci.*, 26:1–15, 2011.

[15] Gardner, W. A., Napolitano, A. and L. Paura. Cyclostationarity: Half a century of research. *Signal Processing*, 86:639–697, 2006.

[16] Grenander, U., Rosenblatt, M. and N. Blachman. Statistical analysis of stationary time series. *Physics Today*, 10:47, 1957.

[17] Harrison, J. M and A. J. Lemoine. Limit theorems for queues with periodic input. *Journal of Applied Probability*, 14:566–576, 1977.

[18] Heyman, D. P. and W. Whitt. The asymptotic behavior of queues with time-varying arrival rates. *Journal of Applied Probability*, 21:143–156, 1984.

[19] Hurd, H. L. and A. Miamee. *Periodically Correlated Random Sequences: Spectral Theory and Practice.* Wiley, New York, 2007.

[20] Jacobs, P. and P. Lewis. Discrete time series generated by mixtures. I: Correlational and runs properties. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 94–105, 1978.

[21] Jacobs, P. and P. Lewis. Discrete time series generated by mixtures II: asymptotic properties. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 222–228, 1978.

[22] Jacobs, P. and P. Lewis. Stationary discrete autoregressive-moving average time series generated by mixtures. *Journal of Time Series Analysis*, 4:19–36, 1983.

[23] Jagers, P. and O. Nerman. Branching processes in a periodically varying environment. *Annals of Probability*, 13:254–268, 1985.

[24] Kachour, M. and J. F. Yao. First-order rounded integer-valued autoregressive (RINAR(1)) process. *Journal of Time Series Analysis*, 30:417–448, 2009.

[25] Karlin, S. and H. M. Taylor. *A First Course in Stochastic Processes.* Academic Press, New York, 1975.

[26] Latour, A. The multivariate GINAR(p) process. *Advances in Applied Probability*, pages 228–248, 1997.

[27] Lemoine, A. J. On queues with periodic poisson input. *Journal of Applied Probability*, 18:889–900, 1981.

[28] Lund, R. A dam with seasonal input. *Journal of Applied Probability*, 31:526–541, 1994.

[29] Lund, R. Renewal count time series. *NSF grant proposal*, 2012.

[30] Lund, R. and I. V. Basawa. Modeling for periodically correlated time series. *Asymptotics, Nonparametrics, and Time Series*, pages 37–62, 1999.

[31] Lund, R. and R. L. Tweedie. Geometric convergence rates for stochastically ordered markov chains. *Mathematics of Operations Research*, 21:182–194, 1996.

[32] Lund, R., Meyn, S. P. and R. L. Tweedie. Computable exponential convergence rates for stochastically ordered markov processes. *The Annals of Applied Probability*, 6:218–237, 1996.

[33] McKenzie, E. Autoregressive moving-average processes with negative-binomial and geometric marginal distributions. *Advances in Applied Probability*, pages 679–705, 1986.

[34] McKenzie, E. Some ARMA models for dependent sequences of Poisson counts. *Advances in Applied Probability*, pages 822–835, 1988.

[35] McKenzie, E. Discrete variate time series. In *Stochastic Processes: Modeling and Simulation*, volume 21 of *Handbook of Statistics*, pages 573–606. North-Holland, Amsterdam, 2003.

[36] Pedeli, X. and D. Karlis. A bivariate INAR(1) process with application. *Statistical Modelling*, 11:325–349, 2011.

[37] Pedeli, X. and D. Karlis. Some properties of multivariate INAR(1) processes. *Computational Statistics & Data Analysis*, 2013.

[38] Phatarfod, R. M. The bottomless dam with seasonal inputs. *Australian and New Zealand Journal of Statistics*, 22:212–217, 1980.

[39] Resnick, S. I. *Adventures in Stochastic Processes*. Birkhäuser, Boston, 1992.

[40] Ross, S. M. *Stochastic Processes*. Wiley, New York, second edition, 1996.

[41] Smith, W. L. Renewal theory and its ramifications. *Journal of the Royal Statistical Society, Series B*, 20:243–302, 1958.

[42] Steutel, F. and K. Van Harn. Discrete analogues of self-decomposability and stability. *The Annals of Probability*, pages 893–899, 1979.

[43] Thorisson, H. Periodic regeneration. *Stochastic Processes and their Applications*, 20:85–104, 1985.

[44] Yahav, I. and G. Shmueli. On generating multivariate poisson data in management science applications. *Applied Stochastic Models in Business and Industry*, 28:91–102, 2012.