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Domination and Decomposition in Multiobjective Programming

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DOMINATION AND DECOMPOSITION IN MULTIOBJECTIVE PROGRAMMING

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ABSTRACT

During the last few decades, multiobjective programming has received much attention for both its numerous theoretical advances as well as its continued success in modeling and solving real-life decision problems in business and engineering. In extension of the traditionally adopted concept of Pareto optimality, this research investigates the more general notion of domination and establishes various theoretical results that lead to new optimization methods and support decision making.

After a preparatory discussion of some preliminaries and a review of the relevant literature, several new findings are presented that characterize the nondominated set of a general vector optimization problem for which the underlying domination structure is defined in terms of different cones. Using concepts from linear algebra and convex analysis, a well known result relating nondominated points for polyhedral cones with Pareto solutions is generalized to nonpolyhedral cones that are induced by positively homogeneous functions, and to translated polyhedral cones that are used to describe a notion of approximate nondominance. Pareto-oriented scalarization methods are modified and several new solution approaches are proposed for these two classes of cones. In addition, necessary and sufficient conditions for nondominance with respect to a variable domination cone are developed, and some more specific results for the case of Bishop-Phelps cones are derived.

Based on the above findings, a decomposition framework is proposed for the solution of multi-scenario and large-scale multiobjective programs and analyzed in terms of the efficiency relationships between the original and the decomposed subproblems. Using the concept of approximate nondominance, an interactive decision making procedure is formulated to coordinate tradeoffs between these subproblems and applied to selected problems from portfolio optimization and engineering design.

Some introductory remarks and concluding comments together with ideas and research directions for possible future work complete this dissertation.

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Finally, it is beyond doubt that any other recognition falls behind the deep gratefulness that I owe to my family, my siblings, and especially my parents. Their caring love and unfailing support are my steady source of strength, and to make them proud and to know they are happy is ample motivation for always moving on.

DEDICATION

> *Das einzig Wichtige im Leben sind die Spuren von Liebe, die wir hinterlassen, wenn wir ungefragt weggehen und Abschied nehmen müssen.*
>
> *Albert Schweitzer, 1875 – 1965*

There is nothing more important than the assurance of good friends and family to love and care for one another, and the stronger these ties the harder when such relationships finally come to an end. In deepest love to my own parents, I like to honor and dedicate my dissertation to the very special parents of my mother and my father. My grandparents have seen me grow and leave home to study and learn, and I wish they knew that they will always be missed and never leave my heart.

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CHAPTER 1

INTRODUCTION

Whether a doctoral graduate chooses his first professional position, a university professor a research emphasis for her next grant proposal, or a business manager an investment strategy for last year's financial returns, making the final decision in each case requires a thorough consideration of multiple and, in general, conflicting criteria. Does the graduate strive for new challenges and a career in industry or enjoy the further commitment to research and education in academia? Does the professor plan to spend more time to prove a promising recent conjecture or continue her engagement in more profitable interdisciplinary collaboration? And does the manager gear his actions towards the short-term satisfaction of stakeholders or choose to invest the capital into more sustainable reserves? Without a doubt, for each of these three situations one can easily think of many additional alternatives and objectives, and clearly, based upon personal priorities and preferences, different people would make different decisions.

In fact, every day during both our private and professional activities, we make choices that very often not only affect our own lives but also the lives of our family, friends, customers, co-workers, and essentially anybody else with whom we interact or, as a result of our decision, do not interact. Furthermore, the presence of several objectives and potential uncertainties usually does not allow us to identify a universal best alternative so that we need to weigh or trade off between the different consequences to reach a consensus and arrive at a best compromise decision. This step can be facilitated in two ways. First, we should immediately exclude all those alternatives from further consideration that are decidedly worse than, or *dominated* by another alternative. Second, to reduce the challenges arising with a large number of objectives we may divide, or *decompose* the different objectives into smaller groups to iteratively select potential candidates that are brought into agreement in the form of a preferred decision for our initial problem. These two ideas of *domination* and *decomposition* form the heart of this dissertation.

The discipline of multiple criteria decision making deals with the analysis of decision problems like the ones described above and provides decision makers with theoretical and methodological support on how to structure a problem, describe available alternatives, identify essential criteria, deal with uncertainties, reflect on preferences, and many other issues related to the overall decision making process. Combining aspects from a great variety of different fields such as economics, sociology, or psychology, the number of possible applications is long and includes a nearly endless number of problems from business and management, financial, industrial and economic planning, location, transportation, scheduling, and all areas of engineering. Because a decision is usually associated with some qualitative action, it is not surprising at all that mathematics and especially operations research play a substantial role as well in deriving underlying mathematical models that formulate both the decisions and objectives in a quantitative manner to enable a subsequent numerical comparison between the different resulting consequences.

In this text, we study the class of mathematical models for which each of a finite number of objectives is described in terms of a real-valued mathematical function, thereby giving rise to a multiobjective programming problem. We refer to the underlying domain as the *decision set*, and based on its properties or characteristics of the objective functions we usually distinguish between discrete and continuous, or linear and nonlinear programs, respectively. For the most part of our discussion, we consider the continuous nonlinear case so that both the decision set and its image under the multiple objective functions are continuous sets which usually consist of an infinite number of points. We call this image the set of *outcomes*, which consequently describe the consequences of each decision, and under the assumption of objective minimization we are interested in identifying those decisions which are mapped to the minimal elements in the outcome set.

For mathematical programs with a single objective, this notion of minimality is well defined based on the order of real numbers, and in principle we can use any applicable optimization technique from linear, nonlinear, or integer programming to find solutions for the corresponding optimization problems. For multiple objectives, however, each outcome is a vector for which the individual components correspond to the several objective functions so that we need to give a more precise statement of what we mean with vector minimality. Based on the assumption of an underlying partial order that can be described by certain classes of cones, the concept that is discussed in this text is the concept of domination which, similar to its informal introduction above, is used to characterize the preference relationships between the different alternatives and outcomes.

In general, however, the partial order, or equivalently, the implied preference or domination structure does not provide the means to completely capture all preferences and identify a unique outcome that dominates every other outcome, but results in a set of *efficient* decisions whose outcomes are merely *nondominated*, or not dominated by any other outcome. In extension of the existing theory of multiobjective or vector optimization, which deals with the characterization of these efficient and nondominated sets and the development of applicable methods for their generation or approximation, in this dissertation we study various of these previous results and methods and investigate possible generalizations for three new classes of domination cones. Furthermore, we analyze the effects of objective decompositions and use our results to formulate several interactive decision making procedures to eventually identify a *preferred* decision as the unique solution for a multiobjective decision or programming problem. These procedures are illustrated on selected applications from financial optimization and engineering design.

For further guidance, we now give a brief outline of the general organization of this dissertation. In preparation for the original discussion in this text, we begin in Chapter 2 with a review of some relevant mathematical concepts and literature and conclude with a concise statement of our specific research objectives. In particular, in Section 2.1 we introduce some elementary notation and properties of functions, define the fundamental notions of cones and binary relations, and highlight the connection between – and relevance of – the former and latter in terms of partial orders.

Of central importance in this chapter is Section 2.2 in which we define a domination structure and the nondominated set in order to subsequently formulate the multiobjective program with the corresponding concept of efficiency. We also highlight the two concepts of Pareto and epsilon-efficiency, which are prominently discussed in later chapters, and present several typical solution methods. For Section 2.3, we change our style and more informally address some of the previous aspects in the more practical context of multiple criteria decision making with particular focus on the existing literature that describes preferences based on domination structures and discusses interactive decision making or decomposition. Based on that discussion, Section 2.4 contains the research statement of this dissertation.

In Chapter 3, we study the nondominated set of a multiobjective program for three different domination structures that generalize the previous notions of domination cones, extend several existing results from the literature and derive some new characterizations of – and methods for – the specific domination concepts defined. In particular, in Section 3.1 we show how every cone can be induced by a positively homogeneous function, use this new representation to extend some known results from the polyhedral to the nonpolyhedral case, and modify several solution methods from Pareto to more general domination cones. In Section 3.2, we formulate the new preference assumption of ideal-symmetry for which the domination structure cannot anymore be described by only one cone but is defined by a collection of domination cones. We illustrate some benefits of the resulting variable-cone model, address the problem of finding the corresponding nondominated set, and derive some optimality conditions for its characterization. Highlighting the concept of epsilon-nondominance, we define translated cones for a possible characterization of approximate nondominance in Section 3.3 and derive some specific representation results for the case of translated polyhedral cones. To provide the means of computing epsilon-efficient decisions, we extend several optimality conditions from traditional to translated domination cones and develop some new problem formulations that also allow the volitional generation of a merely approximate solution.

Several examples are given throughout the discussion, and further research that is stimulated by these results is proposed in Section 3.4.

For multiobjective programs and the concept of Pareto efficiency, in Chapter 4 we examine the effects of objective decompositions and formulate three interactive coordination methods to facilitate the selection of a preferred decision in the context of multiple criteria decision making. The efficiency relationships between the original and the decomposed subproblems are analyzed in Section 4.1 and provide the theoretical foundation for the procedures that we explain in detail in Section 4.2. Being aware that for practical or discrete problems actual optimization is not always possible, two optimization-based procedures that make use of nonhierarchical and hierarchical decision making schemes are complemented by an additional approach that can also be applied to a mere selection problem. In Section 4.3, we select one of the former procedures for an initial demonstration on a mathematical programming example before summarizing our results and addressing remaining research directions in Section 4.4.

Choosing four applications from financial and engineering optimization, we highlight the practical use of each of the three procedures in Chapter 5. The optimization-based nonhierarchical coordination method is illustrated in Section 5.1 on an investment selection problem from portfolio optimization for which the final investment strategy needs to compromise between different subproblems that are defined based on different stock market scenarios. For the hierarchical procedure, in Section 5.2 we describe an application of truss topology design in structural optimization and, similar to the previous case, show how we can identify a truss design that performs well for multiple loading conditions. In Section 5.3, we present a modification of this approach to solve a discrete selection problem that requires selecting a single layout from among a finite set of possible layout configurations for a medium-sized truck vehicle. Finally, using another discrete data set provided courtesy of the Ford Motor Company, in Section 5.4 we also comment on several implications of the optimization-free procedure.

Some concluding remarks and our final thoughts are offered in Chapter 6.

CHAPTER 2

LITERATURE REVIEW AND RESEARCH STATEMENT

2.1 Basic Concepts and Notation

We first introduce some basic notation, clarify our general conventions, and provide a number of concepts that are used extensively throughout the complete following text. Starting in Section 2.1.1 with the elementary definition of a function and the associated notions of injectivity, additivity, linearity, and convexity, the definitions and propositions regarding convex sets and cones in Sections 2.1.2 and binary relations and partial orders in Section 2.1.3 are taken in large part from the fundamental works by Rockafellar (1970, 1997) on general convex analysis and Sawaragi et al. (1985) and Yu (1985) on some more specific results underlying the theory of multiobjective and vector optimization.

2.1.1 Real and Vector-Valued Functions

We denote the set of real numbers by $\mathbb{R}$ and, for any positive integers l, m and n, let

$$\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n)^T : x_i \in \mathbb{R} \text{ for all } i = 1, 2, \ldots, n\} \tag{2.1a}$$

be the set of n-dimensional real vectors and

$$\mathbb{R}^{l \times m} = \{A = (a^1, a^2, \ldots, a^l)^T : a^i \in \mathbb{R}^m \text{ for all } i = 1, 2, \ldots, l\} \tag{2.1b}$$

be the set of real $l \times m$ matrices.

Definition 2.1.1 (Function, image and domain)**.** Let $X \subseteq \mathbb{R}^n$ be a nonempty set. A *function*

$$f : X \to \mathbb{R}^m \tag{2.2}$$

is a mapping that assigns to each element x in the *domain* X a unique element $y \in \mathbb{R}^m$, denoted by $y = f(x)$. The *image* of X under f is denoted by

$$Y := f(X) := \{y \in \mathbb{R}^m : y = f(x) \text{ for some } x \in X\} \tag{2.3}$$

If $m = 1$, then f is a *real-valued* function, otherwise it is *vector valued* and can also be written in terms of its *component functions* $f = (f_1, \dots, f_m)$, where each $f_i : X \to \mathbb{R}$ is a real-valued function. If $f = (f_1, \dots, f_m)$ is vector valued, then we write

$$f(x^1) \leqq f(x^2) \text{ if and only if } f_i(x^1) \leq f_i(x^2) \text{ for all } i = 1, \dots, m \tag{2.4a}$$

$$f(x^1) \leq f(x^2) \text{ if and only if } f_i(x^1) \leqq f_i(x^2) \text{ and } f(x^1) \neq f(x^2) \tag{2.4b}$$

$$f(x^1) < f(x^2) \text{ if and only if } f_i(x^1) < f_i(x^2) \text{ for all } i = 1, \dots, m \tag{2.4c}$$

and use $\geqq$, $\geq$ and $>$ accordingly.

In general, two different elements $x^1 \neq x^2$ in the domain X of f can be mapped to the same element $y = f(x^1) = f(x^2) \in \mathbb{R}^m$.

Definition 2.1.2 (Injective function)**.** A function $f : X \to \mathbb{R}^m$ is said to be *injective* if

$$f(x^1) = f(x^2) \text{ if and only if } x^1 = x^2 \text{ for all } x^1, x^2 \in X \tag{2.5}$$

Remark 2.1.3 (Sufficient condition for injectivity)**.** For a vector-valued function $f = (f_1, \dots, f_m)$, injectivity of some f_i is sufficient but, in general, not necessary for injectivity of f.

Definition 2.1.4 (Subadditive, superadditive, and additive function)**.** A function $f : X \to \mathbb{R}^m$ is said to be *subadditive* if

$$f(x^1 + x^2) \leqq f(x^1) + f(x^2) \text{ for all } x^1, x^2 \in X \tag{2.6}$$

Furthermore, f is said to be *superadditive* if the inequality holds in reverse, and *additive* if it is both subadditive and superadditive.

In particular, if f is subadditive, then $-f$ is superadditive and vice versa.

Definition 2.1.5 (Positively homogeneous, sublinear, superlinear, and linear function)**.** A function $f : X \to \mathbb{R}^m$ is said to be *positively homogeneous* if

$$f(\lambda x) = \lambda f(x) \text{ for all } x \in X \text{ and } \lambda > 0 \tag{2.7}$$

In this case, f is said to be *sublinear* if it is also subadditive, *superlinear* if it is superadditive, and *linear* if it is additive.

As before, if f is sublinear, then $-f$ is superlinear and vice versa. In particular, a positively homogeneous function is linear if and only if it is both sublinear and superlinear.

Definition 2.1.6 (Convex and concave function)**.** A function $f : X \to \mathbb{R}^m$ is said to be *convex* if

$$f\left(\lambda x^1 + (1-\lambda)x^2\right) \leqq \lambda f(x^1) + (1-\lambda)f(x^2) \text{ for all } x^1, x^2 \in X \text{ and } 0 \leq \lambda \leq 1 \tag{2.8}$$

Furthermore, f is said to be *concave* if the inequality holds in reverse.

Again, if f is convex, then $-f$ is concave and vice versa.

Proposition 2.1.7 (Rockafellar and Wets (1998))**.** *A positively homogeneous function $f : X \to \mathbb{R}^m$ is convex if and only if it is sublinear, and concave if and only if it is superlinear. In the former case*

$$f\left(\sum_{i=1}^{k} \lambda_i x^i\right) \leqq \sum_{i=1}^{k} \lambda_i f(x^i) \textit{ for all } x^i \in X \textit{ and } \lambda_i > 0,\ i = 1, \ldots, k \tag{2.9}$$

and in the latter the above holds with the inequality in reverse.

In particular, a positively homogeneous function is linear if and only if it is both convex and concave.

Throughout this whole text, we consider $\mathbb{R}^n$ and $\mathbb{R}^m$ as Euclidean vector spaces equipped with the standard Euclidean *inner product*, *norm*, and *metric*

$$\langle x^1, x^2 \rangle := \sum_{i=1}^{n} x_i^1 x_i^2,\ \|x\| := \sqrt{\langle x, x \rangle},\ \text{and } d(x^1, x^2) = \|x^1 - x^2\| \tag{2.10a}$$

$$\langle y^1, y^2 \rangle := \sum_{i=1}^{m} y_i^1 y_i^2,\ \|y\| := \sqrt{\langle y, y \rangle},\ \text{and } d(y^1, y^2) = \|y^1 - y^2\| \tag{2.10b}$$

unless specified otherwise. In particular, this provides us with a topology of open sets with the standard notions of continuity and differentiability. If the function $f = (f_1, \ldots, f_m)$ is differentiable at $x \in X$, then we write

$$\nabla f_i(x) = \left(\frac{\partial f_i(x)}{\partial x_1}, \ldots, \frac{\partial f_i(x)}{\partial x_n} \right)^T \in \mathbb{R}^n \tag{2.11a}$$

for the *gradient* of f_i for all $i = 1, \ldots, m$ and denote the *Jacobian* of f at x by

$$\nabla f(x) = (\nabla f_1(x), \ldots, \nabla f_m(x))^T \in \mathbb{R}^{m \times n} \tag{2.11b}$$

As we will make only very rare explicit use of these concepts, we do not repeat the precise definitions which can be found in any beginning textbook on advanced calculus (Fulks, 1961) or real analysis (Royden, 1963; Rudin, 1976). Furthermore, although much of the material presented in this text is similarly valid for any other real linear or topological space, we choose the conceptual simplification of Euclidean spaces to avoid any additional notational burden and, therefore, for the benefit of a more coherent presentation.

2.1.2 Elements from Convex Analysis

Let $\mathbb{R}^m$ be a Euclidean space. We denote the *interior*, the *boundary*, and the *closure* of a set $Y \subset \mathbb{R}^m$ by $\operatorname{int} Y$, $\operatorname{bd} Y$, and $\operatorname{cl} Y$, respectively. For any two sets Y and Z in $\mathbb{R}^m$, we let

$$Y + Z := \{y + z : y \in Y \text{ and } z \in Z\} \tag{2.12a}$$

be the *Minkowski sum*, and for $z \in \mathbb{R}^m$ a single element, we also write $Y + z$ instead of $Y + \{z\}$. Moreover, for $\lambda \in \mathbb{R}$ a real number, we denote

$$\lambda Y := \{\lambda y : y \in Y\} \text{ and } -Y := \{-y : y \in Y\} \tag{2.12b}$$

Definition 2.1.8 (Convex set)**.** A set $Y \subset \mathbb{R}^m$ is said to be *convex* if

$$\lambda Y + (1-\lambda)Y \subseteq Y \text{ for all } 0 \leq \lambda \leq 1 \tag{2.13}$$

Definition 2.1.9 (Cone)**.** A set $C \subset \mathbb{R}^m$ is called a *cone* if

$$\lambda C \subseteq C \text{ for all } \lambda > 0 \tag{2.14}$$

A nonempty cone is also said to be *proper*, and in this text we assume that every cone is a proper cone.

Remark 2.1.10 (Cone and origin)**.** Following the classic definition in Rockafellar (1970) but different from Rockafellar and Wets (1998) and other authors, according to Definition 2.1.9 a cone may contain the origin or not. Compare Remark 2.1.20.

Proposition 2.1.11 (Convex cone)**.** *A cone $C \subset \mathbb{R}^m$ is convex if and only if*

$$C + C \subseteq C \tag{2.15}$$

Definition 2.1.12 (Pointed cone)**.** A cone $C \subset \mathbb{R}^m$ is said to be *pointed* if

$$\sum_{i=1}^{k} c^i = 0 \text{ if and only if } c^i = 0 \text{ for all } c^i \in C,\ i = 1, \dots, k \tag{2.16}$$

Proposition 2.1.13 (Pointed convex cone)**.** *A convex cone $C \subset \mathbb{R}^m$ is pointed if and only if*

$$C \cap -C \subseteq \{0\} \tag{2.17}$$

Definition 2.1.14 (Cone convexity, closedness, boundedness, and compactness)**.** Let $C \subset \mathbb{R}^m$ be a cone. A set $Y \subset \mathbb{R}^m$ is said to be

(i) *C-convex* if $Y + C$ is convex,

(ii) *C-closed* if $Y + \operatorname{cl} C$ is closed,

(iii) C-*bounded* if $Y \subset z + \operatorname{cl} C$ for some $z \in \mathbb{R}^m$, and

(iv) C-*compact* if $(y - \operatorname{cl} C) \cap Y$ is compact for all $y \in Y$.

Proposition 2.1.15 (Cone convexity)**.** *A set $Y \subset \mathbb{R}^m$ is convex if and only if it is $\{0\}$-convex. Moreover, if Y is convex, then Y is C-convex for any convex cone $C \subset \mathbb{R}^m$.*

While the notion of C-boundedness in Definition 2.1.14 (iii) is given in Tanino and Sawaragi (1980), Sawaragi et al. (1985) provide an alternative characterization for convex sets based on the concept of a recession cone.

Definition 2.1.16 (Recession cone)**.** Let $Y \subset \mathbb{R}^m$ be a convex set. The set

$$\operatorname{rec} Y := \{z \in \mathbb{R}^m : y + \lambda z \in Y \text{ for all } y \in Y \text{ and } \lambda > 0\} \tag{2.18}$$

is called the *recession cone* of Y.

The recession cone is a convex cone and contains the origin.

Proposition 2.1.17 (Cone boundedness)**.** *A nonempty closed convex set $Y \subset \mathbb{R}^m$ is bounded if and only if $\operatorname{rec} Y = \{0\}$. Moreover, if Y is C-bounded, then*

$$\operatorname{rec} Y \cap (-\operatorname{cl} C) = \{0\} \tag{2.19}$$

The latter is used as definition for C-boundedness in Sawaragi et al. (1985) and equivalent with Definition 2.1.14 (iii) if C is pointed, closed, and convex.

Proposition 2.1.18 (Cone closedness, boundedness, and compactness)**.** *Let $C \subset \mathbb{R}^m$ be a pointed closed convex cone. A convex set $Y \subset \mathbb{R}^m$ is C-compact if and only if it is C-closed and C-bounded, or equivalently, if $\operatorname{rec} Y \cap (-C) = \{0\}$.*

To prevent any confusion, we emphasize that the next definition is correct.

Definition 2.1.19 (Dual and strict dual cone)**.** Let $C \subset \mathbb{R}^m$ be a set (!). The sets

$$C^* := \{d \in \mathbb{R}^m : \langle c, d \rangle \geq 0 \text{ for all } c \in C\} \tag{2.20a}$$

$$C_s^* := \{d \in \mathbb{R}^m : \langle c, d \rangle > 0 \text{ for all } c \in C \setminus \{0\}\} \tag{2.20b}$$

are called the *dual* and *strict dual cone* of C, respectively. If $C^* = C$, then C is said to be *self-dual*.

Remark 2.1.20 (Convexity of dual cones)**.** Dual and strict dual cones are closed convex and convex cones, respectively. In particular, a strict dual cone does not contain the origin, and $C_s^* \cup \{0\}$ is a closed convex cone. Compare Remark 2.1.10.

Proposition 2.1.21 (Dual and strict dual cone)**.** *Let C, C_1, $C_2 \subset \mathbb{R}^m$ be sets (!). If $C_1 \subseteq C_2$, then $C_2^* \subseteq C_1^*$. Moreover*

$$(\operatorname{int} C)^* = C^* = (\operatorname{cl} C)^* \text{ and } (\operatorname{int} C)_s^* = C^* \setminus \{0\} \tag{2.21}$$

Definition 2.1.22 (Hyperplane and halfspace)**.** A set $H \subset \mathbb{R}^m$ is called a *hyperplane* if there exist $a \in \mathbb{R}^m \setminus \{0\}$ and $b \in \mathbb{R}$ so that

$$H = H(a, b) := \{y \in \mathbb{R}^m : \langle a, y\rangle = b\} \tag{2.22}$$

In this case, the vector a is called a *normal vector* to H. Moreover, the sets

$$H(a, b)^+ := \{y \in \mathbb{R}^m : \langle a, y\rangle \geq b\} \tag{2.22a}$$

$$H(a, b)^- := \{y \in \mathbb{R}^m : \langle a, y\rangle \leq b\} \tag{2.22b}$$

are called the *positive* and *negative halfspaces* associated with H, respectively.

Definition 2.1.23 (Supporting hyperplane)**.** Let $Y \subset \mathbb{R}^m$ be a set. A hyperplane $H \subset \mathbb{R}^m$ is called a *supporting hyperplane* to Y if

$$Y \subseteq H^+ \text{ (or } Y \subseteq H^-\text{) and } \operatorname{cl} Y \cap H \neq \emptyset \tag{2.23}$$

Theorem 2.1.24 (Supporting hyperplane theorem)**.** *Let $Y \subset \mathbb{R}^m$ be a convex set and $z \in \operatorname{bd} Y$ be a boundary point of Y. Then there exists a supporting hyperplane $H \subset \mathbb{R}^m$ to Y with normal vector $a \in \mathbb{R}^m \setminus \{0\}$ so that*

$$\langle a, y - z\rangle \geq 0 \text{ for all } y \in Y \tag{2.24}$$

Similar to halfspaces in Definition 2.1.22, we define polyhedral sets and cones.

Definition 2.1.25 (Polyhedral set and cone)**.** A set $D \subset \mathbb{R}^m$ is called a *polyhedral set* if there exist a matrix $A \in \mathbb{R}^{l\times m} \setminus \{0\}$ and a vector $b \in \mathbb{R}^l$ so that

$$D = D(A, b) := \{y \in \mathbb{R}^m : Ay \geqq b\} \tag{2.25a}$$

If $b = 0$, then

$$C = C(A) := D(A, 0) = \{y \in \mathbb{R}^m : Ay \geqq 0\} \tag{2.25b}$$

is called a *polyhedral cone.*

Proposition 2.1.26 (Polyhedral set)**.** *A set $D \subset \mathbb{R}^m$ is a polyhedral set if and only if it is the intersection of a finite number of halfspaces*

$$D = \{y \in \mathbb{R}^m : \langle a^i, y\rangle \geq b_i,\ i = 1, 2, \ldots, l\} \tag{2.26}$$

In particular, if $A = (a^1, a^2, \ldots, a^l)^T \in \mathbb{R}^{l\times m}$ and $b = (b_1, b_2, \ldots, b_l)^T \in \mathbb{R}^l$, then $D = D(A, b)$.

Definition 2.1.27 (Linear independence and matrix rank)**.** A set of real vectors $\{a^1, a^2, \ldots, a^l\} \subset \mathbb{R}^m$ is said to be *linearly independent* if

$$\sum_{i=1}^{l} \lambda_i a^i = 0 \text{ if and only if } \lambda_i = 0 \text{ for all } i = 1, \ldots, l \tag{2.27}$$

The *rank* of a matrix $A = (a^1, a^2, \ldots, a^l)^T \in \mathbb{R}^{l\times m}$ is the maximal number of linearly independent vectors from $\{a^1, a^2, \ldots, a^l\}$ and denoted by $\operatorname{rank} A$.

Proposition 2.1.28 (Pointed convex polyhedral cone)**.** *A polyhedral set $D = D(A, b) \subset \mathbb{R}^m$ is always convex. Moreover, a polyhedral cone $C = C(A) \subset \mathbb{R}^m$ is pointed if and only if* $\operatorname{rank} A = m$.

Proposition 2.1.29 (Farkas' lemma)**.** *Let $C = C(A) \subset \mathbb{R}^m$ be a polyhedral cone with matrix $A \in \mathbb{R}^{l\times m}$. Then*

$$C^* = \{d \in \mathbb{R}^m : d = \lambda^T A^T, \lambda \in \mathbb{R}^m, \lambda \geq 0\} \tag{2.28}$$

The next result states the relationship between matrices and linear functions.

Proposition 2.1.30 (Linear function and matrix). *A function $f : \mathbb{R}^m \to \mathbb{R}^l$ is linear if and only if there exists a matrix $A \in \mathbb{R}^{l \times m}$ such that $f(x) = Ax$, and f is injective if and only if* $\operatorname{rank} A = m$.

2.1.3 Binary Relations and Partial Orders

We first define the notion of a binary relation on an arbitrary set Z and collect some of the most important properties.

Definition 2.1.31 (Binary relation). A *binary relation* $\mathcal{R}$ on a set Z is a subset of the Cartesian product $Z \times Z$. It is said to be

(i) *reflexive* if $(z, z) \in \mathcal{R}$ for all $z \in Z$,

(ii) *irreflexive* if $(z, z) \notin \mathcal{R}$ for all $z \in Z$,

(iii) *symmetric* if $(z^1, z^2) \in \mathcal{R} \Leftrightarrow (z^2, z^1) \in \mathcal{R}$ for all $z^1, z^2 \in Z$,

(iv) *asymmetric* if $(z^1, z^2) \in \mathcal{R} \Rightarrow (z^2, z^1) \notin \mathcal{R}$ for all $z^1, z^2 \in Z$,

(v) *antisymmetric* if (z^1, z^2) and $(z^2, z^1) \in \mathcal{R} \Rightarrow z^1 = z^2$ for all $z^1, z^2 \in Z$,

(vi) *transitive* if (z^1, z^2) and $(z^2, z^3) \in \mathcal{R} \Rightarrow (z^1, z^3) \in \mathcal{R}$ for all $z^1, z^2, z^3 \in Z$,

(vii) *negatively transitive* if (z^1, z^2) and $(z^2, z^3) \notin \mathcal{R} \Rightarrow (z^1, z^3) \notin \mathcal{R}$ for all $z^1, z^2, z^3 \in Z$,

(viii) *weakly connected* if $(z^1, z^2) \notin \mathcal{R}$ and $z^1 \neq z^2 \Rightarrow (z^2, z^1) \in \mathcal{R}$ for all $z^1, z^2 \in Z$, and

(ix) *(strongly) connected or complete* if $(z^1, z^2) \notin \mathcal{R} \Rightarrow (z^2, z^1) \in \mathcal{R}$ for all $z^1, z^2 \in Z$.

Different from the above definitions which are taken from Sawaragi et al. (1985), Yu (1985) calls a binary relation *irreflexive* or *asymmetric* if it is not reflexive or not symmetric, respectively, and *nontransitive* if it is not transitive.

Proposition 2.1.32 (Relationships between properties of binary relations). *Let $\mathcal{R} \subseteq Z \times Z$ be a binary relation on a set Z.*

(i) If $\mathcal{R}$ is asymmetric, then it is irreflexive.

(ii) If $\mathcal{R}$ is transitive and irreflexive, then it is asymmetric.

(iii) If $\mathcal{R}$ is negatively transitive and asymmetric, then it is transitive.

(iv) If $\mathcal{R}$ is transitive, irreflexive, and weakly connected, then it is negatively transitive.

Definition 2.1.33 (Equivalence relation and orders)**.** A binary relation $\mathcal{R} \subseteq Z \times Z$ on a set Z is called

(i) an *equivalence relation* if it is reflexive, symmetric, and transitive,

(ii) a *preorder* if it is reflexive and transitive,

(iii) a *total preorder* if it is reflexive, transitive, and connected,

(iv) a *partial order* if it is reflexive, transitive, and antisymmetric,

(v) a *strict partial order* if it is irreflexive and transitive,

(vi) a *weak order* if it is asymmetric and negatively transitive, and

(vii) a *total order* if it is asymmetric, transitive, and weakly connected.

Proposition 2.1.34 (Strict partial, weak, and total order)**.** *Let $\mathcal{R} \subseteq Z \times Z$ be a binary relation on a set Z. If $\mathcal{R}$ is a total order, then it is is a weak order, and if it is a weak order, then it is a strict partial order.*

If $\mathcal{R}$ is a binary relation and $(z^1, z^2) \in \mathcal{R}$, then we also write $z^1 \mathcal{R} z^2$. In particular, if $\mathcal{R}$ is a partial order on Z, then $(Z, \mathcal{R})$ is called a *partially ordered set*, and we write $z^1 \preceq z^2$ and $z^1 \npreceq z^2$ instead of $(z^1, z^2) \in \mathcal{R}$ and $(z^1, z^2) \notin \mathcal{R}$, respectively. Furthermore, in this case

$$z^1 \prec z^2 :\Longleftrightarrow z^1 \preceq z^2 \text{ and } z^2 \npreceq z^1 \tag{2.29a}$$

$$z^1 \sim z^2 :\Longleftrightarrow z^1 \preceq z^2 \text{ and } z^2 \preceq z^1 \tag{2.29b}$$

are a strict partial order and an equivalence relation, respectively, and we call $\prec$ the *strict partial order* associated with $\preceq$ on $(Y, \mathcal{R})$.

Remark 2.1.35 (Componentwise order)**.** The binary relations $\geqq$ and $\leqq$ in Equation 2.4 are preorders on $\mathbb{R}^m$, and $\geq$ and $\leq$ are the associated strict partial orders, respectively. Furthermore, $>$ and $<$ are also strict partial orders.

In this text, we use the concept of binary relations to introduce orders onto sets $Y \subset \mathbb{R}^m$, and for convenience we usually define these orders on the complete Euclidean space $\mathbb{R}^m$. In particular, this allows the definition of two further properties based on the operations of real vector addition and scalar multiplication.

Definition 2.1.36 (Compatibility with addition and scalar multiplication)**.** A binary relation $\mathcal{R}$ on $\mathbb{R}^m$ is said to be *compatible with*

(i) *addition* if $(y^1, y^2) \in \mathcal{R} \Rightarrow (y^1 + z, y^2 + z) \in \mathcal{R}$ for all $y^1, y^2, z \in \mathbb{R}^m$, and

(ii) *scalar multiplication* if $(y^1, y^2) \in \mathcal{R} \Rightarrow (\lambda y^1, \lambda y^2) \in \mathcal{R}$ for all $y^1, y^2 \in \mathbb{R}^m$ and $\lambda > 0$.

The next few results relate the concept of binary relations to cones.

Definition 2.1.37 (Cone relation)**.** Let $C \subset \mathbb{R}^m$ be a cone. The binary relation

$$\mathcal{R}_C := \left\{(y^1, y^2) \in \mathbb{R}^m \times \mathbb{R}^m : y^2 - y^1 \in C\right\} \tag{2.30}$$

is called the *cone relation* on $\mathbb{R}^m$ induced by C.

Proposition 2.1.38 (Cone relation)**.** *Let $C \subset \mathbb{R}^m$ be a cone and $\mathcal{R}_C$ be the induced cone relation. Then $\mathcal{R}_C$ is compatible with addition and scalar multiplication and*

(i) reflexive if and only if $0 \in C$,

(ii) transitive if and only if C is convex, and

(iii) antisymmetric if and only if C is pointed.

In particular, from Definition 2.1.33 and Proposition 2.1.38, we obtain that for a pointed convex cone C, the cone relation $\mathcal{R}_C$ is a partial or strict partial order if and only if $0 \in C$ or $0 \notin C$, respectively. Hence, a pointed convex cone is also called an *ordering cone*, and

$$y^1 \preceq_C y^2 :\Longleftrightarrow y^2 - y^1 \in C \tag{2.31a}$$

$$y^1 \prec_C y^2 :\Longleftrightarrow y^2 - y^1 \in C \setminus \{0\} \tag{2.31b}$$

are called the *partial* and *strict partial order* induced by C, respectively.

We can reverse the results in Proposition 2.1.38 under the assumption that $\mathcal{R}$ is compatible with addition and scalar multiplication.

Proposition 2.1.39 (Relation cone)**.** *Let $\mathcal{R}$ be a binary relation on $Y \subset \mathbb{R}^m$ that is compatible with scalar multiplication. Then*

$$C_{\mathcal{R}} := \left\{y^2 - y^1 \in \mathbb{R}^m : (y^1, y^2) \in \mathcal{R}\right\} \tag{2.32}$$

is a cone. In this case, if $\mathcal{R}$ is also compatible with addition, then

(i) $0 \in C_{\mathcal{R}}$ *if and only if* $\mathcal{R}$ *is reflexive,*

(ii) $C_{\mathcal{R}}$ *is convex if and only if* $\mathcal{R}$ *is transitive, and*

(iii) $C_{\mathcal{R}}$ *is pointed if and only if* $\mathcal{R}$ *is antisymmetric.*

In particular, using the componentwise orders from Equation 2.4 or Remark 2.1.35, the *nonnegative*, *nonzero*, and *positive orthant* of $\mathbb{R}^m$

$$\mathbb{R}^m_{\geqq} := C_{\leqq} = \{y \in \mathbb{R}^m : y \geqq 0\} \tag{2.33a}$$

$$\mathbb{R}^m_{\geq} := C_{\leq} = \{y \in \mathbb{R}^m : y \geq 0\} \tag{2.33b}$$

$$\mathbb{R}^m_{>} := C_{<} = \{y \in \mathbb{R}^m : y > 0\} \tag{2.33c}$$

are pointed convex cones and play an important role in the following discussion of multiobjective programming and optimization. For reasons that become apparent later, we name these cones *Pareto cones.*

2.2 Multiobjective Programming and Optimization

In this section, we use the concepts of partial orders and ordering cones so far introduced in the previous discussion to define the notions of domination and nondominance for multiobjective programming and real-vector optimization problems. For clarity of our presentation, in Section 2.2.1 we initially restrict our attention to the latter and wait until we formulate the multiobjective program in Section 2.2.2 to address the conceptual relationship between efficiency in the domain and nondominance in the image of a vector-valued objective function. Several commonly used optimization methods that are based on objective scalarization are then reviewed in Section 2.2.3 and characterized in terms of their ability to generate efficient or nondominated solutions and their associated tradeoffs. While we also mention some more specific references throughout the following text, most of the presented material can similarly be found in various standard monographs on multiobjective programming or optimization (Chankong and Haimes, 1983; Sawaragi et al., 1985; Yu, 1985; Steuer, 1986; Miettinen, 1999; Jahn, 2004; Ehrgott, 2005, among others).

2.2.1 Domination and the Nondominated Set

We first introduce the fundamental concepts of domination sets and domination structures that are adopted in this text and address their relationships to pointed convex cones as established by Yu (1974). Furthermore, we define the nondominated set, compare our notion to the original definition in the former reference, present some basic results and then point to several extensions and related concepts that can be found in the literature. As before, all given results are also contained and proven in Sawaragi et al. (1985) or Yu (1985).

Definition 2.2.1 (Domination set and domination structure)**.** Let $Y \subset \mathbb{R}^m$ be a nonempty set, $\mathcal{R} \subset Y \times Y$ be a partial order on Y, and $\prec$ be the strict partial order associated with $\mathcal{R}$. For every $y \in Y$, the set

$$D(y) := \{d \in \mathbb{R}^m : y - d \prec y\} \tag{2.34a}$$

is called the *domination set* or set of *dominated directions* at y, and the collection

$$\mathcal{D} := \{D(y) \subset \mathbb{R}^m : y \in Y\} \tag{2.34b}$$

is called the associated *domination structure* on Y.

If $y^1 \prec y^2$, then y^1 is said to *dominate* y^2 or, equivalently, y^2 is said to *be dominated* by y^1. With the same meaning, Yu (1974) also says that y^1 is *preferred* to y^2, but we postpone the notion of preference until later in this chapter.

Remark 2.2.2 (Domination set in the sense of Yu (1974))**.** In the original definition by Yu (1974), the domination set $D(y)$ is defined as

$$D(y) := \{d \in \mathbb{R}^m : y \prec y + d\} \tag{2.35}$$

which coincides with $D(y + d)$ rather than with $D(y)$ in Definition 2.2.1. However, if the strict partial order $\prec$ is compatible with addition, then $y - d \prec y \Leftrightarrow y = y + d - d \prec y + d$ and thus $D(y) = D(y + d)$ for all $d \in \mathbb{R}^m$. In particular, in this case $D(y) = D(y')$ for all $y' \in Y$, and Definition 2.2.1 is equivalent with the definition by Yu (1974) because all domination sets are identical.

If the sets of dominated directions in Definition 2.2.1 vary for different $y \in Y$, so that $D(y) \neq D(y')$ for $y \neq y'$, in general, then we call the collection $\mathcal{D} = \{D(y) : y \in Y\}$ a *variable* domination structure for Y. Otherwise, if $D(y) = D$ for all $y \in Y$, then we usually write $\mathcal{D} = D$ instead of $\mathcal{D} = \{D\}$ and say that the domination structure is *constant*. In particular, the following special case for a constant domination structure is implied by Proposition 2.1.39.

Proposition 2.2.3 (Domination cone)**.** *Let $\mathcal{R}$ be a partial order that is compatible with addition and scalar multiplication, and $\prec$ be the strict partial order associated with $\mathcal{R}$. Then*

$$D(y) = C_{\prec} := \{d \in \mathbb{R}^m : 0 \prec d\} \text{ for all } y \in Y \tag{2.36}$$

In particular, $C_{\prec}$ is a pointed convex cone that does not contain the origin.

In this case, we also say that the constant domination structure $\mathcal{D}$ is a *constant cone* and call $C_{\prec}$ the *domination cone*. In particular, for the special case of the componentwise order $\mathcal{D}$ reduces to the nonzero Pareto cone $\mathbb{R}^m_{\geq}$.

Based on Proposition 2.1.38, we can also reverse Proposition 2.2.3 and use any pointed convex cone to induce a constant domination structure that is defined by the underlying cone relation from Equation 2.31.

Proposition 2.2.4 (Domination cone)**.** *Let $C \subset \mathbb{R}^m$ be a pointed convex cone. If $D(y) \subset \mathbb{R}^m$ is defined by the strict partial order induced by C, then $D(y) = C \setminus \{0\}$ for all $y \in Y$.*

In this case, $\mathcal{D} = C \setminus \{0\}$ is again a constant cone, and if $y^1 \prec_C y^2$, then y^1 is also said to dominate y^2, or y^2 is said to be dominated by y^1 *with respect to C*.

Definition 2.2.5 (Nondominated and weakly nondominated set)**.** Let $Y \subset \mathbb{R}^m$ be a nonempty set, and $\mathcal{D} = \{D(y) : y \in Y\}$ be a domination structure on Y. The sets

$$N(Y, \mathcal{D}) := \{\hat{y} \in Y : (\hat{y} - D(\hat{y}) \setminus \{0\}) \cap Y = \emptyset\} \tag{2.37a}$$

$$N_w(Y, \mathcal{D}) := \{\hat{y} \in Y : (\hat{y} - \operatorname{int} D(\hat{y}) \setminus \{0\}) \cap Y = \emptyset\} \tag{2.37b}$$

are called the *nondominated* and *weakly nondominated set* of Y with respect to $\mathcal{D}$, respectively.

We note that $N_w(Y, \mathcal{D}) = N(Y, \mathcal{D}^\circ)$, where $\mathcal{D}^\circ := \{\operatorname{int} D(y) : y \in Y\}$. In particular, if $\mathcal{D} = C$ is a constant cone, then $N_w(Y, C) = N(Y, \operatorname{int} C)$ so that, for large parts of this text, we can restrict our discussion to the nondominated set only. Moreover, since Definition 2.2.5 explicitly excludes the zero vector from each domination set, we do not need to further distinguish whether C does or does not contain the origin. Finally, if $C = \mathbb{R}^m_{\geqq}$ is the Pareto cone, then we usually simplify notation and call

$$N(Y) := N(Y, \mathbb{R}^m_{\geq}) = N(Y, \mathbb{R}^m_{\geqq}) \tag{2.38}$$

the *Pareto set* of Y. For illustration, Figure 2.1 depicts the Pareto set and the nondominated set for a cone $C \subset \mathbb{R}^2$, corresponding to the highlighted bold curves, for the indicated set Y as subset of $\mathbb{R}^2$.

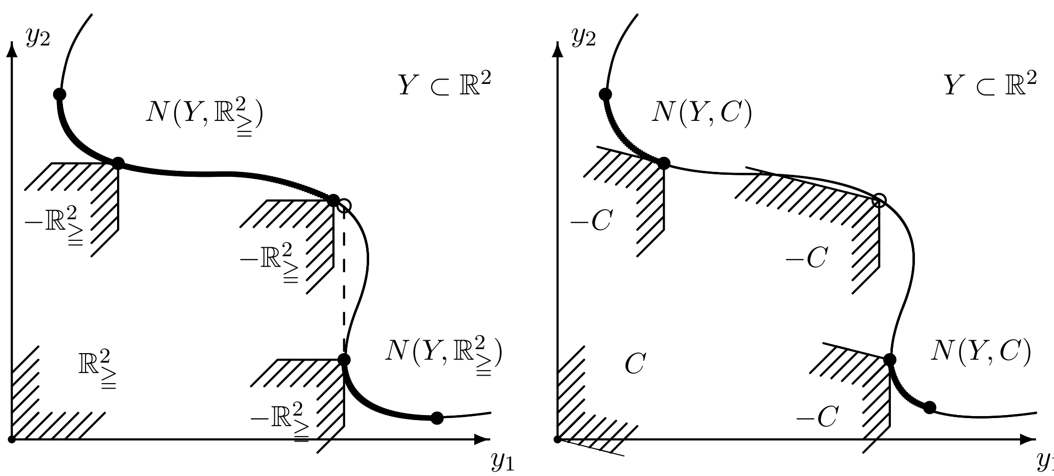


Figure 2.1 Pareto set $N(Y, \mathbb{R}^2_{\geqq})$ and nondominated set $N(Y, C)$ for a cone $C \subset \mathbb{R}^2$

For nondominance with respect to constant domination cones, Figure 2.1 indicates that a smaller set of dominated directions produces a larger nondominated set, and it turns out that this observation holds as a general result.

Proposition 2.2.6 (Nondominance relationship for two domination structures)**.** *Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D}_1 = \{D_1(y) : y \in Y\}$ and $\mathcal{D}_2 = \{D_2(y) : y \in Y\}$ be two domination structures on Y. Then*

$$D_1(y) \subseteq D_2(y) \text{ for all } y \in Y \Longrightarrow N(Y, \mathcal{D}_2) \subseteq N(Y, \mathcal{D}_1) \tag{2.39}$$

In many proofs in this text, we make implicit use of the following equivalent statements for the nondominance of $\hat{y} \in Y$ and usually choose the most convenient notion without any further explanation.

Proposition 2.2.7 (Equivalent conditions for nondominance)**.** *Let $\hat{y} \in Y$ and $D(\hat{y}) \in \mathcal{D}$ be the domination set at $\hat{y}$. The following conditions are equivalent:*

(i) $(\hat{y} - D(\hat{y}) \setminus \{0\}) \cap Y = \emptyset$*;*

(ii) there does not exist $y \in Y$ *such that* $\hat{y} - y \in D(\hat{y}) \setminus \{0\}$*;*

(iii) there does not exist $y \in Y \setminus \{\hat{y}\}$ *and* $d \in D(\hat{y}) \setminus \{0\}$ *such that* $\hat{y} = y + d$*.*

Remark 2.2.8 (Nondominance in the sense of Yu (1974))**.** As emphasized in Remark 2.2.2, for a variable domination structure $\mathcal{D} = \{D(y) : y \in Y\}$, the concept of domination adopted in this text differs from the original notion in Yu (1974) who defines the nondominated set similar to the conditions in Proposition 2.2.7 as

$$N(Y, \mathcal{D}) := \{\hat{y} \in Y : \text{ there is no } y \in Y \text{ such that } \hat{y} \in y + D(y) \setminus \{0\}\} \tag{2.40}$$

We note that the fundamental difference between our concept of domination and the original definition by Yu (1974) is that, in our case, nondominance of $\hat{y}$ can be verified based on the domination set $D(\hat{y})$ at $\hat{y}$ alone and does not depend on the domination sets $D(y)$ at all the remaining other $y \in Y \setminus \{\hat{y}\}$. Furthermore, if $\mathcal{D} = D$ is a constant domination structure, then both notions again are equivalent and the nondominated set in Remark 2.2.8 coincides with Definition 2.2.5.

Throughout the complete text, we usually assume that both Y and the nondominated set $N(Y, \mathcal{D})$ are nonempty, and for the most typical case in which the domination structure $\mathcal{D}$ is described by a constant cone C, the following result establishes conditions that guarantee that our assumption is satisfied, in general.

Proposition 2.2.9 (Nonemptiness of $N(Y, C)$). *Let $C \subset \mathbb{R}^m$ be a pointed convex cone, and $Y \subset \mathbb{R}^m$ be a nonempty C-compact set. Then $N(Y, C) \neq \emptyset$.*

In addition, several stronger results than Proposition 2.2.9 exist and replace pointedness of C by the weaker assumption of acuteness (Hartley, 1978) or C-compactness of Y by C-semicompactness (Corley, 1980), then relying on Zorn's Lemma, or equivalently, the Axiom of Choice (Zermelo, 1904; Ciesielski, 1997). Other authors derive conditions using the concept of a projecting cone (Benson, 1978) or recession cone (Borwein, 1977; Bitran and Magnanti, 1979; Henig, 1982), and a recent comparative survey of these and many other results is provided by Sonntag and Zalinescu (2000). In addition to the above results, Naccache (1978) and Bitran and Magnanti (1979) also study connectedness and Tanino and Sawaragi (1978, 1980) examine stability properties of the nondominated set. From among the few results for domination structures that are not constant cones, Proposition 2.2.9 is extended by Hazen and Morin (1983a,b, 1984) who also derive some first-order necessary and sufficient conditions for nonconical nondominance.

Two other large parts of the literature deal with questions of duality (pioneered by Isermann, 1978; Tanino and Sawaragi, 1979; Corley, 1981; Bitran, 1981; Jahn, 1983; Hsia and Lee, 1988, and others) with a recent overview provided by Nakayama (1999) and, in extension of earlier definitions by Kuhn and Tucker (1951) and Geoffrion (1968), with various notions of proper nondominance (including Borwein, 1977; Hartley, 1978; Benson, 1979, 1983; Coladas Uría, 1981; White, 1982; Henig, 1982, 1990), some of which are further analyzed in Miettinen and Mäkelä (2001). As before, in the majority of these references, the domination structure is defined as a constant convex cone, and some more specific results for polyhedral cones are obtained by Tamura and Miura (1979), Corley (1985), and Fujita (1996).

A smaller portion also follows the original definition by Yu (1973, 1974) and considers domination structures that are not necessarily cones but convex sets, including some works involving the same author (Yu and Leitmann, 1974; Yu, 1975; Bergstresser et al., 1976; Bergstresser and Yu, 1977) that, together with several of the above results, are comprehensively discussed in Yu (1985). Further extensions of domination structures to more general spaces are proposed by Chew (1979) and Weidner (1985, 1987, 2003) who also investigates domination sets in more general vector optimization. Finally, only two but very recent papers by Chen and Yang (2002) and Chen et al. (2005a) characterize variable domination structures for vector-variational inequalities in the context of generalized quasi-vector equilibrium problems, and some additional references of more specific relevance for our later discussion are mentioned throughout the remaining text.

2.2.2 Multiobjective Programs and Efficiency

With the concept of domination and the definition of the nondominated set at hand, we can define the general multiobjective program for which we assume that the set $Y \subset \mathbb{R}^m$ is the image of some set $X \subseteq \mathbb{R}^m$ under a vector-valued function f.

Definition 2.2.10 (Multiobjective Program)**.** Let $X \subseteq \mathbb{R}^n$ be a nonempty set, $f : X \to \mathbb{R}^m$ be a vector-valued function, and $\mathcal{D} = \{D(y) \subset \mathbb{R}^m : y \in Y\}$ be a domination structure on the image $Y = f(X) \subset \mathbb{R}^m$ of X under f. The triple

$$(X, f, \mathcal{D}) \tag{2.41}$$

is called the *multiobjective program* (MOP) with *feasible decision set* X and *objective function* f.

If $m = 1$ or $m = 2$, then we usually refer to the triple $(X, f, \mathcal{D})$ as *single objective program* (SOP) or *biobjective program* (BOP), respectively. Throughout this text, however, we always assume that $m \geq 2$ and then say that the objective function f maps the set of feasible decisions X from the *decision space* $\mathbb{R}^n$ to the set of *outcomes* $Y = f(X)$ in the *outcome* or *objective space* $\mathbb{R}^m$. We treat the feasible set X in a mostly generic manner and implicitly assume its representation either as a finite or countable list

$$X = \{x^1, x^2, \ldots x^N\} \text{ or } X = \{x^1, x^2, x^3, \ldots\} \tag{2.42a}$$

for some of our practical considerations or, for all theoretical discussion, as a constrained set

$$X = \{x \in \mathbb{R}^n : g(x) \leqq 0, h(x) = 0\} \tag{2.42b}$$

where the functions $g = (g_1, \ldots, g_k) : \mathbb{R}^n \to \mathbb{R}^k$ and $h = (h_1, \ldots, h_l) : \mathbb{R}^n \to \mathbb{R}^l$ describe k inequality and l equality constraints, respectively.

Definition 2.2.11 (Efficiency)**.** Let $(X, f, \mathcal{D})$ be MOP, $Y = f(X)$ be its outcome set, and $N(Y, \mathcal{D})$ be the nondominated set of Y with respect to $\mathcal{D}$. The sets

$$E(X, f, \mathcal{D}) := \{\hat{x} \in X : f(\hat{x}) \in N(Y, \mathcal{D})\} \tag{2.43a}$$

$$E_w(X, f, \mathcal{D}) := \{\hat{x} \in X : f(\hat{x}) \in N_w(Y, \mathcal{D})\} \tag{2.43b}$$

are called the *efficient* and *weakly efficient set* for MOP, respectively.

Hence, a feasible decision $\hat{x} \in X$ is *(weakly) efficient* for MOP if and only if its outcome $\hat{y} = f(\hat{x})$ is *(weakly) nondominated* with respect to the domination structure $\mathcal{D}$. In particular, then $E(X, f, \mathcal{D}) \neq \emptyset$ if and only if $N(Y, \mathcal{D}) \neq \emptyset$, and for $\mathcal{D} = C$ a constant pointed convex cone, Corley (1980) shows that Proposition 2.2.9 can also be formulated for the efficient set if $X \neq \emptyset$ is compact and if f is C-semicontinuous (for details, see the original reference or Sawaragi et al., 1985).

For the special case where $\mathcal{D} = \mathbb{R}^m_{\geqq}$ is the Pareto cone and $N(Y)$ is the Pareto set, we usually reformulate Definition 2.2.11 for the definition of Pareto efficiency.

Definition 2.2.12 (Pareto efficiency)**.** Let $X \subseteq \mathbb{R}^n$ be a nonempty feasible set, $f : X \to \mathbb{R}^m$ be an objective function, and $Y = f(X) \subset \mathbb{R}^m$ be the set of outcomes of X under f. A feasible decision $\hat{x} \in X$ is said to be

(i) *strictly Pareto efficient* if there is no $x \in X \setminus \{\hat{x}\}$ such that $f(x) \leqq f(\hat{x})$,

(ii) *Pareto efficient* if there does not exist $x \in X$ such that $f(x) \leq f(\hat{x})$, and

(iii) *weakly Pareto efficient* if does not exist $x \in X$ such that $f(x) < f(\hat{x})$.

In these cases, the outcome $\hat{y} = f(\hat{x}) \in Y$ is said to be a *(strict) Pareto*, *Pareto*, or *weak Pareto outcome*, respectively.

This definition essentially repeats Proposition 2.2.7 for the three Pareto cones defined in Equation 2.33. Moreover, it is clear (otherwise use Proposition 2.2.6) that strict Pareto efficiency implies Pareto efficiency, and that Pareto efficiency implies weak Pareto efficiency. In particular, if f is injective, then Pareto and strict Pareto efficiency are equivalent, and in general, this distinction becomes irrelevant for Pareto outcomes in the objective space.

Since Definition 2.2.12 does not depend on an explicit domination structure or domination cone but the componentwise order, in this case we also denote MOP as the pair (X, f) and usually formulate

$$\text{MOP: Minimize } f(x) \text{ subject to } x \in X \tag{2.44}$$

Similarly, and analogous to Definitions 2.2.5 and 2.2.11, in this case we denote the sets of *strictly Pareto*, *Pareto*, and *weakly Pareto efficient decisions* by $E_s(X, f)$, $E(X, f)$, and $E_w(X, f)$, and let $N(Y)$ and $N_w(Y)$ be the sets of *Pareto* and *weak Pareto outcomes*, respectively.

The predominant literature on multiobjective programming is based on the minimization (or an equivalent maximization) formulation in (2.44) which allows the definition of numerous additional concepts of which we only introduce the few needed within the context of our own results. In particular, by separately solving the single objective programs

$$\text{Minimize } f_i(x) \text{ subject to } x \in X \tag{2.45}$$

for all $i = 1, \ldots, m$ and combining the optimal objective function values, we obtain the ideal point of MOP.

Definition 2.2.13 (Ideal and utopia point)**.** Let (X, f) be MOP with feasible decision set $X \subseteq \mathbb{R}^n$ and objective function $f : X \to \mathbb{R}^m$. The point $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ with

$$z_i = \inf\{f_i(x) : x \in X\} \text{ for all } i = 1, \ldots, m \tag{2.46a}$$

is called the *ideal point*, and any $r \in \mathbb{R}^m$ with

$$r \leq f(x) \text{ for all } x \in X \tag{2.46b}$$

is called a *utopia point* of MOP.

Whenever needed, we assume that the ideal point exists and is finite, or equivalently, that the set of utopia points is nonempty.

Remark 2.2.14 (Existence of the ideal point)**.** If $Y = f(X)$ is $\mathbb{R}^m_{\geqq}$-compact, then the ideal point exists and is finite. In particular, in this case the infimum in (2.46a) can be replaced by the minimum and a point $r \in \mathbb{R}^m$ is utopia if and only if $r \leq z$.

It is clear that if $z = f(x)$ for some $x \in X$, then x is optimal for all single objective programs in (2.45) and, consequently, Pareto efficient for MOP. Since the different objectives are typically in conflict with each other, however, the minimum values z_i in (2.46a) are usually attained by different $x^i \in X$ for which $f_j(x^i) > f_j(x^j) = z_j$ for all $j \neq i$, in general. Therefore, to find any decision $x \in X$ that further reduces the objective value of x^i with respect to f_j to some $f_j(x) < f_j(x^i) = z_i$, we need to accept a corresponding increase in f_i to some new value $f_i(x) > f_i(x^i)$. The ratio between such two changes defines the important general notion of a tradeoff which is further discussed in Chankong and Haimes (1983).

Definition 2.2.15 (Tradeoff and tradeoff rate)**.** Let (X, f) be MOP and $x^1, x^2, \hat{x} \in X$ be feasible decisions. If $f_j(x^2) - f_j(x^1) \neq 0$, then the ratio

$$T_{ij}(x^1, x^2) := \frac{f_i(x^1) - f_i(x^2)}{f_j(x^2) - f_j(x^1)} \tag{2.47a}$$

is called the *tradeoff* between x^1 and x^2 involving f_i and f_j. If f is continuously differentiable at $\hat{x}$, then

$$T_{ij}(\hat{x}) := -\left.\frac{\partial f_i(x)}{\partial f_j}\right|_{x=\hat{x}} \tag{2.47b}$$

is called the *tradeoff rate* at $\hat{x}$ involving f_i and f_j.

Kaliszewski (1994) generalizes this definition to the notion of a global tradeoff, and Miettinen (1999) draws the connection between these concepts and the marginal rate of substitution from economics. Furthermore, the extension of tradeoff rates to tradeoff directions based on cones is investigated for convex MOP by Henig and Buchanan (1997), followed by Lee and Nakayama (1997) for the differentiable case and Miettinen and Mäkelä (2002) for the nondifferentiable case.

Finally, various other notions refine the definition of Pareto efficiency and include the concepts of proper efficiency by Kuhn and Tucker (1951) and Geoffrion (1968) to guarantee tradeoff rates that are bounded, and several concepts of approximate or ε-efficiency originally defined by Kutateladze (1979). While the former are of no particular relevance for our discussion and therefore omitted, the latter play an important role in many following parts of this text so that we present two possible definitions and review significant parts of the relevant literature in further detail.

Definition 2.2.16 ((Additive) Epsilon efficiency)**.** Let (X, f) be MOP and $\varepsilon \in \mathbb{R}^m$, $\varepsilon \geqq 0$, be a nonnegative vector. A feasible decision $\hat{x} \in X$ is said to be

(i) *strictly ε-Pareto efficient* if there is no $x \in X \setminus \{\hat{x}\}$ such that $f(x) \leqq f(\hat{x}) - \varepsilon$,

(ii) *ε-Pareto efficient* if there is no $x \in X$ such that $f(x) \leq f(\hat{x}) - \varepsilon$, and

(iii) *weakly ε-Pareto efficient* if there is no $x \in X$ such that $f(x) < f(\hat{x}) - \varepsilon$.

In these cases, the outcome $\hat{y} = f(\hat{x}) \in Y$ is said to be a *strict ε-Pareto*, *ε-Pareto* or *weak ε-Pareto outcome*, respectively.

Similar to Definition 2.2.11, we immediately obtain that strict ε-Pareto efficiency implies ε-Pareto efficiency, and that ε-Pareto efficiency implies weak ε-Pareto efficiency. In particular, if $\varepsilon = 0$, then these two conditions coincide and ε-Pareto efficiency reduces to the previous notion of Pareto efficiency. We denote the sets of *strictly ε-Pareto*, *ε-Pareto*, and *weakly ε-Pareto efficient decisions* of X under f by $E_s(X, f, \varepsilon)$, $E(X, f, \varepsilon)$, and $E_w(X, f, \varepsilon)$, and let $N(Y, \varepsilon)$ and $N_w(Y, \varepsilon)$ be the sets of *ε-Pareto* and *weak ε-Pareto outcomes*, respectively.

If $f(x) \geqq 0$ for all $x \in X$, then the following version of ϵ-efficiency is also possible and motivated by a similar notion in Papadimitriou and Yannakakis (2000).

Definition 2.2.17 ((Multiplicative) Epsilon efficiency)**.** Let (X, f) be MOP with $f(x) \geqq 0$ for all $x \in X$ and $0 \leq \epsilon \leq 1$. A feasible decision $\hat{x} \in X$ is said to be

(i) *strictly ϵ-Pareto efficient* if there is no $x \in X \setminus \{\hat{x}\}$ s.t. $f(x) \leqq (1-\epsilon)f(\hat{x})$,

(ii) *ϵ-Pareto efficient* if there is no $x \in X$ such that $f(x) \leq (1-\epsilon)f(\hat{x})$, and

(iii) *weakly ϵ-Pareto efficient* if there is no $x \in X$ such that $f(x) < (1-\epsilon)f(\hat{x})$.

In these cases, the outcome $\hat{y} = f(\hat{x}) \in Y$ is said to be a *strict ϵ-Pareto*, *ϵ-Pareto* or *weak ϵ-Pareto outcome*, respectively.

In this text and the following review of the literature, we restrict our attention to the first notion of (additive) ε-efficiency, which is originally defined by Kutateladze (1979) and independently introduced into multiobjective programming first by Loridan (1984) and later by White (1986) who also studies six alternative definitions and establishes their corresponding relationships. Following either Loridan or White, related definitions or examinations of these concepts are given by Lemaire (1992), Helbig and Pateva (1994), Tanaka (1996), Yokoyama (1996, 1999) and Li and Wang (1998) for multiobjective programs and extended to more general vector optimization problems by Vályi (1985), Németh (1989), Tammer (1994), Loridan et al. (1999) and Rong and Wu (2000).

In particular, Yokoyama (1992, 1994) and Deng (1997) derive several optimality conditions for convex MOP and Liu (1996) for nondifferentiable MOP based on the use of penalty functions. Rong (1997) proposes a notion for proper ε-efficiency, and Kazmi (2001) derives conditions for the existence of epsilon-minima. Similar to the latter, Dutta and Vetrivel (2001) introduce quasi-epsilon-weak minima and also establish necessary and sufficient optimality conditions for their existence. Liu and Yokoyama (1999) and recently Gupta et al. (2005) investigate ε-efficiency for multiobjective fractional programs, and Gutiérrez et al. (2006a,b,c) study ε-efficient solutions in the context of scalarization approaches that we introduce in Section 2.2.3 and later address in Section 3.3.3.

In addition to these mostly theoretical results, the concept of ε-efficiency is also used in a number of practical applications, including the approximation of biobjective Pareto sets and the formulation of approximation algorithms for minimum cost flows (Ruhe and Fruhwirth, 1990), as well as the study of problems in transportation (White, 1998a), finance (White, 1998b), location (Blanquero and Carrizosa, 2002), and scheduling (Angel et al., 2003). Finally, Fadel et al. (2002) employ the concept of ε-efficiency as a measure of sensitivity to examine the curvature of Pareto curves of biobjective problems for applications in engineering design.

2.2.3 Scalarization Approaches and Tradeoffs

A great variety of solution methods exist for generating Pareto efficient decisions and are extensively discussed in many monographs on multiobjective programming and optimization (Miettinen, 1999; Jahn, 2004; Ehrgott, 2005, and others). In particular, Ehrgott and Wiecek (2005) provide a recent comprehensive survey of many of these methods and propose their classification into scalarization methods and nonscalarizing approaches. The methods that we here present are chosen as a subset of the former and provide a parametric characterization of the set of efficient decisions by combining the individual objective function components into one real-valued function which attains its optimal value if and only if the associated decision is (weakly) efficient for the original problem.

Definition 2.2.18 (Increasing function)**.** Let $Y \subset \mathbb{R}^m$ be nonempty. A real-valued function $s : Y \to \mathbb{R}$ is said to be *increasing* on Y if

$$y^1 \leq y^2 \Longrightarrow s(y^1) \leq s(y^2) \quad \text{and} \quad y^1 < y^2 \Longrightarrow s(y^1) < s(y^2) \tag{2.48a}$$

for all $y^1, y^2 \in Y$, and it is said to be *strictly increasing* if

$$y^1 \leq y^2 \Longrightarrow s(y^1) < s(y^2) \tag{2.48b}$$

For the purpose of this text, we call any increasing function s on Y or a suitable subset $T \subseteq Y$ a *scalarization function* for MOP. To change the domain of s from the outcome set Y in $\mathbb{R}^m$ to the feasible set X in $\mathbb{R}^n$, we introduce the function composition $s \circ f : \mathbb{R}^n \to \mathbb{R}$ as the associated *value function* of s on $\mathbb{R}^n$. Finally, if Π is a set of additional scalarization parameters and $s = s(\pi)$ for some $\pi \in \Pi$, then we write $s(\pi, y)$ instead of $s(\pi)(y)$ or also drop the parameter π as in Equation 2.48 if the specific choice of $\pi \in \Pi$ is fixed or irrelevant.

Definition 2.2.19 (Scalarized multiobjective program)**.** Let (X, f) be MOP with feasible set $X \subseteq \mathbb{R}^n$ and objective function $f : X \to \mathbb{R}^m$. Let $S \subseteq X$, $T = f(S) \subseteq Y$, and $s(\pi) : T \to \mathbb{R}$ be a scalarization function for MOP with scalarization parameter $\pi \in \Pi$. The parametric single objective program

$$\text{SOP}(\pi)\text{: Minimize } s(\pi, f(x)) \text{ subject to } x \in S \subseteq X \tag{2.49}$$

is called the associated *scalarization problem* for MOP.

Since SOP is a single objective program, the notions of optimality and approximate optimality are standard and follow from the canonical order of real numbers. The latter is also discussed in Loridan (1982) and, although not needed in our current context, introduced here for our later discussion in Section 3.3.3.

Definition 2.2.20 (Epsilon-Optimality and Optimality)**.** Let SOP be given, and let $\epsilon \in \mathbb{R}$, $\epsilon \geq 0$. A solution $\hat{x} \in S \subseteq X$ is called

(i) *strictly ϵ-optimal* for SOP if $s(f(\hat{x})) < s(f(x)) + \epsilon$ for all $x \in X \setminus \{\hat{x}\}$, and

(ii) *ϵ-optimal* for SOP if $s(f(\hat{x})) \leq s(f(x)) + \epsilon$ for all $x \in X$.

If $\epsilon = 0$, then $\hat{x}$ is also called *strictly optimal* or *optimal*, respectively.

By definition, we note that each optimal solution for SOP is in particular strictly ϵ-optimal, and each strictly ϵ-optimal solution is also ϵ-optimal. In particular, if $0 \leq \epsilon^1 \leq \epsilon^2$, then optimality implies ϵ^1-optimality, and ϵ^1-optimality implies ϵ^2-optimality. However, for now we postpone all further discussion of ϵ-optimality until our later investigation of approximate nondominance and summarize various results that relate optimality and efficiency for SOP and MOP, respectively.

Proposition 2.2.21 (Relationship between SOP optimality and MOP efficiency)**.** *Let (X, f) be MOP, $Y = f(X)$ be its set of outcomes, $s : Y \to \mathbb{R}$ be an increasing scalarization function, and $\hat{x} \in X$ be optimal for the associated SOP.*

(i) If $\hat{x} \in X$ is strictly optimal for SOP, then $\hat{x} \in E_s(X, f)$.

(ii) If s is strictly increasing, then $\hat{x} \in E(X, f)$.

In any case, $\hat{x} \in E_w(X, f)$.

This result is similarly established in Wierzbicki (1986) and its proof therefore omitted. Furthermore, if $s = s(\pi)$ depends on some additional scalarization parameter, then for different choices of $\pi \in \Pi$, in general we obtain different optimal solutions for SOP and, thus, different efficient decisions for MOP. While some scalarization methods are capable to generate every efficient decision and thus completely characterize the efficient set, others also depend on certain properties of the underlying outcome set and are restricted to generate only a smaller subset. For questions addressing these issues we refer to the discussion in Wierzbicki (1986).

The probably most common scalarization method combines all objective functions in the form of a nonnegative linear combination or weighted sum and is attributed variously to Gass and Saaty (1955), Zadeh (1963), or Geoffrion (1968).

Definition 2.2.22 (Weighted-sum scalarization)**.** Let (X, f) be MOP and $w \in \mathbb{R}^m, w \geq 0$. The single objective program

$$\text{WS}(w)\text{: Minimize } \sum_{i=1}^{m} w_i f_i(x) \text{ subject to } x \in X \tag{2.50}$$

is called the *weighted-sum scalarization* (WS) of MOP with weighting parameter w.

The geometric interpretation of this method is that it finds that boundary point of the outcome set Y at which the weighting vector w is normal to a supporting hyperplane to Y.

Proposition 2.2.23 (Weighted-sum sufficient condition for efficiency)**.** *Let (X, f) be MOP and $\hat{x} \in X$ be optimal for WS(w) with weighting parameter $w \in \mathbb{R}^m$, $w \geq 0$.*

(i) If $\hat{x}$ is strictly optimal, then $\hat{x} \in E_s(X, f)$.

(ii) If $w > 0$, then $\hat{x} \in E(X, f)$.

In any case, $\hat{x} \in E_w(X, f)$.

Based on its geometric interpretation, this method is limited to finding only those efficient decisions whose associated outcomes occur in convex regions of the Pareto set. If Y is $\mathbb{R}^m_{\geqq}$-convex, however, then there exists a supporting hyperplane at every Pareto outcome by Theorem 2.1.24 (supporting hyperplane theorem), and the next result follows.

Proposition 2.2.24 (Weighted-sum necessary condition for efficiency). *Let (X, f) be MOP and $Y = f(X)$ be $\mathbb{R}^m_{\geqq}$-convex.*

(i) If $\hat{x} \in E_w(X, f)$, then there exists $w \geq 0$ such that $\hat{x}$ is optimal for WS(w).

(ii) If $\hat{x} \in E(X, f)$, then there exists $w > 0$ such that $\hat{x}$ is optimal for WS(w).

Li et al. (1999) establish the following result for the tradeoff rate from Definition 2.2.15 at an optimal solution for the weighted-sum scalarization, which can also be derived as special case of the tradeoff rates for the general hyperplane method proposed by Sakawa and Yano (1990).

Proposition 2.2.25 (Weighted-sum tradeoff rates). *Let (X, f) be MOP and $\hat{x} \in X$ be optimal for WS(w). If f is continuously differentiable at $\hat{x}$ and $w_i \neq 0$, then*

$$T_{ij}(\hat{x}) := -\left.\frac{\partial f_i(x)}{\partial f_j}\right|_{x=\hat{x}} = \frac{w_j}{w_i} \tag{2.51}$$

Another important scalarization method is the constrained-objective scalarization, also called the *epsilon-constraint method*, which is originally introduced by Haimes et al. (1971) and well known from the detailed discussion in Chankong and Haimes (1983). Instead of combining the individual objectives, this method converts all objectives into constraints, imposes additional objective bounds, and then chooses only a single objective for actual minimization.

Definition 2.2.26 (Constrained-objective scalarization). Let (X, f) be MOP, f_k be any single objective of $f = (f_1, \ldots, f_m)$, and $b \in \mathbb{R}^m$ be a vector. The single objective program

$$\text{CO}_k(b)\text{: Minimize } f_k(x) \text{ subject to } f(x) \leqq b \text{ and } x \in X \tag{2.52}$$

is called *constrained-objective scalarization* (CO) of MOP with objective bound b.

While the original problem formulation drops the kth constraint $f_k(x) \leq b_k$ from $f(x) \leqq b$, for later convenience we can also keep this constraint without changing any of the original results.

Proposition 2.2.27 (Constrained-objective necessary condition for efficiency)**.** *Let (X, f) be MOP, $b \in \mathbb{R}^m$, and $\hat{x} \in X$ be optimal for $CO_k(b)$.*

(i) If $\hat{x}$ is strictly optimal, then $\hat{x} \in E_s(X, f)$.

(ii) If $\hat{x}$ is optimal for $CO_k(b)$ for all $k = 1, \ldots, m$, then $\hat{x} \in E(X, f)$.

In any case, $\hat{x} \in E_w(X, f)$.

Other than a weighted sum, the constrained-objective scalarization is not limited by convexity but can be used to find any efficient decision for MOP.

Proposition 2.2.28 (Constrained-objective necessary and sufficient condition for efficiency)**.** *Let (X, f) be MOP and $\hat{x} \in X$. Then $\hat{x} \in E(X, f)$ if and only if $\hat{x}$ is optimal for $CO_k(f(\hat{x}))$ for all $k = 1, \ldots, m$.*

Based on the well known sensitivity theorem from nonlinear programming (Luenberger, 1973, 1984), Chankong and Haimes (1983) also establish a corresponding result for the tradeoff rate at an optimal solution $\hat{x}$ for the constrained-objective scalarization. For a precise definition of the concepts stated in the assumptions of the following proposition, we refer to the original references.

Proposition 2.2.29 (Constrained-objective tradeoff rates)**.** *Let (X, f) be MOP, f be twice continuously differentiable, $b \in \mathbb{R}^m$, and $\hat{x} \in X$ be optimal for $CO_k(b)$. If*

(i) $\hat{x}$ is a regular point of the constraints $f(x) \leqq b$ in $CO_k(b)$,

(ii) the second-order sufficiency conditions are satisfied at $\hat{x}$, and

(iii) there are no degenerate constraints at $\hat{x}$, then

$$T_{ki}(\hat{x}) := -\left.\frac{\partial f_k(x))}{\partial f_i(x)}\right|_{x=\hat{x}} = \lambda_{ki} \text{ for all } i = 1, \ldots, m, i \neq k \tag{2.53}$$

where the λ_{ki} are the Lagrangean multipliers associated with each of the constraints $f_i(\hat{x}) \leq b_i$ of $CO_k(b)$.

We can also combine the weighted-sum method and the constrained-objective scalarization into the following hybrid method which is similarly introduced by Wendell and Lee (1977), Corley (1980), and Guddat et al. (1985).

Definition 2.2.30 (Hybrid scalarization)**.** Let (X, f) be MOP, $w \in \mathbb{R}^m, w \geq 0$, and $b \in \mathbb{R}^m$. The single objective program

$$\text{HB}(w, b)\text{: Minimize } \sum_{i=1}^{m} w_i f_i(x) \text{ subject to } f(x) \leqq b \text{ and } x \in X \tag{2.54}$$

is called the *hybrid scalarization* (HB) of MOP with weighting parameter w and objective bound b.

In particular, Guddat et al. (1985) choose the parameter b as an outcome $f(x^\circ)$ for some feasible decision $x^\circ \in X$ with an underlying interpretation as aspiration of reference point. The following results, however, hold in further generality and essentially repeat the statements of Propositions 2.2.23 and 2.2.28.

Proposition 2.2.31 (Hybrid necessary condition for efficiency)**.** *Let (X, f) be MOP and $\hat{x} \in X$ be optimal for HB(w, b) with weighting vector $w \in \mathbb{R}^m, w \geq 0$, and objective bound $b \in \mathbb{R}^m$.*

(i) If $\hat{x}$ is strictly optimal, then $\hat{x} \in E_s(X, f)$.

(ii) If $w > 0$, then $\hat{x} \in E(X, f)$.

In any case, $\hat{x} \in E_w(X, f)$.

Proposition 2.2.32 (Hybrid necessary and sufficient condition for efficiency)**.** *Let (X, f) be MOP and $w \in \mathbb{R}^m, w \geq 0$. Then $\hat{x} \in E(X, f)$ if and only if $\hat{x}$ is optimal for HB$(w, f(\hat{x}))$.*

Remark 2.2.33 (Relationship between hybrid, weighted-sum, and constrained-objective scalarization)**.** If $f(x) \leqq b$ for all $x \in X$, then the constraints in HB(w, b) are always satisfied and the hybrid scalarization reduces to the weighted-sum scalarization WS(w). Furthermore, if $w = e^k$ is the kth unit vector, then $\sum_{i=1}^{m} w_i f_i(x) = f_k(x)$ and HB(e^k, b) reduces to the constrained-objective scalarization CO$_k(b)$.

Hence, the hybrid scalarization generalizes both the weighted-sum and the constrained-objective scalarization methods. Furthermore, if we let $w = (1, \ldots, 1)^T$ be the vector with all components equal to one and choose $b = f(x^\circ)$ for some feasible decision $x^\circ \in X$, then we obtain the scalarization method that is originally introduced by Benson (1978) in the following formulation.

Definition 2.2.34 (Benson scalarization)**.** Let (X, f) be MOP and $x^\circ \in X$. The single objective program

$$\text{B}(f(x^\circ))\text{: Maximize } \sum_{i=1}^{m} l_i \text{ subject to } f(x^\circ) - l = f(x) \in f(X) \text{ and } l \geqq 0 \quad (2.55)$$

is called the *Benson scalarization* (B) of MOP with reference point $f(x^\circ)$.

Proposition 2.2.35 (Benson condition for efficiency)**.** *Let (X, f) be MOP, $x^\circ \in X$ and $(\hat{l}, \hat{x})$ be optimal for B($f(x^\circ)$). Then $\hat{x} \in E(X, f)$, and in particular, $x^\circ \in E(X, f)$ if and only if $l = 0$.*

We note that the geometric idea behind this method is to start from the reference point $f(x^\circ)$ in the outcome set $Y = f(X)$ and then project this point along a nonnegative direction l of maximum length onto the Pareto set $N(Y)$. Based on a similar geometric interpretation, first Roy (1971), then Gembicki and Haimes (1975), and later Pascoletti and Serafini (1984) use a similar direction-based approach for the generation of efficient decisions with respect to a general pointed convex cone.

Definition 2.2.36 (Pascoletti-Serafini scalarization)**.** Let (X, f, C) be MOP with $C \subset \mathbb{R}^m$ a pointed convex cone, $r \in \mathbb{R}^m$ be a vector, and $v \in \operatorname{int} C$ be an element of the cone interior. The single objective program

$$\text{PS}(r, v)\text{: Minimize } \mu \text{ subject to } r + \mu v - c \in f(X),\ c \in C \text{ and } \mu \in \mathbb{R} \quad (2.56)$$

is called the *Pascoletti-Serafini scalarization* (PS) of MOP with reference point r and reference direction v.

Proposition 2.2.37 (Pascoletti-Serafini condition for efficiency)**.** *Let (X, f, C) be MOP. If $\hat{\mu}$ is optimal for PS(r, v) and $\hat{y} = r + \hat{\mu} v - c \in Y$, then*

(i) $\hat{y} \in N_w(Y, C)$, *and*

(ii) there exists $\hat{c} \in C$ *such that* $\hat{y} = r + \hat{\mu} v - \hat{c} \in N(Y, C)$.

For the special case in which $C = \mathbb{R}^2_{\geqq}$ is the Pareto cone, Figure 2.2 illustrates the principle idea of the Pascoletti-Serafini method with four different choices of the reference point $r \in \mathbb{R}^m$ and two different direction vectors $v \in \mathbb{R}^m_{>}$.

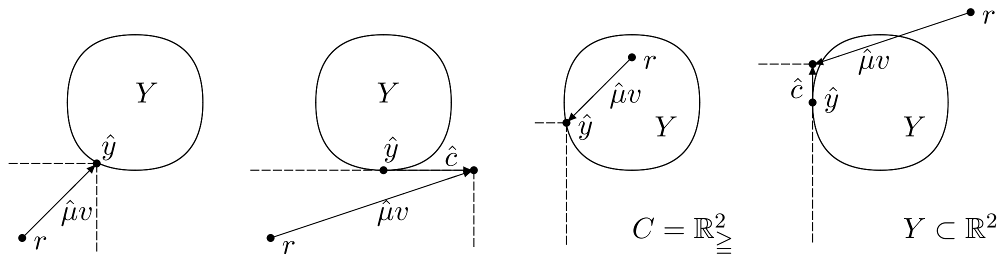


Figure 2.2 The Pascoletti-Serafini method $\mathrm{PS}(r, v)$ for $Y \subset \mathbb{R}^2$ and the Pareto cone $\mathbb{R}^2_{\geqq}$ with four choices of reference point $r \in \mathbb{R}^2$ and direction vector $v \in \mathbb{R}^2_{>}$

The Pascoletti-Serafini scalarization method can also be derived as a special application of the results in Gerth and Weidner (1990) who investigate separation theorems in general topological spaces and is analyzed in some further detail in Schandl (1999). Besides a generalization of the weighted-sum scalarization for closed convex cones (Gearhart, 1983; Jahn, 1984; Sawaragi et al., 1985, among others), however, we are not aware of any other common scalarization method that is formulated for a general cone and propose some possible extensions also for the constrained-objective, hybrid, and Benson scalarization in Section 3.1.3.

For the Pareto case, we present two further scalarization methods of relevance for this text that belong to the important class of approaches that utilize weighted-ℓ_p norms and are originally introduced by Zeleny (1973) to approximate the ideal point in the context of compromise programming. In particular, we only consider the special case of the probably most important weighted-ℓ_∞, or equivalently, the weighted-Chebyshev norm scalarization commonly accredited to Bowman (1976), Choo and Atkins (1983), or Steuer (1986).

Definition 2.2.38 (Weighted-Chebyshev norm scalarization)**.** Let (X, f) be MOP, $r \in \mathbb{R}^m$ with $r \leq f(x)$ for all $x \in X$ be a utopia point, and $w \in \mathbb{R}^m, w \geq 0$ be a nonnegative vector of weights. The single objective program

$$\mathrm{CN}(r, w)\text{: Minimize } \max_{i=1,\ldots,m} \{w_i(f_i(x) - r_i)\} \text{ subject to } x \in X \tag{2.57}$$

is called the *weighted-Chebyshev norm scalarization* (CN) of MOP with reference point r and weighting parameter w.

Proposition 2.2.39 (Weighted-Chebyshev norm necessary and sufficient condition for (weak) efficiency)**.** *Let (X, f) be MOP and $\hat{x} \in X$. Then $\hat{x} \in E_w(X, f)$ if and only if there exists $w > 0$ such that $\hat{x}$ is optimal for CN(r, w).*

Hence, we can use the weighted-Chebyshev norm scalarization method to find all efficient and weakly efficient decisions for any MOP. Furthermore, Steuer and Choo (1983) and Steuer (1986) also propose an augmented-Chebyshev norm and Kaliszewski (1987) formulates a modified-Chebyshev norm to guarantee that all generated solutions are also always efficient.

Remark 2.2.40 (Weighted-Chebyshev norm tradeoff rates)**.** The weighted-Chebyshev norm scalarization CN(r, w) can equivalently be written as

$$\text{Minimize } \alpha \text{ subject to } \alpha \geq w_i(f_i(x) - r_i) \text{ for all } i = 1, \ldots, m \text{ and } x \in X \tag{2.58}$$

so that, under conditions corresponding to those for the constrained-objective method in Proposition 2.2.29, the tradeoff rate at an optimal solution $\hat{x}$ is given by

$$T_{ij}(\hat{x}) := -\frac{\partial f_i(x)}{\partial f_j}\bigg|_{x=\hat{x}} = \frac{\lambda_j w_j}{\lambda_i w_i} \tag{2.59}$$

where λ_i and λ_j are the Lagrangean multipliers associated with the constraints $\alpha \geq w_i(f_i(x) - r_i)$ and $\alpha \geq w_j(f_j(x) - r_j)$ in (2.58), respectively.

In further generalization of weighted-ℓ_p norms, Schandl et al. (2002a) introduce the notion of oblique norms and develop norm-based approximation methods for both BOP and MOP (Schandl et al., 2001, 2002b). For the general class of scalarization approaches that are based on norm minimization, Lin (2005) investigates the consequences on the generated solutions when choosing different locations for the reference point. In particular, if $r = 0$ and $w = 1$, then the weighted-Chebyshev norm scalarization CN$(0, 1)$ reduces to the max-norm scalarization, which is also known as the max-ordering approach (Kouvelis and Yu, 1997; Ehrgott, 2005).

Definition 2.2.41 (Max-norm scalarization)**.** Let (X, f) be MOP. The single objective program

$$\text{MN: Minimize} \max_{i=1,\ldots,m} \{f_i(x)\} \text{ subject to } x \in X \tag{2.60}$$

is called the *max-norm scalarization* (MN) of MOP.

Proposition 2.2.42 (Max-norm sufficient condition for weak efficiency)**.** *Let (X, f) be MOP. If $\hat{x} \in X$ is optimal for MN, then $\hat{x} \in E_w(X, f)$.*

In particular, analogous to Remark 2.2.40 we can obtain the tradeoff rate at an optimal solution $\hat{x}$ for MN as

$$T_{ij}(\hat{x}) := -\frac{\partial f_i(x)}{\partial f_j}\bigg|_{x=\hat{x}} = \frac{\lambda_j}{\lambda_i} \tag{2.61}$$

In summary, we conclude that a large number of scalarization approaches exist which can be defined based on linear combinations of objectives in the form of weighted sums, objective constraints and bounds, reference points, directions, and various norms and norm variants. As we mention at the very beginning, of course there are also many other scalarization approaches in addition to the ones described, but we decide to limit our list to the ones that actually reappear in later parts of this dissertation and defer to the provided references for any additional information.

2.3 Critique of Multiple Criteria Decision Making

As indicated in the introduction and formalized in Definition 2.2.10, a multiobjective program MOP is a mathematical model that describes the decisions and objectives of a multiobjective optimization or decision problem as a set of real vectors $X \subseteq \mathbb{R}^n$ and a vector-valued function $f : X \to \mathbb{R}^m$, respectively. Moreover, to enable the partial comparison of different decisions in terms of their resulting outcomes $y \in Y = f(X) \subset \mathbb{R}^m$, we can additionally specify a domination structure $\mathcal{D} = \{D(y) : y \in Y\}$ to equip the associated outcome space with a notion of domination or, by default, adopt an underlying concept of objective minimization as the special case of Pareto optimality. In any case, throughout this text we constantly assume that X and f are given explicitly, and we are primarily interested in the characterization, generation, or approximation of the efficient and nondominated sets $E(X, f, \mathcal{D})$ and $N(Y, \mathcal{D})$ associated with different concepts of domination.

Clearly, in doing so we in large part ignore that for practical decision making problems many additional steps are required before we obtain a model in the form of a multiobjective program, including the preliminary problem analysis, data collection, handling of uncertainties, and the final model formulation. Being well aware of the frequent difficulties and crucial importance of all these aspects, however, we decide not to expose any of these additional details, which are extensively discussed in numerous monographs on multiple criteria decision making (Keeney and Raiffa, 1976; Zeleny, 1982; Chankong and Haimes, 1983; Yu, 1985) and multiobjective optimization (Sawaragi et al., 1985; Steuer, 1986; Miettinen, 1999; Jahn, 2004; Ehrgott, 2005). In particular, two very recent and comprehensive collections of bibliographic state-of-the-art surveys (Ehrgott and Gandibleux, 2002; Figueira et al., 2005) provide excellent overviews over the general field, and many other works exist that also focus on more specific aspects of multiobjective programming models in the engineering and management sciences (Goicoechea et al., 1982; Osyczka, 1984, 1985; Stadler, 1984, 1988; Eschenauer et al., 1990; Stadler and Dauer, 1992; Collette and Siarry, 2003, and many others).

2.3.1 Optimization versus Decision Making

In critical review of some existing ideas in the literature, in this section we briefly address some of the more practical aspects of multiobjective programming as they pertain to multiple criteria decision making and the general discussion within this text. Henceforth assuming that the decision problem is modeled by a feasible decision set and vector-valued objective function, we first take the traditional and still typical viewpoint and consider the resulting multiobjective program with respect to Pareto efficiency. In this case, the literature of multiobjective programming and optimization contains a vast amount of theoretical results that characterize the efficient or the Pareto set and, moreover, provides us with a great variety of different solution methods for its generation. From a practical point of view, however, it is not immediately clear what the solution of a multiobjective programming problem is actually meant to be.

For a mathematical program with a single objective, solving this problem is clearly understood as the process of either finding a decision for which the corresponding scalar outcome attains a minimal value, or coming to the conclusion that such an outcome does not exist. In particular, although in certain instances still a very challenging task, we can frequently use an appropriate technique from linear or nonlinear programming or optimization to actually compute or at least sufficiently closely approximate such an optimal solution. For a multiobjective program, however, there usually does not exist a unique optimal outcome, but a nondominated set which consists of many and possibly infinitely many points. Hence, while it is theoretically possible to generate each nondominated outcome based on the scalarization approaches discussed in Section 2.2.3, for practical purposes this is clearly impossible so that we need to restrict the computation to only a subset of the complete nondominated set in the form of a discrete representation (Armann, 1989; Benson and Sayin, 1997; Sayin, 2000; Mattson et al., 2004) or finite approximations (for a recent survey, see Ruzika and Wiecek, 2005).

In principle, there exist two different approaches to finding suitable representations of the nondominated set of a multiobjective programming problem. The first approach relies on scalarization and uses a parametrized real-valued function so that, by varying the associated scalarization parameters and solving the series of resulting single objective programs, different nondominated outcomes and efficient decisions can be obtained. In particular, the arguably best known scalarization method is to aggregate all multiple objectives into a single objective in the form of a linear combination or weighted sum, which we introduce and characterize in Definition 2.2.22 and Propositions 2.2.23 and 2.2.24, respectively. Although seemingly convenient to handle and therefore still the most widely used method in business and engineering applications, several drawbacks of this method are recognized (Das and Dennis, 1997; Fliege, 2004; Scott and Antonsson, 2005) which most prominently include its failure to generate points in nonconvex regions of the Pareto set. Many of the alternative methods that we describe in Section 2.2.3, however, remove these limitations and, in principle, can be used to find representations of the complete nondominated set to essentially any desired accuracy.

Of course, in the above discussion we again implicitly assume that the solution of each scalarized single objective program is sufficiently simple so to enable the generation of many and possibly a quite large number of efficient decisions and nondominated outcomes. For many realistic problems, however, we also face the challenge of high computational complexities that frequently impede the successful use of the aforementioned scalarization approaches. In particular, and especially in engineering applications, the computation of objective values is often based on underlying finite element or simulation codes which impose severe restrictions on the number of permissible function evaluations and, thus, on the number and type of optimization problems that we can actually solve. Consequently, a recent alternative trend to generate nondominated points is the development of new heuristic approaches such as multiobjective evolutionary and genetic algorithms (Deb, 2001; Zitzler et al., 2001; Coello Coello et al., 2002; Coello Coello and Romero, 2002; Yen, 2003) that can also be applied if the evaluation of function values or gradients is computationally too expensive to allow the use of scalarization methods that are based on traditional optimization. Although typically very problem dependent and without a rigorous theoretical foundation, these approaches frequently show very good performance in practice (Narayanan and Azarm, 1999; Gunawan et al., 2004; Deb and Tiwari, 2005; Gantovnik et al., 2006, and many more).

In any case, from an *optimization* point of view, we can conclude that the most common interpretation of solving a multiobjective program is the computation of a suitable subset of the corresponding efficient and nondominated sets, subject to certain properties or characteristics such as representation accuracy or approximation quality (Ruzika and Wiecek, 2005). From the perspective of a *decision maker*, however, this optimization stage and the resulting set of potential candidates is again only a preliminary step to prepare the selection of a final preferred decision which then constitutes the overall solution to the multiobjective programming model and, after translation into the real-life problem context, the original decision making problem. In particular, while the solution set in the optimization sense can be clearly defined based on rigorous mathematical concepts, decision making by nature involves a decision maker with subjective preferences, priorities, expectations and personal aspirations which are often not easily described or readily articulated in terms of the chosen mathematical model.

Hence, we conclude that in the case of decision making, finding a final solution can still be quite difficult if preferences are not completely modeled or known and if the numbers of potential candidates and objectives are too large to make use of existing enumeration or visualization techniques. In remedy of these two difficulties, we essentially see two possible ways out which we address in the following two Sections 2.3.2 and 2.3.3 with reference to several related approaches in the literature. The first idea is to refine the concept of Pareto optimality by new preference assumptions and to introduce new domination structures to include some of the decision maker's preferences already into the optimization stage, thereby reducing the set of candidate decisions that are presented to the decision maker for further consideration. The second idea is to support the decision maker in the progressive articulation of preferences by interactive decision making procedures and several decomposition strategies that facilitate the overall conceptual perception of a large-scale multiobjective program. Based on some of the approaches that we review in the following discussion, we initiate our own contributions with our research statement in the concluding section of this chapter.

2.3.2 Preference Principles and Domination

The notion and modeling of preferences play an important role in many aspects of economics, sociology, psychology, and clearly, multiple criteria decision making, and are extensively researched during the past century (for a comprehensive recent survey, see Öztürk et al., 2005) and discussed at great length in numerous fundamental monographs (Fishburn, 1964, 1970; Krantz et al., 1971; Keeney and Raiffa, 1976; Roberts, 1979; Roubens and Vincke, 1985; Roy, 1996; Aleskerov and

Monjardet, 2002). Without going into any particular detail, however, again we only provide a rather informal discussion of some of these general concepts and then point to some more specific approaches that explicitly use domination structures to model preferences in the context of multiobjective programming.

In the previous section, we describe how to use a parametrized scalarization function to generate a representation of the nondominated set by varying the associated scalarization parameters and solving the resulting collection of single objective programs. These parameters are usually weights, reference points, or bounds on objectives which, as indicated in several results in Section 2.2.3, also carry an inherent meaning of tradeoffs or relative importances between the different criteria. Hence, the maybe most elegant approach to model the decision maker's preferences is in terms of a real-valued function $s : Y \to \mathbb{R}$ so that

$$y^1 \prec y^2 \text{ if and only if } s(y^1) < s(y^2) \text{ for all } y^1, y^2 \in Y \tag{2.62}$$

For deterministic choice problems, such a function is usually called a *value function*, whereas the term *utility function* is commonly used for decision problems that also involve risks and uncertainties. Consequently, the theory that is concerned with this approach is typically known as value and utility theory (Fishburn, 1964, 1970; Keeney and Raiffa, 1976) and due to its significant interdisciplinary impacts still a very active field of study (compare part IV in Figueira et al., 2005).

The formulation of a value or utility function has the main advantage that it enables the direct assessment of the complete preference structure of the decision maker and thus does not require the aforementioned distinction between optimization and decision making. However, it is also clear that the construction of such a function is usually quite difficult if not impossible, as the decision maker is usually not capable to precisely specify all his preferences in terms of the required scalarization parameters. Therefore, another approach is to merely assume some more generic preference assumptions to yield a set of potential candidates from among which the decision maker may choose based on additional preferences revealed during the decision process or further criteria not included in the original optimization model. Many of these assumptions have their origin in the study of social wealth, economic welfare, and the distribution of incomes, and we restrict our discussion to only three of the most common principles with an immediate relevance to multiobjective programming and some later aspects in this text.

Already highlighted in Definition 2.2.12, the traditional and still most common solution concept in multiobjective programming is that of Pareto efficiency which stems from the economic *Edgeworth-Pareto Principle*

$$y^1 \prec y^2 \text{ if } y_i^1 < y_i^2 \text{ for some } 1 \leq i \leq m \text{ and } y_j^1 \leq y_j^2 \text{ for all } 1 \leq j \leq m \tag{2.63}$$

that is independently introduced into the economic literature by Edgeworth (1881) and Pareto (1896). The predominant use of this principle indicates its wide acceptance and validity for multiobjective programming, optimization, and decision making, although it usually requires further refinement for the latter as it usually does not imply a unique preferred decision but only a set of efficient decisions or nondominated outcomes. In particular, since this principle equates overall improvement with improvement in any individual component, it does not model any tradeoff between objectives, say that a relatively large reduction in one objective may be able to compensate a small increase in some other objective component.

One possible refinement of the Edgeworth-Pareto Principle is commonly known as the *Principle of Transfers* and independently formulated by Pigou (1912, 1932) and Dalton (1920). This principle states that given an outcome y with $y_i < y_j$ for two different indices i and j, then

$$y + \delta e_i - \delta e_j \prec y \text{ for all } 0 < \delta < y_j - y_i \tag{2.64}$$

with the interpretation that any outcome with two unequal objective components can be improved by transferring a certain amount from the larger to the smaller objective so to reduce the difference between these two components. In particular, if all objective components are identical, then further improvement with respect to this principle is not possible so that we usually combine the Edgeworth-Pareto Principle with the Principle of Transfers. Recently, this approach is also studied in the context of multiobjective programming as the concept of equitable efficiency (Kostreva and Ogryczak, 1999; Kostreva et al., 2004; Baatar and Wiecek, 2006).

Very similar to the max-norm scalarization, Rawls (1958, 1971) essentially formulates the same idea as a *Principle of Justice*

$$y^1 \prec y^2 \text{ if } \max\{y_i^1 : 1 \leq i \leq m\} < \max\{y_i^2 : 1 \leq i \leq m\} \tag{2.65}$$

so that an outcome y^1 is preferred over y^2 as long as the maximal component of y^1 is smaller than the maximal component of y^2. The partial order associated with this principle is also called the *max-order* and can further be modified to the *lexicographic order* introduced by Debreu (1954) and Georgescu-Roegen (1954)

$$\begin{aligned} y^1 \prec y^2 \text{ if and only if } y_i^1 < y_i^2 &\text{ for some } 1 \leq i \leq m \\ \text{and } y_j^1 = y_j^2 &\text{ for all } 1 \leq j \leq i-1 \end{aligned} \tag{2.66}$$

which, in fact, establishes a total order on the set of outcomes. While preference with respect to the max-order is defined based on the smaller of the maximum components, for the lexicographic order we usually assume that all objectives are ranked according to their relative importances so that we can identify a preferred outcome $\hat{y}$ based on a sequential elimination of outcomes using *lexicographic optimization*

$$\hat{y}_i := \min\{y_i : y_j = \hat{y}_j \text{ for all } 1 \leq j \leq i-1 \text{ and } y \in Y\} \text{ for all } i = 1, \ldots, m \tag{2.67}$$

In particular, similar to the approach using a value or utility function, a lexicographic order gives rise to a unique preferred outcome, and some of the relationships between utility theory and lexicographic orders are discussed in Fishburn (1974). Recently, Ehrgott (2005) also combines lexicographic optimization with the max-order to define a lexicographic max-ordering approach for multiobjective programming, and many other variants of objective ranking and elimination methods exist (compare part III in Figueira et al., 2005) but are not needed in the context of this text.

The existing literature on other preference models in multiple criteria decision making is extensive and also includes many recent developments for more general decision rules (Greco et al., 2000; Fortemps et al., 2004), rough sets (Greco et al., 1999; Fortemps et al., 2004), and fuzzy measures (Słowiński, 1998; Greco et al., 1999; Fodor et al., 2000; Grabisch and Labreuche, 2005). Nevertheless, we also omit the exposition of any of these approaches as they are of no immediate relevance to our discussion. Instead, we now briefly highlight the few existing works that model preferences based on the concept of domination or domination structures in an interplay of multiple criteria decision making with multiobjective programming.

Following his original definition of a domination structure and the closer investigation of the special case of cones, Yu (1973, 1974, 1975) also describes some specific applications of either completely or partially known domination cones for decision making under certainty and uncertainty, respectively. In further generalization, Bergstresser et al. (1976) investigate the consequences of extending domination structures from convex cones to convex sets and derive conditions that require the underlying domination structure be constant, and Lin (1976) compares these new concepts to traditional Pareto optimality and shows how domination structures can also be used to generalize the notion of utility functions. The relevance of domination structures for solution concepts in multicriteria games is examined in Bergstresser and Yu (1977), and Tanino et al. (1980) again study domination cones for preferences in the context of cardinal utility theory and group decision making.

In the general context of fuzzy set theory, Yu and Leitmann (1977) discuss relationships between domination and confidence structures for Bayesian decision making, whereas the new notions of fuzzy cones and fuzzy dual cones are introduced by Takeda and Nishida (1980) to model preferences that are not known precisely but only fuzzily determined. Also for the case of unknown preferences, Ramesh et al. (1988, 1989) and independently Köksalan (1989) extend an earlier deterministic method (Korhonen et al., 1984; Köksalan et al., 1988) that uses pairwise comparisons and sequential elimination to construct a convex domination cone that describes the underlying preference structure of the decision maker. Several other modifications to this approach are later proposed by Prasad et al. (1997) to improve its convergence and further reduce some of the required preference information that need to be obtained from or specified by the decision maker.

Based on the notion of tradeoffs between and relative importances of the different objectives, Noghin (1997) proposes the use of convex polyhedral cones to refine the traditional concept of Pareto optimality, and following his work Hunt and Wiecek (2003) and Hunt (2004) formulate and study two general classes of convex polyhedral cones to model both preferences and tradeoffs as specified by the decision maker. Also within the more recent literature, Wu (2004) again uses convex cones to describe preferences for fuzzy multiobjective programs, and Yun et al. (2004) finally suggest a new model that incorporates various preference structures of decision makers in the context of data envelopment analysis.

In the majority of these reviewed papers that use the concept of domination structures for modeling preferences of the decision maker, we find that the chosen model is described by a constant domination set and typically by a constant convex cone. Based on the results in Section 2.1.3, however, this implies that the set of dominated directions is identical for every outcome which usually imposes quite severe restrictions on the underlying preference assumptions. In particular, the principles of both transfers (2.64) and justice (2.65) cannot be modeled by a constant cone (Baatar and Wiecek, 2006), and the occurrence of variable or changing preferences is long recognized also in many other economic choice and practical decision making situations (Basmann, 1956; Hammond, 1976; Grout, 1982; Sprumont, 1996; Gul and Pesendorfer, 2005, among others).

To our best knowledge, however, we are aware of only two current papers that explicitly address variable domination structures for multiobjective programming, namely in the context of nonlinear scalarization for multicriteria decision making problems and variational inequalities (Chen and Yang, 2002; Chen et al., 2005a). These papers, however, do not discuss the possible role of domination structures for the modeling of preferences, and we also do not know of any other paper that can directly be related to preference modeling using a variable domination structure. Hence, this situation motivates one of our own particular research objectives that is further addressed at the end of this chapter in Section 2.4.

2.3.3 Interactive and Decomposition Methods

We emphasize in the earlier sections that the ultimate task in solving a multiple criteria decision making problem is the selection of a final alternative which usually depends on the subjective evaluation and preferences of a decision maker. We also point out that this selection can be quite difficult if this preference information is not articulated or described in a quantitative manner, possibly hindered by too many decisions to choose from or too many objectives to simultaneously take into account. With reference to various parts of the decision making and engineering literature, we now give a brief overview of two different approaches that are suggested to facilitate the articulation of preferences and enable the consideration of multiple criteria using interactive and decomposition methods, respectively.

Based on the presentation in Miettinen (1999, 2002) and Korhonen (2005), we generally distinguish the following four cases for the collection of preferences from the decision maker. First, several methods exist that do no depend on any subjective input but find a final decision based on a completely determined set of decision rules or a globally valid optimality concept. In fact, some of these methods are used quite commonly and include the important methods of compromise programming (Yu, 1972; Zeleny, 1973) to approximate the ideal point or the approaches based on max-norms and max-orders (Ehrgott, 2005) which we present in Section 2.3.2. While not of primary interest for practical decision making, however, these methods are particularly useful for implementation into computer software and automated decision making when preferences are not available or completely unknown.

On the other hand, if the decision maker is fully knowledgeable about all his preferences, then we can formulate an associated value function and directly find the preferred decision without further decision making but based on the optimal solution of the corresponding single objective program. Since, in this case, the complete preference information is revealed before we actually generate any efficient decision, these methods are also known as *a priori methods*. Alternatively, although based on the solution of multiple optimization problems, this class also includes several outranking methods such as lexicographic max-ordering or lexicographic optimization

(Ehrgott, 2005) together with approaches based on the analytic hierarchy process (Saaty, 1980) or goal programming (Charnes and Cooper, 1977; Ignizio, 1983; Lee and Olson, 1999). Clearly, these methods are convenient and computationally efficient because we do not need to generate any decision that is not preferred by the decision maker, but usually not very useful in practice if preferences are partially unknown or not readily articulated by the decision maker.

Hence, a more common approach is to merely assume a set of general preference principles, model the underlying preference or domination structure, and then find a finite representation of the corresponding efficient or nondominated set for further consideration by the decision maker, essentially following our discussion in Section 2.3.1. In these cases, we usually assume that the decision maker is capable to further specify his preferences after the preliminary optimization stage based on pairwise comparisons, ranking procedures, or suitable visualization techniques, provided their practicality for a reasonable number of generated alternatives. Consequently, since the decision making stage now succeeds the optimization, in distinction to the former we also call these approaches *a posteriori methods*. These methods, however, often share the common drawback that many solutions may be generated that eventually are of no real interest to the decision maker.

Several procedures that are based on a progressive articulation of preferences attempt to overcome the drawbacks of a priori and a posteriori methods by gathering partial preference information already during the optimization stage. In these *interactive methods*, only one or very few initial solutions are generated and, possibly together with additional information such as their associated tradeoffs, presented to the decision maker for his subsequent consideration. Since the number of solutions is kept sufficiently small, we usually assume that the decision maker is able to evaluate and select the most preferred among these solutions and possibly specify further preference information revealed during this decision process. Based on the current and past choices of the decision maker, we then generate a new set of solutions and iterate the procedure until the decision maker finds a satisfying solution or decides to otherwise terminate the overall decision making process.

One of the major advantages of this interactive decision making scheme is that the decision maker does not need to specify all preferences a priori, but can still influence the generation of efficient decisions and to a certain degree explore the nondominated set by progressively revealing preferences after learning more about the proposed alternatives and the overall problem in general. Some of the classic and best known interactive methods include the Geoffrion-Dyer-Feinberg or GDF method by Geoffrion et al. (1972), the interactive surrogate worth tradeoff method by Chankong and Haimes (1978, 1983), the methods by Zionts and Wallenius (1983) for linear problems and by Steuer and Choo (1983) and Steuer (1986) who make prominent use of the weighted-Chebyshev norm. Jaszkiewicz and Słowiński (1995) formulate an approach based on the analogy of a light beam search, and the methods by Narula et al. (1994) and Wierzbicki (1999) also make use of reference directions and points, respectively. Highlighting the proximal-bundle method originally introduced by Miettinen and Mäkelä (1995), most recently Hakanen (2006) also reviews several other interactive methods (see Miettinen, 1999, 2002; Korhonen, 2005, for many more) and examines their potential for real-life applications with a particular focus on chemical process design.

Hence, we summarize that the main idea of interactive methods is to organize the complete decision making process as a sequence of more manageable selection problems with only a small number of possible alternatives to support decision makers in both learning about and articulation of their individual preferences when initially unknown or only difficult to reveal. Instead of reducing the number of alternatives, however, we can also choose to restrict the number of objectives that we consider simultaneously by adopting a suitable decomposition technique. While many different decomposition approaches exist, the common underlying idea again is to simplify an original and relatively difficult problem as a collection of auxiliary and comparably more tractable subproblems that can be solved separately and properly coordinated so so obtain an optimal solution for the original problem.

Besides the conceptual simplification of the original problem and its overall reduction in dimensionality, Michelena and Papalambros (1997) also enumerate several other reasons for decomposing a general problem which primarily include various computational benefits in terms of both reliability and algorithm speed, the enhancement of parallel and distributed computing, reduced programming and debugging effort, and the possibility of employing different solution techniques for the different decomposed subproblems. To review some of the existing methods and address their relevance to decision problems with multiple criteria, we now span a wide variety of decomposition techniques from mathematical and multiobjective programming as well as design optimization and systems engineering. Since the literature for especially the latter is extremely vast, however, we only highlight some of the methods of most relevance to our own work and, at the same time, point to a few selected papers that emphasize the general versatility of decomposition approaches in multiobjective programming, decision making, and engineering design.

In the context of single objective mathematical programming, most decomposition approaches are based on the early results by Dantzig and Wolfe (1961) for linear and Benders (1962) for mixed-variable programs and utilize a special structure of the underlying decision set to decompose the problem with respect to its decision variables or constraints (Conejo et al., 2006). While Burkard et al. (1985) provide a fundamental exposition of general decomposition principles which also apply to multiobjective programs and decision problems (Bogetoft and Tind, 1989), the specific cases of Dantzig-Wolfe and Benders decomposition are further extended to multiobjective programs with block diagonal structure (Yang et al., 1988) and cone constraints (Csergőffy, 2001, among others). For decision making in a fuzzy environment (Bellman and Zadeh, 1970), these two methods are also combined with interactive decision making procedures (Sakawa and Kato, 1997, 1998), genetic algorithms (Kato and Sakawa, 1997, 1998), and multilevel optimization (see Sakawa, 2002, and references therein). Finally, a more general decomposition technique based on a hybrid method that combines the principle of optimality for dynamic programming with fuzzy set theory is discussed in Hussein and Abo-Sinna (1993).

While the above mentioned approaches are motivated from single objective programming for which decompositions are naturally restricted to the decision space, for a multiobjective program we can also decompose its outcome and preference space and, additionally, an associated parameter space if we solve the problem using a parametrized scalarization function. For the latter, Zeleny (1974) studies multiparametric decompositions and develops a weight decomposition method for linear multiobjective programs which Solanki and Cohon (1989) combine with the noninferior set estimation method proposed by Cohon (1978). Parameter decompositions for generalized Chebyshev norms are studied by Dauer and Osman (1985) for convex problems and by Kassem (1997) in the context of fuzzy multiobjective programming. Furthermore, Lam and Choo (1995) employ a linear goal programming method to determine objective weights used for a decomposition of the preference space, and Benson and Sun (2000, 2002) again consider parameter decompositions for weighted sums that result in corresponding decompositions of the outcome space.

Another classification of decomposition approaches for multiobjective programs distinguishes multilevel or hierarchical decomposition from nonhierarchical optimization based on the coordination or linking mechanisms and coupling relationships between the different decomposed subproblems (for an overview, see Lieberman, 1992). In hierarchical coordination approaches, all subproblems are solved independently as separate optimization problems and then communicate their solutions to a coordinating master problem that achieves an overall optimal solution by determination of linking or coupling variables (Mesarović et al., 1970a,b). In nonhierarchical methods, two or more subproblems typically coordinate with each other by bidirectional interventions but without a superior master problem (Shankar et al., 1993), and, in addition, several combinations of these two approaches exist and include the hierarchical overlapping coordination by Macko and Haimes (1978) and its later modification proposed by Shima (1991).

In the context of multiple criteria decision making, Lazimy (1986) proposes an interactive decomposition scheme that decomposes the original problem into a collection of biobjective subproblems that are coordinated by a standard linear program as master problem to facilitate preference assessments and tradeoffs. In a fundamental research monograph, Haimes et al. (1990) describe and examine a variety of different hierarchical decomposition and coordination strategies for multi-objective programs and large-scale systems. As one possible application, they focus on organizational decompositions that involve multiple decision makers for which different subproblems model the individual subunits with their private objectives, while the coordination of the overall solution is assigned to a superior unit which harmonizes the behavior of the subordinate subsystems. Based on its practical relevance for many real organizations with departmental decision making hierarchies and various other large-scale systems in many different disciplines, much attention is dedicated to these particular approaches (Findeisen et al., 1980; Onana, 1989; Haimes et al., 1990; Gómez et al., 2001; Caballero et al., 2002).

In addition, a countless number of ad-hoc decomposition strategies is proposed for specific applications in management, including resource allocation (Lee and Rho, 1985), transportation (Korchemkin, 1986), or plant location (Fernández and Puerto, 2003), as well as in engineering, for example, machine tool spindle systems (Montusiewicz and Osyczka, 1990), proprotor aerodynamics (Tadghighi, 1998), or wing design (Wrenn and Dovi, 1988; Tribes et al., 2005). Other decomposition approaches exist in the context of structural and multidisciplinary optimization (Sobieszczanski-Sobieski et al., 1985; Sobieszczanski-Sobieski and Kodiyalam, 2001; Blouin et al., 2004; Mehr and Tumer, 2006) and include traditional collaborative optimization (Kroo, 1996; Tappeta and Renaud, 1997; Chen et al., 2005b; Haftka and Watson, 2006; Rabeau et al., 2006) or, more recently, concurrent subspace optimization (Huang and Bloebaum, 2004) and analytical target cascading (Lassiter et al., 2005). For problems in engineering design, a model-based decomposition is proposed in distinction to object, aspect, and sequential decomposition (Michelena and Papalambros, 1997; Wagner and Papalambros, 1999) and uses a decomposition analysis to find an optimal decomposition based on sets of criteria that can be computed in parallel or need to be evaluated in sequence (Yoshimura et al., 2003).

In the same context, we also find some decomposition methods for which the subsequent coordination is accomplished in the form of an interactive decision making procedure (Tappeta and Renaud, 1999; Tappeta et al., 2000; Azarm and

Narayanan, 2000; Verma et al., 2005) or enhanced by providing preference information such as tradeoffs (Kasprzak and Lewis, 2000) obtained from a sensitivity analysis (Gunawan and Azarm, 2005; Li et al., 2005). Furthermore, several related studies emphasize the importance of visualizing the optimization process (Messac and Chen, 2000), the design data (Eddy and Lewis, 2002; Stump et al., 2002, 2003, 2004) and the Pareto frontiers (Agrawal et al., 2004, 2005; Lotov et al., 2004, 2005; Mattson and Messac, 2005) for facilitated problem perception and decision making. Three surveys on general coordination approaches are given by Coates et al. (2000), Whitfield et al. (2000), and with a special focus on decentralized design by Whitfield et al. (2002). In case of the latter, the issue of convergence is addressed in Chanron and Lewis (2005) and Chanron et al. (2005) who use game theoretic concepts to model and analyze the competing interest of the different decision makers.

For many interactive and decomposition methods, however, convergence is frequently not addressed among the major topics of interest as especially the former depend on subjective input by the decision maker who might also change his mind and adjust his preferences throughout the overall decision making process. Hence, since the final preferred decision is in large part problem dependent, the only conditions that are usually imposed on any decision making procedure is that every *proposed* solution is in fact *efficient* for the overall problem and that, in principle, every *efficient* decision can be *found* by suitable coordination and based on the input received by the decision maker. These two requirements stimulate our later investigation of objective decompositions for multiobjective programs and the corresponding research objective now outlined in the concluding section of this chapter.

2.4 Research Statement and Objectives

After the acquisition of some fundamentals from multiobjective programming, the discussion of its relevance for practical decision making, and a broad review of several ideas within the mathematical, economic, and engineering literature, we can draw the first conclusion that in spite of numerous advances in all these fields the selection of preferred alternatives for decision problems with multiple criteria remains subject to various persistent challenges. As we all know from our personal experience and exposure to the real world, making an ultimate decision among several possibilities is still a very demanding task and often complicated by the fact that we are not fully aware of our underlying preferences for a too large number of competing objectives to be considered simultaneously. We develop our research objectives from these two observations and their extensive motivation in our preceding critique of multiple criteria decision making in Section 2.3.

In this dissertation, we devote one chapter each to our ideas in theory, methodology, and selected applications of multiobjective programming. Whenever suitable, we intend to first formulate a mathematical framework that provides a rigorous foundation for our approaches and then use or combine concepts and results from convex analysis, linear algebra, and mathematical programming to derive our own and original contributions. Throughout the text, we also offer some additional discussion and remarks to draw the connections of our results to the aforementioned challenges in decision making, which we now highlight in further detail to prepare the precise statement of the research questions that we subsequently plan to address.

Because preferences, in general, are not precisely known, there is a great need for the formulation of simplified yet realistic preference models which enable the reduction of an initial set of alternatives to a smaller subset of potential candidates of particular interest to the decision maker. Based on our discussion in Section 2.2.2, the economic literature provides a series of possible preference principles in extension of the classical Pareto concept that can be used to induce new domination structures in further refinement of the traditional Pareto cones. Related approaches in the mathematical literature, however, are currently restricted to the use of constant convex cones and, in particular, polyhedral cones and, therefore, motivate our investigation of several new domination structures pursued in Chapter 3.

In particular, our objective in Section 3.1 is the general exploration of nonpolyhedral cones in multiobjective programming with respect to possible cone representations and properties as well as the characterization and generation of the corresponding nondominated sets. In further extension of constant nonpolyhedral cones, in Section 3.2 we first study some of the inherent shortcomings of current models that are based on domination cones that are constant, set out to examine the implications of replacing constant by variable cones, and then derive one specific variable domination cone in remedy of the previously recognized model limitations. Finally, and based on the high relevance of approximations, our focus in Section 3.3 is on the investigation of domination structures that can be used to also describe a notion of approximate nondominance and the development of a wide variety of approaches for the computational approximation of both the resulting efficient and nondominated sets.

We anticipate that the accomplishment of these goals can successfully enhance current decision making by providing new and more flexible means to model preferences and finding more relevant representations or approximations for those outcomes of actual interest to the decision maker. Furthermore, and as discussed in Section 2.2.3, the selection of the final decision can then be achieved by any suitable enumeration, exploration, or visualization technique in possible combination with an interactive or decomposition-based decision making procedure to further reduce burden associated with a large number of decision criteria. Our inquiry in Chapter 4 is placed within this general framework.

In particular, our objective in Section 4.1 is to examine the consequences of decomposing a multiobjective program with respect to its objective function and the analysis of the corresponding efficiency relationships between the original and resulting decomposed subproblems. In Section 4.2, we investigate how to use our obtained results for the formulation of different interactive decision making procedures that are proposed to support decision makers in finding preferred solutions to potentially large-scale problems by merely solving the collection of smaller-sized subproblems. Finally, we assess the practicality of our methods on an example in Section 4.3 and several real-life applications in Chapter 5.

In the concluding Chapter 6 we view our objectives in retrospect.

CHAPTER 3

DOMINATION CONES AND THE NONDOMINATED SET

In this chapter, we study the concept of domination in multiobjective programming and focus on the characterization and generation of the nondominated set when the underlying domination structure is defined in terms of different cones. In particular, in each of the three following sections we investigate one particular generalization of polyhedral and constant domination cones, namely to nonpolyhedral cones in Section 3.1, to variable cones in Section 3.2, and to domination sets that are translated cones in Section 3.3. The material in the first two sections is adapted and in parts amended from the similar presentation in Engau and Wiecek (2006) and Engau (2006), respectively, and in the third section we combine several aspects of approximate nondominance that are initially addressed in Engau (2004) and further developed in Engau and Wiecek (2007b,c,d). At the end of this chapter, we offer a summarizing discussion of our findings together with an outlook on possible further work in the concluding Section 3.4.

3.1 Constant Polyhedral and Nonpolyhedral Cones

Based on the fundamental observation in Section 3.1.1 that every cone can be described by a positively homogeneous function, we derive several results that characterize properties of cones based on the corresponding properties of the underlying function that can be sublinear or superlinear. We then show in Section 3.1.2 how this new cone representation can be used to establish relationships between nondominated points with respect to a general cone and Pareto points, in generalization of a well known result for the polyhedral case. In Section 3.1.3, we examine possible modifications of several scalarization methods that are originally formulated for finding Pareto points to also generate nondominated points for a general polyhedral or nonpolyhedral cone, and we illustrate our findings on three biobjective examples in Section 3.1.4. The last Section 3.1.5 establishes some further results for an alternative cone representation that is formulated as a special case of our previous characterization and, in this vein, also provides examples for multiobjective programs with more than two objectives.

3.1.1 General Cone Representation and Characterization

We begin with the fundamental theorem in this section and establish the relationship between cones and positively homogeneous functions.

Theorem 3.1.1 (Cone representation theorem)**.** *A set $C \subset \mathbb{R}^m$ is a cone if and only if there exists a positively homogeneous function $\Gamma : \mathbb{R}^m \to \mathbb{R}^l$ such that*

$$C \cup \{0\} = C(\Gamma) := \{c \in \mathbb{R}^m : \Gamma(c) \geqq 0\} \tag{3.1}$$

Proof. We first derive that $C(\Gamma)$ is indeed a cone, and then we show how every cone that contains the origin can be described by a positively homogeneous function.

($\Leftarrow$) Let $\Gamma : \mathbb{R}^m \to \mathbb{R}^l$ be a positively homogeneous function and $c \in C(\Gamma)$. It follows that $\Gamma(c) \geqq 0$ by definition of $C(\Gamma)$ and $\lambda\Gamma(c) = \Gamma(\lambda c) \geqq 0$ for all $\lambda > 0$ because Γ is positively homogeneous. Hence, we obtain that $\lambda c \in C(\Gamma)$, showing that $C(\Gamma)$ is a cone.

($\Rightarrow$) Let $C \subset \mathbb{R}^m$ be a cone and define $\Gamma = (\Gamma_1, \ldots, \Gamma_m) : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\Gamma_i(c) := \begin{cases} |c_i| & \text{if } c \in C \\ -|c_i| & \text{if } c \notin C \end{cases} \tag{3.2a}$$

for all $i = 1, \ldots, m$. The function Γ is positively homogeneous, because

$$\Gamma_i(\lambda c) = \begin{cases} |\lambda c_i| = \lambda|c_i| = \lambda\Gamma_i(c) & \text{if } c \in C \\ -|\lambda c_i| = -\lambda|c_i| = \lambda\Gamma_i(c) & \text{if } c \notin C \end{cases} \tag{3.2b}$$

and thus $\Gamma(\lambda c) = \lambda\Gamma(c)$ for all $\lambda > 0$. In particular, $\Gamma(c) \geqq 0$ if and only if $c \in C \cup \{0\}$, and thus, $C(\Gamma) = C \cup \{0\}$. □

After introducing the notion of a cone in Definition 2.1.9, we indicate in Remarks 2.1.10 and 2.1.20 why we do not require that a cone necessarily contains the origin. As a consequence of this decision, however, we now need to explicitly add the origin to every cone in Theorem 3.1.1 because, for any positively homogeneous function Γ, it follows that

$$\Gamma(0) = \Gamma(\lambda \cdot 0) = \lambda\Gamma(0) \text{ for all } \lambda > 0 \tag{3.3}$$

and thus $\Gamma(0) = 0$. While a different cone definition would result in a somewhat more elegant cone representation theorem, however, this distinction is essentially irrelevant for all our remaining discussion.

Definition 3.1.2 (Cone induced by a positively homogeneous function)**.** Let $\Gamma : \mathbb{R}^m \to \mathbb{R}^l$ be a positively homogeneous function. The set

$$C(\Gamma) := \{c \in \mathbb{R}^m : \Gamma(c) \geqq 0\} \tag{3.4}$$

is called the *cone induced by* Γ.

Remark 3.1.3 (Nonuniqueness of cone representation)**.** If the cone $C = C(\Gamma)$ is induced by a positively homogeneous function Γ, then $C = C(\lambda\Gamma)$ for all $\lambda > 0$.

Following Definition 3.1.2, we highlight four specific classes of cones that we study in further detail throughout this complete chapter. While the first two are also introduced in (2.33a) and (2.25b) of Definition 2.1.25, respectively, for completeness and further comparison we again repeat their definition.

Definition 3.1.4 (Pareto cone)**.** Let $\Gamma : \mathbb{R}^m \to \mathbb{R}^m$ be the identity $\Gamma(c) = c$. The induced cone $C(\Gamma)$ is denoted by

$$\mathbb{R}^m_{\geqq} := \{c \in \mathbb{R}^m : c \geqq 0\} \tag{3.5}$$

and called the m-dimensional *Pareto cone*.

Definition 3.1.5 (Polyhedral cone)**.** Let $A \in \mathbb{R}^{l\times m}$ be a matrix and $\Gamma : \mathbb{R}^m \to \mathbb{R}^l$ be the linear function $\Gamma(c) = Ac$. The induced cone $C(\Gamma)$ is denoted by

$$C(A) := \{c \in \mathbb{R}^m : Ac \geqq 0\} \tag{3.6}$$

and called the *polyhedral cone* $C(A)$ induced by A.

In particular, for $A = I^m \in \mathbb{R}^{m\times m}$ the m-dimensional identity matrix, Definition 3.1.5 reduces to Definition 3.1.4 and $C(I^m) = \mathbb{R}^m_{\geqq}$ is the Pareto cone.

Definition 3.1.6 (Pth-order cone)**.** Let $\Gamma : \mathbb{R}^m \to \mathbb{R}$ be defined by $\Gamma(c) = c_1 - \|c_{-1}\|_p$ where $c_{-1} := (c_2, \dots, c_m)^T \in \mathbb{R}^{m-1}$. The induced cone $C(\Gamma)$ is denoted by

$$C_p^m := \{c \in \mathbb{R}^m : c_1 \geq \|c_{-1}\|_p\} \tag{3.7}$$

and called the m-dimensional *pth-order cone*.

For $p = 2$, Stern and Wolkowicz (1991) refer to the second-order cone C_2^m as *ice-cream cone*, and Ben-Tal and Nemirovski (2001) call C_2^m a *Lorentz cone* and investigate its approximation by cones that are polyhedral.

The next cone is originally introduced by Bishop and Phelps (1963) and Phelps (1974) and studied by Bednarczuk (1996) in the context of vector optimization in more general topological spaces.

Definition 3.1.7 (Bishop-Phelps cone)**.** Let $d \in \mathbb{R}^m \setminus \{0\}$ be a nonnegative vector, $0 < \gamma < 1$ be a real number, and $\Gamma : \mathbb{R}^m \to \mathbb{R}$ be defined by $\Gamma(c) = \langle c, d\rangle - \gamma\|c\|\|d\|$. The induced cone $C(\Gamma)$ is denoted by

$$C_{\gamma,d} := \{c \in \mathbb{R}^m : \langle c, d\rangle \geq \gamma\|c\|\|d\|\} \tag{3.8}$$

and called the *Bishop-Phelps cone* over d with parameter γ.

In the following results, we first establish sufficient conditions for convexity and pointedness of $C(\Gamma)$ and, since the pointedness conditions is not satisfied for C_p^m and $C_{\gamma,d}$, we then show separately that both pth-order and Bishop-Phelps cones are also convex and pointed.

Proposition 3.1.8 (Sufficient conditions for convexity and pointedness)**.** *Let* $\Gamma : \mathbb{R}^m \to \mathbb{R}^l$ *be a positively homogeneous function and* $C(\Gamma) \subset \mathbb{R}^m$ *be the cone induced by* Γ*. Then*

(i) $C(\Gamma)$ *is convex if* Γ *is superlinear, and*

(ii) $C(\Gamma)$ *is pointed if* Γ *is superlinear and* $\Gamma(c) = 0 \Leftrightarrow c = 0$.

Proof. Let $\Gamma : \mathbb{R}^m \to \mathbb{R}^l$ be a positively homogeneous function and $C(\Gamma) \subset \mathbb{R}^m$ be its induced cone.

For (i), let $c^1 \in C(\Gamma)$ and $c^2 \in C(\Gamma)$, so $\Gamma(c^1) \geqq 0$ and $\Gamma(c^2) \geqq 0$. Since Γ is superlinear, it follows that

$$\Gamma(c^1 + c^2) \geqq \Gamma(c^1) + \Gamma(c^2) \geqq 0 \tag{3.9a}$$

and thus $c^1 + c^2 \in C(\Gamma)$, showing that $C(\Gamma)$ is convex.

For (ii), let $\Gamma(c) = 0 \Leftrightarrow c = 0$ and $\sum_{i=1}^k c^i = 0$ with $c^i \in C$, so $\Gamma(c^i) \geqq 0$ for all $i = 1, \ldots, k$. Since Γ is superlinear, it follows that

$$0 = \Gamma(0) = \Gamma\left(\sum_{i=1}^k c^i\right) \geqq \sum_{i=1}^k \Gamma(c^i) \geqq 0 \tag{3.9b}$$

so $\Gamma(c^i) = 0$ and all $c^i = 0$, showing that $C(\Gamma)$ is pointed. □

Remark 3.1.9 (Equivalent pointedness condition for polyhedral cones)**.** The condition $\Gamma(c) = 0 \Leftrightarrow c = 0$ is always satisfied if the function Γ is injective. If Γ is linear and thus described by a matrix, then these conditions are also equivalent and Proposition 3.1.8 reduces to the well known characterization of pointed polyhedral cones in Proposition 2.1.28.

Hence, it again follows that all polyhedral cones $C(A)$ are convex, and pointed if and only if $\operatorname{rank} A = m$, or equivalently, if the linear function $\Gamma(c) = Ac$ is injective. In particular, we obtain that the Pareto cone $\mathbb{R}^m_{\geqq}$ is convex and pointed, and the same holds true for all pth-order and Bishop-Phelps cones. In these later cases, however, only convexity follows immediately from Proposition 3.1.8 because both $\Gamma(c) = c_1 - \|c_{-1}\|_p$ and $\Gamma(c) = \langle c, d\rangle - \gamma\|c\|\|d\|$ are superlinear, whereas in general $\Gamma(c) = 0 \not\Rightarrow c = 0$. Nevertheless, since C_p^m and $C_{\gamma,d}$ are convex, we can easily show their pointedness using Proposition 2.1.13.

Proposition 3.1.10 (Convexity and pointedness of pth-order and Bishop-Phelps cones)**.** *Every pth-order cone $C_p^m \subset \mathbb{R}^m$ and every Bishop-Phelps cone $C_{\gamma,d} \subset \mathbb{R}^m$ is convex and pointed.*

Proof. Based on our previous remarks, we only need to show that C_p^m and $C_{\gamma,d}$ are pointed. Hence, to show that C_p^m is pointed, let $c \in C_p^m$ and $-c \in C_p^m$, so

$$c_1 \geq \|c_{-1}\|_p \geq 0 \text{ and } -c_1 \geq \|-c_{-1}\|_p = \|c_{-1}\|_p \geq 0 \tag{3.10a}$$

It follows that $c_1 = 0$ and $c_{-1} = 0$ which implies $c = 0$, showing that C_p^m is pointed.

To show that $C_{\gamma,d}$ is pointed, let $c \in C_{\gamma,d}$ and $-c \in C_{\gamma,d}$, so

$$\langle c, d\rangle \geq \gamma\|c\|\|d\| \geq 0 \text{ and } \langle -c, d\rangle = -\langle c, d\rangle \geq \gamma\|c\|\|d\| \geq 0 \tag{3.10b}$$

In this case, it follows that $\langle c, d\rangle = 0$ and, thus, $\|c\| = 0$ which implies $c = 0$, showing that $C_{\gamma,d}$ is pointed. □

Proposition 3.1.11. *Let $\Gamma : \mathbb{R}^m \to \mathbb{R}^l$ be a positively homogeneous function. If Γ is sublinear, then*

$$-C(-\Gamma) \subseteq C(\Gamma) \tag{3.11}$$

Furthermore, the above inclusion holds in reverse if Γ is superlinear, and with equality if Γ is a linear function.

Proof. Let $\Gamma : \mathbb{R}^m \to \mathbb{R}^l$ be sublinear and $c \in -C(-\Gamma)$, so $-\Gamma(-c) \geqq 0$. It follows that

$$0 = \Gamma(0) = \Gamma(c - c) \leqq \Gamma(c) + \Gamma(-c) \tag{3.12}$$

which implies $\Gamma(c) \geqq -\Gamma(-c) \geqq 0$, showing that $c \in C(\Gamma)$. If, instead, Γ is superlinear, then $-\Gamma$ is sublinear and, thus, $-C(\Gamma) \subseteq C(-\Gamma)$, or $C(\Gamma) \subseteq -C(-\Gamma)$. If Γ is linear, then Γ is both sublinear and superlinear and the result follows. □

3.1.2 Mapping Theorems for the Nondominated Set

In the following theorem, we establish the relationships between the image of the nondominated set $N(Y, C(\Gamma))$ under the mapping Γ and the Pareto set of the image $\Gamma(Y)$, based on the properties of the positively homogeneous function Γ.

Theorem 3.1.12 (Nondominance mapping theorem)**.** *Let $Y \subset \mathbb{R}^m$ be a nonempty set, $\Gamma : \mathbb{R}^m \to \mathbb{R}^l$ be a positively homogeneous function, and $C(\Gamma) \subset \mathbb{R}^m$ be the cone induced by Γ.*

(i) If Γ is sublinear, then

$$\Gamma[N(Y, C(\Gamma))] \subseteq N(\Gamma[Y], \mathbb{R}^l_{\geqq}) \tag{3.13a}$$

(ii) If Γ is superlinear and injective, then

$$\Gamma[N(Y, C(\Gamma))] \supseteq N(\Gamma[Y], \mathbb{R}^l_{\geqq}) \tag{3.13b}$$

Both inclusions become equality if Γ is linear and injective.

Proof. We only show the first two statements which then immediately imply the third. Hence, for (i), let $\hat{u} \in \Gamma[N(Y, C(\Gamma))]$, so $\hat{u} = \Gamma(\hat{y})$ with $\hat{y} \in N(Y, C(\Gamma))$. Then $Y \cap (\hat{y} - C(\Gamma) \setminus \{0\}) = \emptyset$, and hence, there does not exist $y \in Y \setminus \{\hat{y}\}$ such that $\hat{y} - y \in C(\Gamma)$, or $\Gamma(\hat{y} - y) \geqq 0$. Now suppose by contradiction that $\hat{u} \notin N(\Gamma[Y], \mathbb{R}^l_{\geqq})$, then there exists $u = \Gamma(y)$, $u \neq \hat{u}$, so $y \neq \hat{y}$, with $u \in \Gamma(Y) \cap (\hat{u} - \mathbb{R}^l_{\geqq})$, thus $\hat{u} - u \in \mathbb{R}^l_{\geqq}$, or $\Gamma(\hat{y}) - \Gamma(y) \geqq 0$. However, by sublinearity of Γ,

$$\Gamma(\hat{y} - y) + \Gamma(y) \geqq \Gamma(\hat{y}) \Longrightarrow \Gamma(\hat{y} - y) \geqq \Gamma(\hat{y}) - \Gamma(y) \geqq 0 \tag{3.14a}$$

in contradiction to the non-existence of $y \in Y \setminus \{\hat{y}\}$ with $\Gamma(\hat{y} - y) \geqq 0$.

For (ii), let $\hat{u} = \Gamma(\hat{y}) \in N(\Gamma[Y], \mathbb{R}^l_{\geqq})$. Then $\Gamma[Y] \cap (\hat{u} - \mathbb{R}^l_{\geqq} \setminus \{0\}) = \emptyset$, and hence, there does not exist $u \in \Gamma[Y] \setminus \{\hat{u}\}$ such that $\hat{u} - u \in \mathbb{R}^l_{\geqq}$, or $\hat{u} - u \geqq 0$. Since Γ is injective, $\Gamma(y) \neq \Gamma(\hat{y})$ if and only if $y \neq \hat{y}$, and so there does not exist $y \in Y \setminus \{\hat{y}\}$ such that $\Gamma(\hat{y}) - \Gamma(y) \geqq 0$. Now suppose by contradiction that $\hat{u} = \Gamma(\hat{y}) \notin \Gamma[N(Y, C(\Gamma))]$, or $\hat{y} \notin N(Y, C(\Gamma))$. Then there exists $y \neq \hat{y}$ with $y \in Y \cap (\hat{y} - C(\Gamma))$, thus $\hat{y} - y \in C(\Gamma)$, or $\Gamma(\hat{y} - y) \geqq 0$. However, by superlinearity of Γ,

$$\Gamma(\hat{y}) \geqq \Gamma(\hat{y} - y) + \Gamma(y) \Longrightarrow \Gamma(\hat{y}) - \Gamma(y) \geqq \Gamma(\hat{y} - y) \geqq 0 \tag{3.14b}$$

in contradiction to the non-existence of $y \in Y \setminus \{\hat{y}\}$ with $\Gamma(\hat{y}) - \Gamma(y) \geqq 0$. □

In particular, if Γ is a linear function, then $\Gamma(c) = Ac$ for some matrix $A \in \mathbb{R}^{l \times m}$ and Theorem 3.1.12 reduces to a well known result for polyhedral cones that is widely established throughout the literature (see Yu, 1985; Sawaragi et al., 1985; Weidner, 1990; Noghin, 1997; Cambini et al., 2003; Hunt and Wiecek, 2003, among several others).

Theorem 3.1.13 (Nondominance mapping theorem for polyhedral cones)**.** *Let $Y \subset \mathbb{R}^m$ be a nonempty set, $A \in \mathbb{R}^{l\times m}$ be a real $l \times m$ matrix, and $C(A) \subset \mathbb{R}^m$ be the polyhedral cone induced by A. Then*

$$A[N(Y, C(A))] \subseteq N(A[Y], \mathbb{R}^l_{\geqq}) \tag{3.15a}$$

If $C(A)$ is pointed, or equivalently, if $\operatorname{rank} A = m$ *or $c \to Ac$ is injective, then*

$$A[N(Y, C(A))] = N(A[Y], \mathbb{R}^l_{\geqq}) \tag{3.15b}$$

Hence, the problem of finding the nondominated set of Y with respect to a pointed polyhedral cone $C(A)$ is equivalent to finding the Pareto set of $A[Y]$ which, in principle, can be accomplished using a suitable scalarization method from Section 2.2.3. In general, however, if the cone $C(\Gamma)$ is not polyhedral, then equality in Theorem 3.1.12 cannot be expected so that we need to find alternative or more direct means to generate nondominated outcomes for a general cone.

3.1.3 Scalarization Methods for General Cones

In this part, we discuss several of the scalarization methods from Section 2.2.3 that are originally formulated for finding Pareto outcomes and show how these methods can be modified to also allow for a general (polyhedral or nonpolyhedral) cone C. In particular, we first establish this generalization for the hybrid scalarization method and then derive the corresponding results for the weighted-sum, the constrained-objective, and the Benson method as special cases.

Proposition 3.1.14 (Hybrid scalarization)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a convex cone, $w \in C^* \setminus \{0\}$ (a nonzero element of the dual cone in Definition 2.1.19), $b \in \mathbb{R}^m$, and $\hat{y} \in Y$ be optimal for the hybrid scalarization*

$$\textit{HB}(w,b)\textit{: Minimize } \langle w, y\rangle \textit{ subject to } b - y \in C \textit{ and } y \in Y \tag{3.16}$$

(i) *If $\hat{y}$ is strictly optimal, then $\hat{y} \in N(Y, C)$.*

(ii) *If $w \in C_s^*$, then $\hat{y} \in N(Y, C)$.*

In any case, $\hat{y} \in N_w(Y, C)$.

Proof. Let $\hat{y} \in Y$ be optimal for $\mathrm{HB}(w, b)$ with $w \in C^* \setminus \{0\}$ and, by contradiction, assume that $\hat{y} \notin N(Y, C)$. Then there exists $y \in (\hat{y} - C \setminus \{0\}) \cap Y$, so $\hat{y} = y + c$ for some $c \in C \setminus \{0\}$. The point y is feasible for $\mathrm{HB}(w, b)$, because $b - y = b - \hat{y} + c \in C$ by feasibility of $\hat{y}$ for $\mathrm{HB}(w, b)$ and convexity of C, but

$$\langle w, \hat{y} \rangle = \langle w, y + c \rangle = \langle w, y \rangle + \langle w, c \rangle \begin{cases} \geq \langle w, y \rangle & \text{in case (i) as } w \in C^* \\ > \langle w, y \rangle & \text{in case (ii) as } w \in C_s^* \end{cases} \tag{3.17}$$

in contradiction to the (in case (i): strict) optimality of $\hat{y}$ for $\mathrm{HB}(w, b)$.

In any case, if $w \in C^* \setminus \{0\}$ and $\hat{y}$ is optimal for $\mathrm{HB}(w, b)$, then $w \in (\operatorname{int} C)_s^*$ by Proposition 2.1.21 and $\hat{y} \in N(Y, \operatorname{int} C) = N_w(Y, C)$ by (ii). □

After dropping the constraints $b - y \in C$ in the hybrid method, we obtain the corresponding result for the weighted-sum scalarization.

Proposition 3.1.15 (Weighted-sum scalarization)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$, $w \in C^* \setminus \{0\}$, and $\hat{y} \in Y$ be optimal for the weighted-sum scalarization*

$$\textit{WS(w): Minimize } \langle w, y \rangle \textit{ subject to } y \in Y \tag{3.18}$$

(i) *If $\hat{y}$ is strictly optimal, then $\hat{y} \in N(Y, C)$.*

(ii) *If $w \in C_s^*$, then $\hat{y} \in N(Y, C)$.*

In any case, $\hat{y} \in N_w(Y, C)$.

This result is similarly established in Sawaragi et al. (1985), and its proof follows as for the hybrid scalarization in Proposition 3.1.14. In particular, while the previous result requires that C is a convex cone to guarantee feasibility of the contradicting outcome y for the additional cone constraints $b - y \in C$, we drop these constraints for the weighted-sum method so that C does not need to be convex anymore and, moreover, can also be a general domination set.

For specific choices of the weighting parameter w, we can also derive the corresponding results for the constrained-objective and the Benson method as special cases, again under the assumption that C is a convex cone.

Proposition 3.1.16 (Constrained-objective scalarization)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a convex cone, $e^k \in C^*$ be the kth unit vector, $b \in \mathbb{R}^m$, and $\hat{y} \in Y$ be optimal for the constrained-objective scalarization*

$$CO_k(b)\text{: Minimize } y_k \text{ subject to } b - y \in C \text{ and } y \in Y \tag{3.19}$$

(i) If $\hat{y}$ is strictly optimal, then $\hat{y} \in N(Y, C)$.

(ii) If $e^k \in C_s^*$, then $\hat{y} \in N(Y, C)$.

In any case, $\hat{y} \in N_w(Y, C)$.

The proof is clear from Proposition 3.1.14 with $w = e^k$. In particular, we note that the conditions $e^k \in C^*$ and $e^k \in C_s^*$ are equivalent to $c_k \geq 0$ and $c_k > 0$ for all $c \in C$, respectively. Similarly, for $w = (1, \ldots, 1)^T \in C^*$, or $\sum_{i=1}^m c_i \geq 0$ for all $c \in C$, and if $b = y^\circ \in Y$ is chosen as reference point, then the hybrid method reduces to the scalarization proposed by Benson but for a general convex cone C.

Proposition 3.1.17 (Benson scalarization)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a convex cone, $w = (1, \ldots, 1)^T \in C^*$ be the vector with all components equal to 1, $y^\circ \in Y$, and $\hat{c} \in C$ be optimal for the Benson scalarization*

$$B(y^\circ)\text{: Maximize } \sum_{i=1}^m c_i \text{ subject to } y^\circ - c = y \in Y \text{ and } c \in C \tag{3.20}$$

(i) If $\hat{c}$ is strictly optimal, then $\hat{y} = y^\circ - \hat{c} \in N(Y, C)$.

(ii) If $w \in C_s^*$, then $\hat{y} = y^\circ - \hat{c} \in N(Y, C)$.

In any case, $\hat{y} = y^\circ - \hat{c} \in N_w(Y, C)$.

Proof. Since the constraints $y^\circ - c = y \in Y$ and $c \in C$ can equivalently be written as $y^\circ - y = c \in C$ and $y \in Y$, and because the maximization of

$$\sum_{i=1}^m c_i = \sum_{i=1}^m (y_i^\circ - y_i) = \sum_{i=1}^m y_i^\circ - \sum_{i=1}^m y_i \tag{3.21}$$

is equivalent to the (negative) minimization of $\sum_{i=1}^{m} y_i$, Proposition 3.1.17 follows from Proposition 3.1.14 with $w = (1, \ldots, 1)^T$. $\square$

Hence, in addition to the Pascoletti-Serafini scalarization in Definition 2.2.36, we now can also use the the hybrid, the constrained-objective, and the Benson method to generate nondominated outcomes for a general domination cone C, or the weighted-sum method for an arbitrary domination set D.

3.1.4 Graphical Examples for Biobjective Programs

We illustrate the use of the nondominance mapping theorems in Section 3.1.2 and the scalarization methods in Section 3.1.3 for finding the nondominated set of the unit disk $Y = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 1\}$ for the three choices of $C \subset \mathbb{R}^2$ depicted in Figure 3.1, namely the two-dimensional Pareto cone $C^1 = \mathbb{R}^2_{\geqq}$ (on the left), the polyhedral cone $C^2 = C(A)$ with matrix $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ (in the center), and any two-dimensional pth-order cone $C^3 = C_p^2 = \{c = (c_1, c_2) \in \mathbb{R}^2 : c_1 \geq \|c_2\|_p\}$ (on the right). Furthermore, the boundary of Y is the unit circle which we denote by $\operatorname{bd} Y = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 = 1\}$ for later convenience.

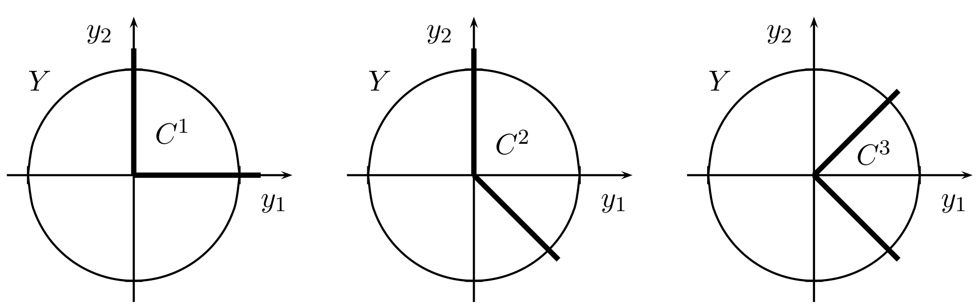


Figure 3.1 The unit disk as outcome set $Y \subset \mathbb{R}^2$ and the two-dimensional Pareto (left), polyhedral (center), and pth-order cone (right) in Examples 3.1.18–3.1.20

Example 3.1.18 (Pareto cone). Let $\Gamma^1 : \mathbb{R}^2 \to \mathbb{R}^2$ with $\Gamma^1(c) = c$ be the identity, so $C(\Gamma^1) = C(I^2) = \mathbb{R}^2_{\geqq}$ be the Pareto cone depicted in Figure 3.1 (left). Then

$$\Gamma^1[N(Y, C(\Gamma^1))] = N(Y, C(I^2)) = N(Y, \mathbb{R}^2_{\geqq}) \tag{3.22a}$$

and Theorems 3.1.12 and 3.1.13 apply trivially. In particular, in this case it immediately follows that

$$N(Y, C(\Gamma^1)) = N(Y, \mathbb{R}^2_{\geqq}) = \{y \in \operatorname{bd} Y : y \leq 0\} \tag{3.22b}$$

as shown in Figure 3.2 (left). Furthermore, since the Pareto cone is both convex and self-dual so that, in particular, $e^k \in C^* = \mathbb{R}^m_{\geqq}$ for all $k = 1, \ldots, m$ and $w = (1, \ldots, 1)^T \in \mathbb{R}^m_{\geqq}$, Propositions 3.1.14 (hybrid), 3.1.15 (weighted-sum), 3.1.16 (constrained-objective) and 3.1.17 (Benson) apply to all four scalarization methods in their original formulation for the Pareto case. For further illustration, Figure 3.3 (left) again depicts the Pareto cone together with three possible choices for the weighting vector for both the hybrid and the weighted-sum scalarization method.

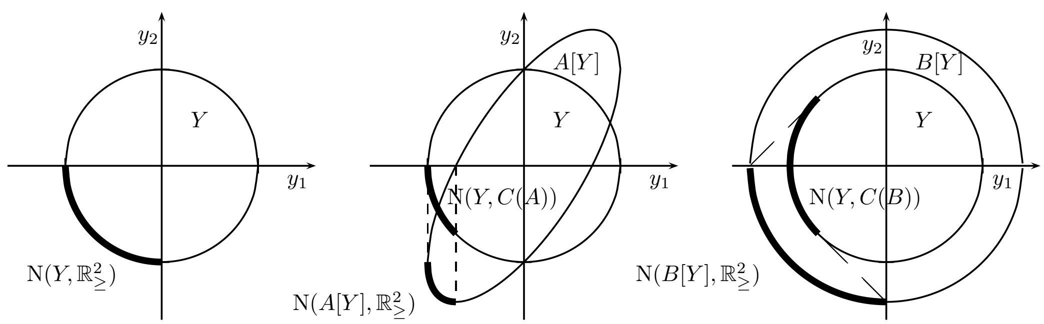


Figure 3.2 Nondominance mapping of $N(Y, C(\Gamma))$ and $N(\Gamma[Y], \mathbb{R}^2_{\geqq})$ for Pareto (left), polyhedral (center), and pth-order cone (right) in Examples 3.1.18–3.1.20

Example 3.1.19 (Polyhedral cone)**.** Let $\Gamma^2 : \mathbb{R}^2 \to \mathbb{R}^2$ with $\Gamma^2(c) = Ac$ for $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ be a linear function, so $C(\Gamma^2) = C(A)$ be the induced polyhedral cone shown in Figure 3.1 (center). Since $\operatorname{rank} A = 2$, it follows that $C(A)$ is pointed and Γ^2 is injective, so that both Theorem 3.1.12 and 3.1.13 apply and give that

$$\Gamma^2[N(Y, C(\Gamma^2))] = A[N(Y, C(A))] = N(A[Y], \mathbb{R}^2_{\geqq}) \tag{3.23a}$$

In particular, mapping these sets under the inverse $A^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ of Γ^2 we find that

$$N(Y, C(\Gamma^2)) = A^{-1}[N(A[Y], \mathbb{R}^2_{\geqq})] = \{y \in \operatorname{bd} Y : y_2 \leq 0 \text{ and } y_1 - y_2 \leq 0\} \quad (3.23b)$$

as shown in Figure 3.2 (center). Furthermore, since $C(A)$ is convex with dual cone $C(A)^* = C(A^*)$ for $A^* = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$, Propositions 3.1.14 and 3.1.15 apply respectively for the hybrid and the weighted-sum scalarization with possible weighting vectors $w \in C(A^*) \setminus \{0\}$ as illustrated in Figure 3.3 (center). However, since $A^*e^1 = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \not> 0$ and $A^*e^2 = A^*1 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \not> 0$, only (i) of Propositions 3.1.16 and 3.1.17 apply to the constrained-objective and Benson method, respectively. In particular, the formulation of the hybrid scalarization becomes

$$\text{Minimize } \langle w, y \rangle \text{ subject to } A(y^\circ - y) \geqq 0 \text{ and } y \in Y \quad (3.24)$$

and thus, together with the constrained-objective and Benson method as special cases, can be solved as a linear program in the outcome space.

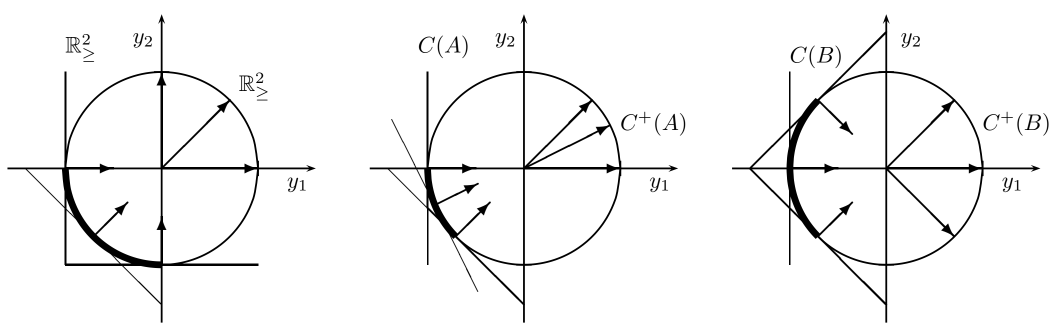


Figure 3.3 Dual cones and possible choices of the weighting vector for the Pareto (left), polyhedral (center), and pth-order cone (right) in Examples 3.1.18–3.1.20

Example 3.1.20 (Pth-order cone). Let $\Gamma^3 : \mathbb{R}^2 \to \mathbb{R}^2$ with $\Gamma^3(c) = c_1 - \|c_2\|_p = c_1 - |c_2|$ be a superlinear function and $C(\Gamma^3) = C^2_p$ be the induced pth-order cone, shown in Figure 3.1 (right). Since $\Gamma^3(c_1, c_2) = \Gamma^3(c_1, -c_2)$ for all $c = (c_1, c_2)^T \in \mathbb{R}^2$, however, Γ^3 is not injective so that both Theorems 3.1.12 and 3.1.13 do not apply to the pth-order cone C^2_p when induced by the superlinear function Γ^3.

However, since $c_1 - |c_2| \geq 0$ can equivalently be written as $c_1 - c_2 \geq 0$ and $c_1 + c_2 \geq 0$, let $\Gamma^4(c) = Bc$ with $B = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ be a linear function, so $C(\Gamma^3) = C(\Gamma^4) = C(B)$. Then $\operatorname{rank} B = 2$, $C(B)$ is pointed and, in this case, Γ^4 is also injective so that both Theorems 3.1.12 and 3.1.13 apply and give that

$$\Gamma^4[N(Y, C(\Gamma^4))] = B[N(Y, C(B))] = N(B[Y], \mathbb{R}^2_{\geqq}) \tag{3.25a}$$

Now mapping these sets under the inverse $B^{-1} = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ of Γ^4, it follows that

$$\begin{aligned} N(Y, C(\Gamma^3)) = N(Y, C(\Gamma^4)) &= B^{-1}[N(B[Y], \mathbb{R}^2_{\geqq})] \\ &= \{y \in \operatorname{bd} Y : y_1 + y_2 \leq 0 \text{ and } y_1 - y_2 \leq 0\} \end{aligned} \tag{3.25b}$$

as shown in Figure 3.2 (right). Furthermore, since $C(B)$ is convex and self-dual, Propositions 3.1.14 and 3.1.15 apply to hybrid method and weighted-sum with $w \in C(B) \setminus \{0\}$, illustrated in Figure 3.3 (right). However, since $Be^1 = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \not> 0$ and $Be^2 = B1 = \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right) \not> 0$, again only part (i) of Propositions 3.1.16 and 3.1.17 apply to the constrained-objective and Benson method, while the hybrid method becomes

$$\text{Minimize } \langle w, y \rangle \text{ subject to } B(y^\circ - y) \geqq 0 \text{ and } y \in Y \tag{3.26}$$

and, as before, can be solved as a linear program.

We note that since all cones in the three above examples eventually turn out to be polyhedral, in each case the general nondominance mapping theorem in Theorem 3.1.12 reduces to the special case for polyhedral cones in Theorem 3.1.13. In fact, Eichfelder (2006) recently shows that every closed convex cone $C \subset \mathbb{R}^2$ is polyhedral, and while we already know from Remark 3.1.3 that cone representations are not unique, in general, we can further conclude that every closed convex cone C in $\mathbb{R}^2$ can also be induced by a linear function or matrix. In particular, it follows that both Theorems 3.1.12 and 3.1.13 can be applied to all biobjective programs for which the underlying domination cone is closed and convex and suitably described in terms of an injective linear function or matrix with full column rank $m = 2$.

Consequently, to also illustrate situations in which Theorem 3.1.12 provides a true generalization of Theorem 3.1.13, it is not sufficient to only discuss biobjective programs but we need to study multiobjective programs $N(Y, C)$ with $Y \subset \mathbb{R}^m$ and $m > 2$. Since in these cases, however, a convenient graphical illustration is usually not possible, we choose an alternative path and derive some further analytical results for truly nonpolyhedral cones in the following section.

3.1.5 Other Cone Representations and Further Results

In conclusion of our discussion of polyhedral and nonpolyhedral cones, we derive some more specific results for the class of cones that are induced by positively homogeneous functions $\Gamma = (\Gamma_1, \ldots, \Gamma_l) : \mathbb{R}^m \to \mathbb{R}^l$ of the form

$$\Gamma_i(c) = \langle a^i, c\rangle - \alpha_i \|B^i c\|_p \tag{3.27}$$

where the $\alpha_i \in \mathbb{R}$ are real numbers, $a^i \in \mathbb{R}^m$ are real vectors, and $B^i \in \mathbb{R}^{\cdot \times m}$ are real matrices with m columns for all $i = 1, \ldots, l$.

Remark 3.1.21 (Conditions for sublinearity, superlinearity, and linearity)**.** It is clear that Γ is sublinear if $\alpha_i \leq 0$ and superlinear if $\alpha_i \geq 0$ for all $i = 1, \ldots, l$. In particular, if all $\alpha_i = 0$, then Γ is linear and its induced cone $C(\Gamma) = C(A)$ with $A = (a^1, \ldots, a^l)^T \in \mathbb{R}^{l \times m}$ is polyhedral.

Furthermore, if we choose $\Gamma : \mathbb{R}^m \to \mathbb{R}$ as a real-valued function with $a = e^1$ the first unit vector, $\alpha = 1$, and $B = (0, I^{m-1}) \in \mathbb{R}^{(m-1)\times m}$ with $0 \in \mathbb{R}^{m-1}$ a vector of zeros and $I^{m-1} \in \mathbb{R}^{(m-1)\times(m-1)}$ the $(m-1)$-dimensional identity matrix, then

$$\Gamma(c) = \langle a, c\rangle - \alpha\|Bc\|_p = \langle e^1, c\rangle - \|(0, I^{m-1})c\|_p = c_1 - \|c_{-1}\|_p \tag{3.28}$$

and $C = C_p^m$ is the m-dimensional pth-order cone. Hence, we can use the new cone representation in (3.27) to induce all pth-order and polyhedral cones, including the Pareto cone as special case.

Our first result is an immediate consequence of the nondominance mapping theorem for general cones and Remark 3.1.21.

Proposition 3.1.22 (Nondominance mapping for new cone representation I)**.** *Let $\alpha_i \in \mathbb{R}$, $a^i \in \mathbb{R}^m$, and $B^i \in \mathbb{R}^{\cdot\times m}$ for $i = 1, \ldots, l$, and let $C(\Gamma) \subset \mathbb{R}^m$ be the* *nonpolyhedral cone induced by* $\Gamma = (\Gamma_1, \ldots, \Gamma_l) : \mathbb{R}^m \to \mathbb{R}^l$ *with* $\Gamma_i(c) = \langle a^i, c\rangle - \alpha_i \|B^i c\|_p$. *If all* $\alpha_i \leq 0$, *then*

$$\Gamma[N(Y, C(\Gamma))] \subseteq N(\Gamma[Y], \mathbb{R}^l_{\geqq}) \tag{3.29}$$

The proof follows directly from Theorem 3.1.12 and Remark 3.1.21 and is omitted. Moreover, if all $\alpha_i \geq 0$, then the reverse inclusion in Proposition 3.1.22 holds if the function Γ is also injective, or if $C(\Gamma)$ is pointed. For special cases, this can be verified easily.

Proposition 3.1.23 (Condition for convexity and pointedness)**.** *Let* $\alpha \geq 0$ *and* $C(\Gamma) \subset \mathbb{R}^m$ *be the nonpolyhedral cone induced by* $\Gamma = (\Gamma_1, \ldots, \Gamma_m) : \mathbb{R}^m \to \mathbb{R}^m$ *with* $\Gamma_i(c) = c_i - \alpha\|c\|_p$. *If* $\alpha m^{1/p} < 1$, *then the function* Γ *is injective, and the cone* $C(\Gamma)$ *is convex and pointed.*

Proof. To show that Γ is injective, let $\Gamma(c) = \Gamma(d)$ and, without loss of generality, assume that $\varepsilon = \alpha\left(\|c\|_p - \|d\|_p\right) \geq 0$ (otherwise switch c and d). Then $c_i - \alpha\|c\|_p = d_i - \alpha\|d\|_p$, or $c_i - d_i = \alpha\left(\|c\|_p - \|d\|_p\right) = \varepsilon$, and thus, $c = d + \varepsilon 1$. Hence, $\|c\|_p = \|d + \varepsilon 1\|_p \leq \|d\|_p + \varepsilon\|1\|_p$, or $\varepsilon = \alpha\left(\|c\|_p - \|d\|_p\right) \leq \varepsilon\alpha\|1\|_p = \varepsilon\alpha m^{1/p}$, and thus, $\varepsilon = 0$ and $c = d$, showing that Γ is injective. As Γ, in particular, is superlinear, Proposition 3.1.8 then gives that $C(\Gamma)$ is convex and pointed. □

Corollary 3.1.24 (Nondominance mapping for new cone representation II)**.** *Let* $\alpha \geq 0$, $\alpha m^{1/p} < 1$, *and* $C(\Gamma) \subset \mathbb{R}^m$ *be the nonpolyhedral cone induced by* $\Gamma = (\Gamma_1, \ldots, \Gamma_m) : \mathbb{R}^m \to \mathbb{R}^m$ *with* $\Gamma_i(c) = c_i - \alpha\|c\|_p$. *Then*

$$N(\Gamma[Y], \mathbb{R}^m_{\geqq}) \subseteq \Gamma[N(Y, C(\Gamma))] \tag{3.30}$$

Our last result pertains to the possible application of scalarization methods based on the conditions for nondominance that we establish in Propositions 3.1.14 (hybrid), 3.1.15 (weighted-sum), 3.1.16 (constrained-objective), and 3.1.17 (Benson). Since the validation of these conditions requires to know (at least a subset of) the dual cone $C(\Gamma)^*$, the next result provides a partial characterization of the dual cone associated with a cone that is slightly more general than the one discussed in Proposition 3.1.23 and Corollary 3.1.24.

Proposition 3.1.25 (Dual cone for new cone representation)**.** *Let $\alpha_i \geq 0$ for all $i = 1, \ldots, m$, and $C(\Gamma) \subset \mathbb{R}^m$ be the convex nonpolyhedral cone induced by $\Gamma = (\Gamma_1, \ldots, \Gamma_m) : \mathbb{R}^m \to \mathbb{R}^m$ with $\Gamma_i(c) = c_i - \alpha_i \|c\|_p$. Then $C(\Gamma)$ belongs to its dual*

$$C(\Gamma) \subseteq C(\Gamma)^* \tag{3.31}$$

Moreover, $e^k \in C(\Gamma)^*_s$ if $\alpha_k > 0$ for $k = 1, \ldots, m$, and $1 \in C(\Gamma)^*_s$ if some $\alpha_i > 0$.

Proof. To show that $C(\Gamma)$ is contained in its dual cone, let $w \in C(\Gamma)$ and choose any $c \in C(\Gamma)$, so $w_i - \alpha_i \|w\|_p \geq 0$ and $c_i - \alpha_i \|c\|_p \geq 0$. Then

$$\langle w, c \rangle = \sum_{i=1}^{m} w_i c_i \geq \sum_{i=1}^{m} \alpha_i^2 \|w\|_p \|c\|_p \geq 0 \tag{3.32a}$$

yielding $w \in C(\Gamma)^*$ and thus $C(\Gamma) \subseteq C(\Gamma)^*$. Furthermore, if $c \neq 0$ and $\alpha_k > 0$, then

$$\langle e^k, c \rangle = c_k \geq \alpha_k \|c\|_p > 0 \tag{3.32b}$$

and it follows that $e^k \in C(\Gamma)^*_s$. Similarly, if some $\alpha_i > 0$, then

$$\langle 1, c \rangle = \sum_{i=1}^{m} c_i \geq \sum_{i=1}^{m} \alpha_i \|c\|_p > 0 \tag{3.32c}$$

which again implies that $1 \in C(\Gamma)^*_s$ to conclude the proof. $\square$

Hence, in this case Propositions 3.1.14 and 3.1.15 apply to the hybrid and the weighted-sum scalarization with $w \in C(\Gamma) \setminus \{0\}$, and Propositions 3.1.16 and 3.1.17 apply to the constrained-objective and Benson method if $\alpha_k > 0$ or some $\alpha_i > 0$, respectively. In particular, the hybrid method becomes

$$\text{Minimize } \langle w, y \rangle \text{ subject to } y_i^\circ - y_i \geq \alpha_j \|y^\circ - y\|_p \text{ for } i = 1, \ldots, m,\ y \in Y \tag{3.33}$$

and, in this case, can be solved as a pth-order cone program.

Some consequences of these results and possible topics for further investigation are summarized in the discussion section that concludes this chapter.

3.2 Variable and Ideal-Symmetric Convex Cones

In this section, we return to the original notion of domination in Definition 2.2.1 and inquire both relevance and consequences of domination structures that are variable and, thus, defined by a collection of different domination sets. Together with the new discussion in Section 3.2.1 that revists the relationships between partial orders and cones, we highlight some inherent model limitations of constant cones and then introduce the new notion of ideal-symmetry to derive a variable-cone model in remedy of these recognized shortcomings in Section 3.2.2. Based on the subsequent generalization of scalarization approaches for variable domination cones, the characterization of the nondominated set with respect to this new model is addressed in Section 3.2.3, and some alternative conditions are established for biobjective cases in Section 3.2.4 and applied to three examples in Section 3.2.5.

3.2.1 Assumptions and Limitations of Constant Cones

We begin our investigation by adding some more details to the discussion in Section 2.1.3 in which we describe the relationship between convex cones and partial orders. In particular, in Propositions 2.1.38 and 2.1.39 we establish the equivalence between a general cone and binary relations that are compatible with scalar multiplication and show that this equivalence can be extended to pointed convex cones and (strict) partial orders under the additional assumption of compatibility with addition. However, in this case Proposition 2.2.3 implies that the induced domination structure is described by a constant pointed convex cone, and in order to also study variable cones we first relax the latter of these two assumptions by the similar notion of additivity. Hence, different from the partial and strict partial orders $\preceq$ and $\prec$ in Section 2.1.3 which distinguish between reflexive and irreflexive binary relations, we denote the underlying order or relation without this distinction by $\preccurlyeq$ and, for further convenience, let this order be defined on the complete space $\mathbb{R}^m$.

Definition 3.2.1 (Additivity)**.** A binary relation $\preccurlyeq$ on $\mathbb{R}^m$ is said to be *additive* if

$$y^1 \preccurlyeq y^3 \text{ and } y^2 \preccurlyeq y^4 \Longrightarrow y^1 + y^2 \preccurlyeq y^3 + y^4 \text{ for all } y^1, y^2, y^3, y^4 \in \mathbb{R}^m \tag{3.34}$$

Remark 3.2.2 (Generalized additivity)**.** If $y^i \in \mathbb{R}^m$ for $i = 1, \ldots, 2k$ and $y^j \preccurlyeq y^{j+k}$ for $j = 1, \ldots, k$, then additivity implies that also

$$\sum_{j=1}^{k} y^j \preccurlyeq \sum_{j=1}^{k} y^{j+k} \tag{3.35}$$

In particular, if $y^1 = y^2 = \ldots = y^k$ and $y^{k+1} = y^{k+2} = \ldots = y^{2k}$, then additivity includes compatibility with scalar multiplication for all integer scalars $\lambda = k$.

Proposition 3.2.3 (Additivity, reflexivity, and transitivity)**.** *If a binary relation is additive and reflexive on $\mathbb{R}^m$, then it is compatible with addition and transitive.*

Proof. To show that an additive and reflexive binary relation is compatible with addition, let $y^1 \preccurlyeq y^2$ and $z \in \mathbb{R}^m$. It follows that $z \preccurlyeq z$ by reflexitivity, and then additivity implies that $y^1 + z \preccurlyeq y^2 + z$, showing compatibility with addition.

Similarly, to show transitivity, let $y^1 \preccurlyeq y^2$ and $y^2 \preccurlyeq y^3$. Again $-y^2 \preccurlyeq -y^2$ by reflexivity, and then additivity implies that $y^1 + y^2 - y^2 \preccurlyeq y^2 + y^3 - y^2$ and thus $y^1 \preccurlyeq y^3$, showing transitivity and concluding the proof. □

Hence, we find that a reflexive additive binary relation is also compatible with addition and transitive and thus, under the additional assumption of compatibility with scalar multiplication, that $C_{\preccurlyeq}$ is a convex cone from Proposition 2.1.39. This result, however, can also be derived directly from the assumption of additivity.

Proposition 3.2.4 (Relation cone under additivity)**.** *If the binary relation $\preccurlyeq$ is additive and compatible with scalar multiplication, then*

$$C_{\preccurlyeq} := \{y^2 - y^1 \in \mathbb{R}^m : y^1 \preccurlyeq y^2\} \tag{3.36}$$

is a convex cone.

Proof. To show that the set $C_{\preccurlyeq}$ is a cone, let $c \in C_{\preccurlyeq}$ and $\lambda > 0$. Then there exist $y^1, y^2 \in \mathbb{R}^m$ so that $c = y^2 - y^1$ and $y^1 \preccurlyeq y^2$, and compatibility with scalar multiplication implies that also $\lambda y^1 \preccurlyeq \lambda y^2$ and thus $\lambda y^2 - \lambda y^1 = \lambda(y^2 - y^1) = \lambda c \in C_{\preccurlyeq}$ showing that $C_{\preccurlyeq}$ is a cone.

To show that the cone $C_\preccurlyeq$ is convex, let $c^1, c^2 \in C_\preccurlyeq$. Then there exist $y^1, y^2, y^3, y^4 \in \mathbb{R}^m$ so that $c^1 = y^3 - y^1$, $c^2 = y^4 - y^2$ and $y^1 \preccurlyeq y^3$, $y^2 \preccurlyeq y^4$, and additivity implies that also $y^1 + y^2 \preccurlyeq y^3 + y^4$ and thus $y^3 + y^4 - (y^1 + y^2) = (y^3 - y^1) + (y^4 - y^2) = d^1 + d^2 \in C_\preccurlyeq$ showing convexity and concluding the proof. $\square$

By dropping the assumption of compatibility with addition, however, Proposition 2.2.3 does not imply anymore that the induced domination structure is necessarily constant and thus motivates to introduce the following notions to subsequently define the corresponding domination sets $D(y)$ separately at every outcome y.

Definition 3.2.5 (Additivity and compatibility with scalar multiplication at a point)**.** A binary relation is said to be *additive* and *compatible with scalar multiplication at* $y \in \mathbb{R}^m$ if respectively

$$y - d^1 \preccurlyeq y \text{ and } y - d^2 \preccurlyeq y \Longrightarrow y - (d^1 + d^2) \preccurlyeq y \text{ for all } d^1, d^2 \in \mathbb{R}^m \tag{3.37a}$$

$$y - d \preccurlyeq y \Longrightarrow y - \lambda d \preccurlyeq y \text{ for all } d \in \mathbb{R}^m \text{ and } \lambda > 0 \tag{3.37b}$$

Based on this definition, the next result essentially repeats Proposition 3.2.4.

Proposition 3.2.6 (Relation cone at a point)**.** *If the binary relation $\preccurlyeq$ is additive and compatible with scalar multiplication at $y \in \mathbb{R}^m$, then*

$$D(y) := \{d \in \mathbb{R}^m : y - d \preccurlyeq y\} \tag{3.38}$$

is a convex cone.

Proof. To show that the set $D(y)$ is a cone, let $d \in D(y)$ and $\lambda > 0$. If $d = 0$, then $\lambda d = 0 \in D(y)$, otherwise $y - d \preccurlyeq y$ and compatibility with scalar multiplication implies that also $y - \lambda d \preccurlyeq y$ and, thus, $\lambda d \in D(y)$, showing that $D(y)$ is a cone.

To show that the cone $D(y)$ is convex, let $d^1, d^2 \in D(y)$. If $d^1 = 0$ or $d^2 = 0$, then $d^1 + d^2 \in D(y)$, otherwise $y - d^1 \preccurlyeq y$, $y - d^2 \preccurlyeq y$ and additivity implies that also $y - (d^1 + d^2) \preccurlyeq y$ and, thus $d^1 + d^2 \in D(y)$, showing that the cone $D(y)$ is convex and concluding the proof. $\square$

Following the above discussion, however, the cones $D(y)$ in Proposition 3.2.6 do not necessarily need to be identical so that the resulting domination structure can also be defined as a *variable* cone. Before we continue to pursue this observation, we first introduce the additional notion of monotonicity.

Definition 3.2.7 (Monotonicity)**.** A binary relation is said to be *monotonic* if

$$y - e^i \preccurlyeq y \text{ for all } i = 1, \dots, m \text{ and } e^i \in \mathbb{R}^m \text{ the } i\text{th unit vector} \tag{3.39}$$

The interpretation of monotonicity in the context of multiobjective programming is immediate: provided we are interested in objective minimization and reduce any single component of some original outcome y, then the resulting outcome is preferred to that original outcome. In particular, the assumption of monotonicity is equivalent to the Edgeworth-Pareto principle that we introduce in (2.63) in Section 2.3.2, and under this additional assumption we now show that the convex cone in Proposition 3.2.6 always contains all (nonzero) elements of the Pareto cone.

Proposition 3.2.8 (Pareto compatibility)**.** *If a monotonic binary relation is additive and compatible with scalar multiplication at $y \in \mathbb{R}^m$, then*

$$D(y) := \{d \in \mathbb{R}^m : y - d \preccurlyeq y\} \cup \{0\} \tag{3.40}$$

is a convex cone that contains the Pareto cone.

Proof. The proof that $D(y)$ is a convex cone follows exactly as in Proposition 3.2.6. Hence, to show that the convex cone $D(y)$ contains the Pareto cone, let $d \in \mathbb{R}^m_{\geqq}$, so $d = \sum_{i=1}^m d_i e^i = (d_1, \dots, d_m) \geqq 0$. From monotonicity, $y - e^i \preccurlyeq y$ and, thus, $d = y - (y - e^i) = e^i \in D(y)$. If $d_i = 0$, then $d_i e^i = 0 \in D(y)$, otherwise $d_i > 0$ and $d_i e^i \in D(y)$ also, because $D(y)$ is a cone. Convexity of $D(y)$ then implies that $d = \sum_{i=1}^m d_i e^i \in D(y)$, showing that $D(y)$ contains the Pareto cone. □

In Proposition 3.2.8, if a domination set or cone $D(y)$ contains the Pareto cone, then we say that $D(y)$ is *Pareto compatible*. Hence, the assumptions on $\preccurlyeq$ in Proposition 3.2.8 imply that the underlying domination structure can be described by a collection of Pareto compatible convex cones but, so far, do not impose any condition that also guarantees that this domination structure is in fact variable.

In preparation of such a condition for the definition of our variable-cone model in the next section, we now present two examples that do not only highlight several of the model limitations of a constant cone but also provide intuitive insight into the later formulation of the corresponding variability assumption.

Example 3.2.9 (Model limitation of constant convex cones)**.** Let $Y = \{y \in \mathbb{R}^2 : y_1 + y_2 \geq 1,\ y_1 \geq 0,\ y_2 \geq 0\}$ be as depicted in Figure 3.4 and $C \subset \mathbb{R}^2$ be a Pareto compatible convex cone. In particular, denote $z^1 = (1,0)$, $z^2 = (0,1)$, and let $c^1 = z^2 - z^1 = (-1,1)$ and $c^2 = z^1 - z^2 = (1,-1)$. Then we can distinguish the following four cases that are also shown in Figure 3.4.

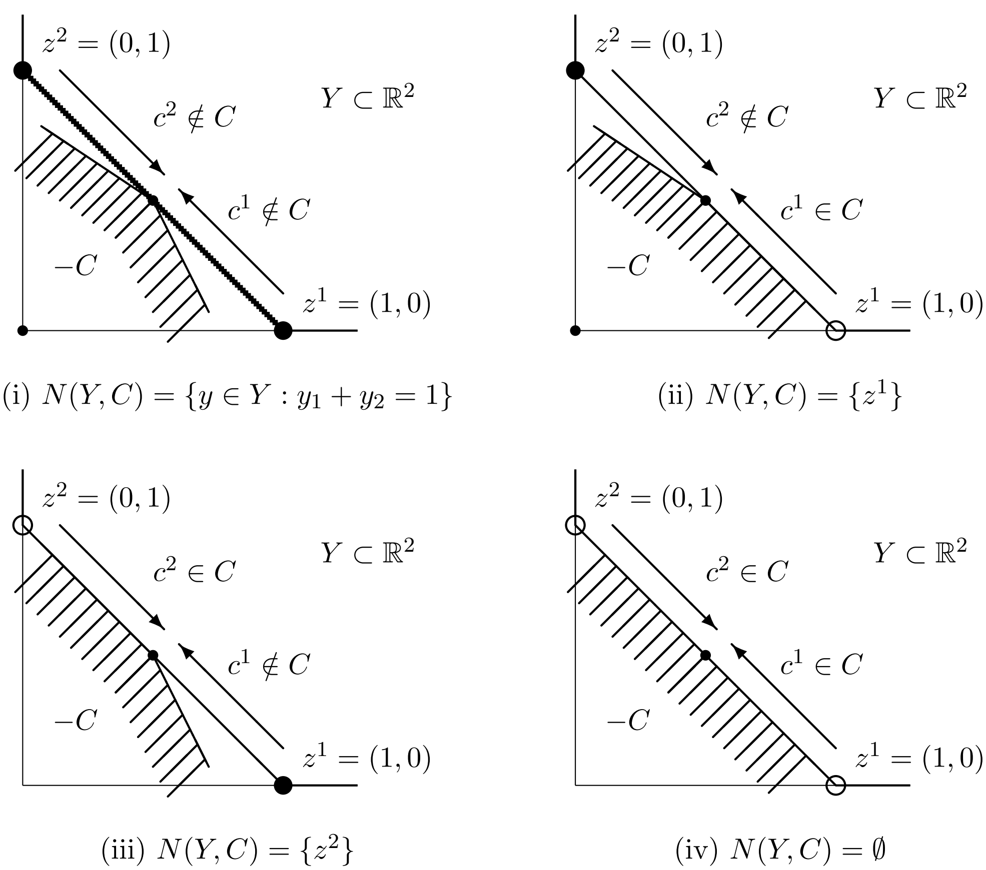


Figure 3.4 Nondominated set $N(Y,C)$ in each of the four cases in Example 3.2.9

(i) If c^1 and $c^2 \notin C$, then $N(Y, C) = \{y \in Y : y_1 + y_2 = 1\}$ is the Pareto set.

(ii) If $c^1 \in C$ and $c^2 \notin C$, then $N(Y, C) = \{z^1\}$ reduces to a singleton.

(iii) Similarly, if $c^1 \notin C$ and $c^2 \in C$, then $N(Y, C) = \{z^2\}$.

(iv) Finally, if c^1 and $c^2 \in C$, then $N(Y, C) = \emptyset$.

Hence, in this case, the nondominated set of Y is either empty, a singleton, or the complete Pareto set $N(Y, \mathbb{R}^2_{\geqq}) = \{y \in \mathbb{R}^2 : y_1 + y_2 = 1,\ y_1 \geq 0,\ y_2 \geq 0\}$. In particular, it is not possible to obtain a nondominated set that excludes the two extreme points z^1 and z^2 or parts of the extreme ends while maintaining a set of nondominated outcomes in the middle region of the Pareto set.

The previous example illustrates that constant cone models are limited in that directions are either dominated or nondominated at all outcomes $y \in Y$ so that, in particular, the vectors c^1 and c^2 in Figure 3.4 are either contained in all cones or do not belong to any domination cone $D(y) = C$. Consequently, in this case it is not possible that a Pareto outcome in the middle region of the Pareto set dominates both z^1 and z^2 without also being dominated by z^2 and z^1, respectively, which furthermore precludes to impose a preference principle of transfers (2.64) or justice (2.65), or a concept of equitable efficiency as described in Section 2.3.2.

Figure 3.5, on the other hand, illustrates how a *variable* cone, in principle, is capable to remove this current limitation by changing its orientation and shape and, thus, the set of dominated directions for every individual outcome.

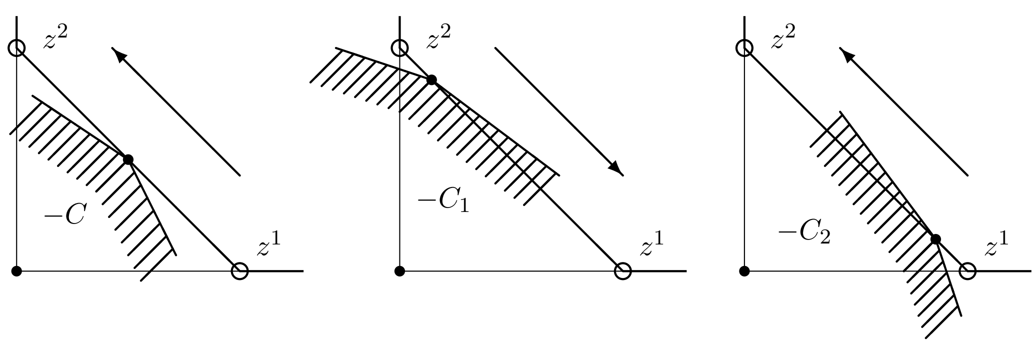


Figure 3.5 A variable cone in remedy of the model limitation in Example 3.2.9

In the next example, we address another shortcoming of preference models that are defined based on constant cones, similar to the discussion in Example 3.2.9.

Example 3.2.10 (Another model limitation of constant cones)**.** Let $Y \subset \mathbb{R}^2$ and y^1, y^2, y^3, $y^4 \in Y$ with $y^1 + y^4 = y^2 + y^3$ be as depicted in Figure 3.6 (left). Restricting consideration to these four outcomes, we find that y^1, y^2, and y^3 are nondominated with respect to the Pareto cone, while y^4 is dominated by y^1 but neither dominated by nor preferred to y^2 and y^3. Although arguable, in principle we expect that in a practical decision making context, y^1 is preferred to y^2 and y^3 and, thus, the overall best outcome. Hence, an underlying preference model should enable that y^1 is preferred to all the three other outcomes, but it should not introduce any additional preference relationships between y^2, y^3 and y^4.

Using a constant cone $C \subset \mathbb{R}^2$, however, $y^1 \preccurlyeq y^2$ and $y^1 \preccurlyeq y^3$ are equivalent with $y^2 - y^1$ and $y^3 - y^1 \in C$ and, thus, also imply that $y^4 - y^2 = y^3 - y^1$ and $y^4 - y^3 = y^2 - y^1 \in C$, or $y^2 \preccurlyeq y^4$ and $y^3 \preccurlyeq y^4$, respectively. In particular, it is not possible to define a preference model that allows to individually specify one or both of the preference relationships $y^1 \preccurlyeq y^2$ and $y^1 \preccurlyeq y^3$ between y^1, y^2, and y^3, without also affecting those between y^2, y^3 and y^4.

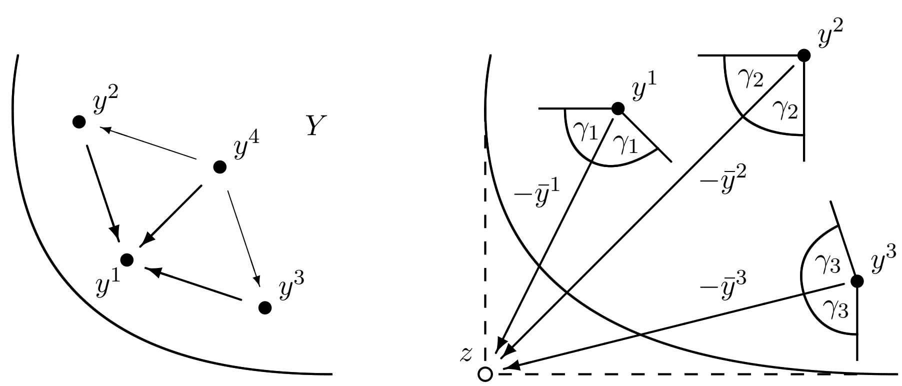


Figure 3.6 Four outcomes discussed in Example 3.2.10 (left) and the geometric interpretation of ideal-symmetry in Definition 3.2.14 and Remark 3.2.15 (right)

Hence, to remove the shortcoming in Example 3.2.10, again we need to allow for a variable cone to specify the dominated directions and thus the corresponding preference relationships separately for every individual or pair of outcomes, respectively. Similar to Figure 3.5 and our earlier discussion, Figure 3.6 (right) indicates how we may define such a variable cone and also provides the pictorial intuition for the derivation of our variable-cone model now pursued in the following section.

3.2.2 Ideal-Symmetry Assumption and Model Derivation

Motivated by our previous discussion, we derive a variable domination structure that can be described by a collection of convex cones for which we introduce variability by the technical assumption that the set of dominated directions at any outcome $y \in Y$ is symmetric with respect to the direction pointing to (or leading from) the ideal point. The general idea behind this assumption is illustrated in Figure 3.6 and made precise in the following definitions.

Definition 3.2.11 (Ideal point and partially ideal outcome)**.** Let $Y \subset \mathbb{R}^m$ be a nonempty set. The point $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ with

$$z_i = \inf\{y_i : y \in Y\} \text{ for all } i = 1, \ldots, m \tag{3.41}$$

is called the *ideal point* of Y. An outcome $y \in Y$ with $y_i = z_i$ for some index i is said to be *partially ideal*.

Note that this definition is the exact equivalent to Definition 2.2.13, and as before, we again assume that the ideal point $z \in \mathbb{R}^m \setminus Y$ exists and is finite but not itself contained in the outcome set. In this case, we define the ideal domination vector at every outcome y as shown in Figure 3.6.

Definition 3.2.12 (Ideal domination vector)**.** Let $Y \subset \mathbb{R}^m$ be nonempty and $z \in \mathbb{R}^m$ be the ideal point of Y. For every $y \in Y$, the vector

$$\bar{y} := y - z \in \mathbb{R}^m \tag{3.42}$$

is called the *ideal domination vector* at y.

It is clear that the ideal domination vector at every outcome $y \in Y$ is nonnegative, and positive if and only if y is not partially ideal.

Proposition 3.2.13 (Nonnegativity of ideal domination vector)**.** *Let $Y \subset \mathbb{R}^m$ be nonempty and $z \in \mathbb{R}^m$ be the ideal point of Y. Then*

(i) $\bar{y} \geq 0$ for all $y \in Y$, and

(ii) $\bar{y} > 0$ if and only if $y \in Y$ is not partially ideal.

Next, we give the fundamental definition of your model.

Definition 3.2.14 (Ideal-symmetry)**.** Let $Y \subset \mathbb{R}^m$ be nonempty and $z \in \mathbb{R}^m$ be the ideal point of Y. A set $D \subset \mathbb{R}^m$ is said to be *ideal-symmetric* at $y \in Y$ if

$$\langle d^1, \bar{y} \rangle = \langle d^2, \bar{y} \rangle, \|d^1\| = \|d^2\| \Rightarrow d^1 \in D \text{ if and only if } d^2 \in D \text{ for all } d^1, d^2 \in \mathbb{R}^m \tag{3.43}$$

Remark 3.2.15 (Geometric interpretation of ideal-symmetry)**.** In Definition 3.2.14, if $d^1, d^2 \neq 0$, $\langle d^1, \bar{y} \rangle = \langle d^2, \bar{y} \rangle$, and $\|d^1\| = \|d^2\|$, then

$$\frac{\langle d^1, \bar{y} \rangle}{\|d^1\| \|\bar{y}\|} = \frac{\langle d^2, \bar{y} \rangle}{\|d^2\| \|\bar{y}\|} \tag{3.43a}$$

and concepts from analytic geometry provide the possible interpretation of each ratio as the cosine of the angle formed by one of the vectors d^1 or d^2 and $\bar{y}$, so

$$\cos \measuredangle(d^1, \bar{y}) = \cos \measuredangle(d^2, \bar{y}) \text{ or } \measuredangle(d^1, \bar{y}) = \measuredangle(d^2, \bar{y}) \tag{3.43b}$$

This geometric intuition is depicted for the biobjective case ($m = 2$) in Figure 3.6, remains valid for three objectives ($m = 3$) as the angle interpretation of the ratios in Equation 3.43a still holds in spatial analytic geometry, and is naturally generalized for $m > 3$ by the definition chosen.

Furthermore, similar to the notion of ideal-symmetry, we call a cone $C \subset \mathbb{R}^m$ *symmetric* over $s \in \mathbb{R}^m$ if $c \in C$ implies that $c' \in C$ for all $c' \in \mathbb{R}^m$ with $\langle c, s \rangle = \langle c', s \rangle$ and $\|c\| = \|c'\|$. For notational brevity, and although slightly ambivalent to its previous meaning, in the following lemma we use the parameter γ to denote the cosine of any corresponding angle $\gamma_1, \gamma_2, \gamma_3$ in Figure 3.6.

Lemma 3.2.16 (Symmetric cone)**.** *Let $\gamma \in \mathbb{R}$, $s \in \mathbb{R}^m$, $s \geq 0$, and define*

$$C_{\gamma,s} := \left\{ c \in \mathbb{R}^m \setminus \{0\} : \frac{\langle c, s\rangle}{\|c\|\|s\|} \geq \gamma \right\} \cup \{0\} \tag{3.44}$$

Then $C_{\gamma,s}$ is a cone that is symmetric with respect to s.

(i) If $\gamma \geq 0$, then $C_{\gamma,s}$ is convex.

(ii) If $\gamma > 0$, then $C_{\gamma,s}$ is convex and pointed.

(iii) If $\gamma \leq \min_i\{s_i\}\|s\|^{-1}$, then $C_{\gamma,s}$ contains the Pareto cone.

Proof. To show that $C_{\gamma,s}$ is a cone, let $c \in C_{\gamma,s}$ and $\lambda > 0$. If $c = 0$, then $\lambda c = 0 \in C_{\gamma,s}$, otherwise

$$\frac{\langle \lambda c, s\rangle}{\|\lambda c\|\|s\|} = \frac{\lambda\langle c, s\rangle}{\lambda\|c\|\|s\|} = \frac{\langle c, s\rangle}{\|c\|\|s\|} \geq \gamma \tag{3.44a}$$

and thus $\lambda c \in C_{\gamma,s}$, showing that $C_{\gamma,s}$ is a cone.

To show that the cone $C_{\gamma,s}$ is symmetric with respect to s, let $c \in C_{\gamma,s}$ and $c' \in \mathbb{R}^m$ with $\langle c, s\rangle = \langle c', s\rangle$ and $\|c\| = \|c'\|$. If $c = 0$, then $c' = 0 \in C_{\gamma,s}$, otherwise

$$\frac{\langle c', s\rangle}{\|c'\|\|s\|} = \frac{\langle c, s\rangle}{\|c\|\|s\|} \geq \gamma \tag{3.44b}$$

and thus $c' \in C_{\gamma,s}$, showing the the cone $C_{\gamma,s}$ is symmetric with respect to s.

For (i), let $\gamma \geq 0$. To show that the cone $C_{\gamma,s}$ is convex, let $c^1, c^2 \in C_{\gamma,s}$. If $c^1 = c^2 = 0$, then $c^1 + c^2 = 0 \in C_{\gamma,s}$, otherwise

$$\frac{\langle c^1 + c^2, s\rangle}{\|c^1 + c^2\|\|s\|} \geq \frac{\langle c^1, s\rangle + \langle c^2, s\rangle}{(\|c^1\| + \|c^2\|)\|s\|} \geq \frac{\gamma\|c^1\| + \gamma\|c^2\|}{(\|c^1\| + \|c^2\|)} = \gamma \tag{3.44c}$$

and thus $c^1 + c^2 \in C_{\gamma,s}$, showing that the cone $C_{\gamma,s}$ is convex.

For (ii), let $\gamma > 0$, then $\gamma \geq 0$ and the cone $C_{\gamma,s}$ is convex. To show that the convex cone $C_{\gamma,s}$ is pointed, let $c \in C_{\gamma,s} \setminus \{0\}$, then

$$\frac{\langle -c, s\rangle}{\|-c\|\|s\|} = -\frac{\langle c, s\rangle}{\|c\|\|s\|} \leq -\gamma < \gamma \tag{3.44d}$$

and thus $-c \notin C_{\gamma,s}$, $c \notin -C_{\gamma,s}$, or $C_{\gamma,s} \cap -C_{\gamma,s} \subset \{0\}$, showing that the convex cone $C_{\gamma,s}$ is pointed.

For (iii), let $\gamma \leq \min_i\{s_i\}\|s\|^{-1}$. To show that the cone $C_{\gamma,s}$ contains the Pareto cone, let $c = \sum_{i=1}^m c_i e^i = (c_1, \ldots, c_m) \geq 0$. If $c = 0$, then $c \in C_{\gamma,s}$, otherwise

$$\frac{\langle c, s\rangle}{\|c\|\|s\|} = \frac{\langle \sum c_i e^i, s\rangle}{\|\sum c_i e^i\|\|s\|} \geq \frac{\sum c_i \langle e^i, s\rangle}{\sum c_i \|e^i\|\|s\|} = \frac{\sum c_i s_i}{\sum c_i \|s\|} \geq \frac{\min_i\{s_i\}}{\|s\|} \geq \gamma \tag{3.44e}$$

and thus $c \in C_{\gamma,s}$, showing that the cone $C_{\gamma,s}$ contains the Pareto cone. □

Remark 3.2.17 (Relationship to Bishop-Phelps cone)**.** If $0 < \gamma < 1$, then the symmetric cone $C_{\gamma,s}$ in Lemma 3.2.16 is in particular a Bishop-Phelps cone as defined in Definition 3.1.7, and convexity and pointedness also follow from Proposition 3.1.10.

Based on Lemma 3.2.16, we now replace s by $\bar{y}$ to define our variable-cone model as a collection of ideal-symmetric convex cones that contain the Pareto cone. In particular, and mostly for further notational convenience, we denote $\bar{y}_{\min} := \min_i\{\bar{y}_i\}$ so that $\bar{y}_{\min} \geq 0$ for all $y \in Y$, and $\bar{y}_{\min} = 0$ if and only if y is partially ideal from Proposition 3.2.13. Moreover, we choose the parameter γ so to replace the previous term $\gamma\|s\|$ with $0 \leq \gamma \leq s_{\min}\|s\|^{-1}$ by $\gamma \cdot \bar{y}_{\min}$ with $0 \leq \gamma \leq 1$.

Proposition 3.2.18 (Variable-cone model)**.** *Let $Y \subset \mathbb{R}^m$ be a nonempty set, $z \in \mathbb{R}^m$ be the ideal point of Y, and $\mathcal{D} := \{D_\gamma(y) : y \in Y\}$ be a variable domination structure with domination sets*

$$D_\gamma(y) := \{d \in \mathbb{R}^m : \langle d, \bar{y}\rangle \geq \gamma\|d\|\bar{y}_{\min}\} \textit{ for all } y \in Y \tag{3.45}$$

where $\bar{y} := y - z$, $\bar{y}_{\min} := \min_i\{\bar{y}_i\}$, and $0 \leq \gamma \leq 1$. For every $y \in Y$, $D_\gamma(y)$ is an ideal-symmetric convex cone that contains the Pareto cone. In particular, $D_\gamma(y)$ is pointed if and only if y is not partially ideal and $\gamma > 0$.

The proof is clear with Lemma 3.2.16.

3.2.3 Scalarization Methods for Variable-Cone Model

In Section 3.1.3, we show that several scalarization methods that are originally formulated for the Pareto case can be modified to also find nondominated outcomes with respect to general cones. In this section, we investigate some related results when the domination structure is described by a variable cone and, at the same time, derive some more specific results for the variable-cone model that we derive in the previous section and which is completely defined in Proposition 3.2.18.

We begin with the weighted-sum method for a variable domination structure. While the following statement essentially repeats Proposition 3.1.15 in Section 3.1.3, which there we do not prove but derive from the result for the hybrid scalarization in Proposition 3.1.14, we now include the proof and then point to a subtle difference in the formulation of these two results.

Proposition 3.2.19 (Weighted-sum scalarization for a variable domination structure)**.** *Let $Y \subset \mathbb{R}^m$ be nonempty, $\mathcal{D} = \{D(y) : y \in Y\}$ be a domination structure, $w \in \mathbb{R}^m \setminus \{0\}$ be a nonzero weighting parameter, and $\hat{y} \in Y$ be optimal for the weighted-sum scalarization*

$$\textit{WS(w): Minimize } \langle w, y\rangle \textit{ subject to } y \in Y \tag{3.46}$$

(i) If $w \in D(\hat{y})^*$ and $\hat{y}$ is strictly optimal, then $\hat{y} \in N(Y, \mathcal{D})$.

(ii) If $w \in D(\hat{y})^*_s$, then $\hat{y} \in N(Y, \mathcal{D})$.

(iii) If $w \in D(\hat{y})^*$, then $\hat{y} \in N_w(Y, \mathcal{D})$.

Proof. Let $\hat{y} \in Y$ be optimal for WS(w), so $\langle w, \hat{y}\rangle \leq \langle w, y\rangle$ for all $y \in Y$, or $\langle w, \hat{y}\rangle < \langle w, y\rangle$ for all $y \in Y \setminus \{\hat{y}\}$ if $\hat{y}$ is strictly optimal.

For (i) and (ii), suppose by contradiction that $\hat{y} \notin N(Y, \mathcal{D})$. Then there exists $y \in (\hat{y} - D(\hat{y}) \setminus \{0\}) \cap Y$, or equivalently, $\hat{y} - y = d \in D(\hat{y}) \setminus \{0\}$, and thus

$$\langle w, \hat{y}\rangle = \langle w, y + d\rangle = \langle w, y\rangle + \langle w, d\rangle \begin{cases} \geq \langle w, y\rangle & \text{in case (i) as } w \in D(\hat{y})^* \\ > \langle w, y\rangle & \text{in case (ii) as } w \in D(\hat{y})^*_s \end{cases} \tag{3.47}$$

in contradiction to the (in case (i): strict) optimality of $\hat{y}$ for WS(w).

Furthermore for (iii), if $w \in D(\hat{y})^* \setminus \{0\}$ and $\hat{y}$ is optimal for WS(w), then $w \in (\operatorname{int} D(\hat{y}))^*_s$ by Proposition 2.1.21 and $\hat{y} \in N_w(Y, \mathcal{D})$ by (ii). $\square$

While apparently the same result as Proposition 3.1.15, we note that the conditions on the weighting parameter w now also depend on the corresponding solution $\hat{y}$ which, however, itself depends on the weighting vector w. Hence, in practice, we need to first choose w and, after finding $\hat{y}$, we still need to explicitly verify if any of the conditions in Proposition 3.2.19 is actually satisfied, for example by solving the additional single objective cone program

$$\text{Minimize } \langle w, d\rangle \text{ subject to } d \in D(\hat{y}) \setminus \{0\} \tag{3.48}$$

In this case, if the optimal objective function value is positive or nonnegative, we can conclude that $w \in D(\hat{y})^*_s$ or $D(\hat{y})^*$ and thus $\hat{y} \in N(Y, \mathcal{D})$ or $N_w(Y, \mathcal{D})$, respectively.

For our variable-cone model in Proposition 3.2.18, the following corollary to Proposition 3.2.19 is possible based on the observation that, for every outcome $y \in Y$, the ideal domination vector $\bar{y}$ is always contained in the dual cone $D(y)^*$.

Corollary 3.2.20 (Weighted-sum scalarization for variable-cone model)**.** *Let $Y \subset \mathbb{R}^m$ be nonempty, $\mathcal{D} = \{D_\gamma(y) : y \in Y\}$ be defined by $D_\gamma(y) = \{d \in \mathbb{R}^m : \langle d, \bar{y}\rangle \geq \gamma\|d\|\bar{y}_{\min}\}$ with $\bar{y} = y - z$, $\bar{y}_{\min} = \min_i\{\bar{y}_i\}$, and $0 < \gamma \leq 1$ for all $y \in Y$. Let $y^\circ \in Y$, and $\bar{y}^\circ = y^\circ - z \in \mathbb{R}^m$ be the weighting vector for the weighted-sum scalarization*

$$\textit{WS}(\bar{y}^\circ)\textit{: Minimize } \langle \bar{y}^\circ, y\rangle \textit{ subject to } y \in Y \tag{3.49}$$

(i) If $y^\circ \in Y$ is strictly optimal for WS($\bar{y}^\circ$), then $y^\circ \in N(Y, \mathcal{D})$.

(ii) If $y^\circ \in Y$ is optimal for WS($\bar{y}^\circ$) and not partially ideal, then $y^\circ \in N(Y, \mathcal{D})$.

(iii) If $y^\circ \in Y$ is optimal for WS($\bar{y}^\circ$), then $y^\circ \in N_w(Y, \mathcal{D})$.

Proof. Let $y^\circ \in Y$ be optimal for the weighted-sum method WS($\bar{y}^\circ$) with weighting parameter $\bar{y}^\circ \in \mathbb{R}^m$. Since, by definition, $\langle d, \bar{y}^\circ\rangle \geq \gamma\|d\|\bar{y}^\circ_{\min} \geq 0$ for all $d \in D(y^\circ)$, this shows that $\bar{y}^\circ \in D(y^\circ)^*$. Moreover, if y° is not partially ideal, then $\bar{y}^\circ_{\min} > 0$ by Proposition 3.2.13 and, thus, $\langle d, y^\circ\rangle \geq \gamma\|d\|\bar{y}^\circ_{\min} > 0$ for all $d \in D(y^\circ) \setminus \{0\}$, showing that $\bar{y}^\circ \in D(y^\circ)^*_s$. The proof now follows from Proposition 3.2.19. □

Hence, to verify nondominance of $y^\circ \in Y$ under our variable-cone model $\mathcal{D}$, the problem WS($\bar{y}^\circ$) can be solved and, if y° is a unique optimal solution, or if y° is optimal and not partially ideal, then $y^\circ \in N(Y, \mathcal{D})$. In general, however, these conditions are only sufficient, but not necessary and, in particular, satisfied only if the ideal domination vector $\bar{y}$ at y coincides with the normal vector of a supporting hyperplane to Y at y°.

Furthermore, for the generalization of the hybrid method, the constrained-objective and the Benson method, it turns out that after the modification of the cone constraints $b - y \in C$ to *variable* cone constraints $b - y \in D(y)$, the proof of Proposition 3.1.14 does not hold anymore unless we explicitly choose the cone constraints at the optimal solution $\hat{y} \in Y$. In this case, however, we also obtain a necessary condition for nondominance with respect to a variable domination structure, similar to the result in Proposition 2.2.32.

Proposition 3.2.21 (Hybrid scalarization for a variable domination cone)**.** *Let $Y \subset \mathbb{R}^m$ be a nonempty set, $\mathcal{D} = \{D(y) \subset \mathbb{R}^m : y \in Y\}$ be a domination structure for which each $D(y)$ is a convex cone, $w \in \mathbb{R}^m \setminus \{0\}$ be a nonzero weighting parameter, and $\hat{y} \in Y$ be optimal for the hybrid scalarization*

$$\text{HB}(w, b)\text{: Minimize } \langle w, y \rangle \text{ subject to } b - y \in D(\hat{y}) \text{ and } y \in Y \tag{3.50}$$

(i) If $w \in D(\hat{y})^*$ and $\hat{y}$ is strictly optimal, then $\hat{y} \in N(Y, \mathcal{D})$.

(ii) If $w \in D(\hat{y})^*_s$, then $\hat{y} \in N(Y, \mathcal{D})$.

(iii) If $w \in D(\hat{y})^*$, then $\hat{y} \in N_w(Y, \mathcal{D})$.

Moreover, if $\hat{y} \in N(Y, \mathcal{D})$, then $\hat{y}$ is (strictly) optimal for HB($w, \hat{y}$) and any $w \in \mathbb{R}^m$.

Proof. The proof for (i), (ii) and (iii) follows as the proof of Proposition 3.2.19 in which the contradicting solution $y \in Y$ is feasible for HB(w, b) because

$$b - y = b - \hat{y} + d \in D(\hat{y}) \tag{3.51}$$

by feasibility of $\hat{y}$ for HB(w, b) and convexity of the cone $D(\hat{y})$.

Moreover, if $\hat{y} \in N(Y, \mathcal{D})$ is also chosen as the reference point of the hybrid scalarization, then there does not exist $y \in Y$ such that $\hat{y} - y \in D(\hat{y}) \setminus \{0\}$ which implies that $\hat{y}$ is the unique and, thus, a strict optimal solution for HB$(w, \hat{y})$. $\square$

The modification of the constrained-objective and Benson method can be accomplished accordingly and is omitted. Instead, similar to Corollary 3.2.20, we derive a corresponding result for the variable-cone model from the previous section.

Corollary 3.2.22 (Hybrid scalarization for the variable-cone model)**.** *Let $Y \subset \mathbb{R}^m$ be a nonempty set, and $\mathcal{D} = \{D_\gamma(y) : y \in Y\}$ be defined by $D_\gamma(y) = \{d \in \mathbb{R}^m : \langle d, \bar{y} \rangle \geq \gamma \|d\| \bar{y}_{\min}\}$ with $\bar{y} = y - z$, $\bar{y}_{\min} = \min_i\{\bar{y}_i\}$, and $0 < \gamma \leq 1$ for all $y \in Y$. Let $y^\circ \in Y$, and $\bar{y}^\circ = y^\circ - z \in \mathbb{R}^m$ be the weighting vector for the hybrid scalarization*

$$\text{HB}(\bar{y}^\circ, b)\text{: Minimize } \langle \bar{y}^\circ, y \rangle \text{ subject to } b - y \in D(\bar{y}^\circ) \text{ and } y \in Y \tag{3.52}$$

(i) If $y^\circ \in Y$ is strictly optimal for HB($\bar{y}^\circ, b$), then $y^\circ \in N(Y, \mathcal{D})$.

(ii) If $y^\circ \in Y$ is optimal for HB($\bar{y}^\circ, b$) and not partially ideal, then $y^\circ \in N(Y, \mathcal{D})$.

(iii) If $y^\circ \in Y$ is optimal for HB($\bar{y}^\circ, b$), then $y^\circ \in N_w(Y, \mathcal{D})$.

Moreover, $y^\circ \in N(Y, D)$ if and only if y° is (strictly) optimal for HB($\bar{y}^\circ, y^\circ$).

Hence, we conclude that the hybrid method can still be used to verify if any outcome $y \in Y$ is nondominated under the variable-cone model $\mathcal{D}$. In particular, based on the Pareto compatibility of $\mathcal{D}$, Proposition 2.2.6 implies that all nondominated outcomes can be found within the Pareto set.

Proposition 3.2.23. *Let $Y \subset \mathbb{R}^m$ be a nonempty set, and $\mathcal{D} = \{D(y) : y \in Y\}$ be a domination structure for which each domination set $D(y)$ contains the Pareto cone, $\mathbb{R}^m_{\geqq} \subset D(y)$ for all $y \in Y$. Then*

$$N(Y, \mathcal{D}) \subseteq N(Y, \mathbb{R}^m_{\geqq}) \tag{3.53}$$

Hence, to find the nondominated set for our variable-cone model $\mathcal{D}$, the previous discussion suggests to first find the Pareto set $N(Y, \mathbb{R}^m_{\geqq})$ and then check which Pareto points $\hat{y} \in N(Y, \mathbb{R}^m_{\geqq})$ remain nondominated with respect to $D(\hat{y})$. While the latter, in principle, can be accomplished using the necessary and sufficient condition in Corollary 3.2.22, in the next section we derive two alternative optimality conditions based on the geometric interpretation of the underlying ideal-symmetry assumption provided in Remark 3.2.15 and Figure 3.6.

3.2.4 Optimality Conditions for Biobjective Cases

For the special case of convex and concave biobjective programs, we derive two alternative optimality conditions for nondominance with respect to the variable-cone model derived in Section 3.2.2, based on the geometric interpretation of ideal-symmetry in Remark 3.2.15 and Figure 3.6. For the convex case, we use the following version of Theorem 2.1.24 (Rockafellar, 1970).

Lemma 3.2.24 (Supporting hyperplane theorem)**.** *Let $Y \subset \mathbb{R}^m$ be $\mathbb{R}^m_{\geqq}$-convex and $y^\circ \in N(Y, \mathbb{R}^m_{\geqq})$. Then there exists a supporting hyperplane to Y at y° with normal vector $n \in \mathbb{R}^m \setminus \{0\}$ so that $\langle n, y - y^\circ \rangle \geq 0$ for all $y \in Y$.*

Based on this result, our first theorem uses the existence of a supporting hyperplane and its associated normal vector at every Pareto outcome $y^\circ \in N(Y, \mathbb{R}^2_{\geqq})$.

Theorem 3.2.25 (Nondominance conditions for convex biobjective case)**.** *Let $Y \subset \mathbb{R}^2$ be nonempty, and $\mathcal{D} = \{D_\gamma(y) : y \in Y\}$ be defined by $D_\gamma(y) = \{d \in \mathbb{R}^2 : \langle d, \bar{y} \rangle \geq \gamma \|d\| \bar{y}_{\min}\}$ with $\bar{y} = y - z$, $\bar{y}_{\min} = \min_i \{\bar{y}_i\}$, and $0 \leq \gamma \leq 1$ for all $y \in Y$. Let $N(Y, \mathbb{R}^2_{\geqq})$ be $\mathbb{R}^2_{\geqq}$-convex, $y^\circ \in N(Y, \mathbb{R}^2_{\geqq})$, and $n \in \mathbb{R}^m \setminus \{0\}$ be the normal vector of a supporting hyperplane to Y at y° with $\langle n, y - y^\circ \rangle \geq 0$ for all $y \in Y$.*

(i) If $\langle n, \bar{y}^\circ \rangle^2 + \gamma^2 \|n\|^2 \bar{y}^{\circ 2}_{\min} > \|n\|^2 \|\bar{y}^\circ\|^2$, then $y^\circ \in N(Y, \mathcal{D})$.

(ii) If $\langle n, \bar{y}^\circ \rangle^2 + \gamma^2 \|n\|^2 \bar{y}^{\circ 2}_{\min} \geq \|n\|^2 \|\bar{y}^\circ\|^2$, then $y^\circ \in N_w(Y, \mathcal{D})$.

Furthermore, let n satisfy that $\langle n, y - y^\circ \rangle > 0$ for all $y \in Y \setminus \{y^\circ\}$.

(iii) If $\langle n, \bar{y}^\circ \rangle^2 + \gamma^2 \|n\|^2 \bar{y}^{\circ 2}_{\min} \geq \|n\|^2 \|\bar{y}^\circ\|^2$, then $y^\circ \in N(Y, \mathcal{D})$.

Proof. Let $N(Y, \mathbb{R}^2_{\geqq})$ be $\mathbb{R}^2_{\geqq}$-convex and $y^\circ \in N(Y, \mathbb{R}^2_{\geqq})$. As shown in Figure 3.7, let $\eta = \measuredangle(n, \bar{y}^\circ)$ be the positive angle between n and $\bar{y}^\circ$, so

$$0 \leq \cos \eta = \cos \measuredangle(n, \bar{y}^\circ) = \frac{\langle n, \bar{y}^\circ \rangle}{\|n\| \|\bar{y}^\circ\|} \leq 1 \tag{3.54a}$$

and $0 \leq \sin \eta \leq 1$. Let $\delta = \max\{\measuredangle(d, \bar{y}^\circ) : d \in D_\gamma(y^\circ) \setminus \{0\}\}$ be the maximal positive angle between any $d \in D_\gamma(y^\circ) \setminus \{0\}$ and $\bar{y}^\circ$, so

$$0 \leq \cos \delta = \min \left\{ \frac{\langle d, \bar{y}^\circ \rangle}{\|d\| \|\bar{y}^\circ\|} : \langle d, \bar{y}^\circ \rangle \geq \gamma \|d\| \bar{y}^\circ_{\min}, d \neq 0 \right\} = \frac{\gamma \bar{y}^\circ_{\min}}{\|\bar{y}^\circ\|} \leq 1 \tag{3.54b}$$

and $0 \leq \sin\delta \leq 1$. Finally, let $\mu = \max\{\measuredangle(n, d) : d \in D_\gamma(y^\circ) \setminus \{0\}\}$ be the maximal positive angle between n and any $d \in D_\gamma(y^\circ) \setminus \{0\}$ at y, so $\mu = \eta + \delta$ and

$$\cos\mu = \min\left\{\frac{\langle n, d\rangle}{\|n\|\|d\|} : d \in D_\gamma(y^\circ) \setminus \{0\}\right\} \tag{3.54c}$$

For (i), the assumption $\langle n, \bar{y}^\circ\rangle^2 + \gamma^2\|n\|^2\bar{y}^{\circ 2}_{\min} > \|n\|^2\|\bar{y}^\circ\|^2$ is equivalent to

$$\frac{\langle n, \bar{y}^\circ\rangle^2}{\|n\|^2\|\bar{y}^\circ\|^2} + \frac{\gamma^2\bar{y}^{\circ 2}_{\min}}{\|\bar{y}^\circ\|^2} = \cos^2\eta + \cos^2\delta > 1 \tag{3.55a}$$

yielding $\cos^2\eta > 1 - \cos^2\delta = \sin^2\delta$ and $\cos^2\delta > 1 - \cos^2\eta = \sin^2\eta$. In particular, this implies that $\cos\eta > \sin\delta$ and $\cos\delta > \sin\eta$, and thus

$$\cos\eta\cos\delta - \sin\eta\sin\delta = \cos(\eta + \delta) = \cos\mu > 0 \tag{3.55b}$$

and equivalence follows from repeating the same argument with $<$ instead of $>$. Hence, by definition of μ, we first conclude that $\langle n, \bar{y}^\circ\rangle^2 + \gamma^2\|n\|^2\bar{y}^{\circ 2}_{\min} > \|n\|^2\|\bar{y}^\circ\|^2$ if and only if

$$\cos\mu = \min\left\{\frac{\langle n, d\rangle}{\|n\|\|d\|} : d \in D_\gamma(y^\circ) \setminus \{0\}\right\} > 0 \tag{3.55c}$$

or equivalently, $\langle n, d\rangle\|n\|^{-1}\|d\|^{-1} > 0$ and, thus, $\langle n, d\rangle > 0$ for all $d \in D_\gamma(y^\circ) \setminus \{0\}$. Since, by assumption, $\langle n, y - y^\circ\rangle \geq 0$ for all $y \in Y$, or $\langle n, y^\circ - y\rangle \leq 0$, it therefore follows that $y^\circ - y \notin D_\gamma(y^\circ) \setminus \{0\}$, showing that $y^\circ \in N(Y, \mathcal{D})$.

For (ii), repeating the same arguments as above yields that the condition $\langle n, \bar{y}^\circ\rangle^2 + \gamma^2\|n\|^2\bar{y}^{\circ 2}_{\min} \geq \|n\|^2\|\bar{y}^\circ\|^2$ is equivalent to $\langle n, d\rangle \geq 0$ for all $d \in D_\gamma(y^\circ)$ and, thus, $\langle n, d\rangle > 0$ for all $d \in \operatorname{int} D_\gamma(y^\circ)$. Analogous to the proof in (i), it now follows that $y^\circ - y \notin \operatorname{int} D_\gamma(y^\circ) \setminus \{0\}$, showing that $y^\circ \in N_w(Y, \mathcal{D})$.

Furthermore for (iii), if $\langle n, y - y^\circ\rangle > 0$ for all $y \in Y \setminus \{y^\circ\}$, or $\langle n, y^\circ - y\rangle < 0$, then the same conclusion as in (i) already follows if $\langle n, d\rangle \geq 0$ for all $d \in D_\gamma(y^\circ)$ as direct implication of the derivation in (ii). $\square$

Remark 3.2.26 (Extension to multiple objectives)**.** Mainly using trigonometric concepts from basic analytic geometry, the proof of Theorem 3.2.25 readily extends to the case with three objectives for which the interpretation of angles remains valid and, in particular, preserves the same geometric meaning as in the proven biobjective case. For $m > 3$, however, the argumentation loses this geometric interpretation and, thus, the theorem might not hold anymore.

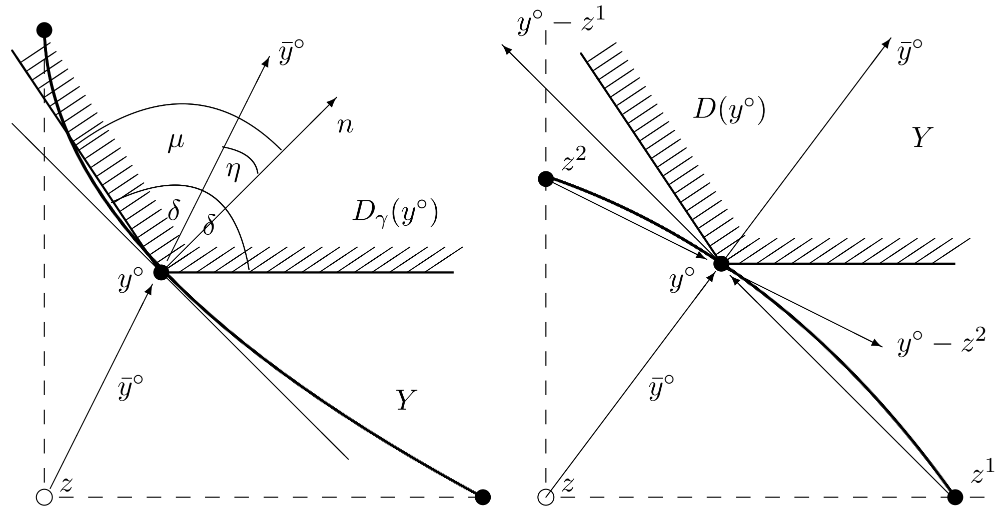


Figure 3.7 Illustration of the nondominance conditions for biobjective programs in Theorems 3.2.25 (left) and 3.2.27 (right) under the variable-cone model

We derive a similar result for the concave biobjective case for the particular choice $D_\gamma(y) = D_1(y)$ and under the additional assumption that Y is $\mathbb{R}^2_{\geqq}$-compact. In this case, the ideal point $z = \{z_1, z_2\} \in \mathbb{R}^2$ can be defined using the minimum, $z_i = \min\{y_i : y \in Y\}$ for $i = 1, 2$, so that the two optimal lexicographic solutions

$$z^1 = (z_1^1, z_2) \text{ where } z_1^1 := \min\{y_1 : y_2 = z_2, y \in Y\} \tag{3.56a}$$

$$z^2 = (z_1, z_2^2) \text{ where } z_2^2 := \min\{y_2 : y_1 = z_1, y \in Y\} \tag{3.56b}$$

belong both to the outcome set and the Pareto set, as depicted in Figure 3.7 (right).

Theorem 3.2.27 (Nondominance condition for concave biobjective case)**.** *Let $Y \subset \mathbb{R}^2$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^2 : \langle d, \bar{y}\rangle \geq \|d\|\bar{y}_{\min}\}$ with $\bar{y} = y - z$ and $\bar{y}_{\min} = \min_i\{\bar{y}_i\}$ for all $y \in Y$. Let Y be $\mathbb{R}^2_{\geqq}$-compact, $N(Y, \mathbb{R}^2_{\geqq})$ be $\mathbb{R}^2_{\geqq}$-concave (i.e., $-\mathbb{R}^2_{\geqq}$-convex), $y^\circ \in N(Y, \mathbb{R}^2_{\geqq})$, and $k \in \{1, 2\}$ be so* *that* $\bar{y}^\circ_k = \bar{y}^\circ_{\min}$. *Let* $z = \{z_1, z_2\} \in \mathbb{R}^2$ *be the ideal point of* Y *and* z^1 *and* $z^2 \in Y$ *be as in (3.56). Then*

$$y^\circ \in N(Y, \mathcal{D}) \text{ if and only if } y^\circ - z^k \notin D(y^\circ) \tag{3.57}$$

Proof. Let Y be $\mathbb{R}^2_{\geqq}$-compact. As outlined in the discussion preceding the statement of the theorem, it follows that $z^1, z^2 \in N(Y, \mathbb{R}^2_{\geqq})$, and $z^1 \neq z^2$ as the ideal point $z \notin Y$. Moreover, since $\bar{z}^1 = (z^1_1 - z_1, 0) \geq 0$, it follows that $D(z^1) = \{d = (d_1, d_2) \in \mathbb{R}^2 : d_1 \geq 0\}$ and, thus, $z^2 \in (z^1 - D(z^1)) \cap Y \setminus \{z^1\}$. This shows that $z^1 \notin N(Y, \mathcal{D})$ and, by repeating the analogous argument for z^2, that $z^2 \notin N(Y, \mathcal{D})$.

($\Leftarrow$) Now let $N(Y, \mathbb{R}^2_{\geqq})$ be $\mathbb{R}^2_{\geqq}$-concave and $y^\circ \in N(Y, \mathbb{R}^2_{\geqq}) \setminus \{z^1, z^2\}$, so, in particular, $z_1 < y^\circ_1 < z^1_1$ and $z_2 < y^\circ_2 < z^2_2$. By $\mathbb{R}^2_{\geqq}$-concavity, then there does not exist $y \in Y$ that falls below the two line segments from y° to z^1 and z^2, as indicated in Figure 3.7. In particular, if we can show that $y^\circ - z^1, y^\circ - z^2 \notin D(y^\circ)$, then there does not exist $y \in (y^\circ - D(y^\circ)) \cap Y \setminus \{y^\circ\}$ which implies that $y^\circ \in N(Y, \mathcal{D})$.

Hence, without loss of generality, let $\bar{y}^\circ_{\min} = \bar{y}^\circ_1 \leq \bar{y}^\circ_2$, then $y^\circ - z^1 \notin D(y^\circ)$ by assumption and it only remains to show that $y^\circ - z^2 \notin D(y^\circ)$. But this follows, because

$$\langle y^\circ - z^2, \bar{y}^\circ \rangle - \|y^\circ - z^2\| \bar{y}^\circ_1 \tag{3.58a}$$

$$= \langle y^\circ - z^2, y^\circ - z \rangle - \|y^\circ - z^2\| \cdot \bar{y}^\circ_1 \tag{3.58b}$$

$$= (y^\circ_1 - z_1)^2 + (y^\circ_2 - z^2_2)(y^\circ_2 - z_2) - \sqrt{(y^\circ_1 - z_1)^2 + (y^\circ_2 - z^2_2)^2} \cdot \bar{y}^\circ_1 \tag{3.58c}$$

$$= \bar{y}^{\circ 2}_1 + (y^\circ_2 - z^2_2)\bar{y}^\circ_2 - \sqrt{\bar{y}^{\circ 2}_1 + (y^\circ_2 - z^2_2)^2} \cdot \bar{y}^\circ_1 \tag{3.58d}$$

$$\leq \bar{y}^{\circ 2}_1 + (y^\circ_2 - z^2_2)\bar{y}^\circ_2 - \bar{y}^{\circ 2}_1 \tag{3.58e}$$

$$= (y^\circ_2 - z^2_2)\bar{y}^\circ_2 \tag{3.58f}$$

$$= (y^\circ_2 - z^2_2)(y^\circ_2 - z_2) < 0 \tag{3.58g}$$

from $z_2 < y^\circ_2 < z^2_2$. The reverse direction ($\Rightarrow$) is clear because $z^1, z^2 \in Y$. $\square$

The point $(z^1_1, z^2_2) \in \mathbb{R}^2$ in Theorem 3.2.27 is also known as the *nadir point* (see Miettinen, 1999, among others), and more general, for a set $Y \subset \mathbb{R}^m$, the nadir point can defined by $z^N = \{z^N_1, \ldots, z^N_m\}$ with $z^N_i := \sup\{y_i : y \in N(Y, \mathbb{R}^m_{\geqq})\}$.

Remark 3.2.28 (Extension to multiple objectives)**.** The proof of Theorem 3.2.27 only holds for the biobjective case $m = 2$, based on the exploited characterization of the nadir point using the two optimal lexicographic solutions in (3.56). For $m > 2$, however, the nadir point cannot be computed by lexicographic optimization but must be found through optimization over the Pareto set, so that this point is not readily available (Yamamoto, 2002; Ehrgott and Tenfelde-Podehl, 2003), in general.

3.2.5 Model Illustration on Biobjective Examples

We illustrate the possible application of Theorems 3.2.25 and 3.2.27 for finding the nondominated set for each of the three sets $Y \subset \mathbb{R}^2$ depicted in Figure 3.8. In each case, the ideal point lies at the origin so that $z = (0, 0) \in \mathbb{R}^2$ and $\bar{y} = y$ for all $y \in Y$. Furthermore, to apply both Theorem 3.2.25 and 3.2.27, we choose $\gamma = 1$ and, thus, let the domination structure $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^2 : \langle d, y\rangle \geq \|d\| y_{\min}\}$ for all $y \in Y$.

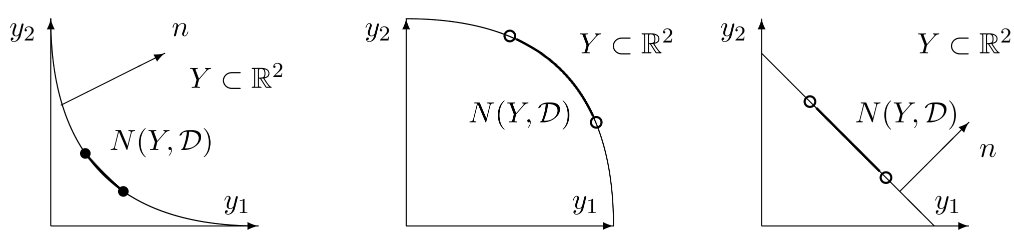


Figure 3.8 Nondominated sets $N(Y, \mathcal{D})$ for the variable-cone model and convex (left), concave (center), and linear (right) Pareto sets as in Examples 3.2.29–3.2.31

Example 3.2.29 (Convex case)**.** Let $Y = \{y \in \mathbb{R}^2 : (1 - y_1)^2 + (1 - y_2)^2 \leq 1\}$ be as depicted in Figure 3.8 (left). Then $N(Y, \mathbb{R}^2_{\geqq}) = \{y \in Y : (1 - y_1)^2 + (1 - y_2)^2 = 1,\ y_1 \leq 1,\ y_2 \leq 1\}$ is $\mathbb{R}^2_{\geqq}$-convex, and Theorem 3.2.25 can be used to find $N(Y, \mathcal{D})$. Hence, let $y \in N(Y, \mathbb{R}^2_{\geqq})$, so

$$(1 - y_1)^2 + (1 - y_2)^2 = 1 \text{ for } 0 \leq y_1 \leq 1 \text{ and } 0 \leq y_2 \leq 1 \tag{3.59a}$$

and, without loss of generality, assume that $y_{\min} = y_1 \leq y_2$, so $0 \leq y_1 \leq 1 - \frac{1}{2}\sqrt{2}$ or $1 - y_1 \geq \frac{1}{2}\sqrt{2} > 0$. Since the supporting hyperplane at y has the normal vector $n = (1-y_1, 1-y_2) \in \mathbb{R}^2$ that satisfies $\|n\|^2 = 1$ and $\langle n, y-y' \rangle > 0$ for all $y' \in Y \setminus \{y\}$, the condition in Theorem 3.2.25 becomes

$$\begin{aligned}
& \langle n, y\rangle^2 + \|n\|^2 y_{\min}^2 - \|n\|^2\|y\|^2 \\
= \quad & [(1-y_1)y_1 + (1-y_2)y_2]^2 + y_1^2 - (y_1^2 + y_2^2) \\
= \quad & (1 - y_1 - y_2)^2 - y_2^2 \\
= \quad & (1-y_1)^2 - 2(1-y_1)y_2 \\
= \quad & (1-y_1)(1-y_1-2y_2) \geq 0 \\
\Leftrightarrow \quad & (1-y_1) \geq 2y_2 \\
\Leftrightarrow \quad & (1-y_1)^2 = 1 - (1-y_2)^2 = 2y_2 - y_2^2 \geq 4y_2^2 \\
\Leftrightarrow \quad & y_1 \leq y_2 \leq \tfrac{2}{5}
\end{aligned}$$

Using Equation 3.59a, y_1 and y_2 can equivalently be restricted from below by $\frac{1}{5}$, so that $\frac{1}{5} \leq y_1 \leq y_2 \leq \frac{2}{5}$ and thus, by symmetry of Y in y_1 and y_2

$$N(Y, \mathcal{D}) = \left\{y \in Y : (1-y_1)^2 + (1-y_2)^2 = 1, \tfrac{1}{5} \leq y_1, y_2 \leq \tfrac{2}{5}\right\} \tag{3.59b}$$

as shown in Figure 3.8 (left).

If Y is $\mathbb{R}^2_{\geqq}$-concave, we can use Theorem 3.2.27 instead of Theorem 3.2.25.

Example 3.2.30 (Concave case)**.** Let $Y = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 \geq 1,\ y_1 \geq 0,\ y_2 \geq 0\}$ be as depicted in Figure 3.8 (center). Then $N(Y, \mathbb{R}^2_{\geqq}) = \{y \in Y : y_1^2 + y_2^2 = 1\}$ is $\mathbb{R}^2_{\geqq}$-concave, and Theorem 3.2.27 can be used to find $N(Y, \mathcal{D})$. Hence, let $y \in N(Y, \mathbb{R}^2_{\geqq})$

$$y_1^2 + y_2^2 = 1 \text{ for } 0 \leq y_1 \leq 1 \text{ and } 0 \leq y_2 \leq 1 \tag{3.60a}$$

and, without loss of generality, assume that $y_{\min} = y_1 \leq y_2$, so $0 \leq y_1 \leq \frac{1}{2}\sqrt{2}$ or $1 - y_1 \geq 1 - \frac{1}{2}\sqrt{2}$. Since $z^1 = (1, 0)$ and $z^2 = (0, 1)$, the condition in Theorem 3.2.27 becomes

$$\begin{aligned}
& \langle y - z^1, y\rangle - \|y - z^1\| y_1 \\
= \quad & (y_1 - 1)y_1 + y_2^2 - \sqrt{(y_1-1)^2 + y_2^2} \cdot y_1 \\
= \quad & 1 - y_1 - \sqrt{2 - 2y_1} \cdot y_1 \\
= \quad & \sqrt{1 - y_1}(\sqrt{1 - y_1} - \sqrt{2} \cdot y_1) < 0 \\
\Leftrightarrow \quad & 2y_1^2 > 1 - y_1 \quad \Leftrightarrow \quad \tfrac{1}{2} < y_1 \leq y_2
\end{aligned}$$

Using Equation 3.60a, y_1 and y_2 can equivalently be restricted from above by $\frac{1}{2}\sqrt{3}$ so that $\frac{1}{2} < y_1 \leq y_2 < \frac{1}{2}\sqrt{3}$ and thus, by symmetry of Y in y_1 and y_2

$$N(Y, \mathcal{D}) = \left\{ y \in Y : y_1^2 + y_2^2 = 1, \tfrac{1}{2} < y_1, y_2 < \tfrac{1}{2}\sqrt{3} \right\} \tag{3.60b}$$

again shown in Figure 3.8 (center).

The concluding example considers the same set previously discussed in Example 3.2.9 and finally shows how the new variable-cone model resolves the limitations highlighted in our earlier discussion.

Example 3.2.31 (Linear case)**.** Let $Y = \{y \in \mathbb{R}^2 : y_1 + y_2 \geq 1,\ y_1 \geq 0,\ y_2 \geq 0\}$ be as depicted in Figure 3.8 (right). Then $N(Y, \mathbb{R}^2_{\geqq}) = \{y \in Y : y_1 + y_2 = 1\}$ is both $\mathbb{R}^2_{\geqq}$-convex and $\mathbb{R}^2_{\geqq}$-concave so that both Theorems 3.2.25 and 3.2.27 can be used to find $N(Y, \mathcal{D})$. Hence, let $y \in N(Y, \mathbb{R}^2_{\geqq})$, so

$$y_1 + y_2 = 1 \text{ for } 0 \leq y_1 \leq 1 \text{ and } 0 \leq y_2 \leq 1 \tag{3.61a}$$

and, without loss of generality, assume that $y_{\min} = y_1 \leq y_2$, so $0 \leq y_{\min} = y_1 \leq \frac{1}{2} \leq y_2$. Since the supporting hyperplane at y has the normal vector $n = (1, 1)$ that satisfies $\langle n, y - y'\rangle \geq 0$ for all $y' \in Y$, the condition in Theorem 3.2.25 becomes

$$\begin{aligned}
& \langle n, y\rangle^2 + \|n\|^2 y_{\min}^2 - \|n\|^2 \|y\|^2 \\
= \quad & (y_1 + y_2)^2 + 2y_1^2 - 2(y_1^2 + y_2^2) \quad = \quad 1 - 2y_2^2 \quad > \quad 0 \\
\Leftrightarrow \quad & \qquad\qquad\qquad y_1 \quad \leq \quad y_2 \quad < \quad \tfrac{1}{2}\sqrt{2}
\end{aligned}$$

Using Equation 3.61a, y_1 and y_2 can equivalently be restricted from below by $1-\frac{1}{2}\sqrt{2}$ so that $1-\frac{1}{2}\sqrt{2} < y_1 \leq y_2 < \frac{1}{2}\sqrt{2}$ and thus, by symmetry of Y in y_1 and y_2

$$N(Y,\mathcal{D}) = \left\{y \in Y : y_1 + y_2 = 1, 1-\tfrac{1}{2}\sqrt{2} < y_1, y_2 < \tfrac{1}{2}\sqrt{2}\right\} \tag{3.61b}$$

as shown in Figure 3.8 (right). Alternatively, with $z^1 = (1,0)$ and $z^2 = (0,1)$, the condition in Theorem 3.2.27 becomes

$$\begin{aligned}
& \langle y - z^1, y\rangle - \|y - z^1\| y_1 \\
= \quad & (y_1 - 1)y_1 + y_2^2 - \sqrt{(y_1-1)^2 + y_2^2} \cdot y_1 \\
= \quad & (y_1 - 1)y_1 + (1-y_1)^2 - \sqrt{(y_1-1)^2 + (1-y_1)^2} \cdot y_1 \\
= \quad & (y_1 - 1)(2y_1 - 1) - \sqrt{2}(1-y_1)y_1 \\
= \quad & (y_1 - 1)(2y_1 - 1 + \sqrt{2}y_1) < 0 \\
\Leftrightarrow \quad & 2y_1 - 1 + \sqrt{2} > 0 \\
\Leftrightarrow \quad & y_1 > \tfrac{1}{2+\sqrt{2}} = \tfrac{2-\sqrt{2}}{2} = 1 - \tfrac{1}{2}\sqrt{2}
\end{aligned}$$

yielding the same answer as Theorem 3.2.25.

In conclusion, we observe that the variable-cone model derived and studied in this section principally removes the model limitation highlighted in Example 3.2.9. Since the underlying assumption of ideal-symmetry, however, is only one among many possibilities to introduce variability into the domination structure of a multiobjective program, many further research directions are possible and addressed as part of the concluding discussion at the end of this chapter.

3.3 Translated Cones and Approximate Nondominance

The purpose of this section is to characterize a notion of approximate nondominance, namely the concept of epsilon-nondominance (Kutateladze, 1979; Loridan, 1984; White, 1986), using the framework of cones and to develop various solution methods for the generation of epsilon-nondominated outcomes. In preparation for the former and based on the observation that the underlying domination structure cannot be described by a cone anymore, but needs to be defined as a cone that is shifted from the origin, we begin our investigation in Section 3.3.1 with the definition and possible representations of translated cones. In Sections 3.3.2 and 3.3.3, we reexamine several of our previous findings for this new class of cones, including the nondominance mapping theorem from Section 3.1.2 and the scalarization methods from Section 3.1.3, respectively, and we derive various additional results in this new context of approximate nondominance. Sections 3.3.4, 3.3.5 and 3.3.6 finally address the question of purposely generating ε-nondominated outcomes from a theoretical, methodological, and practical point of view, respectively.

3.3.1 Representation of Translated Cones

We first introduce the notion of a translated cone and then study some general properties and possible representations of translated cones as polyhedral sets, using some of the elementary concepts from linear algebra and convex analysis that are reviewed in Section 2.1.2.

Definition 3.3.1 (Translated cone)**.** A nonempty set $D \subset \mathbb{R}^m$ is called a *translated cone* if there exists a cone $C \subset \mathbb{R}^m$ and a vector $e \in \mathbb{R}^m$ such that

$$D = C_e := C + e \tag{3.62}$$

In this case, the vector e is called a *translation vector* for D.

Following the classical definition by Rockafellar (1970), a translated cone $C_e \subset \mathbb{R}^m$ is not necessarily a cone anymore but has its vertex shifted from the origin to the new point $e \in \mathbb{R}^m$. In this case, however, a more general notion presented in Luenberger (1969) also allows for a cone with an arbitrary vertex and corresponds to what Rockafellar calls a skew orthant or generalized m-dimensional simplex with one ordinary vertex and $m-1$ directions (or vertices at infinity). Other than the above two monographs, Nachbin (1996) introduces affine cones to describe mappings between convex vector spaces, and Bauschke (2003) mentions translated cones in the context of duality results for Bregman projections onto linear constraints. None of these references, however, contributes any specific results to the representation of translated cones as now pursued in the remaining section.

First, if D is a translated cone, then there exists a unique cone C such that $D = C_e$, and the vector e is unique if C is also pointed.

Proposition 3.3.2 (Uniqueness of cone and translation vector)**.** *Let $C, C^1, C^2 \subset \mathbb{R}^m$ be cones and $e^1, e^2 \in \mathbb{R}^m$.*

(i) If $C^1_{e^1} = C^2_{e^2}$, then $C^1 = C^2$.

(ii) If $C_{e^1} = C_{e^2}$ and C is pointed, then $e^1 = e^2$.

Proof. For (i), suppose $C^1_{e^1} = C^2_{e^2}$ and let $d \in C^1$. Then $d + e^1 \in C^1_{e^1} = C^2_{e^2}$ and thus $d + e^1 - e^2 \in C^2$. Since C^2 is a cone, we also have that $\frac{1}{2}(d + e^1 - e^2) \in C^2$, thus $e^2 + \frac{1}{2}(d + e^1 - e^2) \in C^2_{e^2} = C^1_{e^1}$ and $e^2 + \frac{1}{2}(d + e^1 - e^2) - e^1 = \frac{1}{2}(d + e^2 - e^1) \in C^1$. The fact that C^1 is a cone now gives that also $d + e^2 - e^1 \in C^1$, and we conclude $d + e^2 - e^1 + e^1 = d + e^2 \in C^1_{e^1} = C^2_{e^2}$ and finally $d + e^2 - e^2 = d \in C^2$. Hence $C^1 \subseteq C^2$, and by interchanging the roles of C^1 and C^2 we obtain the result.

For (ii), let $C^\circ = C \cup \{0\}$, then also $C^\circ_{e_1} = C^\circ_{e_2}$ and, in particular, $e^1 \in C_{e^1} = C_{e^2} \ni e^2$. Then there must exist vectors $d^1, d^2 \in C$ such that $e^1 = d^2 + e^2$ and $e^2 = d^1 + e^1$, yielding that $d^1 = -d^2$ and thus $d^1 = d^2 = 0$ as C is pointed, showing that $e^1 = e^2$ and concluding the proof. □

If C is not pointed, then the translation vector is not unique, in general.

Example 3.3.3 (Translated cone with multiple translation vectors)**.** If we let

$$C = \{d = (d_1, d_2)^T \in \mathbb{R}^2 : d_1 \geq 0\} \tag{3.63}$$

then $C = C_e$ for all vectors $e \in \{(e_1, e_2)^T \in \mathbb{R}^2 : e_1 = 0\}$.

Although a translated cone is not necessarily a cone, we also say that $D = C_e$ is pointed if the underlying cone C is pointed. In particular, if $C = C(A) = \{d \in \mathbb{R}^m : Ad \geqq 0\}$ is a polyhedral cone induced by some matrix $A \in \mathbb{R}^{l \times m}$, then we know from Proposition 2.1.28 that $C(A)$ is pointed if and only if $\operatorname{rank} A = m$. An equivalent condition uses the concept of the kernel of a polyhedral set or cone which we introduce in the following definition and then utilize to characterize the set of suitable translation vectors for translated polyhedral cones.

Definition 3.3.4 (Kernel)**.** Let $D(A,b) \subset \mathbb{R}^m$ be the polyhedral set described by the matrix $A \in \mathbb{R}^{l\times m}$ and vector $b \in \mathbb{R}^m$. The set

$$\ker D(A,b) := \{y \in \mathbb{R}^m : Ay = b\} \tag{3.64}$$

is called the *kernel* of $D(A,b)$.

If A and b are clear from the context, then we simply write $\ker D$ instead of $\ker D(A,b)$. However, since the representation $D(A,b)$ of a polyhedral set is not unique, in general, the kernel also depends on the specific choices of the matrix $A \in \mathbb{R}^{l\times m}$ and the vector $b \in \mathbb{R}^l$.

Example 3.3.5 (Dependence of kernel on representation)**.** If we let $A = (1,1)^T \in \mathbb{R}^{2\times 1}$, $b = (1,0)^T \in \mathbb{R}^2$, and

$$D^1 := D(A,b) = \{d \in \mathbb{R} : d \geq 1, d \geq 0\} = \{d \in \mathbb{R} : d \geq 1\} \tag{3.65}$$

be the associated polyhedral set (in this case: cone), then $\ker D^1 = \emptyset$. However, if we let $D^2 := D(1,1)$, then $D^1 = D^2$ but $\ker D^2 = \{1\}$.

Next, we formulate the fundamental result for the characterization and representation of translated polyhedral cones as polyhedral sets.

Theorem 3.3.6 (Translated polyhedral cones)**.** *Let $C = D(A)$ be the polyhedral cone induced by some matrix $A \in \mathbb{R}^{l\times m}$.*

(i) Let $e \in \mathbb{R}^m$ be a translation vector for a translated polyhedral cone C_e. Let $b = Ae \in \mathbb{R}^l$ and $D = D(A,b)$ be the polyhedral set induced by A and b. Then

$$C_e = D \tag{3.66a}$$

(ii) Let $b \in \mathbb{R}^l$ and $D = D(A,b)$ be the polyhedral set induced by A and b. Let $\ker D(A,b) \neq \emptyset$ and $e \in \ker D(A,b)$ be any kernel element of D. Then

$$D = C_e \tag{3.66b}$$

In both cases (i) and (ii), C and D are pointed if and only if $\operatorname{rank} A = m$, or equivalently, if $\ker C = \{0\}$.

Proof. For (i), we have to show that $C_e = D$ where $C = D(A)$ and $D = D(A,b)$ with $b = Ae$. For the first inclusion, let $d = c+e \in C_e$ for some $c \in C$. Then $Ac \geqq 0$ and it follows that

$$Ad = A(c+e) = Ac + Ae \geqq 0 + b = b \tag{3.67a}$$

which implies that $d \in D$, showing that $C_e \subseteq D$. For the reversed inclusion, let $d \in D$. Then $Ad \geqq b = Ae$ and it follows that

$$Ad - Ae = A(d-e) \geqq 0 \Longrightarrow d - e \in C \Longrightarrow d \in C + e = C_e \tag{3.67b}$$

which shows that also $D \subseteq C_e$ and, thus, $C_e = D$.

For (ii), we have to show that if $D = D(A,b)$ and $\ker D(A,b) \neq \emptyset$, then $D = C_e$ where $C = D(A)$ and $e \in \ker D(A,b)$. Hence, let $D = D(A,b)$ and $e \in \ker D(A,b)$. Then $Ae = b$ and it follows that

$$d \in D \Longleftrightarrow Ad \geqq b = Ae \Longleftrightarrow Ad - Ae = A(d-e) \geqq 0 \Longleftrightarrow d - e \in C \tag{3.67c}$$

which implies that $D - e = C$ and therefore $D = C_e$.

By definition, the translated cone $D = C_e$ is pointed if and only if C is pointed, or equivalently, if $\operatorname{rank} A = m$ or $Ac = 0$ if and only if $c = 0$. Clearly , in this case $\ker C = \{0\}$ and the proof is complete. $\square$

Hence, we conclude that every translated polyhedral cone $D = C_e$ with $C = C(A)$ can be described as a polyhedral set $D = D(A,b)$, and that a polyhedral set $D = D(A,b)$ describes a translated polyhedral cone if its kernel $\ker D(A,b)$ is nonempty. In this case, we also call D a *polyhedral translated cone.*

Based on the interpretation of $\ker D(A,b)$ as the solution set of the inhomogeneous system of linear equations $Ad = b$, we can also formulate the following alternative characterization as our concluding corollary.

Corollary 3.3.7 (Polyhedral translated cones)**.** *Let $A \in \mathbb{R}^{l \times m}$ be a real $l \times m$ matrix, $b \in \mathbb{R}^l$ be a real vector, and $D = D(A,b) \subset \mathbb{R}^m$ be the associated polyhedral set. Then D is a* *(i) polyhedral translated cone if* $\operatorname{rank} A = \operatorname{rank}(A, b)$*, and a*

(ii) pointed polyhedral translated cone if $\operatorname{rank} A = \operatorname{rank}(A, b) = m$*.*

In both cases, $\ker D(A, b) \neq \emptyset$ *and* $D = C_e$ *with* $C = C(A)$ *the polyhedral cone induced by* A *and* $e \in \ker D(A, b)$ *a (in case (ii): the unique) translation vector.*

In particular, we obtain that the polyhedral cone $C = C(A)$ is always unique from Proposition 3.3.2.

3.3.2 Cone Characterizations of Epsilon-Nondominance

We first introduce the notion of ε-nondominance for general cones in extension of the concept of additive ε-efficiency in Definition 2.2.16 and similar to the definition in Helbig and Pateva (1994). We then highlight the relationship to translated cones, provide various characterizations for the associated ε-nondominated set, and use the representation of translated cones as polyhedral sets to finally derive some more specific results for ε-nondominance with respect to a polyhedral cone.

Definition 3.3.8 (Epsilon-Nondominance)**.** Let $Y \subset \mathbb{R}^m$ be a nonempty set, $C \subset \mathbb{R}^m$ be a cone, and $\varepsilon \in C$ be a vector. The sets

$$N_s(Y, C, \varepsilon) := \{y \in Y : (y - \varepsilon - C \cup \{0\}) \cap Y = \emptyset\} \tag{3.68a}$$

$$N(Y, C, \varepsilon) := \{y \in Y : (y - \varepsilon - C \setminus \{0\}) \cap Y = \emptyset\} \tag{3.68b}$$

$$N_w(Y, C, \varepsilon) := \{y \in Y : (y - \varepsilon - \operatorname{int} C) \cap Y = \emptyset\} \tag{3.68c}$$

are called the *strictly ε-nondominated*, *ε-nondominated*, and *weakly ε-nondominated set* of Y with respect to C, respectively.

We immediately see that $N_s(Y, C, \varepsilon) \subseteq N(Y, C, \varepsilon) \subseteq N_w(Y, C, \varepsilon)$. Moreover, for $\varepsilon = 0$, we note that $N(Y, C, 0) = N(Y, C)$ and $N_w(Y, C, 0) = N_w(Y, C)$ reduce to the *nondominated* and *weakly nondominated set*, respectively, while the set of strictly (0-)nondominated outcomes is empty and, thus, originally not defined in Definition 2.2.5. Similarly, in the special case that $C = \mathbb{R}^m_{\geqq}$ is the Pareto cone, $N(Y, \mathbb{R}^m_{\geqq}, \varepsilon) = N(Y, \varepsilon)$ and $N_w(Y, \mathbb{R}^m_{\geqq}, \varepsilon) = N_w(Y, \varepsilon)$ correspond to the sets of *ε-Pareto* and *weak ε-Pareto outcomes* that we first introduce after Definition 2.2.16.

Furthermore, we note that the sets $C_\varepsilon = C + \varepsilon$ and $\operatorname{int} C_\varepsilon = \operatorname{int}(C+\varepsilon) = \operatorname{int} C + \varepsilon$ are translated cones with translation vector ε, so that the ε-nondominated set with respect to a cone C can also be described as a nondominated set with respect to the translated cone C_ε.

Proposition 3.3.9 (Translated cone nondominance)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a pointed convex cone, and $\varepsilon \in C \setminus \{0\}$. Then*

(i) $N_s(Y, C, \varepsilon) = N(Y, D)$ for $D = C \cup \{0\} + \varepsilon$,

(ii) $N(Y, C, \varepsilon) = N(Y, D)$ for $D = C \setminus \{0\} + \varepsilon$,

(iii) $N_w(Y, C, \varepsilon) = N(Y, D)$ for $D = \operatorname{int} C + \varepsilon$, or alternatively

(iv) $N_w(Y, C, \varepsilon) = N_w(Y, D)$ for $D = C + \varepsilon$.

Proof. Since $N(Y, D) := \{y \in Y : (y - D \setminus \{0\}) \cap Y = \emptyset\}$ by Definition 2.2.5, the proof for the first three cases (i), (ii), and (iii) follows immediately from Definition 3.3.8 if $D = D \setminus \{0\}$, or if we show that the domination sets D do not contain the origin. In particular, since

$$\operatorname{int} C + \varepsilon \subseteq C \setminus \{0\} + \varepsilon \subseteq C \cup \{0\} + \varepsilon \tag{3.69}$$

it is sufficient to show that $0 \notin C \cup \{0\} + \varepsilon$. Hence, let $\varepsilon \in C \setminus \{0\}$, then it follows from Proposition 2.1.13 that $-\varepsilon \notin C \cup \{0\}$ because C is a pointed convex cone. In particular, we obtain that $0 \notin C \cup \{0\} + \varepsilon$ and the result follows.

Similarly for (iv), we have $N_w(Y, D) = N(Y, \operatorname{int} D)$ and the proof follows directly because $\operatorname{int}(C + \varepsilon) = \operatorname{int} C + \varepsilon$ and $0 \notin \operatorname{int} C + \varepsilon$ from the above. □

The following result is an immediate consequence of the previous proposition and highlights the particular role of the translated cone $C_\varepsilon = C + \varepsilon$.

Corollary 3.3.10 (Translated cone nondominance)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a pointed convex cone, $\varepsilon \in C$, and $D = C_\varepsilon := C + \varepsilon$.*

(i) If $0 \in C$ and $\varepsilon \neq 0$, then $N(Y, D) = N_s(Y, C, \varepsilon)$.

(ii) If $0 \notin C$ (so $\varepsilon \neq 0$), then $N(Y, D) = N(Y, C, \varepsilon)$.

In any case, $N(Y, \operatorname{int} D) = N_w(Y, D) = N_w(Y, C, \varepsilon)$.

The previous two results emphasize that we need to be very specific if we decide to include or exclude the origin from the domination cone C, based on which the translated cone nondominance in Proposition 3.3.9 and Corollary 3.3.10 results in slightly different notions of approximate nondominance. We continue this discussion after collecting one additional statement that describes the relationship between two ε-nondominated sets for two different choices of ε and is based on Proposition 2.2.6.

Proposition 3.3.11. *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a pointed convex cone, and $\varepsilon^1, \varepsilon^2 \in C$ be two vectors so that $\varepsilon^2 - \varepsilon^1 \in C$. Then*

$$N(Y, C, \varepsilon^1) \subseteq N(Y, C, \varepsilon^2) \tag{3.70}$$

and the same inclusion holds for the strictly and weakly ε-nondominated sets.

Proof. Since all three proofs work essentially the same, we only give details for the ε-nondominated case. Hence, let $D^1 = C \setminus \{0\} + \varepsilon_1$ and $D^2 = C \setminus \{0\} + \varepsilon_2$. If $\varepsilon^2 - \varepsilon^1 \in C$, or $\varepsilon^2 \in C + \varepsilon^1$, then it also follows that

$$C \setminus \{0\} + \varepsilon^2 \subseteq C \setminus \{0\} + C + \varepsilon^1 \subseteq C \setminus \{0\} + \varepsilon^1 \tag{3.71}$$

since C is a pointed convex cone. Hence, we see that $D^2 \subseteq D^1$, and now Proposition 2.2.6 implies that $N(Y, D^1) \subseteq N(Y, D^2)$ and, together with Proposition 3.3.9, that $N(Y, C, \varepsilon^1) \subseteq N(Y, C, \varepsilon^2)$. The remaining proofs for the strictly and weakly ε-nondominated cases now follow analogously. $\square$

In particular, since $\varepsilon \in C$ implies that $(1 - \lambda)\varepsilon \in C$ for all $0 \leq \lambda < 1$ and $(\mu - 1)\varepsilon \in C$ for all $\mu > 1$, we conclude from the previous result that

$$N(Y, C) \subseteq N(Y, C, \lambda\varepsilon) \subseteq N(Y, C, \varepsilon) \subseteq N(Y, C, \mu\varepsilon) \tag{3.72}$$

for every $\varepsilon \in C$ and all $0 \leq \lambda \leq 1$ and $\mu \geq 1$.

Based on Proposition 3.3.9 and Corollary 3.3.10, we mention earlier that the three different notions of ε-nondominance distinguish between translated cones that keep or remove the origin from the original underlying cone. For the further study of this subtle difference, we introduce the following useful definition.

Definition 3.3.12 (Epsilon-translated nondominated set)**.** Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a cone, and $\varepsilon \in C$ be a vector. The sets

$$N_\varepsilon(Y, C) := (N(Y, C) + \varepsilon) \cap Y \tag{3.73a}$$

$$N_{w\varepsilon}(Y, C) := (N_w(Y, C) + \varepsilon) \cap Y \tag{3.73b}$$

are called the *ε-translated* and *weakly ε-translated nondominated set* of Y with respect to C, respectively.

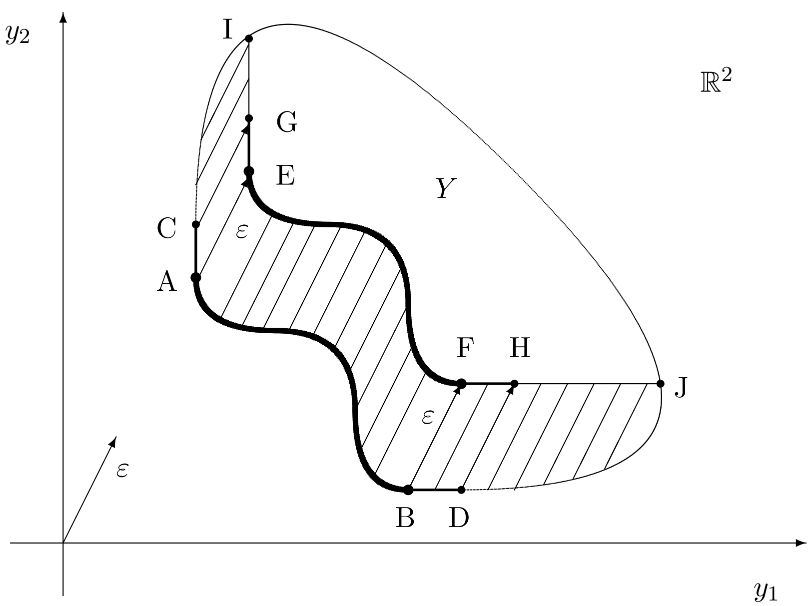


Figure 3.9 Sets of (weakly) Pareto, ε-Pareto, and ε-translated Pareto outcomes

Figure 3.9 illustrates these two notions together with the three previous concepts of (strictly and weakly) ε-nondominance for the depicted set Y and the particular case of Pareto nondominance. First, the set $N(Y, \mathbb{R}^2_{\geqq})$ of Pareto outcomes is given by the curve connecting points A and B, while the set $N_w(Y, \mathbb{R}^2_{\geqq})$ of all weak Pareto outcomes corresponds to the extended curve from C to D. If we translate these two sets by the specified translation vector ε, then we obtain the (weakly) ε-translated Pareto sets $N_\varepsilon(Y, \mathbb{R}^2_{\geqq})$ and $N_{w\varepsilon}(Y, \mathbb{R}^2_{\geqq})$ as the curves connecting points

E and F, and G and H, respectively. In this case, the set $N_s(Y, \mathbb{R}^2_{\geqq}, \varepsilon)$ of strict ε-Pareto outcomes is the shaded area enclosed by all marked points without the curve connecting I, G, E, F, H and J, the ε-Pareto set $N(Y, \mathbb{R}^2_{\geqq}, \varepsilon)$ additionally includes the curve from E to F, and the weakly ε-Pareto set $N_w(Y, \mathbb{R}^2_{\geqq}, \varepsilon)$ also contains the complete remaining curve from I to J.

In particular, all outcomes in the (weakly) ε-translated Pareto set in Figure 3.9 belong to those outcomes that are (weakly) ε-Pareto but not strictly ε-Pareto, respectively, and it turns out that this characterization holds as a general result.

Proposition 3.3.13 (Relationship between ε-nondominance concepts)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a cone, and $\varepsilon \in C$ be a vector. Then*

$$N_\varepsilon(Y, C) \subseteq N(Y, C, \varepsilon) \setminus N_s(Y, C, \varepsilon) \tag{3.74a}$$

$$N_{w\varepsilon}(Y, C) \subseteq N_w(Y, C, \varepsilon) \setminus N_s(Y, C, \varepsilon) \tag{3.74b}$$

Proof. For the first inclusion, let $\tilde{y} \in N_\varepsilon(Y, C)$, so $\tilde{y} = \hat{y} + \varepsilon$ for some $\hat{y} \in N(Y, C)$. Then there does not exist $y \in Y$ such that $\hat{y} - y \in C \setminus \{0\}$, or $\tilde{y} - \varepsilon - y \in C \setminus \{0\}$ which implies that $\tilde{y} \in N(Y, C, \varepsilon)$. Moreover, since $\tilde{y} - \hat{y} - \varepsilon = 0 \in C \cup \{0\}$, it is clear that $\tilde{y} \notin N_s(Y, C, \varepsilon)$, and the result follows. As before, the second statement is proven analogously by replacing the cone C by its interior. $\square$

Figure 3.9 also suggests that outcomes that are ε-Pareto but not strictly ε-Pareto can be characterized as subset of those outcomes among the ε-Pareto set that are nondominated with respect to the negative Pareto cone, with an underlying interpretation of objective maximization rather than minimization. More general, we show in the following result that the set difference between the ε-nondominated set and strictly ε-nondominated set constitutes a subset of those outcomes in the ε-nondominated set (all with respect to C) that are nondominated with respect to the negative cone $-C$.

Theorem 3.3.14. *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a cone, and $\varepsilon \in C$. Then*

$$N(Y, C, \varepsilon) \setminus N_s(Y, C, \varepsilon) \subseteq N(N(Y, C, \varepsilon), -C) \tag{3.75}$$

Proof. If $\hat{y} \in N(Y,C,\varepsilon) \setminus N_s(Y,C,\varepsilon)$, then there does not exist $y \in Y$ such that $\hat{y} - y - \varepsilon \in C \setminus \{0\}$, but $\hat{y} - y - \varepsilon \in C \cup \{0\}$ for some $y \in Y$ which, together with the former, implies that $\hat{y} = y + \varepsilon$. To show that $\hat{y} \in N(N(Y,C,\varepsilon), -C)$, we then need that there does not exist $\tilde{y} \in N(Y,C,\varepsilon)$ such that $\hat{y} - \tilde{y} \in -C \setminus \{0\}$, or equivalently, that $\tilde{y} - \hat{y} \in C \setminus \{0\}$ only if $\tilde{y} \notin N(Y,C,\varepsilon)$. Hence, let $\tilde{y} - \hat{y} \in C \setminus \{0\}$ and the result now follows since $\hat{y} = y + \varepsilon$ implies that $\tilde{y} - y - \varepsilon \in C \setminus \{0\}$ and, thus, $\tilde{y} \notin N(Y,C,\varepsilon)$. $\square$

Combining the previous result with Proposition 3.3.13, we can infer the following concluding corollary.

Corollary 3.3.15. *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a cone, and $\varepsilon \in C$. Then*

$$N_\varepsilon(Y,C) \subseteq N(Y,C,\varepsilon) \setminus N_s(Y,C,\varepsilon) \subseteq N(N(Y,C,\varepsilon), -C) \tag{3.76}$$

Hence, Theorem 3.3.14 and Corollary 3.3.15 provide a partial characterization of the solution set for a modified optimization problem over the set of ε-nondominated outcomes. While we illustrate the relevance of this particular result in the later Section 3.3.4, several parts of the literature also study other optimization problems over the nondominated or efficient set (Benson and Sayin, 1994; Tu, 2000; Yamamoto, 2002; Jorge, 2005), including the computation of nadir points (Ehrgott and Tenfelde-Podehl, 2003) previously mentioned in Remark 3.2.28.

Other than the previous two results that describe the relationship between the (ε-)nondominated sets with respect to a cone C and its negative $-C$, we now address the relationship between ε-nondominated and approximate Pareto outcomes in further generalization of the nondominance mapping theorem initially discussed in Section 3.1.2. In particular, using the representation of translated cones as polyhedral sets in Theorem 3.3.6, we first extend Theorem 3.1.13 from polyhedral cones to polyhedral sets and then show how this new result can be used to derive the corresponding relationship between approximate nondominance with respect to a polyhedral cone and approximate Pareto nondominance.

Theorem 3.3.16 (Nondominance mapping theorem for polyhedral sets)**.** *Let $Y \subset \mathbb{R}^m$ be a set, and $D = D(A, b) \subset \mathbb{R}^m$ be the polyhedral set for some matrix $A \in \mathbb{R}^{l \times m}$ and some vector $b \in \mathbb{R}^l$. Then*

$$A[N(Y, D)] \subseteq N(A[Y], \mathbb{R}^l_{\geqq b}) \tag{3.77a}$$

where $\mathbb{R}^l_{\geqq b} := \mathbb{R}^l_{\geqq} + b = \{v \in \mathbb{R}^l : v \geqq b\}$. If D is pointed, or equivalently, if $\operatorname{rank} A = m$, *then*

$$A[N(Y, D)] = N(A[Y], \mathbb{R}^l_{\geqq b}) \tag{3.77b}$$

Proof. Let $\hat{u} \in A[N(Y, D)]$ with $\hat{u} = A\hat{y}$, $\hat{y} \in N(Y, D)$, i.e., there do not exist $y \in Y, d \in D, d \neq 0$ such that $\hat{y} = y + d$. Now suppose by contradiction that $\hat{u} \notin N(A[Y], \mathbb{R}^l_{\geqq b})$, then $\hat{u} \geqq u + b$ for some $u = Ay \in A[Y], u \neq \hat{u}$ and thus $y \neq \hat{y}$. It follows that

$$\hat{u} - u = A\hat{y} - Ay = A(\hat{y} - y) \geqq b \tag{3.78a}$$

and setting $d = \hat{y} - y$ gives $\hat{y} = y + d$, $d \in D$, $d \neq 0$ in contradiction to the above. For the opposite direction, let $\hat{u} \in N(A[Y], \mathbb{R}^l_{\geqq b})$ with $\hat{u} = A\hat{y}, \hat{y} \in Y$. Then there does not exist $u = Ay \in A[Y], Ay \neq A\hat{y}$ such that $A\hat{y} \geqq Ay + b$, or $A(\hat{y} - y) \geqq b$. Now suppose by contradiction that $\hat{u} \notin A[N(Y, D)]$, or $\hat{y} \notin N(Y, D)$. Then there exist $y \in Y, d \in D, d \neq 0$ such that $\hat{y} = y + d$ and hence

$$A(\hat{y} - y) = Ad \geqq b \tag{3.78b}$$

with $A\hat{y} \neq Ay$ if $\operatorname{rank} A = m$. This yields the contradiction. □

The next result provides an alternative condition on the vector $b \in \mathbb{R}^l$ to guarantee equality in Theorem 3.3.16.

Proposition 3.3.17 (Alternative condition for equality)**.** *Let $Y \subset \mathbb{R}^m$ be a set, and $D = D(A, b) \subset \mathbb{R}^m$ be the polyhedral set for $A \in \mathbb{R}^{l \times m}$ and $b \in \mathbb{R}^l$. If $b \notin -\mathbb{R}^l_{\geqq}$, then*

$$A[N(Y, D)] = N(A[Y], \mathbb{R}^l_{\geqq b}) \tag{3.79}$$

Proof. The inclusion $A[N(Y,D)] \subseteq N(A[Y], \mathbb{R}^l_{\geqq b})$ follows as in Theorem 3.3.16, and for the reversed inclusion, the final contradiction follows from

$$A(\hat{y} - y) = Ad \geqq b \tag{3.80}$$

since $b \notin -\mathbb{R}^l_{\geqq}$ implies that $Ad \neq 0$ and , thus, $A\hat{y} \neq Ay$. $\square$

Based on the above results, we now employ Proposition 3.3.9 to relate ε-nondominated outcomes with respect to a polyhedral cone $C = C(A)$ to outcomes that are nondominated with respect to the polyhedral set $D = D(A,b)$ describing the translated cone $C_\varepsilon = C + \varepsilon$. Using this relationship, we derive the ε-nondominance mapping theorem for polyhedral cones as concluding result in this section.

Theorem 3.3.18 (Epsilon-nondominance mapping theorem for polyhedral cones)**.** *Let $Y \subset \mathbb{R}^m$ be a set and $C = C(A) \subset \mathbb{R}^m$ be the pointed polyhedral cone induced by a matrix $A \in \mathbb{R}^{l \times m}$ with full* rank $A = m$*. Let $\varepsilon \in C \setminus \{0\}$ and $b = A\varepsilon$. Then*

$$A[N_s(Y, C, \varepsilon)] = N(A[Y], \mathbb{R}^l_{\geqq b}) \tag{3.81}$$

Proof. We combine several of the results derived in this and the previous section. First, we obtain from Theorem 3.3.6 that the translated polyhedral cone $D = C_\varepsilon$ with $C = D(A)$ and translation vector ε can be described by the polyhedral set $D = D(A,b)$ for $b = A\varepsilon$ as stated in the assumptions. Hence, it follows that

$$D = D(A,b) = C_\varepsilon = C + \varepsilon \tag{3.81a}$$

where D is pointed because C is pointed, or equivalently, because $\operatorname{rank} A = m$. In particular, since $0 \in C(A)$, Proposition 3.3.9 and Corollary 3.3.10 imply that

$$N(Y, D) = N_s(Y, C, \varepsilon) \tag{3.81b}$$

and therefore we can finally apply Theorem 3.3.16 to conclude that

$$A[N_s(Y, C, \varepsilon)] = N(A[Y], \mathbb{R}^l_{\geqq b}) \tag{3.81c}$$

This completes the proof and section. $\square$

3.3.3 Scalarization Methods and Epsilon-Optimality

After deriving several cone characterizations of ε-nondominance in the previous section, we now investigate the relationships between ε-nondominance and ϵ-optimality when solving a multiobjective program based on scalarization approaches. In particular, for each method introduced in Section 2.2.3 we establish the relationship between $\epsilon \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}^m$ that guarantees that an ϵ-optimal solution for the single objective program SOP is also ε-efficient for MOP. Most recently, some of these results are also and independently derived by Gutiérrez et al. (2006a,b).

Our first result generalizes Lemma 2.2.21 and is formulated for an arbitrary increasing and superadditive but otherwise generic scalarization function in the particular case of Pareto nondominance.

Proposition 3.3.19 (Sufficient conditions for approximate Pareto nondominance). *Let $Y \subset \mathbb{R}^m$ be a nonempty set, and $s : \mathbb{R}^m \to \mathbb{R}$ be an increasing and superadditive scalarization function. Let $\varepsilon \in \mathbb{R}^m$, $\varepsilon \geqq 0$ and $\epsilon \in \mathbb{R}$, $\epsilon \geq 0$ with $s(\varepsilon) \geq \epsilon$, and $\hat{y} \in Y$ be ϵ-optimal for the single objective program*

$$\textit{SOP: Minimize } s(y) \textit{ subject to } y \in Y \tag{3.82}$$

(i) If $\tilde{y}$ is strictly ϵ-optimal, or if $\epsilon < s(\varepsilon)$, then $\tilde{y} \in N_s(Y, \varepsilon)$.

(ii) If s is strictly increasing, then $\tilde{y} \in N(Y, \varepsilon)$.

In any case, $\hat{y} \in N_w(Y, \varepsilon)$

Proof. For (i), if $\tilde{y} \in Y$ is strictly ϵ-optimal for SOP, or $s(\tilde{y}) < s(y) - \varepsilon$ for all $y \in Y$, let $s(\varepsilon) \geq \epsilon$ and suppose by contradiction that $\tilde{y} \notin N_s(Y, \varepsilon)$. Then there exists $y \in Y$ such that $y \leqq \tilde{y} - \varepsilon$, or $y + \varepsilon \leqq \tilde{y}$, and thus

$$s(\varepsilon) \leq s(y + \varepsilon) - s(y) \leq s(\tilde{y}) - s(y) < \epsilon \tag{3.83a}$$

since s is superadditive, increasing, and because $\tilde{y}$ is strictly ϵ-optimal, in contradiction to $s(\varepsilon) \geq \epsilon$. Moreover, if $\epsilon < s(\varepsilon)$, then the contradiction still follows if $\tilde{y}$ is only ϵ-optimal for which all inequalities in (3.83a) may also hold with equality.

For (ii), if s is strictly increasing, let $s(\varepsilon) \geq \epsilon$ and suppose by contradiction that $\tilde{y} \notin N(Y, \varepsilon)$. Then there exists $y \in Y$ such that $y \leq \tilde{y} - \varepsilon$, or $y + \varepsilon \leq \tilde{y}$, and

$$s(\varepsilon) \leq s(y + \varepsilon) - s(y) < s(\hat{y}) - s(y) \leq \epsilon \tag{3.83b}$$

because s is strictly increasing, in contradiction to $s(\varepsilon) \geq \epsilon$.

Finally, again let $s(\varepsilon) \geq \epsilon$ and suppose by contradiction that $\tilde{y} \notin N_w(Y, \varepsilon)$. Then there exists $y \in Y$ such that $y < \tilde{y} - \varepsilon$, or $y + \varepsilon < \tilde{y}$, and the contradiction to $s(\varepsilon) \geq \epsilon$ again follows from the inequality chain in (3.83b). □

Since weighted sums are linear and, in particular, additive and superadditive, we can immediately use Proposition 3.3.19 to derive the corresponding result for the weighted-sum scalarization, which is similarly given in White (1986), Deng (1997), and Dutta and Vetrivel (2001), for the special case of Pareto nondominance.

Choosing an alternative path and following our discussion in Section 3.1.3 in which we extend several scalarization methods for general cones, we first derive the corresponding result for the hybrid scalarization method and then derive those for the weighted-sum, constrained-objective, and Benson method as special cases.

Proposition 3.3.20 (Approximate hybrid scalarization)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a convex cone, $b \in \mathbb{R}^m$, $w \in C^* \setminus \{0\}$, $\varepsilon \in C$, $\epsilon \leq \langle w, \varepsilon \rangle$, and $\tilde{y} \in Y$ be ϵ-optimal for the hybrid scalarization*

$$\textit{HB}(w, b)\textit{: Minimize } \langle w, y \rangle \textit{ subject to } b - y \in C \textit{ and } y \in Y \tag{3.84}$$

(i) If $\tilde{y}$ is strictly ϵ-optimal, or if $\epsilon < \langle w, \varepsilon \rangle$, then $\tilde{y} \in N_s(Y, C, \varepsilon)$.

(ii) If $w \in C_s^*$, then $\tilde{y} \in N(Y, C, \varepsilon)$.

In any case, $\tilde{y} \in N_w(Y, C, \varepsilon)$.

Proof. Let $\tilde{y}$ be ϵ-optimal for $\mathrm{HB}(w, b)$, so

$$\langle w, \tilde{y} \rangle \leq \langle w, y \rangle - \epsilon \text{ for all } y \in Y \tag{3.84a}$$

that satisfy $b - y \in C$, and $\langle w, \tilde{y} \rangle < \langle w, y \rangle - \epsilon$ if $\tilde{y}$ is strictly ϵ-optimal.

For (i), if $\tilde{y}$ is strictly ϵ-optimal, let $w \in C^* \setminus \{0\}$, $\epsilon \leq \langle w, \varepsilon \rangle$, and by contradiction suppose that $\tilde{y} \notin N_s(Y, C, \varepsilon)$. Then there exists $y \in Y$ such that $\tilde{y} - y - \varepsilon \in C \cup \{0\}$, and thus

$$b - y = (b - \tilde{y}) + (\tilde{y} - y - \varepsilon) + \varepsilon \in C \tag{3.84b}$$

by feasibility of $\tilde{y}$ for HB(w, b), $\varepsilon \in C$, and convexity of C, showing that y is also feasible for HB(w, b). Furthermore, it also follows that $\langle w, \tilde{y} - y - \varepsilon \rangle \geq 0$ as $w \in C^*$ which implies that

$$\langle w, \tilde{y} \rangle \geq \langle w, y \rangle + \langle w, \varepsilon \rangle \geq \langle w, y \rangle + \epsilon \tag{3.84c}$$

in contradiction to strict ϵ-optimality of $\tilde{y}$ for HB(w, b). Moreover, if $\epsilon < \langle w, \varepsilon \rangle$, then $\langle w, \tilde{y} \rangle > \langle w, y \rangle + \epsilon$ and the contradiction persists if $\tilde{y}$ is only ϵ-optimal.

For (ii), if $w \in C_s^*$, suppose by contradiction that $\tilde{y} \notin N(Y, C, \varepsilon)$. Then there exists $y \in Y$ with $\tilde{y} - y - \varepsilon \in C \setminus \{0\}$, and feasibility of y for HB(w, b) follows as in (i). However, now $w \in C_s^*$ implies that $\langle w, \tilde{y} - y - \varepsilon \rangle > 0$, or $\langle w, \tilde{y} \rangle > \langle w, y \rangle + \epsilon$ again in contradiction to ϵ-optimality of $\tilde{y}$ for HB(w, b).

In any case, $(\operatorname{int} C)_s^* = C^* \setminus \{0\}$ from Proposition 2.1.21 which implies that $\tilde{y} \in N(Y, \operatorname{int} C, \varepsilon) = N_w(Y, C, \varepsilon)$ from (ii). □

In particular, analogous to the discussion in Section 3.1.3, we can immediately derive the corresponding results for the weighted-sum, the constrained-objective, and the Benson scalarization method. The proofs are, therefore, omitted.

Proposition 3.3.21 (Approximate weighted-sum scalarization)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a cone, $w \in C^* \setminus \{0\}$, $\varepsilon \in C$, $\epsilon \leq \langle w, \varepsilon \rangle$, and $\tilde{y} \in Y$ be ϵ-optimal to the weighted-sum scalarization*

$$\text{WS}(w)\text{: Minimize } \langle w, y \rangle \text{ subject to } y \in Y \tag{3.85}$$

(i) If $\tilde{y}$ is strictly ϵ-optimal, or if $\epsilon < \langle w, \varepsilon \rangle$, then $\tilde{y} \in N_s(Y, C, \varepsilon)$.

(ii) If $w \in C_s^*$, then $\tilde{y} \in N(Y, C, \varepsilon)$.

In any case, $\tilde{y} \in N_w(Y, C, \varepsilon)$.

Proposition 3.3.22 (Approximate constrained-objective scalarization). *Let $Y \subset \mathbb{R}^m$ be a set, $C \subset \mathbb{R}^m$ be a convex cone, $b \in \mathbb{R}^m$, $e^k \in C^*$ be the kth unit vector, $\varepsilon \in C$, $\epsilon \leq \varepsilon_k$, and $\tilde{y} \in Y$ be ϵ-optimal to the constrained-objective scalarization*

$$CO_k(b)\text{: Minimize } y_k \text{ subject to } b - y \in C \text{ and } y \in Y \tag{3.86}$$

(i) If $\tilde{y}$ is strictly ϵ-optimal, or if $\epsilon < \varepsilon_k$, then $\tilde{y} \in N_s(Y, C, \varepsilon)$.

(ii) If $e^k \in C_s^*$, then $\tilde{y} \in N(Y, C, \varepsilon)$.

In any case, $\tilde{y} \in N_w(Y, C, \varepsilon)$.

Proposition 3.3.23 (Approximate Benson scalarization). *Let $Y \subset \mathbb{R}^m$ be a set, $y^\circ \in Y$, $C \subset \mathbb{R}^m$ be a convex cone, $w = (1, \ldots, 1)^T \in C^*$ be the vector with all components equal to one, $\varepsilon \in C$, $\epsilon \leq \sum_{i=1}^{j} \varepsilon_i$, and $\tilde{y} = y^\circ - \tilde{c} \in Y$ where $\tilde{c} \in C$ be ϵ-optimal to the Benson scalarization*

$$B(y^\circ)\text{: Maximize } \sum_{i=1}^{m} c_i \text{ subject to } y^\circ - c = y \in Y \text{ and } c \in C \tag{3.87}$$

(i) If $\tilde{c}$ is strictly ϵ-optimal, or if $\epsilon < \sum_{i=1}^{m} \varepsilon_i$, then $\tilde{y} \in N_s(Y, C, \varepsilon)$.

(ii) If $w = (1, \ldots, 1)^T \in C_s^*$, then $\tilde{y} \in N(Y, C, \varepsilon)$.

In any case, $\tilde{y} \in N_w(Y, C, \varepsilon)$.

For the Pascoletti-Serafini scalarization method, we can prove the following result which, like the original method, is also formulated for a convex cone.

Proposition 3.3.24 (Approximate Pascoletti-Serafini scalarization). *Let $Y \subset \mathbb{R}^m$ be a nonempty set, $C \subset \mathbb{R}^m$ be a convex cone, $\varepsilon \in C$, $\epsilon \in \{\delta : \varepsilon - \delta v \in C\}$, and $\tilde{y} = r + \tilde{\mu} v - \tilde{c} \in Y$ where $(\tilde{\mu}, \tilde{c})$ be ϵ-optimal for the Pascoletti-Serafini scalarization*

$$PS(r, v)\text{: Minimize } \mu \text{ subject to } r + \mu v - c = y \in Y,\ c \in C \text{ and } \mu \in \mathbb{R} \tag{3.88}$$

If $\tilde{\mu}$ is strictly ϵ-optimal for PS(r, v), then $\tilde{y} \in N_s(Y, C, \varepsilon)$, and $\tilde{y} \in N_w(Y, C, \varepsilon)$ in any case.

Proof. For $\epsilon \in \{\delta : \varepsilon - \delta v \in C\}$, denote $c_\epsilon := \varepsilon - \epsilon v \in C$. Then let $\tilde{\mu}$ with $\tilde{c} \in C$ be strictly ϵ-optimal for PS(r, v), so $\tilde{\mu} \leq \mu + \epsilon$ whenever $r + \mu v - c \in Y$ for some

$c \in C$, and suppose by contradiction that $\tilde{\mu} \notin N_s(Y, C, \varepsilon)$. Then there exists $y \in Y$ such that $\tilde{y} - y - \varepsilon \in C \cup \{0\}$, or $y = \tilde{y} - c - \varepsilon$ for some $c \in C \cup \{0\}$ yielding

$$y = r + \tilde{\mu}v - \tilde{c} - c - \varepsilon \tag{3.88a}$$

$$= r + \tilde{\mu}v - \tilde{c} - c - (\epsilon v + c_\epsilon) \tag{3.88b}$$

$$= r + (\tilde{\mu} - \epsilon)v - (\tilde{c} + c + c_\epsilon) \tag{3.88c}$$

where $\tilde{c} + c + c_\epsilon \in C$ as C is a convex cone. Hence, it follows that $\mu = \tilde{\mu} - \epsilon$, or $\tilde{\mu} = \mu + \epsilon$ in contradiction to strict ϵ-optimality of $\tilde{\mu}$ for $\mathrm{PS}(r, v)$.

Next, let $\tilde{\mu}$ with $\tilde{c} \in C$ be ϵ-optimal for $\mathrm{PS}(r, v)$, so $\tilde{\mu} \leq \mu + \epsilon$ whenever $r + \mu v - c \in Y$ for some $c \in C$, and suppose by contradiction that $\tilde{y} \notin N_w(Y, C, \varepsilon)$. Then there exists $y \in Y$ such that $\tilde{y} - y - \varepsilon \in \operatorname{int} C$, or $y = \tilde{y} - c - \varepsilon$ for some $c \in \operatorname{int} C$. Now, since $c \in \operatorname{int} C$, there also exists $\gamma > 0$ such that $c_\gamma = c - \gamma v \in C$ which implies that

$$y = r + \tilde{\mu}v - \tilde{c} - c - \varepsilon \tag{3.88d}$$

$$= r + \tilde{\mu}v - \tilde{c} - (\gamma v + c_\gamma) - (\epsilon v + d) \tag{3.88e}$$

$$= r + (\tilde{\mu} - \gamma - \epsilon)v - (\tilde{c} + c_\gamma + c_\epsilon) \tag{3.88f}$$

where $\tilde{c} + c_\gamma + c_\epsilon \in C$ as C is a convex cone. Hence, it now follows that $\mu = \tilde{\mu} - \gamma - \epsilon$, or $\tilde{\mu} = \mu + \gamma + \epsilon > \mu + \epsilon$ in contradiction to optimality of $\tilde{\mu}$ for $\mathrm{PS}(r, v)$. $\square$

Finally, we derive two further results for the weighted-Chebyshev norm scalarization and, as a special case, the max-norm scalarization method which again are both formulated for the Pareto case.

Proposition 3.3.25 (Approximate weighted-Chebyshev norm scalarization)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $r \in \mathbb{R}^m$, $w \in \mathbb{R}^m$, $w \geq 0$, $\varepsilon \in \mathbb{R}^m, \varepsilon \geqq 0$, $\epsilon \leq \min_{i=1,\ldots,m}\{w_i \varepsilon_i\}$, and $\tilde{y}$ be ϵ-optimal for the weighted-Chebyshev norm scalarization*

$$\text{CN}(r, w)\text{: Minimize } \max_{i=1,\ldots,m} \{w_i(y_i - r_i)\} \text{ subject to } x \in X \tag{3.89}$$

If $\tilde{y} \in Y$ is strictly ϵ-optimal, or if $\epsilon < \min_{i=1,\ldots,m}\{w_i \varepsilon_i\}$, then $\tilde{y} \in N_s(Y, \varepsilon)$, and $\tilde{y} \in N_w(Y, \varepsilon)$ in any case.

Proof. First, if $\tilde{y}$ is strictly ϵ-optimal, so $\max_i\{w_i(\tilde{y}_i - r_i)\} < \max_i\{w_i(y_i - r_i)\} + \epsilon$ for all $y \in Y$, let $\epsilon \leq \min_i\{w_i\varepsilon_i\}$ and suppose by contradiction that $\tilde{y} \notin N_s(Y, \varepsilon)$. Then there exists $y \in Y$ such that $\tilde{y} \geqq y + \varepsilon$ which implies that

$$\max_{i=1,\ldots,m}\{w_i(\tilde{y}_i - r_i)\} \geq \max_{i=1,\ldots,m}\{w_i(y_i - r_i + \varepsilon_i)\} \tag{3.89a}$$

$$\geq \max_{i=1,\ldots,m}\{w_i(y_i - r_i)\} + \min_{i=1,\ldots,m}\{w_i\varepsilon_i\} \tag{3.89b}$$

$$\geq \max_{i=1,\ldots,m}\{w_i(y_i - r_i)\} + \epsilon \tag{3.89c}$$

in contradiction to strict ϵ-optimality of $\tilde{y}$ for $\mathrm{CN}(r, w)$. Moreover, if $\epsilon < \min_i\{w_i\varepsilon_i\}$, then (3.89c) holds strictly and a contradiction persists also if $\tilde{y}$ is only ϵ-optimal.

Second, suppose by contradiction that $\tilde{y} \notin N_w(Y, \varepsilon)$, then there exists $y \in Y$ such that $y < \tilde{y} + \varepsilon$, or $\tilde{y} > y - \varepsilon$, and in this case the inequality in (3.89a) is strict because $w \geq 0$ and, thus, $w_i > 0$ for at least one i, yielding the same contradiction as before and concluding the proof. $\square$

In particular, with $r = 0 \in \mathbb{R}^m$ and $w_i = 1 > 0$ for all $i = 1, \ldots, m$, we obtain the corresponding max-norm result as concluding statement of this section.

Proposition 3.3.26 (Approximate max-norm scalarization)**.** *Let $Y \subset \mathbb{R}^m$ be a set, $\varepsilon \in \mathbb{R}^m$, $\varepsilon \geq 0$, $\epsilon \leq \min_i\{\varepsilon_i\}$, and $\tilde{y}$ be ϵ-optimal for the max-norm scalarization*

$$\textit{MN: Minimize } \max_{i=1,\ldots,m}\{y_i\} \textit{ subject to } y \in Y \tag{3.90}$$

If $\tilde{y} \in Y$ is strictly ϵ-optimal, or if $\epsilon < \min_i\{\varepsilon_i\}$, then $\tilde{y} \in N_s(Y, \varepsilon)$, and $\tilde{y} \in N_w(Y, \varepsilon)$ in any case.

3.3.4 Exact Generation of Approximate Solutions

The previous results for approximate scalarization methods in principle distinguish the generation of strictly ε-nondominated, ε-nondominated, and weakly ε-nondominated outcomes. However, since strictly ε-nondominated outcomes are also (weakly) ε-nondominated, these results do not provide the means to more specifically characterize cases in which a generated outcome is only (weakly) ε-nondominated but not strictly ε-nondominated.

Consequently, and motivated by the prominent role of ε-translated nondominated outcomes in Proposition 3.3.13 that enable the proper distinction between (weakly) ε-nondominated outcomes and strict ε-nondominance, we now investigate several approaches for the exact generation of an ε-translated nondominated outcome. Restricting our attention to the Pareto case, throughout the following discussion we assume that we already know a Pareto or weak Pareto outcome $\hat{y} \in N(Y, \mathbb{R}^m_{\geqq})$ or $N_w(Y, \mathbb{R}^m_{\geqq})$ but that we want to relax this outcome by some specified $\varepsilon \in \mathbb{R}^m$ to obtain the corresponding ε-translated outcome $\tilde{y} = \hat{y} + \varepsilon$. In particular, for convenience we usually assume that $\tilde{y} \in Y$, although $\tilde{y} \notin Y$ is also possible, in general.

While we still formulate all following statements with respect to the outcome set $Y \subset \mathbb{R}^m$, for which some of the following methods and results are obvious as they are explicitly constructed to yield $\tilde{y} = \hat{y} + \varepsilon$ as optimal solution, we emphasize that the main difficulty remains to solve the underlying optimization problem with respect to feasible set X in the decision space $\mathbb{R}^n$. We further address some of the resulting practical or computational issues in the two subsequent sections and, therefore, refrain from the exposition of any such details already at this point.

The first method utilizes an arbitrary norm and is probably the most intuitive method to generate a particular solution $\tilde{y} = \hat{y} + \varepsilon$.

Proposition 3.3.27 (Exact norm generating method)**.** *Let $\hat{y} \in Y$, $\varepsilon \in \mathbb{R}^m$, and $\tilde{y} = \hat{y} + \varepsilon \in Y$. Then $\tilde{y}$ is optimal for the single objective norm-minimization problem*

$$\textit{Minimize } \|\hat{y} - \varepsilon - y\| \textit{ subject to } y \in Y \tag{3.91}$$

We do not need to formally prove the method proposed in Proposition 3.3.27, as its validity is immediately clear from the assumption that $\tilde{y} = \hat{y} + \varepsilon \in Y$. To solve the problem in (3.91), we can choose any suitable norm, including the weighted-ℓ_p or pth-order norm

$$\|\hat{y} - \varepsilon - y\|_p := \left(\sum_{i=1}^{m} w_i (|\hat{y}_i - \varepsilon_i - y_i|)^p \right)^{1/p} \tag{3.92}$$

hence also including the weighted-Chebyshev norm for $p = \infty$ as a special case.

If our previous assumption that $\tilde{y} \in Y$ is not satisfied, then one conceptual drawback of the above norm method is that the identified optimal solution may either be greater or smaller than $\tilde{y}$ and, therefore, does not necessarily belong to the ε-nondominated set $N(Y, \varepsilon)$ unless we explicitly introduce the additional constraint that $y \leqq \hat{y} + \varepsilon$. In this case, however, we can more generally formulate the following multiobjective program.

Proposition 3.3.28 (Exact constraint generating method)**.** *Let $\hat{y} \in Y$, $\varepsilon \in \mathbb{R}^m$, and assume that $\tilde{y} = \hat{y} + \varepsilon \in Y$. Then $\tilde{y}$ is the unique nondominated outcome to the multiobjective programming problem*

$$\text{Minimize } -y \text{ subject to } y \leqq \hat{y} + \varepsilon \text{ and } y \in Y \tag{3.93}$$

Again, correctness of Proposition 3.3.28 is immediate from the assumption that $\tilde{y} = \hat{y} + \varepsilon \in Y$, and a formal proof can be omitted. Furthermore, if our assumption that $\tilde{y} \in Y$ is not satisfied, then any other solution to the problem in (3.93) is still ε-nondominated and can now be found using any suitable scalarization technique. If $\tilde{y} \in Y$, however, then the imposed constraints in (3.93) enforce optimality of $\tilde{y}$ by restricting the original outcome set Y such that $\tilde{y}$ lies at the boundary, and more precisely, at an extreme point of the set $\tilde{y} - \mathbb{R}^m_{\geqq}$.

In the remaining part of this section, we investigate an alternative approach to modify the original problem so that interior points in the original outcome set become boundary points of a modified outcome set. However, in this case we do not introduce any additional constraints but an additional component into the objective function vector, based on the concept of an outcome space augmentation. The underlying idea for this method is to define a suitable augmentation function $a : Y \to \mathbb{R}$ and then consider Y as new set $\mathcal{Y} = \{(y, z)) \in Y \times \mathbb{R} : z = a(y)\}$ in the *augmented* outcome space $\mathbb{R}^{m+1}$ in which all interior points of $Y \subset \mathbb{R}^m$ become boundary points of $\mathcal{Y} \subset \mathbb{R}^{m+1}$. We study the associated augmented MOP for two scalarization methods, namely the weighted-sum and the weighted-Chebyshev norm scalarization, and we show how ε-translated Pareto outcomes for MOP can be identified as nondominated outcomes for the respective AMOP.

Definition 3.3.29 (Augmented Multiobjective Program)**.** Let $Y \subset \mathbb{R}^m$ and $a : Y \to \mathbb{R}$ be a real-valued function. The multiobjective program

$$\text{AMOP: Minimize } [y, a(y)]^T \text{ subject to } y \in Y \tag{3.94}$$

is called the *augmented multiobjective program* (AMOP), and a is called the *augmentation function* of AMOP.

Of course, to relate nondominance for AMOP with ε-nondominance for MOP we need to find some suitable choices of the augmentation function a. In particular, in the following results we define some specific augmentation functions in terms of an optimal solution $\hat{y}$ for a corresponding SOP with scalarization function s.

Definition 3.3.30 (Augmentation function associated with scalarization function)**.** Let $Y \subset \mathbb{R}^m$, $s : Y \to \mathbb{R}$ be a scalarization function, and $\hat{y} \in Y$ be optimal for the associated single objective program

$$\text{SOP: Minimize } s(y) \text{ subject to } y \in Y \tag{3.95a}$$

The function $a : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ with

$$a(y) := (s(y) - s(\hat{y}))^{-1} \tag{3.95b}$$

is called the augmentation function *associated with s and $\hat{y}$.*

Following common convention, we define $0^{-1} := \infty$ and derive some further properties of the resulting augmentation functions in the following proposition.

Proposition 3.3.31 (Properties of augmentation functions associated with scalarization functions)**.** *Let $Y \subset \mathbb{R}^m$, $s : Y \to \mathbb{R}$ be an increasing scalarization function, $\hat{y} \in Y$ be optimal for the corresponding SOP, and $a : Y \to \mathbb{R} \cup \{\infty\}$ be the augmentation function associated with s and $\hat{y}$. Then* *(i)* $a(y) \geq 0$ *for all* $y \in Y$*,*

(ii) $a(y) = \infty$ *for all* y *that are optimal for SOP, and*

(iii) a *is decreasing:* $y^1 \leqq y^2 \Rightarrow a(y^1) \geq a(y^2)$ *for all* $y^1, y^2 \in Y$*.*

Proof. Since $\hat{y}$ is optimal for SOP, it follows that $s(\hat{y}) \leq s(y)$, or $s(y) - s(\hat{y}) \geq 0$ and thus $a(y) = (s(y) - s(\hat{y}))^{-1} \geq 0$ for all $y \in Y$, showing (i). In particular, if $y \in Y$ is optimal to SOP, then $s(y) = s(\hat{y})$ and $a(y) = \infty$, showing (ii). For (iii), let $y^1, y^2 \in Y$ with $y^1 \leqq y^2$, then $s(y^1) \leq s(y^2)$ since s is an increasing scalarization function and $0 \leq s(y^1) - s(\hat{y}) \leq s(y^2) - s(\hat{y})$ from (i). Taking inverses gives $(s(y^1) - s(\hat{y}))^{-1} \geq (s(y^2) - s(\hat{y}))^{-1}$ and thus implies that $a(y^1) \geq a(y^2)$. $\square$

The two concluding theorems show that by solving AMOP with an augmentation function associated with either a weighted-sum or a weighted-Chebyshev norm and an optimal solution $\hat{y}$, the corresponding ε-translated outcome $\tilde{y} = \hat{y} + \varepsilon$ for MOP can be obtained as an optimal solution to the respective scalarized AMOP. For a further motivation of this approach, we refer to Figure 3.10 which also illustrates our first result for the weighted-sum augmentation method.

Theorem 3.3.32 (Weighted-sum augmentation)**.** *Let* $Y \subset \mathbb{R}^m$ *and* $\hat{y} \in Y$ *be optimal for the weighted-sum scalarization WS(w) with scalarization function*

$$s(w, y) := \langle w, y \rangle \tag{3.96a}$$

with weighting parameter $w \in \mathbb{R}^m$*,* $w \geq 0$*, and*

$$a(w, y) := (s(w, y) - s(w, \hat{y}))^{-1} = \langle w, y - \hat{y} \rangle^{-1} \tag{3.96b}$$

be the augmentation function for AMOP. Let $\varepsilon \in \mathbb{R}^m$*,* $\varepsilon \geq 0$*, and assume that* $\tilde{y} = \hat{y} + \varepsilon \in Y$*. Let* $\epsilon = s(w, \varepsilon) = \langle w, \varepsilon \rangle$*. Then* $\tilde{y}$ *is optimal for the weighted-sum scalarization WS*$\Big(\big[w, \epsilon^2\big]^T\Big)$ *of AMOP with weighting parameter* $(w, \epsilon^2)^T \in \mathbb{R}^{m+1}$*.*

Proof. Consider the AMOP and the related weighted-sum scalarization

$$\text{AMOP:}\;\; \text{Minimize } \big[y, \langle w, y - \hat{y} \rangle^{-1}\big]^T \text{ subject to } y \in Y \tag{3.97a}$$

$$\text{WS}\Big(\big[w, \epsilon^2\big]^T\Big)\text{:}\;\; \text{Minimize } \langle w, y \rangle + \epsilon^2 \langle w, y - \hat{y} \rangle^{-1} \text{ subject to } y \in Y \tag{3.97b}$$

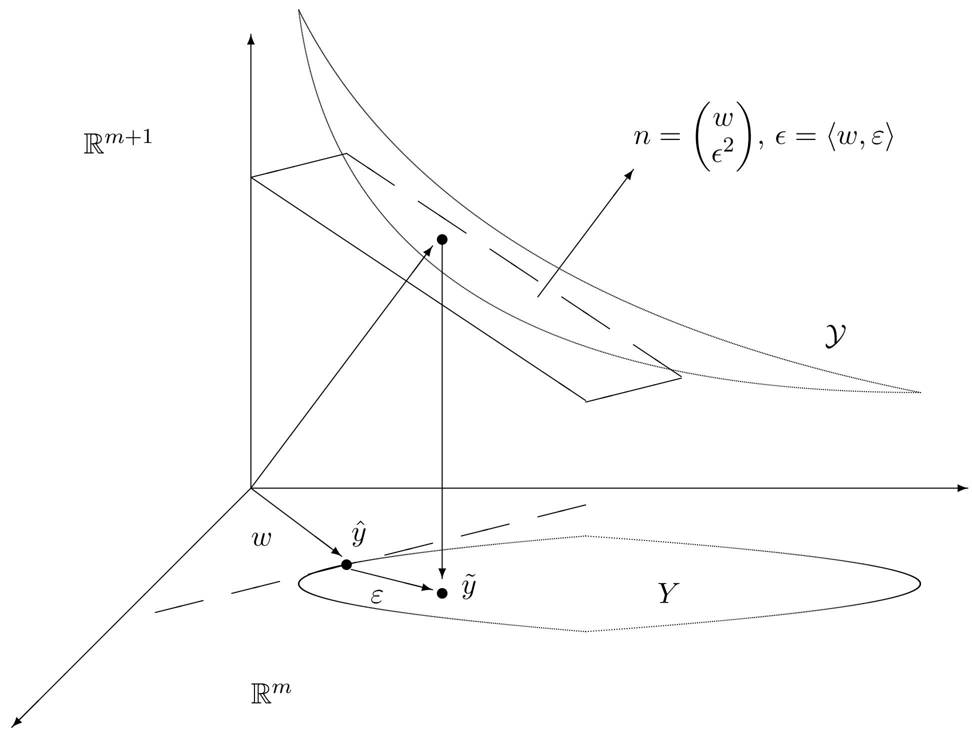


Figure 3.10 Illustration of the weighted-sum augmentation in Theorem 3.3.32

and substitute $t = s(w, y) = \langle w, y \rangle$ and $\hat{t} = s(w, \hat{y}) = \langle w, \hat{y} \rangle$. Then the objective function for $\mathrm{WS}\Big(\big[w, \epsilon^2\big]^T\Big)$ becomes $t + \epsilon^2 \left(t - \hat{t}\right)^{-1}$ and attains its minimum for $\tilde{t} = \hat{t} + \epsilon = s(w, \hat{y}) + \epsilon$, or

$$\tilde{t} = s(w, \hat{y}) + \epsilon = \langle w, \hat{y} \rangle + \langle w, \varepsilon \rangle = \langle w, \hat{y} + \varepsilon \rangle = s(w, \hat{y} + \varepsilon) = s(w, \tilde{y}) \qquad (3.98)$$

Hence, it follows that $\tilde{y}$ is optimal for $\mathrm{WS}\Big(\big[w, \epsilon^2\big]^T\Big)$. $\square$

The final theorem uses the weighted-Chebyshev norm scalarization function and is formulated to find the (weak) ε-translated Pareto outcome $\tilde{y} = \hat{y} + \varepsilon$ for any (weak) Pareto outcome $\hat{y} \in N(Y)$, but under the additional assumption that $\varepsilon > 0$.

Theorem 3.3.33 (Weighted-Chebyshev norm augmentation). *Let $Y \subset \mathbb{R}^m$ and $\hat{y} \in N(Y, \mathbb{R}^m_{\geqq})$ be a Pareto outcome. Let*

$$s(w, \hat{y}, y) := \max_{i=1,\ldots,m} \{w_i(y_i - \hat{y}_i)\} \tag{3.99a}$$

be the weighted-Chebyshev norm scalarization function with weighting parameter $w \in \mathbb{R}^m, w \geq 0$ and reference point $\hat{y}$, and

$$a(w, \hat{y}, y) := (s(w, \hat{y}, y) - s(w, \hat{y}, \hat{y}))^{-1} = s(w, \hat{y}, y)^{-1} \tag{3.99b}$$

be the augmentation function for AMOP. Let $\varepsilon \in \mathbb{R}^m, \varepsilon > 0$, and assume that $\tilde{y} = \hat{y} + \varepsilon \in Y$. Let $w_i = \varepsilon_i^{-1}$ for all $i = 1, \ldots, m$. Then $\tilde{y}$ is optimal for the weighted-Chebyshev norm method $CN\big([w, 1]^T, [\hat{y}, 0]^T\big)$ of AMOP with weighting parameter $(w, 1)^T \in \mathbb{R}^{m+1}$ and reference point $(\hat{y}, 0)^T \in \mathbb{R}^{m+1}$.

Proof. Consider AMOP and the related weighted-Chebyshev norm scalarization

$$\text{AMOP: Minimize } \left[y, s\left(w, \hat{y}, y\right)^{-1}\right]^T \text{ subject to } y \in Y \tag{3.100a}$$

$$\text{CN}\big([w, 1]^T, [\hat{y}, 0]^T\big)\text{: Minimize } \max_{i=1,\ldots,m} \left\{w_i(y_i - \hat{y}_i), s\left(w, \hat{y}, y\right)^{-1}\right\} \text{ subject to } y \in Y \tag{3.100b}$$

and substitute $t = s(w, \hat{y}, y) = \max_{i=1,\ldots,m}\{w_i(y_i - \hat{y}_i)\} \geq 0$. Then the objective function for $\text{CN}\big([w, 1]^T, [\hat{y}, 0]^T\big)$ becomes $\max\{t, t^{-1}\}$ with $t \geq 0$ and attains its minimum for $\tilde{t} = 1$, or

$$\tilde{t} = s(w, \hat{y}, y) = \max_{i=1,\ldots,m} \{w_i(y_i - \hat{y}_i)\} = 1 \tag{3.101a}$$

In particular, for $\tilde{y} = \hat{y} + \varepsilon$ it follows that

$$s(w, \hat{y}, \tilde{y}) = \max_{i=1,\ldots,m} \{w_i(\tilde{y}_i - \hat{y}_i)\} = \max_{i=1,\ldots,m} \{w_i \varepsilon_i\} = 1 \tag{3.101b}$$

by choice of $w_i = \varepsilon_i^{-1}$, and hence, $\tilde{y} = \hat{y} + \varepsilon$ is optimal for $\text{CN}([w, 1]^T, [\hat{y}, 0]^T)$. $\square$

In the following two sections, we first contrast these new generating methods with the previous approximate scalarization approaches and then examine their applicability on some concluding practical example.

3.3.5 Computation of Approximate Solutions

We now address the more practical question of finding approximate solutions using different techniques from single and multiobjective programming and computational optimization. Based on our theoretical results in Sections 3.3.4 and 3.3.5, we propose two different approaches for the generation of ε-nondominated outcomes for a multiobjective program MOP, which are depicted in Figure 3.11 and further described in the following paragraphs.

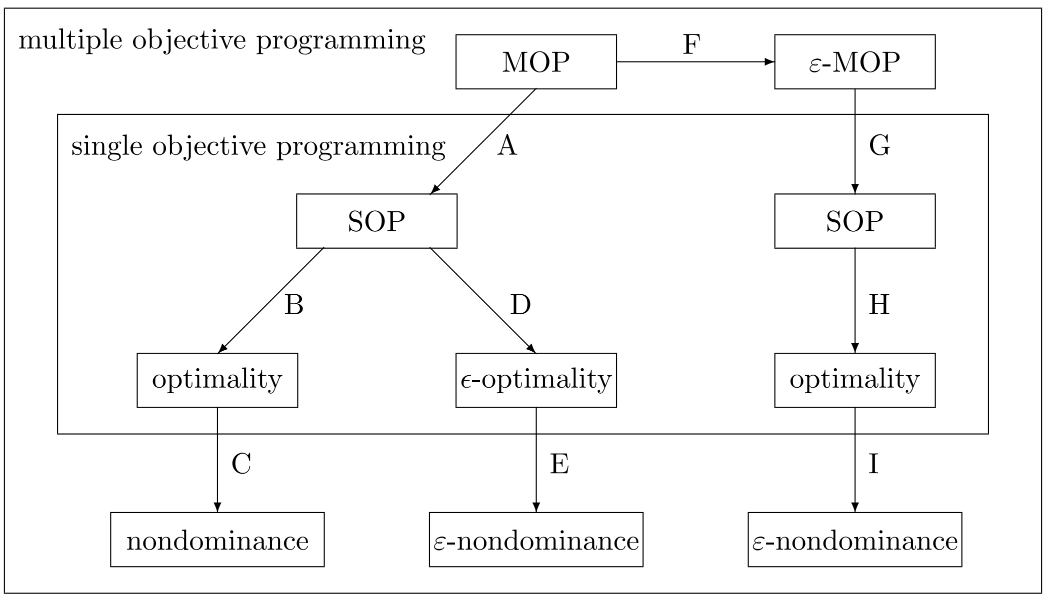


Figure 3.11 Relationships between (ϵ-)optimality for SOP and (ε-)nondominance for MOP for the approximate (ABCDE) and exact (FGHI) generating methods

Usually, if we intend to find the nondominated set for MOP, we choose a suitable scalarization method and formulate an associated single objective program SOP (A) which is solved to optimality (B) using any appropriate linear or nonlinear optimization technique. In this case, the established scalarization results that we review in Section 2.2.2 provide us with relationships between the optimal solutions for SOP and nondominated outcomes (or efficient decisions) for MOP (C). In addition, using the new relationships between ϵ-optimality for $\epsilon \in \mathbb{R}$ and ε-nondominance for

$\varepsilon \in \mathbb{R}^m$ derived in the previous Section 3.3.3, we can also relate ϵ-optimal solutions for SOP (D) to ε-nondominated outcomes for MOP (E). Hence, we accordingly call these methods *approximate methods*, as the generation of ε-nondominated outcomes is based on an approximate solution of SOP.

Procedure 3.3.34 (Approximate method)**.** Let MOP be given.

1.	**Choice of SOP:**	Choose a scalarization problem SOP.
2.	**Choice of $\boldsymbol{\pi}$:**	Choose a scalarization parameter $\pi \in \Pi$.
3.	**Choice of $\boldsymbol{\varepsilon}$:**	Choose an admissible vector $\varepsilon \in \mathbb{R}^m$.
4.	**Computation of $\boldsymbol{\epsilon}$:**	Compute the corresponding $\epsilon \in \mathbb{R}$.
5.	**Solution of SOP:**	Solve SOP for an ϵ-optimal solution.

This generic procedure assumes that based on prior knowledge and experience with the underlying MOP, we are able to choose a suitable scalarization problem SOP in Step 1 together with an appropriate scalarization parameter $\pi \in \Pi$ and the desired vector parameter and $\varepsilon \in \mathbb{R}^m$ in Steps 2 and 3, respectively. Clearly, in practice each of these choices depends on the concrete problem to be solved and the specific reasons for which an approximate solution is generated.

For now, however, we postpone this discussion and directly proceed to Step 4 in which we compute the corresponding scalar parameter $\epsilon = \epsilon(\pi, \varepsilon)$ based on the results derived in the previous section. In particular, we note that each SOP but the Pascoletti-Serafini method provides us with an explicit upper bound on the set of suitable values for $\epsilon \in \mathbb{R}$ for which an ϵ-optimal solution for SOP is still guaranteed to also be ε-nondominated for MOP.

For the most typical case of Pareto nondominance, however, we can also derive the corresponding upper bound on the set $\{\delta : \varepsilon - \delta v \in C\}$ as specified for the approximate Pascoletti-Serafini scalarization in Proposition 3.3.24. In particular, if we choose $C = \mathbb{R}^m_{\geqq}$ to be the Pareto cone, then we readily obtain that

$$\epsilon = \max\{\delta : \varepsilon - \delta v \in \mathbb{R}^m_{\geqq}\} = \max\{\delta : \varepsilon \geqq \delta v\} \tag{3.102a}$$

$$= \max\{\delta : \varepsilon_i \geq \delta v_i \text{ for all } i = 1, \ldots, m\} \tag{3.102b}$$

$$= \max\{\delta : \varepsilon_i / v_i \geq \delta \text{ for all } i = 1, \ldots, m\} = \min_{i=1,\ldots,m} \{\varepsilon_i / v_i\} \tag{3.102c}$$

This result is also listed in Table 3.1 which again summarizes all required scalarization parameters $\pi \in \Pi$, the admissible $\varepsilon \in \mathbb{R}^m$ and the corresponding upper bounds of $\epsilon \in \mathbb{R}$ for all scalarization approaches examined in Section 3.3.3. In particular, since the max-norm scalarization does not depend on any additional parameters, in this case, the second step (choice of π) of Procedure 3.3.34 does not apply.

Table 3.1 Parameter choices for the approximate methods in Procedure 3.3.34

SOP(π)	$\pi \in \Pi$	$\varepsilon \in \mathbb{R}^m$	$\epsilon = \epsilon(\pi, \varepsilon)$
Hybrid HB(w, b)	$w \geq 0$, $b \in \mathbb{R}^m$		$\epsilon = \langle w, \varepsilon \rangle$
Weighted-sum WS(w)	$w \geq 0$		$\epsilon = \langle w, \varepsilon \rangle$
Constrained-objective CO$_k$(b)	$b \in \mathbb{R}^m$		$\epsilon = \varepsilon_k$
Benson B(y°)	$y^\circ \in Y$	$\varepsilon \geq 0$	$\epsilon = \sum_i \varepsilon_i$
Pascoletti-Serafini PS(r, v)	$r \in \mathbb{R}^m$, $v \geq 0$		$\epsilon = \min_i\{\varepsilon_i / v_i\}$
Chebyshev-norm CN(r, w)	$r \in \mathbb{R}^m$, $w \geq 0$		$\epsilon = \min_i\{w_i \varepsilon_i\}$
Max-norm MN	no parameter		$\epsilon = \min_i\{\varepsilon_i\}$

Finally, in Step 5 of Procedure 3.3.34 we only need to solve SOP for an ϵ-optimal solution in which case our previous results ensure that the associated outcome is also ε-nondominated for MOP. In principle, we suggest that the approximate solution of SOP is best accomplished by a suitable modification of the termination criteria of the chosen optimization routine, although possibly challenging due to inaccurate or unavailable optimality bounds and unclear or hidden black-box implementations of the different stopping conditions in commercial optimization software. We illustrate some of our own difficulties together with the discussion of our preliminary computational results in the subsequent section.

Another potential drawback of the above approximate methods is that an ϵ-optimal solution for SOP, in general, does not guarantee the full relaxation specified by the vector parameter ε which, in certain situations that we also discuss later, might be of primary interest when intentionally generating such approximate solutions. In order to resolve this current limitation, in Section 3.3.4 we investigate several alternative approaches for the exact generation of ε-translated nondominated outcomes, and for its description again see Figure 3.11. These alternative methods modify MOP and formulate an associated ε-MOP (F) so that optimal solutions (H) for its associated scalarized problem (G) yield associated ε-nondominated outcomes (I) for the original MOP. In particular, we propose two exact methods based on norms and additional constraints in Propositions 3.3.27 and 3.3.28, respectively, and the two augmentation approaches in Theorems 3.3.32 and 3.3.33 which are again summarized in the following generic procedure and Table 3.2.

Procedure 3.3.35 (Augmentation method)**.** Let MOP be given.

1.	**Choice of SOP:**	Choose a scalarization function s for SOP.
2.	**Choice of π:**	Choose a scalarization parameter π for SOP.
3.	**Solution of SOP:**	Solve SOP for a Pareto outcome $\hat{y} \in N(Y)$.
4.	**Choice of ε:**	Choose $\varepsilon \geq 0$ and let $\tilde{y} = \hat{y} + \varepsilon \in N_\varepsilon(Y)$.
5.	**AMOP Formulation:**	Formulate the AMOP associated with s and $\hat{y}$.
6.	**AMOP Scalarization:**	Formulate a scalarization method for AMOP.
7.	**AMOP Solution:**	Solve the scalarization problem for $\tilde{y} \in N_\varepsilon(Y)$.

In order to examine the practical applicability of these generating methods for the actual computation of ε-nondominated outcomes, the following section discusses some of our numerical results that are obtained from implementing the approximate method, both the exact norm and exact constraint generating methods, as well as the weighted-sum augmentation approach.

Table 3.2 Parameter choices for the augmentation methods in Procedure 3.3.35

1. Choice of SOP	Weighted-sum	Weighted-Chebyshev norm
2. Choice of $\boldsymbol{\pi}$	$s(w, y) = \langle w, y \rangle$	$s(w, \hat{y}, y) = \max_i\{w_i(y_i - \hat{y}_i)\}$
3. Solution of SOP	$\hat{y} \in N(Y)$	$\hat{y} \in N(Y)$
4. Choice of $\boldsymbol{\varepsilon}$	$\varepsilon \geq 0$	$\varepsilon \geq 0$
5. AMOP formulation	$a(w, y) = \langle w, y - \hat{y} \rangle^{-1}$	$a(w, \hat{y}, y) = \max_i\{w_i(y_i - \hat{y}_i)\}$
6. AMOP scalarization	$\mathrm{WS}([w, \epsilon^2])$, $\epsilon = \langle w, \varepsilon \rangle$	$\mathrm{CN}([w, 1]^T, [\hat{y}, 0]^T)$, $w_i = \varepsilon_i^{-1}$
7. AMOP solution	$\tilde{y} = \hat{y} + \varepsilon \in N_\varepsilon(Y)$	$\tilde{y} = \hat{y} + \varepsilon \in N_\varepsilon(Y)$

3.3.6 Example and Computational Results

We present some selected computational results that are obtained from implementing the previously derived approximate and exact generating methods for finding approximate and ε-translated Pareto outcomes for a multiobjective program. In particular, for our illustration we choose an engineering problem from truss topology design which is frequently used as prominent test case also for many other applications (Stadler and Dauer, 1992; Coello Coello and Lamont, 2004) and algorithms (Ben-Tal and Nemirovski, 1997; Jarre et al., 1998; Coello and Christiansen, 2000, among many others).

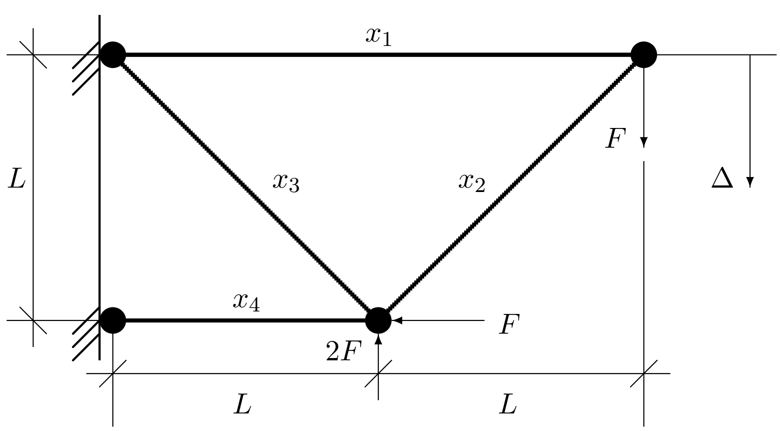


Figure 3.12 A four-bar plane truss with cross-sectional areas x_1, x_2, x_3 and x_4

The specific problem that we select is taken from Koski (1988) who formulates a mathematical model for designing a four-bar plane truss, shown in Figure 3.12, as a biobjective program with the two conflicting objectives of minimizing both the volume V of the truss (f_1) and the displacement Δ of the joint connecting bars 1 and 2 (f_2), subject to given physical restraints on the feasible cross-sectional areas x_1, x_2, x_3, x_4 of the four bars. The stress on the truss is caused by three forces of magnitudes F and $2F$ as depicted in Figure 3.12, and the length L of each bar, the Young's modulus of elasticity E and the only nonzero stress component σ are modeled as constants. The corresponding formulation of this problem is given as

$$\text{Minimize } f_1(x) = L(2x_1 + \sqrt{2}x_2 + \sqrt{2}x_3 + x_4) \tag{3.103a}$$

$$f_2(x) = \frac{FL}{E}\left(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} - \frac{2\sqrt{2}}{x_3} + \frac{2}{x_4}\right) \tag{3.103b}$$

$$\text{subject to} \quad (F/\sigma) \leq x_1 \leq 3(F/\sigma) \tag{3.103c}$$

$$\sqrt{2}(F/\sigma) \leq x_2 \leq 3(F/\sigma) \tag{3.103d}$$

$$\sqrt{2}(F/\sigma) \leq x_3 \leq 3(F/\sigma) \tag{3.103e}$$

$$(F/\sigma) \leq x_4 \leq 3(F/\sigma) \tag{3.103f}$$

where $F = 10$ kN, $E = 2 \times 10^5$ kN/cm^2, $L = 200$ cm and $\sigma = 10$ kN/cm^2.

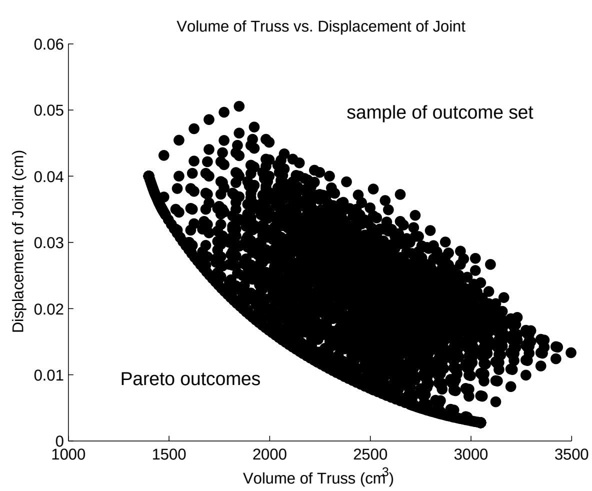


Figure 3.13 Outcome set for the four-bar plane truss problem in 3.103

A sample of the outcome set for the problem in (3.103) is depicted in Figure 3.13 and shows that the possible truss volumes range between about 1500 cm^3 and 3500 cm^3 with the corresponding nodal displacements at the joint connecting bars 1 and 2 less than 0.05 cm. Hence, due to this significant difference in magnitude, for all following computation we first normalize these objectives and then generate one hundred Pareto outcomes for an initial representation of the actual Pareto set, also highlighted in Figure 3.13.

We observe that the generated Pareto set is convex so that, in principle, we can use the weighted-sum scalarization to generate all Pareto outcomes for this problem. Consequently, we also choose this method for the following illustration of the approximate method, following the scheme outlined in the previous section. First, based on the determined ranges of truss volume and joint displacement, we specify four different relaxation or tolerance vectors ε and, for varying choices of the weighting vector w, compute the associated scalar $\epsilon = w_1\varepsilon_1 + w_2\varepsilon_2$. Then, in the actual computation step, we solve the corresponding weighted-sum scalarization for an ϵ-optimal solution by suitable modification of the underlying stopping criterion. The obtained results are shown in Figure 3.14.

As expected, in each case we find that the newly generated outcomes are ε-Pareto for the original problem, based on the specified values for ε indicated for each of the four individual plots. However, while we choose one hundred different combinations for the weighting vector in each case, the numbers of distinct outcomes that are actually generated are quite different and, in general, decrease with increasing ε. We believe that the main reason for this behavior is that a modified stopping criterion enables the optimization routine to terminate early and, thus, with intermediate solutions that do not yet reach different regions of the outcome set. In other words, the larger the admissible relaxation, the earlier we can stop the optimization algorithm and the more often we are still at the same intermediate solution, resulting in fewer different outcomes generated and providing a possible explanation for the above observation. A similar argumentation also explains why

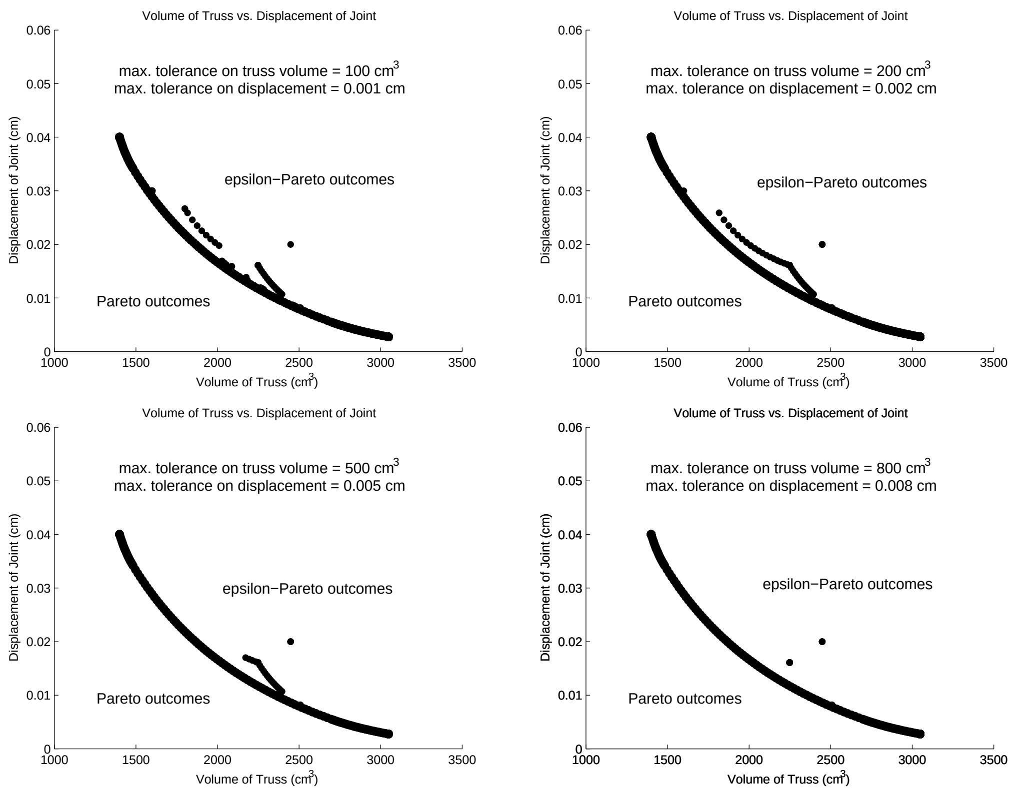


Figure 3.14 Pareto and ε(-translated) Pareto outcomes for the four-bar truss problem generated by the approximate weighted-sum method with four choices of ε

the generated ε-Pareto outcomes are more or less concentrated in the same area and, in this case, all fall into the middle region of the Pareto curve.

Moreover, we can also identify numerous other issues that may influence which ε-Pareto outcomes are generated when using an approximate method that is based on scalarization, including the chosen optimization routine, the underlying algorithm and its implementation, the selection of initial points, and of course the imposed stopping criterion. Consequently, much further work and investigation is possible and necessary to completely analyze this approach which, however, is far beyond the purpose of this discussion. Hence, as only one further consequence of the above dependencies and numerical difficulties, at the moment we conclude that approximate methods do not seem suitable to generate a reasonable representation of the set of ε-Pareto outcomes and especially of those ε-translated Pareto outcomes that achieve the full relaxation specified by the parameter ε.

In remedy of this last shortcoming and to contrast the approximate methods with the exact generating methods derived in Section 3.3.5, we also present some of our results for the exact norm, the exact constraint, and the weighted-sum augmentation method. Starting from the identical set of one hundred Pareto outcomes initially generated by the weighted-sum scalarization for the original problem, we use the same four choices of ε as before to formulate the corresponding new problems for finding the associated ε-translated outcomes. In particular, Figures 3.15 and 3.16 show our results for the exact norm and constraint generating method, respectively, and as intended both approaches yield a reasonable set of fully ε-translated Pareto outcomes for each of the four different values of the parameter ε.

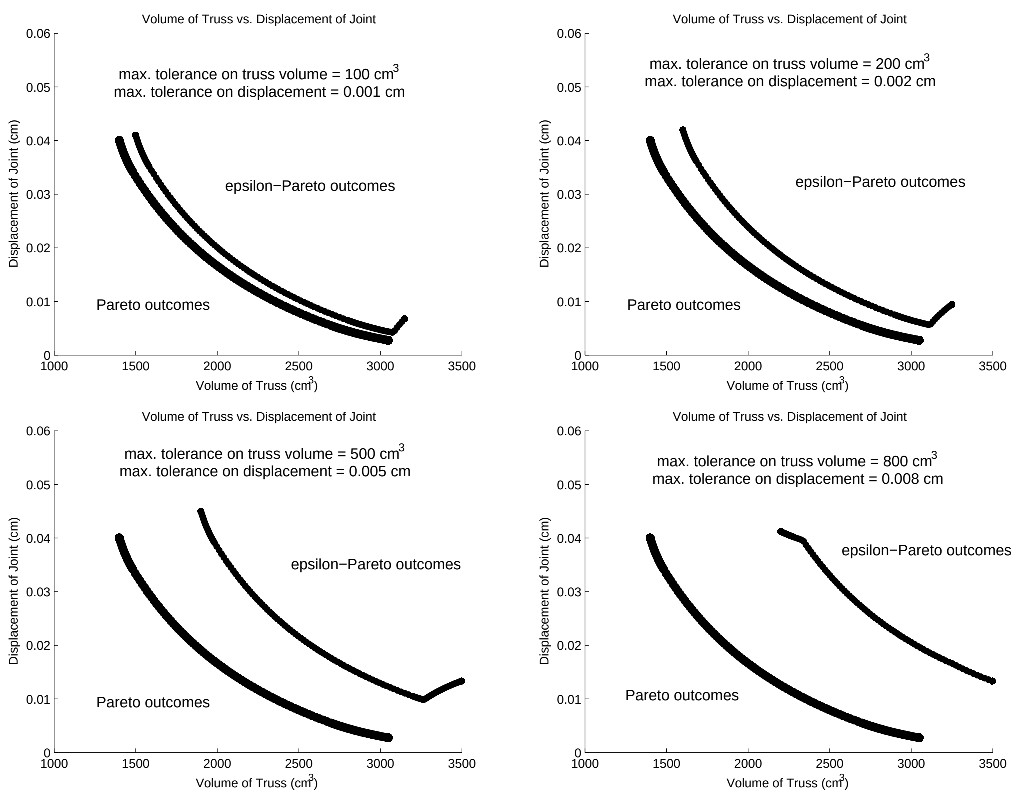


Figure 3.15 Pareto and ε(-translated) Pareto outcomes for the four-bar truss problem generated by the exact norm generating method with four choices of ε

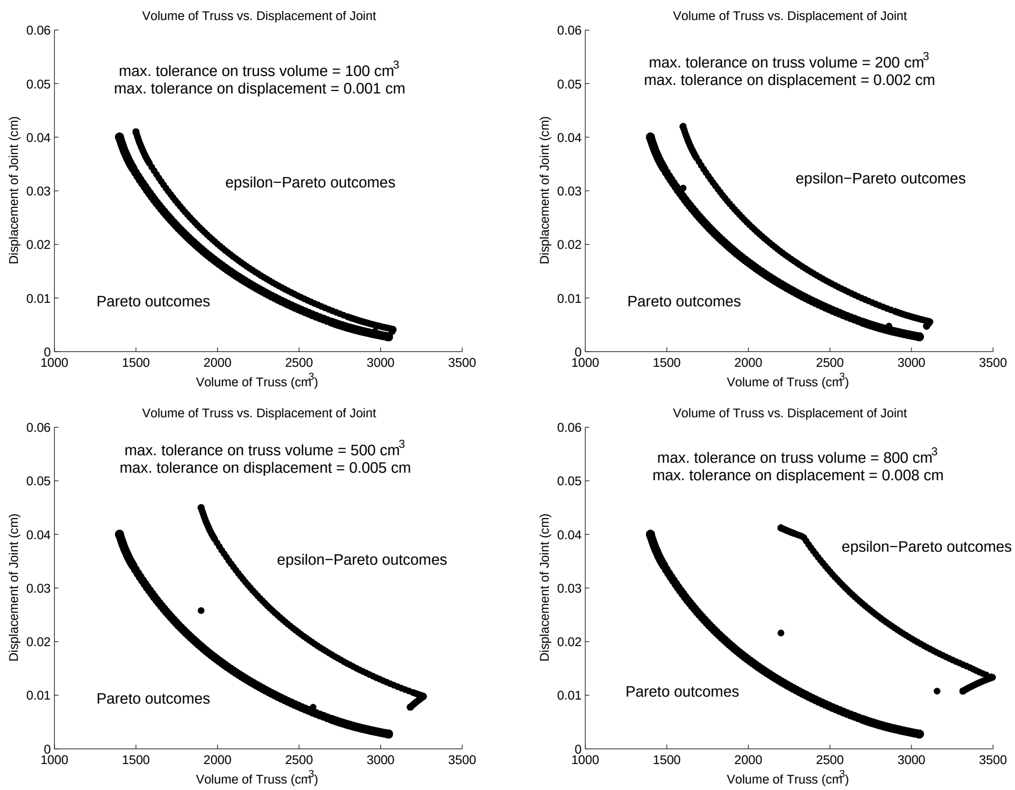


Figure 3.16 Pareto and ε(-translated) Pareto outcomes for the four-bar truss problem generated by the exact constraint generating method with four choices of ε

While the generated sets are almost identical, we emphasize one subtle difference between these two methods which is also mentioned during the initial discussion of these two approaches in Section 3.3.5. We recall that a norm minimization method merely attempts to minimize the deviation from the currently specified ε-translated outcome, thereby possibly resulting in a final outcome that is slightly greater and, therefore, not itself ε-nondominated, in general. The introduction of additional constraints for the second approach, however, guarantees that all generated outcomes also satisfy the additionally imposed constraints and, thus, are always ε-nondominated, in particular. This observation also explains the different behavior of these two methods in the lower right regions of Figures 3.15 and 3.16 in which the corresponding ε-translated outcomes do not anymore belong to the outcome set, opposite to the assumption in our previous theoretical discussion.

Moreover, another difference between these two approaches is that only the exact norm generating method is a single objective program whereas the constraint method is still formulated as a multiobjective program and, thus, dependent on a scalarization method for its solution. Based upon the scalarization function used, this might lead to additional numerical difficulties possibly explaining the few outliers in Figure 3.16 that do not occur for the exact norm method in Figure 3.15. In general, however, we can conclude that both approaches seem to resolve the previously recognized shortcomings of the approximate method and yield an almost complete representation of the set of fully relaxed or ε-translated Pareto outcomes for various choices of ε over the complete outcome set of the original problem.

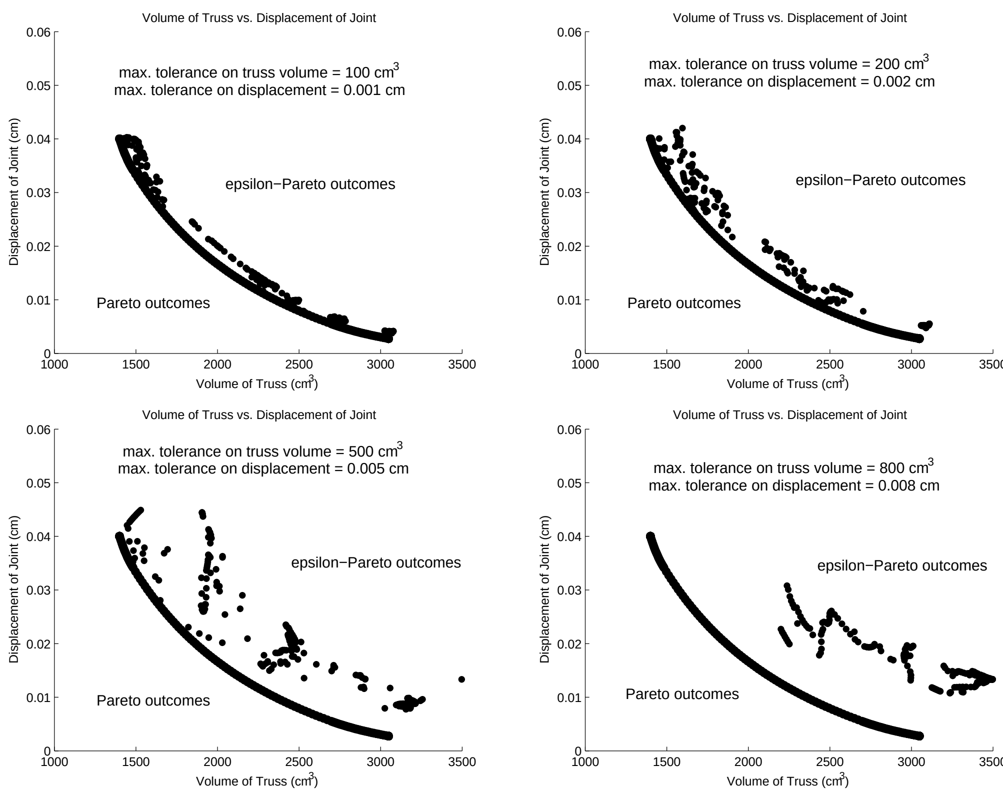


Figure 3.17 Pareto and ε(-translated) Pareto outcomes for the four-bar truss problem generated by the exact weighted-sum augmentation with four choices of ε

Finally, the results shown in Figure 3.17 are obtained from implementing the weighted-sum augmentation approach and reveal that this method, although conceptually appealing, is subject to various numerical issues and difficulties. In particular, we believe that due to the potential division by zero, this formulation becomes very sensitive and, especially in comparison to the results for the two previous methods, therefore does not seem suited for actual implementation in practice. Nevertheless, again much further research is possible and may lead to other findings, further improvements, or new and better approaches for the generation of ε-nondominated outcomes in multiobjective programming.

3.4 Discussion and Further Research

In this chapter, we study the concept of domination in multiobjective programming, and based on the prominent role played by domination cones, we focus in large part on domination structures that can be defined in terms of cones. While we emphasize as part of our literature review in Section 2.2.1 that the established theory of multiobjective or vector optimization includes various results that describe properties of the nondominated set for a general convex cone, the use of cones in multiobjective programming is frequently limited to polyhedral cones and the development of generating methods in the large majority of cases typically restricted to finding Pareto optimal solutions.

Consequently, we begin in Section 3.1 with the investigation of general cones while taking a closer look at possible cone representations and appropriate modifications of several existing scalarization methods. In analogy to the definition of polyhedral cones that are described by linear functions, we find in Theorem 3.1.1 (cone representation theorem) that an arbitrary cone can be induced by a function that is positively homogeneous, and based on the additional property of superlinearity we establish some first sufficient conditions for pointedness and convexity in Proposition 3.1.8. Also motivated by a corresponding result for polyhedral cones, in Theorem 3.1.12 (nondominance mapping theorem) we establish relationships between the image of the nondominated set with respect to a general cone and the

Pareto set of the image of the complete outcome set under that cone-inducing mapping. In addition, independent of the specific cone representation, we show in the series of Propositions 3.1.14–3.1.17 how the hybrid, constrained-objective and Benson scalarization methods can be modified for a general cone, and the weighted-sum method for a general domination set. Along with this discussion, we illustrate our main results for four special types of cones, namely Pareto, polyhedral, pth-order and Bishop-Phelps cones, among which the first three are special cases of an alternative cone representation that is examined separately at the end of that section.

We believe that this material offers several interesting research questions worthwhile of further pursuit. First, other than for polyhedral cones, the characterization of pointedness and the nondominance mapping theorem in the general case are currently valid only as a sufficient condition and set inclusion, respectively, and further assumptions or a completely different cone representation may be derived to also obtain the corresponding or new results in stronger form. Second, although the modified scalarization methods, in principle, allow the generation of nondominated outcomes for a general cone, except for the weighted-sum we need to replace the initially linear inequality constraints by new and in general nonlinear cone constraints, so that the resulting scalarized problems become mathematical cone programs which are considerably more difficult to solve or may also require the development of completely new solution methods and algorithms in extension of semidefinite and second-order cone programming (Alizadeh and Goldfarb, 2003; Pataki, 2003). Finally, although the literature uses the concept of Bishop-Phelps cones for applications in nonlinear analysis (Hyers et al., 1997) and reports on various applications of especially second-order cones in engineering and robust optimization, including portfolio optimization, signal processing, and truss topology design (Lobo et al., 1998), the question remains if there is also any practical relevance of general cones for the purpose of domination in multiobjective programming.

In a way following this very last question, our investigation of variable domination cones in Section 3.2 provides at least a partial answer as the cones underlying the derived variable-cone model in Proposition 3.2.18 can be described as Bishop-Phelps cones, based on the assumption of ideal-symmetry that is introduced in Definition 3.2.14 to steer the variability of the associated domination structure. Placing our discussion into the general framework of preferences and preference modeling, we lead to this assumption with two motivating examples that illustrate some of the limitations of preference models that are described by constant cones. In particular, in Example 3.2.9 we find that, for linear cases, it is not possible to describe a preference model that excludes extreme or corner points while maintaining a set of nondominated outcomes in the middle region of the Pareto frontier, and Example 3.2.10 shows that a constant cone does not enable the individual specification of preferences between two arbitrarily selected outcomes without also affecting the preference relationships among the remaining other points. After examining some of the consequences and, in fact, difficulties of variable cones in the formulation of generating methods, we instead use the geometric interpretation of the ideal-symmetry assumption in Remark 3.2.15 to derive two alternative optimality conditions in Theorem 3.2.25 for the convex and Theorem 3.2.27 for the concave biobjective case, that are subsequently used to illustrate how our new variable-cone model removes the previously recognized shortcomings of constant cones.

Due to the scarce existing literature but the seemingly high potential for significant impacts on the advancement of domination and preference modeling, we expect that further inquiry of variable domination structures and variable cones, in particular, opens a rich and rewarding avenue for many following research activities. In continuation of our own work, the maybe most imperative issue is the formulation of more general conditions for nondominance that can either emerge from our preliminary discussion of scalarization methods for variable cones or extend Theorems 3.2.25 and 3.2.27 from biobjective to multiobjective cases by removing the restricting geometric character currently employed in their proofs. Moreover, while the three chosen Examples 3.2.29–3.2.31 are simple enough to find the nondominated sets based on the above results combined with our perceptual intuition and some basic algebra, we do not only need to derive new conditions but eventually also develop suitable optimization methods for the algorithmic or computational generation of nondominated outcomes with respect to variable cones. Finally, our assumption of ideal-symmetry can be replaced by several other mechanisms that guarantee variability of the associated domination structure which, in particular, can also be derived directly from an underlying preference assumption such as the principle of transfers (2.64) or justice (2.65) in Section 2.3.2 to eventually produce a variety of new approaches to variable preference modeling of immediate relevance also to multiple criteria decision making.

Another notion of high importance in both theory and applications of multiobjective programming and optimization is the concept of approximation or approximate nondominance that we initially address in Section 2.3.1 and investigate with respect to its underlying domination structure, possible cone representations, and generating methods in Section 3.3. Highlighting the concepts of (strict and weak) epsilon-nondominance that we introduce in Definition 3.3.8, we first realize that the associated domination structures cannot be described by a cone anymore but now correspond to translated cones (Luenberger, 1969), and we show in Theorem 3.3.6 how translated polyhedral cones have an equivalent representation in terms of polyhedral sets. We use this result to generalize the nondominance mapping in Theorem 3.1.13 to also hold for approximate Pareto and epsilon-nondominance in Theorem 3.3.18, and based on the definition of epsilon-translated nondominated outcomes in Definition 3.3.12 we establish several other relationships between the (strict and weak) epsilon-nondominated sets which are summarized in Corollary 3.3.15. Finally, we propose two different approaches for the generation of approximate solutions based on scalarization methods, illustrate their application on an example and already provide a detailed assessment of these methods and some selected computational results along with the discussion in the foregoing text.

While possible directions for future work certainly include a similar examination of other approximation concepts than the ones chosen in this text, we can also think of several research questions that directly build on our above results obtained for epsilon-nondominance. First, while we present some initial results on translated cones and focus on representations for the polyhedral case in terms of a linear system of inequalities, analogous characterizations may be derived for translated nonpolyhedral cones based on our earlier cone representation in Theorem 3.1.1 when using systems of inequalities that are nonlinear. Second, we encourage closer investigation of the proposed generating methods together with their further improvement and application for the development of new approximation methodologies for difficult or large-scale multiobjective programs, in which the generation of the nondominated set is practically impossible or at least computationally too expensive. Additionally, and opposed to the common belief that epsilon-nondominance is a concept of approximation that primarily accounts for modeling limitations or computational inaccuracies and thus is tolerable rather than desirable, we also promote the relevance and application of this concept in practical decision making situations, including its prominent role in the study of decomposition and coordination methods now pursued in the following next chapter.

CHAPTER 4

DECOMPOSITION AND COORDINATION METHODS

In remedy of the challenges resulting from a high number of objectives in multiobjective programming and multiple criteria decision making, in this chapter we choose to decompose the vector objective function of a large-scale multiobjective program to define a collection of smaller-sized subproblems and investigate several solution methods and decision making procedures for their subsequent coordination. The associated function and problem decomposition is introduced and analyzed in Section 4.1 which establishes the theoretical foundation for the formulation of an interactive decision making scheme that we describe in Section 4.2 and illustrate on an example in Section 4.3. The first two sections are extended versions of the similar treatment in Engau and Wiecek (2007e), while the particular discussion in Section 4.2.2 and the third section is adjusted from Engau and Wiecek (2007a) originally addressing the same subject matter but in the more applied context of multidisciplinary optimization and engineering design. Section 4.4 concludes this chapter with a brief summary and suggestions for both further research and numerous applications in direct transition also to the next following chapter.

4.1 The Decomposed Multiobjective Program

The first definition introduces the adopted notion of decomposition and also serves to clarify the associated notation that is used throughout this chapter.

Definition 4.1.1 (Decomposition and Partition)**.** Let $X \subseteq \mathbb{R}^n$ be a set, $f : X \to \mathbb{R}^m$ be a vector-valued function, and $I := \{1, 2, \ldots, m\}$ be the index set of f

$$f = (f_1, f_2, \ldots, f_m) =: (f_i)_{i \in I} \tag{4.1a}$$

A collection $\{I_j \subset I : j = 1, \ldots, M\}$ is called a *decomposition* of I, and

$$F := \{f^j = (f_i)_{i \in I_j} : j = 1, \ldots, M\} \tag{4.1b}$$

is called the associated *function decomposition* of f. It is said to be

(i) *complete* if $\bigcup_{j=1}^{M} I_j = I$, and

(ii) *pairwise disjoint* if $I_i \cap I_j = \emptyset$ for all $i \neq j \in I$.

A complete and pairwise disjoint decomposition is also called a *partition*.

In this complete chapter, we restrict our discussion to the Pareto case and study the multiobjective program (X, f) with respect to the feasible set X and objective function f rather than with respect to the associated outcome set $Y = f(X)$. Since the function decomposition of f is conceptually equivalent to projecting Y onto a collection of different subspaces of the outcome space $\mathbb{R}^m$, the former is much more convenient as it does not require to distinguish nondominance in these different subspaces, but allows to restrict our discussion to a common underlying feasible decision set X in the decision space $\mathbb{R}^n$.

Definition 4.1.2 (Decomposed Multiobjective Program)**.** Let (X, f) be MOP and $F = \{f^j = (f_i)_{i \in I_j} : j = 1, \ldots, M\}$ with $|I_j| = m_j$ be a function decomposition of f. The collection of all multiobjective programs

$$\text{MOP}_j\text{: Minimize } f^j(x) = [f_1^j(x), \ldots, f_{m_j}^j(x)] \text{ subject to } x \in X \tag{4.2}$$

is called the *decomposed MOP* (DMOP), and each MOP_j is also called a *subproblem* of DMOP. It is said to be *complete* if the function decomposition F is complete.

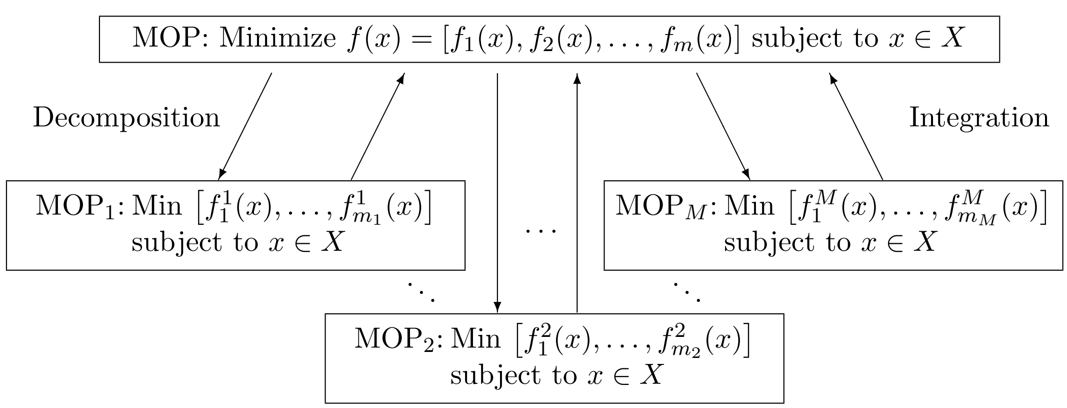


Figure 4.1 Decomposition of MOP into collection of subproblems MOP_j (DMOP)

Definition 4.1.2 is illustrated in Figure 4.1. The original MOP with decision set X and objective function $f = (f_1, f_2, \ldots, f_m)$ is decomposed into a collection of M subproblems MOP_j with the same feasible set X but new objective functions whose components $f_1^j, \ldots, f_{m_j}^j$ are only a subset of those contained in the original objective function f. After solving these subproblems, we refer to the reversed process of relating solutions for MOP_j to solutions for MOP as *integration*.

A problem formulation that is very similar to this decomposition approach also arises in multiscenario multiobjective programming (Singh, 2001; Fadel et al., 2005; Wiecek et al., 2006) which we introduce next.

Definition 4.1.3 (Multiscenario multiobjective program)**.** Let $S = \{1, 2, \ldots, M\}$ be a set of *scenarios*, $X^s \subseteq \mathbb{R}^n$ be a feasible set and $f^s : X^s \to \mathbb{R}^{m_s}$ be a vector-valued objective function for each single scenario $s \in S$. Let $X := \bigcap_{s \in S} X^s \neq \emptyset$ be the common feasible set for all scenarios. The collection

$$\text{MSMOP: Minimize } \{f^s(x) : s \in S\} \text{ subject to } x \in X \tag{4.3}$$

is called the *multiscenario multiobjective program* (MSMOP).

Hence, one possible application of DMOP is that it allows to treat MSMOP in terms of its individual scenarios as subproblems MOP_s that are restricted to the common feasible set X. In particular, in this case any feasible decision for a subproblem is also feasible for the original problem, and the integration of decisions from the subproblem level to the original MOP can be accomplished by simply evaluating those objective functions that have not yet been considered in the respective subproblem. The following result now establishes the relationships between efficient decisions for a subproblem MOP_j of DMOP and the original problem MOP and is similarly established by Singh (2001) in the context of MSMOP.

Proposition 4.1.4 (Efficiency for MOP_j implies efficiency for MOP)**.** *Let MOP and DMOP be given, and $\hat{x} \in X$ be a feasible decision.*

(i) If $\hat{x}$ is weakly efficient for MOP_j, then $\hat{x}$ is weakly efficient for MOP

$$E_w(X, f^j) \subseteq E_w(X, f) \tag{4.4a}$$

(ii) If $\hat{x}$ is efficient for MOP$_j$ and if f^j is injective, then $\hat{x}$ is efficient for MOP

$$E(X, f^j) \subseteq E(X, f) \tag{4.4b}$$

Proof. For (i), let $\hat{x} \in E_w(X, f^j)$ and, by contradiction, assume that $\hat{x} \notin E_w(X, f)$. Then there exists $x \in X$ such that $f(x) < f(\hat{x})$, or equivalently, $f_i(x) < f_i(\hat{x})$ for all $i \in I$. In particular, this implies that $f_i(x) < f_i(\hat{x})$ for all $i \in I_j \subset I$, or equivalently, $f^j(x) < f^j(\hat{x})$ in contradiction to $\hat{x} \in E_w(X, f^j)$.

For (ii), let $\hat{x} \in E(X, f^j)$ and, by contradiction, assume that $\hat{x} \notin E(X, f)$. Then there exists $x \in X$ such that $f(x) \leq f(\hat{x})$, or equivalently, $f_i(x) \leq f_i(\hat{x})$ for all $i \in I$ and $f(x) \neq f(\hat{x})$, so $x \neq \hat{x}$. In particular, this implies that $f_i(x) \leq f_i(\hat{x})$ for all $i \in I_j \subset I$ and $f^j(x) \neq f^j(\hat{x})$ because f^j is injective, or equivalently, $f^j(x) \leq f^j(\hat{x})$ in contradiction to $\hat{x} \in E(X, f^j)$. $\square$

Hence, any (weakly) efficient decision for a subproblem of DMOP is also (weakly) efficient for the original MOP. The following example illustrates that the reversed statement, however, in general does not hold true.

Example 4.1.5 (Efficiency for MOP does not imply efficiency for MOP$_j$)**.** Let $X = \{x^1, x^2, x^3, x^4\}$ be a feasible set with only four decisions, and the objective function $f = (f_1, f_2, f_3) : X \to \mathbb{R}^3$ be composed of three single objectives that are to be minimized. The objective function values $f(x)$ of the four decisions are listed in Table 4.1, and it is easily verified that all four decisions are efficient for MOP with objective function $f = (f_1, f_2, f_3)$. However, if we define the three subproblems MOP$_1$, MOP$_2$, and MOP$_3$ with objective functions $f^1 = (f_1, f_2)$, $f^2 = (f_1, f_3)$, and $f^3 = (f_2, f_3)$, respectively, then we find that in each case only one of the decisions x^1, x^2, or x^3 remains efficient while the respective other two are merely weakly efficient. In particular, the decision x^4 is neither efficient nor weakly efficient for any of the three subproblems, although apparently the best compromise decision.

Although Example 4.1.5 illustrates that efficient decisions for the original problem are not necessarily efficient for any subproblem, it is clear that, in fact, every feasible decision is ε-efficient for MOP and ε^j-efficient for MOP$_j$.

Table 4.1 Objective function values for the four decisions in Example 4.1.5

decision	f_1	f_2	f_3	observation
x^1	1	1	9	efficient for MOP and MOP$_1$ with $f^1 = (f_1, f_2)$
x^2	1	9	1	efficient for MOP and MOP$_2$ with $f^2 = (f_1, f_3)$
x^3	9	1	1	efficient for MOP and MOP$_3$ with $f^3 = (f_2, f_3)$
x^4	2	2	2	efficient for MOP but not for any subproblem

Proposition 4.1.6 (Epsilon-efficiency of feasible solutions)**.** *Let MOP and DMOP be given, and $\tilde{x} \in X$ be a feasible decision. Then $\tilde{x}$ is ε-efficient for MOP for some $\varepsilon \in \mathbb{R}^m$, $\varepsilon \geqq 0$, and ε^j-efficient for MOP$_j$ for some $\varepsilon^j \in \mathbb{R}^{m_j}$, $\varepsilon^j \geqq 0$.*

Proof. Let $r \in \mathbb{R}^m$ with $r \leqq f(x)$ for all $x \in X$ be a utopia or the ideal point for MOP and set $\varepsilon := f(\tilde{x}) - r \geqq 0$. Then there does not exist $x \in X$ such that $f(x) \leq r = f(\tilde{x}) - \varepsilon$ and, thus, $\tilde{x} \in E(X, f, \varepsilon)$. The proofs of $\tilde{x} \in E(X, f^j, \varepsilon^j)$ follow analogously. $\square$

The ε found in the proof of Proposition 4.1.6 is usually only a very weak upper bound. To improve this bound, we can also solve the Benson scalarization

$$\mathrm{B}(f(\tilde{x})): \text{ Maximize } \sum_{i=1}^{m} \varepsilon_i \text{ subject to } f(\tilde{x}) - \varepsilon \in f(X) \text{ and } \varepsilon \geqq 0 \tag{4.5}$$

for an optimal solution $\hat{\varepsilon}$ to obtain an efficient decision $\hat{x} \in E(X, f)$ with $f(\hat{x}) = f(\tilde{x}) - \varepsilon$. In particular, then $f(\tilde{x}) = f(\hat{x}) + \hat{\varepsilon}$ is an $\hat{\varepsilon}$-translated Pareto outcome and $\tilde{x} \in E(X, f, \hat{\varepsilon})$ but $\tilde{x} \notin E(X, f, \varepsilon)$ for any $\varepsilon \leq \hat{\varepsilon}$ by Proposition 3.3.13.

Corollary 4.1.7 (Efficiency relationships between MOP and DMOP)**.** *Let MOP and DMOP be given. Then there exist $\varepsilon^j \in \mathbb{R}^{m_j}$, $\varepsilon^j \geqq 0$, such that*

$$\bigcup_{j=1,\ldots,M} E_w(X, f^j) \subset E_w(X, f) \subset \bigcap_{j=1,\ldots,M} E(X, f^j, \varepsilon^j) \tag{4.6}$$

Furthermore, if f^j is injective for all $j = 1, \ldots, M$, then the weakly efficient sets can be replaced by efficient sets.

4.1.1 Subproblem Scalarization

Starting from an initial decomposition of MOP into the collection DMOP of subproblems MOP_j, in this section we first use the concept of scalarization functions to combine the multiple objectives of each individual subproblem into a single objective and then consider the collection of all these objectives as a new multiobjective program. We then investigate several relationships between scalarizations of this new problem and scalarizations of the original MOP.

Definition 4.1.8 (MSOP)**.** Let MOP and DMOP be given, and for all $j = 1, \ldots, M$ let $s_j(\pi^j) : \mathbb{R}^{m_j} \to \mathbb{R}$ be a scalarization function for MOP_j with scalarization parameter $\pi^j \in \Pi^j$. The collection of all single objective programs

$$\text{SOP}_j\text{: Minimize } s_j(\pi^j, f^j(x)) \text{ subject to } x \in X \tag{4.7}$$

is called the *multiple SOP* (MSOP) of DMOP.

By dropping the scalarization parameters $\pi^j \in \Pi^j$ and changing from the scalarization functions $s_j : \mathbb{R}^{m_j} \to \mathbb{R}$ to the associated collection of value functions $v_j : X \to \mathbb{R}$ on the decision set X

$$v_j(x) = s_j(\pi^j, f^j(x)) \text{ for all } j = 1, \ldots, M \tag{4.7a}$$

we note that MSOP can equivalently be written as the multiobjective program

$$\text{MSOP: Minimize } v(x) = [v_1(x), \ldots, v_M(x)] \text{ subject to } x \in X \tag{4.7b}$$

In this formulation, now each subproblem is represented by a separate value function v_j which can be chosen either based on analytical properties such as convexity or, in a more applied context, according to the underlying physical interpretation of the objectives contained in MOP_j. The relationships between efficient decisions for MSOP and the original MOP are derived in the following result.

Proposition 4.1.9 (Efficiency relationships between MSOP and MOP)**.** *Let MOP and MSOP be given, and $\hat{x} \in X$ be a feasible decision.*

(i) If $\hat{x}$ is weakly efficient for MSOP, then $\hat{x}$ is weakly efficient for MOP

$$E_w(X,v) \subseteq E_w(X,f) \tag{4.8a}$$

(ii) If $\hat{x}$ is efficient for MSOP, v is strictly increasing (with respect to $f(x)$) and DMOP complete, then $\hat{x}$ is efficient for MOP

$$E(X,v) \subseteq E(X,f) \tag{4.8b}$$

Proof. For (i), let $\hat{x} \in E_w(X,v)$ and, by contradiction, assume that $\hat{x} \notin E_w(X,f)$. Then there exists $x \in X$ such that $f(x) < f(\hat{x})$, or equivalently, $f^j(x) < f^j(\hat{x})$ for all $j = 1,\ldots,M$. In particular, this implies that $v_j(x) < v_j(\hat{x})$ for all $j = 1,\ldots,M$, or equivalently, $v(x) < v(\hat{x})$ in contradiction to $\hat{x} \in E_w(X,v)$.

For (ii), let $\hat{x} \in E(X,v)$ and, by contradiction, assume that $\hat{x} \notin E(X,f)$. Then there exists $x \in X$ such that $f(x) \leq f(\hat{x})$, or equivalently, $f^j(x) \leqq f^j(\hat{x})$ for all $j = 1,\ldots,M$ and $f^k(x) \leq f^k(\hat{x})$ for some $k \in \{1,\ldots,M\}$ (as the objective function component with strict inequality is contained in some f^k because DMOP is complete). In particular, this implies that $v_j(x) \leq v_j(\hat{x})$ for all $j = 1,\ldots,M$ and $v_k(x) < v_k(\hat{x})$ because v is strictly increasing, or equivalently, $v(x) \leq v(\hat{x})$ in contradiction to $\hat{x} \in E(X,v)$. $\square$

Hence, analogous to the result in Proposition 4.1.4, every (weakly) efficient decision for MSOP is also (weakly) efficient for the original MOP. Following the previous notational conventions, we abbreviate a scalarization of MSOP by SSOP and characterize the resulting single objective functions for the weighted-sum and the weighted-Chebyshev norm scalarization in the following two examples.

Example 4.1.10 (Weighted-sum SSOP)**.** Let MOP and DMOP with $f^j = (f_i)_{i \in I_j}$ and $|I_j| = m_j$ for all $j = 1,\ldots,M$ be given. Let $w^j = (w_i^j)_{i \in I_j} \in \mathbb{R}^{m_j}$, $w^j \geq 0$, and $u \in \mathbb{R}^M$, $u \geq 0$, be weighting parameters for weighted sums $s_j(w^j, f^j(x)) : \mathbb{R}^{m_j} \to \mathbb{R}$ for SOP$_j$ and $s(u, v(x)) : \mathbb{R}^M \to \mathbb{R}$ be a weighted-sum for SSOP, then

$$\begin{aligned} s(u,v(x)) &= \sum_{j=1}^{M} u_j v_j(x) = \sum_{j=1}^{M} u_j s_j(w^j, f^j(x)) \\ &= \sum_{j=1}^{M} u_j \sum_{i \in I_j} w_i^j f_i(x) = \sum_{j=1}^{M} \sum_{i \in I_j} u_j w_i^j f_i(x) = \sum_{i=1}^{m} \sum_{j : i \in I_j} u_j w_i^j f_i(x) \end{aligned} \tag{4.9}$$

Example 4.1.11 (Weighted-Chebyshev norm SSOP)**.** Let MOP and DMOP be given as in Example 4.1.10, $r \in \mathbb{R}^m$ be a reference point and $w^j = (w_i^j)_{i \in I_j} \in \mathbb{R}^{m_j}$, $w^j \geq 0$ be the weighting parameters for weighted-Chebyshev norms $s_j(r, w^j, f^j(x)) : \mathbb{R}^{m_j} \to \mathbb{R}$ for SOP$_j$ for all $j = 1, \ldots, M$. Let $0 \in \mathbb{R}^m$ be the reference point and $u \in \mathbb{R}^M$, $u \geq 0$ be the weighting parameter for the weighted-Chebyshev norm $s(0, u, v(x)) : \mathbb{R}^M \to \mathbb{R}$ for SSOP, then

$$\begin{aligned} s(0, u, v(x)) &= \max_{j=1,\ldots,M} \{u_j v_j(x)\} = \max_{j=1,\ldots,M} \{u_j s_j(r, w^j, f^j(x))\} \\ &= \max_{j=1,\ldots,M} \{u_j \max_{i \in I_j} \{w_i^j (f_i(x) - r_i)\}\} = \max_{j=1,\ldots,M} \max_{i \in I_j} \{u_j w_i^j (f_i(x) - r_i)\} \\ &= \max_{i=1,\ldots,m} \max_{j: i \in I_j} \{u_j w_i^j (f_i(x) - r_i)\} = \max_{i=1,\ldots,m} \{\max_{j: i \in I_j} \{u_j w_i^j\} (f_i(x) - r_i)\} \end{aligned} \tag{4.10}$$

The previous two examples reveal the following relationship between SSOP and the corresponding scalarizations of the original MOP.

Proposition 4.1.12 (Equivalence between SOP and SSOP)**.** *Let MOP and DMOP be given, and let DMOP be complete. If SOP, all SOP$_j$ and SSOP are defined using either weighted-sum or weighted-Chebyshev norm scalarization functions, then SSOP is equivalent to SOP.*

Proof. (Weighted sum $\Rightarrow$) Let SSOP be defined using weighted sums, then its objective function takes the form

$$s(u, v(x)) = \sum_{i=1}^{m} \sum_{j: i \in I_j} u_j w_i^j f_i(x) \tag{4.11a}$$

from Example 4.1.10. In particular, if $w_i = \sum_{j: i \in I_j} u_j w_i^j$ for all $i \in I$, then $s(u, v(x)) = \sum_{i=1}^m w_i f_i(x)$ is also a weighted sum for SOP.

(Weighted sum $\Leftarrow$) For the converse, let SOP be defined using a weighted sum with weighting parameter $w \in \mathbb{R}^m$, $w \geq 0$, and let DMOP be complete. Then $|\{j : i \in I_j\}| \neq \emptyset$ for all $i = 1, \ldots, m$, and with $w_i^j = w_i |\{j : i \in I_j\}|^{-1}$ for all $i \in I_j$ and $u = 1$ as the weighting parameters for SOP$_j$ and SSOP, it follows analogous to Example 4.1.10 that

$$s(1, v(x)) = \sum_{j=1}^{M} v_j(x) = \sum_{j=1}^{M} \sum_{i \in I_j} w_i^j f_i(x) = \sum_{i=1}^{m} \sum_{j:i \in I_j} w_i^j f_i(x) = \sum_{i=1}^{m} w_i f_i(x) \quad (4.11b)$$

showing that SSOP is equivalent to SOP.

(Chebyshev norm $\Rightarrow$) If SSOP is defined using weighted-Chebyshev norms, then its objective function takes the form

$$s(0, u, v(x)) = \max_{i=1,\ldots,m} \{\max_{j:i \in I_j} \{u_j w_i^j\} (f_i(x) - r_i)\} \quad (4.12a)$$

from Example 4.1.11. In particular, if $w_i = \max_{j:i \in I_j} \{u_j w_i^j\}$ for all $i \in I$, then $s(0, u, v(x)) = \max_i \{w_i(f_i(x) - r_i)\}$ is also a weighted-Chebyshev norm for SOP.

(Chebyshev norm $\Leftarrow$) For the converse, let SOP be defined using a weighted-Chebyshev norm with weighting parameter $w \in \mathbb{R}^m, w \geq 0$ and reference point $r \in \mathbb{R}^m$, and let DMOP be complete. Let $w_i^j = w_i$ for all $i \in I_j$ and $u = 1$ be the weighting parameters and r and $0 \in \mathbb{R}^M$ be the reference points for SOP_j and SSOP, respectively, then it follows analogous to Example 4.1.11 that

$$\begin{aligned} s(0, 1, v(x)) &= \max_{j=1,\ldots,M} \{v(x)\} = \max_{j=1,\ldots,M} \{s_j(w^j, r, f^j(x))\} \quad (4.12b) \\ &= \max_{j=1,\ldots,M} \{\max_{i \in I_j} \{w_i^j (f_i(x) - r_i)\}\} = \max_{i=1,\ldots,m} \{w_i(f_i(x) - r_i)\} \end{aligned}$$

again showing that SSOP is equivalent to SOP. □

In particular, since every (weakly) efficient decision for MOP can be generated as an optimal solution for a weighted-Chebyshev norm scalarization function (Steuer, 1986, or Proposition 2.2.39), one consequence of the equivalence between SOP and SSOP in Proposition 4.1.12 is that through a suitable scalarization of the subproblems in DMOP, any (weakly) efficient solution for MOP is also (weakly) efficient for MSOP. This observation is summarized in the following corollary.

Corollary 4.1.13 (Equivalence between MOP and MSOP)**.** *Let MOP and DMOP be given, and $\hat{x} \in X$ be a feasible decision. If $\hat{x}$ is weakly efficient for MOP, then there exist a collection of scalarization functions and associated value functions for all MOP_j such that $\hat{x}$ is weakly efficient for MSOP.*

4.1.2 Subproblem Coordination

Other than for the previous subproblem scalarization, we now do not combine the objective functions from different subproblems into a new single objective program but modify each MOP_j by introducing the objectives from other subproblems as additional constraints into an associated coordination problem.

Definition 4.1.14 (Coordination problem)**.** Let MOP and DMOP be given, and $x^j \in X$ be a feasible decision and $\varepsilon^j \in \mathbb{R}^{m_j}$, $\varepsilon^j \geqq 0$ be a nonnegative vector for all $j = 1, \ldots, m$. The multiobjective program

$$\mathrm{COP}_k\text{: Minimize } f^k(x) \tag{4.13}$$

$$\text{subject to } f^j(x) \leqq f^j(x^j) + \varepsilon^j \text{ for all } j = 1, \ldots, M \tag{4.13a}$$

$$\text{and } x \in X \tag{4.13b}$$

is called the kth *coordination problem* (COP_k), and x^j and ε^j are called the *reference point* and *coordination parameter* for COP_k from COP_j, respectively.

For sake of brevity and notational convenience, we also write

$$X(f^j, x^j, \varepsilon^j) := \{x \in X : f^j(x) \leqq f^j(x^j) + \varepsilon^j\} \tag{4.14}$$

for the associated *coordination constraints* and then denote the feasible set for COP_k as $\bigcap_{j=1}^M X(f^j, x^j, \varepsilon^j)$. Similar to Propositions 4.1.4 and 4.1.9, the relationships between efficient decisions for COP_k and MOP can be described as follows.

Proposition 4.1.15 (Efficiency for COP_k implies efficiency for MOP)**.** *Let MOP and DMOP be given, and $\hat{x} \in X$ be a feasible decision.*

(i) If $\hat{x}$ is weakly efficient for COP_k, then $\hat{x}$ is weakly efficient for MOP

$$E_w\left(\bigcap_{j=1}^M X(f^j, x^j, \varepsilon^j), f^k\right) \subseteq E_w(X, f) \tag{4.15a}$$

(ii) If $\hat{x}$ is efficient for COP_k and if f^k is injective, then $\hat{x}$ is efficient for MOP

$$E\left(\bigcap_{j=1}^M X(f^j, x^j, \varepsilon^j), f^k\right) \subseteq E(X, f) \tag{4.15b}$$

(iii) If $\hat{x}$ is efficient for MOP and if DMOP is complete, then $\hat{x}$ is efficient for all COP$_k$ with $x^j = \hat{x}$ and $\varepsilon^j = 0$ for all $j = 1, \ldots, M$

$$\hat{x} \in E\left(\bigcap_{j=1}^{M} X(f^j, \hat{x}, 0), f^k\right) \text{ for all } k = 1, \ldots, M \tag{4.15c}$$

In any case, if $\hat{x}$ is feasible for COP$_k$ and if x^j is (weakly) ε-efficient for MOP$_j$, then $\hat{x}$ is (weakly) $(\varepsilon^j + \varepsilon)$-efficient for MOP$_j$.

Proof. For (i), let $\hat{x} \in E_w(\bigcap_{j=1}^{M} X(f^j, x^j, \varepsilon^j), f^k)$ and, by contradiction, assume that $\hat{x} \notin E_w(X, f)$. Then there exists $x \in X$ such that $f(x) < f(\hat{x})$, or equivalently, $f_i(x) < f_i(\hat{x})$ for all $i \in I$. In particular, this implies that $f_i(x) < f_i(\hat{x})$ for all $i \in I_j \subset I$, or equivalently, $f^j(x) < f^j(\hat{x}) \leqq f^j(x^j) + \varepsilon^j$ for all $j = 1, \ldots, M$, showing that also $x \in \bigcap_{j=1}^{M} X(f^j, x^j, \varepsilon^j)$. In particular, $f^k(x) < f^k(\hat{x})$, in contradiction to $\hat{x} \in E_w(\bigcap_{j=1}^{M} X(f^j, x^j, \varepsilon^j), f^k)$.

For (ii), let $\hat{x} \in E(\bigcap_{j=1}^{M} X(f^j, x^j, \varepsilon^j), f^k)$ and, by contradiction, assume that $\hat{x} \notin E(X, f)$. Then there exists $x \in X$ such that $f(x) \leq f(\hat{x})$, and, analogous to the above, it follows that $x \in \bigcap_{j=1}^{M} X(f^j, x^j, \varepsilon^j)$. Moreover, then $f(x) \neq f(\hat{x})$, so $x \neq \hat{x}$ and $f^k(x) \neq f^k(\hat{x})$ as f^k is injective. In particular, this implies that $f^k(x) \leq f^k(\hat{x})$ in contradiction to $\hat{x} \in E(\bigcap_{j=1}^{M} X(f^j, x^j, \varepsilon^j), f^k)$.

For (iii), let $\hat{x} \in E(X, f)$, and assume that $\hat{x} \notin E(\bigcap_{j=1}^{M} X(f^j, \hat{x}, 0), f^k)$ by contradiction. Then there exists $x \in \bigcap_{j=1}^{M} X(f^j, \hat{x}, 0)$ such that $f^k(x) \leq f^k(\hat{x})$, or equivalently, $f^j(x) \leqq f^j(\hat{x})$ for all $j = 1, \ldots, M$ and $f^k(x) \leq f^k(\hat{x})$ (as the objective function component with strict inequality is contained in some f^k because DMOP is complete). In particular, this implies that $f_i^j(x) \leq f_i^j(\hat{x})$ for all $i \in I_j \subset I$ and $f_l^k(x) < f_l^k(\hat{x})$ for some $l \in I_k \subset I$, so $f_i(x) \leq f_i(\hat{x})$ for all $i \in I$ and $f_l(x) < f_l(\hat{x})$ for some $l \in I$, or equivalently, $f(x) \leq f(\hat{x})$ in contradiction to $\hat{x} \notin E(X, f)$.

Finally, let $x^j \in E_w(X, f^j, \varepsilon)$ and, of course by contradiction, assume that $\hat{x} \notin E_w(X, f^j, \varepsilon^j + \varepsilon)$. Then there exists $x \in X$ such that $f^j(x) < f^j(\hat{x}) - \varepsilon^j - \varepsilon \leqq f^j(x^j) - \varepsilon$, in contradiction to $x^j \in E_w(X, f^j, \varepsilon)$. If $x^j \in E(X, f^j, \varepsilon)$, the proof of $\hat{x} \in E(X, f^j, \varepsilon^j + \varepsilon)$ follows analogously. □

Hence, we conclude that through the modification of the initial subproblems MOP_j by coordination constraints and a suitable choice of the coordination parameters ε^j, any efficient solution for MOP can be identified by solving a corresponding coordination problem COP_k. In general, however, the set of efficient decisions for COP_k depends on the particular choices of ε^j, and the following problem can be used to examine some of these underlying dependencies.

Definition 4.1.16 (Tradeoff problem)**.** Let MOP, DMOP, and COP_k with reference points $x^j \in X$ and coordination parameters $\varepsilon^j \in \mathbb{R}^{m_j}$ for all $j = 1, \ldots, m$ be given. The single objective program

$$\text{TOP}_{kl}\text{: Minimize } f_l^k(x) \tag{4.16}$$

$$\text{subject to } f^j(x) \leqq f^j(x^j) + \varepsilon^j \text{ for all } j = 1, \ldots, M \tag{4.16a}$$

$$\text{and } x \in X \tag{4.16b}$$

is called the lth *tradeoff problem* (TOP_{kl}) for COP_k involving objective f_l^k.

The following result establishes the relationship between efficient decisions for COP_k and optimal solutions for TOP_{kl}.

Proposition 4.1.17 (Efficiency for COP_k implies optimality for TOP_{kl})**.** *Let MOP and DMOP be given, and let $\hat{x} \in X$.*

(i) If $\hat{x}$ is efficient for COP_k, then $\hat{x}$ is optimal for TOP_{kl} with $x^k = \hat{x}$ and $\varepsilon^k = 0$.

(ii) If $\hat{x}$ is efficient for MOP and if DMOP is complete, then $\hat{x}$ is optimal for all TOP_{kl} with $x^j = \hat{x}$ and $\varepsilon^j = 0$ for all $j = 1, \ldots, M$.

Proof. For (i), let $\hat{x} \in E(\bigcap_{j=1}^{M} X(f^j, x^j, \varepsilon^j), f^k)$ and, by contradiction, assume that $\hat{x}$ is not optimal for TOP_{kl} with $x^k = \hat{x}$ and $\varepsilon^k = 0$. Then there exists $x \in \bigcap_{j=1}^{M} X(f^j, x^j, \varepsilon^j)$ with $x^k = \hat{x}$ and $\varepsilon^k = 0$ such that $f_l^k(x) < f_l^k(\hat{x})$, or equivalently, $f^j(x) \leqq f^j(x^j) + \varepsilon^j$ for all $j = 1, \ldots, M$, so $x \in \bigcap_{j=1}^{M} X(f^j, x^j, \varepsilon^j)$ and, for $j = k$, $f^k(x) \leq f^k(\hat{x})$, in contradiction to $\hat{x} \in E(\bigcap_{j=1}^{M} X(f^j, x^j, \varepsilon^j))$.

For (ii), let DMOP be complete, so $\bigcup_{j=1}^{M} I_j = I$, and $\hat{x} \in E(X, f)$. By Proposition 4.1.15, it follows that $\hat{x} \in E(\bigcap_{j=1}^{M} X(f^j, \hat{x}, 0), f^k)$ for all $k = 1, \ldots, m$, and the proof follows immediately from (i). □

Remark 4.1.18 (Tradeoff rates associated with coordination and tradeoff problems)**.** Let $\hat{x} \in X$ be efficient for COP_k. By Proposition 4.1.17, it then follows that $\hat{x}$ is also optimal for TOP_{kl}, and using Definition 2.2.15 the tradeoff between $\hat{x}$ and x^j involving f_l^k and f_i^j is given by

$$T_{ji}^{kl}(\hat{x}, x^j) := \frac{f_l^k(\hat{x}) - f_l^k(x^j)}{f_i^j(x^j) - f_i^j(\hat{x})} \tag{4.17a}$$

Furthermore, if the conditions in Proposition 2.2.29 (constrained-objective tradeoff rates) are satisfied for TOP_{kl}, then the tradeoff rate between f_l^k and f_i^j at $\hat{x}$ is

$$T_{ji}^{kl}(\hat{x}) := -\frac{\partial f_l^k(x)}{\partial f_i^j(x)}\bigg|_{x=\hat{x}} = -\frac{\partial f_l^k(x)}{\partial \varepsilon_i^j}\bigg|_{x=\hat{x}} = \lambda_{ji}^{kl} \tag{4.17b}$$

where λ_{ij}^{kl} denotes the Lagrangean multiplier of the coordination constraint $f_i^j(x) \leq f_i^j(\hat{x}) + \varepsilon_i^j$ in TOP_{kl}. Hence, for an infinitesimal small increase δ of ε_i^j and thus in $f^j(\hat{x})$, the resulting decrease in $f_l^k(\hat{x})$ equals $\delta\lambda_{ij}^{kl}$.

The above results are used in the following decision making procedures.

4.2 Interactive Procedures and Decision-Making

Based on the previous theoretical results, we now present three different procedures that can be used by a decision maker to identify a preferred solution for a multiobjective programming problem using an initial objective decomposition as proposed in Section 4.1. In each case, the decision making steps involved are first summarized in a generic procedure or flowchart and then explained in some more detail in the subsequent discussion.

4.2.1 Optimization-Based Nonhierarchical Coordination

We initially assume that the decision problem and the underlying multiobjective program is a classical optimization problem with functional descriptions for all objectives and constraints. In particular, in this case we can use any suitable optimization technique from linear and nonlinear programming to solve the corresponding subproblems or coordination problems.

Procedure 4.2.1 (Nonhierarchical coordination)**.** Let MOP and DMOP be given.

1. Select an initial $\hat{x} \in X$, and let $x^j = \hat{x}$ and $\varepsilon^j = 0$ for all $j = 1, \ldots, M$.

2a. If $\hat{x}$ is a preferred solution for MOP, stop.

2b. Otherwise select MOP_k for improvement of $\hat{x}$ with respect to f^k.

3. Solve COP_k for a new solution $\tilde{x}$, and update $x^k = \tilde{x}$ and $\varepsilon^k = 0$.

4a. If $f^k(\tilde{x})$ is satisfying, adjust $\varepsilon^k \geqq 0$, set $\hat{x} = \tilde{x}$ and go back to Step 2.

4b. Otherwise compute the tradeoffs $T^{k\cdot}_{j\cdot}$ at $\tilde{x}$ from TOP_{kl}.

5. Update some or all ε^j and go back to Step 3.

The validity of Procedure 4.2.1 is implied by Proposition 4.1.15 which guarantees that all efficient decisions for COP_k that we select in Step 3 are at least weakly efficient for the original MOP. Moreover, as every efficient decision for MOP can in principle be found as an efficient decision to some coordination problem, based on the decision maker's input the procedure is capable to also find every potentially preferred solution for the overall problem. We now give some more details and provide some helpful discussion for each step of the procedure and its application to solving a multiobjective program

$$\text{MOP: Minimize } f(x) = [f_1(x), f_2(x), \ldots, f_m(x)] \text{ subject to } x \in X \tag{4.18a}$$

The procedure starts with MOP and assumes that the decision maker is able to define a suitable decomposition by selecting subsets of the initial objective function components $(f_1, f_2, \ldots, f_m)$ which are then used to formulate the collection of associated subproblems

$$\text{MOP}_j\text{: Minimize } f^j(x) = [f^j_1(x), f^j_2(x), \ldots, f^j_{m_j}(x)] \text{ subject to } x \in X \tag{4.18b}$$

that form the decomposed MOP (DMOP). Clearly, in practice this step depends in large part on the specific problem that the decision maker wants to solve and may also influence the choice of intermediate and thus the final preferred solution. As general guideline, we suggest to decompose MOP into subproblems that have a sufficiently small number of objectives and, ideally, into biobjective problems to enable the immediate visualization of the Pareto sets in the form of two-dimensional Pareto curves, providing various perceptual benefits for the further selection of efficient decisions and reference points for the individual coordination problems. For now, however, we postpone the further discussion of these and other related issues until we show some actual applications of this procedure in the next chapter.

Henceforth assuming that the decomposition DMOP and all MOP_j are defined, in Step 1 of Procedure 4.2.1 we select a first feasible decision $\hat{x}$, which is also used as the first reference point for all subproblems, and initialize the coordination parameters $\varepsilon^j = 0$ for all $j = 1, \ldots, m$. To examine if $\hat{x}$ is already preferred for MOP, in Step 2 we evaluate $f(\hat{x})$ for the complete MOP and, provided this outcome is already satisfying, terminate the procedure with $\hat{x}$ as the preferred solution. Otherwise, if the current outcome is not satisfying, then there exists at least one other subproblem MOP_k, or equivalently, some objective function f^k which we would like to further improve in the subsequent Step 3.

Since the different objectives in different subproblems are usually conflicting, in general we expect that improvement with respect to f^k is only possible at a corresponding decay in some other objective. Initially, however, since the first feasible decision $\hat{x}$ is chosen arbitrarily and not necessarily efficient, we might also improve upon $f^k(\hat{x})$ without further decay in any other $f^j(\hat{x})$, and therefore, we formulate the corresponding kth coordination problem

$$\text{COP}_k\text{: Minimize } f^k(x) = [f_1^k(x), \ldots, f_{m_k}^k(x)] \tag{4.19}$$

$$\text{subject to } f_i^j(x) \leq f_i^j(x^j) + \varepsilon_i^j \text{ for all } i \in I_j,\ j = 1, \ldots, M \tag{4.19a}$$

$$x \in X \tag{4.19b}$$

with initial reference points $x^j = \hat{x}$ and coordination parameters $\varepsilon^j = 0$ for all $j = 1, \ldots, M$. We then solve this problem using any applicable optimization technique, for example a scalarization method, and select a new solution $\tilde{x}$ that now, from Proposition 4.1.15, is also at least weakly efficient for the original MOP.

Next, we need to evaluate the corresponding outcome $f^k(\tilde{x})$ of the new decision $\tilde{x}$ in MOP$_k$ in comparison to the previous outcome $f^k(\hat{x})$. If we are satisfied with the improvement that we achieved over $f^k(\hat{x})$, then we can immediately accept $\tilde{x}$ as new solution to replace $\hat{x}$ and start with the next iteration of the procedure at Step 2. In other instances, we might also be overly satisfied with the improvement achieved by $f^k(\tilde{x})$, in which case we would still use $\tilde{x}$ to replace the previous solution $\hat{x}$ but, additionally, increase the corresponding coordination parameter ε^k so to allow some later decay in f^k to possibly enable further improvement in some other subproblem during subsequent iterations of the procedure. Otherwise, if we are not satisfied with the achieved improvement in f^k, then we need to accept a further decay in at least one other objective f^j in another subproblem by increasing its corresponding coordination parameter ε^j.

To help the decision maker on this tradeoff and to decide which ε^j to modify, we propose to compute the associated sensitivities from the corresponding single objective tradeoff problems

$$\text{TOP}_{kl}\text{: Minimize } f_l^k(x) \tag{4.20}$$

$$\text{subject to } f_i^j(x) \leq f_i^j(x^j) + \varepsilon_i^j \quad \text{for all } i \in I_j, j \neq k \tag{4.20a}$$

$$f_i^k(x) \leq f_i^k(\tilde{x}) \qquad \text{for all } i \in I_k \tag{4.20b}$$

$$x \in X \tag{4.20c}$$

From Proposition 4.1.17, we know that since $\tilde{x}$ is efficient for COP$_k$, it is also efficient for the corresponding TOP$_{kl}$. In particular, if the conditions in Proposition 2.2.29 (constrained-objective tradeoff rates) hold at $\tilde{x}$, we can compute the Lagrangean multipliers λ_{ji}^{kl} associated with each constraint $f_i^j(x) \leq f_i^j(x^j) + \varepsilon_i^j$ in TOP$_{kl}$. Following from Remark 4.1.18, a positive $\lambda_{ji}^{kl} > 0$ then indicates that the corresponding coordination constraint is active at the current decision $\tilde{x}$ so that an increase of the corresponding coordination parameter ε_i^j may yield further improvement in f_l^k. Hence, after a corresponding update of some or all of these ε_i^j in Step 5, we can again solve the coordination problem COP_k with new coordination parameters in Step 3 to find a new solution $\tilde{x}$ and repeat Step 4 as described above.

Note that the computation of the Lagrangean multipliers can be accomplished rather conveniently based on the fact that the efficient decision $\tilde{x}$ for COP_k is already known to be optimal for the tradeoff problem TOP_{kl}. Since almost all common optimization software provides a final solution together with its associated Lagrangean multipliers, we therefore suggest to use any suitable optimization routine and specify the optimal solution $\tilde{x}$ as the initial point for TOP_{kl}, which then should almost immediately confirm optimality of $\tilde{x}$ and return all λ_{ji}^{kl} in one or at least a comparably small number of total iterations.

We finally remark that convergence of this procedure, in general, cannot be established due to its dependency on input by the decision maker. Nevertheless, based on the underlying theoretical results every decision proposed by the procedure is at least (weakly) efficient for MOP, and in principle, every efficient solution can be reached. If a preferred decision, however, cannot be found in a desired number of iterations, then the decision maker may allow more iterations, choose a different initial solution, vary the decomposition and, thus, DMOP, or eventually modify the initial aspirations and expectations.

4.2.2 Optimization-Based Hierarchical Coordination

One potential drawback of the previous nonhierarchical coordination procedure is that for problems with many objectives or decompositions into many subproblems, the number of additional constraints in the coordination problems becomes quite large and renders the simultaneous update of several coordination parameters ε^j difficult. Furthermore, if there are several subproblems for which the current decision is not satisfactory, it might not always be easy to decide which subproblem to choose for further improvement. To facilitate these two difficulties, we propose an alternative procedure that, instead of working with all subproblems simultaneously, uses a hierarchical ranking of all subproblems. As illustrated in Figure 4.2, this new procedure initially considers only MOP_1 as first coordination problem COP_1 and then sequentially modifies the subsequent MOP_j into new COP_j by introducing only the objectives from the previous iterations as additional constraints. In particular, note that the coordination problems in this case are slightly different from the ones in the previous approach and limited to those constraints of earlier subproblems, so that we do not need to select all ε^j and reference points x^j immediately, but can specify all our choices in a sequential manner.

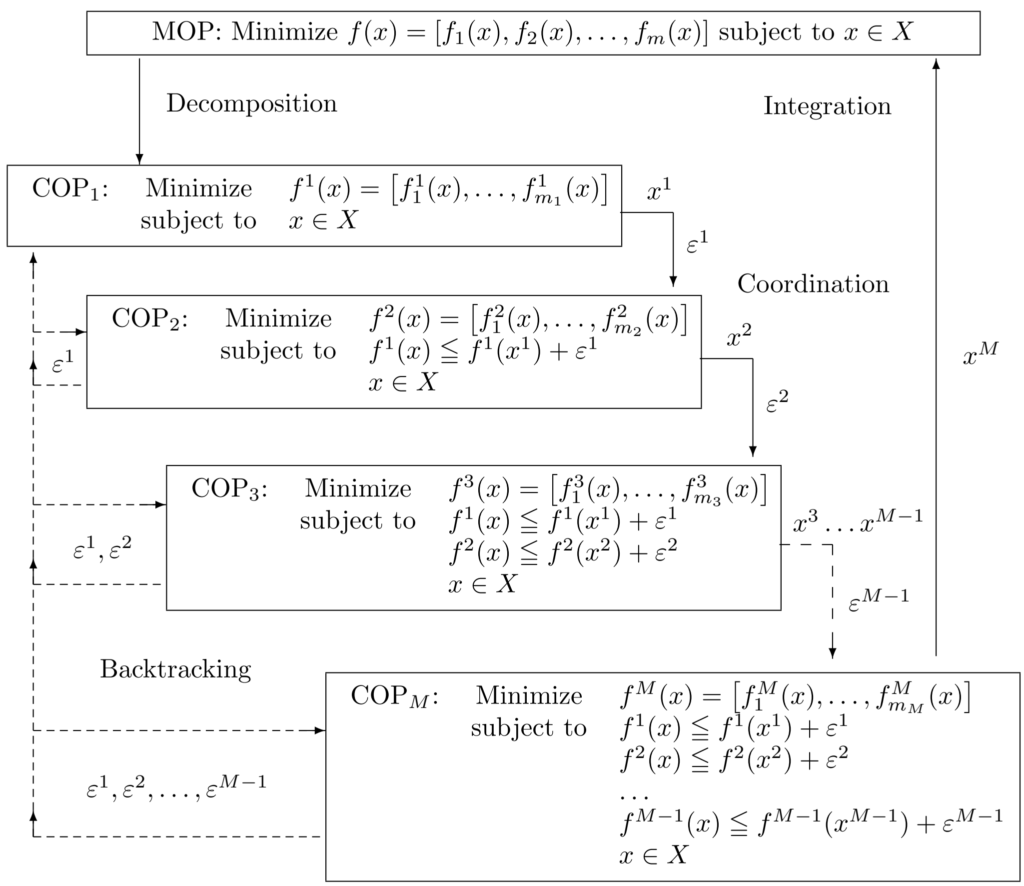


Figure 4.2 Optimization-based hierarchical decomposition and coordination

Clearly, other than for the nonhierarchical procedure, in this case the decision maker must decide not only on the particular decomposition of the original MOP into subproblems, but also on the subproblem hierarchy. While this step is again problem specific and therefore addressed in some more detail in context of the later applications, the similar problem also arises for general lexicographic orders for either single (Fishburn, 1974; Rentmeesters et al., 1996) or multiobjective programs (Ying, 1983). Some other research, however, exists to provide the decision maker with further guidance to establish a ranking among criteria (Matsumoto et al., 1993; Dym et al., 2002) or related works that focus on the decomposition of multiobjective programs based on criteria influence (Yoshimura et al., 2002, 2003).

Once the decomposition and the subproblem hierarchy have been defined, the procedure works as follows. In each step, the decision maker solves a coordination problem COP_k and selects a corresponding decision x^k that is used as the reference point for a corresponding coordination constraint introduced into all subsequent coordination problems. Additionally, the decision maker may also specify an additional coordination parameter ε^k that describes the maximal acceptable deviation for subsequent solutions from the current outcome $f^k(x^k)$. Of course, the decision maker might also initially set all $\varepsilon^j = 0$ and later change any of these values using a backtracking mechanism as indicated in Figure 4.2.

For the update of these ε^j and similar to the previous procedure, the decision maker can again utilize the sensitivity information obtained from the corresponding tradeoff problems. For further illustration, we assume that the decision maker has solved COP_1 through COP_{k-1} by selecting decisions $x^1, \ldots, x^{k-1}$ and, mainly to simplify the following arguments and without loss of generality, has initially decided to set $\varepsilon^j = 0$ for all $j = 1, \ldots, k-1$. In particular, this implies that all coordination constraints $f^j(x) \leqq f^j(x^j) + \varepsilon^j$ in COP_k reduce to $f^j(x) \leqq f^j(x^j)$ and, hence, that every feasible decision x^k for COP_k is also efficient for the previous COP_j. While the decision maker should now select a new decision x^k and proceed to the next problem, it may also happen that the decision maker is not satisfied with any of the available outcomes $f^k(\tilde{x})$ for COP_k and thus is willing to change some or all of the previous ε^j to gain further improvement in one of the objectives of f^k.

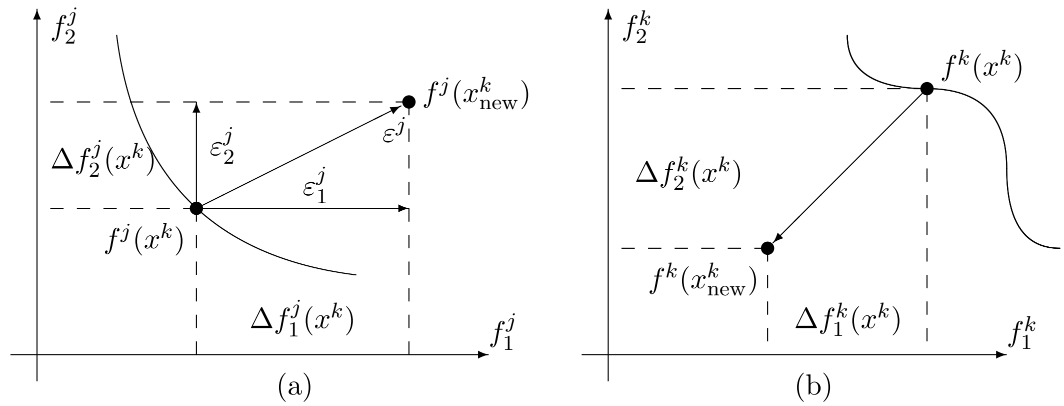


Figure 4.3 On the left (a), Pareto curve for COP_j with $f^j = (f_1^j, f_2^j)$, and on the right (b), its image for COP_k with $f^k = (f_1^k, f_2^k)$ [not the Pareto curve for COP_k]

This situation is depicted in Figure 4.3 in which we assume for further simplicity that both COP_j and COP_k are biobjective programs with objectives $f^j = (f_1^j, f_2^j)$ and $f^k = (f_1^k, f_2^k)$, respectively. In particular, since each x^k is also efficient for COP_j, $f^j(x^k)$ lies on the Pareto curve for COP_j, depicted in Figure 4.3(a), and on the corresponding image of the efficient set for COP_j in COP_k, shown in Figure 4.3 (b). Hence, to improve the current objective $f^k(x^k)$, we need to increase the coordination parameter $\varepsilon^j = (\varepsilon_1^j, \varepsilon_2^j)$ to also allow for ε^j-efficient decisions for COP_j, for example the decision x^k_{new} as indicated in Figure 4.3. In this case, the corresponding tradeoffs between x^k and x^k_{new} involving f^k and f^j are given by

$$T_{j1}^{k1}(x^k, x^k_{new}) = \frac{\Delta f_1^k(x^k)}{\Delta f_1^j(x^k)} = \frac{f_1^k(x^k) - f_1^k(x^k_{\mathrm{new}})}{f_1^j(x^k_{\mathrm{new}}) - f_1^j(x^k)} = \frac{f_1^k(x^k) - f_1^k(x^k_{\mathrm{new}})}{\varepsilon_1} \tag{4.21a}$$

$$T_{j1}^{k2}(x^k, x^k_{new}) = \frac{\Delta f_2^k(x^k)}{\Delta f_1^j(x^k)} = \frac{f_2^k(x^k) - f_2^k(x^k_{\mathrm{new}})}{f_1^j(x^k_{\mathrm{new}}) - f_1^j(x^k)} = \frac{f_2^k(x^k) - f_2^k(x^k_{\mathrm{new}})}{\varepsilon_1} \tag{4.21b}$$

$$T_{j2}^{k1}(x^k, x^k_{new}) = \frac{\Delta f_1^k(x^k)}{\Delta f_2^j(x^k)} = \frac{f_1^k(x^k) - f_1^k(x^k_{\mathrm{new}})}{f_2^j(x^k_{\mathrm{new}}) - f_1^2(x^k)} = \frac{f_1^k(x^k) - f_1^k(x^k_{\mathrm{new}})}{\varepsilon_2} \tag{4.21c}$$

$$T_{j2}^{k2}(x^k, x^k_{new}) = \frac{\Delta f_2^k(x^k)}{\Delta f_2^j(x^k)} = \frac{f_2^k(x^k) - f_2^k(x^k_{\mathrm{new}})}{f_2^j(x^k_{\mathrm{new}}) - f_2^j(x^k)} = \frac{f_2^k(x^k) - f_2^k(x^k_{\mathrm{new}})}{\varepsilon_2} \tag{4.21d}$$

and as before, to decide which of the ε^j to modify, we can use the sensitivities from the corresponding tradeoff problem

$$\text{TOP}_{kl}\text{: Minimize } f_l^k(x) \tag{4.22}$$

$$\text{subject to } f_i^j(x) \le f_i^j(x^j) + \varepsilon_i^j \text{ for all } i \in I_j,\ j = 1, \ldots, k-1 \tag{4.22a}$$

$$f_l^k(x) \le f_l^k(x^k) \qquad \text{for all } l \in I_k \tag{4.22b}$$

$$x \in X \tag{4.22c}$$

In this case, however, and according to the defined subproblem hierarchy, we only introduce those coordination constraints that are associated with the subproblems that are solved in previous iterations. Again assuming that we can compute the Lagrangean multipliers for each of these constraints, we obtain that

$$T_{ji}^{kl}(x^k) = -\left.\frac{\partial f_l^k(x)}{\partial f_j^j(x)}\right|_{x=x^k} = \lambda_{ji}^{kl} \text{ for all } l \in I_k \text{ and } i \in I_j,\ j = 1, \ldots, k-1 \tag{4.23}$$

and at least for very small changes in ε_i^j, we may use these tradeoff rates to approximate the actual tradeoffs similar to the ratios listed in (4.21) as

$$\frac{\Delta f_l^k(x^k)}{\Delta f_i^j(x^k)} = \frac{\Delta f_l^k(x^k)}{\varepsilon_i^j} \approx -\lambda_{ji}^{kl} \tag{4.23a}$$

$$\Longleftrightarrow \quad \Delta f_l^k(x^k) \approx -\lambda_{ji}^{kl}\varepsilon_i^j \text{ for all } l \in I_k \text{ and } i \in I_j,\ j = 1, \ldots, k-1 \tag{4.23b}$$

As mentioned earlier, however, in general these estimates are only valid for very small and, in fact, infinitesimal small changes in ε_i^j. Merely as a rule of thumb, we state that the larger the magnitude of a computed Lagrangean multiplier, the larger the tradeoff that we should expect. For example, if $\lambda_{ji}^{kl} \gg 1$, then we infer that already a small additional increase in ε_i^j may yield a significant improvement of $f_l^k(x^k)$. On the other hand, if $\lambda_{ji}^{kl} < 1$, then the improvement in $f_l^k(x^k)$ is relatively small compared to the increase ε_i^j. For practical purposes, we therefore suggest to initially choose a tradeoff threshold of 1 at which we do not wish to further increase any ε_i^j, then suggesting to terminate the procedure with the current decision as final and preferred solution for the overall problem. In particular, in this case we do not need to solve all subproblems, while Proposition 4.1.15 still guarantees that the final decision is also efficient for the original MOP, although possibly in favor of the objectives contained in those subproblems $\mathrm{COP}_1, \ldots, \mathrm{COP}_k$ so far participating in the coordination process, compared to $\mathrm{COP}_{k+1}, \ldots, \mathrm{COP}_M$ that are omitted due to early termination. Finally, and similar to the previous procedure, if a preferred decision cannot be revealed in a desired number of iterations, or after solving the last subproblem COP_M, the decision maker again may decide to allow more iterations, use backtracking to change some of the reference points x^j or coordination parameters ε^j, modify the proposed tradeoff threshold, or eventually restart the procedure with a newly chosen decomposition or subproblem hierarchy.

4.2.3 Optimization-Free Coordination

Due to the high complexity in many real-life optimization models or the limited number of admissible objective function evaluations because of cost-intensive computational requirements, traditional optimization methods often are not applicable and must be replaced by heuristic approaches such as a genetic algorithm or simulation and sampling techniques. In these cases, a (possibly still quite large) set of feasible decisions is generated and presented to the decision maker, who then needs to make a final decision from among these candidates based on (possibly quite many) objectives that are used for their evaluation. The procedure in this section assumes that once this set of preliminary candidate solutions has been found, no further optimization is possible so that the decision problem essentially reduces to a mere selection problem. However, the selection of a preferred decision may still be a very difficult task for large numbers of decisions and objectives and can be facilitated by a similar decomposition and coordination mechanism as proposed for the previous two procedures.

The alternative approach in this case is based on the observation in Proposition 4.1.6 that every efficient decision for MOP is ε^j-efficient in every subproblem MOP_j for some ε^j, giving rise to the relationship

$$\bigcup_{j=1}^{M} E(X, f^j) \subseteq E(X, f) \subseteq \bigcap_{j=1}^{M} E(X, f^j, \varepsilon^j) \qquad (4.24)$$

in the corresponding Corollary 4.1.7. Hence, the idea of the following procedure is to iteratively modify the choices of ε^j until we obtain a nonempty intersection of the sets of ε^j-efficient decisions $E(X, f^j, \varepsilon^j)$ for all MOP$_j$ with only a reasonably small number of potentially preferred decisions for which the final selection becomes a more manageable task. In this case, the proper coordination of this process should guarantee that we also compromise between the different ε^j to eventually reach preferred compromise solutions for the different subproblems in DMOP.

Procedure 4.2.2 (Optimization-free coordination)**.** Let MOP and DMOP be given.

1. Select initial $\varepsilon^j \in \mathbb{R}^{m_j}$, $\varepsilon^j \geqq 0$, for all $j = 1, \ldots, M$.
2. Find $E(X, f^j, \varepsilon^j)$ for MOP$_j$ for all $j = 1, \ldots, M$.

3a. If $\bigcap_{j=1}^{M} E(X, f^j, \varepsilon^j) \neq \emptyset$, select a solution $\hat{x}$ from the intersection.

3b. Otherwise change some or all ε^j and go back to Step 2.

4a. If $\hat{x}$ is a preferred solution, stop.

4b. Otherwise change some or all ε^j and go back to Step 2.

In principle, this procedure can be used in two different ways. As a first possibility, the decision maker may choose to start with comparably large values for all ε^j so that, initially, there still exist many solutions in the common intersection of the ε^j-efficient sets for MOP$_j$. In this case, however, it may still be difficult to identify a preferred decision so that the decision maker may sequentially reduce one or more of the coordination parameters ε^j until only a reasonable small number of decisions remain in the new intersection. Under the assumption that the changes in ε^j agree with the preferences and expectations of the decision maker, a final preferred solution then can be chosen from among the set of decisions remaining in the common intersection of the ε^j-efficient decisions for MOP$_j$.

On the other hand, the decision maker may also prefer to start with very small values for ε^j, or initially set $\varepsilon^j = 0$ for all $j = 1, \ldots, M$. In this case, however, the intersection of the ε^j-efficient sets in Step 3 is usually empty, so that the decision maker needs to sequentially increase one or more of the ε^j until finding one or a small number of decisions that belong to the common intersection and thus establish reasonable compromise decisions, again provided that the choices of ε^j reflect the specific preferences of the decision maker.

Without any underlying optimization, however, it is clear that the mere intersection of the ε-efficient sets in Step 3 of this procedure in general does not provide the means to guarantee that the final selected solution is actually efficient for MOP. However, and as mentioned before, the advantage of this coordination procedure over the two previous approaches is that it is also applicable to a discrete solution set of a multiobjective program as found by, for example, a sampling technique or a genetic algorithm, for which a further optimization is computationally too expensive or for other reasons not possible.

4.3 Mathematical Programming Example

Before we apply each of the three coordination procedures introduced in the previous section to a real-life application in the next chapter, we first demonstrate the proposed hierarchical decision making procedure and the underlying coordination mechanism on an example from mathematical programming. Thereby adopting the role of a hypothetical decision maker, by the nature of this approach it is unavoidable that all our decisions remain subjective and, in practice, would also depend on the actual decision maker's expertise, preferences and expectations.

The chosen problem consists of four quadratic objective functions f_1, f_2, f_3 and f_4 which need to be minimized, subject to three inequality constraints g_1, g_2 and g_3 in two variables x_1 and x_2

$$\begin{array}{llcccccccl}
\text{Minimize} & f_1(x_1,x_2) & = & (x_1-2)^2 & + & (x_2-1)^2 & & & & \\
& f_2(x_1,x_2) & = & x_1^2 & + & (x_2-3)^2 & & & & \\
& f_3(x_1,x_2) & = & (x_1-1)^2 & + & (x_2+1)^2 & & & & \\
& f_4(x_1,x_2) & = & (x_1+1)^2 & + & (x_2-1)^2 & & & & \\
\text{subject to} & g_1(x_1,x_2) & = & x_1^2 & - & x_2 & & & & \leq 0 \\
& g_2(x_1,x_2) & = & x_1 & + & x_2 & - & 2 & & \leq 0 \\
& g_3(x_1,x_2) & = & -x_1 & & & & & & \leq 0
\end{array}$$

We denote the feasible set for this problem by $X = \{x \in \mathbb{R}^2 : g_i(x) \leq 0,\ i = 1, 2, 3\}$ and, since we consider all decisions and objectives without any underlying physical interpretation, priorities or relative importances, we group the four objectives into the two canonical pairs $f^1 = (f_1^1, f_2^1) = (f_1, f_2)$ and $f^2 = (f_1^2, f_2^2) = (f_3, f_4)$. Additionally, to reduce any further notational burden, we also replace all double indices $_{11}$, $_{12}$, $_{21}$ and $_{22}$ by $_1$, $_2$, $_3$ and $_4$, respectively.

To find a preferred decision to the above problem using the proposed hierarchical procedure, we now start by solving

$$\text{COP}_1\text{: Minimize } f^1 = [f_1(x), f_2(x)] \text{ subject to } x \in X \tag{4.25}$$

for ten Pareto solutions that are depicted on the left in Figure 4.4 and then select the highlighted middle point as first solution and reference point $x^1 = (0.447, 1.553)$. In a practical context, this choice can be justified by the fact that this decision yields comparable outcome levels for $f_1(x^1) = 2.717$ and $f_2(x^1) = 2.294$ and, thus, is one of the best compromise decisions for COP_1. Furthermore, for later reference, we also calculate the two other objective values $f_3(x^1) = 6.823$ and $f_4(x^1) = 2.340$.

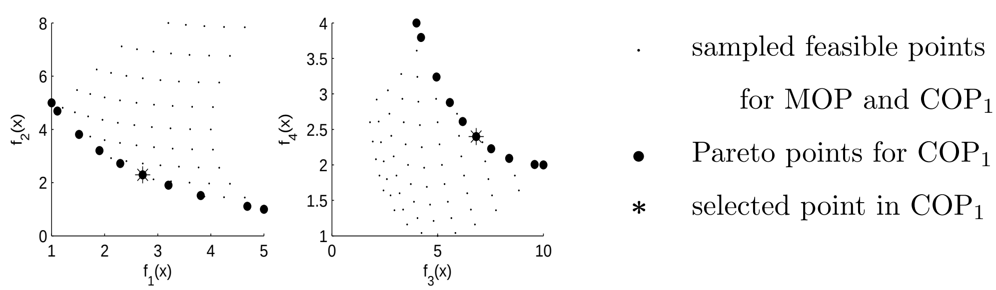


Figure 4.4 On the left, ten Pareto outcomes for COP_1 with $f^1 = (f_1, f_2)$, and on the right, their corresponding images for $f^2 = (f_3, f_4)$

However, we remark that, so far, our decision is merely based on the first subproblem with respect to objectives f_1 and f_2. The plot on the right of Figure 4.4 shows the Pareto outcomes for COP_1 with respect to the two other objectives f_3

and f_4 and is provided here for convenience only. In addition, both plots also depict a sample of the complete feasible set which show how all Pareto outcomes that are found as solutions for the first subproblem belong to the worst outcomes with respect to the second subproblem. In this context, also compare Figure 4.4 with the previous illustration in Figure 4.3.

Before we proceed to the next coordination problem COP_2, we investigate the tradeoffs at the current decision x^1 with respect to the two objectives f_3 and f_4 by computing the Lagrangean multipliers for the corresponding tradeoff problems

$$\begin{aligned} \text{TOP}_3\text{:}\ \text{Minimize}\quad & f_3(x) \\ \text{subject to}\quad & f_1(x) \le f_1(x^1) + \varepsilon_1 \\ & f_2(x) \le f_2(x^1) + \varepsilon_2 \\ & f_4(x) \le f_4(x^1) \\ & x \in X \end{aligned} \qquad \begin{aligned} \text{TOP}_4\text{:}\ \text{Minimize}\quad & f_4(x) \\ \text{subject to}\quad & f_1(x) \le f_1(x^1) + \varepsilon_1 \\ & f_2(x) \le f_2(x^1) + \varepsilon_2 \\ & f_4(x) \le f_4(x^1) \\ & x \in X \end{aligned}$$

where, initially, $\varepsilon_1 = \varepsilon_2 = 0$. Then choosing the optimal decision x^1 as initial point in our optimization routine, this immediately (after one iteration) confirms optimality of x^1 for both TOP_3 and TOP_4 and provides us with the tradeoff information

$$T_{31}(x^1) = -\left.\frac{\partial f_3(x)}{\partial f_1(x)}\right|_{x=x^1} = \lambda_{31} = 0.171 \qquad T_{41}(x^1) = -\left.\frac{\partial f_4(x)}{\partial f_1(x)}\right|_{x=x^1} = \lambda_{41} = 1.171$$

$$T_{32}(x^1) = -\left.\frac{\partial f_3(x)}{\partial f_2(x)}\right|_{x=x^1} = \lambda_{32} = 1.829 \qquad T_{42}(x^1) = -\left.\frac{\partial f_4(x)}{\partial f_2(x)}\right|_{x=x^1} = \lambda_{42} = 0.829$$

Hence, we see that only two tradeoff values are greater than 1, thus suggesting that improvement in f_3 or f_4 is best achieved by increasing the coordination parameter ε_2 for f_2 or ε_1 for f_1, respectively, before solving the second coordination problem

$$\text{COP}_2\text{:}\ \text{Minimize}\ f^2 = [f_3(x), f_4(x)] \tag{4.26}$$

$$\text{subject to}\ f_1(x) \le f_1(x^1) + \varepsilon_1 = 2.717 + \varepsilon_1 \tag{4.26a}$$

$$f_2(x) \le f_2(x^1) + \varepsilon_2 = 2.294 + \varepsilon_2 \tag{4.26b}$$

$$x \in X \tag{4.26c}$$

In particular, if we decide to focus on the more promising tradeoff between f_3 and f_2, we might choose a new coordination parameter $\varepsilon_2 = 1$, and after solving COP_2

with $\varepsilon = (\varepsilon_1, \varepsilon_2) = (0,\ 1)$, we select the new decision $x_{\text{new}} = (0.440,\ 1.239)$ with its corresponding outcome $f(x_{\text{new}}) = (2.490,\ 3.294,\ 5.328,\ 2.131)$ as new solution. Hence, in this case, the actually achieved tradeoff between x^1 and x_{new} involving f_3 and f_2 can now be computed according to the ratios in Equation 4.21 as

$$T_{32}(x^1, x_{\text{new}}) = \frac{\Delta f_3(x^1)}{\Delta f_2(x^1)} = \frac{f_3(x^1) - f_3(x_{\text{new}})}{f_2(x_{\text{new}}) - f_2(x^1)} = \frac{6.823 - 5.328}{3.294 - 2.294} = 1.495 \qquad (4.27)$$

Here we note that although the chosen value $\varepsilon_2 = 1$ is rather large compared to the previous outcome value of $f^2(x^1) = 2.294$, the computed tradeoff $\lambda_{32} = 1.829$ at x^1 provides a quite reasonable estimate for the actual tradeoff of 1.495. However and as emphasized before, these tradeoff values should not be mistaken as precise predictions for the tradeoffs that are usually achieved, especially when we decide to simultaneously change both coordination parameters ε_1 and ε_2.

For illustration of this last remark, assume that we now decide to set $\varepsilon = (\varepsilon_1, \varepsilon_2) = (1,\ 2)$. Based on the initial tradeoff values computed at x^1, we might expect to gain improvements of $\Delta f_3(x^1) \approx -\lambda_{32}\varepsilon_2 = 1.829 \cdot 2 = 3.658$ in f_3 and $\Delta f_4(x^1) \approx -\lambda_{41}\varepsilon_1 = 1.171 \cdot 1 = 1.171$ in f_4. Then solving COP$_2$ for, say, five new efficient decisions, we obtain the five corresponding Pareto outcomes depicted on the right of Figure 4.5 which, together with their new tradeoff values, are listed in Table 4.2. For convenience, we also circle all those sampled outcomes from Figure 4.4 that satisfy the coordination constraints $f_1(x) \leq f_1(x^1) + \varepsilon_1 = 3.717$ and $f_2(x) \leq f_2(x^1) + \varepsilon_2 = 4.294$ and, thus, form the underlying set of feasible points for COP$_2$.

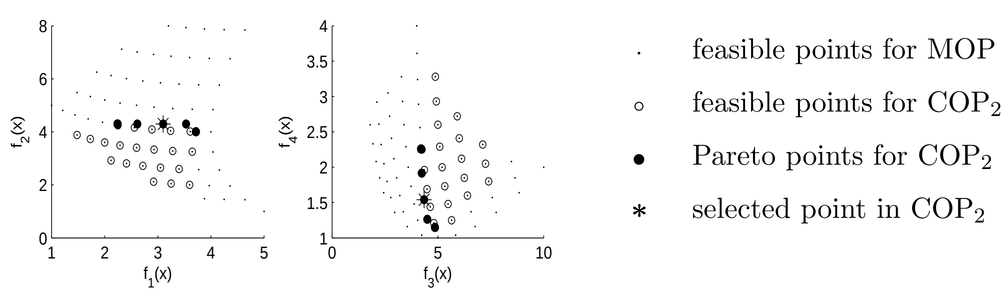


Figure 4.5 On the right, five Pareto outcomes for COP$_2$ with $f^2 = (f_2, f_3)$ and $\varepsilon^1 = (\varepsilon_1, \varepsilon_2) = (1, 2)$, and on the left, their corresponding images for $f^1 = (f_1, f_2)$

Table 4.2 Efficient decisions for COP_2 and their objective and tradeoff values

#	x_1	x_2	f_1	f_2	f_3	f_4	λ_{31}	λ_{32}	λ_{41}	λ_{42}
1	0.503	0.990	2.242	4.294	4.206	2.258	0	0.990	1.003	0
2	0.384	0.964	2.613	4.294	4.236	1.916	0	0.990	0.856	0
3	0.240	0.942	3.101	4.294	4.348	1.541	0	0.990	0.704	0
4	0.122	0.931	3.532	4.294	4.501	1.264	0	0.990	0.596	0
5	0.072	1.000	3.717	4.005	4.861	1.149	0	1.060	0.556	0

We find that the maximal improvements with respect to f^3 and f^4

$$\Delta_{\max} f^3(x^1) = f^3(x^1) - f^3(0.503, 0.990) = 6.823 - 4.206 = 2.617 \text{ (expected: 3.658)}$$

$$\Delta_{\max} f^4(x^1) = f^4(x^1) - f^4(0.072, 1.000) = 2.400 - 1.149 = 1.251 \text{ (expected: 1.171)}$$

are achieved by two different of the five efficient decisions for COP_2 and thus not achievable simultaneously so that, in this case, the choice of a preferred solution must also compromise between these different possible improvements. Again selecting the best compromise decision among these five solutions, we choose the third decision in Table 4.2 as our second solution $x^2 = (0.240, 0.942)$, which is also highlighted in Figure 4.5 with its new objective function values $f(x^2) = (3.101, 4.294, 4.348, 1.541)$. In particular, we note that all tradeoff values at this decision reduce to values below the suggested tradeoff threshold of 1, and consequently, we terminate the procedure with this decision as our final solution.

To conclude this example, we also observe that while our final and all other highlighted outcomes in Figure 4.5 are Pareto for the second subproblem COP_2, so that all underlying decisions are (at least weakly) efficient for the original MOP, none of these outcomes actually lies on the Pareto curve for either the first or second subproblem. Hence, and as mentioned earlier, the decision making procedures proposed in this chapter are capable to find decisions that truly compromise between the different subproblems and that cannot be found using other decomposition approaches that do not properly coordinate the chosen decomposition.

4.4 Discussion and Further Research

In close interplay of theoretical investigation and methodological development, in this chapter we study a decomposition strategy for large-scale multiobjective programs and establish various results that subsequently lead to the formulation of three interactive coordination and decision making procedures. Based on our explicit goal to remedy the challenges resulting from a high number of objectives, we choose to decompose the original objective vector into several subsets of objective components that can be evaluated independently, thus giving rise to a collection of separate and smaller-sized subproblems. In particular, the objective spaces in every subproblem are of lower dimension than for the original problem, and therefore, we expect that solving these subproblems in the sense of choosing preferred decisions becomes a more manageable task for the decision maker.

In the characterization of the efficiency relationships between the solutions for the original and the decomposed subproblems, we show in Proposition 4.1.4 that all (weakly) efficient decisions for subproblems are also (weakly) efficient for the original problem, for which the stronger result holds true under the additional assumption that the decomposed objective function is injective. For the converse relationship, however, we find in Example 4.1.5 that efficient decisions for the original problem are not necessarily efficient for any subproblem, but merely epsilon-efficient for a suitable choice of the associated relaxation parameter.

We contrast our decomposition approach with traditional scalarization techniques and verify in a series of results preceding Corollary 4.1.13 that any efficient decision that can be generated by scalarizing the complete problem can also be found by separate scalarizations of the decomposed subproblems. In particular, as one additional advantage of the proposed objective decomposition, we mention the possibility to group objectives with similar analytical properties or a common physical interpretation into the same subproblem and, for each of these, use different scalarization functions or parameters to achieve additional flexibility compared to scalarization approaches that combine all objectives at once.

To identify efficient solutions by other means than scalarization, we propose an alternative approach that coordinates between different subproblems by adding their objectives as additional constraints into a modified coordination problem. In this case, Proposition 4.1.15 essentially maintains the result of Proposition 4.1.4 that all (weakly) efficient decisions for the coordination problems are also (weakly) efficient for the original problem but, in addition, now every efficient decision for the original problem also solves a suitable coordination problem. In particular, although otherwise similar to the constrained-objective scalarization in Definition 2.2.26, each coordination problem is still a multiobjective program so that not only the specification of the associated parameters but also the selection of its corresponding solution can be influenced by preferences and input of the decision maker.

In further modification of the above coordination problems, we introduce a collection of single objective programs that can be used to assess the tradeoffs and sensitivities associated with each objective function component and coordination constraint at any selected efficient decision. In particular, based on our result in Proposition 4.1.17 that guarantees optimality of efficient decisions also for these tradeoff problems, we can forgo their explicit solution and address in Remark 4.1.18 and the following discussion how we can compute all relevant sensitivities in a straightforward manner from the associated set of Lagrangean multipliers.

Following the above results, we formulate three interactive decision making procedures that are proposed to support decision makers in organizing the coordination between subproblems and to facilitate the required specification of coordination parameters and the selection of preferred decisions for the individual subproblems. The first Procedure 4.2.1 and the second in Figure 4.2 differ in whether this coordination is accomplished in a nonhierarchical or hierarchical fashion and are based on optimization, whereas the unique feature of Procedure 4.2.2 is its formulation as a pure selection method. Especially in view of our earlier discussion in Section 3.4, we again emphasize the prominent role played by the concept of epsilon-efficiency to reflect compromises and tradeoffs based on the preferences of the decision maker but essentially independent of its previous consideration as concept of approximation.

Choosing the optimization-based hierarchical procedure to solve a mathematical programming example with four objectives, we illustrate how we can conveniently select an overall preferred decision by working only with one pair of biobjective problems and effectively use visualizations of the Pareto curves for each individual problem which is impossible in the original four-dimensional outcome space. Moreover, in our discussion we focus in large part on the proposed sensitivity analysis and find that although the provided tradeoff information, in general, does not provide reliable estimates for the actually achieved improvement, it can be a helpful indication for the suggested termination of the overall decision making procedure. In addition, we expect that the proposed decomposition-coordination scheme is also well suited for decentralized decision processes and provides a general framework that enables the independent participation of multiple decision makers.

Nevertheless, much further work is possible and necessary to achieve or further advance the practicality of the proposed decision making procedures for real-life applications and again comes along with several interesting research ideas also of interest from a theoretical point of view. First, the current limitations of the proposed tradeoff and sensitivity analysis urge to take a closer look at its underlying assumptions and to derive conditions under which we can more precisely predict the resulting changes in the corresponding objective function values. Second, while the current tradeoff analysis utilizes a sensitivity result from single objective programming, namely the sensitivity theorem from nonlinear programming (Luenberger, 1984) that is based on Lagrangean duality, the question remains if there also exists a multiobjective analogon and motivates a further investigation of general vector duality and sensitivity. Third, by replacing Pareto by general cone efficiency, it is not necessarily clear how to properly decompose the overall domination cone for the individual subproblems so to obtain similar results to those presented for the Pareto case, and we propose such inquiry as another interesting research problem.

From among the numerous methodological issues that need to be addressed, we only highlight the apparent dependency of the decision making process and, thus, of the final preferred decision on the chosen decomposition. While several of our results, namely Propositions 4.1.9, 4.1.12, 4.1.15 and 4.1.17, require a complete decomposition that includes all original objectives in at least one subproblem, at the moment we do not impose any other restriction on either the number of objectives per subproblem nor the number of subproblems themselves. Hence, although we generally recommend to decompose the original problem into sufficiently small and preferably biobjective programs to enable the visualization of the individual Pareto curves, the examination of other consequences may lead to new insights and enhance the subsequent decision making already in the preparatory decomposition step.

In most practical cases, however, we believe that a suitable decomposition is usually problem-dependent and best chosen based on the specific objectives that are modeled in the underlying multiobjective program. In particular, emerging interests in the formulation of multiscenario multiobjective programs as given in Definition 4.1.3 provide a host of potential applications that imply a natural problem decomposition into different scenarios (Wiecek et al., 2006). For example, Kouvelis and Yu (1997) examine single objective programs for which the different scenarios are defined by different data instances, and they are interested in finding solutions that are robust, or comparably preferred in all associated subproblems. In structural optimization and engineering design, different scenarios are frequently defined as different loading conditions, and for multidisciplinary optimization a different subproblem can be associated with each individual discipline (Sobieszczanski-Sobieski and Haftka, 1997). Other possible applications include product platform design (Fellini et al., 2002, 2005) for which the scenarios represent different products that are produced from a common set of shared components resulting in potential sacrifices in individual product performance, and typical scenarios in vehicle configuration design include the same vehicle operating in various steering maneuvers under different road or driving conditions (Gobbi and Mastinu, 2002; Fadel et al., 2005).

Without going into any further detail, we summarize that the number of possible applications of the proposed decomposition and coordination methods is essentially unlimited, and we choose only four particular problems for the further illustration of our decision making procedures in the now following chapter.

CHAPTER 5

APPLICATIONS IN FINANCE AND ENGINEERING DESIGN

In this chapter, we apply the decomposition approach proposed in the previous Chapter 4 to selected real-life applications from finance and engineering design and show how each of the three interactive decision making procedures can be used to coordinate between the resulting subproblems by setting tradeoffs and resolving conflicts for the identification of a final and overall preferred decision. In particular, we extend the classical Roy-Markowitz model for portfolio optimization in Section 5.1 to allow for multiple estimates of investment returns and risks, thereby giving rise to a multiscenario multiobjective program, and we similarly solve a truss design problem with multiple loading conditions in Section 5.2 using the two optimization-based procedures. Their modification for a selection problem is described in Section 5.3 for a vehicle configuration problem in the context of packaging optimization, and we finally use the originally proposed optimization-free procedure to select a vehicle system component in the context of highway safety in the concluding Section 5.4.

5.1 Investment Selection in Portfolio Optimization

A common problem in portfolio optimization asks an investor to distribute a financial capital among n assets to maximize the portfolios's return over a fixed holding period. For the following discussion, we let $x_a \in \mathbb{R}$, $0 \leq x_a \leq 1$, denote the investment proportion into asset $a = 1, \dots n$, and $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ be the investment vector. Since asset returns, in general, are not known at the beginning of the holding period, we let r_a be a real-valued random variable for the return of asset a, and $r(x) = \sum_{a=1}^{n} r_a x_a$ be the portfolio return of the investment vector or investment strategy x. With this notation, the resulting *portfolio optimization problem* (POP) can be formulated as the stochastic program

$$\text{POP:}\ \ \text{Maximize } r(x) = \sum_{a=1}^{n} r_a x_a \text{ subject to } \sum_{a=1}^{n} x_a = 1,\ x \geqq 0 \qquad (5.1)$$

To solve POP, Roy (1952) and Markowitz (1952) independently introduce an equivalent deterministic biobjective program by replacing the random return maximization $r(x)$ with maximization of the expected return $E[r(x)]$ and minimization of its variance $V[r(x)]$. Following their approach, we let $\mu_a = E[r_a]$ be the expected return and $\sigma_{aa}^2 = V[r_a]$ be the variance of asset $a = 1, \ldots, n$, and for $a \neq b = 1, \ldots, n$, we let $\sigma_{ab}^2 = Cov[r_a(x), r_b(x)]$ be the covariance between r_a and r_b. Modern portfolio theory assumes that μ_a, σ_{aa}^2, and σ_{ab}^2 are known for all $a, b = 1, \ldots, n$, although, in reality, expected returns, variances and covariances are unknown and can only be estimated from historical data (Elton et al., 2003). Hence, in general, the selected and presumingly optimal portfolio is usually biased depending on the particularly chosen estimator, a fact commonly ignored in the discussion of possible investments.

To remove this drawback and to find an investment strategy that performs well for a variety of different estimators, in this example, we study the Roy-Markowitz mean-variance model and allow the simultaneous consideration of three different estimates $j = 1, 2, 3$ for return expectations μ_a^j, variances σ_{aa}^{2j} and covariances σ_{ab}^{2j}. More precisely, given historical data over D days, we let r_{ad} denote the return of asset a on day $d = 1, \ldots, D$ and then estimate the average return μ_a as

$$\mu_a^j = E^j(r_a) = \frac{\sum_{d=1}^{D} w_d^j r_{ad}}{\sum_{d=1}^{D} w_d^j} \tag{5.2a}$$

where the parameters w_d^j are weights that may vary for different days d to put different emphases on, say older or more recent data. In particular, with $w_d^1 = 1$ for all $d = 1, \ldots, D$, we use the sample mean $\bar{r}_a$ as estimate μ_a^1 and, furthermore, choose $w_d^2 = d$ and $w_d^3 = d^2$ to obtain first and second order linearly weighted averages as μ_a^2 and μ_a^3, respectively, thereby assigning higher weights to data that is more recent. In all three cases, the variances and covariances are computed accordingly as

$$\sigma_{ab}^{2j} = E^j[(r_{ad} - E^j(r_a))(r_{bd} - E^j(r_b))] = \frac{\sum_{d=1}^{D} w_d^j (r_{ad} - \mu_a^j)(r_{bd} - \mu_b^j)}{\sum_{d=1}^{D} w_d^j} \tag{5.2b}$$

Finally replacing positive return maximization by the equivalent negative return minimization, the resulting collection of biobjective programs

$$\text{BOP}_j\text{:}\quad \text{Minimize } (-E^j[r(x)], V^j[r(x)]) \text{ subject to } \sum_{a=1}^{n} x_a = 1, x \geqq 0 \tag{5.3}$$

with $E^j[r(x)] = \sum_{a=1}^n \mu_a^j x_a$ and $V^j[r(x)] = \sum_{a=1}^n \sum_{b=1}^n \sigma_{ab}^{2j} x_a x_b$ is a multiscenario multiobjective program in the same form as DMOP and, thus, can be solved using any of the two optimization-based procedures proposed in the previous chapter. In particular, since the expectation terms are linear and the variances are quadratic in the investment vector x, all objective functions are convex and twice continuously differentiable which, in principle, allows to compute the Lagrangean multipliers from the tradeoff problems associated with each of the three subproblems BOP_j.

Based on recent one-year data of twenty assets included in the S&P500 stock market index, we first use the two formulas in (5.2a) and (5.2b) to compute expected returns μ_a^j, variances σ_{aa}^{2j} and covariances σ_{ab}^{2j} for all assets $a, b = 1, \ldots, 20$ and $j = 1, 2, 3$, respectively. Furthermore, we normalize all data to magnitudes less than 1 by dividing each return vector and covariance matrix by the maximum absolute value of its entries, before starting with Step 1 of Procedure 4.2.1.

Step 1: We select an initial strategy $\hat{x}$ of equal investment proportions $\hat{x}_a = 1/20$ for every asset $a = 1, \ldots, 20$ and initialize $x^j = \hat{x}$ and $\varepsilon^j = (\varepsilon_E^j, \varepsilon_V^j) = (0, 0)$ for $j = 1, 2, 3$. The associated expected returns and variances correspond to point A in Figure 5.1, that also shows the individual Pareto sets obtained from separately solving each of the three subproblems BOP_j. Moreover, the numerical values of all returns and variances are listed in Table 5.1, and to simplify the later discussion we also denote the current portfolio $\hat{x}$ as x^A.

Step 2b: The initial investment x^A naturally shows a considerable diversification and, therefore, already provides very small variances of at most 0.13, strongly favoring the minimization of the portfolio's risk over its return maximization. Consequently, this initial solution is not preferred, and we can select any and, specifically, choose the first subproblem for its further improvement.

Step 3: We formulate and solve the problem

$$\text{COP}_1\text{: minimize } (-E^1[r(x)], V^1[r(x)]) \tag{5.4}$$

$$\text{subject to } -E^j[r(x)] \leq -E^j[r(x^j)] + \varepsilon_E^j \tag{5.4a}$$

$$V^j[r(x)] \leq V^j[r(x^j)] + \varepsilon_V^j \quad \text{for } j = 1, 2, 3 \tag{5.4b}$$

where $x^j = x^A$ and $\varepsilon_E^j = \varepsilon_V^j = 0$ for all j from Step 1. Using the weighted-sum scalarization approach, we identify the Pareto set for COP_1, depicted in Figure 5.1, and then select the solution corresponding to point B as our new portfolio $\hat{x} = x^B$. In particular, this solution yields the most improvement with respect to expected returns, while meeting or even further reducing the variances and, thus, the risk associated with the initial investment strategy x^A, which can be inferred by comparing the numerical values in Table 5.1. Moreover, the new point B now is Pareto for the first subproblem, although it is not Pareto for the second or third subproblem. Since still $\varepsilon^1 = 0$, we only update $x^1 = x^B$ before proceeding to Step 4.

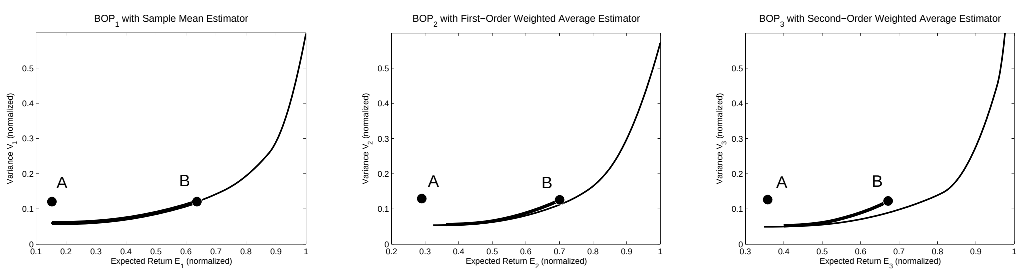


Figure 5.1 Individual Pareto sets for each BOP_j with bold curves corresponding to the Pareto set for COP_1 (left) in Step 3 and its image for BOP_2 and BOP_3

Step 4b: Figure 5.1 indicates that the variance constraints $V^j[r(x)] \leq V^j[r(x^A)] + \varepsilon_V^j$ with $\varepsilon_V^j = 0$ in COP_1 prevent a further improvement in the expected return unless we decide to further relax the values for ε_V^j. Since the V^1 constraint in COP_1 is active at the current solution x^B, we can formulate the associated tradeoff problem TOP_{11} and minimize the negative expected return $-E^1[r(x)]$ under the same constraints as in COP_1 to compute that $T_{12}^{11} = \lambda_{12}^{11} = 3.35$, where λ_{12}^{11} is the Lagrangean multiplier associated with the constraint $V^1[r(x)] \leq V^1[r(x^1)]$ at the optimal solution x^B of TOP_{11}. Although we note from Table 5.1 that the two constraints for V^2 and V^3 are not active, the slack is sufficiently small to similarly approximate $T_{22}^{11} = \lambda_{22}^{11} = 3.15$ and $T_{32}^{11} = \lambda_{32}^{11} = 3.27$ from neighboring points of x^B. An interpretation of these values implies that for very small increases in ε_V^j or, equivalently, V^j, we expect a threefold improvement in the expected return E^1.

Step 5: To restrict the acceptable variances to values of, say less than 0.3, we specify all $\varepsilon_V^j = 0.18$ before resolving COP_1 in Step 3. While this should guarantee the significant further improvement in E^1, however, this relaxation is clearly too large to result in an accurate threefold improvement obtained as current tradeoff in Step 4b, valid only for much smaller relaxations of ε_V^j.

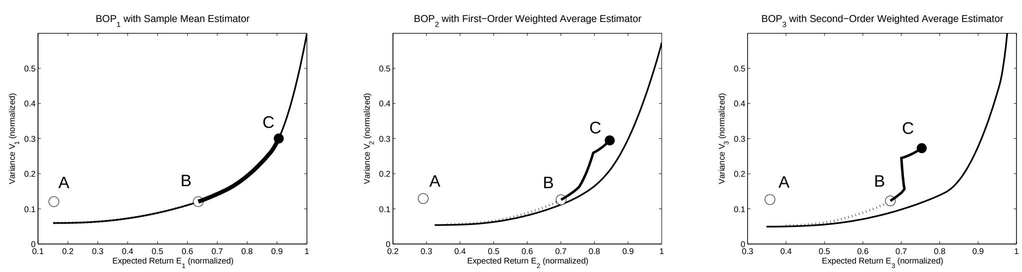


Figure 5.2 Bold curves from point B to point C correspond to the Pareto set for COP_1 (left) in Step 3(2) and its image for BOP_2 (center) and BOP_3 (right)

Step 3(2): Figure 5.2 shows the new Pareto set obtained by solving COP_1 with $x^1 = x^B$ from the previous Step 3, initial strategies $x^2 = x^3 = x^A$, $\varepsilon_E^j = 0$ and $\varepsilon_V^j = 0.18$ for $j = 1, 2, 3$. In particular, while all solutions are still Pareto for the first subproblem, now they deviate significantly from the individual Pareto curves for the second and third subproblem. Hence, after selecting the point C as new solution $\hat{x} = x^C$, we update $x^1 = x^C$, reset $\varepsilon_V^1 = 0$, and again proceed to Step 4.

Step 4a: Overly satisfied with the achieved return $E^1 = 0.9$ of the current portfolio x^C, which is listed with the other values in Table 5.1, we adjust $\varepsilon_E^1 = 0.1$ to merely maintain an expected return $E_1 \geq 0.8$ for the first subproblem, but possibly gain a better solution in view of the other two subproblems.

Step 2b(2) While the current portfolio x^C is still reasonable for the second subproblem but rather unacceptable for the third subproblem, simply judged based on the respective distances from point C to the Pareto curves in Figure 5.2, we now focus on further improvement on the expected return E^3.

Step 3(3): Consequently, we formulate and solve the problem

$$\text{COP}_3\text{: minimize } (-E^3[r(x)], V^3[r(x)]) \tag{5.5}$$

$$\text{subject to } -E^j[r(x)] \leq -E^j[r(x^j)] + \varepsilon_E^j \tag{5.5a}$$

$$V^j[r(x)] \leq V^j[r(x^j)] + \varepsilon_V^j \quad \text{for } j = 1, 2, 3 \tag{5.5b}$$

where $x^1 = x^C$ is the current solution, $x^2 = x^3 = x^A$ is still the initial point, $\varepsilon_E^1 = 0.1$, $\varepsilon_V^1 = 0$, $\varepsilon_E^j = 0$ and $\varepsilon_V^j = 0.18$ for $j = 2, 3$. The resulting Pareto set for COP_3 is depicted in Figure 5.3, and the values for the three highlighted points D, E, and F are listed in Table 5.1. Note that all solutions for COP_3 relax E^1 by the maximum specified amount of $\varepsilon_E^1 = 0.1$ and achieve improvement not only in the optimized third subproblem, but also in the second subproblem. In particular, after selecting the solution x^E that corresponds to the middle point E in Figure 5.3 as the best compromise solution $\hat{x}$ for COP_3, we set $x^3 = x^E$, $\varepsilon^3 = 0$, and do not desire further improvement of either E^2 or V^2.

Step 4a and 2a: Satisfied with the achieved objective function values in the third and, moreover, all other subproblems, we do not need to further adjust ε^3 and now go to Step 2a to terminate the procedure with the current $\hat{x} = x^E$ as the preferred final solution and investment strategy.

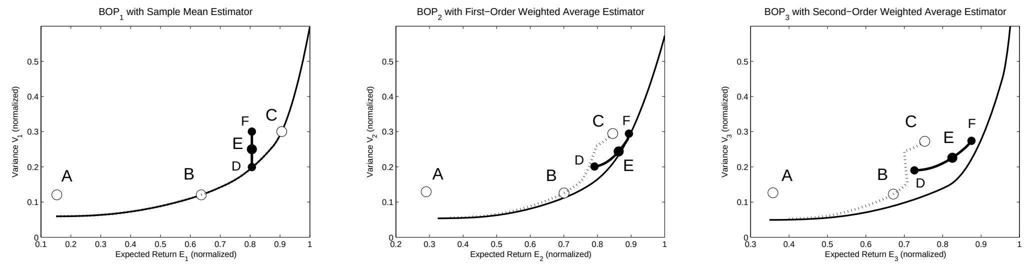


Figure 5.3 Bold curves from point D to point F correspond to the Pareto set for COP_3 (right) in Step 3(3) and its image for BOP_1 (left) and BOP_2 (center)

Table 5.1 Return and variance estimates for portfolios in Figures 5.1, 5.2, and 5.3

	BOP_1		BOP_2		BOP_3	
	$E^1[r(\hat{x})]$	$V^1[r(\hat{x})]$	$E^2[r(\hat{x})]$	$V^2[r(\hat{x})]$	$E^3[r(\hat{x})]$	$V^3[r(\hat{x})]$
A	0.152906	0.120385	0.290292	0.129330	0.357932	0.126309
B	0.636321	0.120385	0.700503	0.125709	0.671444	0.122688
C	0.905929	0.300385	0.845338	0.294723	0.753173	0.272746
D	0.805929	0.199371	0.791192	0.201070	0.726274	0.190173
E	0.805929	0.250229	0.863149	0.243992	0.825127	0.225964
F	0.805929	0.300385	0.893574	0.294483	0.875052	0.274021

This example can be modified in several ways. Instead of changing the weights for the chosen return and variance estimates, different scenarios can also be defined based on data of varying histories, for example, 12-month or 3-year performances (Ehrgott et al., 2004). In addition, various new objectives such as dividends, social responsibility, or liquidity can be included into the portfolio selection process (Spronk et al., 2005; Steuer et al., 2005), thereby necessitating to decompose the resulting problem without a physical interpretation of different scenarios. The application of the procedure, however, will essentially remain the same.

5.2 Truss Topology Design in Structural Optimization

Structural design optimization deals with the development and engineering of complex systems and structures such as cars, airplanes, spaceships or satellites. Based on tremendous gain in experience and knowledge, together with the rapid progress in available computing technologies, the underlying mathematical models and design simulations become better and better and provide designers with growing amounts of data that need to be analyzed for choosing a final optimal design. In particular, the steadily increasing number of design specifications, measures and criteria used to evaluate the performance of the simulated designs lead to cumbersome and sometimes unachievable tradeoff analyses, thus resulting in complex and very difficult, if not unsolvable, decision making problems.

Consequently, structural design is also a common field for application of multiobjective optimization. By nature, in a majority of such problems one criterion is typically the structural volume or weight which needs to be minimized, while loading conditions or other structural performance specifications may give rise to the additional criteria. In particular, in our discussion we investigate the design of the four-bar plane truss structure already described in Section 3.3.6 and various other places in the literature (Koski, 1984, 1985; Koski and Silvennoinen, 1987; Koski, 1988; Stadler and Dauer, 1992; Coello and Christiansen, 2000; Coello Coello, 2001; Coello Coello and Lamont, 2004) for which the original mathematical model is a biobjective program with the two conflicting objectives of minimizing both the volume V of the truss and the displacement d_1 of the node joining bars 1 and 2 under the loading condition depicted for the leftmost truss in Figure 5.4.

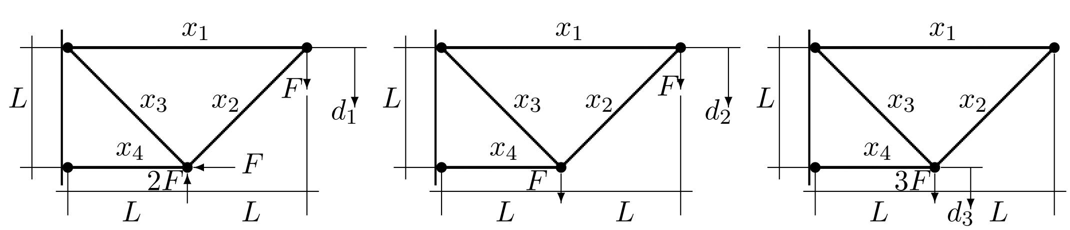


Figure 5.4 Three different loading conditions on a four-bar plane truss structure

In our particular application, however, we also use the two additional loading conditions from Koski (1984) that are depicted for the middle and right truss in Figure 5.4 and analyzed with respect to their associated displacements d_2 and d_3 by Blouin (2004). The length $L = 200$ cm of the structure, the acting force $F = 10$ kN, Young's modulus of elasticity $E = 2 \times 10^5$ kN/cm^2 and the only nonzero stress component $\sigma = 10$ kN/cm^2 are again assumed to be constant, and the cross-sectional areas x_1, x_2, x_3 and x_4 of the four bars again are subject to several additional physical restraints that yield the feasible design or decision set

$$X = \left\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \tfrac{F}{\sigma} \leq x_1, x_4 \leq 3\tfrac{F}{\sigma}, \sqrt{2}\tfrac{F}{\sigma} \leq x_2, x_3 \leq 3\tfrac{F}{\sigma}\right\} \tag{5.6}$$

The overall problem then becomes to select a feasible preferred design that minimizes the structural volume $V(x)$ and joint displacements d_1, d_2, d_3 for the three different loading conditions in Figure 5.4

$$\text{Minimize } V(x) = L(2x_1 + \sqrt{2}x_2 + \sqrt{2}x_3 + x_4) \tag{5.7a}$$

$$d_1(x) = \frac{FL}{E}\left(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} - \frac{2\sqrt{2}}{x_3} + \frac{2}{x_4}\right) \tag{5.7b}$$

$$d_2(x) = \frac{FL}{E}\left(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} + \frac{4\sqrt{2}}{x_3} + \frac{6}{x_4}\right) \tag{5.7c}$$

$$d_3(x) = \frac{FL}{E}\left(\frac{6\sqrt{2}}{x_3} + \frac{3}{x_4}\right) \text{ subject to } x \in X \tag{5.7d}$$

This problem can also be viewed as a multiscenario multiobjective program (Fadel et al., 2005) as introduced in Definition 4.1.3, with each loading condition acting as one of three possible scenarios. Based on the assumption that the first loading scenario occurs most often and the third scenario only very rarely, we decompose the overall problem into the three associated criteria pairs $f^1 = (f_1^1, f_2^1) = (d_1, V)$, $f^2 = (f_1^2, f_2^2) = (d_2, V)$ and $f^3 = (f_1^3, f_2^3) = (d_3, V)$. In particular, we note that the volume criterion participates in every scenario and, thus, is repeated in each subproblem. Moreover, by including additional scenarios this example readily extends to problems with five or any other number of objectives as well, and the restriction to three subproblems is merely for the ease of demonstration. Similar to our previous discussion of the mathematical programming example in Section 4.2.2, we use the optimization-based hierarchical procedure from Figure 4.2 to find a common design that is preferred for all given scenarios. Consequently, we start by solving

$$\text{COP}_1\text{: Minimize } f^1(x) = [d_1(x), V(x)] \text{ subject to } x \in X \tag{5.8}$$

to obtain the ten Pareto solutions that are highlighted in Figure 5.5, from among which we subsequently select the highlighted point ($*$) as preferred compromise solution for COP$_1$. The corresponding truss design $x^1 = (1.7459, 2.4732, 1.4142, 2.4730)$ is illustrated in Figure 5.7 and has performance values of $V(x^1) = 2292.5$ cm^3 for the truss volume and $d_1(x^1) = 0.0110$ cm, $d_2(x^1) = 0.0721$ cm and $d_3(x^1) = 0.0872$ cm for the three node displacements.

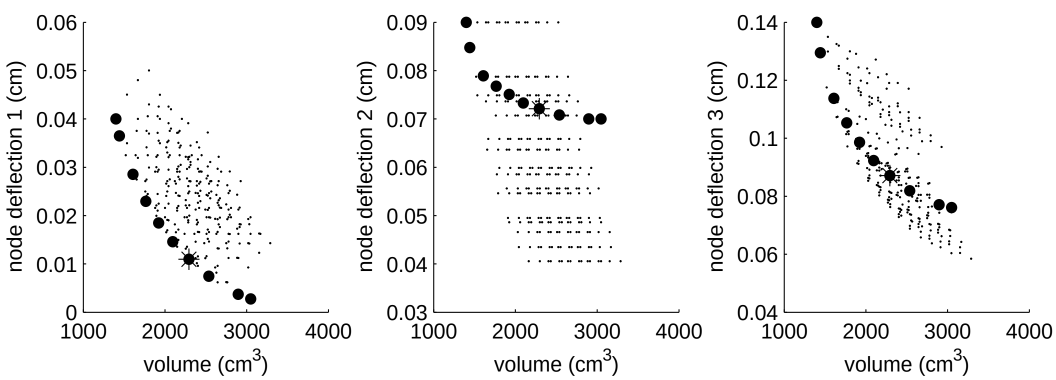


Figure 5.5 On the left, ten Pareto solutions for COP$_1$ and the first scenario, and centered and on the right, their respective images for the second and third scenario

Before we solve the next coordination problem, we formulate the two tradeoff problems that are associated with the second and third scenario

$$\begin{aligned} \text{TOP}_{21}\text{: Minimize } & d_2(x) \\ \text{subject to } & d_1(x) \le d_1(x^1) + \varepsilon_{d_1} \\ & V(x) \le V(x^1) + \varepsilon_V \\ & x \in X \end{aligned} \qquad \begin{aligned} \text{TOP}_{31}\text{: Minimize } & d_3(x) \\ \text{subject to } & d_1(x) \le d_1(x^1) + \varepsilon_{d_1} \\ & V(x) \le V(x^1) + \varepsilon_V \\ & x \in X \end{aligned}$$

but, for conceptual simplicity, restrict our analysis to the tradeoff between the different deflection criteria. Solving the two tradeoff problems TOP_{21} and TOP_{31} with x^1 as initial design, we obtain the associated Lagrangean multipliers

$$\lambda_{21} = \left.\frac{\partial d_2(x)}{\partial d_1(x)}\right|_{x=x^1} = 413.77 \quad \text{and} \quad \lambda_{31} = \left.\frac{\partial d_3(x)}{\partial d_1(x)}\right|_{x=x^1} = 538.03 \tag{5.9}$$

in which case the very large magnitudes of these values clearly indicate that the currently selected design x^1 should be further improved. However, as emphasized before and quite obvious at this point, these values do not give us an accurate prediction on the improvement that we should expect to actually obtain.

Considering that the current first node deflection of $d_1(x^1) = 0.0110$ cm is significantly smaller than the second and third displacement, $d_2(x^1) = 0.0721$ cm and $d_3(x^1) = 0.0872$ cm, we assume that we are still willing to accept a design that yields a first node deflection of up to 0.03 cm, provided a reasonable tradeoff or improvement with respect to d_2 or d_3. Thus, while setting $\varepsilon_V = 0$ to maintain the current volume of the truss, we only change the tolerance value $\varepsilon_{d_1} = 0.02$ cm and then solve the next coordination problem

$$\mathrm{COP}_2\text{: Minimize } f^2 = [d_2(x), V(x)] \tag{5.10}$$

$$\text{subject to } d_1(x) \leq d_1(x^1) + \varepsilon_{d_1} = 0.0310 \tag{5.10a}$$

$$V(x) \leq V(x^1) + \varepsilon_V = 2292.5 \tag{5.10b}$$

$$x \in X \tag{5.10c}$$

By solving COP_2 for the second scenario, we now find a corresponding new set of Pareto optimal designs, that is depicted in the middle plot of Figure 5.6 and for which the corresponding performances for the first and third scenario are again depicted in the left and in the right plot, respectively.

In particular, we note from the left plot that all these new solutions, in fact, meet the specified upper performance bound on d_1 for the first subproblem, while resulting in a tradeoff between the volume and second node deflection in COP_2. Assuming that our main incentive is still the improvement with respect

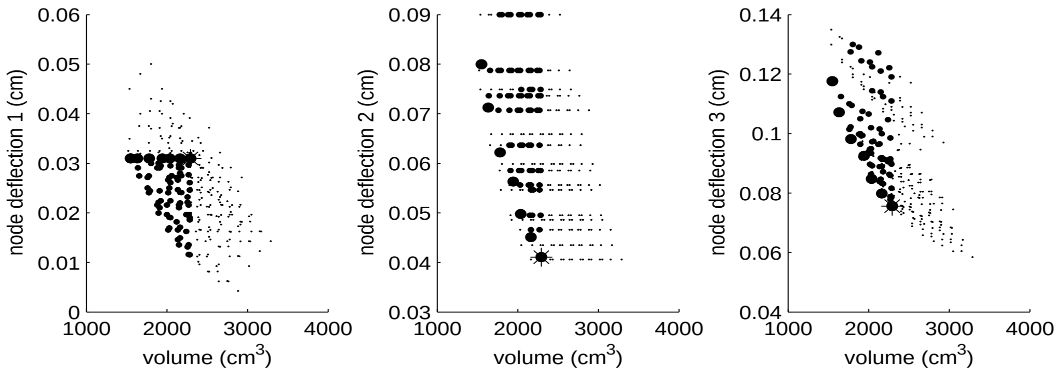


Figure 5.6 In the center, Pareto solutions for COP_2 and the second scenario, and on the left and right, their respective images for the first and third scenario

to the second node deflection d_2, we decide not to further improve the structural volume and, thus, select the highlighted bottommost point as the improved second design $x^2 = (1.1754, 1.6582, 2.7837, 2.8298)$, depicted in Figure 5.7.

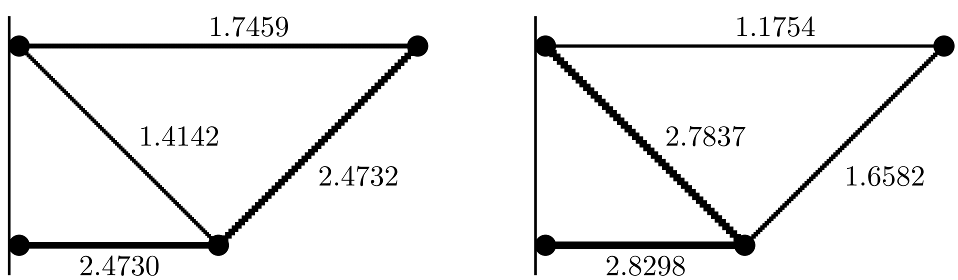


Figure 5.7 Selected Pareto designs x^1 for COP_1 (left) and x^2 for COP_2 (right)

The performances of this new design are given as $V(x^2) = 2292.5$ cm^3, $d_1(x^2) = 0.0310$ cm, $d_2(x^2) = 0.0411$ cm and $d_3(x^2) = 0.0756$ cm, and hence, the actually achieved tradeoffs can now be computed as

$$\frac{\Delta d_2(x)}{\Delta d_1(x)} = \frac{d_2(x^1) - d_2(x^2)}{d_1(x^2) - d_1(x^1)} = \frac{0.0721 - 0.0411}{0.0310 - 0.0110} = 1.55 \tag{5.11a}$$

$$\frac{\Delta d_3(x)}{\Delta d_1(x)} = \frac{d_3(x^1) - d_3(x^2)}{d_1(x^2) - d_1(x^1)} = \frac{0.0872 - 0.0756}{0.0310 - 0.0110} = 0.58 \tag{5.11b}$$

As expected, the tradeoff between d^1 and d^2 achieves a value greater than 1 and, thus, is a favorable one. The smaller tradeoff value between d_1 and d_3 is not surprising either as we only solve COP_2 which, in fact, does not minimize with respect to d_3. Nevertheless, from the right plot in Figure 5.6 we see that the current design x^2 already gives a reasonable compromise solution with respect to the third loading scenario that also involves d_3. In particular, upon computing the updated tradeoff ratios λ_{21} from TOP_{21} at the new design x^2 and including the additional constraint $d_2(x) \leq d_2(x^2) + \varepsilon_{d_2}$ with $(\varepsilon_{d_1}, \varepsilon_{d_2}, \varepsilon_V) = (0.02,\ 0,\ 0)$ into a new TOP_{31}

$$\text{TOP}_{31}\text{: Minimize } d_3(x) \tag{5.12}$$

$$\text{subject to } d_1(x) \leq d_1(x^1) + \varepsilon_{d_1} = d_1(x^2) = 0.0310 \tag{5.12a}$$

$$d_2(x) \leq d_2(x^2) + \varepsilon_{d_2} = 0.0411 \tag{5.12b}$$

$$V(x) \leq V(x^2) + \varepsilon_V = 2292.5 \tag{5.12c}$$

$$x \in X \tag{5.12d}$$

we obtain a set of new Lagrangean multipliers, yielding the updated tradeoff ratios

$$T_{21}(x^2) = -\left.\frac{\partial d_2(x)}{\partial d_1(x)}\right|_{x=x^2} = \lambda_{21} = 0.7877 \tag{5.13a}$$

$$T_{31}(x^2) = -\left.\frac{\partial d_3(x)}{\partial d_1(x)}\right|_{x=x^2} = \lambda_{31} = 0 \tag{5.13b}$$

$$T_{32}(x^2) = -\left.\frac{\partial d_3(x)}{\partial d_2(x)}\right|_{x=x^2} = \lambda_{32} = 0.2481 \tag{5.13c}$$

Since all these values are now less than a possible tradeoff threshold of 1, we may conclude that solving the third coordination problem is unlikely to yield significant further improvement, and thus, our previous discussion suggests to terminate the solution of this problem with x^2 from Figure 5.7 as the final preferred design.

5.3 Vehicle Layout Configuration in Packaging Optimization

Packaging optimization deals with the spatial arrangement of a given set of objects into some specified volume. We personally face such problems when packing our book bag or briefcase to get ready for school or work, or when fitting clothes and other belonging into a suitcase and that suitcase into our car's trunk to get ready for our next vacation. Clearly, these problems also occur in many industrial settings, for example when a manufacturer of cat food needs to put the produced cans into boxes, a grocery store supplier loads those boxes onto a truck, and a worker at the local store takes the cans out of their boxes and arranges their display onto shelves for their final sale to a customer. In an engineering context, packaging optimization is also related to layout and configuration design, and the problem that we address in the following discussion is taken from the particular case of vehicle configuration design following the treatment in Yi (2005).

In this particular application, we are interested in the layout of a set of vehicle components for a medium-sized truck that needs to be designed for possible use in both civilian and military situations and environments, thus imposing some very specific and critical design objectives onto the final truck configuration. As depicted in Figure 5.8, the components include engine, reformer, transmission, accumulator, pump, reservoir, fuel tank, and auxiliary power unit (APU), and a total of 36 decision variables models the respective location and rotational orientation of these components as well as their overall packaging sequence.

The objectives that we consider are the truck's ground clearance, its vehicle dynamics, as well as component maintainability and survivability. A high ground clearance is important to enable the truck's flexibility in a variety of different terrains and is measured as the maximum angle that the truck can climb without interference with the ground. As vehicle dynamics objective, we employ the maximum lateral acceleration of the truck before wheel-lift, which must be avoided due to the potential fatal consequence of a truck roll-over. It is clear, however, that the possibility of a roll-over increases with the truck's ground clearance, and consequently, we already expect to see a conflict between these first two objectives in our later analysis.

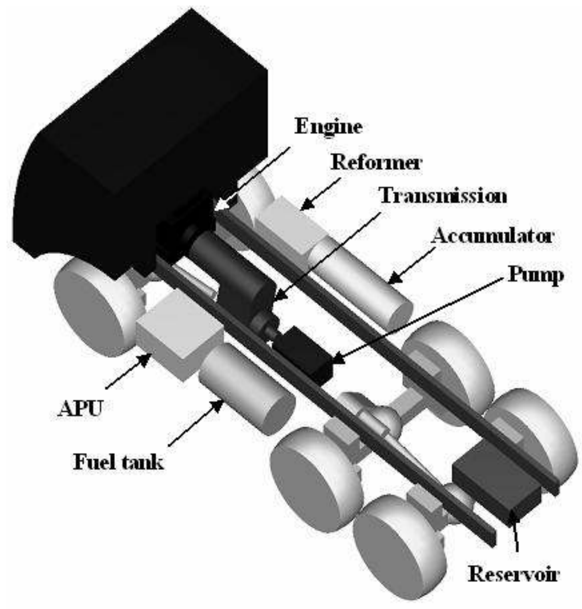


Figure 5.8 Vehicle components for configuration design of a medium-size truck

The component maintainability is defined as index for the ease of maintenance of the truck, for which we require that those components that need to be handled more frequently, such as the reservoir or fuel tank, are easily accessible. Finally, with the objective of survivability we refer to the specific military requirement that the truck must be capable to resist the attack of bullets and explosives during combat, so that especially the fuel tank and power unit should be blocked by other items to be protected from a direct external access. As before, we note that the objectives of maintainability and survivability are usually in conflict.

A very detailed discussion of this model together with an extensive explanation of the underlying computation and simulation codes is given in the original reference (Yi, 2005). In particular, due to the high complexity in evaluating the different objectives for a given packaging sequence and design configuration, this problem is originally solved by a genetic algorithm that provides the designer or decision maker with a set of one hundred candidate designs and their corresponding outcomes from among which a final truck layout is to be chosen. Clearly, this selection still puts a significant burden on the decision maker who now needs to consider and suitably compare all these candidates to ultimately select a final and preferred layout design. In particular, since this problem consists of four objectives so that the outcome space is four dimensional, a visualization of these outcomes is only possible if we choose a decomposition into biobjective subproblems, using the general objective decomposition approach as proposed in the previous chapter.

In Figure 5.9, we show this decomposition of the four-dimensional outcome space into the two subproblems with ground clearance and vehicle dynamics in the first plot on the left and maintainability and survivability in the second subproblem plotted on the right. While the physical interpretation of ground clearance and vehicle dynamics is given as maximum angle (measured in degrees) and acceleration (measured in meters per square seconds), the other two objectives are normalized between 0 and 1 so that their values can be interpreted as percental achievement of the respective single objective optimum. We also note that, in both cases, the two objectives exhibit the conflict that we expect from our earlier discussion.

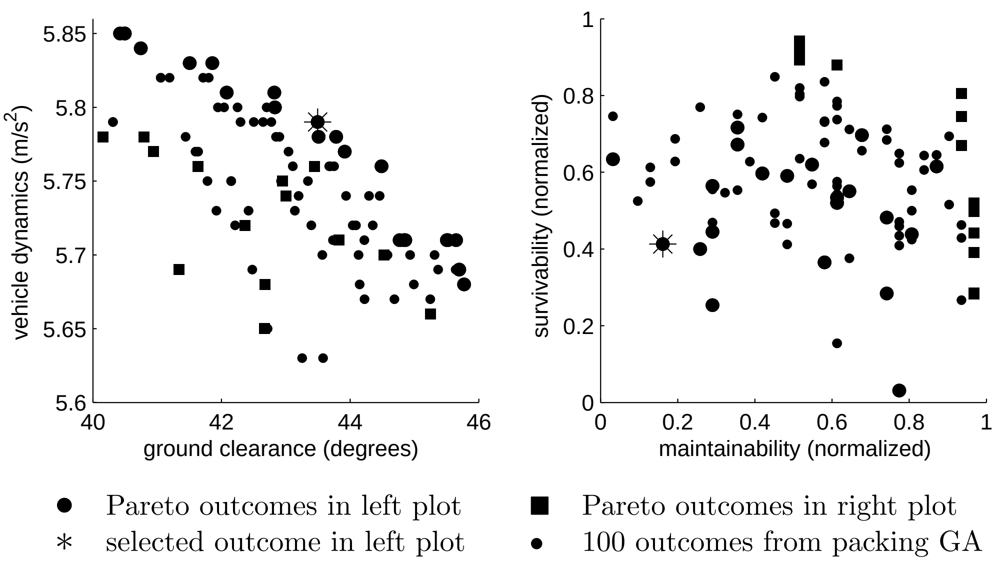
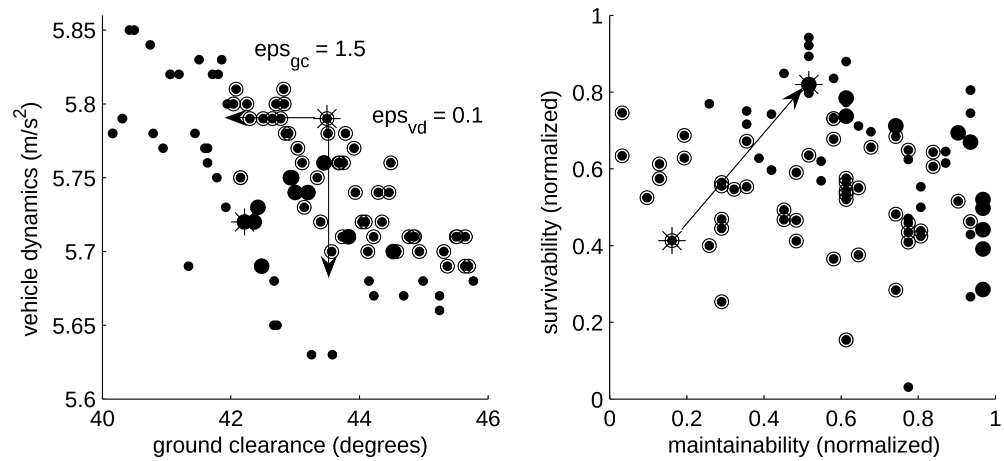


Figure 5.9 Decomposition of the four-dimensional outcome space of the truck layout problem into biobjective subproblems with their individual Pareto sets

Since in this application all outcomes are readily available, we do not need to formally solve any subproblem or coordination problem but can immediately select a preferred outcome ($*$) from the Pareto set of the first subproblem. Here note that, because all objectives are subject to maximization, the Pareto sets correspond to the upper right fronts and are highlighted by black bullets ($\bullet$) and squares ($\blacksquare$) for the first and second subproblem, respectively. Similar to the discussion for the previous examples and Figures 4.3 and 4.4, we also note that the individual Pareto outcomes for each subproblem are again among the worst outcomes for the respective other problem, justifying the subsequent coordination step to find a design whose outcome establishes a better compromise between all four objectives by trading off ground clearance and vehicle dynamics with maintainability and survivability.

To decide on this tradeoff, we also investigate the objective values at the current design with a ground clearance of 43.5 degrees, a vehicle dynamics index of 5.79 m/s^2, and maintainability and survivability values of 16% and 41%, respectively. Based on further expert opinion, we can now specify maximal deviations from each of these values and, as indicated in Figure 5.10, select $\varepsilon_{\text{gc}} = 1.5$ degrees for ground clearance and $\varepsilon_{\text{vd}} = 0.8$ m/s^2 for vehicle dynamics. The resulting set of ε-Pareto outcomes for the first subproblem and the correspondingly improved outcomes for the second subproblem are highlighted as circled points in Figure 5.10.

In view of the high importance of survivability, we now select the new highlighted point ($*$) in Figure 5.10 as the new and final outcome, yielding the new objective values of 42.21 degrees for ground clearance and 5.72 m/s^2 for vehicle dynamics as well as 52% for maintainability and 81% for survivability. Hence, the decay of 0.07 m/s^2 in vehicle dynamics and 1.29 degrees in ground clearance has enabled an improvement of 36% in maintainability and 40% in survivability. While the comparison and interpretation of these changes with respect to the actual truck layouts, depicted in Figure 5.11, is difficult due to complex dependencies between the different vehicle characteristics, we only note how the cylinder-shaped fuel tank has moved from the very exposed side position for the first to a more hidden position for the second truck layout. Clearly, based on the critical role of a well protected fuel



○ ε-Pareto outcomes in left and corresponding images in right plot
● Pareto outcomes in right plot *among* the above ε-Pareto outcomes (○)
• 100 outcomes from GA $\quad *$ previously and newly selected outcome

Figure 5.10 Subproblem tradeoff and selected outcomes for truck layout problem

tank for the overall survivability of both the truck's driver and its passengers, this might be one of the reasons for the significant improvement in survivability of the second over the first truck layout. Any further discussion, however, again requires specific expert knowledge on this particular application.

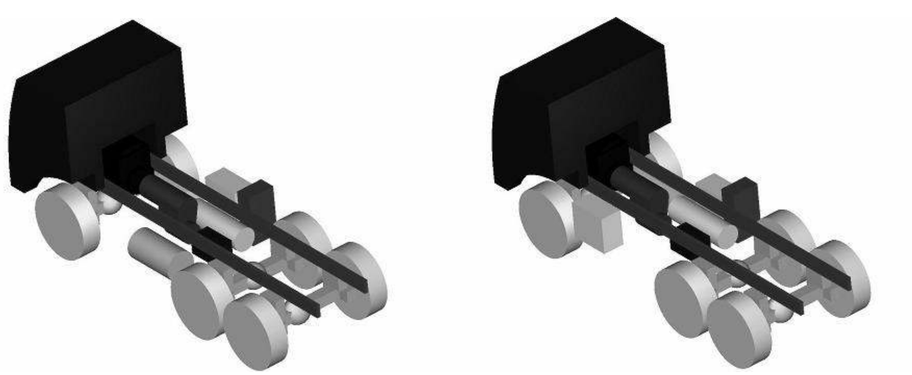


Figure 5.11 Vehicle configurations of outcomes selected for truck layout problem

5.4 Seat and Head Restraint Selection in Vehicle System Design

According to the most recent data released by the U.S. Department of Transportation, 43,443 people died in motor vehicle crashes in the year 2005 (Fatality Analysis Reporting System (FARS), 2006). To reduce the deaths and injuries that occur in car and other traffic accidents, the Vehicle Research Center (VRC) of the Insurance Institute for Highway Safety (IIHS) conducts research on crash testing to supply consumers with information about car crashworthiness and car manufacturers with recommendations on improving their adopted design specifications.

In this context, we consider the selection of a seat and head restraint system following the recent Research Council for Automobile Repair (RCAR) standard for evaluating and rating the ability of seats and head restraints to prevent neck injury in moderate and low-speed car crashes (Insurance Insitute for Highway Safety, 2006). The procedures and criteria were developed by the International Insurance Whiplash Prevention Group (IIWPG), and the data underlying our analysis is provided by courtesy of Ford Research & Advance Engineering Laboratories (Fu, 2006).

This data consists of 510 design solutions that are evaluated by four performance criteria as developed by the IIWPG, namely the forward acceleration of the seat occupant's torso (T1 acceleration), the time to head restraint contact, the neck shear force, and the neck tension force. The latter two are considered jointly and combined by an equally weighted vector sum, thereby yielding a multiobjective program with three objectives. Since the performance evaluation of any additional design would necessitate to conduct a new crash test and, thus, is not possible anymore at this stage of the design process, an actual optimization is replaced by the mere selection of the preferred available design solution, using the objective decomposition and coordination outlined in Procedure 4.2.2 in Section 4.2.3.

Let $I = \{1, 2, 3\}$, and $f = (f_1, f_2, f_3)$ denote the vector criterion including T1 acceleration (f_1), time to head restraint contact (f_2), and neck forces (f_3), all normalized to values between 0 and 1. We completely decompose I into $I_1 = \{1, 2\}$, $I_2 = \{1, 3\}$, $I_3 = \{2, 3\}$, and, accordingly, let $f^1 = (f_1, f_2)$, $f^2 = (f_1, f_3)$, $f^3 = (f_2, f_3)$, and $\varepsilon^1 = (\varepsilon_1, \varepsilon_2)$, $\varepsilon^2 = (\varepsilon_1, \varepsilon_3)$, $\varepsilon^3 = (\varepsilon_2, \varepsilon_3)$. Finally, we let $X = \{x^1, \ldots, x^{510}\}$ be the set of designs and denote $\mathcal{E} = |E(X, f, \varepsilon)|$, $\mathcal{E}_j = |E(X, f^j, \varepsilon^j)|$ for $j = 1, 2, 3$, $\mathcal{E}_{jk} = |E(X, f^j, \varepsilon^j) \cap E(X, f^k, \varepsilon^k)|$ for $j \neq k = 1, 2, 3$, and $\mathcal{E}_{123} = |\bigcap_{j=1}^{3} E(X, f^j, \varepsilon^j)|$. Table 5.2 shows these cardinalities for different choices of ε, and Figure 5.12 depicts all 510 designs evaluated for MOP and the three biobjective subproblems BOP_1, BOP_2, and BOP_3 to provide illustration of our following discussion.

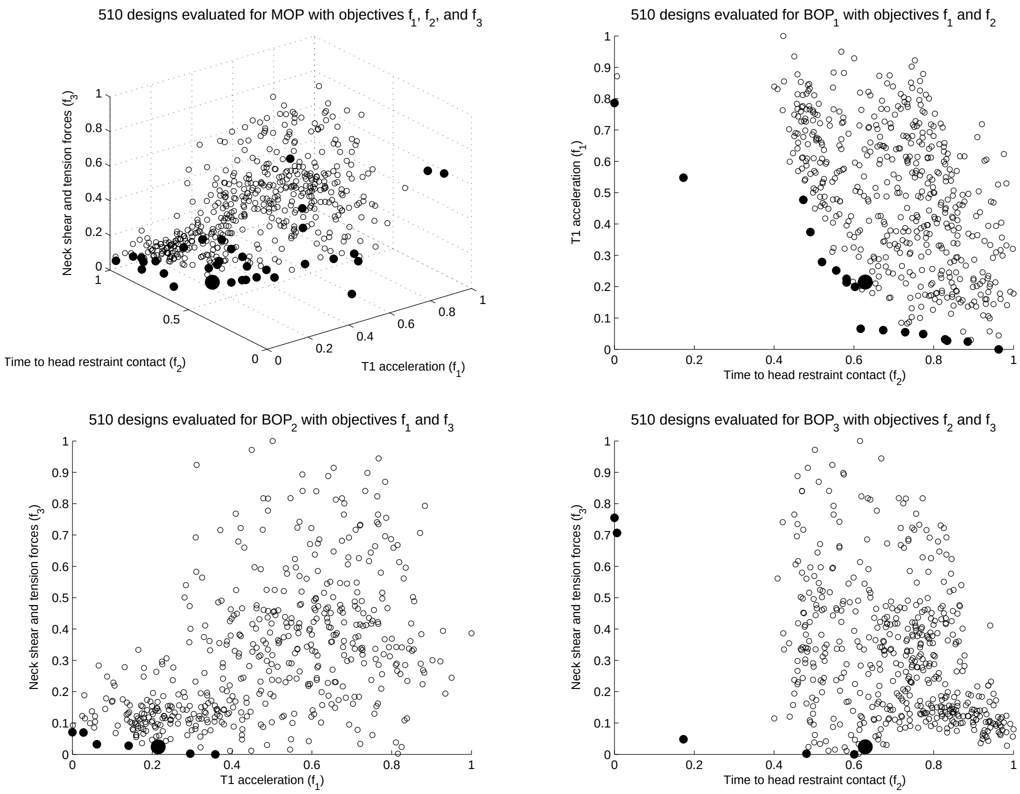
Table 5.2 Solution set sizes of MOP and DMOP for design selection problem

	ε_1	ε_2	ε_3	$\mathcal{E}$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_{12}$	$\mathcal{E}_{13}$	$\mathcal{E}_{23}$	$\mathcal{E}_{123}$
1	0.0	0.0	0.0	36	17	7	5	3	2	1	0
2	0.1	0.0	0.0	84	54	27	5	20	3	1	0
3	0.5	0.0	0.0	384	380	263	5	263	5	2	2
4	0.0	0.1	0.0	94	69	7	13	6	8	2	2
5	0.0	0.5	0.0	410	399	7	183	7	173	4	4
6	0.0	0.0	0.1	118	17	97	73	9	7	69	6
7	0.0	0.0	0.5	432	17	422	425	15	16	421	15
8	0.0	0.1	0.1	160	69	97	78	24	22	69	18
9	0.1	0.1	0.1	205	129	119	78	55	35	71	32
10	0.03	0.03	0.03	77	45	28	16	11	4	11	1

Step 1: In this initialization step, we select $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ so that all coordination parameters for each of the subproblems are initially equal to zero.

Step 2: Based on pairwise comparisons, we find the true efficient sets $E(X, f^j, 0) = E(X, f^j)$ with set cardinalities as listed in Row 1 in Table 5.2. Note that we obtain 36 efficient solutions for MOP, of which $\mathcal{E}_1 = 17$, $\mathcal{E}_2 = 7$, and $\mathcal{E}_3 = 5$ are also efficient for BOP_1, BOP_2, and BOP_3, respectively. In particular, this means that 7 solutions are efficient for the complete, but not for any subproblem.

Step 3a: While the intersections of the solution sets for any two subproblems contain $\mathcal{E}_{12} = 3$, $\mathcal{E}_{13} = 2$, or $\mathcal{E}_{23} = 1$ solutions and, in particular, are nonempty, there does not exist a common design that is efficient for all three subproblems simultaneously, $\mathcal{E}_{123} = 0$. Hence, we need to relax at least one of the ε_j, and since



○ outcomes for MOP ● Pareto outcomes for BOP_j ● final selected outcome

Figure 5.12 Decomposition of the three-dimensional outcome space of the design selection problem into biobjective subproblems with their individual Pareto sets

the smallest set contains $\mathcal{E}_3 = 5$ solutions, obtained for the third subproblem BOP_3 that combines the second and third objective, we subsequently choose either ε_2 or ε_3. For completeness, however, we also note that upon relaxing $\varepsilon_1 = 0.1$, the intersection of all three solution sets remains empty, and even for the relatively large value of $\varepsilon_1 = 0.5$, we only find $\mathcal{E}_{123} = 2$ designs as potential candidate solutions. The corresponding other cardinalities are listed in Rows 2 and 3 of Table 5.2.

Step 2(2): Rows 4/5 and 6/7 in Table 5.2 show the number of new solutions when relaxing ε_2 and ε_3, respectively, for repeating values of 0.1 and 0.5. In particular, now the number $\mathcal{E}_{123}$ of common solutions in the intersection of all three subproblems increases from 2 and 4 for relaxation of ε_2 to 6 and 15 for ε_3.

Step 3b and 4a: A remaining drawback of these solutions, however, is the relatively large relaxation of only one ε_j, whereas better compromise solution might exist for smaller but simultaneous relaxation of several ε_j. Hence, we once more repeat Step 2, before finally concluding this discussion.

Step 2(3): Row 8 in Table 5.2 shows that a small simultaneous relaxation of both $\varepsilon_2 = \varepsilon_3 = 0.1$ is sufficient to find $\mathcal{E}_{123} = 18$ and, thus, more solutions than for a separate relaxation by 0.5 in either one. Moreover, although the individual relaxation of $\varepsilon_1 = 0.1$ in Row 2 did not result in an increase in the number of common solutions for all three subproblems, in combination with ε_2 and ε_3 it does and now almost doubles this number from earlier 18 to now 32 solutions.

Step 3b and 4b: Hence, to select a final solution, we can further reduce the current values $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0.1$ until finding a unique solution in the intersection of the three individual subproblems. In particular, by choosing $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0.03$, only one of the previously 18 solutions remains in the common solution set, as given in Row 10 in Table 5.2. Therefore, we terminate this exemplary illustration of our procedure and the optimization-free coordination method with this solution, corresponding to the highlighted point in Figure 5.12, as preferred final design.

In general, Steps 3b and 4b of Procedure 4.2.2 do not require that the final solution is always obtained as a singleton. In fact, for most decision making situations we recommend that this procedure only be used to identify a (suitably small) number of candidate solutions for ultimate consideration based on specific problem knowledge and expert opinion. Without this expertise, however, we decide to conclude this example without any additional discussion.

CHAPTER 6

CONCLUSION

During the last few decades, multiobjective programming has received much attention for both its numerous theoretical advances as well as its continued success in modeling real-life optimization and decision problems with multiple criteria in various fields including the management and engineering sciences. A common characteristic of these problems is that, in general, there does not exist a unique optimal solution, but a set of efficient decisions or nondominated outcomes from among which a decision maker chooses based on personal preferences or additional criteria not included in the original optimization model. While a great variety of approaches exist to generate parts of these two sets, however, choosing a final preferred solution still remains difficult, especially if the numbers of solutions and objectives are too large to allow the effective use of currently existing decision making techniques. The research that we here present can in various ways be related to this observation, and its main accomplishments fall into the following two major parts.

At first, we pursue the mathematical investigation of multiobjective programs that usually elides the role of a decision maker and merely focuses on the theoretical characterization of the sets of efficient decisions or nondominated outcomes and the development of different optimization techniques for their generation, based on the nature of the underlying feasible set, objective functions, and the chosen concept of optimality. Based on the equivalence between partial orders and certain classes of convex cones, much attention has been devoted to the case of cone efficiency where, in large part, the cone is polyhedral and described by a linear system of inequalities as induced by a linear function or matrix. General domination structures, however, are a more flexible means to describe preference or domination relationships and provide the fundamental framework for this part of our work.

We present a diverse collection of new results that characterize the nondominated set of a multiobjective program when the underlying domination structure is defined in terms of different cones. In particular, we extend the case of a constant polyhedral cone to constant and variable nonpolyhedral cones, and to translated polyhedral cones that can be used to also describe epsilon-nondominated outcomes. In each of these three cases, we start with an analysis of the associated cone representation, discuss some related cone properties and investigate possible generalizations of several results originally known for only polyhedral cones. Some of our main results include that every cone can be described by a positively homogeneous function, that the defined variable cones belong to the class of Bishop-Phelps cones, and that translated polyhedral cones can also be characterized as polyhedral sets. For a further characterization and the generation of corresponding nondominated outcomes, we modify scalarization methods that are originally formulated for the Pareto case and extend the associated optimality conditions for domination by general cones and epsilon-nondominance. We derive some more specific conditions for the variable-cone model and develop several new approaches that also allow the intentional generation of a merely approximate solution. Wherever possible, we illustrate our findings on some analytical, graphical, or practical example and offer an outlook to topics for possible further research in the concluding discussion.

While advancing various theoretical aspects of multiobjective programming, we envision that the previous results also have several potential implications for practical decision making and preference modeling, in particular. While it is undoubtful that preferences of decision makers in general cannot be described completely and therefore must remain at least partially unmodeled, our variable-cone model illustrates that several limitations of the current restriction to constant polyhedral cones can be removed if we also allow for variable domination. Some possible further research into this direction is proposed along with the relevant discussion within the text. Furthermore, we believe that the importance of epsilon-nondominance is usually underestimated as simply a concept of approximation or suboptimality, in spite of its high relevance to objective decompositions, decision tradeoffs and compromising between scenarios as discussed in the second part of this dissertation.

Therein we continue with the investigation of Pareto multiobjective programs and employ the concepts of function and problem decompositions to examine the consequences of decomposing the original objective function into a collection of new functions that consist of only a subset of the original objective function components. In particular, in analyzing the corresponding efficiency relationships between the original problem and the decomposed subproblems, we observe that efficient decisions for the latter are also efficient for the former and that this relation, in general, cannot be reversed unless we relax efficiency to epsilon-efficiency. Several further results are derived for either scalarization of the original and subproblem objectives or for adding objectives as additional constraints to allow the overall coordination of efficiency tradeoffs, and we obtain that both approaches are capable to completely characterize the efficient set of the original problem.

Based on the latter result and reintroducing participation of a decision maker, we formulate a first interactive decision making procedure that utilizes the above problem decomposition and subproblem coordination to support the solution of a large-scale problem by only working with the collection of smaller-sized subproblems. The procedure assumes that the decision maker is capable to articulate his preferences for both the selection of intermediate solutions and the specification of additional coordination parameters that are used for trading off between the different subproblems. In addition to the possibility of assessing these tradeoffs through a sensitivity analysis for every intermediate decision, we further enhance the initially nonhierarchical procedure by a hierarchical coordination to reduce some of the preference information required from the decision maker, and we demonstrate its use on a mathematical programming example. Also taking into account that many real-life decision problems are limited in computational resources or simply too complex to enable a traditional subproblem optimization, we finally suggest a third procedure for the pure selection of a final solution from a given candidate set or population which may be generated by sampling and simulation techniques or other heuristic approaches such as an evolutionary or genetic algorithm.

To illustrate applications for each of the three procedures, we describe four real-life problems from financial optimization and engineering design and provide some careful discussion in support of our theoretical findings. The first two problems are drawn from portfolio and structural optimization and formulated as multiscenario programs which are solved using the optimization-based hierarchical and nonhierarchical coordination method, respectively. In both cases, we show that the procedures provide convenient methods to arrive at final solutions that establish reasonable tradeoffs between the different scenarios. The other two examples are motivated from vehicle design and described as selection problems for which we need to choose a single final outcome from a finite set that is obtained through simulations and a genetic algorithm. Using a modified version of the previous procedures in one and the third procedure in the other case, we illustrate how a decision maker can arrive at a preferred decision for these two problems as well.

While the first two applications reveal that the additional information obtained from the proposed sensitivity analysis, at the moment, cannot provide the decision maker with accurate predictions on remaining tradeoffs between subproblems, we are confident that a further analysis in continuation of our current efforts can bring new insights and lead to additional improvements of our proposed methodology. As of now, however, we believe that our procedures are already well suited for practical decision making by offering the benefits of a simplified problem perception, enhanced preference articulation, and facilitated decision making due to the overall reduction of dimensionality. In particular, for decompositions into biobjective subproblems, our methods enable the simultaneous consideration of multiple objectives without their aggregation or scalarization but, at the same time, maintaining the capability to visualize Pareto curves, which is of vital importance for decision makers without strong analytical training. A method validation for large-scale problems, however, together with several other remaining issues as outlined in the respective discussions is postponed for inquiry in the near or later future.

For now, we decide to conclude this text with the following final remark. Without a doubt, the preparation and writing of this dissertation is influenced by a series of different decisions which reflect numerous compromises and tradeoffs in selection and arrangement of the included materials. If this outcome supports understanding, achieves appreciation, stimulates interest or motivates continuation of any of the work presented, then we meet our primary objective and are content to come to an end.

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INDEX

Numbers in *italics* refer to pages where items are defined (or first introduced).

Symbols and Notation

$\mathbb{R}^m_{\geqq}$. *see* Pareto cone/nonneg. orthant
$\mathbb{R}^m_{\geq}$ . . . . *see* nonzero orthant
$\mathbb{R}^m_{>}$ . . . . *see* positive orthant
$\|\cdot\|$ . . . . *see* Euclidean norm
$\|\cdot\|_p$ . . . . *see* pth-order norm
$\langle\cdot,\cdot\rangle$ . . . . *see* inner product
bd . . . . *see* boundary
cl . . . . *see* closure
int . . . . *see* interior
rec . . . . *see* recession cone
$\sim$ . . . . *see* equivalence relation
$\prec$ . . . . *see* strict partial order
$\preceq$ . . . . *see* partial order
$\mathcal{R}$ . . . . *see* binary relation
$\mathcal{R}_C$ . . . . *see* cone relation
$C_{\mathcal{R}}$ . . . . *see* relation cone
C . . . . *see* cone
C^* . . . . *see* dual cone
C^*_s . . . . *see* strict dual cone
C^m_p . . . . *see* pth-order cone
C_e . . . . *see* translated cone
$C_{\gamma,d}$ . . . . *see* Bishop-Phelps cone
$C(A)$ . . . . *see* polyhedral cone
$D(A,b)$ . . . . *see* polyhedral set
$D(y)$ . . . . *see* domination set
$\mathcal{D}$ . . . . *see* domination structure
f . . . . *see* (objective) function
X . . . . *see* (feasible) decision set
$Y = f(X)$ *see* (image or) outcome set
$(X, f, \mathcal{D})$. *see* multiobjective program
$E(X, f, \mathcal{D})$ . . . . *see* efficient set
$E_w(X, f, \mathcal{D})$ . . . *see* weakly efficient set
$E(X, f)$ . . *see* (set of) Pareto eff. dec.
$E_s(X, f)$ . . *see* strictly Pareto eff. dec.
$E_w(X, f)$. *see* weakly Pareto eff. dec.
$E(X, f, \varepsilon)$ . . . *see* epsilon-Par. eff. dec.
$E_s(X, f, \varepsilon)$ . . *see* strictly eps.-eff. dec.
$E_w(X, f, \varepsilon)$ . . *see* weakly eps.-eff. dec.
$N(Y, \mathcal{D})$ . . . . *see* nondominated set
$N_w(Y, \mathcal{D})$ . . *see* weakly nondomin. set
$N(Y)$ . . . . *see* Pareto set
$N_w(Y)$ . . . . *see* weak Pareto set
$N(Y, D, \varepsilon)$ . . . . . . *see* epsilon-nond. set
$N_s(Y, C, \varepsilon)$. *see* strictly eps.-nond. set
$N_w(Y, D, \varepsilon)$ *see* weakly eps.-nond. set
$N_\varepsilon(Y, D)$ *see* epsilon-transl. nond. set
$N_{w\varepsilon}(Y, D)$ *see* eps.tr.weakly nond. set
$T_{ij}(x^1, x^2)$ . . . . *see* tradeoff
$T_{ij}(x)$ . . . . *see* tradeoff rate
s . . . . *see* scalarization function
π . . . . *see* scalarization parameter
a . . . . *see* augmentation function

Problem Abbreviations

AMOP . . . . *see* augmented MOP
B . . . . *see* Benson scalarization
BOP . . . . *see* biobjective program
CN . . . . *see* weighted-Chebyshev norm
CO . . . . *see* constrained-objective scal.
COP . . . . *see* coordination problem
DMOP . . . . *see* decomposed MOP
HB . . . . *see* hybrid scalarization
MN . . . . . . *see* max-norm scalarization
MOP . . . . . *see* multiobjective program
MSMOP . . . . . *see* multiscenario MOP
POP . . . . *see* portfolio optim. problem
PS . . . . *see* Pascoletti-Serafini scal.
SOP . . . . *see* scalarization problem
TOP . . . . *see* tradeoff problem
WS . . . *see* weighted-sum scalarization

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