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Equitable Efficiency in Multiple Criteria Optimization

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EQUITABLE EFFICIENCY IN MULTIPLE CRITERIA OPTIMIZATION

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ABSTRACT

Equitable efficiency in multiple criteria optimization was introduced mathematically in the middle of nineteen-nineties. The concept tends to strengthen the notion of Pareto efficiency by imposing additional conditions on the preference structure defining the Pareto preference. It is especially designed to solve multiple criteria problems having commensurate criteria where different criteria values can be compared directly.

In this dissertation we study some theoretical and practical aspects of equitably efficient solutions. The literature on equitable efficiency is not very extensive and provides very limited number of ways of generating such solutions. After introducing some relevant notations, we develop some scalarization based methods of generating equitably efficient solutions. The scalarizations developed do not assume any special structure of the problem. We prove an existence result for linear multiple criteria problems.

Next, we show how equitably efficient solutions arise in the context of a particular type of linear complementarity problem and matrix games. The set of equitably efficient solutions, in general, is a subset of efficient solutions. The multiple criteria alternative of the linear complementarity problem dealt in our dissertation has identical efficient and equitably efficient solution sets.

Finally, we demonstrate the relevance of equitable efficiency by applying it to the problem of regression analysis and asset allocation.

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INTRODUCTION

1.1 Multiple Criteria Optimization and Equitable Efficiency

Decision making is an integral and indispensable part of human life. Every day one has to make decisions of some type or the other. The decision process is relatively easier when there is a single criterion or objective in mind. The process gets complicated when one has to make decisions in the presence of more than one criteria to judge the decisions. In such circumstances, a single decision might not exist which optimizes all the criteria simultaneously.

Multi-criteria optimization is a mathematical modeling approach to decision making. It deals with optimization problems governing the optimization of more than one criterion with or without constraints on the decision variables. Contrary to intuition, results of single-criteria optimization do not naturally extend to the realm of multi-criteria optimization.

Roots of multi-criteria optimization can be associated with the seminal work of Pareto (1909), who theorized that society was in the 'best state' when no one's state of life could be improved without deteriorating someone else's state of life.

In traditional multi-criteria optimization, the criteria are considered incomparable. Different criteria might represent different physical outcomes and/or entities having different units of measurement. Available methods then tend to find solutions to the problem without delving into the practical interpretations of the solutions obtained. In practice, there are cases where the criteria to be optimized can be compared directly in the sense that they represent the same entity and all of them have the same units of measurement. In such a case, interest lies not only in finding optimal solutions to all the criteria but do this in an equitable way, in a way where all the criteria are treated 'impartially', so as to improve all the criteria as much as possible. Impartial treatment of criteria brings the issue of ‘equity’ among solutions to the fore. In equitable multi-criteria optimization(Kostreva and Ogryczak, 1997), the focus is on the distribution among the solutions rather to the solutions themselves.

A problem most frequently encountered in classical multi-criteria optimization is that the set of solutions considered by the optimization process is an infinite set, making the selection of a unique preferred decision quite difficult. Considering models with equitable efficiency relieves some of the burden from the decision maker by shrinking the solution set. It turns out that the set of equitably efficient solutions is contained within the set of efficient solution for the same problem. It is noteworthy that the concept of equitable efficiency does not contradict or devalue the notion of Pareto efficiency; in fact, it strengthens the latter by imposing additional constraints on the principles governing efficiency.

As noted earlier, multi-criteria optimization is concerned with the optimization of more than one objective function simultaneously. Any optimization model becomes operational only when it clearly defines what optimization means. The concept of optimality is easily comprehensible for single-criteria optimization problems. In such problems, two solutions can be easily compared on the basis of their numerical or scalar values. In the multi-criteria case, a unique solution optimizing all the criteria seldom exists and it becomes impossible to improve upon the value of one criterion without worsening the value of at least one other criteria. As for single-criteria optimization problems, solutions are judged according to their criteria values and this defines a preference structure in the criteria space. Solution concepts are developed from an axiomatic point of view and defined in terms of the corresponding preference model Vincke (1992). The preference model corresponding to standard Pareto efficiency assumes a preference relation defined on the criteria space to satisfy the properties of reflexivity, strict monotonicity and transitivity. Additionally, if it also satisfies the principles of symmetry or anonymity and inequality reduction or the principle of transfers, it becomes an equitable preference relation. Solutions obtained on the assumptions of an equitable preference relation are said to be equitably efficient. It is clear that an equitable preference relation is a also Pareto preference and hence equitably efficient solutions are Pareto efficient too.

The notion of equitable efficiency in multi-criteria optimization came up in the late 90's,(Kostreva and Ogryczak, 1997). The concept is especially tailored and designed for problems having uniform criteria, for example the case where different criteria might represent allocation of resources or location of facilities at specific point. In such cases, it is desirable to come up with an equitable distribution of resources to serve the public equitably.

1.2 Research Motivation and Goal

In multi-criteria optimization, the primary goal is to obtain a subset of the efficient designs. Generally all the existing methods intend to find efficient solutions in two stages. In the first stage, all the criteria are aggregated into a single criterion by some aggregation function and in the second stage, techniques of solving single-criteria problem are applied to the aggregated problem. In aggregating these different and conflicting criteria, one does not pay much attention to their nature. The criteria are aggregated even if they are incomparable. This process usually involves scaling, rendering the criteria unit less.

Motivation for the current research stems from the need to treat a particular class of multi-criteria optimization problems, namely, one in which all the criteria to be optimized are uniform, differently than what has been done in the literature. The objectives of this research are to study the theory and methodology of equitable efficiency in multi-criteria optimization, devise new methods to generate equitably efficient solutions, stress the importance of equitably efficient solutions by identifying solutions of certain problems as equitably efficient solutions of related problems and demonstrate the relevance of equitable efficiency by looking at practical applications in some areas. Chapter 2, which is a literature survey on equitable efficiency in multi-criteria optimization, gives an overview of related research done in this area by various authors. The remaining text is organized as follows.

Chapter 3 presents the formulation of multi-criteria optimization problems in general. Notations and terminology required in subsequent chapters are introduced. Equitable efficiency is defined, both from an axiomatic as well as a vector point of view. The relation between equitable efficiency of a multi-criteria optimization problem and Pareto efficiency of an associated problem is presented.

Chapter 4 presents some of the existing results related to the generation of equitably efficient solutions. Scalarization techniques, often used to compute efficient solutions for a multi-criteria optimization problem, with additional restrictions on the parameters, can be applied to generate equitably efficient solutions. We develop some equitable-scalarization-based methods of finding equitably efficient solutions. Methods for finding equitably efficient solutions of a multi-criteria optimization problem depend on ordering of objective functions in a monotonic order. Ordering of functions makes their implementation hard. We develop some equitable-scalarization-based methods to generate equitably efficient solutions. These methods act on the objective functions directly without taking any ordering into account. Furthermore, we present the two-phase minimax method of equitable efficiency. All the methods developed in this chapter make no assumption about the specific structure of the optimization problem. We apply the Benson's criteria to prove nonexistence of equitably efficient solutions for a particular class of problems.

Chapter 5 stresses the relevance of equitable efficiency by noticing how equitably efficient solutions naturally arise as solutions of certain problems. We prove that the solution of a matrix game can be characterized as an equitably efficient solution of a related multi-criteria problem.

Linear complementarity problems have been a subject of active research for more than 40 years. In(Kostreva and Wiecek, 1993; Kostreva and Yang, 2004), the authors propose a unifying approach toward the seemingly different topics of multi-criteria optimization and a linear complementary problems.We prove that if all the components of a nondominated vector are equal, then the solution is equitably nondominated and use the result to relate the solutions of a given linear complementarity problem to the equitably efficient solutions of a particular type of multi-criteria optimization problem. Existence/non-existence of solutions of linear complementarity problems are proved in terms of equitably efficient solutions of a related multi-criteria optimization problem.

Chapters 6 and 7 present the applicability of equitable efficiency to problems where it is desirable to treat all the criteria uniformly or impartially. For such problems, looking for solutions with smaller differences between the component criteria than those obtained from Pareto efficiency is desirable. In the literature on equitable efficiency, the notion is applied to problems of location and portfolio analysis. In chapter 6, we present the regression problem of parameter estimation as a multi-criteria optimization problem where the residuals are considered as criteria functions to be minimized. Classical methods of regression analysis are suitable under certain assumptions of the statistical distribution of the given data. Equitable efficiency is applied without any assumptions on the data set. Equitably efficient solutions for the multi-criteria regression problem are obtained. We also show how the solution of the ordinary least squares problem can be obtained as a limiting solution of our multi-criteria problem.

In multi-criteria portfolio analysis, most of the available models, find efficient portfolios for a bi-criteria problem, treating the expected return and variance of the stocks as the two criteria. It is noteworthy that these two criteria are incommensurate in terms of the units involved in expressing the two criteria values. Treating the two criteria separately seems more plausible in such a scenario.

In Chapter 7, we build a multi-class asset allocation problem where all stocks within a particular class are assumed to have some common features. Grouping the expected return from different classes gives rise to a uniform multi-criteria asset allocation problem where each class is treated uniformly. We maximize the expected return of each asset class subject to bounds on the covariance of the assets. A two-phase method of finding equitably efficient allocations is described. Solutions obtained are analyzed with respect to solutions obtained without considering equity of efficient solutions.

Chapter 8 concludes the dissertation with some suggestions for further research.

CHAPTER 2

LITERATURE REVIEW

Multi-criteria optimization problems, in general, are concerned with finding Pareto efficient or non-dominated solutions. Such solutions have the property that one can not improve the value of a criterion without deteriorating the value of at least one other criterion. Such problems have been studied extensively since the publication of the seminal work of Pareto (1909) and detailed references on some of the methods related to various multi-criteria optimization processes can be found in Evans (1984) and more recently in Miettinen (1999).

Multi-criteria optimization forms a part of the more general theory of multi-criteria decision making discussed by Zeleny (1982), Chankong and Haimes (1983) and Yu (1985). The last step toward a decision making process is to provide the decision maker with some efficient solutions of the related multi-criteria problem. There may be instances of certain problems where the decision maker is not content with mere efficiency of solutions but requires some kind of parity among the various criteria values. It is in such a context that the notion of equitably efficient solution gains importance.

Equity has been a topic of intense interest for sociologists and economists. Some prevalent measures of inequality can be found in Atknison (1970) and Allison (1978). Fandel and Gal (2001) and Luss (1999) emphasize the issue of equity within different areas of applications.

Increasing interest in equity issues resulted in new methodologies in the areas of operations research. Luss (1999) reviews a variety of resource allocation problems where it is desirable to allocate limited resources equitably among several activities. He discusses the lexicographic minimax approach to find equitable solutions for such problems.

In a recent article, Lemaitre et al. (2003) discuss the exploitation of earth observing satellites by different groups of users from the multi-criteria optimization perspective. The authors attempt to find equitable and efficient allocation of resources resulting from the co-exploitation of a satellite by several agents. Four different approaches for selecting the best allocation are proposed. The first approach, allocating satellite revolutions to each agent in turn, is perfectly equitable but lacks efficiency. The second approach leans towards equity of allocation where the individual utilities of the agents are combined linearly into a collective utility function, but the weighting coefficients are chosen in a way so as to favor equity. In their third approach, the authors adopt a bi-criteria approach, allowing comparison of allocations over two criteria, equity and efficiency, with equity, in turn, being based on inequality indices. The fourth approach considers a unique collective utility of the agents to characterize equitable and efficient allocations. It is further argued that each of the four approaches has its own way of tackling the equity/efficiency dilemma.

It well known that any multi-criteria optimization problem starts usually with an assumption that the criteria are incomparable, i.e., different criteria may have different units and physical interpretations. Many applications, however, arise from situations where the criteria are comparable; every criterion has the same physical interpretation and their values can be compared directly. In such situations, one is interested in solutions that are not only efficient but equitable as far as the distribution of criteria values is concerned.

The literature available on equitable solutions related to operations research is rather scarce and is mostly related to location and portfolio analysis problems.

The minimax approach is one of the common approaches to solving a multi-criteria problem. Ogryczak (1997) argues that such an approach, applied to locating public facilities, does not comply with the principles of efficiency and equity modeling. In this article he focuses on the lexicographic minimax method, a refinement of the minimax method to solve such problems. He shows that this method complies with the Pareto efficiency principle and the principle of transfers, ideas essential for equity; while the minimax approach may violate both these principles.

Kostreva and Ogryczak (1999) model location problems in a geographic information systems environment where the geographical space itself acts as the decision as well as the criteria space. A multi-criteria problem is formed where an individual objective function is associated with each spatial unit. Most classical location studies focus on some aspects of two major approaches; the center or the median approach. These two concepts minimize the maximum distance and the average distance respectively. Under certain conditions these two methods generate efficient solutions, but the solution concepts do not comply with the principle of transfers, and hence, the solutions are not equitably efficient. Lexicographic median and lexicographic center approaches are used to obtain equitably efficient solutions of the multi-criteria location problem.

Kostreva and Ogryczak (1997) propose a new solution concept, called equitable efficiency, to solve linear optimization problems with multiple equitable criteria. The criteria are equitable in the sense that they measure the same physical or abstract entity. They have shown equitable efficiency to be a refinement of Pareto efficiency by adding, to the reflexivity, strict monotonicity and transitivity of the Pareto preference order, the requirements of impartiality and satisfaction of the principle of transfers. Finding equitably efficient solutions of such problems is shown to be equivalent to finding the Pareto efficient solutions of an associated problem obtained by applying equitable aggregations to the original problem. Based on this equivalence, several techniques are developed to approximate the equitably efficient set. By means of an example, the authors explore the domination structure for equitable efficiency in two-dimensions. It has been geometrically shown that the domination structure for equitable efficiency is larger than that of Pareto efficiency. This in turn implies that the set of equitably efficient solutions is smaller than the set of efficient solutions, and, in fact, the equitably efficient set is contained within the efficient set. The article stresses the fact that unlike Pareto domination, where the domination structure is a convex cone, the domination structure for equitable efficiency is neither a cone nor convex. It is proved that if a linear multi-criteria optimization problem has an efficient solution, then it has an equitably efficient solution too. The equitably efficient set is proved to be connected.

Kostreva et al. (2004) present the theory of equitable efficiency in greater generality. They discuss two different approaches for obtaining the equitably efficient set for a general, possibly nonlinear multi-criteria optimization problem. In the first approach they transform the problem to a single criterion optimization problem by aggregating the functions using equitable aggregations. Any mapping from the criteria space to the set of reals is said to form an equitable aggregation if it is strictly monotonic in each criterion value, is indifferent or symmetric with respect to the criterion values and satisfies the principle of transfers. They show that any optimal solution to the single-criteria aggregated problem is equitably efficient for the original multi-criteria problem. The second approach is based on the concept of Ordered Weighted Averaging of criteria; a concept developed by Yager (1988). In this approach, the criteria are arranged in a non-increasing order, thereby justifying the name ordered; a cumulative ordering map is then applied to this ordered set of criteria vectors. The cumulative ordering map is a function from the criteria space to the criteria space. The components of the vector obtained after applying this map represent the sum of the largest criterion value, the sum of the largest two criteria values and so forth. Finally, another multi-criteria problem is formed whose criteria are the coordinates of the cumulatively ordered criteria values. It is shown that an efficient solution for this problem is equitably efficient for the original problem. The authors apply two methods, namely the weighting and the lexicographic minimax methods to the transformed problem to find equitably efficient solutions for the original problem. Applying weights to the cumulatively ordered criteria results in the single criterion problem referred to as the ordered weighted averaging aggregation (OWA) problem. Further, it is proved that for strictly decreasing and positive weights, every optimal solution of the OWA problem is an equitably efficient of the original multi-criteria optimization problem. A limiting case of the OWA is one where the difference among the weights tend to infinity. In this case, the OWA problem reduces to the lexicographic minimax problem, where one must minimize the largest criteria, the second largest criteria, and so on, of the ordered criteria vector. It is shown that any efficient solution of the lexicographic minimax problem is equitably efficient for the original multi-criteria problem. The authors apply their results to a case studied by Fandel and Gal (2001) and show that the solutions obtained are more equitable than those obtained by Fandel and Gal (2001).

Though seemingly simple and theoretically well developed, the Ordered Weighted Averaging approach of generating equitably efficient solutions is not simple to implement in practice. The first step in this approach is to order criteria vectors, a collection of functions, and there is no easy way to do so. The ordering map sorts the criteria vector in a non-decreasing order. Such an ordering tends to introduce nonlinearity into the problem. An otherwise linear multi-criteria problem, due to the effect of the ordering, may become piecewise linear, thereby bringing the issue of non-differentiability to the fore.

Baatar and Wiecek (2006) use matrix approach to develop a preference structure related to equitable efficiency. The authors present a two-step method of generating equitably efficient solutions.

Ogryczak (2000) considers location problems as a multi-criteria optimization problem, where for each client, there is defined an individual criterion function, which measures the effect of a location pattern with respect to client satisfaction. The author suggests a bi-criteria mean-equity approach as a simplified alternative to the OWA approach of obtaining equitably efficient solutions. In this model, efficiency of solutions is related to the minimization of mean criteria, a linear function of the criteria, and equity is associated with the minimization of some inequity measure, which again, is a function of the criteria. The author discusses the mean equity model by assuming a trade-off coefficient between the inequity measure and the mean criteria value. The trade-off coefficient is used to convert the bi-criteria mean-equity model to a single criterion problem. An optimal solution to the single criterion model is called a λ-mean-equity solution, λ being the trade-off coefficient. The author presents results relating the λ-mean-equity solution to the equitably efficient solution of the original multi-criteria optimization problem. It is shown that any λ-mean-equity solution, with λ within a certain range, is equitably efficient for the original problem.

Portfolio analysis is another area of application of equitable efficiency. Equitable efficiency is suitable for finding solutions to multi-criteria problems with uniform criteria. Ogryczak (2000) develops a multi-period multi-criteria linear programming model of the classical portfolio problem with a finite set of securities. For each security, the expected return for each period is available from observed or forecasted data. Each period is associated with a criterion measuring the return from investments into different securities. The model is not governed by the usual Pareto preference commonly used to compare criteria vectors and identify efficient solutions. Uniformity of the criteria is utilized to develop a new preference, the equitable preference, which strengthens the properties of Pareto preference with the property of impartiality and the Pigou-Dalton principle of transfers. The solutions obtained are equitably efficient. Based on the theory of choice under risk, the multi-criteria model with equitable preference is shown to be equivalent to a multi-criteria program with modified criteria functions and the usual Pareto preference. Since classical scalarization techniques for multi-criteria programs may generate solutions that are not equitably efficient, the author presents two different approaches of obtaining equitably efficient solutions. The first approach, based on a bi-criteria problem, is analyzed in the context of three different measures of risk. In each case it is shown that the optimal solution of a parameterized single criterion problem is an equitably efficient solution of the portfolio selection problem. The second approach is based on the method of ordered weighted averaging (OWA). In the OWA approach, weights are applied to the criteria vectors after arranging them in a non-decreasing order. Varying the weights allows the identification of equitably efficient solutions of the portfolio selection problem. When differences between weights tend to infinity, the OWA problem yields the lexicographic maximization problem, whose optimal solution is equitably efficient for the portfolio selection problem. For strictly decreasing and positive weights, OWA yields linear programs with a large number of constraints. Duals of these linear programming problems can be efficiently solved using the column generation technique. Optimal solutions to these problems are shown to be equitably efficient for the portfolio selection problem.

CHAPTER 3

NOTATIONS AND TERMINOLOGY

This chapter provides an introduction to the basic terminology used in multi-criteria optimization. Results from literature, relevant to our work are presented.

3.1 Notations

The following notations are followed throughout the text.
Distinct vectors are denoted by superscripts while vector components are denoted by subscripts. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
Let y^1, y^2 be two vectors in $\mathbb{R}^m$,

$$y^1 \leqq y^2 \iff y^1_i \leq y^2_i \quad \text{for all } i = 1, 2, ...m.$$
$$y^1 \leq y^2 \iff y^1 \leqq y^2 \text{ and not } y^2 \leqq y^1.$$

Equivalently,

$$y^1 \leq y^2 \iff y^1_i \leq y^2_i \text{ for all } i = 1, ..., m$$

and

$$y^1_j < y^2_j \text{ for some } j \in \{1, ..., m\}.$$
$$y^1 < y^2 \iff y^1_i < y^2_i \text{ for all } \ i = 1, 2, ...m.$$
$$y^1 = y^2 \iff y^1_i = y^2_i \text{ for all } i = 1, 2, ...m.$$

The relations $\geqq$, $\geq$ and $>$ are defined in an analogous manner. Further, we denote the nonnegative and positive orthant of $\mathbb{R}^m$ by

$$\mathbb{R}^m_{\geqq} = \{y \in \mathbb{R}^m : y \geqq 0\}$$
$$\mathbb{R}^m_{>} = \{y \in \mathbb{R}^m : y > 0\}.$$

Let $A,\ B\ \subseteq\ \mathbb{R}^m$. Set addition of A and B is given by the set $C \in \mathbb{R}^m$, defined as

$$C = A + B = \{x + y : x \in X \text{ and } y \in Y\}.$$

Similarly for $y \in \mathbb{R}^m$ and $A \subseteq \mathbb{R}^m$, we define the set $y + A \subseteq \mathbb{R}^m$ as

$$y + A = \{y + x : x \in A\}.$$

3.2 Binary Relations and Orders

Binary relations are ways of describing relationships between two entities. Mathematical theory of binary relations plays an important role in multi-criteria optimization. Any decision process requires comparing different alternatives. Binary relations define the relationship between pairs of alternatives of a given set.

Definition 3.1. Let S be a set. A *binary relation* on S is a subset R of $S \times S$, where $S \times S = \{(s^1, s^2) \mid s^1, s^2 \in S\}$. If $(s^1, s^2) \in R$, we write $s^1 R\, s^2$.

Binary relations possessing certain properties are defined accordingly.

Let R be a binary relation on a set S. R is called

1. Reflexive, if sRs for all $s \in S$,
2. Irreflexive, if not sRs for all $s \in S$,
3. Symmetric, if $s^1 R\, s^2 \implies s^2 R\, s^1$ for all $s^1, s^2 \in S$,
4. Asymmetric, if $s^1 R\, s^2 \implies$ not $s^2 R\, s^1$ for all s^1, $s^2 \in S$,
5. Antisymmetric, if $s^1 R\, s^2$ and $s^2 R\, s^1 \implies s^1 = s^2$ for all $s^1, s^2 \in S$,
6. Transitive, if $s^1 R\, s^2$ and $s^2 R\, s^3 \implies s^1 R\, s^3$ for all $s^1, s^2, s^3 \in S$,
7. Connected, if $s^1 R\, s^2$ or $s^2 R\, s^1$ for all $s^1, s^2 \in S, s^1 \neq s^2$,
8. Strongly connected(total), if $s^1 R\, s^2$ or $s^2 R\, s^1$ for all $s^1, s^2 \in S$.

In the definitions that follow, R is a binary relation on a set S.

Definition 3.2. R is an *equivalence relation* if it is reflexive, symmetric and transitive.

Definition 3.3. R is a *preorder* if it is reflexive and transitive.

Definition 3.4. R is a *weak order* if it is reflexive, symmetric and connected.

Definition 3.5. R is a *partial order* if it is reflexive, antisymmetric and transitive.

Definition 3.6. R is a *strict partial order* if it is asymmetric and transitive.

3.3 Formulation of the Multi-criteria Optimization Problem

A multi-criteria problem, in general, is concerned with the optimization of two or more criterion (objective) functions subject to certain constraints on the decision variables. The term 'optimization'subsumes both maximization and minimization problems. We consider the problem of optimizing functions f_i, where the f_i are m real-valued functions called criteria or objective functions and $m \geq 2$. Maximizing a function is equivalent to minimizing the negative of the same function. Hence, without loss of generality, we can assume a multi-criteria optimization problem to be one in which all the criteria are to be minimized. A multi-criteria optimization problem, then, can be written in the form

$$\text{(MCOP)}: \textit{minimize } \{f_1(x), \ldots, f_m(x)\}$$
$$\textit{subject to } x \in X \subseteq \mathbb{R}^n,$$

where $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are m real-valued functions defined on the domain $X \subseteq \mathbb{R}^n$. If

$$f = (f_1, ..., f_m),$$

the above problem can be represented in vector form as

$$\text{(MCOP)}: \textit{minimize } f(x)$$
$$\textit{subject to } x \in X \subseteq \mathbb{R}^n.$$

We refer to either of the above problems as MCOP.

Definition 3.7. A vector $x \in X$ is called a *feasible decision*, X is called the set of *feasible decisions* and $\mathbb{R}^n$ is referred as the *decision space.* Analogously, the set Y given by

$$Y \ = \ f(X) \ = \{y \in \mathbb{R}^m \mid y = f(x) \text{ for some } x \in X\}$$

denotes the set of *attainable outcomes* in the *criteria space* $\mathbb{R}^m$.

3.4 Minimizing Multiple Criteria

In single objective minimization problems, we compare the objective values at different feasible decisions to select the best decision. Decisions are ranked according to the objective values at those decisions and the decision resulting in the least smallest objective value is the most preferred decision.
Similarly, to make the multi-criteria optimization model operational, one needs to assume certain solution concepts specifying what it means to minimize multiple criteria functions. Attainable outcomes belong to the criteria space $\mathbb{R}^m$; accordingly, solution concepts are defined by the choice of a preference (order) relation on $\mathbb{R}^m$. The preference relation allows comparisons of different outcome vectors. More detailed study on classification and properties of multi-criteria optimization can be found in Ehrgott (1997, 1998).

The preference structure in the criteria space, associated with the underlying preference relation, is characterized by the relation of weak preference denoted by $\preceq$,(Chankong and Haimes, 1983).

Closely related to $\preceq$ are two more binary relations, $\prec$, the relation of strict preference and $\cong$, the relation of indifference. For vectors y^1 and $y^2 \in \mathbb{R}^m$,

$$y^1 \prec y^2 \iff y^1 \preceq y^2 \text{ and not } y^2 \preceq y^1,$$
$$y^1 \cong y^2 \iff y^1 \preceq y^2 \text{ and } y^2 \preceq y^1.$$

Pareto preference, lexicographic preference and the max-ordering preference are some of the important preference relations used to describe minimization of multiple criteria functions. Pareto preference is the one that is commonly used.

Definition 3.8. A binary relation $\preceq$ defined on $\mathbb{R}^m$ is called a *rational(Pareto)* preference if it satisfies the properties of

1. reflexivity, $y \preceq y$ for all $y \in \mathbb{R}^m$, meaning y is at least as preferred as y;

2. transitivity, $y^1 \preceq y^2$ and $y^2 \preceq y^3 \implies y^1 \preceq y^3$, for all $y^1, y^2, y^3 \in \mathbb{R}^m$, which, in terms of preferences, means that if y^1 is at least as preferred as y^2 and y^2 is at least as preferred as y^3, then y^1 is at least as preferred as y^3; and

3. strict monotonicity, $y - \epsilon e^i \prec y$ for all $y \in \mathbb{R}^m$, where $\epsilon > 0$ and $e^i \in \mathbb{R}^m$ is a unit vector with 1 at the i-th positions and 0 elsewhere.

Note that the principle of strict monotonicity implies that reducing one of the components by a positive quantity results in a preferred vector.

Definition 3.9. The outcome vector $y^1 \in Y$ *rationally(Pareto) dominates* $y^2 \in Y$ if and only if $y^1 \prec y^2$ for all rational preference relations $\preceq$. Analogously, a feasible decision $x \in X$ is a *Pareto optimal(efficient)* solution of MCOP if and only if $y = f(x)$ is rationally nondominated.

To make it practical, rational preference is defined in terms of vector inequalities.

Definition 3.10. Let $y^1, y^2 \in Y$, $y^1 \prec y^2 \iff y^1 \leq y^2$. Analogously, a feasible decision x^0 is a *Pareto optimal(efficient)* solution of MCOP if and only if $\nexists x \in X, x^0 \neq x$, such that $f(x) \leq f(x^0)$.

The literature on multi-criteria optimization has several other equivalent definitions of efficiency and nondominance(Ehrgott, 2005). We give the following definition as it will be used as an alternative when needed.

Definition 3.11. Let $x^* \in X$. x^* is an efficient solution of MCOP if $f(x) \leqq f(x^*)$ for some $x \in X$ implies $f(x) = f(x^*)$.

3.5 Equitable Efficiency of Solutions

As already mentioned in the introduction, classical multi-criteria optimization assumes the criteria to be incomparable. There are practical instances where the criteria can be compared directly in the sense that each criterion measures the same physical outcome. In such cases it is more desirable to treat all the criteria uniformly than to assign more/less importance to selected ones. Uniform treatment of criteria tends to look for preferences that are symmetrical in the outcome vector. Equitability of solutions leads to the quest for solutions that lie close to the absolute-equity hyperplane in the criteria space, $\mathbb{R}^m$. On this hyperplane, all the criteria achieve the same value. These two features of the desired preference relation ensure that one does not purposely prefer one criterion over the remaining ones. We formalize this discussion mathematically.

Let $\preceq$ be a preference relation defined on $\mathbb{R}^m$.

Definition 3.12. (Kostreva and Ogryczak, 1999). $\preceq$ is said to be *impartial* if

$$(y_{\pi(1)}, \ldots, y_{\pi(m)}) \cong (y_1, \ldots, y_m) \text{ for all } y \in Y \subset \mathbb{R}^m, \text{where}$$
$$\pi \in \Pi \text{ and } \Pi = \{\pi \mid \pi \text{ is a permutation of the index set } I = \{1, \ldots, m\}\}.$$

Note that $\cong$, the relation of indifference can be related to $\preceq$ in the following manner,

$$(y_{\pi(1)}, \ldots y_{\pi(m)}) \cong (y_1, \ldots, y_m) \text{ if and only if}$$
$$(y_{\pi(1)}, \ldots y_{\pi(m)}) \preceq (y_1, \ldots, y_m) \text{ and } (y_1, \ldots, y_m) \preceq (y_{\pi(1)}, \ldots y_{\pi(m)}).$$

Definition 3.13. (Marshall and Olkin, 1979). $\preceq$ is said to satisfy the '*principle of transfers*,' if $y_i > y_j \implies y - \epsilon e^i + \epsilon e^j \prec y$, for $0 < \epsilon < y_i - y_j$, where, $y = (y_1, ., y_i, ., y_j, .., y_m) \in \mathbb{R}^m$ and $e^i \in \mathbb{R}^m$ is a unit vector whose i-th component is 1.

Definition 3.14. (Kostreva and Ogryczak, 1999). A binary relation $\preceq$ defined on $\mathbb{R}^m$ is called an *equitable preference relation* if it is reflexive, transitive, strictly monotonic, impartial and satisfies the principle of transfers.

Here after, we shall denote an equitable preference relation on the outcome space Y by $\preceq_e$.

Efficient solutions for a MCOP are defined in terms of the Pareto preference relation. Solutions for MCOP defined in terms of equitable preference relations are called *equitably efficient solutions.*

Definition 3.15. Let y^1, $y^2 \in \mathbb{R}^m$ be two attainable outcomes. y^1 is said to *equitably dominate* y^2 if and only if $y^1 \prec_e y^2$ for all equitable preference relations, $\preceq_e$, where $y^1 \prec_e y^2$ if and only if $y^1 \preceq_e y^2$ and not $y^2 \preceq_e y^1$.

An outcome vector y is equitably nondominated if and only if there does not exist another outcome vector y' such that y' equitably dominates y. Analogously, a feasible decision $x^0 \in X$ is called an equitably efficient solution of the MCOP if and only if there does not exist $x \in X$ such that $f(x) \prec_e f(x^0)$ for all equitable preference relations.

Similar to efficiency, to make it practical, equitable efficiency too can be defined in terms of vector inequalities. In order to do that, we define certain mappings.

Definition 3.16. Let $\Theta : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a mapping defined as

$$\Theta(y) = (\theta_1(y), ..., \theta_m(y)), \text{ where } \theta_1(y) \geq \theta_2(y) \geq ...., \geq \theta_m(y)$$

and

$$\theta_i(y) = y_{\pi(i)}, \text{ where } \pi \text{ is some permutation of the set } I = \{1, ..., m\}.$$

Note that Θ is an *ordering map* that sorts the components of y in a non-increasing order.

Definition 3.17. Define the *cumulative ordering map* on $\mathbb{R}^m, \bar{\Theta} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ as

$$\bar{\Theta}(y) = (\bar{\theta}_1(y), ..., \bar{\theta}_m(y)), \text{ where}$$
$$\bar{\theta}_i(y) = \sum_{j=1}^{i} \theta_j(y), \text{ for } i = 1, ..., m.$$

Note that the i^{th} coefficient of the vector $\bar{\Theta}(y)$ represents the sum of i largest components of the vector y.

Defining equitable efficiency from the preference point of view is too abstract. As mentioned in the previous section, equitable efficiency is made operational by relating the preference relation to vector inequalities in the outcome space.

Theorem 3.1. *(Kostreva et al., 2004). Let y^1, y^2 be two outcome vectors in $\mathbb{R}^m$. y^1 equitably dominates y^2 if and only if $\bar{\Theta}(y^1) \leq \bar{\Theta}(y^2)$.*

By Definition 3.17 and Theorem 3.1, it is clear that finding equitably efficient solutions of MCOP is equivalent to finding efficient solutions of the problem

$$\text{(COMCOP): } \textit{minimize} \;\; (\bar{\theta}_1(f(x), ..., \bar{\theta}_m(f(x))$$
$$\textit{subject to} \;\; x \in X \subseteq \mathbb{R}^n,$$

or

$$\text{(COMCOP): } \textit{minimize} \;\; \bar{\Theta}(f(x))$$
$$\textit{subject to} \;\; x \in X \subseteq \mathbb{R}^n.$$

We refer to either of the above problem as COMCOP, the cumulatively ordered multi-criteria optimization problem.

Corollary 3.1. *(Kostreva et al., 2004). A feasible solution $x \in X$ is an equitably efficient solution of MCOP if and only if it is an efficient solution of COMCOP.*

In the following chapter we develop characterizations of equitably efficient solutions of MCOP in terms of solutions of single objective optimization problems.

CHAPTER 4

SOME CHARACTERIZATIONS OF EQUITABLY EFFICIENT SOLUTIONS

Solving any multi-criteria optimization problem implies finding a subset of the efficient decisions. While there is extensive amount of literature on finding efficient solutions of such problems, the literature on finding equitably efficient solutions of such problems is not very extensive. Kostreva and Ogryczak (1999) develop a weighting method of finding equitably efficient solutions of a linear MCOP.

In this chapter we develop some scalarization-based methods to generate equitably efficient solutions. It is well known that a solution of a minimax optimization problem is, in general, only weakly efficient for the corresponding multi-criteria problem(Ehrgott, 2005). We show that under uniqueness condition, the solution is equitably efficient. We present the two-phase minimax method of finding equitably efficient solutions regardless of uniqueness.

Questions regarding existence of solutions to any optimization problem are of primary concern. We prove conditions under which equitably efficient solutions may not exist for linear multi-criteria problems.

4.1 Scalarizations and Equitable Efficiency

Scalarization is one of the most common approaches used to solve a MCOP. As discussed earlier, finding an equitably efficient solution of a MCOP is equivalent to finding an efficient solution to its related COMCOP. Scalarizing functions are used to transform a given MCOP into a single criterion optimization problem, here after referred to as SCOP, by aggregating the criteria of a MCOP into a single criterion. Efficient solutions of the MCOP are then studied in terms of the optimal solution(s) of the SCOP.

In order to guarantee consistency of the aggregated problem with minimization of all the criteria of the MCOP, the scalarizing function must be strictly increasing coordinatewise.

Definition 4.1. (Ehrgott, 2005). Let $Y \subseteq \mathbb{R}^m$. Any function $g : Y \to \mathbb{R}$ is called a *scalarizing function* for Y. Let $y^1, y^2 \in Y$. The scalarizing function g is *strongly increasing* if

$$y^1 \leq y^2 \Longrightarrow g(y^1) < g(y^2).$$

Note that strongly increasing functions are strictly increasing componentwise.

Definition 4.2. Let the MCOP with feasible set $X \in \mathbb{R}^n$, criteria functions $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$ and the strongly increasing scalarizing function g be given. The SCOP associated with the MCOP is given by

$$\text{(SCOP)} : \textit{minimize} \ \{g(y) : y \in Y = f(X)\},$$

where $f = (f_1, \ldots, f_m)$.

As the feasible set Y in the criteria space is not given explicitly, one deals with the above problem with feasible decisions in the decision space. In such a case, the above problem is equivalent to

$$\text{(SCOP)} : \textit{minimize} \ \{g(f(x)) : x \in X\}.$$

Definition 4.3. Let the SCOP associated with the MCOP be given. A feasible decision $x^0 \in X$ is said to be an *optimal solution* of SCOP if

$$g(f(x^0) \leq g(f(x) \ \text{ for all } x \in X.$$

Let g be a strongly increasing scalarizing function. If $\hat{x} \in X$ is an optimal solution of SCOP, then $\hat{x}$ is an efficient solution of MCOPEhrgott and Wiecek (2005).

4.1.1 Weighting-sum scalarization

The weighting sum method is one of the most common ways of finding efficient solutions of MCOP. Details of the method can be found in Geoffrion (1968).

Definition 4.4. The weighting sum scalarization of the MCOP is defined as

$$\text{W(w): } \textit{minimize} \ \sum_{i=1}^{m} w_i f_i(x)$$
$$\textit{subject to} \ \ x \in X,$$

where $w \in \mathbb{R}^m_{\geq}$ is any given weighting vector.

Note that in this case the scalarizing function g is given by

$$g(f(x)) = \sum_{i=1}^{m} w_i f_i(x).$$

Analogously, the weighting method for the COMCOP is given by

$$\text{W(w): } \textit{minimize} \ \sum_{i=1}^{m} w_i \bar{\theta}_i(f(x))$$
$$\textit{subject to} \ \ x \in X,$$

where $w \in \mathbb{R}^m_{\geq}$ is any given weighting vector.

Due to Definitions (3.16) and (3.17) of Θ and $\bar{\Theta}$ respectively, the above problem is equivalent to

$$P(\lambda): \quad \textit{minimize} \ \sum_{i=1}^{m} \lambda_i \theta_i(f(x))$$
$$\textit{subject to} \ \ x \in X,$$

where $\lambda_i = \sum_{j=1}^{i} w_i$ for $i = 1, \ldots, m$.

Proposition 4.1. *(Kostreva et al., 2004). For any sequence of strictly decreasing and positive weights* $\{\lambda_i\}_{i=1}^{m}$*, each optimal solution of* $P(\lambda)$ *is an equitably efficient solution of the MCOP*

Proposition 4.2. *(Kostreva and Ogryczak, 1999). Suppose the criteria function* $f_i, i = 1, \ldots, m$ *are linear and the feasible set* X *of MCOP is defined by a system of inequalities and equalities. A feasible solution* x^0 *is equitably efficient if and only if, there exists a sequence of strictly decreasing and positive weights* $\lambda_1 > \lambda_2 > \ldots > \lambda_m > 0$*, such that* x^0 *is an optimal solution of problem* $P(\lambda)$.

Proposition 4.3. *(Kostreva and Ogryczak, 1999). Suppose the multi-criteria optimization problem is a linear one, in which the criteria functions are linear and the feasible set* X *is defined by a system of linear inequalities. If the problem has an efficient solution, it has an equitably efficient solution.*

4.1.2 Minimax scalarization

One of the most common approaches to building a single-criterion optimization problem from a given MCOP is by using some kind of norm function as the scalarizing function. Based on certain properties of the norm defining the scalarization, we are able to generate a weekly efficient or an efficient decision for the MCOP. The minimax scalarization corresponds to the l_p norm of a vector when $p \longrightarrow \infty$.

Definition 4.5. For the MCOP, its associated min-max scalarization is defined as

$$\text{(MINIMAX): } \textit{minimize} \quad \max_{i=1,\ldots,m} f_i(x)$$
$$\textit{subject to} \quad x \in X.$$

In case the MINIMAX problem has alternate optimal solutions, one of these solutions is efficient. In case the solution is unique, it is efficient for the MCOP, Kouvelis and Yu (1997). We show that in case of an unique optimal solution for the MINIMAX problem, the solution is equitably efficient for the MCOP.

Proposition 4.4. *Let x^* be the unique optimal solution of MINIMAX, then x^* is an equitably efficient solution of MCOP.*

Proof. : Let $z = \max_{i=1,\ldots,m} f_i(x)$, then problem MINIMAX is equivalent to

$$(MINIMAX) : minimize\ \{z \,:\, x \in X,\ f_i(x) \leq z,\ i = 1, \ldots, m\}.$$

Let (x^*, z^*) uniquely solve MINIMAX, then

$$z^* = minimize\ \{z \,:\, x \in X,\ f_i(x) \leq z,\ i = 1, \ldots, m\}.$$

Hence

$$f_i(x^*) \leq z^* \text{ for all } i = 1, \ldots, m \tag{4.1}$$

and

$$f_j(x^*) = z^* \text{ for some } j \in \{1, \ldots, m\}.$$

Assume x^* is not an equitably efficient solution of MCOP, then there exists some x^0 in X, $x^0 \neq x^*$, such that

$$\bar{\theta}_i(f(x^0)) \leq \bar{\theta}_i(f(x^*))$$

for all $i = 1, \ldots, m$ with strict inequality for at least one i.

In particular, consider

$$\bar{\theta}_1(f(x^0)) \leq \bar{\theta}_1(f(x^*)).$$

If

$$\bar{\theta}_1(f(x^0)) = \bar{\theta}_1(f(x^*)),$$

then by definitions (3.16) and (3.17) of θ_i and $\bar{\theta}_i$ respectively and from (4.1), we get

$$\begin{aligned} \max_{i=1,\dots,m} f_i(x^0) &= \max_{i=1,\dots,m} f_i(x^*) \\ &\leq z^* \\ &= \min z \\ &\leq z. \end{aligned} \tag{4.2}$$

If

$$\bar{\theta}_1(f(x^0) < \bar{\theta}_1 f(x^*)$$

then

$$\begin{aligned} \max_{i=1,\dots,m} f_i(x^0) &< \max_{i=1,\dots,m} f_i(x^*) \\ &\leq z^* \\ &= \min z \\ &\leq z. \end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we have

$$\begin{aligned} \max_{i=1,\dots,m} f_i(x^0) &\leq z^* \\ &\leq z. \end{aligned}$$

But

$$\max_{i=1,\dots,m} f_i(x^0) \leq z^* \Longrightarrow f_i(x^0) \leq z \tag{4.4}$$

for all $i = 1, \dots, m$ and $x^0 \in X$.

equation (4.4) implies that x^0 is feasible for MINIMAX.

Let z^0 be the value of MINIMAX at decision x^0, then

$$z^0 = \min z,$$

where

$$f_i(x^0) \leq z^0$$

for all $i = 1, \ldots, m$ and

$$f_j(x^0) = z^0 \tag{4.5}$$

for some $j \in \{1, \ldots, m\}$.

From (4.4) and (4.5), we get

$$\begin{aligned} f_j(x^0) &= z^0 \\ &\leq \max_{i=1,\ldots,m} f_i(x^0) \\ &\leq z^*. \end{aligned}$$

$z^0 \leq z^*$ implies that x^* is not the unique optimal solution of MINIMAX, a contradiction. Hence x^* is an equitably efficient solution of the MCOP.

□

Proposition (4.4) can be used to find an equitably efficient solution provided the solution of the related minimax problem is known to be unique. An instance, where uniqueness is guaranteed is when the objective functions of the multi-criteria optimization problem are all strictly convex and the feasible region is convex. For a bicriteria problem(Yu, 1973) derives conditions for Pareto optimality in the criteria space under which the minimax problem has a unique solution. In general, however, uniqueness can not be verified(Marler and Arora, 2004). Regardless of uniqueness, in the proposition that follows, we prove that the two-phase method(Kouvelis and Yu, 1997) can be utilized to find an equitably efficient solution of the related multi-criteria problem. We prove a lemma which is needed in the proof of the proposition that follows.

Lemma 4.1. *Let* $y^1, y^2 \in \mathbb{R}^m$*. If* $\max_{i=1,\ldots,m} y_i^1 = \max_{i=1,\ldots,m} y_i^2$ *and* $\bar{\Theta}(y^1) \leq \bar{\Theta}(y^2)$*, then* $\bar{\theta}_m(y^1) < \bar{\theta}_m(y^2)$*.*

Proof. : Without loss of generality, let

$$\begin{aligned} &y^1 = (y_1^1, \ldots, y_m^1) \text{ and } y^2 = (y_1^2, \ldots, y_m^2) \text{ where} \\ &y_1^1 \geq \ldots \geq y_m^1 \text{ and } y_1^2 \geq \ldots \geq y_m^2. \end{aligned}$$

By assumption

$$y_1^1 = \max_{i=1,\ldots,m} y_i^1 = y_1^2 = \max_{i=1,\ldots,m} y_i^2. \tag{4.6}$$

If $j = 2$, then

$$\begin{aligned} \bar{\theta}_j(y^1) \leq \bar{\theta}_j(y^2) &\implies y_1^1 + y_2^1 \leq y_1^2 + y_2^2 \\ &\implies y_2^1 \leq y_2^2, \text{ by } (4.6). \end{aligned}$$

Proceeding inductively, it is clear that for every $j > 1$ the inequalities

$$\bar{\theta}_j(y^1) \leq \bar{\theta}_j(y^2) \implies y_j^1 \leq y_j^2$$

and

$$\bar{\theta}_k(y^1) < \bar{\theta}_k(y^2) \text{ for some } k \in \{2, \ldots, m\}$$

imply that $y_k^1 < y_k^2$.

The arguments presented above show that under the conditions stated in the lemma

$$\bar{\theta}_m(y^1) = \sum_{i=1}^{m} y_i^1 < \sum_{i=1}^{m} y_i^2 = \bar{\theta}_m(y^2)$$

□

Proposition 4.5. *Let $\tilde{z}$ be the optimal objective value of MINIMAX problem. Consider the problem*

$$\begin{aligned} \textit{(SUM): minimize} \quad & \sum_{i=1}^{m} f_i(x) \\ \textit{subject to} \quad & f_i(x) \le \tilde{z} \\ & x \in X. \end{aligned}$$

If x^* is an optimal solution of problem SUM, then x^* is an equitably efficient solution of MCOP.

Proof. Since $\tilde{z}$ is the optimal objective value of minimax,

$$f_i(x) \le \tilde{z}, \ \forall x \in X \text{ and } i = 1, \ldots, m.$$

$$f_i(x) \le \tilde{z} \Longrightarrow \max_{i=1,\ldots,m} f_i(x) \le \tilde{z}, \ \forall x \in X. \tag{4.7}$$

Let x^* be an optimal solution of SUM. Suppose x^* is not an equitably efficient solution of MCOP, then x^* is not an efficient solution of COMCOP and there exists some $x^0 \neq x^* \in X$ such that $\bar{\Theta}(f(x^0))$ dominates $\bar{\Theta}(f(x^*))$. Hence

$$\bar{\theta}_i(f(x^0)) \le \bar{\theta}_i(f(x^*)) \ \forall i \in \{1, \ldots, m\}. \tag{4.8}$$

In particular, for $i = 1$,

$$\begin{aligned} & \bar{\theta}_1(f(x^0)) \le \bar{\theta}_1(f(x^*)) \\ \Longrightarrow & \max_{i=1,\ldots,m} f_i(x^0) \le \max_{i=1,\ldots,m} f_i(x^*) \\ & \le \tilde{z}, \text{ by } (4.7). \end{aligned}$$

If

$$\max_{i=1,\ldots,m} f_i(x^0) < \max_{i=1,\ldots,m} f_i(x^*) \le \tilde{z}$$

then x^0 provides a smaller objective value to minimax problem than $\tilde{z}$, the optimal objective value. Hence

$$\max_{i=1,\dots,m} f_i(x^0) = \max_{i=1,\dots,} f_i(x^*) \leq \tilde{z}. \tag{4.9}$$

From (4.8), (4.9) and Lemma (4.1), we get

$$\begin{aligned} &\bar{\theta}_m(f(x^0)) < \bar{\theta}_m(f(x^*)) \\ \Longrightarrow &\sum_{i=1}^{m} f_i(x^0) < \sum_{i=1}^{m} f_i(x^*). \end{aligned} \tag{4.10}$$

Inequality (4.10) contradicts the optimality of x^* for SUM.

□

Corollary 4.1. *Every optimal solution of the problem*

$$\begin{aligned} &\textit{minimize} \quad \sum_{i=1}^{m} f_i(x) \\ &\textit{subject to} \quad f_i(x) \leq z^*,\ x \in X, \end{aligned}$$

where $z^* = \min_{x\in X} \max_{i=1,\dots,m} f_i(x)$, *is an equitably efficient solution of MCOP.*

Proof. Follows from Propositions 4.4 and 4.5. □

4.1.3 Benson scalarization

Benson (1978) scalarization is often used to investigate the efficiency of a given feasible decision. In this section we utilize Benson's scalarizations on the ordered objectives to characterize equitably efficient solutions of a multi-criteria optimization problem with the solution of single-criterion problem. Further, we also prove a result which shows that equitably efficient solutions to a multi-criteria problem may not exist under certain assumptions on the criteria functions and the feasible region.

Definition 4.6. Given any feasible decision $x^0 \in X$, Benson's scalarization of the MCOP is defined as

$$\text{(BENSON)} \ : maximize \sum_{i=1}^{m} \epsilon_i$$

$$subject\ to\ f_i(x^0) - f_i(x) = \epsilon_i,\ i = 1, \ldots, m$$

$$\epsilon_i \geq 0,\ x \in X.$$

As in Yager (1988), we call following multi-criteria optimization problem

$$\text{(OWA)} \ : \ minimize \ \{\theta_1(f(x)), \ldots, \theta_m(f(x))\}$$

$$subject\ to\ x \in X$$

where,

$$f = (f_1, \ldots, f_m) : \mathbb{R}^m \longrightarrow \mathbb{R}$$

and

$$\Theta = (\theta_1, \ldots, \theta_m) : \mathbb{R}^m \longrightarrow \mathbb{R},$$

as defined in Definition (3.16), is an ordering map on $\mathbb{R}^m$, OWA, the ordered weighted averaging problem.
Consider the scalarization of OWA given by

$$P(\epsilon) : maximize \sum_{i=1}^{m} \epsilon_i$$

$$subject\ to \quad \theta_i(f(x^0)) - \theta_i(f(x)) = \epsilon_i,\ i = 1, \ldots, m$$

$$\epsilon_i \geq 0,\ x \in X,$$

where x^0 is any given feasible decision in X and θ_i's are as defined in Definition (3.16).

Theorem 4.1. *$x^0 \in X$ is an equitably efficient solution of MCOP if and only if the optimal objective value of $P(\epsilon)$ is zero.*

Proof. : For $x^0 \in X$, $P(\epsilon)$ is feasible with $\epsilon = 0$.

$$\epsilon_i \geq 0, \ \forall i = 1, \ldots, m \Longrightarrow \sum_{i=1}^{m} \epsilon_i \geq 0.$$

Hence, the optimal objective value of $P(\epsilon) \geq 0$.

Let $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$ be such that the vector (x, ϵ) is feasible for $P(\epsilon)$. Feasibility of (x, ϵ) for $P(\epsilon)$ implies $\epsilon_i \geq 0$.

$$\epsilon_i \geq 0 \Longrightarrow \theta_i(f(x^0)) - \theta_i(f(x)) \geq 0, \ \forall i = 1, \ldots, m \tag{4.11}$$

and

$$max \sum_{i=1}^{m} \epsilon_i \geq 0. \tag{4.12}$$

Let the optimal objective value of $P(\epsilon)$ be zero, then, by nonnegativity of $\epsilon, \epsilon_i = 0$ for all $i = 1, \ldots, m$. Due to definition of $P(\epsilon)$,

$$\begin{aligned} \epsilon = 0 &\Longrightarrow \theta_i(f(x^0)) - \theta_i(f(x)) = 0, \text{ for all } i = 1, \ldots, m \\ &\Longrightarrow \theta_i(f(x^0)) = \theta_i(f(x)), \text{ for all } i = 1, \ldots, m. \end{aligned} \tag{4.13}$$

By Definition (3.17)

$$\bar{\theta}_i = \sum_{j=1}^{i} \theta_j, \text{ for } i = 1, \ldots, m.$$

Hence from (4.13), we get

$$\bar{\theta}_i(f(x^0))) = \bar{\theta}_i(f(x)) \text{ for all } i = 1, \ldots, m.$$

So, if there is some $x \in X$ such that $\bar{\theta}_i(f(x) \leqq \bar{\theta}_i(f(x^0))$, then $\bar{\theta}_i(f(x)) = \bar{\theta}_i(f(x^0))$ for all $i = 1, \ldots, m$.

Hence, by Definition (3.17), x^0 is an efficient solution of OWAP and by Corollary (3.17), x^0 is an equitably efficient solution of MCOP.

Conversely, let x^0 be an equitably efficient solution of MCOP. Let $\widehat{\epsilon}$ denote the optimal objective value of $P(\epsilon)$ attained at some optimal decision $\widehat{x}$. Suppose $\widehat{\epsilon} > 0$.

$$\widehat{\epsilon} > 0 \implies max \sum_{i=1}^{m} \epsilon_i > 0 \tag{4.14}$$

$$\implies \sum_{i=1}^{m} \epsilon_i > 0 \tag{4.15}$$

$$\implies \epsilon_i > 0 \text{ for at least one } i \in \{1, \ldots, m\} \tag{4.16}$$

and

$$\epsilon_i \geq 0 \text{ for all other } i. \tag{4.17}$$

Using the definition of $P(\epsilon)$ together with (4.16) and (4.17) we get

$$\theta_i(f(x^0)) - \theta_i(f(\widehat{x})) > 0 \text{ for at least one } i \in \{1, \ldots, m\}$$

and

$$\theta_j(f(x^0)) - \theta_j(f(\widehat{x})) \geq 0 \text{ for all other } i. \tag{4.18}$$

By definition of $\bar{\theta}_i$, $\bar{\theta}_i = \sum_{j=1}^{i} \theta_j$, hence from (4.18) we get

$$\bar{\theta}_i(f(x^0)) - \bar{\theta}_i(f(\widehat{x})) > 0 \text{ for at least one } i \in, \{1, \ldots, m\}$$

and

$$\bar{\theta}_i(f(x^0)) - \bar{\theta}_i(f(\widehat{x})) \geq 0 \text{ for all other } i. \tag{4.19}$$

Now, (4.19) implies that x^0 is not an efficient solution of COMCOP. Hence it is not an equitably efficient solution of MCOP, a contradiction.

□

4.1.4 Existence of equitably efficient

solutions for linear MCOPs

Question of existence of equitably efficient solutions is dealt withinKostreva and Ogryczak (1999). In this article, the authors show that for a linear multi-criteria optimization problem, if the set of efficient solutions is nonempty, then the set of equitably efficient solutions is nonempty too. In the theorem that follows, we show that under unboundedness of the Benson problem, the equitably efficient set of a linear multi-criteria problem is empty.

Theorem 4.2. *Suppose the criteria functions $f_i, i = 1, \ldots, m,$ of the MCOP are linear and the feasible set X is defined by a system of linear equalities and inequalities. If the optimal objective value of $P(\epsilon)$ is not finite, then MCOP has no equitably efficient solution.*

Proof. Since $P(\epsilon)$ is unbounded, for every real number $M \geq 0$, there exists some $x^M \in X$ such that

$$\epsilon_i = \theta_i(f(x^0)) - \theta_i(f(x^M)) \geq 0 \tag{4.20}$$

and

$$\sum_{i=1}^{m} \epsilon_i = \sum_{i=1}^{m} (\theta_i(f(x^0)) - \theta_i(f(x^M))) > M. \tag{4.21}$$

Suppose x' is an equitably efficient solution of MCOP, from Proposition (4.2) there exists a sequence of weights $\lambda_1 > \lambda_2 > \ldots \lambda_m > 0$ such that x' is an optimal solution of

$$\begin{array}{ll} minimize & \sum_{i=1}^{m} \lambda_i \, \theta_i(f(x)) \\ subject\, to & x \in X. \end{array} \tag{4.22}$$

For such a sequence of weights,

$$\sum_{i=1}^{m} \lambda_i \, (\theta_i(f(x)) - \theta_i(f(x'))) \geq 0 \text{ for all } x \in X. \tag{4.23}$$

Since $x^0 \in X$, (4.23) implies that

$$\sum_{i=1}^{m} \lambda_i \left(\theta_i(f(x^0)) - \theta_i(f(x'))\right) \geq 0. \tag{4.24}$$

Let $\lambda = \lambda_m$. Let $M' > 0$ be any fixed but arbitrary real number. Let $M = \frac{M'}{\lambda} \geq 0$. From (4.20) and (4.21), for such an M, there exists some $x^M \in X$ such that

$$\epsilon_i = \theta_i(f(x^0)) - \theta_i(f(x^M)) \geq 0 \tag{4.25}$$

and

$$\sum_{i=1}^{m} \epsilon_i = \sum_{i=1}^{m} \left(\theta_i(f(x^0)) - \theta_i(f(x^M))\right) > M. \tag{4.26}$$

If

$$\sum_{i=1}^{m} (\theta_i(f(x^0)) - \theta_i(f(x^M))) > M$$

then

$$\begin{aligned} \lambda \sum_{i=1}^{m} (\theta_i(f(x^0)) - \theta_i(f(x^M))) &> \lambda M \\ &= \frac{M'}{M} M \\ &= M' \\ &\geq 0. \end{aligned} \tag{4.27}$$

Since $\lambda = \lambda_m < \lambda_i$ for all $i = 1, \ldots, m-1$, (4.27) gives

$$\begin{aligned} \sum_{i=1}^{m} \lambda_i \left(\theta_i(f(x^0)) - \theta_i(f(x^M))\right) &> \lambda \sum_{i=1}^{m} \left(\theta_i(f(x^0)) - \theta_i(f(x^M))\right) \\ &> M' \\ &\geq 0. \end{aligned} \tag{4.28}$$

From (4.24)

$$\sum_{i=1}^{m} \lambda_i \left(\theta_i(f(x^0)) - \theta_i(f(x'))\right) \geq 0.$$

Since $M' \geq 0$ is arbitrary, setting

$$M' = \sum_{i=1}^{m} \lambda_i \left(\theta_i(f(x^0)) - \theta_i(f(x')\right) \geq 0 \text{ in } (4.28)$$

we get

$$\sum_{i=1}^{m} \lambda_i \left(\theta_i(f(x^0)) - \theta_i(f(x'))\right) < \sum_{i=1}^{m} \lambda_i \left(\theta_i(f(x^0)) - \theta_i(f(x^M))\right)$$
$$\Longrightarrow \sum_{i=1}^{m} -\lambda_i \, \theta_i(f(x')) < \sum_{i=1}^{m} -\lambda_i \, \theta_i(f(x^M)). \tag{4.29}$$

Because $\lambda_i > 0$ for all $i = 1, \ldots, m$, from (4.29) we get

$$\sum_{i=1}^{m} \lambda_i \, \theta_i(f(x^M)) < \sum_{i=1}^{m} \lambda_i \, \theta_i(f(x')). \tag{4.30}$$

Then, (4.30) implies that x' is not an optimal solution of (4.22), a contradiction. Hence MCOP does not have an equitably efficient solution. □

4.2 Equitable Scalarizations

Any optimal solution of SCOP is an efficient solution of MCOP if the scalarizing function is strongly increasing(Ehrgott and Wiecek, 2005). We show that scalarizing functions with additional properties may be used to find equitably efficient solutions.

Definition 4.7. Let

$$g : Y \subseteq \mathbb{R}^m \longrightarrow R$$

be a scalarizing function defined on Y.
g is said to be *symmetric* if $g(y_1, \ldots, y_m) = g(y_{\tau(1)}, \ldots, y_{\tau(m)})$
for every permutation τ of the index set $I = \{1, \ldots, m\}$.

Definition 4.8. Let $y = (y_1, \ldots, y_m)$ and $y_i > y_j$ for some $i, j \in \{1, \ldots, m\}$.
g is said to satisfy the *principle of transfers* if
$g(y_1, \ldots, y_i - \epsilon, \ldots, y_j + \epsilon, \ldots, y_m) < g(y_1, \ldots, y_i, \ldots, y_j, \ldots, y_m)$
for $0 < \epsilon < y_i - y_j$.

Definition 4.8 expresses the fact that transferring a small amount from a better to a worse component results in a more preferred outcome vector.

If g is symmetric and satisfies the principle of transfers, then it is called strictly *Schur-convex*. Scalarizations of MCOP defined in terms of functions that are strongly increasing and strictly Schur-convex are 'equitable scalarizations'. In case g is an equitable scalarization function, every optimal solution of SCOP is an equitably efficient solution of MCOP(Kostreva et al., 2004).

In this section, we present some equitable scalarizations and prove that they define strictly Schur-convex scalarizing functions.

4.2.1 Exponential Schur scalarization

For any $\mathbf{y} = (y_1, \ldots, y_m) \in Y$, consider the scalarizing function defined as

$$\text{(ESS): } ;\, g(y) = \sum_{i=1}^{m} e^{ay_i}$$

where

$$y_i = f_i(x),\ x \in X \text{ and } a > 0.$$

We show that ESS defines an equitable scalarization on Y.

1. Let $y^1 \leq y^2$, where

$$y^1 = (y_1^1, \ldots, y_m^1) \text{ and } y^2 = (y_1^2, \ldots, y_m^2) \in Y,$$
$$y^1 \leq y^2 \implies y_i^1 \leq y_i^2 \ \ \forall\, i \in \{1, \ldots, m\} \text{ and } y^1 \neq y^2. \tag{4.31}$$

Since $a > 0$, from (4.31) we get

$$ay_i^1 \leq ay_i^2 \text{ for all } i \in \{1, \ldots, m\} \text{ and}$$
$$ay_j^1 < ay_j^2 \text{ for some } j \in \{1, \ldots, m\}. \tag{4.32}$$

From (4.32)

$$e^{ay_i^1} \leq e^{ay_i^2} \text{ for all } i \in \{1, \dots, m\} \text{ and}$$
$$e^{ay_j^1} < e^{ay_j^2} \text{ for some } j \in \{1, \dots, m\}.$$

Summing (4.32) over the index set $\{1, \dots, m\}$, we get

$$\sum_{i=1}^{m} e^{ay_i^1} < \sum_{i=1}^{m} e^{ay_i^2}$$
$$\Longrightarrow g(y^1) < g(y^2). \tag{4.33}$$

Inequality (4.33) implies that g is strongly increasing.

2. Let T be the set of permutations of the set $I = \{1, \dots, m\}$. For any $\tau \in T$

$$\begin{aligned} g(y_{\tau(1)}, \dots, y_{\tau(m)}) &= \sum_{i=1}^{m} e^{ay_{\tau(i)}} \\ &= \sum_{i=1}^{m} e^{ay_i} \\ &= g(y_1, \dots, y_m). \end{aligned} \tag{4.34}$$

Inequality (4.34) shows that g is symmetric. Finally, we show that g satisfies the principle of transfers.

3. Let $y^1 = (y_1^1, \dots, y_m^1) \in Y$ where $y_i^1 > y_j^1$ for some i, $j \in \{1, \dots, m\}$.
Since g is symmetric, we can assume $y_i^1 = y_1^1$ and $y_j^1 = y_2^1$.
Let $0 < \epsilon < y_1^1 - y_2^1$.
Consider the vector $\boldsymbol{y}^2$ obtained by transferring ϵ from y_1^1 to y_2^1 keeping other components the same,

$$y^2 = (y_1^1 - \epsilon, y_2^1 + \epsilon, \dots, y_m^1).$$

By assumption, $0 < \epsilon < y_1^1 - y_2^1$. Since $a > 0$, $0 < a\epsilon < a(y_1^1 - y_2^1)$. Then

$$
\begin{aligned}
a(y_1^1 - y_2^1) > a\epsilon &\implies e^{a(y_1^1 - y_2^1)} > e^{a\epsilon} \\
&\implies \frac{e^{ay_1^1}}{e^{ay_2^1}} > e^{a\epsilon} \\
&\implies \frac{e^{ay_1^1}}{e^{a\epsilon}} > e^{ay_2^1}
\end{aligned} \tag{4.35}
$$

and

$$
\begin{aligned}
a\epsilon > 0 &\implies e^{a\epsilon} > e^0 = 1 \\
&\implies 1 - ea^{\epsilon} < 0.
\end{aligned} \tag{4.36}
$$

Multiplying (4.35) by $(1 - e^{a\epsilon)}$ and using (4.36) we get

$$
\begin{aligned}
&\frac{e^{ay_1^1}}{e^{a\epsilon}}(1 - e^{a\epsilon}) < e^{ay_2^1}(1 - e^{a\epsilon}) \\
\implies & e^{ay_1^1}(e^{-a\epsilon} - 1) < e^{ay_2^1}(1 - e^{a\epsilon}) \\
\implies & e^{a(y_1^1 - \epsilon)} - e^{ay_1^1} < e^{ay_2^1} - e^{a(y_2^1 + \epsilon)} \\
\implies & e^{a(y_1^1 - \epsilon)} + e^{a(y_2^1 + \epsilon)} < e^{ay_1^1} + e^{ay_2^1}.
\end{aligned} \tag{4.37}
$$

Inequality (4.37) implies

$$
\begin{aligned}
& e^{a(y_1^1 - \epsilon)} + e^{a(y_2^1 + \epsilon)} + \\
& (e^{ay_3^1} + \ldots + e^{ay_m^1}) < e^{ay_1^1} + e^{ay_2^1} + \\
& (e^{ay_3^1} + \ldots + e^{ay_m^1}) \\
\implies & g(y_1^1 - \epsilon, y_2^1 + \epsilon, \ldots, y_m^1) < g(y_1^1, y_2^1, \ldots, y_m^1) \\
\implies & g(y^2) < g(y^1).
\end{aligned} \tag{4.38}
$$

Inequality (4.38) implies that g satisfies the principle of transfers. Thus g is strongly increasing, symmetric and satisfies the principle of transfers. Hence, it defines an equitable scalarization on the outcome space $\boldsymbol{Y}$.

4.2.2 Squared-sum Schur scalarization

Let

$$a_i = \min_{x \in X} f_i(x)$$

and

$$a < \min_{i=1,\ldots,m} a_i.$$

For any $\mathbf{y} = (y_1, \ldots, y_m) \in Y$, consider the scalarizing function defined as

$$(SSSS): \quad g(y_1, \ldots, y_m) = \sum_{i=1}^{m} (y_i - a)^2, where \;\; y_i = f_i(x), \quad x \in X.$$

We prove that (SSSS) is an equitable scalarization.

1. Let $y^1 \leq y^2$ where

$$y^1 = (y_1^1, \ldots, y_m^1) \text{ and } y^2 = (y_1^2, \ldots, y_m^2) \in Y$$

and

$$y^1 \leq y^2 \implies y_i^1 \leq y_i^2 \;\; \forall\, i \in \{1, \ldots, m\} \text{ and } y^1 \neq y^2. \tag{4.39}$$

By assumption

$$a < y_i^1 \text{ and } a < y_i^2 \; \forall i \in \{1, \ldots, m\}. \tag{4.40}$$

Inequality (4.40) implies

$$0 < y_i^1 - a \text{ and } 0 < y_i^2 - a \; \forall i \in \{1, \ldots, m\}. \tag{4.41}$$

From (4.39) and (4.41)

$$(y_i^1 - a)^2 \leq (y_i^2 - a)^2 \; \forall i \in \{1, \ldots, m\}$$

and

$$(y_j^1 - a)^2 < (y_j^2 - a)^2 \text{ for some } j \in \{1, \ldots, m\}. \tag{4.42}$$

Summing both sides of (4.42) yields

$$\sum_{i=1}^{m}(y_i^1-a)^2 < \sum_{i=1}^{m}(y_i^2-a)^2$$
$$\Longrightarrow g(y_1^1,\ldots,y_m^1) < g(y_1^2,\ldots,y_m^2)$$
$$\Longrightarrow g(y^1) < g(y^2) \qquad (4.43)$$

Inequality (4.43) implies g is strongly increasing.

2. Let τ be the vector denoting any permutation of the index set $I=\{1,\ldots,m\}$, then

$$\begin{aligned} g(y_{\tau(1)},\ldots,y_{\tau(m)}) &= \sum_{i=1}^{m}(y_{\tau(i)}-a)^2 \\ &= \sum_{i=1}^{m}(y_i-a)^2 \\ &= g(y_1,\ldots,y_m). \end{aligned} \qquad (4.44)$$

Inequality (4.44) implies that g is symmetric.

3. Let $y^1 = (y_1^1,\ldots,y_m^1) \in Y$ where $y_i^1 > y_j^1$ for some $i,\ j \in \{1,\ldots,m\}$. Without loss of generality, let $i > j,\ i,j \in \{1,\ldots,m\}$ be such that $y_i^1 > y_j^1$. Let $0<\epsilon<y_i^1-y_j^1$ and y^2 be the vector obtained from y^1 by transferring ϵ from y_i^1 to y_j^1 keeping other components the same,

$$y^2 = (y_1^1,\ldots,y_i^1-\epsilon,\ldots,y_j^1+\epsilon,\ldots,y_m^1).$$

Note that $y_i^1-y_j^1 = -(y_j^1-y_i^1) = -((y_j^1-a)-(y_i^1-a))$. Then

$$y_i^1-y_j^1 = -((y_j^1-a)-(y_i^1-a) > \epsilon$$
$$\Longrightarrow -2((y_j^1-a)-(y_i^1-a)) > 2\epsilon.$$

Since $\epsilon > 0$, the above inequality implies

$$\begin{aligned} -2\epsilon((y_j^1 - a) - (y_i^1 - a)) &> 2\epsilon^2 \\ \Longrightarrow 2\epsilon^2 + 2\epsilon\{(y_j^1 - a) - (y_i^1 - a)\} &< 0 \\ \Longrightarrow 2\epsilon^2 + 2\epsilon(y_j^1 - a) - 2\epsilon(y_i^1 - a) &< 0. \end{aligned} \tag{4.45}$$

From (4.45)

$$\begin{aligned} 2\epsilon^2 + 2\epsilon(y_j^1 - a) - 2\epsilon(y_i^1 - a) \; + \\ (y_1^1 - a)^2 + \ldots + (y_m^1 - a)^2 \quad < \quad (y_1^1 - a)^2 + \ldots \\ +(y_m^1 - a)^2. \end{aligned} \tag{4.46}$$

Regrouping terms in (4.46), keeping together terms involving y_i^1 and y_j^1 gives

$$\begin{aligned} (y_1^1 - a)^2 + \ldots + (2\epsilon^2 + 2\epsilon(y_j^1 - a) - \\ 2\epsilon(y_i^1 - a) + (y_i^1 - a)^2 + (y_j^1 - a)^2) + \\ \ldots + (y_m^1 - a)^2 \quad &< \quad (y_1^1 - a)^2 + \ldots + (y_m^1 - a)^2 \\ \Longrightarrow \; (y_1^1 - a)^2 + \ldots + (y_j^1 - a + \epsilon)^2 + \\ (y_i^1 - a - \epsilon)^2 + \ldots + (y_m^1 - a)^2 \quad &< \quad (y_1^1 - a)^2 + \\ \ldots (y_m^1 - a)^2. \end{aligned} \tag{4.47}$$

Equivalently,

$$\begin{aligned} (y_1^1 - a)^2 + \ldots + (y_i^1 - \epsilon - a)^2 + \\ (y_j^1 + \epsilon - a)^2 + \ldots + (y_m^1 - a)^2 \quad < \\ (y_1^1 - a)^2 + \ldots (y_m^1 - a)^2 \\ \Longrightarrow g(y_1^1 - a, \ldots, y_i^1 - \epsilon - a, \ldots + \\ y_j^1 + \epsilon - a + \ldots + y_m^1 - a)^2 \quad < \\ g(y_1^1 - a, \ldots, y_m^1 - a). \end{aligned} \tag{4.48}$$

From (4.48)

$$g(y^2) \; < g(y^1), \tag{4.49}$$

g satisfies the principle of transfers.

Proposition 4.6. *If $a > 0$, then every optimal solution of*

$$\begin{array}{ll} \textit{minimize} & \sum_{i=1}^{m} e^{af_i(x)} \\ \textit{subject to} & x \in X \end{array}$$

is an equitably efficient solution of MCOP.

Proof. From section 4.2.1, $\sum_{i=1}^{m} e^{af_i(x)}$ is an equitable scalarization defined on the outcome space Y. Hence every optimal solution of

$$\textit{minimize} \sum_{i=1}^{m} e^{af_i(x)}, \text{ subject to } x \in X$$

is an equitably efficient solution of MCOP(Kostreva et al., 2004). □

Proposition 4.7. *Every optimal solution of*

$$\textit{minimize} \quad \sum_{i=1}^{m} (f_i(x) - a)^2 \tag{4.50}$$

$$\textit{subject to} \quad x \in X \tag{4.51}$$

where $a < \min_{i=1,\dots,m} a_i$ and $a_i = \min_{x \in X} f_i(x)$, is an equitably efficient solution of MCOP.

Proof. For a defined according to the conditions in the proposition, $\sum_{i=1}^{m} (f_i(x) - a)^2$ defines an equitable scalarization on the outcome space Y, section 4.2.2. Hence every optimal solution of

$$\textit{minimize} \sum_{i=1}^{m} (f_i(x) - a)^2, \quad \textit{subject to } x \in X$$

is an equitably efficient solution of MCOP(Kostreva et al., 2004). □

Stressing the relevance of equitably efficient solutions is the focus of the next chapter. In this chapter we present equitably efficient solutions in relation to solutions of matrix game and linear complementarity problems.

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CHAPTER 5

MATRIX GAMES, LINEAR COMPLEMENTARITY PROBLEMS AND EQUITABLE EFFICIENCY

Matrix games or two-person zero-sum games are games with only two participants in which one participant wins what the other loses. The first attempt to formalize a theory of such games was made by Borel (1921, 1924, 1927). Later a strong foundation of the theory was laid by Neumann (1928) who gave the celebrated "Minimax Theorem" for such games. Linear Complementarity problem, on the other hand, is a general problem which unifies linear and quadratic programs and bimatrix games(Murty, 1988).

In this chapter we present multi-criteria formulations of matrix games and linear complementarity problems and study the relationships between solutions of the latter two problems and equitably efficient solutions of the related multi-criteria problem . For the matrix game, we show that a solution of the game can be obtained by finding an equitably efficient solution of the multi-criteria problem. Solutions of the linear complementarity problem, on the other hand, can be identified as equitably efficient solutions that have specific values for the objective functions.

5.1 Matrix Games and Equitable Efficiency

A matrix game τ is defined by a real $m \times n$ matrix A along with the cartesian product $X \times Y$, where $X = \{x \in \mathbb{R}^{\mathbf{m}} \,|\, x_i \geq 0, \sum_{i=1}^{m} x_i = 1\}$ and $Y = \{y \in \mathbb{R}^{\mathbf{n}} \,|\, y_i \geq 0, \sum_{j=1}^{n} y_j = 1\}$. $x \in X$ is called a mixed strategy for the row player and $y \in Y$ a mixed strategy for the column player. For $i = 1, \ldots, m$, the mixed strategy with 1 at the i^{th} position and 0 elsewhere is called the i^{th} pure strategy for the row player. A similar definition of pure strategy holds for the column player.

Definition 5.1. If $(x, y) \in X \times Y$, then the payoff associated with the strategy (x, y) is given by $E(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j$.

Definition 5.2. A solution of the matrix game τ is a pair of mixed strategies $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m), \bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)$ and a real number v such that $xA\bar{y}^t \leq \bar{x}A\bar{y}^t = v \leq \bar{x}Ay^t$. $\bar{x}$ is called an optimal strategy for the row player and $\bar{y}$, an optimal strategy for the column player. v is called the value of the game.

Theorem 5.1. *(Chvátal, 2000). For any matrix game τ defined by the $m \times n$ matrix A, there exist optimal strategies x^*, y^* for the row and column players respectively such that*

$$\min_y x^* A y^t = \max_x xAy^{*t}$$

with the minimum taken over all $y \in Y$ and maximum taken over all $x \in X$.

As shown in Chvátal (2000), finding an optimal strategy to the matrix game is equivalent to solving the following linear programming problems:

$$\begin{aligned} (LP) \quad & maximize \ v \\ s.t. \ v - \sum_{i=1}^{m} x_i a_{ij} & \leq 0, \ j = 1, \ldots, n, \\ \sum_{i=1}^{m} x_i & = 1, \\ x_i & \geq 0, \ i = 1, \ldots, m \end{aligned}$$

and

$$\begin{aligned} (LD) \quad & minimize \ w \\ s.t. \ w - \sum_{j=1}^{n} a_{ij} y_j & \geq 0, \ i = 1, \ldots, m, \\ \sum_{j=1}^{n} y_j & = 1, \\ y_j & \geq 0, \ j = 1, \ldots, n. \end{aligned}$$

Note that LP and LD are duals of one another.

We formulate the matrix game problem as expressed by LP and LD as a biobjective problem and look at the relationship of solutions between the two problems. Toward this end, consider the biobjective optimization problem formulated as

$$\begin{aligned}
(BOP) \quad & minimize\{-v, w\} \\
s.t.\ v - \sum_{i=1}^{m} x_i a_{ij} & \leq 0\,,\ j = 1, \ldots, n, \\
w - \sum_{j=1}^{n} a_{ij} y_j & \geq 0\,,\ i = 1, \ldots, m, \\
\sum_{i=1}^{m} x_i & = 1, \\
x_i & \geq 0,\ i = 1, \ldots, m, \\
\sum_{j=1}^{n} y_j & = 1, \\
y_j & \geq 0,\ j = 1, \ldots, n.
\end{aligned}$$

Let χ denote the feasible set of the above BOP.

Theorem 5.2. *Let (x^*, v^*, y^*, w^*), where $(x^*, v^*) \in \mathbb{R}^{m+1}$ and $(y^*, w^*) \in \mathbb{R}^{n+1}$ solve the matrix game τ, then (x^*, v^*, y^*, w^*) is an equitably efficient solution of BOP.*

Proof. If (x^*, v^*, y^*, w^*) solves the matrix game A, then (x^*, v^*) solves LP and (y^*, w^*) solves LD.

If (x^*, v^*) solves LP, then by feasibility

$$v^* \leq \sum_{i=1}^{m} x_i^* a_{ij},\ \forall j = 1, \ldots, n, \tag{5.1}$$

$$\sum_{i=1}^{m} x_i^* = 1,\ x_i^* \geq 0,\ \forall i = 1, \ldots, m \tag{5.2}$$

and as (y^*, w^*) solves LD

$$w^* \geq \sum_{j=1}^{n} a_{ij} y_j^*,\ \forall i = 1, \ldots, m, \tag{5.3}$$

$$\sum_{j=1}^{n} y_j^* = 1,\ y_j^* \geq 0,\ \forall j = 1, \ldots, n. \tag{5.4}$$

Inequalities (5.1)-(5.4) imply that (x^*, v^*, y^*, w^*) is a feasible solution of BOP. Suppose (x^*, v^*, y^*, w^*) is not an equitably efficient solution of BOP, then, by Corollary

3.1, (x^*, v^*, y^*, w^*) is not an efficient solution of the problem

$$minimize\, \{\bar{\theta}_1(-v, w),\, \bar{\theta}_2(-v, w) \,|\, x \in \chi\}.$$

Hence, there exists some $(x, v, y, w) \in \chi$ such that $(-v, w)$ dominates $(-v^*, w^*)$ in the objective space.
If $(-v, w)$ dominates $(-v^*, w^*)$ then

$$\bar{\theta}_i(-v, w) \le \bar{\theta}_i(-v^*, w^*) \text{ for } i = 1, 2 \tag{5.5}$$

and

$$\bar{\theta}_j(-v, w) < \bar{\theta}_j(-v^*, w^*) \text{ for at least one } j \in \{1, 2\}. \tag{5.6}$$

By definition of $\bar{\theta}, \bar{\theta}_i(f(x)) = \sum_{j=1}^{i} \theta_j(f(x))$, where the vector $\theta(f(x))$ denotes the components of the vector $f(x)$ sorted in a nonincreasing order. Applying this definition to the vectors $(-v, w)$ and $(-v^*, w^*)$, we get

$$\bar{\theta}_1(-v, w) = max\{-v, w\}, \tag{5.7}$$

$$\bar{\theta}_1(-v^*, w^*) = max\{-v^*, w^*\}, \tag{5.8}$$

$$\bar{\theta}_2(-v, w) = -v + w \tag{5.9}$$

$$\bar{\theta}_2(-v^*, w^*) = -v^* + w^*. \tag{5.10}$$

Since (x^*, v^*, y^*, w^*) solves the matrix game, (x^*, v^*) and (y^*, w^*) solve LP and LD respectively. As LP and LD are dual to each other, by the strong duality theorem, $v^* = w^*$. With $v^* = w^*$, 5.10 gives $\bar{\theta}_2(-v^*, w^*) = 0$. If strict inequality in (5.6) holds for $j = 2$, then combining (5.6) and (5.10), we get

$$\bar{\theta}_2(-v, w) = -v + w < \bar{\theta}_2(-v^*, w^*) = 0$$

$$\implies w < v. \tag{5.11}$$

If (x, v, y, w) is feasible to BOP, then (x, v) and (y, w) are feasible to LP and LD respectively. So we have feasible solutions to two dual LPs such that the objective value of the min problem is strictly less than the objective value of the max problem.

This contradicts the weak duality theorem. Hence, strict inequality in (5.6) can not hold for $j = 2$.

If $j = 1$ in (5.6), then from (5.6)-(5.8), we get

$$\bar{\theta}_1(-v, w) = max\{-v, w\} < \bar{\theta}_1(-v^*, w^*) = max\{-v^*, w^*\}. \tag{5.12}$$

In (5.12), we have the following possibilities:

Case 1.

$$max\{-v, w\} = -v, \; max\{-v^*, w^*\} = -v^*, \text{then}$$
$$-v < -v^* \implies v > v^*.$$

We have a feasible point (x, v) of LP at which the objective value is strictly greater than the objective value at (x^*, v^*). This contradicts the optimality of (x^*, v^*) for LP.

Case 2.

$$max\{-v, w\} = -v, \; max\{-v^*, w^*\} = w^*, \text{then}$$
$$w \leq -v < w^* \implies w < w^*.$$

Now we have a feasible point (y, w) of LD at which the objective value is strictly less than the objective value at (y^*, w^*). This contradicts the optimality of (y^*, w^*) for LD.

Case 3.

$$max\{-v, w\} = w, \; max\{-v^*, w^*\} = -v^*, \text{then}$$
$$-v \leq w < -v^* \implies v > v^*.$$

Again we arrive at a contradiction to (x^*, v^*) being an optimal solution of LP.

Case 4.

$$max\{-v, w\} = w, \; max\{-v^*, w^*\} = w^*, \text{then}$$
$$-v \leq w < w^* \implies w < w^*.$$

The above contradicts the optimality of (y^*, w^*) for LD.

In all the possible cases, we arrive at contradictions. Hence, the inequality in 5.6 can not hold for $j = 1$ too and our assumption about (x^*, v^*, y^*, w^*) can not be true. $\square$

Theorem 5.3. *If (x^*, v^*, y^*, w^*) is an equitably efficient solution of BOP, then it solves the matrix game τ.*

Proof. Suppose (x^*, v^*, y^*, w^*) is an equitably efficient solution of BOP but is not a solution of the matrix game, then there exists some $(\tilde{x}, \tilde{v})$ feasible to LP and $(\tilde{y}, \tilde{w})$ feasible to LD respectively such that

$$-\tilde{v} \leq -v^* \text{ and } \tilde{w} \leq w^* \text{ with strict inequality for at least one of them.} \tag{5.13}$$

Inequality (5.13) implies that

$$-\tilde{v} + \tilde{w} < -v^* + w^*. \tag{5.14}$$

If $(\tilde{x}, \tilde{v})$ is feasible to LP and $(\tilde{y}, \tilde{w})$ is feasible to LD, then the point $(\tilde{x}, \tilde{v}, \tilde{y}, \tilde{w})$ is a feasible point of the BOP.

By definition of $\bar{\theta}$,

$$\bar{\theta}_1(-\tilde{v}, \tilde{w}) = \theta_1(-\tilde{v}, \tilde{w}) = \max\{-\tilde{v}, \tilde{w}\}, \tag{5.15}$$

$$\bar{\theta}_2(-\tilde{v}, \tilde{w}) = \theta_1(-\tilde{v}, \tilde{w}) + \theta_2(-\tilde{v}, \tilde{w}) = -\tilde{v} + \tilde{w}, \tag{5.16}$$

$$\bar{\theta}_1(-v^*, w^*) = \theta_1(-v^*, w^*) = \max\{-v^*, w^*\} \text{ and} \tag{5.17}$$

$$\bar{\theta}_2(-v^*, w^*) = \theta_1(-v^*, w^*) + \theta_2(-v^*, w^*) = -v^* + w^*. \tag{5.18}$$

Combining (5.16) and (5.18) with (5.14), we get

$$\bar{\theta}_2(-\tilde{v}, \tilde{w}) < \bar{\theta}_2(-v^*, w^*). \tag{5.19}$$

Consider the following possible cases for $\bar{\theta}_1(-\tilde{v},\tilde{w})$ and $\bar{\theta}_1(-v^*,w^*)$:

Case 1.

$$\bar{\theta}_1(-\tilde{v},\tilde{w}) = \max\{-\tilde{v},\tilde{w}\} = -\tilde{v}, \tag{5.20}$$

$$\bar{\theta}_1(-v^*,w^*) = \max\{-v^*,w^*\} = -v^*. \tag{5.21}$$

Inequalities (5.13), (5.20) and (5.21) together imply that

$$\bar{\theta}_1(-\tilde{v},\tilde{w}) \leq \bar{\theta}_1(-v^*,w^*). \tag{5.22}$$

Case 2.

$$\bar{\theta}_1(-\tilde{v},\tilde{w}) = \max\{-\tilde{v},\tilde{w}\} = -\tilde{v}, \tag{5.23}$$

$$\bar{\theta}_1(-v^*,w^*) = \max\{-v^*,w^*\} = w^*. \tag{5.24}$$

$$\max\{-v^*,w^*\} = w^* \Longrightarrow -v^* \leq w^*,$$

but from (5.13), $-\tilde{v} \leq -v^*$. Hence, $-\tilde{v} \leq -v^* \leq w^*$.

$$-\tilde{v} \leq w^* \Longrightarrow \bar{\theta}_1(-\tilde{v},\tilde{w}) \leq \bar{\theta}_1(-v^*,w^*). \tag{5.25}$$

Case 3.

$$\bar{\theta}_1(-\tilde{v},\tilde{w}) = \max\{-\tilde{v},\tilde{w}\} = \tilde{w}, \tag{5.26}$$

$$\bar{\theta}_1(-v^*,w^*) = \max\{-v^*,w^*\} = -v^*. \tag{5.27}$$

Inequality (5.26) implies $-\tilde{v} \leq \tilde{w}$ and inequality (5.27) implies $w^* \leq -v^*$. Combining these two inequalities with (5.13) we get

$$-\tilde{v} \leq \tilde{w} \leq w^* \leq -v^*.$$

$$\tilde{w} \leq -v^* \Longrightarrow \bar{\theta}_1(-\tilde{v},\tilde{w}) \leq \bar{\theta}_1(-v^*,w^*). \tag{5.28}$$

Case 4.

$$\bar{\theta}_1(-\tilde{v},\tilde{w}) = \max\{-\tilde{v},\tilde{w}\} = \tilde{w}, \tag{5.29}$$

$$\bar{\theta}_1(-v^*,w^*) = \max\{-v^*,w^*\} = w^*. \tag{5.30}$$

From (5.13), $\tilde{w} \leq w^*$, which together with 5.29 and 5.30 implies

$$\bar{\theta}_1(-\tilde{v}, \tilde{w}) \leq \bar{\theta}_1(-v^*, w^*). \tag{5.31}$$

From (5.22), (5.25), (5.28) and (5.31) we see that

$$\bar{\theta}_1(-\tilde{v}, \tilde{w}) \leq \bar{\theta}_1(-v^*, w^*) \tag{5.32}$$

for all possible choices of $\bar{\theta}_1(-\tilde{v}, \tilde{w})$ and $\bar{\theta}_1(-v^*, w^*)$. Finally, combining (5.19) and (5.32), we get

$$\bar{\theta}_i(-\tilde{v}, \tilde{w}) \leq \bar{\theta}_i(-v^*, w^*) \tag{5.33}$$

where strict inequality holds for $i = 2$.

Inequality (5.33) implies that the point (x^*, v^*, y^*, w^*) is dominated by the point $(\tilde{x}, \tilde{v}, \tilde{y}, \tilde{w})$ in the objective space. Hence (x^*, v^*, y^*, w^*) can not be an efficient point of the problem

$$minimize\,\{\bar{\theta}_1(-v, w), \bar{\theta}_2(-v, w) \mid x \in \chi\}$$

and equivalently, by Corollary 3.1, it can not be an equitably efficient solution of MOP, a contradiction.

Hence, our assumption that (x^*, v^*, y^*, w^*) does not solve the matrix game can not be true. □

5.2 Linear Complementarity Problems and Equitable Efficiency

Relationships between solutions of linear complementarity problems(LCP) and multiple criteria optimization problems(MOP) have been described in the works of several authors. In the literature, the first approach to unify the two seemingly different concepts seems to be that of Kostreva and Wiecek (1993) where the authors propose the concept of zero-efficient solutions to study LCPs. In(Isac et al., 1995), the authors propose multiple objective approximation to certain feasible but unsolvable LCPs. Ebiefung (1995) explores the connections between generalized LCPs and MOP.

In this section, we develop conditions by which solutions to a LCP can be identified as equitably efficient solutions to a related MOP. The results allow us to view zero efficient points as nothing else but equitably efficient solutions of MOP with particular objective values.

Given an $n \times n$ matrix M and an $n \times 1$ column vector q, $LCP(q, M)$ is to determine $x \in \mathbb{R}^n$ satisfying

$$
\begin{aligned}
& y = Mx + q \\
& y_i \geq 0 \\
& x_i \geq 0 \\
& y_i x_i = 0, \ \forall i = 1, \ldots, n.
\end{aligned}
$$

Associated with the LCP, we formulate the following multi-objective problem(MOP) as in Kostreva and Wiecek[1993].

$$
\begin{aligned}
\text{(MOP)} \qquad & \textit{minimize } \{y_1 x_1, \ldots, y_n x_n\} \\
& \textit{subject to } x \in X,
\end{aligned}
$$

where,

$$X = \{x \in \mathbb{R}^n | x \geq 0, y = Mx + q \geq 0\}.$$

In addition, let,

$$f_i(x) = y_i x_i, \ \forall i = 1, \ldots, n.$$

Remark 5.1. Note that in the above MOP formulation, the objective functions are uniform in the sense that they have the same form. Each function can be thought of as expressing the same physical outcome. For such a problem, equitably efficient solutions are more justified than the efficient solutions.

In(Kostreva et al., 2004) the authors show that if $\bar{x}$ is an efficient solution of the multiple criteria problem

$$\textit{minimize} \{f_1(x), \ldots, f_m(x), \ s.t. \, x \, \in \, X\}$$

and $s_i \,:\, \mathrm{R} \longrightarrow \mathrm{R}$ are strictly increasing scaling functions satisfying the requirement

$$s_1(f_1(\bar{x})) \,=\, \ldots \,=\, s_m(f_m(\bar{x})),$$

then $\bar{x}$ is an equitably efficient solution of the scaled problem

$$minimize\,\{s_1(f_1(x)), \ldots, s_m(f_m(x)),\ s.t.\, x \,\in\, X\}.$$

We prove a similar result relating efficient and equitably efficient solutions of a general MOP without imposing any conditions on the criteria functions. The result will later be used to prove some other theorems in the chapter.

Theorem 5.4. *If there exists an efficient solution $x^0 \in X$ such that $f_1(x^0) = f_2(x^0) = \ldots = f_m(x^0)$, then x^0 is an equitably efficient solution of MCOP.*

Proof. Let $f_1(x^0) \,=\, f_2(x^0) \,=\, \ldots \,=\, f_m(x^0) \,=\, a$. If $x^0 \in X$ is an efficient solution of MCOP then

$$f(x^0) \,=\, (f_1(x^0),\, \ldots,\, f_m(x^0)) \,=\, (a,\, \ldots,\, a) \tag{5.34}$$

is a nondominated solution in the criteria space. Suppose x^0 is not an equitably efficient solution of MCOP, then by Theorem 3.1, x^0 is not an efficient solution of the multiple criteria problem

$$minimize\,\{\,\bar{\Theta}(f(x))\, s.\,t.\; x \,\in\, X\},$$

where

$$\begin{aligned}
\bar{\Theta}(f(x)) \,&=\, (\bar{\theta}_1(f(x)), \ldots, \bar{\theta}_m(f(x))),\\
\bar{\theta}_i(f(x)) \,&=\, \sum_{j=1}^{i} \theta(f(x)),\ \forall i \,=\, 1, \ldots, m;\ x \,\in\, X,\\
\theta_1(f(x)) &\geq \theta_2(f(x)) \geq \ldots \geq \theta_m(f(x)) \text{ and}\\
\theta_i(f(x)) \,&=\, f_{\tau(i)}(x) \text{ for some permutation } \tau \text{ of } \{1, 2, \ldots, m\}.
\end{aligned}$$

Hence, there exists some $x \in X$, $x \neq x^0$ such that $\bar{\Theta}(f(x))$ dominates $\bar{\Theta}(f(x^0)$. If $\bar{\Theta}(f(x))$ dominates $\bar{\Theta}(f(x^0))$ then

$$\begin{aligned}
&\bar{\theta}_i(f(x)) \leq \bar{\theta}_i(f(x^0)) \, \forall \, i = 1, 2, \ldots, m \text{ and} \\
&\bar{\theta}_j(f(x)) < \bar{\theta}_j(f(x^0)) \text{ for at least one } j \in \{1, \ldots, m\}. \qquad (5.35)
\end{aligned}$$

By Definition 3.16,

$$\theta_1(f(x)) \geq \ldots \geq \theta_m(f(x)),$$

where

$$\theta_i(f(x)) = f_{\tau(i)}(x), \text{ for some permutation } \tau \text{ of the set } \{1, \ldots, m\}.$$

$$f_i(x^0) = a \Longrightarrow \theta_i(f(x^0)) = \theta_i(a, a, \ldots, a) = a, \; \forall i = 1, \ldots, m.$$

By Definition 3.17,

$$\begin{aligned}
\bar{\theta}_i(f(x^0)) &= \sum_{j=1}^{i} \theta_j(f(x^0)) \\
&= ia, \; \forall i = 1, \ldots, m. \qquad (5.36)
\end{aligned}$$

From (5.35) and (5.36) we get

$$\begin{aligned}
&\bar{\theta}_i(f(x)) \leq ia \text{ for all } i = 1, \ldots, m \text{ and} \\
&\bar{\theta}_j(f(x)) < ja \text{ for at least one } j \in \{1, \ldots, m\}. \qquad (5.37)
\end{aligned}$$

In particular, for $i = 1$, inequality (5.37) gives

$$\bar{\theta}_1(f(x)) = \theta_1(f(x)) \leq a, \qquad (5.38)$$

but

$$\theta_1(f(x)) = \max_{i=1,\ldots,m} f(x).$$

Hence,

$$\max_{i=1,\ldots,m} f_i(x) \le a.$$

$$\max_{i=1,\ldots,m} f_i(x) \le a \Longrightarrow f_i(x) \le a \text{ for all } i = 1,\ldots,m. \tag{5.39}$$

By assumption of the theorem

$$f_i(x^0) = a \text{ for all } i = 1,\ldots,m,$$

so from inequality (5.39) we get

$$f_i(x) \le f_i(x^0) \text{ for all } i = 1,\ldots,m. \tag{5.40}$$

From inequality (5.35)

$$\begin{aligned}\sum_{j=1}^{i} \theta_j(f(x)) &= \bar{\theta}_i(f(x)) \\ &< \bar{\theta}_i(f(x^0)) \\ &= ia \text{ for at least one } i \in \{1,\ldots,m\},\end{aligned}$$

which is possible only if

$$f_i(x) < f_i(x^0) \text{ for at least one } i \text{ in inequality 5.40.} \tag{5.41}$$

Combining inequalities 5.40 and 5.41, we get

$$\begin{aligned} f_i(x) &\le f_i(x^0) \text{ for all } i \in \{1,\ldots,m\} \text{ and} \\ f_j(x) &< f_j(x^0) \text{ for at least one } j \in \{1,\ldots,m\}. \end{aligned} \tag{5.42}$$

Inequality (5.42) implies that the point $f(x)$ dominates $f(x^0)$, a contradiction. Hence our assumption that x^0 is not an equitably efficient solution can not be true. □

Theorem 5.5. *If x^0 solves LCP then x^0 is an equitably efficient solution of MOP with $f_i(x^0) = 0$, $\forall i = 1, \ldots, n$.*

Proof. If x^0 solves LCP, then

$$x_i^0 \geq 0,\ y_i^0 \geq 0 \text{ and} \tag{5.43}$$

$$y_i^0 x_i^0 = 0,\ \forall i = 1, \ldots, n,\ \text{ where} \tag{5.44}$$

$$y_i = M_i x^0 + q_i,\ M_i \text{ being the } i^{th} \text{ row of the matrix } M.$$

Inequalities (5.43) and (5.44) imply that x^0 is a feasible solution of MOP with criteria function values $f_i(x^0) = 0$, for all $i = 1, \ldots, m$.
We show that x^0 is an efficient solution of MOP.
Suppose x^0 is not an efficient solution of MOP, then $\exists$ some $x \neq x^0 \in X$ such that

$$f_i(x) \leq f_i(x^0) = 0,\ \forall i \in \{1, \ldots, n\} \text{ and}$$

$$f_j(x) < f_j(x^0) = 0 \text{ for at least one } j \in \{1, \ldots, n\}. \tag{5.45}$$

$$x \in X \implies x_i \geq 0,\ y_i \geq 0.$$

$$x_i \geq 0,\ y_i \geq 0 \implies f_i(x) = y_i x_i \geq 0,\ \forall i = 1, \ldots, n. \tag{5.46}$$

Inequalities (5.45) and (5.46) can not hold simultaneously. Hence our assumption that x^0 is not efficient can not be true.
x^0 is an efficient solution of MOP with $f_i(x) = 0$, $\forall i = 1, \ldots, n$; hence by Theorem 5.1, x^0 is an equitably efficient solution of MOP. □

Theorem 5.6. *If x^0 is a feasible solution of MOP with $f_i(x^0) = 0, \forall i = 1, \ldots, n$, then x^0 is an equitably efficient solution of MOP and solves LCP.*

Proof. Let $x^0 \in X$ be such that $f_i(x^0) = y_i^0 x_i^0 = 0$, $\forall i = 1, \ldots, n$.
Suppose x^0 is not equitably efficient for MOP, then x^0 is not an efficient solution of the cumulatively ordered multiple criteria problem

$$minimize\, \{\bar{\theta}_1(f(x)), \ldots, \bar{\theta}_n(f(x)),\ s.t.\, x \in X\}.$$

Hence, there exists some $x \neq x^0 \in X$ such that

$$\bar{\theta}_i(f(x)) \leq \bar{\theta}_i(f(x^0)) \ \forall i \in \{1, \ldots, n\} \text{ and} \tag{5.47}$$

$$\bar{\theta}_j(f(x)) < \bar{\theta}_j(f(x^0)) \text{ for at least one } j \in \{1, \ldots, n\}. \tag{5.48}$$

$$\begin{aligned} f_i(x^0) = 0 &\implies \theta_i(f(x^0)) = 0 \,\forall\, i = 1, \ldots, n \\ &\implies \bar{\theta}_i(f(x^0)) = 0 \,\forall\, i = 1, \ldots, n. \end{aligned} \tag{5.49}$$

Substituting the above values of $\bar{\theta}_i(f(x^0))$ from inequality (5.49) in inequalities (5.47) and (5.48), we get

$$\begin{aligned} &\bar{\theta}_i(f(x)) \leq 0 \,\forall i \in \{1, \ldots, n\} \text{ and} \\ &\bar{\theta}_j(f(x)) < 0 \text{ for at least one } j \in \{1, \ldots, n\}. \end{aligned} \tag{5.50}$$

$$\begin{aligned} x \in X &\implies x_i \geq 0,\ y_i \geq 0, \\ x_i \geq 0, y_i \geq 0 &\implies f_i(x) = y_i x_i \geq 0,\ \forall i = 1, \ldots, n, \\ f_i(x) \geq 0 &\implies \theta_i(f(x)) \geq 0 \implies \bar{\theta}_i(f(x)) \geq 0 \,\forall i = 1, \ldots, n. \end{aligned} \tag{5.51}$$

Inequalities (5.50) and (5.51) can not hold simultaneously. Hence, x^0 is an efficient solution of the cumulatively ordered multiple criteria problem, which by Theorem 3.1 is an equitably efficient solution of the MOP.

Alternatively, using the definition of efficient solutions, it can be easily verified that any $x^0 \in X$ for which $f(x^0) = 0$, is efficient.

This, coupled with the fact that all the criteria values are equal, namely 0, proves that x^0 is an equitably efficient solution of MOP.

Since $x \in X$, $x_i \geq 0$, $y_i \geq 0$, $y_i x_i = 0$, where $y_i = M_i x + q_i, \forall i = 1, \ldots, n$, x^0 solves the LCP. □

Remark 5.2. From Theorems 5.5 and 5.6 it is clear that any feasible solution of MOP at which all the objective values attain the value zero, is equitably efficient.

Corollary 5.1. *$x^0 \in X$ is an equitably efficient solution with $f_i(x^0) = 0, \forall i = 1, \ldots, n$, if and only if x^0 solves LCP.*

Proof. The proof follows from Theorems 5.5, 5.6 and Remark 5.2. □

Remark 5.3. In general, the set of equitably efficient solutions of a MOP is a proper subset of the set of equitably efficient solutions of the MOP. This follows from the fact that equitably efficient solutions are obtained by imposing additional conditions on the Pareto preference that identifies efficient solutions. For the MOP related to the LCP, the set of equitably efficient solutions, if there exists one, is the same as the set of efficient solutions.

Corollary 5.2. *If LCP(q,M) has a solution, then the set of efficient solutions of MOP and the set of equitably efficient solutions of MOP are non-empty and identical. If x is in either of these sets, then $f_i(x) = 0,\ i = 1, \ldots, n$.*

Proof. Follows from Theorem 5.5 and Remark 5.3. □

The following two chapters demonstrate equitably efficient solutions by applying the concept to problems as diverse as linear regression analysis and asset allocation.

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CHAPTER 6

EQUITABLE EFFICIENCY AND REGRESSION ANALYSIS

The problem of parameter estimation in regression analysis is well known and widely studied. Regression analysis models the relationship between a response or the dependent variable, usually named y and one or more predictor or the independent variables, usually named x's. Linear regression assumes the best estimate of the response variable as a linear function of some unknown parameters, the regression parameters, while in nonlinear models, relationship between variables to be analyzed is nonlinear in the parameters. Deviation between the observed or the actual value of the response variable and the value obtained from the regression function is called the error. The goal of regression analysis is to choose the regression parameters so as to minimize some function of these errors. Depending on the data under study, various error functions are used to model the problem, the ones commonly used being the least squares regression function, regression function formed by taking the maximum of the absolute deviations and the one formed by summing the absolute deviations. When relationship between the variables can not be adequately described by linear models, nonlinear models are applied to the data.

In this chapter we study some regression models as multiple criteria optimization models. Since regression is concerned with minimization of some function of the errors, there seems no reason to treat these errors in any way but equitable. We start by defining the problem as is done in statistical literature, provide an alternative equitable multiple criteria optimization formulation and apply methods of multiple criteria optimization to obtain various equitably nondominated errors in the criteria space and correspondingly, equitably efficient solutions in the parameter space. We further notice how the regression parameters of the ordinary statistical model can be identified as an equitably efficient solution of the related multi criteria optimization problem in a limiting sense.

Statistical models start with certain assumptions on the distribution of the underlying data. Irrespective of the statistical distribution of the errors, our multi-criteria model can be used to obtain the parameters resulting in equitable errors.

6.1 Linear Regression Problem

Regression analysis is concerned with the problem of predicting the values of a variable, called the dependent or the response variable, on the basis of information provided by other variables, called the independent, predictor or regression variables.

Let Y, an $n \times 1$ vector, denote the values of the response variable corresponding to X, an $n \times k$ matrix of the values of the regressor variables. Then

$$(RM): \quad Y = \beta_0 \eta + X\beta + \epsilon$$

is the multiple linear regression model, where β_0, a scalar, β, a $k \times 1$ vector of unknown parameters, are the regression parameters, η is a $n \times 1$ vector of ones and ϵ is a $n \times 1$ vector of unobservable random errors. The objective of regression analysis is to estimate the regression parameters in a way so as to minimize the deviations between the actual values of Y from those obtained by the regression model. This 'closeness' between the two Y values is expressed in terms of L_p-norms, resulting in minimization problems of the form

$$minimize \quad (\sum_{i=1}^{n} |e_i|^p)^{\frac{1}{p}}, \; p \geq 1$$

where $\quad e_i = y_i - (b_0 + x_i \cdot b), \; i = 1, \ldots, n.$

y_i, then denotes an estimate of the i^{th} element of the vector Y. x_i is the i^{th} row of the matrix X, b_0 is a scalar and b is a $k \times 1$ vector. b_0 and b_1 give estimates of β_0 and β respectively.

The three most common approaches of estimating β correspond to the cases where $p = 1, 2$ and $p = \infty$.

$p = 1$ represents the L_1-norm estimation problem, commonly referred as $MSAE$, minimizing sum of absolute errors,

$$(MSAE): \quad minimize \sum_{i=1}^{n} |e_i|.$$

For $p = \infty$, the L_∞-norm estimation problem, known as $MMAE$, minimizing maximum of absolute errors, is defined as

$$(MMAE): \quad minimize \max_{i=1,...n} |e_i|.$$

The optimization commonly used in regression analysis corresponds to the case when $p = 2$, called the L_2-norm estimation problem. This problem is more frequently referred as $MSSE$, minimizing sum of squared errors or the least squares regression problem,

$$(MSSE): \quad minimize \ (\sum_{i=1}^{n} (|e_i|)^2)^{\frac{1}{2}}.$$

6.2 Multi-criteria Formulation of the Linear Regression Problem

The MSSE criterion of parameter estimation results in the best linear unbiased estimator of the parameters, Arthanari and Dodge (1981). The criterion is optimal and results in the maximum likelihood estimates only if the errors are independent and follow a normal distribution with mean zero and common variance σ^2, but for non-gaussian distributions, the estimators might be far away from optimal(Andrews, 1974). In(Hogg, 1974), the author demonstrates that the effect of outliers that occur at the extreme values of the predictor variables can be very disruptive. Similar limitations apply for the MSAE estimators too. MSAE estimation procedure is preferred to the MSSE when errors follow contaminated normal distribution(Ekblom, 1974).

The problem of estimating the regression parameters of regression models of the form (RM) via multiple criteria formulations have been studied earlier by some authors. In(Narula and Wellington, 1980, 2002), the authors formulate the multiple regression problem as a multiple criteria problem by considering MSSE, MAAE and

MSAE as three different criteria. They argue that under certain circumstances, choice of a single criterion might not be appropriate for estimating the parameters and propose alternative ways of using combinations of these three different criteria, rather than a single one, to estimate the parameters. In(Narula and Straubel, 1992), the authors propose similar procedure for parameter estimation in nonlinear regression models.

Its noteworthy that the aforesaid criteria are incommensurate in the sense that they are different in nature and hence can not be directly compared, but for any of the given MSSE, MSAE or the MMAE measures, all the errors for any particular measure are commensurate as they represent outcomes that have the same physical interpretation and units.

Let

$$e_i = y_i - (b_0 + x_i \cdot b),\ i = 1, \ldots, n,$$

where (b_0, b) is the vector estimating the vector (β_0, β) of parameters and e_i denotes the error associated with the $i - th$ observation.

Let

$$\rho : \mathbb{R} \longrightarrow \mathbb{R},$$

denote the criterion function associated with the error. The multi-criteria regression model associated with the linear regression problem, then, is defined as the problem

$$(MCRP): \quad minimize \quad \{\rho(e_1), \ldots, \rho(e_n)\}.$$

If

$$\rho(e) = e^2,$$

we have the multi-criteria alternative of the linear L_2-norm estimation problem, given by,

$$(MCLSQ): \quad minimize \quad \{e_1^2, \ldots, e_n^2\}$$

and if

$$\rho(e) = |e|,$$

the multi-criteria model corresponding to the linear L_1-norm estimation is given by

$$(MCAE): \quad minimize \ \{|e_1|, \ldots, |e_n|\}.$$

Note that the (MMAE) model of linear regression can be obtained by applying minimax optimization to the (MCAE) model.

In the next section we apply the (MCLSQ) model to estimate the regression parameters from an equitable point of view. We also see that the solution to the (MSSE) model can be obtained as an equitably efficient solution of the related multi-criteria problems for a particular limiting sequence of weights. The method is applicable to other regression models too.

6.3 Equitably Efficient Solutions of the Multiple Criteria Regression Model

In most cases, choice of a specific regression model to fit a data set depends on the problem under study. For example, ordinary least squares is unsuitable for a data where the errors are not normally distributed. even for data where the errors are normally distributed, the least squares estimation is more sensitive to outliers and are often removed from the data before applying the principles of least squares estimation. In certain cases, for example, in biological data sets, it might not be wise to remove the outliers as they might be the ones being more responsible for a meaningful interpretation of the model. Equitable models can be considered as an alternative way to to study such problems, where it is desirable to retain the outliers and treat the errors uniformly. As the goal of the modeling is to produce equitable solutions, no assumption is superposed on the underlying data set.

6.3.1 Equitable solutions of the MCLSQ model

Consider the MCLSQ problem defined earlier,

$$(MCLSQ): \quad minimize \ \{e_1^2, \ldots, e_n^2\} \ ,$$

where, $e_i = y_i - (b_0 + x_i.b)$. Let $f = \{e_1^2, \ldots, e_n^2\}$.

Let

$$\theta_1(f) \geq \theta_2(f) \geq \ldots \theta_n(f)$$

denote the nonincreasing ordering of the components of f and

$$\bar{\theta}_i(f) = \sum_{j=1}^{i} \theta_j(f),\ i = 1, \ldots, n.$$

From Corollary 3.1, finding equitably efficient solutions of MCLSQ is equivalent to finding efficient solutions of the problem

$$(OWAMCLSQ): \quad minimize\ \{\bar{\theta}_1(f), \ldots, \bar{\theta}_n(f)\}.$$

Ordering the functions in OWAMCLSQ makes it difficult to make the problem implementable, but the above problem can be reduced to an equivalent problem, Kostreva et al. (2004),

$$\begin{aligned} &minimize\ \ \{z_1, \ldots, z_n\} \\ &subject\ \ to\ z_k = kt_k + \sum_{i=1}^{n} d_{ik},\ k = 1, \ldots, n \\ &t_k + d_{ik} \geq e_i^2,\ \ i, k = 1, \ldots, n. \end{aligned} \tag{6.1}$$

We apply the weighted method to the convert the above problem to

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} w_i z_i \\ \text{subject to} \quad & z_k = kt_k + \sum_{i=1}^{m} d_{ik},\ k = 1, \ldots, n, \\ & t_k + d_{ik} \geq e_i^2,\ i, k = 1, \ldots, n, \\ & d_{ik} \geq 0 \text{ for all } i, k = 1, \ldots, n, \end{aligned}$$

where, $w = (w_1, \ldots, w_n),\ w_i > 0$ for all $i = 1, \ldots, n$ is a given weighting vector. Solving the above problem by varying the weights results in various efficient solutions of OWAMCLSQ, which, in turn are equitably efficient solutions of MCLSQ.

In particular, for $0 < \epsilon < 1$, consider the weighting vectors $w \in \mathbb{R}^n$ given by

$$w = (\epsilon, \ldots, \epsilon, 1 - \epsilon).$$

$$0 < \epsilon_i < 1 \implies w_i > 0, \; \forall i = 1, \ldots, n.$$

Every optimal solution of the weighted problem

$$minimize \quad \{ w_1\bar{\theta}_1(f) + w_2\bar{\theta}_2(f) + \ldots + w_n\bar{\theta}_n(f)\},$$

is an efficient solution of OWAMCLSQ, which, by Corollary 3.1, is an equitably efficient solution of MCLSQ.

As a limiting case, the weighting vector given by

$$w = \lim_{\epsilon \to 0} \; (\epsilon, \ldots, 1 - \epsilon)$$

generates an equitably efficient solution of MCLSQ. Note that such a weighting vector tends to minimize the function

$$\begin{aligned} \bar{\theta}_n(f) &= \sum_{j=1}^{n} \theta_j(f) \\ &= e_1^2 + \ldots e_n^2, \end{aligned}$$

which is the objective function of the ordinary least squares problem.

By making ϵ sufficiently close to zero, one is able to generate solutions that are not only equitably efficient but stay close enough to the solution of the ordinary least squares problem. Hence, in addition to obtaining various equitably efficient solutions to the multi-criteria model, the solution of the ordinary least squares problem can be viewed as an equitably efficient solution for a particular limiting sequence of weights.

In the following section, we obtain equitable estimators of a numerical problem.

6.3.2 Example

Consider the data set in table 6.1(Arthanari and Dodge, 1981).

Table 6.1 Example Data

Observation Number(i)	y$_i$	x$_i$
1	2	5
2	5	4
3	4	6
4	8	9
5	3	7

The linear least squares estimators for this data set are obtained as the optimal solution of the problem

$$minimize \quad \{\sum_{i=1}^{5}(y_i - (b_0 + b_1 x_i))^2, \quad s.t. \quad (b_0, b_1) \in \mathbb{R}^2\}.$$

Table 6.2 gives the least squares estimators of the data along along with the squared errors and the sum of squared errors.

Table 6.2 Least Squared Estimators and Corresponding Errors

Estimators(b_i)		Squared Errors(e_i^2)					Sum of Squared Errors
b_0	b_1	e_1^2	e_2^2	e_3^2	e_4^2	e_5^2	$\sum_{i=1}^{5} {e_i}^2$
-0.0416	0.7162	2.3733	4.7336	0.0659	2.5427	3.8926	13.6081

We obtain some some equitably efficient parameters of the MCLSQ model ensuring that the errors are equitably nondominated in the criteria space.

Equitably efficient solutions of MCLSQ problem are obtained as equitably efficient solutions of the problem

$$minimize \quad \{(y_1-(b_0+b_1x_1))^2,\dots,(y_5-(b_0+b_1x_5))^2, \quad s.t. \;\; (b_0,b_1)\in\mathbb{R}^2\},$$

which, by (6.1), can be obtained by finding the efficient solutions of the problem

$$\begin{aligned}
&minimize \;\; \{z_1,\dots,z_5\}\\
&subject\;\; to \;\; z_k = kt_k + \sum_{i=1}^{5} d_{ik}, \quad k=1,\dots,5, \qquad (6.2)\\
&t_k + d_{ik} \geq (y_i-(b_0-b_1x_i)^2, \quad i,k=1,\dots,5,\\
&(b_0,b_1)\in\mathbb{R}^2.
\end{aligned}$$

Efficient solutions of (6.2) can be obtained as optimal solutions of the single-objective weighted problem

$$\begin{aligned}
minimize \quad & \sum_{i=1}^{5} w_k z_k\\
subject\;\; to \;\; z_k &= kt_k + \sum_{i=1}^{5} d_{ik}, \quad k=1,\dots,5,\\
t_k + d_{ik} &\geq (y_i-(b_0+b_1x_i))^2, \quad i,k=1,\dots,5,\\
d_{ik} &\geq 0 \text{ for all } i,k=1,\dots,5,
\end{aligned}$$

for given weighting vectors $(w_1,\dots,w_5)$, where $w_i>0$ for all $i=1,\dots,5$.

Table 6.3 presents some of the equitably efficient solutions for certain vector of weights while Table 6.4 presents the equitably efficient solutions along with the individual squared errors as well as the sum of squared errors.

Table 6.3 Equitable Estimators and Corresponding Weights

Weights(w_i)					Equitable Estimators(b_i)	
w_1	w_2	w_3	w_4	w_5	b_0	b_1
0.0100	0.0100	0.0100	0.0100	0.9600	0.0241	0.7072
0.0100	0.0100	0.0100	0.4600	0.5100	0.0624	0.7054
0.0100	0.0100	0.0100	0.9100	0.0600	0.1111	0.7017
0.0100	0.0100	0.4600	0.0100	0.5100	0.6829	0.6167
0.0100	0.0100	0.4600	0.4600	0.0600	0.7635	0.6052
0.0100	0.0100	0.9100	0.0100	0.0600	0.8000	0.6000
0.0100	0.4600	0.0100	0.0100	0.5100	0.7000	0.6000
0.0100	0.4600	0.0100	0.4600	0.0600	0.7277	0.6000
0.0100	0.4600	0.4600	0.0100	0.0600	0.8000	0.6000

Table 6.4 Equitable Estimators with Errors and Squared Errors

Equitable Estimators(b_i)		Squared Errors(e_i^2)				Sum of Squared Errors	
b_0	b_1	e_1^2	e_2^2	e_3^2	e_4^2	e_5^2	$\sum_{i=1}^{5} e_i^2$
0.0241	0.7072	2.4346	4.6093	0.0716	2.5943	3.8999	13.6097
0.0624	0.7054	2.5257	4.4780	0.0868	2.5257	4.0000	13.6162
0.1111	0.7017	2.6226	4.3356	0.1031	2.4771	4.0917	13.6301
0.6829	0.6167	3.1207	3.4232	0.1469	3.1207	4.0000	13.8114
0.7635	0.6052	3.2026	3.2965	0.1559	3.2026	4.0000	13.8576
0.8000	0.6000	3.2400	3.2400	0.1600	3.2400	4.0000	13.8800
0.7000	0.6000	2.8900	3.6100	0.0900	3.6100	3.6100	13.8100
0.7277	0.6000	2.9851	3.5054	0.1074	3.5054	3.7162	13.8194
0.8000	0.6000	3.2400	3.2400	0.1600	3.2400	4.0000	13.8800

From Table 6.4, we find that the first three solutions have sum of squared errors very close to that obtained for the ordinary least squares regression of Table 6.2. Comparing the range of the individual errors, we see that these equitable solutions are lesser dispersed than the ordinary least squares solution. For the remaining solutions, the sum of squared errors are relatively higher but the solutions are much less dispersed, thereby, reducing inequity. Equity of these solutions can be considered to compensate for the increase in the sum of squares values.

Table 6.5 presents some equitably efficient solutions for limiting sequence of weights and Table 6.6 presents the equitably efficient solutions with individual squared errors and sum of squared errors corresponding to these weights.

Table 6.5 Equitable Estimators and Corresponding Weights

Weights(w_i)					Equitable Estimators(b_i)	
w_1	w_2	w_3	w_4	w_5	b_0	b_1
0.00001	0.00001	0.00001	0.00001	0.99999	-0.0405	0.7162
0.00002	0.00002	0.00002	0.00002	0.99998	-0.0404	0.7162
0.00003	0.00003	0.00003	0.00003	0.99997	-0.0403	0.7163
0.00004	0.00004	0.00004	0.00004	0.99996	-0.0402	0.7162
0.00005	0.00005	0.00005	0.00005	0.99995	-0.0401	0.7162
0.00006	0.00006	0.00006	0.00006	0.99994	-0.0400	0.7162
0.00007	0.00007	0.00007	0.00007	0.99993	-0.0399	0.7162
0.00008	0.00008	0.00008	0.00008	0.99992	-0.0399	0.7162
0.00009	0.00009	0.00009	0.00009	0.99991	-0.0398	0.7161

Table 6.6 Equitable Estimators with errors and Squared Errors

Equitable Estimators(b_i)		Squared Errors(e_i^2)					Sum of Squared Errors
b_0	b_1	e_1^2	e_2^2	e_3^2	e_4^2	e_5^2	$\sum_{i=1}^{5} e_i^2$
-0.0405	0.7162	2.3734	4.7334	0.0659	2.5427	3.8927	13.6081
-0.0404	0.7162	2.3735	4.7332	0.0660	2.5427	3.8928	13.6081
-0.0403	0.7163	2.3736	4.7329	0.0660	2.5426	3.8929	13.6081
-0.0402	0.7162	2.3738	4.7327	0.0660	2.5426	3.8931	13.6081
-0.0401	0.7162	2.3739	4.7324	0.0660	2.5425	3.8932	13.6081
-0.0400	0.7162	2.3741	4.7322	0.0660	2.5424	3.8934	13.6081
-0.0400	0.7162	2.3743	4.7319	0.0661	2.5423	3.8935	13.6081
-0.0399	0.7162	2.3745	4.7316	0.0661	2.5423	3.8937	13.6081
-0.0398	0.7161	2.3747	4.7313	0.0661	2.5422	3.8939	13.6081

From Table 6.6, except for round-off errors, the range of dispersion of equitable errors for any solution are almost similar to the dispersion range of the solution obtained using the ordinary least squares procedure. The sum of squared errors for these solutions are very close to the least squares solution. This seems to justify the applicability of the equitably efficient concept to the least squares regression problem which demands equity in treatment of individual errors.

CHAPTER 7
EQUITABLE ASSET ALLOCATION FOR SINGLE PERIOD INVESTMENT

Portfolio allocation is concerned with investing an initial outlay of capital into a number of assets whose returns are uncertain. In single period allocation, money is invested at the initial time and payoff is attained at the end of the time-period. Uncertainty associated with the returns make the investment risky and the investor is faced with the task of investing judiciously so as to maximize his expected portfolio return with minimum risk involved. Portfolio problem, thus, is a bicriteria problem consisting of two criteria, expected return and risk of returns, the former being maximized and the latter being minimized.

Different risk measures associated with the returns are studied in portfolio literature, variance of returns being the most commonly used. Markowitz (1952) formulated the portfolio selection problem as a parametric quadratic programming problem in which the variance of the returns is minimized subject to the constraint that the required expected return is warranted(Teo and Yang, 2001). Diversifying by investing suitable amounts in different financial instruments tends to reduce the overall risk of the portfolio.

Asset allocation is the term given to portfolio allocation where that the financial instruments under consideration are divided into different asset classes. For example, if we are looking for an investment strategy to invest in stocks, bonds and futures on commodities, it is better to have two asset classes, one composed of stocks and bonds and the other composed of the futures. The underlying principle is based on the correlation between assets in different classes. Futures on commodities tend to be negatively correlated to stocks and bonds and dividing the classes on basis of the correlation tends to reduce the risk.

In this chapter we look at the problem of asset allocation by dividing the assets under suitable classes. The asset returns being random and each state of nature being equally probable, it seems justified to treat each class, or more specifically, the return from each, uniformly. In this way we create an objective function for each class which corresponds to the expected return from that class and look for equitably efficient solutions of this multiple objective problem. This ensures lesser dispersion of returns relative to each other than that one would expect by maximizing the expected return from all the classes grouped together. This might provide added protection in the case the most favored stock goes down drastically.
Our formulation differs from the traditional approach too in the sense that the expected returns from each class solely form the objective functions of the problem while keeping the variance of the assets completely outside the objectives domain. This ensures that we are not comparing two entities that have totally different units and thus avoids the need of any scaling.

7.1 Multiple Criteria Asset Allocation Problem

Consider the set $J = \{S_1, \ldots, S_n\}$ consisting of n assets. Let

$$J = \cup_{i=1}^{m} J_i, \text{ where } J_i \subset J \text{ and } J_i \cap J_j = \phi \text{ for } i \neq j$$

denote the m asset classes in which the n assets are divided. Without loss of generality, let

$$J_1 = \{S_1, \ldots, S_{k_1}\}$$
$$J_2 = \{S_{k_1+1}, \ldots, S_{k_1+1+k_2}\}, \ldots.$$
$$J_m = \{S_{k_{m-1}+1}, \ldots, S_{k_{m-1}+1+k_m}\},\ k_m = n.$$

Let $R = (r_i)_{i=1}^{n}$ be the row vector denoting the random rate of return of stock i, $X = (x_i)_{i=1}^{n}$ be the row vector of the fraction of money invested in asset i. Further, let $E(r_i) = \bar{r}_i$ represent the expected return from asset i. Expected return from the

m asset classes, then, are given by

$$\begin{aligned} E(R_1) &= E(r_1x_1 + \ldots, +r_{k_1}x_{k_1}) \\ &= E(r_1)x_1 + \ldots, +E(r_{k_1})x_{k_1} \\ &= \bar{r}_1x_1 + \ldots, +\bar{r}_{k_1}x_{k_1}, \ldots, \\ E(R_m) &= E(r_{k_{m-1}+1}x_{k_{m-1}+1} + \ldots + r_{k_{m-1}+1+k_m}x_{k_{m-1}+1+k_m}) \\ &= E(r_{k_{m-1}+1})x_{k_{m-1}+1} + \ldots + E(r_{k_{m-1}+1+k_m})x_{k_{m-1}+1+k_m} \\ &= \bar{r}_{k_{m-1}+1}x_{k_{m-1}+1} + \ldots + \bar{r}_{k_{m-1}+1+k_m}x_{k_{m-1}+1+k_m}. \end{aligned}$$

If A denotes the $n \times n$ matrix representing the covariance between the returns from the n assets, then our problem is to maximize the expected returns of each class subject to bounds on the covariance of the asset classes. Mathematically, the multiple-criteria asset allocation is defined as

$$\begin{aligned} (MCAAP): \quad & maximize\{E(R_1), \ldots, E(R_m)\} \\ subject\,to \;\; & C_{min} \leq XAX^t \leq C_{max} \\ & \textstyle\sum_{i=1}^n x_i = 1 \\ & x_i \geq 0, \end{aligned}$$

where C_{min} and C_{max} represent reasonable lower and upper bounds on the risk of investment.

Our goal is to find equitably efficient allocations.

7.2 Two-phase Solution Procedure

Solution of MCAAP depends on the risk level, which, in turn, depends on the covariance matrix A. Choosing the risk level arbitrarily might make the problem infeasible. Reasonable values of C_{min} and C_{max} can be obtained by finding bounds on the portfolio variance.

In phase 1, we solve the following single objective optimization problems:

$$
\begin{aligned}
(SOPI):\quad & maximize \;\; XAX^t \\
& subject\,to \;\; \sum_{i=1}^{n} E(R_i) = z \\
& \qquad\qquad\;\; \sum_{i=1}^{n} x_i = 1 \\
& \qquad\qquad\;\; x_i \geq 0,\; i = 1, \ldots, n
\end{aligned}
$$

and

$$
\begin{aligned}
(SOPII):\quad & minimize \;\; XAX^t \\
& subject\,to \;\; \sum_{i=1}^{n} E(R_i) = z \\
& \qquad\qquad\;\; \sum_{i=1}^{n} x_i = 1 \\
& \qquad\qquad\;\; x_i \geq 0,\; i = 1, \ldots, n,
\end{aligned}
$$

where z is a variable.

Let C_{max} be the optimal objective value of $SOPI$ and C_{min} be the optimal objective value of $SOPII$.

In phase 2 we can use corollary (4.1) or results from sections 4.2.1 and 4.2.2 to find some equitably efficient asset allocations of MCAAP by setting the risk level bounds C_{min} and C_{max} obtained from phase 1. Depending on the method used, phase 2 results in one or several equitably efficient solutions. For instance, if corollary (4.1) is used and minimax problem has a unique solution, then we get one equitably efficient solution. Equitably efficient allocations tend to reduce risk by selecting portfolios with returns less dispersed relative to each other. In the next section we present a numerical example and compare the solutions obtained using the equitable approach to that obtained using the traditional multiple-criteria approach.

7.3 Example

Consider the data in Table 7.1 that gives the monthly returns and standard deviations of commodity futures and bonds from July 1959 to December 2004(Gorton and Rouwenhorst, 2005). The calculated variances are also shown. The correlation between bonds and commodities futures during that period was -0.14. The correlation being negative, its better to take the two instruments as two different asset classes.

Table 7.1 Example Data

	Commodity Futures	Bonds
Average Return($\bar{r}$)	0.89	0.64
Standard Deviation(σ)	3.47	2.45
Variance(σ^2)	12.0409	6.0025

The correlation between to random variables X and Y is given by

$$\rho(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y}.$$

The covariance between commodities futures and bonds is

$$cov = -0.14 * 3.47 * 2.45 = -1.1902.$$

The covariance matrix for returns of futures and bonds is given in Table 7.2.

Table 7.2 Covariance Matrix

	Commodities Futures	Bonds
Commodities Futures	12.0409	-1.1902
Bonds	-1.1902	6.0025

From Tables 7.1 and 7.2 we see that the maximum return on commodities futures is about 40% higher than that on the bonds but the associated risk is almost double. The comparitive higher return comes with a relatively higher risk. The two assets are negatively correlated too. A conservative investor would like to maximize his return without taking too much risk. Under such circumstances it seems more reasonable to look for allocations where the difference between the weights assigned to the two assets is not too huge. Hence, we look for allocations that are efficient as well as equitable. Equitably efficient allocations of the two asset allocation problem are obtained by finding equitably efficient solutions of the problem

$$\begin{aligned}(BCAAP): \quad & maximize\{\bar{r_1}x_1, \bar{r_2}x_2\} \\ subject\,to \quad & C_{min} \leq XAX^t \leq C_{max} \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0,\end{aligned}$$

where $\bar{r_1} = 0.89$ and $\bar{r_2} = 0.64$ represent the expected returns on futures commodities and bonds respectively.

Phase 1 optimal solutions of the single objective problems associated with BCAAP are obtained by solving the problems

$$\begin{aligned} maximize \;\; & XAX^t \\ subject\,to \;\; & \bar{r_1}x_1 + \bar{r_2}x_2 = z \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} minimize \;\; & XAX^t \\ subject\,to \;\; & \bar{r_1}x_1 + \bar{r_2}x_2 = z \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{7.2}$$

where x_1 and x_2 are the fractions invested in commodities futures and bonds respectively.

Optimal solutions of (7.1) and (7.2) along with the objective values are

$$(x_1, x_2, z, XAX^t) = (1.0000, 0.0000, 0.8900, 12.0409)$$

$$(x_1, x_2, z, XAX^t) = (0.3522, 0.6478, 0.7280, 3.4694).$$

In phase 2, we solve BCAAP by setting $C_{min} = 3.47$ and $C_{max} = 12.04$. The two-phase minimax method from Corollary 4.1 yields the equitably efficient asset given in Table 7.3.

Table 7.3 Equitable Assets using 2-phase Minimax

Futures	Bonds	Futures Return	Bonds Return	Variance
x_1	x_2	$E(R_1)$	$E(R_2)$	XAX'
0.3522	0.6478	0.3134	0.4146	3.4694

Several equitably efficient solutions of BCAAP can be obtained by applying equitable scalarizations from section 4.2 or applying the weighted method to the ordered weighted averaging formulation

$$\begin{aligned} maximize \quad & \{\bar{\theta}_1(\bar{r_1}x_1, \bar{r_2}x_2), \bar{\theta}_2(\bar{r_1}x_1, \bar{r_2}x_2)\} \\ subject\,to \quad & C_{min} \leq XAX^t \leq C_{max} \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0. \end{aligned} \tag{7.3}$$

Since

$$\bar{\theta}_1(\bar{r_1}x_1, \bar{r_2}x_2) = maximum\,\{\bar{r_1}x_1, \bar{r_2}x_2\}$$

and

$$\bar{\theta}_2(\bar{r_1}x_1, \bar{r_2}x_2) = \bar{r_1}x_1 + \bar{r_2}x_2,$$

formulation (7.3) is equivalent to

$$\begin{aligned} maximize \;\; & \{y, \bar{r_2}x_2 + \bar{r_2}x_2\} \\ subject\, to \;\; & C_{min} \leq XAX^t \leq C_{max} \\ & \bar{r_1}x_1 \leq y \\ & \bar{r_2}x_2 \leq y \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0. \end{aligned} \tag{7.4}$$

Table 7.4 presents some of the equitably efficient allocations of BCAAP obtained by applying the weighting method to formulation (7.4). Table 7.5 presents efficient allocations of BCAAP.

Table 7.4 Equitably Efficient Assets

Futures	Bonds	Futures Return	Bonds Return	Net Return	Absolute Difference	Variance
x_1	x_2	$E(R_1)$	$E(R_2)$	$E(R_1)+E(R_2)$	$\|E(R_1)-E(R_2)\|$	XAX'
0.4264	0.5736	0.3795	0.3671	0.7466	0.0124	3.5819
0.3487	0.6513	0.3103	0.4169	0.7272	0.1065	3.4697
0.3389	0.6611	0.3017	0.4231	0.7247	0.1214	3.4730
0.3271	0.6729	0.2911	0.4307	0.7218	0.1395	3.4823
0.4152	0.5848	0.3695	0.3743	0.7438	0.0048	3.5505
0.3406	0.6594	0.3031	0.4220	0.7251	0.1189	3.4722
0.3813	0.6187	0.3394	0.3960	0.7353	0.0566	3.4868
0.3408	0.6592	0.3033	0.4219	0.7252	0.1185	3.4721
0.3520	0.6480	0.3133	0.4147	0.7280	0.1014	3.4694
0.3599	0.6401	0.3203	0.4097	0.7300	0.0894	3.4706
0.3830	0.6170	0.3409	0.3949	0.7358	0.0540	3.4889

Table 7.5 Efficient Assets

Futures	Bonds	Futures Return	Bonds Return	Net Return	Absolute Difference	Variance
x_1	x_2	$E(R_1)$	$E(R_2)$	$E(R_1)+E(R_2)$	$\lvert E(R_1)-E(R_2)\rvert$	XAX'
0.5596	0.4404	0.4981	0.2818	0.7799	0.2162	4.3485
0.3742	0.6258	0.3330	0.4005	0.7335	0.0675	3.4793
0.2470	0.7530	0.2198	0.4819	0.7017	0.2621	3.6954
0.2510	0.7490	0.2233	0.4794	0.7027	0.2560	3.6787
0.3589	0.6411	0.3194	0.4103	0.7297	0.0909	3.4703
0.3849	0.6151	0.3425	0.3937	0.7362	0.0511	3.4913
0.2882	0.7118	0.2565	0.4555	0.7121	0.1990	3.5529
0.3042	0.6958	0.2707	0.4453	0.7160	0.1746	3.5165
0.6208	0.3792	0.5525	0.2427	0.7952	0.3098	4.9432
0.6281	0.3719	0.5590	0.2380	0.7970	0.3210	5.0242
0.4273	0.5727	0.3803	0.3665	0.7468	0.0138	3.5848

From Table 7.4 we find that the maximum absolute difference of returns for the equitably efficient allocations is 0.1395 corresponding to allocation 4 and the minimum is 0.0045 corresponding to asset 5. The net returns corresponding to these allocations are 0.7218 with a variance of 3.4283 and 0.7438 with a variance of 3.5505, respectively. The dispersion of the components of the outcome vectors(individual returns) lie between 0.0048 and 0.1315. The variances for all the allocations are reasonably low and are much lower than the individual variances of the two assets. Equitable assets, thus, provide better hedge against investment risk by eliminating allocations with disproportionate returns. Even if one class under performs, we can expect a some what 'balanced' return from the other.

From Table 7.5 we see that most of the allocations are widely dispersed in their returns, compared to the equitably efficient allocations. The allocations with low dispersion have returns and variances similar to the equitable ones. For the assets with wide dispersions, the higher returns are out weighed by the relatively higher variances. such allocations, despite their higher returns, provide reduced hedge against risk in case the class with higher return under performs.

In Table 7.6 we present efficient portfolios for the same data from Table 7.2 without dividing the assets into classes. We apply the classical mean-variance portfolio approach to obtain the solutions. Its evident from the Table that except for the last portfolio, returns and variances for the remaining ones are similar and close to those obtained by asset-class division approach of Table 7.4. In fact, the equitable approach eliminates the last portfolio which has a some what higher return at the expense of reasonably higher variance. Hence finding equitably efficient allocations after carefully dividing into suitable classes seems to reduce the investment risk by assigning allocations with equitable returns.

Figures 7.1 and 7.2 present comparative plots of the equitably efficient assets and efficient assets respectively of our 2-class asset allocation problem. Figure 7.3 is a plot of efficient portfolios without using the class approach and applying the classical mean-variance approach of Markovitz.

Table 7.6 Efficient Mean-Variance Portfolio

Portfolio Return	Variance
E(R)	XAX'
0.7280	3.4694
0.7282	3.4694
0.7284	3.4695
0.7287	3.4696
0.7291	3.4698
0.7296	3.4702
0.7303	3.4712
0.7316	3.4736
0.7342	3.4817
0.7418	3.5314
0.8900	12.0409

Figure 7.1 Equitably Efficient Assets in the Criteria Space

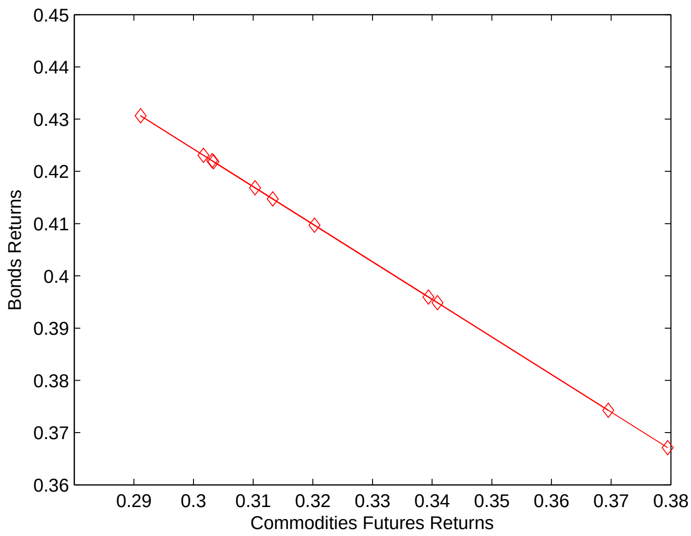


Figure 7.2 Efficient Assets in the Criteria Space

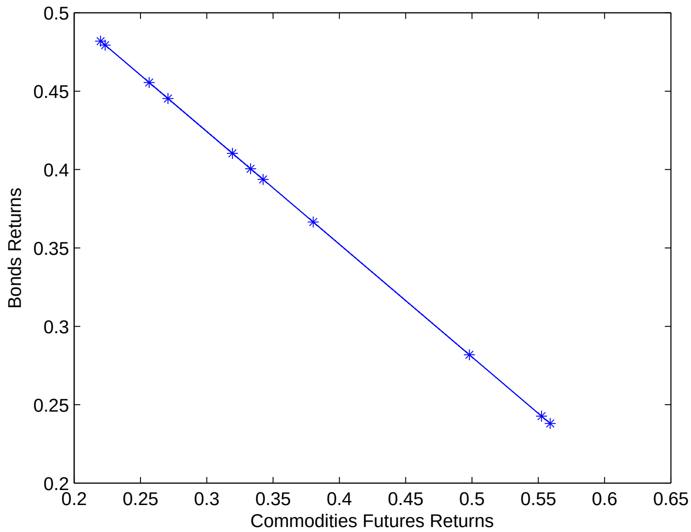
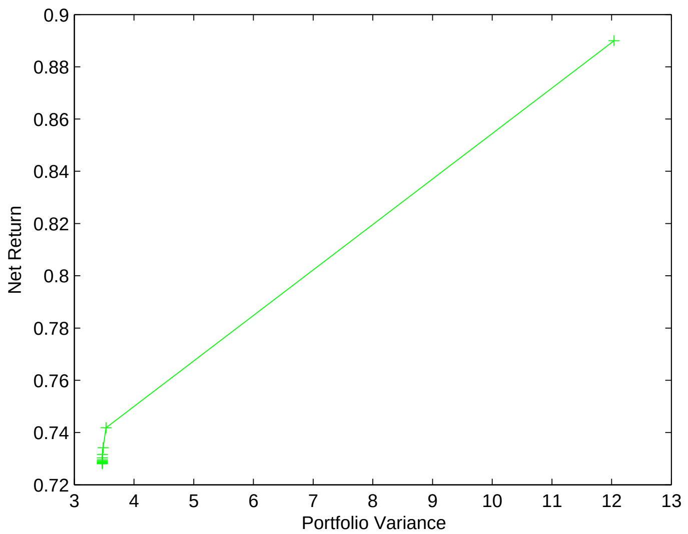


Figure 7.3 Mean-Variance Efficient Portfolio in the Criteria Space



Next chapter is the last chapter of our present research in which we conclude our work with some suggestions for future research.

CHAPTER 8

CONCLUSION AND FUTURE WORK

The objective of our work is to study the theory and methodology of equitable efficiency in multiple-criteria optimization, stress its importance and demonstrate its relevance by means of applications. In the present chapter, we summarize the findings pertaining to our goal and present some directions for future research.

The first two chapters presented a brief introduction and literature review on equitable efficiency in multiple criteria optimization. After introducing the notations and terminology required in our work, we developed some characterizations of equitably efficient solutions. Ordering of objective functions is hard to implement in practice. Toward a theoretical foundation of equitable efficiency, we came up with additional ways of generating such solutions by building equitable aggregations based on strictly-Schur-convex functions. The single-objective optimizations problems resulting from these aggregations being independent of the ordering of the objective functions remove the onus of ordering of objective functions, and thus, focus on ways of generating equitable solutions by working with the objective functions directly. Furthermore, our methods do not assume any special structure on the optimization problem and are able to generate several equitably efficient solutions by varying the parameter.

The minimax procedure is widely discussed in many areas of optimization including multiple-criteria optimization. We proved that the two-phase minimax method can be used to find equitably efficient solutions. We addressed a question related to the existence of equitably efficient solutions for linear MCOPs.

In the next part of our work, we demonstrated the relevance of equitable efficiency by showing its relation to matrix games and linear complementarity problems. We showed how zero-efficient solutions, as defined by some authors in an earlier work, are the same as equitably efficient solutions with specific objective values. Because equitable dominance is a refinement of Pareto dominance, the set of equitably efficient solutions is, in general, a subset of the efficient solutions. Chapter 5 demonstrated the equivalence of equitably efficient and efficient solutions for a MCOP related to a particular class of linear complementarity problems.

In Chapters 6 and 7 we presented the problems of regression analysis and asset allocation where the application of equitable efficiency might be relevant. In the past, regression problems were studied as a multi-criteria optimization problem but none of the previous papers in the area addressed the issue of equitable treatment of regression errors. We considered all the errors to be uniform, which seemed justified too, and thus obtained solutions in which all the errors are treated in an equitable manner. In our view, the equitable asset allocations from Chapter 7 can be justified from a risk-averse investor's view point. A risk averse investor may prefer almost similar returns from two different asset classes than bearing the risk of losing too much in case the preferred asset class performs unexpectedly.

Future work

The issue of equitable efficiency in multiple-criteria optimization is not much explored in the literature. We have developed some theory related to finding equitably efficient solutions by means of equitable aggregations. There is much potential for future work on finding ways to generate equitably efficient solutions using existing scalarizations of finding efficient solutions. The applicability of such scalarizations, for example, the epsilon-constraint scalarization(Chankong and Haimes, 1983), applied to the ordered weighted problem, is one area of concern.

Equitably efficient solutions are contained within the set of efficient solutions. Another direction of research is a two-phase method in which efficient solutions from phase 1 are used to find equitably efficient solutions in phase 2.

We demonstrated the applicability of equitable efficiency to two different problems. Another problem of interest is from stratified sampling. Optimization models have been applied to such problems in the past. (Mulvey, 1983), demonstrates that such problems are special cases of facility location problems. Location problems are well studied in terms of equitable efficiency(Ogryczak, 2000). Analysing the sampling results on the basis of equity deserves attention.

As another area of application, we would like to apply equity models to problems of machine utilization in multi period production systems. For each period we are interested in maximizing the fraction of time the machine is utilized(not idle). For such systems we would like to have solutions that do not result in too disparate utilization from period to period.

There may be many more questions coming up as the work progresses in the future.

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