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Identifying Codes and Domination in the Product of Graphs

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Abstract

An identifying code in a graph is a dominating set that also has the property that the closed neighborhood of each vertex in the graph has a distinct intersection with the set. The minimum cardinality of an identifying code in a graph G is denoted $\gamma^{\mathrm{ID}}(G)$. We consider identifying codes of the direct product $K_n \times K_m$. In particular, we answer a question of Klavžar and show the exact value of $\gamma^{\mathrm{ID}}(K_n \times K_m)$. It was recently shown by Gravier, Moncel and Semri that for the Cartesian product of two same-sized cliques $\gamma^{\mathrm{ID}}(K_n \Box K_n) = \lfloor \frac{3n}{2} \rfloor$. Letting $m \geq n \geq 2$ be any integers, we show that $\gamma^{\mathrm{ID}}(K_n \Box K_m) = \max\{2m - n, m + \lfloor n/2 \rfloor\}$. Furthermore, we improve upon the bounds for $\gamma^{\mathrm{ID}}(G \Box K_m)$ and explore the specific case when G is the Cartesian product of multiple cliques.

Given two disjoint copies of a graph G, denoted G^1 and G^2, and a permutation π of $V(G)$, the permutation graph πG is constructed by joining $u \in V(G^1)$ to $\pi(u) \in V(G^2)$ for all $u \in V(G^1)$. The graph G is said to be a universal fixer if the domination number of πG is equal to the domination number of G for all π of $V(G)$. In 1999 it was conjectured that the only universal fixers are the edgeless graphs. We prove the conjecture.

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Chapter 1

Introduction

Identifying codes and domination have several applications. For example, suppose we want to install motion sensors for a security system in a particular building. In order to guarantee that an intruder is detected no matter which room he enters, we would need to install the sensors in such a way so that each room either contains a sensor or shares a doorway with a room that contains a sensor. Additionally, we may want to require that our security system gives us the specific room that the intruder has entered. In either scenario, minimizing the cost of these sensors is important. Therefore, if we consider the graph G that represents the layout of our building, in the first scenario we seek a dominating set of G, and in the second scenario we seek an identifying code of G. The focus of this thesis is in the area of domination and identifying codes in graph products.

This thesis is organized as follows. In Section 1.1, we give some standard definitions and terminology. In Chapter 2, we give background information for the particular graph products that we study, as well as previous results regarding identifying codes. Chapter 3 is dedicated to identifying codes in the direct product of graphs and Chapter 4 focuses on identifying codes in the Cartesian product of graphs. Chapter 5 focuses on the domination number of a constructed graph which is not necessarily a graph product, but similar to the Cartesian product.

1.1 Definitions

A *simple, undirected graph* G consists of a finite non-empty set of vertices $V(G)$ and a finite set of edges $E(G)$ which are unordered pairs (u, v) of distinct vertices of G, commonly written uv.

If $uv \in E(G)$, we say that u is *adjacent* to v. We say $|V(G)|$ is the *order* of G.

Given a vertex $x \in V(G)$, we let $N(x)$ denote the *open neighborhood* of x, that is, the set of vertices adjacent to x. If x has no neighbors, then we say x is *isolated*. The *closed neighborhood* of x is $N[x] = N(x) \cup \{x\}$. For $S \subseteq V(G)$, the *open neighborhood of* S is $N(S) = \cup_{v \in S} N(v)$. Similarly, the *closed neighborhood of* S is $N[S] = N(S) \cup S$. Given a graph G and $x \in V(G)$, the *degree of* x is defined to be $\deg(x) = |N(x)|$. The *maximum degree of* G is defined to be $\Delta(G) = \max_{v \in V(G)} \deg(v)$. If for some constant $r \in \mathbb{N}$, $\deg(v) = r$ for all $v \in V(G)$, we say that G is r*-regular*. A *complete graph of order* n, denoted K_n, is such that every pair of vertices is adjacent. Thus, K_n is $(n-1)$-regular. We will denote the graph consisting of n isolated vertices as $\overline{K_n}$.

The *path* P_n is the graph whose vertices are $v_1, v_2, \ldots, v_n$ and $E(P_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$. The *length of a path* is the number of edges in the path. The *cycle* C_n is the graph whose vertices are $v_1, v_2, \ldots, v_n$ and $E(C_n) = E(P_n) \cup \{v_1 v_n\}$. The *distance* between two vertices u and v, denoted $d_G(u, v)$, is the length of a shortest path between u and v.

Given a graph G, a graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If a subgraph H of G is such that for every pair of vertices $u, v \in V(H)$ it is the case that $uv \in E(H)$ if and only if $uv \in E(G)$, then H is an *induced subgraph* of G. A *clique* is a complete subgraph of a given graph.

Let G and H be any two graphs. We say that G and H are *isomorphic*, and write $G \cong H$, if there exists a bijection $\phi : V(G) \to V(H)$ where $xy \in E(G)$ if and only if $\phi(x)\phi(y) \in E(H)$ for all $x, y \in V(G)$. Such a map ϕ is an *isomorphism*. An *automorphism* of G is an isomorphism from G to itself. We do not normally distinguish between isomorphic graphs. Thus, it is standard to write $G = H$ rather than $G \cong H$. We say that G is *symmetric* if given any two pairs of adjacent vertices u_1v_1 and u_2v_2, there is an automorphism $\phi : V(G) \to V(G)$ such that $\phi(u_1) = u_2$ and $\phi(v_1) = v_2$.

When analyzing an algorithm, we will use the following notation. Suppose $f(x)$ and $g(x)$ are two functions defined on some subset of $\mathbb{R}$. We write $f(x) = O(g(x))$ if and only if there exists constants N and C such that $|f(x)| \leq C|g(x)|$ for all $x > N$. We write $f(x) = \Theta(g(x))$ if and only if $f(x) = O(g(x))$ and $g(x) = O(f(x))$. Additionally, we write $f(x) = o(g(x))$ if and only if for every $C > 0$ there exists a real number N such that for all $x > N$ we have $|f(x)| < C|g(x)|$.

Chapter 2

Background

2.1 Cartesian Product

Given two graphs G and H, the *Cartesian product of G and H*, denoted $G \square H$, is defined to be the graph with vertex set $V(G) \times V(H)$, in which two vertices (x_1, y_1) and (x_2, y_2) are adjacent if $x_1 = x_2$ and $y_1 y_2 \in E(H)$, or $x_1 x_2 \in E(G)$ and $y_1 = y_2$. Figure 2.1 depicts the Cartesian product of P_3 and K_3.

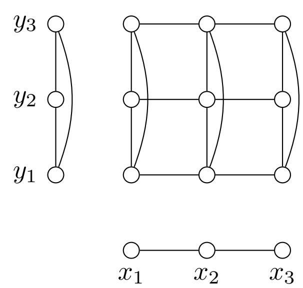


Figure 2.1: $P_3 \square K_3$

Given the Cartesian product $G \square H$, we refer to G as the *first factor* and to H as the *second factor*. In each illustration of a Cartesian product, we represent the first factor horizontally and the second factor vertically.

2.1.1 Applications

The hypercube, or n-cube, is defined to be the graph Q_n whose vertex set consists of all vectors $(v_1, \ldots, v_n) \in \{0,1\}^n$, in which two vertices are adjacent if and only if they differ in precisely one coordinate. Figure 2.2 illustrates Q_3, which is isomorphic to $K_2 \Box K_2 \Box K_2$.

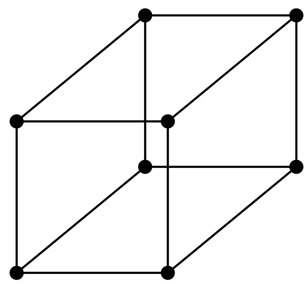


Figure 2.2: Q_3

This particular class of Cartesian products is by far the most widely used in terms of graph products. Notably, the hypercube has been an attractive model for the interconnection network of multicomputer systems due to the fact that the diameter of Q_n is n, which is relatively small considering Q_n contains 2^n vertices. This translates to quick routing time of messages through the network. Moreover, the n-cube is regular and symmetric, which allows for efficiency of algorithm implementation, in that local algorithms work globally.

The grid graphs, or graphs of the form $P_m \Box P_n$, are another class of Cartesian products that have been investigated for decades. Consider the process of Very-Large-Scale integration, or VLSI, which creates integrated circuits by combining thousands of transistors into a single chip. The microprocessor is one example of a VLSI device. Grid graphs model the constraint that within a VLSI layout, wires can only run horizontally and vertically. Thus, routing problems on grid graphs have important applications [30].

In chemical graph theory, the primary goal is to represent a molecule as a graph, and then use graph properties to determine the behavior of the corresponding molecule. Benzenoid graphs represent benzenoid hydrocarbons, which are important components of gasoline, yet have carcinogenic properties. Considering water and soil contamination of this molecule continue to rise, information derived from graph theoretical properties is well received in the chemical community. In 1996, Chepoi [8] showed that benzenoid graphs can be isometrically embedded into the Cartesian product of three trees. This in turn gave rise to a linear time algorithm for computing the Wiener index [9], or the sum of the distances between all pairs of vertices, for these benzenoid graphs.

2.1.2 Algebraic Structure

Let Γ represent the set of all finite, simple graphs, and $\cup$ represent the disjoint union operation for graphs. We first show that Γ with the operations $\cup$ and $\Box$ forms a familiar algebraic structure. The results included in this section are given in the Handbook of Product Graphs [23, pg. 53-54]. We take the time to prove these results as some of the details have been omitted in [23].

Recall that a *monoid* is a set S, together with a binary operation $\cdot$, where each of the following is satisfied.

(a) For all $a, b \in S$, $a \cdot b \in S$. (closure)

(b) For all $a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (associativity)

(c) There exists an element $e \in S$, such that for every $a \in S$, $e \cdot a = a \cdot e = a$.

In terms of Γ and the graph operation $\cup$, it is almost immediate that $(\Gamma, \cup)$ is a monoid. Certainly the disjoint union of two simple, finite graphs is itself a simple, finite graph. Moreover, for any G, H, and K in Γ,

$$(G \cup H) \cup K = G \cup H \cup K = G \cup (H \cup K)\,.$$

Finally, note that the identity element in this case is the empty graph, $\mathcal{O}$. Thus, $(\Gamma, \cup)$ is a monoid.

We next show that Γ is a monoid with respect to the graph operation $\Box$.

Proposition 1. [23] *The set of all simple, finite graphs is a monoid under the graph operation Cartesian product.*

Proof. Closure is immediate from the definition of Cartesian product. Let G, H, and K be graphs in Γ. We wish to show that $(G \Box H) \Box K = G \Box (H \Box K)$. Define

$$\phi : V(G \Box H) \times V(K) \to V(G) \times V(H \Box K)$$

$$\phi(((x, y), z)) = (x, (y, z))$$

for all $x \in V(G)$, $y \in V(H)$, and $z \in V(K)$. Suppose that $((x_1, y_1), z_1)$ is adjacent to $((x_2, y_2), z_2)$

in $(G\Box H)\Box K$, where $x_i \in V(G)$, $y_i \in V(H)$, and $z_i \in V(K)$ for $i \in \{1,2\}$. It follows that either $(x_1, y_1) = (x_2, y_2)$ and $z_1z_2 \in E(K)$ or $(x_1, y_1)(x_2, y_2) \in E(G\Box H)$ and $z_1 = z_2$.

1. Suppose that $(x_1, y_1) = (x_2, y_2)$ and $z_1z_2 \in E(K)$. Thus, $x_1 = x_2$, $y_1 = y_2$, and (y_1, z_1) is adjacent to (y_2, z_2) in $H\Box K$ since $z_1z_2 \in E(K)$. Furthermore, since $x_1 = x_2$, we know $(x_1, (y_1, z_1))$ is adjacent to $(x_2, (y_2, z_2))$ in $G\Box(H\Box K)$.

2. Suppose $z_1 = z_2$ and $(x_1, y_1)(x_2, y_2) \in E(G\Box H)$. This implies that either $x_1 = x_2$ and $y_1y_2 \in E(H)$, or $x_1x_2 \in E(G)$ and $y_1 = y_2$. So assume that $x_1 = x_2$ and $y_1y_2 \in E(H)$. It follows that $(y_1, z_1)(y_2, z_2) \in E(H\Box K)$ since $z_1 = z_2$ and $y_1y_2 \in E(H)$. Thus, $(x_1, (y_1, z_1))$ is adjacent to $(x_2, (y_2, z_2))$ in $G\Box(H\Box K)$. A similar argument shows the same conclusion when $x_1x_2 \in E(G)$ and $y_1 = y_2$.

Thus,

$$((x_1, y_1), z_1)((x_2, y_2), z_2) \in E((G\Box H)\Box K)$$

implies

$$(x_1, (y_1, z_1))(x_2, (y_2, z_2)) \in E(G\Box(H\Box K)),$$

and we have shown that ϕ is a homomorphism. Clearly, ϕ is onto. We next show that ϕ is injective. Suppose that $\phi(((x_1, y_1), z_1)) = \phi(((x_2, y_2), z_2))$ for some $x_i \in V(G), y_i \in V(H), z_i \in V(K), i \in \{1,2\}$. It follows that $(x_1, (y_1, z_1)) = (x_2, (y_2, z_2))$ if and only if $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$. Thus, $((x_1, y_1), z_1) = ((x_2, y_2), z_2)$ and ϕ is a bijection. So ϕ^{-1} exists and the reverse of the above argument shows that ϕ^{-1} is also a homomorphism. Therefore, associativity holds for any G, H, and K in Γ under $\Box$.

Finally, notice that K_1, the graph consisting of a single vertex, is the identity element for Γ under $\Box$. So $(\Gamma, \Box)$ is, indeed, a monoid. $\Box$

Recall that a *semiring* is a set R together with two binary operations, $+$ and $\cdot$, that satisfies the following.

(a) $(R, +)$ is a commutative monoid,

(b) $(R, \cdot)$ is a monoid,

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$ for every $a, b, c \in R$,

(d) The additive identity 0 annihilates R. That is, for every $a \in R$, $a \cdot 0 = 0$.

In the context of Γ with the graph operations $\cup$ and $\Box$, we next show that $(\Gamma, \cup, \Box)$ is a semiring.

Proposition 2. [23] *The set of all simple, finite graphs forms a semiring under the graph operations disjoint union and Cartesian product.*

Proof. We have already shown that $(\Gamma, \cup)$ and $(\Gamma, \Box)$ are monoids. Furthermore, given graphs G and H, $G \cup H = H \cup G$. Thus, $(\Gamma, \cup)$ is a commutative monoid. Next, notice that for any $G \in \Gamma$, $V(G \Box \mathcal{O}) = \emptyset$. Thus, $G \Box \mathcal{O} = \mathcal{O}$.

Finally, let G, H, and K be graphs in Γ and define

$$\psi : V\left(G \Box (H \cup K)\right) \to V\left((G \Box H) \cup (G \Box K)\right)$$
$$\phi((x, y)) = (x, y)$$

for all $x \in V(G)$ and $y \in V(H \cup K)$. Suppose for some $x_1, x_2 \in V(G)$ and $y_1, y_2 \in V(H \cup K)$ that (x_1, y_1) is adjacent to (x_2, y_2) in $G \Box (H \cup K)$. This implies that $x_1 = x_2$ and $y_1 y_2 \in E(H \cup K)$ or $x_1 x_2 \in E(G)$ and $y_1 = y_2$.

1. Suppose that $x_1 = x_2$ and $y_1 y_2 \in E(H \cup K)$. It follows that either y_1 and y_2 are both vertices of H or both vertices of K. Assume first that $y_1, y_2 \in V(H)$, and $y_1 y_2 \in E(H)$. Thus, $(x_1, y_1), (x_2, y_2) \in V(G \Box H)$ and $(x_1, y_1)(x_2, y_2) \in E(G \Box H)$. On the other hand, if $y_1, y_2 \in V(K)$, then $(x_1, y_1)(x_2, y_2) \in E(G \Box K)$.

2. Suppose that $x_1 x_2 \in E(G)$ and $y_1 = y_2$, which implies that $y_1 \in V(H)$ or $y_1 \in V(K)$. If $y_1 \in V(H)$, then $(x_1, y_1)(x_2, y_2) \in E(G \Box H)$. On the other hand, if $y_1 \in V(K)$, then $(x_1, y_1)(x_, y_2) \in E(G \Box K)$.

Thus, ψ is a homomorphism. It is clear that ψ is a bijection so ψ^{-1} exists. Moreover, the reverse argument shows that ψ^{-1} is a homomorphism as well. Therefore, $G \Box (H \cup K) = (G \Box H) \cup (G \Box K)$, and we conclude that $(\Gamma, \cup, \Box)$ is a semiring. $\Box$

There are several important implications of Proposition 2. First note that since $(\Gamma, \Box)$ is a monoid, associativity allows us to drop the parantheses when taking the product of at least three graphs. That is, we may use the notation $G \Box H \Box K$ to denote $(G \Box H) \Box K$ and $G \Box (H \Box K)$, since these two graphs are isomorphic. Furthermore, notice that if we define $f : V(G \Box H) \to V(H \Box G)$ as

$f((x_1, y_1)) = (y_1, x_1)$, then f is an isomorphism. This implies that $(\Gamma, \square)$ is a commutative monoid. In Chapter 4, we consider graphs of the form $K_n \square K_m$ where $2 \leq n \leq m$. Thus, the fact that commutativity holds with respect to the Cartesian product guarantees that we have considered all graphs of the form $K_n \square K_m$ for $n, m \in \mathbb{N}_{\geq 2}$. Finally, the fact that the Cartesian product satisfies $G \square (H \cup K) = (G \square H) \cup (G \square K)$ allows us to consider the Cartesian product of disconnected graphs.

2.2 Direct Product

Given two graphs G and H, the *direct product* $G \times H$ is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edge set is

$$E(G \times H) = \{(x_1, y_1)(x_2, y_2) \mid x_1 x_2 \in E(G) \text{ and } y_1 y_2 \in E(H)\}.$$

Figure 2.3 depicts the direct product of K_2 and K_3. Other names for the direct product that have appeared in the literature are *tensor product, Kronecker product, cardinal product, relational product, cross product, conjunction, weak direct product, Cartesian product,* or *categorical product.*

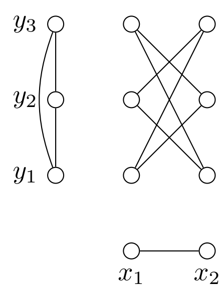


Figure 2.3: $K_2 \times K_3$

2.2.1 Algebraic Structure

Let Γ_0 represent the set of all finite graphs, which may or may not be simple. Similar to the Cartesian product, $(\Gamma_0, \cup)$ is clearly a monoid with identity element $\mathcal{O}$. We now show that $(\Gamma_0, \times)$ is a monoid and $(\Gamma_0, \cup, \times)$ is a semiring. As in Section 2.1.2, the results that follow are stated in [23, pg. 55-56], and we provide the details that have been omitted in [23].

Proposition 3. [23] *The set of all finite graphs is a monoid under the graph operation direct product.*

Proof. Let G, H, and K be graphs in Γ_0. Since G and H are finite, $G \times H$ is also finite. Thus, closure is satisfied. We next show that $(G \times H) \times K = G \times (H \times K)$. Define

$$\phi : V\left((G \times H) \times K\right) \to V\left(G \times (H \times K)\right)$$
$$\phi(((x, y), z)) = (x, (y, z))$$

for all $x \in V(G), y \in V(H)$, and $z \in V(K)$. Suppose that $((x_1, y_1), z_1)$ is adjacent to $((x_2, y_2), z_2)$ in $(G \times H) \times K$, where $x_i \in V(G), y_i \in V(H)$, and $z_i \in V(K)$ for $i \in \{1, 2\}$. It follows that $(x_1, y_1)(x_2, y_2) \in E(G \times H)$ and $z_1z_2 \in E(K)$, which implies that $x_1x_2 \in E(G)$, $y_1y_2 \in E(H)$, and $z_1z_2 \in E(K)$. Thus, (y_1, z_1) is adjacent to (y_2, z_2) in $H \times K$, so that $(x_1, (y_1, z_1))(x_2, (y_2, z_2)) \in E(G \times (H \times K))$. Since the choice of $((x_1, y_1), z_1)$ and $((x_2, y_2), z_2)$ was arbitrary, ϕ is a homomorphism. We next show that ϕ is injective. Suppose that $\phi(((x_1, y_1), z_1)) = \phi(((x_2, y_2), z_2))$ for some $x_i \in V(G), y_i \in V(H), z_i \in V(K), i \in \{1, 2\}$. It follows that $(x_1, (y_1, z_1)) = (x_2, (y_2, z_2))$ if and only if $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$. Thus, $((x_1, y_1), z_1) = ((x_2, y_2), z_2)$ and ϕ is a bijection. So ϕ^{-1} exists and the reverse of the argument above shows that ϕ^{-1} is also a homomorphism. It follows that associativity holds for any G, H, and K in Γ_0 under $\times$. Finally, notice that K_1^s, the graph consisting of a single loop, is the identity element for Γ_0 under $\times$. Thus, $(\Gamma_0, \times)$ is a monoid. $\square$

Proposition 4. [23] *The set of all finite graphs forms a semiring under the graph operations disjoint union and direct product.*

Proof. We have already seen that $(\Gamma_0, \cup)$ and $(\Gamma_0, \times)$ are monoids. Furthermore, $G \cup H = H \cup G$ implies that $(\Gamma_0, \cup)$ is a commutative monoid. Next, note that for any graph $G \in \Gamma_0$, $V(G \times \mathcal{O}) = \emptyset$. Thus, $G \times \mathcal{O} = \mathcal{O}$.

Finally, let G, H, and K be in Γ_0. We need to show that $G \times (H \cup K) = (G \times H) \cup (G \times K)$. Define

$$\phi : V\left(G \times (H \cup K)\right) \to V\left((G \times H) \cup (G \times K)\right)$$
$$\phi((x, y)) = (x, y)$$

for all $x \in V(G)$ and $y \in V(H \cup K)$. Suppose $(x_1, y_1)(x_2, y_2) \in E(G \times (H \cup K))$, where $x_i \in$

$V(G)$ and $y_i \in V(H \cup K)$ for $i \in \{1, 2\}$. By definition of the direct product, $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H \cup K)$. This implies that either $y_1y_2 \in E(H)$ or $y_1y_2 \in E(K)$. If $y_1y_2 \in E(H)$, then $(x_1, y_1)(x_2, y_2) \in E(G \times H)$. Otherwise, $(x_1, y_1)(x_2, y_2) \in E(G \times K)$. So ϕ is a homomorphism. Clearly, ϕ is a bijection so ϕ^{-1} exists. The reverse of the above argument shows that ϕ^{-1} is a homomorphism. It follows that $(\Gamma_0, \cup, \times)$ is a semiring. □

As in Section 2.1.2, the fact that associativity holds with respect to the direct product allows us to write $G \times H \times K$ to denote $(G \times H) \times K$ and $G \times (H \times K)$ as these two graphs are isomorphic. Moreover, if we define $f : V(G \times H) \to V(H \times G)$ as $f((x_1, y_1)) = (y_1, x_1)$, then f is an isomorphism. This implies that $(\Gamma_0, \times)$ is a commutative monoid. Thus, when we consider graphs of the form $K_n \times K_m$ where $3 \leq n \leq m$ in Chapter 3, commutativity guarantees that we have considered all graphs of the form $K_n \times K_m$ where $n, m \in \mathbb{N}_{\geq 3}$.

2.3 Generalized Prisms

Given a graph G and any permutation π of $V(G)$, the *prism of G with respect to π* is the graph πG obtained by taking two disjoint copies of G, denoted G^1 and G^2, and joining every $u \in V(G^1)$ with $\pi(u) \in V(G^2)$. That is, the edges between G^1 and G^2 form a perfect matching in πG.

If π is the identity $\mathbf{1}_G$, then $\pi G = G \Box K_2$. The graph $G \Box K_2$ is often referred to as the *prism of G*. The study of domination in the prism of a graph was initiated by Hartnell and Rall in [24]. In general, πG is not a graph product and appears to have no algebraic structure. In Chapter 5, we resolve a conjecture regarding this particular type of graph.

2.4 Identifying Codes

Given a graph G and a vertex $v \in V(G)$, we define the *ball of radius r centered at v* to be $B_r(v) = \{\ u \in V(G) \ \mid\ d_G(u, v) \leq r\ \}$. A set $C \subseteq V(G)$ is an *r-identifying code of G* if

(i) for every $u \in V(G)$, $B_r(u) \cap C \neq \emptyset$, and

(ii) for every pair of distinct vertices u, v in G, $B_r(u) \cap C \neq B_r(v) \cap C$.

The first to study identifying codes were Karpovsky, Chakrabarty and Levitin [28] who used them to analyze fault-detection problems in multiprocessor systems. For the purpose of this dissertation, we let $r = 1$ and refer to a 1-identifying code as simply an identifying code. Thus, condition (i) becomes $N[u] \cap C \neq \emptyset$ and condition (ii) becomes $N[u] \cap C \neq N[v] \cap C$.

Identifying codes can be restated in the context of another familiar graph parameter. Given a graph G, a set of vertices $D \subseteq V(G)$ is a *dominating set* if $V(G) = N[D]$. That is, D is a dominating set of G if every vertex $v \in V(G) - D$ is adjacent to a vertex $w \in D$. The *domination number*, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. So we can restate the definition of an identifying code as follows: a subset $C \subseteq V(G)$ is an identifying code of G if C is a dominating set and for every pair of distinct vertices u, v in G, $N[u] \cap C \neq N[v] \cap C$.

Given an identifying code, or ID code, C, we call the vertices of C *codewords*. A pair of vertices $u, v \in V(G)$ are *separated* by $x \in C$ if x belongs to exactly one of the closed neighborhoods $N[u]$ or $N[v]$. Given a set $C \subseteq V(G)$, we say C *separates* $V(G)$ if for every pair of vertices $u, v \in V(G)$, there exists a codeword $c \in C$ such that c separates u and v. Therefore, a dominating set C in G is an identifying code if C separates $V(G)$. The black vertices of Figure 2.4 illustrate a minimum identifying code.

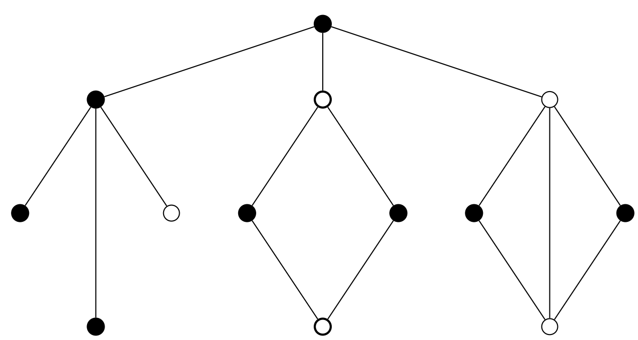


Figure 2.4: Example of a minimum identifying code

Although all graphs possess a dominating set, not all graphs possess an identifying code. For example consider the graph K_3. For any two distinct vertices $u, v \in V(K_3)$, $N[u] = N[v]$. Therefore, no subset of $V(K_3)$ separates $V(K_3)$. Given a graph G, we refer to two distinct vertices $u, v \in V(G)$ such that $N[u] = N[v]$ as *twins*, and we say G is *twin-free* if it contains no such pair. Thus, a graph G admits an identifying code if and only if G is twin-free.

As with dominating sets, we focus on the minimum cardinality of an identifying code, which we denote by $\gamma^{\mathrm{ID}}(G)$. Since any identifying code is also a dominating set, we know $\gamma(G) \leq \gamma^{\mathrm{ID}}(G)$. Notice that constructing an identifying code in G is enough to give an upper bound for $\gamma^{\mathrm{ID}}(G)$. However, establishing a lower bound for $\gamma^{\mathrm{ID}}(G)$ can be significantly harder. That is, proving a dominating set $C \subseteq V(G)$ that separates $V(G)$ is of minimum cardinality can be significantly harder. The majority of Chapters 3 and 4 is dedicated to proving that a proposed upper bound for $\gamma^{\mathrm{ID}}(K_n \times K_m)$ or $\gamma^{\mathrm{ID}}(K_n \square K_m)$ for any $m, n \in \mathbb{N}$ is also a lower bound, thereby giving an exact value. In the remainder of this section, we survey the literature for known lower bounds for $\gamma^{\mathrm{ID}}(G)$ given any graph G.

2.4.1 Lower Bounds

A general lower bound for the size of an identifying code of a twin-free graph was first given in the original paper by Karpovsky, Chakrabarty and Levitin [28].

Theorem 5. [28, Theorem 1, pg. 599] *If G is a twin-free graph on n vertices, then*

$$\gamma^{\mathrm{ID}}(G) \geq \lceil \log_2(n+1) \rceil.$$

The bound given in Theorem 5 is tight, and all graphs achieving this bound are characterized in [35]. Karpovsky et al. [28] also gave a general lower bound of $\gamma^{\mathrm{ID}}(G)$ based on the maximum degree of G. Foucaud [15] later improved this to the lower bound given below.

Theorem 6. [15, Theorem 2.1, pg. 25] *If G is a twin-free, connected graph on n vertices, then*

$$\gamma^{\mathrm{ID}}(G) \geq \frac{2n}{\Delta(G)+2}.$$

Identifying codes have been studied in certain families of graphs, including trees [3], paths [2, 7, 27], and cycles [2, 19, 40, 7, 27]. An excellent, detailed list of references on identifying codes can be found on Antoine Lobstein's webpage [33]. In terms of graph products, a few of the more recent results have been in the study of hypercubes [4, 25, 26, 29, 34], the Cartesian product of cliques [18, 20], and the lexicographic product of two graphs [14]. Additionally, there has been a considerable amount of research on identifying codes in infinite grids. Of the infinite grid graphs that have been studied, the two that bear the closest resemblance to our $K_n \times K_m$ and $K_n \square K_m$

for $m, n \in \mathbb{N}$ are the infinite square grid and the infinite king grid defined below where $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.

1. The infinite square grid G_s has $V(G_s) = \mathbb{Z}^2$ in which two vertices are adjacent if their Euclidean distance is 1.

2. The infinite king graph G_k has $V(G_k) = \mathbb{Z}^2$ in which two vertices are adjacent if their Euclidean distance is 1 or $\sqrt{2}$.

In the context of an infinite graph, bounding the size of an identifying code loses all meaning. If we let $R_n = \{(x, y) \in \mathbb{Z}^2 \ \mid \ |x| \leq n, |y| \leq n \ \}$, then the *density of an identifying code C* is defined as

$$D = \lim_{n\to\infty} \sup \frac{|C \cap R_n|}{|R_n|}.$$

This definition of density is analogous to computing the ratio of code words to the number of vertices in the graph.

Cohen et al. [11] show that the density of any identifying code in the king grid is at least $\frac{2}{9}$. Ben-Haim et al. [1] prove that the minimum density of an identifying code in the square grid is $\frac{7}{20}$. However, the fact that our graphs are neither a king grid graph or a square grid graph, nor are they infinite means much of this work will not be of any use in later chapters.

In 2008, Gravier et al. [20] considered identifying codes in the Cartesian product of two same-sized cliques.

Theorem 7. [20, Theorem 1, pg. 2] *If C is a minimum identifying code of $K_n \square K_n$, then $|C| = \left\lfloor \frac{3n}{2} \right\rfloor$.*

Theorem 7 is the primary motivation behind the work found in Chapters 3 and 4. In Chapter 3, we consider the problem posed by Klavžar at the Bordeaux Workshop on Identifying Codes in 2011 [32] to compute $\gamma^{\mathrm{ID}}(K_n \times K_m)$. In Chapter 4, we go back to the original result in [20] and generalize Theorem 7 to include the Cartesian product of two cliques of any size.

Of course, one should also consider the lower bounds for other relevant graph parameters before continuing. El-Zaher and Pareek [12] observed that $\gamma(G \square H)$ is at least the minimum of the orders of G and H. Assuming $n \leq m$, it follows easily that $\gamma(K_n \square K_m) = n$. Showing that $\gamma(K_n \times K_m) = 3$ is also relatively easy. When we apply the lower bound given in Theorem 6 to $K_n \times K_m$, we get $\gamma^{\mathrm{ID}}(K_n \times K_m) \geq \frac{2mn}{mn-m-n+3}$. Combining the lower bounds for $\gamma^{\mathrm{ID}}(K_n \times K_m)$

from Theorem 6 and from $\gamma^{\mathrm{ID}}(K_n \times K_m) \geq \gamma(K_n \times K_m)$, we get

$$\gamma^{\mathrm{ID}}(K_n \times K_m) \geq \max\left\{\left\lceil \frac{2mn}{(m-1)(n-1)+2} \right\rceil, 3\right\} = 3 \quad \text{for } 3 \leq n \leq m.$$

We show in Chapter 3 that this lower bound is not sharp. In particular, we prove that $\gamma^{\mathrm{ID}}(K_n \times K_m)$ is the closest integer to $\frac{2}{3}(m+n)$ for $6 \leq n \leq m$.

On the other hand, when we apply the lower bound of Theorem 6 to $K_n \square K_m$, we get $\gamma^{\mathrm{ID}}(K_n \square K_m) \geq \frac{2mn}{m+n}$. Since

$$\frac{2mn}{m+n} \geq \frac{2n^2}{2n} = n,$$

and $m \geq n$, the lower bound given in Theorem 6 is equal to $\gamma(K_n \square K_m)$ only when $m = n$ and is better when $m > n$. We show in Chapter 4 that

$$\gamma^{\mathrm{ID}}(K_n \square K_m) = \begin{cases} m + \lfloor n/2 \rfloor & \text{if } m \leq 3n/2 \\ 2m - n & \text{if } m \geq 3n/2. \end{cases}$$

There is another variation of domination that is closely related to identifying codes. A set $C \subseteq V(G)$ is said to be a *locating-dominating set of G* if C is a dominating set and C separates every pair of distinct vertices in $V(G) - C$. Figure 2.5 illustrates a locating-dominating set. The size of a minimum locating-dominating set of a graph G is denoted $\gamma_L(G)$. Since an identifying code of a graph G is necessarily a locating-dominating set, then we have $\gamma^{\mathrm{ID}}(G) \geq \gamma_L(G)$.

(a) locating-dominating set in C_5 (b) ID code in C_5

Figure 2.5: Examples of locating-dominating sets and ID codes in C_5

In [38], Slater proved that for any d-regular graph G on n vertices, $\gamma_L(G) \geq \frac{2n}{d+3}$. This bound is indeed tight, but does not give us any further information about $\gamma^{\mathrm{ID}}(K_n \times K_m)$ or $\gamma^{\mathrm{ID}}(K_n \square K_m)$.

Chapter 3

Identifying codes of the direct product of two cliques

3.1 Introduction

This chapter is an expanded version of [37], where we focus on the cardinality of a minimum identifying code for the direct product of two nontrivial cliques. As all proofs are provided, we do not give specific references to this paper in this chapter.

For a positive integer n we write $[n]$ to denote the set $\{1, 2, \ldots, n\}$, and $[n]$ will be the vertex set of the complete graph K_n. In the direct product $K_n \times K_m$ we refer to a *column* as the set of all vertices having the same first coordinate. A *row* is the set of all vertices with the same second coordinate. In particular, for $i \in [n]$, the i^{th} column is $C_i = \{(i, j) \,|\, j \in [m]\}$. Similarly, for $j \in [m]$ the j^{th} row is the set $R_j = \{(i, j) \,|\, i \in [n]\}$. Using this terminology we see that two vertices of $K_n \times K_m$ are adjacent precisely when they belong to different rows and to different columns. In each figure, rows will be horizontal and columns vertical. For ease of reference in this chapter we refer to K_n as the *first factor* of $K_n \times K_m$ and K_m as the *second factor*. The two product graphs $K_n \times K_m$ and $K_m \times K_n$ are clearly isomorphic under a natural map. Throughout the remainder of this work we always have the smaller factor first.

Let $G = K_n \times K_m$, and suppose that C is a code in G. The *column span of* C is the set of all columns of G that have a nonempty intersection with C. The number of columns in the column span of C is denoted by $\mathrm{cs}(C)$. Similarly, the set of all rows of G that contain at least one member of C is the *row span of* C; its size is denoted $\mathrm{rs}(C)$. For a vertex $v = (i, j)$ of G we say that v is *column-isolated in* C if $C \cap C_i = \{v\}$. Similarly, if $C \cap R_j = \{v\}$ then we say that v is *row-isolated in* C. If v is both column-isolated and row-isolated in C, we simply say v is *isolated in* C. When there is no chance of confusion and the set C is clear from the context we shorten these to column-isolated, row-isolated and isolated, respectively.

3.2 Main Results

We give statements of the main results here, and the proofs will be given in subsequent subsections.

Note that $K_2 \times K_2$ has vertices with identical closed neighborhoods and so has no ID code.

Theorem 8. *For any positive integer $m \geq 5$, $\gamma^{\mathrm{ID}}(K_2 \times K_m) = m - 1$. In addition, if $3 \leq m \leq 4$, $\gamma^{\mathrm{ID}}(K_2 \times K_m) = m$.*

For $3 \leq n \leq 5$ and $n \leq m \leq 2n-1$ the values of $\gamma^{\mathrm{ID}}(K_n \times K_m)$ were computed by computer program and are given in the following table.

$n \backslash m$	3	4	5	6	7	8	9
3	4	4	5				
4		5	6	7	7		
5			6	7	8	9	9

Table 3.1: $\gamma^{\mathrm{ID}}(K_n \times K_m)$ for small n and m

The remaining cases are handled based on the order of the second factor relative to the first factor. Theorem 9 presents this number if both cliques have order at least 3 and one clique is sufficiently large compared to the other; its proof is given in Section 3.4.

Theorem 9. *For positive integers n and m where $n \geq 3$ and $m \geq 2n$,*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = m - 1\,.$$

In all other cases (that is, for $6 \leq n \leq m \leq 2n-1$), the minimum cardinality of an ID code for $K_n \times K_m$ is one of the values $\lfloor 2(n+m)/3 \rfloor$ or $\lceil 2(n+m)/3 \rceil$. The number $\gamma^{\mathrm{ID}}(K_n \times K_m)$

depends on the congruence of $n+m$ modulo 3. It turns out there are only two general cases instead of three, but one of them has an exception to the easily stated formula. The exact values are given in the following results whose proofs are given in Section 3.4.

Theorem 10. *Let n and m be positive integers such that $6 \le n \le m \le 2n-1$. If $n+m \equiv 0 \pmod 3$ or $n+m \equiv 2 \pmod 3$, then*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = \left\lfloor \frac{2m+2n}{3} \right\rfloor .$$

Theorem 11. *For a positive integer $n \ge 6$,*

$$\gamma^{\mathrm{ID}}(K_n \times K_{2n-5}) = 2n-4 .$$

Theorem 12. *Let n and m be positive integers such that $6 \le n \le m \le 2n-2$ and $m \ne 2n-5$. If $n+m \equiv 1 \pmod 3$, then*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = \left\lceil \frac{2m+2n}{3} \right\rceil .$$

3.3 Preliminary Properties

In this section we prove a number of results that will be useful in verifying the minimum size of ID codes in the direct product of two complete graphs. It will be helpful in what follows to remember that a vertex is adjacent to (i,j) in $K_n \times K_m$ precisely when its first coordinate is different from i and its second coordinate is different from j. Also, recall that we are assuming throughout that $n \le m$.

Lemma 13. *If C is an identifying code of $K_n \times K_m$, then $\mathrm{cs}(C) \ge n-1$ and $\mathrm{rs}(C) \ge m-1$. In particular, $|C| \ge m-1$.*

Proof. Suppose that for some $r \ne s$, $C \cap R_r = \emptyset = C \cap R_s$. For any fixed $i \in [n]$, $C \cap N[(i,r)] = C \setminus C_i = C \cap N[(i,s)]$. Since this violates C being an ID code, $K_n \times K_m$ has at most one row disjoint from C. A similar argument shows that $K_n \times K_m$ has no more than one column disjoint from C. Consequently, $|C| \ge m-1$. □

By considering $N[x]$, the following result is obvious but useful. We omit its proof.

Lemma 14. *If $C \subseteq V(K_n \times K_m)$ and $x = (i, r) \in C$, then C separates x from any $y \in (R_r \cup C_i) \setminus \{x\}$.*

Lemma 14 addresses separating two vertices that belong to the same row or to the same column. The next result concerns vertices that are not in a common row or common column, that is, two vertices at opposite "corners" of a two-row and two-column configuration in $K_n \times K_m$.

Lemma 15. *(4-Corners Property) Suppose C is a dominating set of $K_n \times K_m$. For each $(i, r), (j, s) \in K_n \times K_m$ with $i \neq j, r \neq s$, C separates (i, r) and (j, s) if and only if*

$$C \cap (C_i \cup C_j \cup R_r \cup R_s) \not\subseteq \{i, j\} \times \{r, s\}.$$

Proof. Suppose that $i \neq j$ and $r \neq s$, and let C_i, C_j and R_r, R_s be the corresponding columns and rows of $K_n \times K_m$. Write $x = (i, r), y = (j, s), w = (i, s)$ and $z = (j, r)$, and define

$$\begin{aligned} A &= C \setminus (C \cap (C_i \cup C_j \cup R_r \cup R_s)) \\ B &= [C \cap (C_i \cup C_j \cup R_r \cup R_s)] \setminus \{x, y, w, z\}. \end{aligned}$$

Observe that

$$\begin{aligned} C \cap N[x] &= A \cup (C \cap \{x, y\}) \cup (C \cap ((R_s \cup C_j) \setminus \{x, y, w, z\})) \\ C \cap N[y] &= A \cup (C \cap \{x, y\}) \cup (C \cap ((R_r \cup C_i) \setminus \{x, y, w, z\})). \end{aligned}$$

Therefore, C separates x and y if and only if at least one of the two disjoint sets $C \cap ((R_s \cup C_j) \setminus \{x, y, w, z\})$ or $C \cap ((R_r \cup C_i) \setminus \{x, y, w, z\})$ is non-empty. Since B is the union of these 2 sets, it follows that C separates x and y if and only if $B \neq \emptyset$, or equivalently if and only if

$$C \cap (C_i \cup C_j \cup R_r \cup R_s) \not\subseteq \{i, j\} \times \{r, s\}.$$

□

We will say that a dominating set D of $K_n \times K_m$ has the *4-corners property with respect to columns C_i, C_j and rows R_r, R_s* if

$$D \cap (C_i \cup C_j \cup R_r \cup R_s) \not\subseteq \{i, j\} \times \{r, s\}.$$

Hence, if a dominating set D of $K_n \times K_m$ is an ID code, then D has the 4-corners property with respect to every pair of columns and every pair of rows. Each of the next three results follows immediately from this fact.

Corollary 16. *If C is an identifying code of $K_n \times K_m$, then C has no more than one isolated codeword.*

Corollary 17. *Let C be an identifying code of $K_n \times K_m$. If $\mathrm{cs}(C) = n-1$, then there does not exist a column C_j such that $C \cap C_j = \{u, v\}$ where both u and v are row-isolated. Similarly, there is no row R_r containing exactly two codewords each of which is column-isolated if $\mathrm{rs}(C) = m-1$.*

Corollary 18. *If C is an identifying code of $K_n \times K_m$ such that $\mathrm{cs}(C) = n-1$ and $\mathrm{rs}(C) = m-1$, then C has no isolated codeword.*

The next two results will be used to construct ID codes, thereby providing an upper bound for $\gamma^{\mathrm{ID}}(K_n \times K_m)$. Which one is used will depend on the congruence of $n+m$ modulo 3.

Proposition 19. *If $C \subseteq V(K_n \times K_m)$ satisfies the following conditions, then C is an identifying code of $K_n \times K_m$.*

(1) There exist $1 \leq n_1 < n_2 < n_3 \leq n$ and $1 \leq m_1 < m_2 < m_3 \leq m$ such that $(n_1, m_1), (n_2, m_2), (n_3, m_3) \in C$;

(2) C contains at most one isolated vertex, and every other vertex in C is row-isolated or column-isolated; and

(3) $\mathrm{rs}(C) = m$ and $\mathrm{cs}(C) = n$.

Proof. Assume C is as specified. For ease of reference we denote the graph $K_n \times K_m$ by G throughout this proof. By the first assumption above it follows immediately that C dominates G since $\{(n_1, m_1), (n_2, m_2), (n_3, m_3)\}$ does.

We need only to show that C separates every pair x, y of distinct vertices. First assume that x and y are in the same column. If x or y belongs to C, then Lemma 14 shows that C separates them. If neither is in C, then by our assumption that $\mathrm{rs}(C) = m$ and $\mathrm{cs}(C) = n$ we can choose a vertex $z \in C$ from the same row as x. This vertex z separates x and y. Similarly, C separates any two vertices belonging to a common row.

Now, assume $x = (i, r)$ and $y = (j, s)$ where $1 \leq i < j \leq n$ and $1 \leq r < s \leq m$. Any $v = (k, t) \in C$ that is not isolated in C is row-isolated or column-isolated but not both, and it follows that either $|C \cap C_k| \geq 2$ or $|C \cap R_t| \geq 2$.

(a) Suppose $x \in C$ but is not isolated in C. As above, either $|C \cap C_i| \geq 2$ or $|C \cap R_r| \geq 2$. Assume without loss of generality that $|C \cap C_i| \geq 2$. It follows that either $(i, s) \in C$ or there exists $1 \leq t \leq m$ where $t \notin \{r, s\}$ and $(i, t) \in C$. In the first case where we have $(i, s) \in C$, it follows that (i, s) is row-isolated, and thus $y \notin C$. However, each column of G is in the column span of C so there exists $1 \leq p \leq m$ where $p \notin \{r, s\}$ and $(j, p) \in C$ since (i, r) and (i, s) are row-isolated. Thus, $(j, p) \in C \cap N[x]$ but $(j, p) \notin C \cap N[y]$. Hence, C separates x and y. On the other hand, if there exists $1 \leq t \leq m$ where $t \notin \{r, s\}$ and $(i, t) \in C$, then $(i, t) \in C \cap N[y]$ but $(i, t) \notin C \cap N[x]$. Again, this implies that C separates x and y. If we had instead assumed that $|C \cap R_r| \geq 2$, that is we had assumed x is column-isolated and not row-isolated, then a similar argument shows that C separates x and y.

(b) Suppose $x \in C$ and is isolated in C. Since x is both row-isolated and column-isolated, $C = C \cap N[x]$. First assume that $y \notin C$. Since C_j is in the column span of C, there exists $1 \leq t \leq m$ with $t \notin \{r, s\}$ such that $(j, t) \in C$, and (j, t) separates x and y. On the other hand, if $y \in C$, then either $|C \cap C_j| \geq 2$ or $|C \cap R_s| \geq 2$ since y is not isolated. In either case, $C \cap N[y] \neq C$, and therefore C separates x and y.

(c) Suppose $x, y \in V(G) \setminus C$. If we assume that C does not separate x and y, then because each row of G is in the row span of C and each column of G is in the column span of C, it follows that

$$C \cap (C_i \cup C_j \cup R_r \cup R_s) = \{(i, s), (j, r)\}.$$

Thus, by definition, both (i, s) and (j, r) are isolated in C, contradicting the second assumption. Hence, C separates x and y.

Therefore, C separates every pair of distinct vertices, and thus C is an ID code of $K_n \times K_m$. $\square$

Proposition 20. *If $C \subseteq V(K_n \times K_m)$ satisfies the following conditions, then C is an identifying code of $K_n \times K_m$.*

(1) There exist $1 \leq n_1 < n_2 < n_3 \leq n$ and $1 \leq m_1 < m_2 < m_3 \leq m$ such that $(n_1, m_1), (n_2, m_2), (n_3, m_3) \in C$;

(2) *C contains at most one isolated vertex, and every other vertex in C is row-isolated or column-isolated;*

(3) $\mathrm{rs}(C) = m - 1$ *and* $\mathrm{cs}(C) = n$*; and*

(4) *If R_r has the property that every $v \in C \cap R_r$ is column-isolated but not row-isolated, then* $|C \cap R_r| \geq 3$.

Proof. As in the proof of Proposition 19 we see that C dominates $G = K_n \times K_m$.

We show that C separates every pair x, y of distinct vertices in G. Let R_r be the row not in the row span of C. Notice that $G \setminus R_r \cong K_n \times K_{m-1}$ and that C satisfies the hypotheses of Proposition 19 when considered as a subset of $V(G) \setminus R_r$. Thus, C separates x, y if neither is in R_r, and so we may assume that $x \in R_r$, say $x = (i, r)$.

(a) First assume that $y = (j, r)$ with $i \neq j$. Since $\mathrm{cs}(C) = n$, there exists $1 \leq s \leq m$ such that $r \neq s$ and $(i, s) \in C$. This vertex (i, s) separates x and y. Next, assume that $y = (i, t)$ for some $1 \leq t \leq m$ with $t \neq r$. If $y \in C$, then y separates x and y. However, if $y \notin C$, then since each row of G other than R_r is in the row span of C, there exists $1 \leq j \leq n$ with $i \neq j$ such that $(j, t) \in C$. It follows that (j, t) separates x and y.

(b) Next, assume that $y = (j, s)$ where $i \neq j$ and $r \neq s$. If we assume that C does not separate x and y, then C does not satisfy the 4-Corners Property with respect to columns C_i, C_j and rows R_r, R_s. In addition, since R_r is not in the row span of C,

$$C \cap (C_i \cup C_j \cup R_r \cup R_s) \subseteq \{(i, s), (j, s)\}.$$

Since both C_i and C_j are in the column span of C, it follows that $C \cap (C_i \cup C_j \cup R_r \cup R_s) = \{(i, s), (j, s)\}$. This means that R_s contains exactly two members of C and they are both column-isolated, contradicting one of the assumptions. Hence, this case cannot occur either, and it follows that C separates x and y.

Therefore, C is an ID code of $K_n \times K_m$. □

3.4 Proofs of Main Results

In this section we prove all of our main results of this chapter. The general strategy will be to construct an ID code of the claimed optimal size (by employing Propositions 19 and 20) and prove the given direct product has no smaller ID code.

We treat the smallest case first.

Theorem 21. *For any positive integer $m \geq 5$, $\gamma^{\mathrm{ID}}(K_2 \times K_m) = m - 1$. In addition, if $3 \leq m \leq 4$, $\gamma^{\mathrm{ID}}(K_2 \times K_m) = m$.*

Proof. If C is any ID code of $K_2 \times K_3$, then by Lemma 13 it follows that $\mathrm{rs}(C) \geq 2$. No subset of two elements in different rows dominates $K_2 \times K_3$, and so $\gamma^{\mathrm{ID}}(K_2 \times K_3) \geq 3$. It is easy to check that $\{(1,1),(1,2),(1,3)\}$ is an ID code. A similar argument shows that $\gamma^{\mathrm{ID}}(K_2 \times K_4) = 4$.

If $m \geq 5$, it follows from Lemma 13 that $\gamma^{\mathrm{ID}}(K_2 \times K_m) \geq m - 1$, and it is easily checked that $\{(1,1),(1,2)\} \cup \{(2,r) \,|\, 3 \leq r \leq m-1\}$ is an ID code. □

Now we turn our attention to the case when the first factor has order at least 3 and the second factor is sufficiently larger than the first.

Theorem 22. *For positive integers n and m where $n \geq 3$ and $m \geq 2n$,*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = m - 1\,.$$

Proof. Consider the set

$$D = \{(i, 2i-1), (i, 2i) \,|\, i \in [n-1]\} \;\cup\; \{(n, j) \,|\, 2n-1 \leq j \leq m-1\}.$$

Notice that each v in D is row-isolated but not column-isolated, $\mathrm{rs}(D) = m - 1$ and $\mathrm{cs}(D) = n$. Furthermore, $(1,1),(2,3)$ and $(3,5)$ are in D. Thus, Proposition 20 guarantees that D is an ID code, and Lemma 13 gives the desired result. □

We now focus on direct products of the form $K_n \times K_m$ where $6 \leq n \leq m \leq 2n-1$ and prove that in all cases

$$\left\lfloor \frac{2m+2n}{3} \right\rfloor \leq \gamma^{\mathrm{ID}}(K_n \times K_m) \leq \left\lceil \frac{2m+2n}{3} \right\rceil . \tag{3.1}$$

For the remainder of this chapter, when considering any ID code C of $G = K_n \times K_m$ we define

$$A_C = \{v \in C \mid v \text{ is row-isolated in } C\}$$

and

$$B_C = \{v \in C \mid v \text{ is column-isolated in } C\} .$$

Let $|A_C| = x$, and let p denote the number of columns C_i of G such that $|C \cap C_i| \geq 2$ and $C \cap C_i \subseteq A_C$. Similarly, let $|B_C| = y$, and let q represent the number of rows R_r of G such that $|C \cap R_r| \geq 2$ and $C \cap R_r \subseteq B_C$. Notice that C contains at most one isolated codeword, in which case $|A_C \cap B_C| = 1$. Otherwise, $A_C \cap B_C = \emptyset$. Moreover, we always have $|C| \geq |A_C \cup B_C| \geq x + y - 1$.

The approach we take in the proof of Theorem 23, Theorem 24 and Theorem 25 will be to show that any code of cardinality smaller than the claimed value will violate some consequence of the 4-Corners Property. Which consequence will depend on the particular cardinalities of the row span and column span.

Theorem 23. *If n and m are positive integers such that $6 \leq n \leq m \leq 2n - 1$ and $n + m \equiv 0 \pmod 3$ or $n + m \equiv 2 \pmod 3$, then*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = \left\lfloor \frac{2m + 2n}{3} \right\rfloor .$$

Proof. Suppose C is an ID code of $G = K_n \times K_m$ such that $|C| \leq \left\lfloor \frac{2n+2m}{3} \right\rfloor - 1$. We consider four cases based on the possible values of $\mathrm{cs}(C)$ and $\mathrm{rs}(C)$.

1. Suppose $\mathrm{cs}(C) = n$ and $\mathrm{rs}(C) = m$.

   Since $\mathrm{cs}(C) = n$ and $|B_C| = y$, there are $n - y$ columns that each contain at least two codewords. Thus, $|C \setminus B_C| \geq 2(n - y)$, which implies $\frac{2m+2n}{3} - 1 \geq |C| \geq 2n - y$. It follows that $y \geq \frac{4n-2m}{3} + 1$. Similarly, we get $x \geq \frac{4m-2n}{3} + 1$. Together these imply that

   $$\frac{2m + 2n}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m + 2n}{3} + 1 .$$

   This is clearly a contradiction, and hence no such C exists with $\mathrm{cs}(C) = n$ and $\mathrm{rs}(C) = m$.

2. Suppose $\mathrm{cs}(C) = n - 1$ and $\mathrm{rs}(C) = m$.

   Note that since each codeword in B_C is column-isolated and $\mathrm{cs}(C) = n - 1$, there exist at least two codewords in each of the remaining $n-1-y$ columns disjoint from the column span of B_C. However, Corollary 17 guarantees that $|C \cap C_j| \geq 3$ for any column C_j for which $|C \cap C_j| \geq 2$ and $C \cap C_j \subseteq A_C$. Since p represents the number of such columns, $|C \setminus B_C| \geq 2(n-1-y-p) + 3p = 2n-2-2y+p$. Consequently, $|C| \geq 2n-2-y+p$, and it follows that $y \geq \frac{4n-2m}{3} - 1 + p$.

Similarly, since $|A_C| = x$ and $\text{rs}(C) = m$, $|C \setminus A_C| \geq 2(m-x)$, which implies $|C| \geq 2m - x$. From Case 1 we see that this gives $x \geq \frac{4m-2n}{3} + 1$. Moreover, $|C| \geq x + y - 1$ so that

$$\frac{2m+2n}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m+2n}{3} + p - 1\,.$$

Hence, $p = 0$, and we have equality in the above so that

$$\left\lfloor \frac{2m+2n}{3} \right\rfloor - 1 = |C| = x + y - 1.$$

It follows that $C = A_C \cup B_C$. If there exists $v \in C \setminus B_C$, say $v \in C_i$, then C_i contains an additional codeword that is also row-isolated. Hence, p is at least 1. However, this contradicts $p = 0$ since each codeword is either row-isolated or column-isolated. Consequently, $m = \text{rs}(C) \leq |C| = |B_C| \leq n-1 \leq m-1$. This contradiction shows that this case cannot occur.

3. Suppose $\text{cs}(C) = n$ and $\text{rs}(C) = m-1$.

If we interchange the roles of rows and columns in Case 2, then we are led to $q = 0$ and

$$\left\lfloor \frac{2m+2n}{3} \right\rfloor - 1 = |C| = x + y - 1\,.$$

Thus, $C = A_C \cup B_C$. On the other hand, since $\text{cs}(C) = n$ it follows as in Case 1 that

$$y \geq \frac{4n-2m}{3} + 1 \geq \frac{4n - 2(2n-1)}{3} + 1 = \frac{5}{3}\,.$$

Since y is an integer, we see that C has at least two column-isolated codewords. One of these, say v, is isolated since $|C| = x + y - 1$. Let w be a column-isolated codeword with $w \neq v$, and assume that $w \in R_j$. Since w is not isolated but is column-isolated, R_j contains another codeword besides w. All codewords in R_j are therefore in B_C, and thus $q \geq 1$. This contradiction shows that this case cannot occur.

4. Suppose that $\mathrm{cs}(C) = n-1$ and $\mathrm{rs}(C) = m-1$.

   From Case 2 and Case 3, we see that

   $$y \geq \frac{4n-2m}{3} - 1 + p \quad \text{and} \quad x \geq \frac{4m-2n}{3} - 1 + q\,.$$

   Since $\mathrm{cs}(C) = n-1$ and $\mathrm{rs}(C) = m-1$, it follows from Corollary 18 that C does not contain an isolated vertex. It follows that

   $$\frac{2m+2n}{3} - 1 \geq |C| \geq x + y \geq \frac{2m+2n}{3} - 2 + p + q\,.$$

   Hence, $p + q \leq 1$.

   Suppose $p = 1$. Consequently, we have equality throughout the above inequality, and thus $C = A_C \cup B_C$. Suppose there exists $v \in B_C$, say $v \in R_r$. Since $q = 0$ and there are no isolated codewords, it follows that C contains another codeword u in R_r that is not column-isolated. But $u \notin A_C \cup B_C$, which is a contradiction. Therefore, $C = A_C$. Since $p = 1$ we are led to conclude that $\mathrm{cs}(C) = 1$, which is another contradiction.

   To show that $q = 1$ is not possible we simply interchange the roles of A_C and B_C in the above.

   Finally, suppose $p = 0 = q$. Since $p = 0$, any column that contains a row-isolated codeword would also have to contain a codeword that is not row-isolated. Since there can exist at most one of these to guarantee $|C| \leq \lfloor \frac{2m+2n}{3} \rfloor - 1$, there is a column C_i such that $A_C \subseteq C_i$, and for some r, $(i,r) \in C \setminus (A_C \cup B_C)$. Similarly, since $q = 0$, if there exists a row containing a column-isolated codeword, then that row contains a codeword that is not column-isolated. Since $|C \setminus (A_C \cup B_C)| \leq 1$, such a codeword must be (i,r). This implies that $\frac{2m+2n}{3} - 1 \geq |C| \geq m - 1 + n - 2$, and this implies that $n + m \leq 6$, contradicting our assumption.

Therefore, every ID code of $K_n \times K_m$ has cardinality at least $\lfloor \frac{2m+2n}{3} \rfloor$.

An application of Proposition 19 shows that the following sets are ID codes of cardinality $\lfloor \frac{2m+2n}{3} \rfloor$ and finishes the proof. See Figure 3.1 for several specific instances of these constructions.

If $n+m \equiv 0 \pmod 3$, let

$$D_1 = \{(i,2i-1),(i,2i) | 1 \le i \le a\} \cup \{(a+2j-1, 2a+j),(a+2j,2a+j) | 1 \le j \le b\},$$

where $a = \frac{2m-n}{3}$ and $b = \frac{2n-m}{3}$. For $n+m \equiv 2 \pmod 3$ but $m \neq 2n-1$, let $a = \frac{2m-n-1}{3}$, $b = \frac{2n-m-1}{3}$, and

$$D_2 = \{(i,2i-1),(i,2i) \big| 1 \le i \le a\} \cup \{(a+2j-1,2a+j),(a+2j,2a+j) \big| 1 \le j \le b\} \cup \{(n,m)\}.$$

Finally, if $m = 2n-1$, let

$$D_3 = \{(i,2i-1),(i,2i) | i \in [n-1]\} \ \cup \ \{(n,2n-1)\}.$$

$\square$

The following figure illustrates ID codes of optimal order for several of the cases of Theorem 23. The vertices of the direct products in the figure are represented, but the edges are omitted for clarity. Recall that columns are vertical and rows are horizontal. Solid vertices indicate the members of an optimal ID code in each case.

(a) $K_6 \times K_6$

(b) $K_6 \times K_8$

Figure 3.1: Examples of ID codes when $n+m \equiv 0,2 \pmod 3$

For a fixed $n \ge 6$, the lone exception to the formula $\lceil \frac{2m+2n}{3} \rceil$ for $\gamma^{\mathrm{ID}}(K_n \times K_m)$ where $n \le m \le 2n-2$ and $n+m$ congruent to 1 modulo 3 is the instance $m = 2n-5$. We now prove Theorem 24, which shows the correct value is $\lfloor \frac{2(2n-5)+2n}{3} \rfloor$. We restate it here for convenience.

Theorem 24. *For a positive integer $n \geq 6$,*

$$\gamma^{\mathrm{ID}}(K_n \times K_{2n-5}) = 2n - 4\,.$$

Proof. Assume there exists an ID code C for $K_n \times K_{2n-5}$ such that $|C| \leq 2n-5$. Since $\mathrm{rs}(C) \geq 2n-6$, we consider the following two cases.

1. Suppose that $\mathrm{rs}(C) = 2n - 6$.

   Since each codeword in A_C is row-isolated and $\mathrm{rs}(C) = 2n-6$, there exist at least two codewords in each of the remaining $2n-6-x$ rows disjoint from the row span of A_C. However, Corollary 17 guarantees that $|C \cap R_r| \geq 3$ for any row R_r where $C \cap R_r \subseteq B_C$. Since q represents the number of these rows, $|C \setminus A_C| \geq 2(2n-6-x-q)+3q$, which implies $|C| \geq 4n-12-x+q$. Consequently, $2n-5 \geq 4n-12-x+q$, which implies $x \geq 2n-7+q$.

   Similarly, since $\mathrm{cs}(C) \geq n-1$ and each codeword in B_C is column-isolated, there are at least $n-1-y$ columns disjoint from the column span of B_C that each contain at least two codewords. Thus, $|C \setminus B_C| \geq 2(n-1-y)$, which implies that $|C| \geq 2n-2-y$. Therefore, $y \geq 3$. It follows that

   $$2n - 5 \geq |C| \geq x + y - 1 \geq 2n - 5 + q.$$

   Thus, $q = 0$. Moreover, we have equality in the above, and therefore $C = A_C \cup B_C$. On the other hand, $y \geq 3$ and only one of these column-isolated codewords can be isolated. Consequently, $q \geq 1$ since each codeword of C is either row-isolated or column-isolated, which is a contradiction.

2. Suppose $\mathrm{rs}(C) = 2n - 5$.

   Using a similar argument as in Case 1, we have $|C \setminus A_C| \geq 2(2n-5-x)$, which implies $|C| \geq 4n-10-x$. This implies $2n-5 \geq |C| \geq x \geq 2n-5$. Therefore, it follows that $C = A_C$, and thus $\mathrm{cs}(C) = \mathrm{cs}(A_C) \leq \frac{2n-6}{2} + 1 = n-2$, contradicting Lemma 13.

Therefore, no such identifying code C exists with $|C| \leq 2n-5$. It follows that $\gamma^{\mathrm{ID}}(G) \geq 2n-4$.

An application of Proposition 20 shows that the set

$$D = \{(i, 2i-1), (i, 2i) \big| 1 \leq i \leq n-4\} \cup \{(n-3, 2n-7), (n-2, 2n-7), (n-1, 2n-7), (n, 2n-6)\}$$

is an ID code of $K_n \times K_{2n-5}$ of cardinality $2n-4$. $\square$

Theorem 25. *Let n and m be positive integers such that $6 \leq n \leq m \leq 2n-2$ and $m \neq 2n-5$. If $n+m \equiv 1 \pmod 3$, then*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = \left\lceil \frac{2m+2n}{3} \right\rceil .$$

Proof. Notice that $\lceil \frac{2m+2n}{3} \rceil = \frac{2m+2n+1}{3}$. Assume that there exists an ID code C for $K_n \times K_m$ such that $|C| \leq \frac{2n+2m+1}{3} - 1$. We again consider four cases based on the possible values of $\mathrm{cs}(C)$ and $\mathrm{rs}(C)$.

1. Suppose $\mathrm{cs}(C) = n$ and $\mathrm{rs}(C) = m$.

   Using reasoning similar to that in Case 1 of the proof of Theorem 23, we have $|C \setminus B_C| \geq 2(n-y)$. This implies that $|C| \geq 2n-y$, and hence

   $$\frac{2m+2n+1}{3} - 1 \geq |C| \geq 2n - y\,.$$

   It follows that $y \geq \frac{4n-2m+2}{3}$. Similarly, we have that $x \geq \frac{4m-2n+2}{3}$. On the other hand, we know $|C| \geq x+y-1$. Consequently, $\frac{2m+2n+1}{3} - 1 \geq x+y-1 \geq \frac{2m+2n+1}{3}$, which is clearly a contradiction.

2. Suppose $\mathrm{cs}(C) = n-1$ and $\mathrm{rs}(C) = m$.

   Since $|B_C| = y$ and $\mathrm{cs}(C) = n-1$, there exist at least two codewords in each of the remaining $n-1-y$ columns that are disjoint from the column span of B_C. However, Corollary 17 guarantees $|C \cap C_j| \geq 3$ for any such column C_j where $C \cap C_j \subseteq A_C$. Since p represents the number of these columns, $|C \setminus B_C| \geq 2(n-1-y-p) + 3p = 2n-2-2y+p$. As a result it follows that $y \geq \frac{4n-2m-4}{3} + p$.

   Similarly, since $\mathrm{rs}(C) = m$ and $x = |A_C|$ we get $|C \setminus A_C| \geq 2(m-x)$, which implies $|C| \geq 2m-x$. As in Case 1 it follows that $x \geq \frac{4m-2n+2}{3}$. This yields

   $$\frac{2m+2n+1}{3} - 1 \geq |C| \geq x+y-1 \geq \frac{2m+2n+1}{3} + p - 2\,.$$

   Thus, $p \leq 1$. Assume first that $p = 1$. This yields equality in the above, and thus $C = A_C \cup B_C$, $y = \frac{4n-2m-1}{3}$ and $x = \frac{4m-2n+2}{3}$. Furthermore, C contains an isolated codeword, call it v. Since $p = 1$, there exists a column C_i such that $A_C \setminus \{v\} = C \cap C_i$. It follows that $\mathrm{cs}(A_C) = 2$. On the other hand, $\mathrm{cs}(C) = n-1$ so $B_C \setminus \{v\}$ spans the remaining $n-3$ columns. Therefore, $n-3 = \frac{4n-2m-1}{3} - 1$, which contradicts the assumption that $n \leq m$.

Therefore, we conclude that $p = 0$. First assume that C contains no isolated codeword. This implies

$$\frac{2m+2n+1}{3} - 1 \geq |C| \geq x + y \geq \frac{2m+2n+1}{3} + p - 1 \, .$$

Since $p = 0$ we get equality throughout the above, and hence $C = A_C \cup B_C$. As in the proof of Case 2 of Theorem 23 we arrive at a contradiction. Therefore, C contains an isolated codeword, say v. Because $p = 0$, any column that contains a row-isolated codeword other than v would also have to contain a codeword that is not row-isolated. Furthermore, the fact that $p = 0$, together with

$$\frac{2m+2n+1}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m+2n+1}{3} + p - 2 \, ,$$

implies that there exists at most one such codeword that is neither row-isolated nor column-isolated. Note that $x \geq \frac{4m-2n+2}{3} \geq 5$. Therefore, the row-isolated vertices other than v are contained in precisely one column, say C_i. Hence, $A_C \setminus \{v\} \subseteq C \cap C_i$. We let (i,r) denote the codeword that is neither row-isolated nor column-isolated. This means $C = A_C \cup B_C \cup \{(i,r)\}$ and so $y = \frac{4n-2m-4}{3}$. It follows that $\mathrm{cs}(A_C) = 2$. On the other hand, $\mathrm{cs}(C) = n-1$ so $B_C \setminus \{v\}$ spans the remaining $n-3$ columns. Therefore, $n-3 = \frac{4n-2m-4}{3} - 1$, which implies $2m = n+2$, again contradicting the assumption that $n \leq m$.

3. Suppose $\mathrm{cs}(C) = n$ and $\mathrm{rs}(C) = m-1$.

   Since $|A_C| = x$ and $\mathrm{rs}(C) = m-1$, there exist at least 2 codewords in each of the remaining $m-1-x$ rows disjoint from the row span of A_C. However, Corollary 17 guarantees $|C \cap R_r| \geq 3$ for any such row R_r where $C \cap R_r \subseteq B_C$. Since q represents the number of these rows, $|C \setminus A_C| \geq 2(m-1-x-q) + 3q = 2m - 2 - 2x + q$. This implies that $x \geq \frac{4m-2n-4}{3} + q$. Similarly, since $\mathrm{cs}(C) = n$ and $|B_C| = y$ we get $|C \setminus B_C| \geq 2(n-y)$, which implies $|C| \geq 2n - y$. As in Case 1 it follows that $y \geq \frac{4n-2m+2}{3}$. Consequently,

$$\frac{2m+2n+1}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m+2n+1}{3} + q - 2.$$

Thus, $q \leq 1$. Assume first that $q = 1$. This gives equality in the above, and thus $C = A_C \cup B_C$, $y = \frac{4n-2m+2}{3}$ and $x = \frac{4m-2n-1}{3}$. Furthermore, C contains an isolated codeword, call it v. Since $q = 1$, there exists a row R_r such that $B_C \setminus \{v\} = C \cap R_r$. Thus, $\mathrm{rs}(B_C) = 2$. On the other hand, $\mathrm{rs}(C) = m - 1$ so $A_C \setminus \{v\}$ spans the remaining $m - 3$ rows. Therefore, $m - 3 = \frac{4m-2n-1}{3} - 1$, which contradicts the assumption that $m \neq 2n - 5$.

Therefore, $q = 0$. First assume C contains no isolated codeword. Consequently, $C = A_C \cup B_C$ and since $q = 0$, it follows that $C = A_C$. Since $\mathrm{cs}(C) = n$ and no isolated codeword exists, it follows that $|C| \geq 2n$. Therefore, $\frac{2m+2n+1}{3} - 1 \geq 2n$, which implies $m \geq 2n + 1$. Because of this contradiction we conclude that C contains an isolated codeword, say v.

Because $q = 0$, any row that contains a column-isolated codeword other than v would also have to contain a codeword that is not column-isolated.

Furthermore, the fact that $q = 0$, together with

$$\frac{2m + 2n + 1}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m + 2n + 1}{3} + q - 2\,,$$

implies that there exists at most one such codeword that is neither row-isolated nor column-isolated. Note that $y \geq \frac{4n-2m+2}{3} \geq 2$. Therefore, the column-isolated vertices other than v are contained in precisely one row, say R_r, and hence $B_C \setminus \{v\} \subseteq C \cap R_r$. We let (i, r) denote the codeword that is neither row-isolated nor column-isolated. This means $C = A_C \cup B_C \cup \{(i, r)\}$ and so $x = \frac{4m-2n-4}{3}$. It follows that $\mathrm{rs}(B_C) = 2$. On the other hand, $\mathrm{rs}(C) = m - 1$ so $A_C \setminus \{v\}$ spans the remaining $m - 3$ rows. Therefore, $m - 3 = \frac{4m-2n-4}{3} - 1$, which implies $m = 2n - 2$. However, in this specific case $x = 2n - 4$ and $y = 2$. Since $A_C \cap B_C = \{v\}$, it follows that

$$n = \mathrm{cs}(C) \leq \frac{|A_C \setminus \{v\}|}{2} + |B_C| = \frac{2n - 4 - 1}{2} + 2 = n - \frac{1}{2}\,,$$

which is a contradiction.

4. Suppose that $\mathrm{cs}(C) = n - 1$ and $\mathrm{rs}(C) = m - 1$.

   From Case 2 and Case 3, we see that

$$y \geq \frac{4n - 2m - 4}{3} + p \quad \text{and} \quad x \geq \frac{4m - 2n - 4}{3} + q.$$

Since $\mathrm{cs}(C) = n - 1$ and $\mathrm{rs}(C) = m - 1$, it follows from Corollary 18 that C does not contain an isolated codeword. Thus,

$$\frac{2m + 2n + 1}{3} - 1 \geq |C| \geq x + y \geq \frac{2m + 2n + 1}{3} - 3 + p + q.$$

Hence, $p + q \leq 2$.

(i) Suppose that $p = 0$. For each column C_i where $A_C \cap C_i \neq \emptyset$, there will exist another codeword in C_i that is not row-isolated. To guarantee that $\frac{2m+2n+1}{3} - 1 \geq |C|$, C contains at most two such codewords. Therefore, $\mathrm{cs}(A_C) \leq 2$. If $\mathrm{cs}(A_C) = 2$, then $y = \frac{4n-2m-4}{3}$, and it follows that

$$n - 1 = \mathrm{cs}(C) = \mathrm{cs}(A_C) + \mathrm{cs}(B_C) = 2 + \frac{4n - 2m - 4}{3}.$$

This contradicts the assumption that $m \geq n$, and thus $\mathrm{cs}(A_C) < 2$. On the other hand, $x \geq \frac{4m-2n-4}{3} + q \geq \frac{8}{3}$. Hence, C contains precisely one codeword, say v, that is neither row-isolated nor column-isolated. This implies that $\mathrm{cs}(A_C) = 1$, and if we let C_i represent the column containing these row-isolated vertices, then $v \in C_i$ and $\mathrm{cs}(A_C \cup \{v\}) = 1$. Since

$$n - 1 = \mathrm{cs}(C) = \mathrm{cs}(A_C \cup \{v\}) + \mathrm{cs}(B_C) = 1 + \mathrm{cs}(B_C)\,,$$

we know $\mathrm{cs}(B_C) = n-2$. Therefore, $y = n-2$ since each vertex of B_C is column-isolated. On the other hand, to guarantee $\frac{2m+2n+1}{3} - 1 \geq |C|$, it is the case that $y \leq \frac{4n-2m-4}{3} + 1$. Consequently, $n - 2 \leq \frac{4n-2m-4}{3} + 1$, which again implies that $m < n$. This contradiction shows that $p \neq 0$.

(ii) Suppose that $q = 0$. For each row R_r where $B_C \cap R_r \neq \emptyset$, there will exist another codeword in R_r that is not column-isolated. Since $p \neq 0$, C contains at most one such codeword and it follows that $\mathrm{rs}(B_C) \leq 1$. On the other hand, $y \geq \frac{4n-2m-4}{3} + p \geq p \geq 1$. This implies $\mathrm{rs}(B_C) = 1$, and C contains precisely one codeword, say v, that is neither row-isolated nor column-isolated. Since v is in the same row as the vertices of B_C, $\mathrm{rs}(B_C \cup \{v\}) = 1$. This implies

$$m - 1 = \mathrm{rs}(C) = \mathrm{rs}(A_C) + \mathrm{rs}(B_C \cup \{v\}) = \mathrm{rs}(A_C) + 1\,,$$

and consequently $m-2 = \mathrm{rs}(A_C)$. Therefore, $x = m-2$ since each vertex of A_C is row-isolated. On the other hand, since v is not column-isolated and $p=1$, it follows that $\mathrm{cs}(A_C \cup \{v\}) = 2$. Therefore,

$$n-1 = \mathrm{cs}(C) = \mathrm{cs}(A_C \cup \{v\}) + \mathrm{cs}(B_C) = 2 + \mathrm{cs}(B_C)\,,$$

which implies $y = \mathrm{cs}(B_C) = n-3$. Combining these facts we get

$$|C| = |A_C \cup B_C \cup \{v\}| = x + y + 1 = m + n - 4\,.$$

However, $\frac{2m+2n+1}{3} - 1 \geq |C| = m+n-4$, which implies $m+n \leq 10$. This contradicts our assumption that $n \geq 6$.

(iii) Since $p=1$ and $q=1$, then $x \geq \frac{4m-2n-4}{3} + 1$ and $y \geq \frac{4n-2m-4}{3} + 1$. It follows that

$$\frac{2m+2n+1}{3} - 1 \geq |C| \geq x + y \geq \frac{2m+2n+1}{3} - 1.$$

Thus, $C = A_C \cup B_C$. On the other hand, $\mathrm{cs}(A_C) = 1$ since $p=1$. Therefore, B_C spans the remaining $n-2$ columns since $\mathrm{cs}(C) = n-1$. Hence, $n-2 = \frac{4n-2m-4}{3} + 1$, which contradicts $m \geq n$.

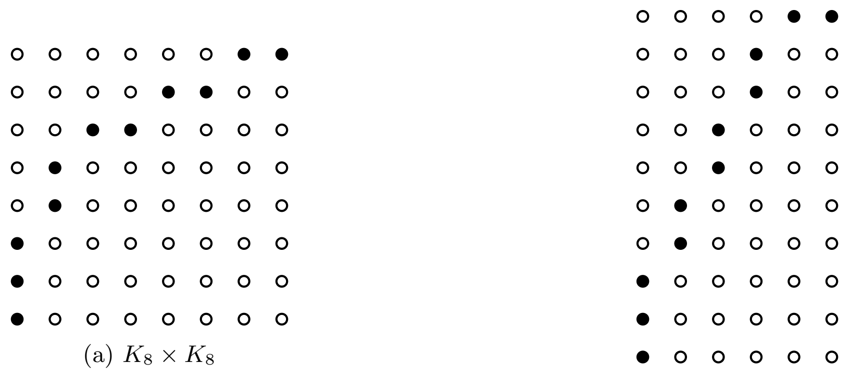
Therefore, every ID code of $K_n \times K_m$ has cardinality at least $\lceil \frac{2m+2n}{3} \rceil$.

We now present ID codes to show that this lower bound is realized. Figure 3.2 contains examples of minimum cardinality ID codes for some cases covered in Theorem 25. As in Figure 3.1 the code consists of the solid vertices.

If $m \neq 2n-2$, let

$$D_1 = \{(1,1)\} \cup \{(i,2i),(i,2i+1) \,\big|\, 1 \leq i \leq a\} \cup \{(a+2j-1, 2a+j+1), (a+2j, 2a+j+1) \big| 1 \leq j \leq b\}\,,$$

where $a = \frac{2m-n-2}{3}$ and $b = \frac{2n-m+1}{3}$. It is straightforward to check that D_1 satisfies the properties of Proposition 19 and is therefore an ID code of $K_n \times K_m$.



(a) $K_8 \times K_8$

(b) $K_6 \times K_{10}$

Figure 3.2: Several ID codes when $n + m \equiv 1 \pmod 3, m \neq 2n - 5$

If $m = 2n - 2$, let

$$D_2 = \{(1,1)\} \cup \{(i, 2i), (i, 2i+1) \,\big|\, 1 \leq i \leq n-2\} \cup \{(n-1, 2n-2), (n, 2n-2)\}\,.$$

Again, one can verify that D_2 satisfies all properties of Proposition 19 and is therefore an ID code of $K_n \times K_{2n-2}$.

Therefore, if $m \neq 2n - 5$ but $n + m \equiv 1 \pmod 3$ and $6 \leq n \leq m \leq 2n - 2$, then

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = \left\lceil \frac{2m + 2n}{3} \right\rceil\,.$$

$\square$

Chapter 4

ID Codes in Cartesian Products of Cliques

4.1 Introduction

In this chapter, we determine the minimum size of an identifying code in the Cartesian product of two general cliques. We then provide upper and lower bounds in regards to the Cartesian product of three or more cliques. For example, we show that the minimum size of an identifying code of the product of three cliques of size m is approximately m^2, and that the minimum size of an identifying code when one clique is much larger than the others is related to the total domination number of the product of the smaller cliques. This chapter is an expanded version of [18]. As all proofs are provided, we do not give specific references to this paper in this chapter.

4.2 Two Cliques

The result for equal cliques was proved by Gravier et al.:

Theorem 26. [20, Theorem 1, pg. 2] *For* $m \geq 1$, $\gamma^{\mathrm{ID}}(K_m \square K_m) = \lfloor 3m/2 \rfloor$.

This result was also reproved in Foucaud et al. [16, Proposition 10], since as we will exploit, the graph $K_m \square K_m$ is the line graph of $K_{m,m}$. We present here the exact result for the Cartesian product of two general cliques.

Theorem 27. *For $2 \le n \le m$, we have*

$$\gamma^{\mathrm{ID}}(K_n \square K_m) = \begin{cases} m + \lfloor n/2 \rfloor & \text{if } m \le 3n/2, \\ 2m - n & \text{if } m \ge 3n/2. \end{cases}$$

We prove this theorem in the next two subsections. We will use $\{1, \ldots, m\}$ for the vertex set of the clique/complete graph K_m.

4.2.1 Lower bound

We will need the edge analogue of ID codes. An *edge-ID code* of G is a set D of edges such that for each edge $e \in E(G)$, the subset of edges of D incident with e is nonempty and distinct. A set of edges D is *edge-dominating* if every edge in G is either in D or incident with an element in D.

We will use the fact that $K_n \square K_m$ is the line graph of $K_{n,m}$ by considering edge-ID codes for $K_{n,m}$. Let the partite sets of $K_{n,m}$ be $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$.

Lemma 28. *If an edge-dominating set D of $K_{n,m}$ is an edge-ID code, then*

(1) in the spanning subgraph H with edge-set D, $|N_H(S)| \ge 2$ for any set S of two vertices in the same partite set;

(2) any set T of two vertices of each partite set is incident with at least three edges of D.

Proof. (1) Suppose $S = \{x_1, x_2\}$ has $|N_H(S)| \le 1$. Say $N_H(S) \subseteq \{y_1\}$. Then D does not separate edges x_1y_1 and x_2y_1, a contradiction.

(2) Suppose $T = \{x_1, x_2, y_1, y_2\}$ is incident with only two edges of D. Then by property (1), these two edges form a matching in T. However, then the two edges of T not in D are not separated. □

Lemma 29. *For integers $2 \le n \le m$, an edge-ID code D of $K_{n,m}$ satisfies*

$$|D| \ge \max\left\{2m - n, m + \left\lfloor \frac{n}{2} \right\rfloor\right\}.$$

Proof. The result is trivial if $m = 2$; so assume $m > 2$. Let D be a minimum edge-ID code of $K_{n,m}$.

Suppose some vertex of X is disjoint from D, say x_1. By property (1), each x_j for $2 \le j \le m$ is incident to at least two edges in D. It follows that $|D| \ge 2m - 2 \ge \max\{2m - n, m + \lfloor n/2 \rfloor\}$, as required. So we may assume every vertex of X is incident with an edge of D. For each $i \in \{1, \ldots, m\}$, choose an edge $e_i \in D$ incident to $x_i \in X$, and let $T = \{e_1, \ldots, e_m\}$.

Consider any $y_i \in Y$. By property (1) applied to each pair of vertices in $N_T(y_i)$, it follows that at most one vertex in $N_T(y_i)$ is not incident with an edge of $D - T$. Thus $|D - T| \geq \sum_{i=1}^{n}(d_T(y_i) - 1) = m - n$, and so $|D| \geq 2m - n$.

It remains to show that $|D| \geq m + \lfloor n/2 \rfloor$. Define

$$Y_1 = \{\, y \in Y \mid y \text{ is incident to exactly 1 edge in } T \,\}, \quad \text{and}$$
$$Y_2 = \{\, y \in Y \mid y \text{ is incident to at least 2 edges in } T \,\}.$$

Let $|Y_1| = n_1$ and $|Y_2| = n_2$. There are two cases:

- Suppose some vertex of Y is disjoint from D, say y_1. By property (1) applied to the set $\{y_i, y_1\}$ for each $y_i \in Y - Y_2 - y_1$, we can conclude D contains at least $2(n - n_1 - n_2 - 1) + n_1 = 2n - n_1 - 2n_2 - 2$ edges disjoint from T. By property (1) applied to one pair of vertices in $N_T(y_i)$ for each $y_i \in Y_2$, it is also true that D contains at least n_2 edges disjoint from T. It follows that:

$$|D| \geq |T| + \left\lceil \frac{(2n - n_1 - 2n_2 - 2) + n_2}{2} \right\rceil = m + \left\lceil \frac{2n - n_1 - n_2 - 2}{2} \right\rceil \geq m + \left\lfloor \frac{n}{2} \right\rfloor .$$

- Suppose every vertex of Y is incident with D. By property (2) applied to $\{y_i, y_j\} \cup N_T(\{y_i, y_j\})$ for each $y_i, y_j \in Y_1$, it follows that at most one vertex of Y_1 is not incident with an edge in $D - T$. At the same time, every vertex in $Y - Y_1 - Y_2$ is incident with an edge of $D - T$. Therefore, D contains at least $(n_1 - 1) + (n - n_1 - n_2) = n - n_2 - 1$ edges disjoint from T. As above, D contains at least n_2 edges disjoint from T. It follows that

$$|D| \geq |T| + \left\lceil \frac{(n - n_2 - 1) + n_2}{2} \right\rceil = m + \left\lfloor \frac{n}{2} \right\rfloor .$$

In any case, an edge-ID code D of $K_{n,m}$ satisfies $|D| \geq \max\{2m - n, m + \lfloor n/2 \rfloor\}$. $\square$

4.2.2 Construction

We now construct ID codes to show the lower bound is also the upper bound. Let $G = K_n \Box K_m$ with $m \geq n$. Gravier et al. [20] constructed ID codes for the case $n = m$. Based on their construction, we define the following sets

$A = \{\, (i, i) \mid 1 \leq i \leq n \,\}$,

$B = \{\, (i, n+i) \mid 1 \leq i \leq m-n \,\}$,

$C = \{\, (n-i+1, i) \mid 1 \leq i \leq \lfloor n/2 \rfloor \,\}$, and

$X = \{\, (1, i), (2, i) \mid \lceil (3n+1)/2 \rceil \leq i \leq m \,\}$.

(Note that $X = \emptyset$ if $m \leq 3n/2$.) Further, define $D = A \cup B \cup C \cup X$. Note that $|D| = n + m - n + \lfloor n/2 \rfloor + m - \lceil (3n+1)/2 \rceil + 1 = 2m - n$ if $m > 3n/2$, and that $|D| = m + \lfloor n/2 \rfloor$ otherwise.

For example, here is the picture for $n = 9$ and $m = 12$.

A								C
	A						C	
		A				C		
			A		C			
				A				
					A			
						A		
							A	
								A
B								
	B							
		B						

Notice that each column and each row of G intersects D, and therefore D dominates G. So let $u = (i, j)$ and $v = (x, y)$ be distinct vertices of G. We need to show that D separates these two vertices.

Start by considering the set A and assume that A does not separate u and v (that is, $N[u] \cap A = N[v] \cap A$), since otherwise we are done. There are two cases.

If $N[u]$ and $N[v]$ contain two vertices of A, then it must be that $i = y$ and $x = j$. But then it is easily seen that u and v are separated by C.

So assume $N[u]$ and $N[v]$ contain exactly one vertex of A. Then it must be that $i = x$, and that $j, y \in \{i\} \cup \{n+1, \ldots, m\}$. If $j = i$, then C separates u and v; so we may assume $y, j > n$.

If $j, y \leq 3n/2$, then u and v are separated by B. On the other hand, if one or both j, y is greater than $3n/2$, then the two vertices are separated by X. Thus D is an ID code, as required.

This concludes the proof of Theorem 27.

4.3 Equal Cliques

We now consider ID codes for the product of multiple copies of equal-sized cliques.

The hypercube is the Cartesian product of K_2's and there has been considerable research on ID codes in hypercubes. Between the original paper by Karpovsky et al. [28], Exoo et al. [13] and Blass et al. [5], the minimum size of an ID code for the d-dimensional hypercube Q_d is now known for small value of d. These values are given in the table below:

d	3	4	5	6	7
$\gamma^{\mathrm{ID}}(Q_d)$	4	7	10	19	32

Between them, it is also shown that γ^{ID} of the d-dimensional hypercube is asymptotically $(2+o(1)) \times 2^d/(d+2)$. (Note that Kim and Kim [31] consider what graph theory calls the Cartesian product of cycles, not hypercubes.)

We start with a simple construction.

Theorem 30. *Let $d \geq 3$ and let G be the Cartesian product of d copies of K_m. Then*

$$\gamma^{\mathrm{ID}}(G) \leq m^{d-1}.$$

Proof. Define D as the set of vertices in G whose coordinates sum to 0 modulo m. That is, D is the set of all $(x_1, x_2, \ldots, x_d)$ with $1 \leq x_i \leq m$ for each $i \in \{1, \ldots, m\}$ and $\sum_{i=1}^{m} x_i$ is a multiple of m. Note that $|D| = m^{d-1}$, since we can freely specify the first $d-1$ coordinates. (In coding theory D is called a *parity code*.)

We claim that D is an independent set. Consider any two vertices v_1 and v_2 in D. If they were adjacent, then v_1 and v_2 would have $d-1$ coordinates in common. However, any $d-1$ coordinates determine the last one. Therefore, D is an independent set.

We claim that D is an ID code. If vertex v is in D, then $N[v] \cap D = \{v\}$. On the other hand, if $u \in V(G) - D$, then u is adjacent to exactly d vertices in D, each of which agrees with u in exactly $d-1$ coordinates. It follows that D certainly separates any two vertices in D, as well as any pair of vertices $u \in V(G) - D$ and $v \in D$.

Thus, we need only to consider pairs of vertices in $V(G) - D$. We claim that $N[u] \cap D$ uniquely determines u for $u \in V(G) - D$. Indeed, we can determine u by simply taking the majority vote in each coordinate. $\square$

In general, the best lower bound we know is the bound from the original paper by Karpovsky et al. [28]. Namely, they showed that $\gamma^{\mathrm{ID}}(G) \geq 2n/(r+2)$ for an r-regular graph G of order n.

Theorem 31. [28, Theorem 1, pg. 599] *Let G be the Cartesian product of d copies of K_m. Then*

$$\gamma^{\mathrm{ID}}(G) \geq \frac{2m^d}{dm - d + 2}.$$

Together with Theorem 30, this shows that for the Cartesian product of d copies of K_m, γ^{ID} is $\Theta(m^{d-1})$. We know almost nothing about the actual values. We can, however, improve the lower bound for three cliques, which we consider next.

4.4 Lower Bound For Three Equal Cliques

We will need some concepts from domination. We say that a set D *open-dominates* vertex x if $N(x) \cap D \neq \emptyset$. We say that set D is a *total dominating set* if it open-dominates every vertex of G. The minimum cardinality of a total dominating set is denoted $\gamma_t(G)$.

Consider the graph $G \square K_m$ and let D be a subset of the vertices. We will use the notation G_i to denote the ith copy of G (that is, all vertices of the form $(?, i)$), and D_i to be the subset of D in G_i. We let X_i denote all vertices in G_i that do not have a neighbor in D_i. (That is, X_i is the vertices of G_i that are not open-dominated by D_i.) Finally, let $\hat{X}_i$ be the subset of $V(G)$ corresponding to X_i (i.e., the projection of X_i onto G).

Lemma 32. *Let D be an ID code of $G \square K_m$ and let X_i be defined as above. Then the $\hat{X}_i$ are disjoint.*

Proof. Suppose that the $\hat{X}_i$ are not disjoint. Say $v \in \hat{X}_i \cap \hat{X}_j$. That is, $(v, i) \in X_i$ and $(v, j) \in X_j$. Then these two vertices are not separated by D, since the intersection of each of their closed neighborhoods with D is precisely the set of vertices of D in the v^{th} copy of K_m. The result follows. $\square$

From this lemma we obtain a lower bound for the product of three cliques:

Theorem 33. $\gamma^{\mathrm{ID}}(K_m \Box K_m \Box K_m) \geq m^2 - o(m^2)$.

Proof. Let $G = K_m \Box K_m$ and let D be an ID code of $G \Box K_m$. Define D_i and X_i as in Lemma 32. From that lemma we have that $\sum_{i=1}^m |X_i| \leq m^2$.

On the other hand, if $|D_i| < m$ then there are at least $m - |D_i|$ rows and columns in G_i not containing a vertex of D_i, and so $|X_i| \geq (m - |D_i|)^2$. It follows that $|D_i| \geq m - \sqrt{|X_i|}$. Hence

$$|D| = \sum_{i=1}^{m} |D_i| \geq m^2 - \sum_{i=1}^{m} \sqrt{|X_i|}.$$

To bound $|D|$ from below, we need to maximize $z = \sum_{i=1}^m \sqrt{|X_i|}$, subject to the constraint $\sum_{i=1}^m |X_i| \leq m^2$. Using the fact that $\sum_{i=1}^m \sqrt{|X_i|}$ is concave, the method of Lagrange for nonlinear programs shows that the optimal value is $z^* = m\sqrt{m}$. Thus, $|D| \geq m^2 - m\sqrt{m}$. $\square$

For $m = 2$ we know from [28] that $\gamma^{\mathrm{ID}}(K_2 \Box K_2 \Box K_2) = \gamma^{\mathrm{ID}}(Q_3) = 4$. When $m = 3$, an exhaustive search by computer shows that $\gamma^{\mathrm{ID}}(K_3 \Box K_3 \Box K_3) = 9$. We conjecture that this pattern continues.

Conjecture 34. *For all $m \geq 1$, $\gamma^{\mathrm{ID}}(K_m \Box K_m \Box K_m) = m^2$.*

4.5 Unequal Cliques

We will need another concept from domination. A set D *doubly dominates* vertex x if $|N[x] \cap D| \geq 2$. We start with a partial converse to Lemma 32.

Lemma 35. *Let D be a subset of the vertices of $G \Box K_m$, and X_i defined as in Lemma 32. If every vertex is doubly dominated from within its copy of K_m, and the projections $\hat{X}_i$ are disjoint, then D is an ID code.*

Proof. The idea is that its neighbors in the copy of K_m determine which copy of K_m a vertex is in, and its neighbors (or lack thereof) in G_i determine which G_i it is in.

To be precise, let (u, i) and (v, j) be two vertices in $G \Box K_m$. If $u \neq v$, then by the hypothesis (u, i) is dominated by at least two vertices of D within its copy of K_m, say (u, k) and (u, ℓ) (where possibly $k, \ell \in \{i, j\}$). Since (v, j) is adjacent to at most one of these vertices, D separates (u, i) and (v, j).

So assume that $u = v$. By the hypothesis, it cannot be that both $(u, i) \in X_i$ and $(u, j) \in X_j$. So say $(u, i) \notin X_i$; that is, (u, i) is dominated by some (w, i) in D; this vertex separates (u, i) from (u, j). Thus D is an ID code. $\square$

We can use Lemmas 32 and 35 to give approximate results for the minimum size of an ID code for the product of cliques where one clique is very large.

Theorem 36. *For any isolate-free graph G of order n, and integer $m \geq 3$,*

$$m\,\gamma_t(G) - n \leq \gamma^{\mathrm{ID}}(G \Box K_m) \leq m\,\gamma_t(G) + 2n - 3\gamma_t(G).$$

Proof. To prove the upper bound, construct set D as follows. Take all the vertices in two copies of G, and take a minimum total dominating set in each of the remaining copies of G except one. Then in the terminology of Lemma 35, each vertex is doubly dominated in its copy of K_m, and each X_i is empty except one, so D is an ID code.

To prove the lower bound, let D be an ID code and let D_i be the intersection of D with the i^{th} copy G_i of G. Since we can form a total dominating set of G_i by adding to D_i one neighbor of each vertex in X_i, it follows that $|D_i| \geq \gamma_t(G) - |X_i|$.

On the other hand, it follows from Lemma 32 that $\sum_{i=1}^{m} |X_i| \leq n$. Thus,

$$|D| = \sum_{i=1}^{m} |D_i| \geq m\,\gamma_t(G) - \sum_{i=1}^{m} |X_i| \geq m\,\gamma_t(G) - n,$$

as required. $\square$

Corollary 37. *For any fixed isolate-free graph G of order n, $\gamma^{\mathrm{ID}}(G \Box K_m) = m\,\gamma_t(G) \pm O(n)$.*

For us, one obvious example is the case where G is the product of equal cliques. The values of $\gamma_t(G)$ do not appear to be known in general for three cliques or more. If one of the cliques is sufficiently large, then it is easy to determine the total domination number of the product of cliques. Indeed, we conclude with the following exact result.

Theorem 38. *Let H be an isolate-free connected graph of order n. If $m > 2\ell$ and $\ell > 2n$, then*

$$\gamma^{\mathrm{ID}}(H \Box K_\ell \Box K_m) = n(m-1).$$

Proof. Let $G = H \square K_\ell$ and $F = G \square K_m$. Define $i \operatorname{MOD} \ell$ to be the unique integer s in the range 1 to ℓ such that $i - s$ is a multiple of ℓ. Let D be the set of vertices of F given by

$$D = \{\, (v, i \operatorname{MOD} \ell, i) \mid v \in V(H), 1 \le i \le m-1 \,\}.$$

Note that $|D| = n(m-1)$.

We claim that D satisfies the hypotheses of Lemma 35. To see this, consider any vertex (v, j, i) in F. Since $m > 2\ell$, there are at least two values $1 \le i_1, i_2 \le m-1$ such that $i_1 \operatorname{MOD} \ell = i_2 \operatorname{MOD} \ell = j$. Thus (v, j, i) is doubly dominated within its copy of K_m. Furthermore, assume $i \ne m$. Then the vertex $(v, i \operatorname{MOD} \ell, i)$ is in D. So if $j \ne i \operatorname{MOD} \ell$, the vertex (v, j, i) has a neighbor in D in its G_i. On the other hand, if $j = i \operatorname{MOD} \ell$, then the vertex (v, j, i) has some neighbor (w, j, i) in D, since H is isolate-free. It follows that all the X_i except X_m are empty, and so the projections $\hat{X}_i$ are disjoint. Thus by Lemma 35, the set D is an ID code.

To prove the lower bound, we use Lemma 32. Let D be a minimum set of vertices of F such that the $\hat{X}_i$ are disjoint. We can assume that we take no more than n vertices from each copy G_i, since taking n vertices is enough to ensure that X_i is empty.

Consider any G_i and let $A_i(v)$ be the copy of K_ℓ whose vertices are of the form $(v, ?, i)$. Assume that $A_i(v)$ does not contain a vertex of D. Then since G_i contains at most n vertices of D, it follows that at least $\ell - n$ vertices of $A_i(v)$ are in X_i. Since $\ell > 2n$, this means that more than half of $A_i(v)$ is in X_i.

Since the $\hat{X}_i$ are disjoint, it follows that for every v, at most one of the $A_i(v)$ does not contain a vertex of D. That is, at least $mn - n$ of the $A_i(v)$ do contain a vertex of D, and so $|D| \ge mn - n$. $\square$

We suspect that the conclusion of the above theorem is true for a much wider range of clique-sizes.

Chapter 5

Edgeless graphs are the only universal fixers

5.1 Introduction

This chapter is an expanded version of [39]. As all proofs are provided, we do not give specific references to this paper in this chapter. Recall from Chapter 2, given a graph G and any permutation π of $V(G)$, the *prism of G with respect to π* is the graph πG obtained by taking two disjoint copies of G, denoted G^1 and G^2, and joining every $u \in V(G^1)$ with $\pi(u) \in V(G^2)$. That is, the edges between G^1 and G^2 form a perfect matching in πG. For any subset $A \subseteq V(G)$, we let $\pi(A) = \cup_{v \in A} \pi(v)$.

If π is the identity $\mathbf{1}_G$, then $\pi G \cong G \Box K_2$, the *Cartesian product* of G and K_2. The graph $G \Box K_2$ is often referred to as the *prism of G*, and the domination number of this graph has been studied by Hartnell and Rall in [24].

One can easily verify that $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$ for all π of $V(G)$. If $\gamma(\pi G) = \gamma(G)$ for some permutation π of $V(G)$, then we say G is a *π-fixer*. If G is a $\mathbf{1}_G$-fixer, then G is said to be a *prism fixer*. Moreover, if $\gamma(\pi G) = \gamma(G)$ for all π, then we say G is a *universal fixer*.

In 1999, Gu [21] conjectured that a graph G of order n is a universal fixer if and only if $G = \overline{K_n}$. Clearly if $G = \overline{K_n}$, then for any π of $V(G)$ we have $\gamma(\pi G) = n = \gamma(G)$. It is the other direction, the question of whether the edgeless graphs are the only universal fixers, that is far more interesting and is the focus of this chapter. Over the past decade, it has been shown that a few classes of graphs do not contain any universal fixers. In particular, given a nontrivial connected graph G, Gibson [17] showed that there exists some π such that $\gamma(G) \neq \gamma(\pi G)$ if G is bipartite. Cockayne, Gibson, and Mynhardt [10] later proved this to be true when G is claw-free. Mynhardt and Xu [36] also showed if G satisfies $\gamma(G) \leq 3$, then G is not a universal fixer. Other partial results can be found in [6, 22]. The purpose of this chapter is to prove Gu's conjecture, which we state as the following theorem.

Theorem 39. *A graph G of order n is a universal fixer if and only if $G = \overline{K_n}$.*

Although the following observation is stated throughout the literature, we give a short proof here for the sake of completeness.

Observation 40. *Let G be a disconnected graph that contains at least one edge. If G is a universal fixer, then every component of G is a universal fixer.*

Proof. Let G be a disconnected graph containing at least one edge, and let $C_1, \cdots, C_k$ represent the components of G where $k \geq 2$. Suppose, for some $j \in \{1, \cdots, k\}$, that C_j is not a universal fixer. There exists a permutation $\pi_j : V(C_j) \to V(C_j)$ such that $\gamma(\pi_j C_j) > \gamma(C_j)$. Now define $\pi : V(G) \to V(G)$ by

$$\pi(x) = \begin{cases} x & \text{if } x \in V(G) \backslash V(C_j) \\ \pi_j(x) & \text{if } x \in V(C_j). \end{cases}$$

Note that πG is a disconnected graph which can be written as the disjoint union

$$\left(\bigcup_{i \neq j} C_i \Box K_2 \right) \cup \ \pi_j C_j.$$

Thus,

$$\begin{aligned} \gamma(\pi G) &= \gamma \left(\bigcup_{i \neq j} C_i \Box K_2 \right) + \gamma(\pi_j C_j) \\ &> \sum_{i \neq j} \gamma\left(C_i \Box K_2\right) + \gamma(C_j) \\ &\geq \gamma(G). \end{aligned}$$

Therefore, if there exists a permutation π of a component C_j of G such that C_j is not a π-fixer, then

G is not a universal fixer. The result follows. □

Observation 40 allows us to consider only nontrivial connected graphs. Therefore, we focus on proving the following theorem.

Theorem 41. *If a connected graph G is a universal fixer, then $G = K_1$.*

5.2 Known Results

In order to study π-fixers, we will make use of the following results. A *γ-set* of G is a dominating set of G of cardinality $\gamma(G)$. The following results were shown by Hartnell and Rall [24], where some statements are in a slightly different form.

Lemma 42. [24, Theorem 4, pg. 395] *Let G be a connected graph of order $n \geq 2$ and π a permutation of $V(G)$. Then $\gamma(\pi G) = \gamma(G)$ if and only if G has a γ-set D such that*

(a) D admits a partition $D = D_1 \cup D_2$ where D_1 dominates $V(G)\backslash D_2$;

(b) $\pi(D)$ is a γ-set of G and $\pi(D_2)$ dominates $V(G)\backslash\pi(D_1)$.

Note that if a graph G is a universal fixer, then G is also a prism fixer. So applying Lemma 42 to the permutation $\mathbf{1}_G$, we get the following type of γ-set.

Definition 43. *A γ-set D of G is said to be symmetric if D admits a partition $D = D_1 \cup D_2$ where*

1. *D_1 dominates $V(G)\backslash D_2$, and*

2. *D_2 dominates $V(G)\backslash D_1$.*

We write $D = [D_1, D_2]$ to emphasize properties 1 and 2 of this partition of D.

Lemma 44. [24, Proposition 6, pg. 398] *If $D = [D_1, D_2]$ is a symmetric γ-set of G, then*

(a) D is independent.

(b) G has minimum degree at least 2.

(c) D_1 and D_2 are maximal 2-packings of G.

(d) For $i \in \{1, 2\}$, $\sum_{x \in D_i} \deg x = |V(G)| - \gamma(G)$.

Theorem 45. [36, Theorem 4] *The conditions below are equivalent for any nontrivial, connected graph G.*

(a) G is a prism fixer.

(b) G has a symmetric γ-set.

(c) G has an independent γ-set D that admits a partition $D = [D_1, D_2]$ such that each vertex in $V(G)\backslash D$ is adjacent to exactly one vertex in D_i for $i \in \{1, 2\}$, and each vertex in D is adjacent to at least two vertices in $V(G)\backslash D$.

We shall add to this terminology that if a symmetric γ-set $D = [D_1, D_2]$ exists such that $|D_1| = |D_2|$, then D is an *even* symmetric γ-set.

5.3 Proof of Theorem 41

The proof of Theorem 41 is broken into three cases depending on the type of symmetric γ-sets a graph possesses. The following property will be useful in each of these cases.

Property 46. *Let $A = [A_1, A_2]$ and $B = [B_1, B_2]$ be symmetric γ-sets of G such that $|A_1| \leq |A_2|$ and $|B_1| \leq |B_2|$.*

(a) If $|A_1| < |B_1|$, then $A_2 \cap B_1 \neq \emptyset$.

(b) If $|B_1| = |A_1| < |A_2|$, then $A_2 \cap B_2 \neq \emptyset$.

Proof. (a) By assumption, $|B_1 \backslash A_1| > 0$ and A_1 dominates $V(G)\backslash A_2$. If $A_2 \cap B_1 = \emptyset$, then by the pigeonhole principle there exists $v \in A_1$ such that v dominates at least two vertices in B_1. This contradicts the fact that B_1 is a 2-packing. Therefore, $A_2 \cap B_1 \neq \emptyset$.

(b) Since $|B_2| = |A_2| > |A_1|$, replacing B_1 with B_2 in the above argument gives the desired result.

□

We call the reader's attention to the fact that any universal fixer is inherently a prism fixer. Therefore, in each of the following proofs, we show that for every nontrivial connected prism fixer G there exists a permutation α such that $\gamma(\alpha G) > \gamma(G)$. Furthermore, the results of Mynhardt and Xu [36] allow us to consider only connected graphs with domination number at least 4.

To prove the next three theorems, we introduce the following notation. Let G be a graph and let π be a permutation of $V(G)$. For each vertex $v \in V(G)$, we let v^1 represent the copy of v in G^1 and v^2 represent the copy of v in G^2; conversely, for $i = 1, 2$, if $v^i \in V(G^i)$, let v be the corresponding vertex of G. If $A \subseteq V(G)$, we define $A^i = \{v^i : v \in A\}$ for $i = 1, 2$. Conversely, if $A^i \in V(G^i)$, then $A = \{v \in V(G) : v^i \in A^i\}$, $i = 1, 2$. If B is a set of vertices in the graph πG, we write $B = X^1 \cup Y^2$, for some symbols X and Y, where $X^1 = B \cap V(G^1)$ and $Y^2 = B \cap V(G^2)$. Thus we navigate between G and πG: the absence of superscripts indicates vertices or sets of vertices in G, and the superscript $i \in \{1, 2\}$ indicates the corresponding vertices or sets of vertices in the subgraph G^i of πG.

Theorem 47. *Let G be a nontrivial connected prism fixer with $\gamma(G) \geq 4$. If G has a symmetric γ-set that intersects every even symmetric γ-set of G nontrivially, then G is not a universal fixer.*

Proof. Let $D = [D_1, D_2]$ be a symmetric γ-set of G that intersects every even symmetric γ-set of G nontrivially. By Lemma 44(c), D_1 and D_2 are 2-packings. Assume without loss of generality that $|D_1| \geq |D_2|$ and let $D_1 = \{x_1, \ldots, x_k\}$. Since D_1 is nonempty and a 2-packing, there exists a vertex $u \in N(x_1)$ such that $u \notin \bigcup_{i=2}^{k} N(x_i)$. Define the permutation α of $V(G)$ by $\alpha(x_i) = x_{i+1}$, $i = 1, \ldots, k-1$, $\alpha(x_k) = u$, $\alpha(u) = x_1$, and $\alpha(v) = v$ for $v \in V(G) \backslash (D_1 \cup \{u\})$. Figure 5.1 illustrates αG with this particular permutation.

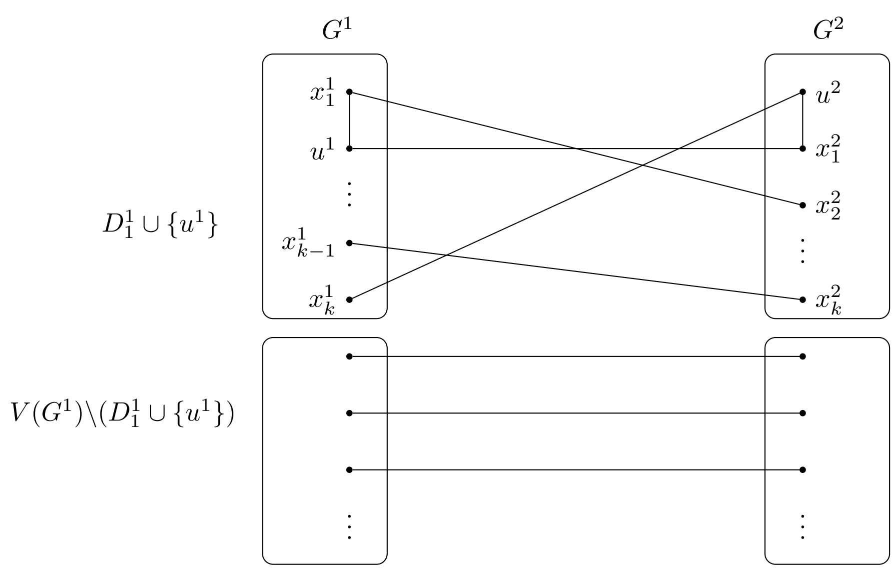


Figure 5.1: αG where D is a symmetric γ-set that nontrivially intersects every even symmetric γ-set of G

Suppose $\gamma(\alpha G) = \gamma(G)$ and let $Q^1 \cup R^2$ be a γ-set of αG. Let S^1 consist of the vertices of G^1 that are not dominated by Q^1. Then S^1 is dominated by R^2, that is, for each $s^1 \in S^1$, $\alpha(s) \in R$ and thus $\alpha(S) \subseteq R$. Suppose $r^2 \in R^2$ is adjacent to $a^1 \in V(G^1) - S^1$. Then $\alpha^{-1}(r) = a$ and Q^1 dominates a^1; hence each vertex of G^1 is dominated by a vertex in Q^1 or a vertex in $R^2 \backslash \{r^2\}$, implying $(Q \cup \alpha^{-1}(R)) \backslash \{a\}$ is a dominating set of G of cardinality less than $\gamma(\alpha G) = \gamma(G)$, which is impossible. Hence the neighbor in G^1 of each vertex in R^2, as determined by α, belongs to S^1, that is $\alpha^{-1}(R) \subseteq S$. It follows that $\alpha(S) = R$. Similarly, if T^2 consists of the vertices of G^2 that are not dominated by R^2, then $\alpha(Q) = T$. Furthermore, S and T are 2-packings, otherwise G would also have a dominating set of cardinality less than $\gamma(G)$. We consider four cases.

Case 1 Assume that $S \cap (D_1 \cup \{u\}) = \emptyset$ and $Q \cap (D_1 \cup \{u\}) = \emptyset$. By definition of α, $\alpha(v) = v$ for each $v \in S \cup Q$. Since $\alpha(S) = R$, $R = S$. Similarly, $Q = T$. Since Q^1 dominates $V(G^1) \backslash S^1$ and R^2 dominates $V(G^2) \backslash T^2$, it follows that T dominates $V(G) \backslash S$ and S dominates $V(G) \backslash T$. Hence $[S, T]$ is a symmetric γ-set of G, where we may assume without loss of generality that $|S| \leq |T|$.

If $|S| = \gamma(G)/2$, then $[S, T]$ is an even symmetric γ-set of G. By the choice of D, $D \cap (S \cup T) \neq \emptyset$. Since $\alpha(v) = v$ for each $v \in S \cup Q = S \cup T$, we know that $D \cap (S \cup T) \subseteq D$. Hence, by the assumptions of Case 1, $D \cap (S \cup T) \subseteq D_2$. Without loss of generality, assume there exists $y \in D_2 \cap T$. By Lemma 44(a), y does not dominate any vertex in D_1. Now each vertex in D_1 is either dominated by a vertex in T or is contained in S. But $S \cap D_1 = \emptyset$ and y does not dominate any vertex in D_1. Hence $T \backslash \{y\}$ dominates D_1. But $|T| = \gamma(G)/2$, so by the choice of D_1, $|T \backslash \{y\}| < \gamma(G)/2 \leq |D_1|$. Therefore D_1 is not a 2-packing, contradicting Lemma 44(c).

Hence assume $|S| < \gamma(G)/2$. Letting S represent A_1 and D_1 represent B_1 in Property 46(a), and recalling that $|D_1| \geq \gamma(G)/2$, we see that $T \cap D_1 \neq \emptyset$. But then $Q \cap D_1 \neq \emptyset$, contrary to the assumption of Case 1. Hence Case 1 cannot occur.

Case 2 Assume that $u \in Q \cup S$. First suppose that $u \in Q$. Then $\alpha(u) = x_1 \in T$. Since $ux_1 \in E(G)$ and T is a 2-packing, $u \notin T$. Hence $u \in N(R)$ by definition of T. Let v be a vertex in R adjacent to u. Since x_1 is the only vertex of D_1 adjacent to u, $\alpha(v) = v$. Since $\alpha(S) = R$, it follows that $v \in S$. But now uv joins $u \in Q$ to $v \in S$, contrary to the definition of S.

Hence we may assume that $u \in S$. Then $\alpha(u) = x_1 \in R$. Since $ux_1 \in E(G)$ and S is a 2-packing, $x_1 \notin S$. Hence $x_1 \in N(Q)$ by definition of S. Let v be a vertex in Q adjacent to

x_1. As above, $\alpha(v) = v$, and since $\alpha(Q) = T$, $v \in T$. Therefore there exists an edge between R and T, contrary to the definition of T.

Case 3 Assume for some $j \in \{2, \ldots, k-1\}$ that $x_j \in Q \cup S$. Suppose we can show that $x_1, u \in Q \cap R$. Since $\alpha(Q) = T$ and $u \in Q$, it fill follow that $\alpha(u) = x_1 \in T$, contrary to the fact that $R \cap T = \emptyset$. Hence this is what we do next.

Since $x_j \in Q \cup S$, $\alpha(x_j) = x_{j+1} \in R \cup T$. Suppose there exists a vertex $v \in R$ such that $vx_j \in E(G)$. By the choice of u, $v \neq u$. Since D_1 is independent, $v \notin D_1$. Therefore $v \in V(G) \backslash (D_1 \cup \{x\})$ and so $\alpha(v) = v$, which implies that $v \in S$. Since S is a 2-packing, $x_j \notin S$, and since no vertex in Q dominates a vertex in S, $x_j \notin Q$, contrary to the assumption of Case 3. Hence no such vertex v exists and thus, by definition of R and T, $x_j \in R \cup T$. Therefore $\alpha^{-1}(x_j) = x_{j-1} \in Q \cup S$. A similar argument shows that x_{j+1} is not adjacent to any vertex in Q and so $x_{j+1} \in Q \cup S$. We can now apply the same argument inductively to $x_{j+1} \in Q \cup S$ and $x_{j-1} \in Q \cup S$ until we arrive at the conclusion that $\{x_1, x_2, \ldots, x_k\} \subseteq (Q \cup S) \cap (R \cup T)$. Then $\alpha^{-1}(x_1) = u \in (Q \cup S)$ and $\alpha(x_k) = u \in (R \cup T)$. Since x_1 and u are adjacent, the definitions of Q and S imply that x_1 and u are both in Q or both in S; but since S is a 2-packing, $x_1, u \in Q$. Similarly, $x_1, u \in R$ and thus $x_1, u \in Q \cap R$, as required.

Case 4 Assume that either x_1 or x_k is in $Q \cup S$. Applying similar arguments as in Case 3 yields the same contradiction. Therefore, this case cannot occur either.

Thus, no such dominating set $Q^1 \cup R^2$ exists for αG and the result follows. □

If a nontrivial connected prism fixer G with $\gamma(G) \geq 4$ has at most one even symmetric γ-set, then the premise of Theorem 47 is true and we immediately obtain the following corollary.

Corollary 48. *Let G be a nontrivial connected prism fixer with $\gamma(G) \geq 4$. If G contains at most one even symmetric γ-set, then G is not a universal fixer.*

Theorem 47 also implies that if a nontrivial connected universal fixer G with $\gamma(G) \geq 4$ exists, then for each even symmetric γ-set D of G, there exists another even symmetric γ-set E of G such that $D \cap E = \emptyset$. We now consider graphs that contain at least two pairwise disjoint even symmetric γ-sets. Note that in this case $\gamma(G)$ is an even integer.

Theorem 49. *Let G be a nontrivial connected prism fixer with $\gamma(G) = 2k$ where $k \geq 2$. If G contains at least two disjoint even symmetric γ-sets, then G is not a universal fixer.*

Proof. Let $D_1, \dots, D_m$ be a maximal set of pairwise disjoint even symmetric γ-sets. Since D_i is symmetric, for each $1 \leq i \leq m$ we can write $D_i = [X_i, Y_i]$ such that X_i dominates $V(G)\backslash Y_i$ and Y_i dominates $V(G)\backslash X_i$. We let $X = \bigcup_i X_i$.

We know that each X_i is a 2-packing of size k. Thus, we can index the vertices of X_i as $x_{i,1}, x_{i,2}, \dots, x_{i,k}$ such that $x_{i+1,j}$ is adjacent to $x_{i,j}$ for $1 \leq i \leq m-1$ and $1 \leq j \leq k$.

In order to define our permutation of $V(G)$, we first assign an additional index to X_m, since we will map X_m to X_1. Note that we have already indexed X_m such that $x_{m,j} \in N(x_{m-1,j})$ for $j = 1, \dots, k$, and this index will be used to map X_{m-1} to X_m. Now for $1 \leq j \leq k$, define a_j such that $x_{m,a_j} \in N(x_{1,j})$, and this index will be used to map X_m to X_1. We may define the following permutation of $V(G)$:

$$\alpha(v) = \begin{cases} x_{i+1,j} & \text{if } v = x_{i,j} \ \text{ for } \ 1 \leq j \leq k \ \text{ and } \ 1 \leq i \leq m-1 \\ x_{1,j+1} & \text{if } v = x_{m,a_j} \ \text{ for } \ 1 \leq j \leq k-1 \\ x_{1,1} & \text{if } v = x_{m,a_k} \\ v & \text{otherwise.} \end{cases}$$

Notice in Figure 5.2 that when we consider the indices of X_m as $x_{m,a_j} \in N(x_{1,j})$, we can write the vertices of X_1 and X_m as a cyclic permutation

$$\beta = (x_{m,a_1}, x_{1,2}, x_{m,a_2}, x_{1,3}, \dots, x_{m,a_k}, x_{1,1}),$$

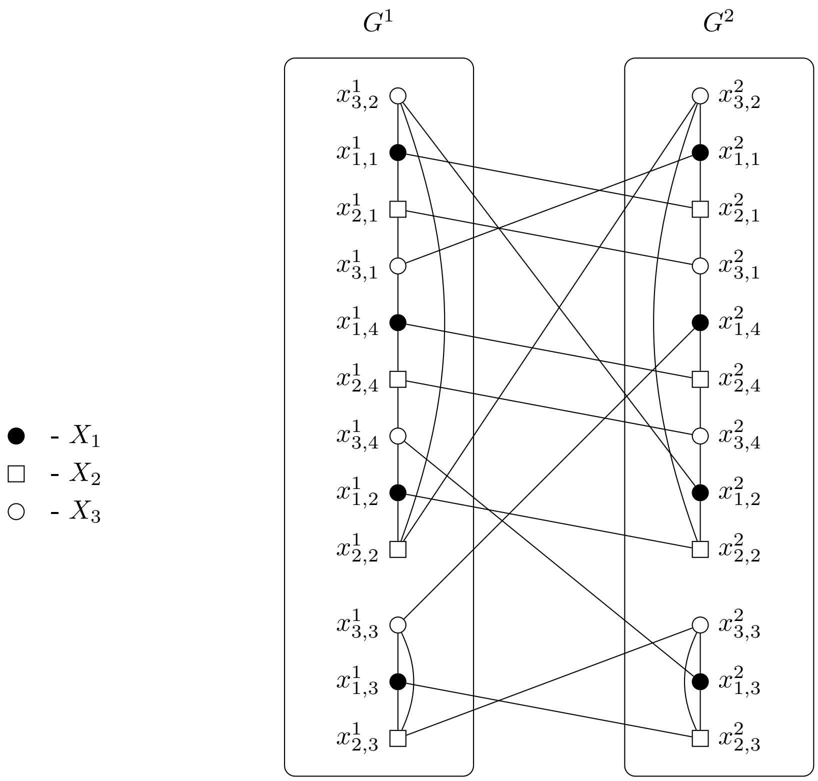
where for each $1 \leq j \leq k$:

(1) $\beta(x_{1,j}) = x_{m,a_j}$; i.e. x_{m,a_j} is adjacent to the vertex immediately preceding it in β, and

(2) $\beta(x_{m,a_j}) = \alpha(x_{m,a_j}) = x_{1,j+1}$; i.e. α maps x_{m,a_j} to the vertex immediately following it in β.

Furthermore, by the definitions of α and a_j, $1 \leq j \leq k$, β cannot be written as a product of subcycles that exhibit the same properties.

Suppose $\gamma(\alpha G) = 2k$ and let $Q^1 \cup R^2$ be a γ-set of αG. Define S^1 and T^2 as in Theorem 47 with all the associated properties.



Note that $\alpha(v) = v$ for all other vertices of G not depicted

Figure 5.2: Specific case when $m = 3$ and $k = 4$

We first claim that $Q \cap X \neq \emptyset$. To see this, suppose neither S nor Q contains a vertex of X. By definition of α, $T = \alpha(Q) = Q$ and $R = \alpha(S) = S$. Thus Q and R are disjoint 2-packings and $[Q, R]$ is a symmetric γ-set of G.

By the symmetry of αG we need only to consider two cases. If $|Q| = k = |R|$, then $[Q, R]$ is an even symmetric γ-set. By the choice of $D_1, \ldots, D_m$, $D_i \cap (Q \cup R) \neq \emptyset$ for some $1 \leq i \leq m$. Because $\alpha(Q \cup R) = Q \cup R$, the definition of α implies that $D_i \cap (Q \cup R) \subseteq Y_i$. Assume without loss of generality that $y_{i,j} \in Q$ for some $1 \leq j \leq k$. Then each vertex of X_i is dominated by a vertex of $Q \backslash \{y_{i,j}\}$ or is contained in S. But by the assumption, $S \cap X = \emptyset$, hence $Q \backslash \{y_{i,j}\}$ dominates X_i. Since $|Q \backslash \{y_{i,j}\}| = k - 1 < |X_i|$, this contradicts X_i being a 2-packing. Therefore either $Q \cap X \neq \emptyset$, and we are done, or $S \cap X \neq \emptyset$. In the latter case, we interchange the labels G^1 and G^2 and obtain $Q \cap X \neq \emptyset$.

On the other hand, if $|Q| < k$, then $S \cap X_i \neq \emptyset$ for each $1 \leq i \leq m$, since each X_i is a 2-packing and every vertex of G is either dominated by Q or is contained in S. This implies for each $1 \leq i \leq m$ that $R \cap X_i \neq \emptyset$ by definition of α. As before, simply relabel G^1 and G^2 so that $|Q| \geq k$ and obtain $Q \cap X \neq \emptyset$.

We next claim that $T \cap X_1 \neq \emptyset$. From above, we may assume $|Q| \geq k$. If $|Q| > k$, then $|R| < k$. This implies that $T \cap X_1 \neq \emptyset$, since X_1 is a 2-packing and every vertex of G is either dominated by R or is contained in T. So assume that $|Q| = k$, and let $x_{i,a} \in Q$ for some $1 \leq i \leq m$ and $1 \leq a \leq k$. If $i = m$, then by definition of α we have $T \cap X_1 \neq \emptyset$. So assume $i \neq m$. Since Y_i is a 2-packing and no vertex of Y_i is adjacent to a vertex of X_i, there exist at least $|Q \cap D_i|$ vertices in $S \cap Y_i$. Moreover, since each vertex of Y_i is mapped to itself under α, we know there exist at least $|Q \cap D_i|$ vertices in $R \cap Y_i$ as well. This, together with the fact that $|Q| = k = |R|$, gives

$$\begin{aligned} |R \backslash Y_i| &\leq k - |Q \cap D_i| \\ &\leq k - 1. \end{aligned}$$

Therefore, since X_i is a 2-packing and each vertex of G is either dominated by R or is contained in T, $T \cap X_i \neq \emptyset$. So assume $x_{i,b} \in T$ for some $1 \leq b \leq k$. If $i = 1$ or if $m = 2$, then we are done with the proof of this claim. So assume $m > 2$ and $i \notin \{1, m\}$. By definition of α, $x_{i-1,b} \in Q$. Applying the above argument inductively, eventually we have $T \cap X_1 \neq \emptyset$. Let $r = |T \cap X_1| > 0$.

We next claim that $r < k$. To see this, suppose that $r = k$. Then $X_1 \subseteq T$. Because X_1

dominates $V(G)\backslash Y_1$, $R \subseteq Y_1$. If $R \subset Y_1$, then T contains X_1 and some vertex $y_{1,j} \in Y_1$. Since Y_1 dominates $V(G)\backslash X_1$, some $x_{1,i}$ and $y_{1,j}$ have a common neighbor in $V(G)\backslash D_1$, contrary to T being a 2-packing. Therefore $R = Y_1$ and so $T = X_1$. Then $Q = \alpha^{-1}(T) = \alpha^{-1}(X_1) = X_m$ and $S = \alpha^{-1}(R) = Y_1$. By the choice of the D_i, $D_1 \cap D_m = \emptyset$. Hence $X_m = Q$ dominates $Y_1 = S$, contradicting the fact that, by definition, $S = V(G)\backslash N[Q]$. Thus, we may conclude that $r < k$.

Let $x_{1,b_1}, x_{1,b_2}, \ldots, x_{1,b_r}$ be the vertices of $T \cap X_1$. There exist exactly r vertices in $Q \cap X_m$; call them $x_{m,c_1}, x_{m,c_2}, \ldots, x_{m,c_r}$. We claim for some $x_{1,b_j} \in T \cap X_1$ that $x_{1,b_j} \notin N(Q \cap X_m)$. So assume not; that is, assume $\{x_{1,b_1}, x_{1,b_2}, \ldots, x_{1,b_r}\} \subset N(Q \cap X_m)$. This implies there exists some relabeling of the b_j's and c_j's such that $x_{m,c_j} \in N(x_{1,b_j})$ and $\alpha(x_{m,c_j}) = x_{1,b_j+1}$ for $b_j \in \{1, \ldots, k-1\}$ and $\alpha(x_{m,c_j}) = x_{11}$ if $c_j = a_k$ where a_k is the index first given to x_m to define α. Consequently, there exists some subcycle of β consisting of the vertices $x_{1,b_1}, x_{1,b_2}, \ldots, x_{1,b_r}, x_{m,c_1}, x_{m,c_2}, \ldots, x_{m,c_r}$ such that for each $1 \leq j \leq r$:

(1) x_{m,c_j} is adjacent to the vertex immediately preceding it within its subcycle; and

(2) x_{m,c_j} is mapped under α to the vertex immediately following it within its subcycle.

However, this contradicts the construction of α unless $r = k$, which we know to be false. Thus, for some $x_{1,b_j} \in T \cap X_1$, $x_{1,b_j} \in S$ or $x_{1,b_j} \in N[Q\backslash X_m]$. If $x_{1,b_j} \in S$, then by definition

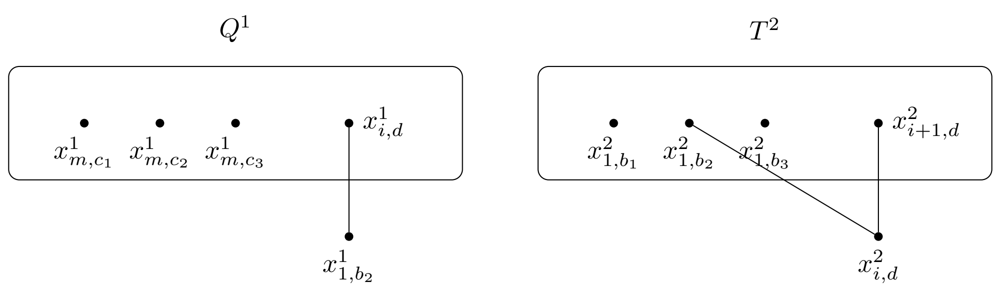


Figure 5.3: Specific case when $|T \cap X_1| = 3$

of α, $x_{2,b_j} \in R$. Since $x_{1,b_j} \in N(x_{2,b_j})$, this implies there exists an edge between R and T. This contradiction shows $x_{1,b_j} \in N[Q\backslash X_m]$. So assume $v \in Q$ where $x_{1,b_j} \in N[v]$. If $\alpha(v) = v$, then v and x_{1,b_j} are both in T, which contradicts T being a 2-packing. On the other hand, if $\alpha(v) \neq v$, then $v = x_{i,d}$ for some $i \neq m$ and $1 \leq d \leq k$.

Case 1 Assume that $i = 1$. Since X_i is a 2-packing, it follows that $v = x_{1,b_j} \in Q$. Thus, $x_{2,b_j} \in T$ by definition of α. But x_{1,b_j} was assumed to be in T, so this violates T being a 2-packing.

Therefore, this case cannot occur.

Case 2 Assume that $i \notin \{1, m\}$. Immediately this implies that $m > 2$. Furthermore, $\alpha(x_{i,d}) = x_{i+1,d}$, and we have $x_{i,d} \in N(x_{1,b_j}) \cap N(x_{i+1,d})$, which contradicts T being a 2-packing, as shown in Figure 5.3. Thus, this case cannot occur either.

Having considered all cases, we have shown such a dominating set $Q^1 \cup R^2$ of αG does not exist of order $2k$. Hence, the result follows. □

We now use the results of this section to prove Theorem 41.

Proof of Theorem 41. Assume that G is a connected universal fixer of order $n \geq 2$. By Mynhardt and Xu [36], we may assume that $\gamma(G) \geq 4$. Since G is a universal fixer, G is a prism fixer. Theorem 47 implies that for every even symmetric γ-set D of G, there exists an even symmetric γ-set D' of G such that $D \cap D' = \emptyset$. However, this contradicts Theorem 49, which states that G cannot contain a pair of disjoint even symmetric γ-sets. Therefore, no such connected universal fixer of order at least 2 exists. That is, if G is a connected universal fixer, then $G = K_1$. □

In conclusion, we know that any component of a universal fixer must be an isolated vertex. It follows that edgeless graphs are the only universal fixers.

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