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Tighter Bounds on Johnson Lindenstrauss Transforms

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ABSTRACT: Johnson and Lindenstrauss (1984) proved that any finite set of data in a high dimensional space can be projected into a low dimensional space with the Euclidean metric information of the set being preserved within any desired accuracy. Such dimension reduction plays a critical role in many applications with massive data. There has been extensive effort in the literature on how to find explicit constructions of Johnson-Lindenstrauss projections. In this poster, we show how algebraic codes over finite fields can be used for fast Johnson-Lindenstrauss projections of data in high dimensional Euclidean spaces.

Johnson Lindenstrauss Transform

Problem

Given data in a high dimensional space, we want to project the data to a low dimensional space so that the pairwise distances are preserved with high probability.

Johnson Lindenstrauss Lemma

- Let n be any positive integer, $0 < \epsilon, \delta < \frac{1}{2}$ and $m = \mathcal{O}(\epsilon^{-2} \log \frac{1}{\delta})$. Then there exists a probabilistic distribution on $A \in \mathbb{R}^{m \times n}$ such that for any vector $x \in \mathbb{R}^n$,
$$\Pr[(1-\epsilon)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\epsilon)\|x\|_2^2] \ge 1 - \delta.$$
  - If $\|x\| = 1$, we have
$$\Pr[|\ \|Ax\|_2^2 - 1\ | > \epsilon] < \delta.$$
  - A transformation matrix A with this property is called a Johnson Lindenstrauss Transformation.

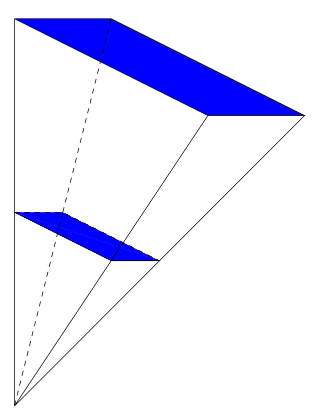
Motivation

Main Motivation

- A dimension reduction technique that preserves pairwise distances and norms.

Applications

- Speeds up algorithm processes:
  - Closest pair
  - Approximate nearest neighbor
  - Finding diameter and minimum spanning tree
- Reduces amount of storage required
  - One-pass streaming algorithms
  - Similarity measures (comparing text documents)



Comments on Parameters

Parameters δ and ϵ

- $\epsilon \in [0, 1/2)$: The desired accuracy
- $1 - \delta \le 1$: The desired probability of success
  - Want $\delta \le \frac{1}{poly(n)}$ in order to compress $poly(n)$ points with a high probability of success.

Parameters n and m

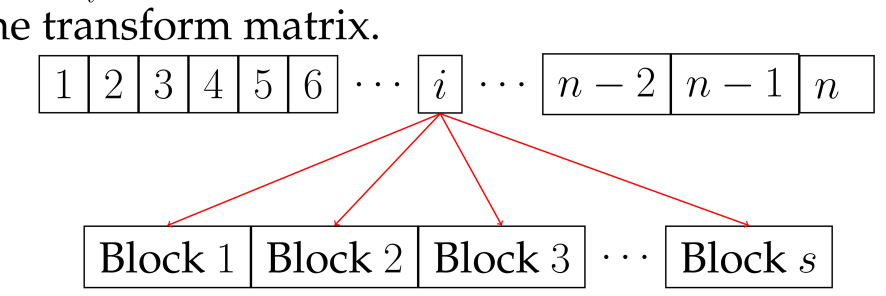
- n: Original dimension
- m: Desired dimension
  - Normally, $n >> m$ where $m \ge \mathcal{O}(\epsilon^{-2} \log \frac{1}{\delta})$

Previous Work: Initial Progress in Construction

- Johnson and Lindenstrauss (1984): Introduced and proved the JL lemma using a complicated transform matrix A with 3 conditions.
- Indyk and Motwani (1998): Dropped two of the conditions: orthogonality and normality of A
  - Construction: Choose independently and randomly $A_{i,j} \sim N(0, 1/n)$.
- Achlioptas (2003): Dropped the spherical symmetry condition of A
  - Relatively Sparse Construction: Choose independently and randomly
$$A_{i,j} = \begin{cases} (n/3)^{-1/2} & p = 1/6 \\ -(n/3)^{-1/2} & p = 1/6 \\ 0 & p = 2/3 \end{cases} \quad \text{with } p \text{ being the probability.}$$

Previous Work: Progress in Faster Computations

- Ailon and Chazelle (2009): Used Fourier in their Fast Johnson Lindenstrauss Transform
- Ailon and Liberty (2009): Used Rademacher and BCH Codes
- Kane and Nelson (2014): Used hash functions
  - Block Construction: Each coordinate of x is hashed to s coordinates where each of these s coordinates lies in one of the s blocks containing m/s entries.
    - Hashing determined by a code C:
      - $C = \{C_1, \ldots, C_n\} \subset [k/s]^s$ with relative distance $1 - O(s/k)$
      - $h : [d] \times [s] \to [k/s]$ can be determined by $(C_i)_j = h(i, j)$
      - The codeword C_i determines the location of the s nonzero elements in the i^{th} column of the transform matrix.



Previous Work: Progress Towards a Tighter Bound

Asymptotic Bounds

Kane's and Nelson's work was one of the first to give attention to the tightness of the success probability of the construction.

- Graph and Block Construction: For sparsity $s = \Omega\left(\epsilon^{-1}\log(1/\delta)\right)$ of matrix A, one may achieve distortion of $1 \pm \epsilon$ with success probability $1 - \delta$.

Specific Bounds

- Kane and Nelson: Provided an upper bound for the block construction:
$$\Pr[|\ \|Ax\|_2^2 - 1\ | > \epsilon] \le \epsilon^{-\ell}\left(64 \max\left\{\frac{\sqrt{l(s-d)}}{s}, \frac{l}{s}\right\}\right)^l,$$
where d is the minimum distance of the code, s is the code length, and l a positive even integer.

Our Results: Tighter Bound

Let s be the length of the code used in the block construction and d the minimum distance.

- Tighter Constraint on Kane's and Nelson's Block Construction:
$$P[|\|Ax\|_2^2 - 1| > \epsilon] < \epsilon^{-\ell} \cdot \left(C_\ell \frac{\sqrt{\ell(s-d))}}{s}\right)^\ell,$$
where $C_\ell \le 64$ for $\ell \le 7564$; more specifically $C_{32} = 4.21$, $C_{64} = 5.92$, and $C_{128} = 8.35$.

Explicit Construction Using AG Codes

AG Code:
(s, k, d)-AG Code over $\mathbb{F}_{q^2}$ from Garcia Stichtenoth Tower (GS tower):

- Code Length: $s = q^u(q^2 - q)$ for some integer $u \ge 1$ and prime power $q \ge 2$
- Dimension: $k < s$, an integer
- Minimum Distance: $d = s - k - g$ where $g = (q^{\lfloor \frac{u+1}{2} \rfloor} - 1)(q^{\lceil \frac{u+1}{2} \rceil} - 1)$ is the genus of the curve.

Bound in Terms of AG Code:
An (s, k, d)-AG Code over $\mathbb{F}_{q^2}$ from GS tower gives
$$\frac{s^2}{s-d} = \frac{(q^{u+2} - q^u)^2}{k + (q^{\lfloor \frac{u+1}{2} \rfloor} - 1)(q^{\lceil \frac{u+1}{2} \rceil} - 1)} \ge 4\epsilon^{-2} \cdot [(2\ell - 1)!!]^{\frac{1}{\ell}}.$$
Hence, $P[|\|Ax\|_2^2 - 1| > \epsilon] < 2^{-\ell}\left((2\ell - 1)!!\right)^{1/2}$.

Parameters

q	u	k	d	g	$s = q^u(q^2 - q)$	$m = s \cdot q^2$	$n = (q^2)^k$	ϵ	$\delta = 0.5^\ell$
2	7	15	16	225	256	1024	1.07×10^{09}	0.42	0.5^{16}
2	10	17	78	1953	2048	8192	1.72×10^{10}	0.21	0.5^{32}
2	10	17	78	1953	2048	8192	1.72×10^{10}	0.30	0.5^{64}
4	2	8	139	45	192	3072	4.29×10^{09}	0.37	0.5^{32}

Consider the third line. If $\delta = .5^{64}$, then we can preserve pairwise distance for $p = 2^{20}$ points with 14% accuracy and by a success probability of $1 - \delta' = 1 - \left(\frac{1}{2}\right)^{25}$.

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