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Application of the Reduced Basis Method to Hyperspectral Diffuse Optical Tomography

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Background

Diffuse Optical Tomography (DOT) is a type of medical imaging in which a laser in the visible to near-infrared range is used as a source, and detectors at the boundary of the tissue record measurements of the scattered photons.

- **Benefits:** uses non-ionizing light, cost-effective, high contrast
- **Disadvantages:** low resolution, tissue highly scattering (use in soft tissue only)
- **Applications:** breast cancer detection, neonatal brain imaging

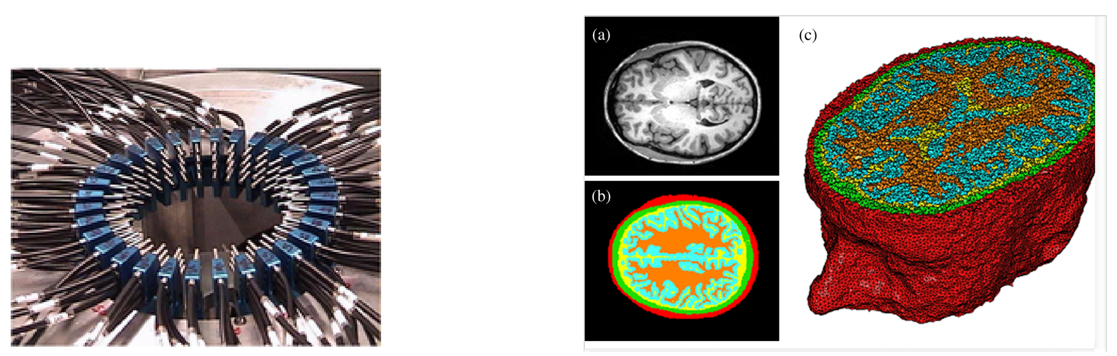

Figure 1: *Diffuse Optical Tomography in practice (from [2], [3])*

Hyperspectral DOT (hyDOT) is when hundreds of optical wavelengths are used in the imaging. A 2-D map is created for each wavelength to make a hypercube of data.

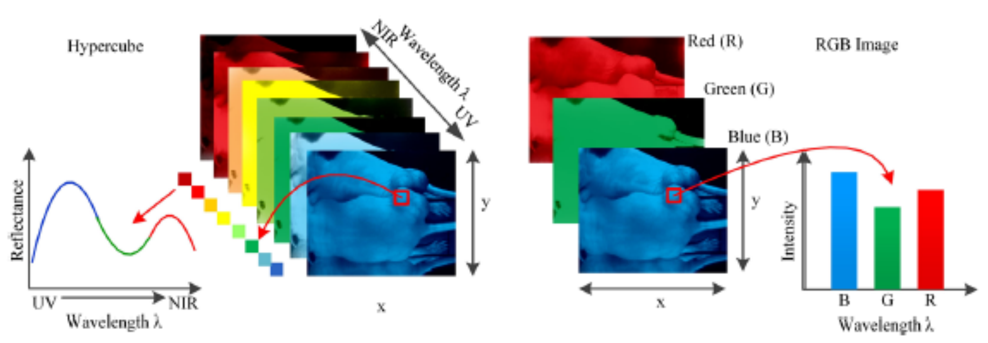

Figure 2: *Hypercube of data from hyperspectral imaging (from [4])*

Problem Overview

In order to construct an image, the photon data is used to construct a spatial map of the optical properties of the tissue such as the values of the absorption and diffusion coefficients.

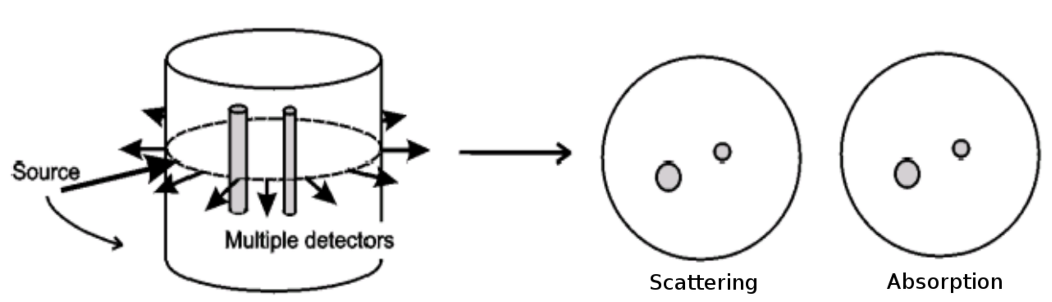

Figure 3: *Reconstruction of optical parameters (from [1])*

To solve the image reconstruction problem, also known as the **inverse problem**, we need to solve the **forward problem** hundreds, if not thousands, of times.

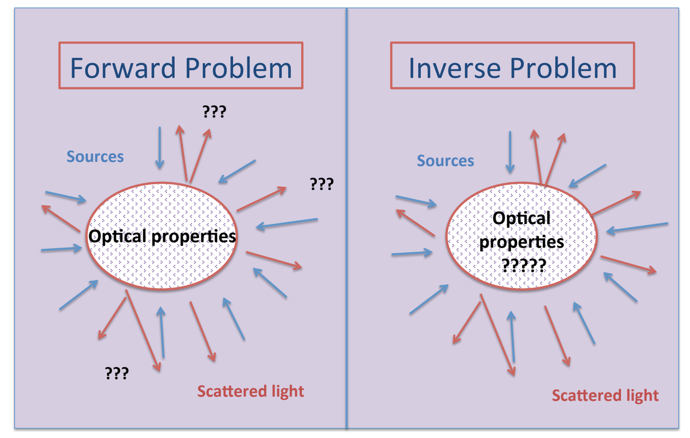

Figure 4: *The forward and inverse problems of DOT*

The forward problem is modeled in the frequency domain by the first order diffusion approximation of the radiative transport equation:

$$-\nabla \cdot (D(\mathbf{x}, \lambda)\nabla u(\mathbf{x})) + (\mu_a(\mathbf{x}, \lambda) + ik)u(\mathbf{x}) = h(\mathbf{x})$$

where $D = \dfrac{1}{3(\mu_a + \mu_s')}$. The dependence of the diffusion and absorption coefficients on the wavelength, λ, add another dimension to the problem.

The Main Question

How does the dependence of the optical parameters on wavelength, λ, affect the image reconstruction (inverse problem)?

This main question is answered by looking at smaller questions:

- **Is the problem computationally feasible (so that the extra information is actually helpful)? How can we make the inverse problem more efficient?**
- In what function space do the optical coefficients lie with respect to the parameter?
- Is there a benefit to considering the problem in the spectral domain?

The Reduced Basis Method

The RBM was developed in the late 1970s and applied to the nonlinear structure analysis of beams and arches. Shortly thereafter it was applied to parametric PDEs.

Main Idea

Find an approximate solution to a PDE on a finite-dimensional manifold induced by its parametric dependence, rather than on the (much) larger-dimensional space associated with the PDE.

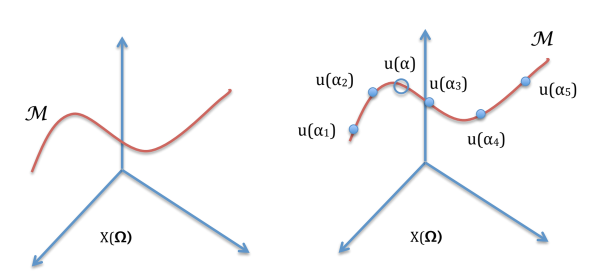

Figure 5: *Approximation on a low-dimensional manifold*

The solution to the weak form of the PDE, given by,

$$a(u(\mu), v; \mu) = f(v) \qquad \forall v \in X$$

where μ is the parameter and X is the finite element approximation space, has reduced basis approximation

$$u_N(\mu) = \sum_{j=1}^{N} \hat{c}_j u_j \in W_N$$

where $W_N = \text{span}\{u_i | i = 1, \dots, N\}$ is the basis space of the manifold. The reduced basis method is only computationally efficient if the bilinear form can be affinely decomposed with respect to the parameter:

$$a(u, v; \mu) = \sum_{q=1}^{Q_a} \Theta^q(\mu) a^q(u, v)$$

Algorithm: Reduced Basis Method

Offline Stage

1. Choose parameter samples : $\mu_1, \dots, \mu_N, N \ll \mathcal{N}$
2. Define $W_N = \text{span}\{u_k, k = 1, \dots, N\}$, where $u_k = \sum_{i=1}^{\mathcal{N}} c_i(\mu_k)\varphi_i$
3. Compute and store $\sum_{i=1}^{N} a^q(u_i, u_j)$, for $q = 1, \dots, Q^a$ (and $\sum_{i=1}^{N} f^{\hat{q}}(u_j)$ for $\hat{q} = 1, \dots, Q_f$, where applicable) for $j = 1, 2, \dots, N$

Online Stage

1. Find a solution with respect to $\hat{c}_i$ for $\sum_{i=1}^{N} \sum_{q=1}^{Q} \hat{c}_i(\mu)\Theta^q(\mu)a^q(u_i, u_j) = f(u_j) \quad \forall j = 1, 2, \dots, N$
2. Reduced Basis approximation: $u_N(\mu) = \sum_{i=1}^{N} \hat{c}_i u_i$

Choosing A Basis

The parameter values chosen to form a manifold basis must be chosen strategically. A greedy algorithm based on *a posteriori* error bounds of the approximation is the most common and effective method (see [3]).

Greedy Sampling Algorithm

for $k = 2 : N_{\max}$ **do**
 $\epsilon^*_{k-1}(\mu) = \left(\sup_{v \in X} \frac{r(v;\mu)}{||v||_X}\right)^2 / \alpha(\hat{\mu})$
 $\mu_k = \arg\max_{\mu \in \Xi} \epsilon^*_{k-1}(\mu)$
 $\varepsilon^*_k = \epsilon^*_{k-1}(\mu_k)$
 if $\varepsilon^*_k \le \epsilon_{tol,min}$ **then**
  $N_{\max} = k - 1$
  exit
 $S_k = S_k \cup \mu_k$
 $W_k = W_k + \{u(\mu_k)\}$

Simple Application

Consider a circular 2-D domain of radius 25, centered at the origin, with a circular tumor of radius 5 at the point (-15,-10). Assume the values of D and μ_a are constant in the healthy tissue Ω_0 and inside the tumor Ω_1, respectively.

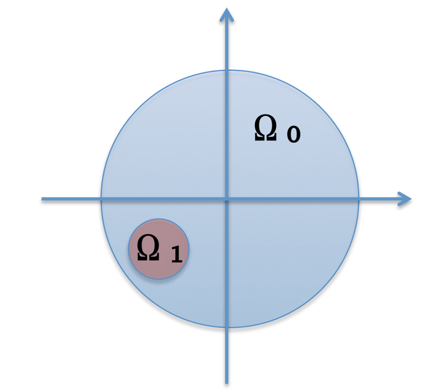

Figure 6: *Domain of the simple example*

The governing equation of the photon diffusion is given by

$$-\nabla \cdot (D(\mathbf{x}, \lambda)\nabla u) + \mu_a(\mathbf{x}, \lambda)u = 0 \qquad \text{in } \Omega$$
$$D(\mathbf{x}, \lambda)\frac{\partial u}{\partial \nu} = f \qquad \text{on } \partial\Omega$$

Results

The relative error between the reduced basis approximation and the finite element solution with respect to the 2-norm was calculated for 100 values of λ and summed to get a total error. The results for a various numbers of basis functions are below:

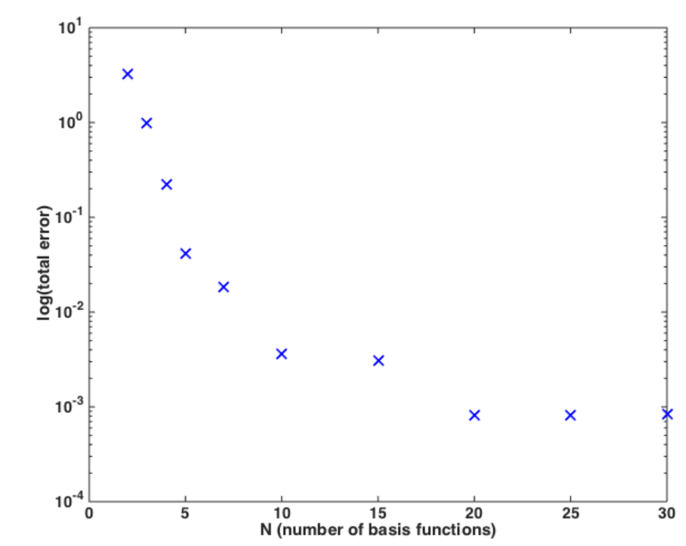

Figure 7: *Total relative error for various basis sizes*

Conclusions

The Reduced Basis Method is successful in accurately (as compared to the finite element method) approximating the solution to the forward problem with a small number of basis functions. Orthogonalization is necessary for the proper conditioning of the matrix representing the bilinear form.

References

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