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W-Peakedness: A Simulation based Study

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ABSTRACT

Kurtosis is a commonly used descriptive statistics. Kurtosis "Coefficient of excess" is critically reviewed in different aspects and is called as, measuring the fatness of the tails of the density functions, concentration towards the central value, scattering away from the target point or degree of peakedness of probability distribution. Kurtosis is referred to the shape of the distribution but many distributions having same kurtosis value may have different shapes while Kurtosis may exist when peak of a distribution is not in existence. Through extensive study of kurtosis on several distributions, Wu (2002) introduced a new measure called "W-Peakedness" that offers a fine capture of distribution shape to provide an intuitive measure of peakedness of the distribution which is inversely proportional to the standard deviation of the distribution. In this paper the work is extended for different others continuous probability distributions. Empirical results through simulation illustrate the proposed method to evaluate kurtosis by W-peakedness

Keywords Kurtosis, W-peakedness, Symmetrical and Asymmetrical distribution

1. INTRODUCTION

The standard fourth moment coefficient of kurtosis is often regarded as a measure of the tail heaviness of a distribution, relative to that of the normal distribution. The developments of the concept of kurtosis, peakedness and tail weight have been reviewed by Blanda and McGillivray (1988). The treatment of kurtosis is often quite short and usually fails to give much understanding. Kendall and Buckland (1971) claimed that kurtosis distinguishes between peakedness and flatness. Darlington (1970) emphasized that the opposite of peaked is bimodal. Chissom (1970) cautioned that a major emphasis must be placed on the tails of a distribution in the determination of fourth moment. They found that tails of a distribution can drastically affect the kurtosis value and it also depends on the peak and tendency towards bimodality, in a single distribution.

Johnson and Kotz (1985) described kurtosis as a measure of deviation from normal depending on the relative frequency of values either near the mean or far from it to values on intermediate distance from the mean. Other

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authors assert that kurtosis measures peakedness near the center of the distribution. Hample (1968) suggested an influence function used to assess the relative importance of tail heaviness and peakedness for the standard kurtosis and related measures. Kurtosis is often thought to measure non normality.

Although $\beta_2 = \mu_4 / \mu_2^2$ is a poor measure of peak or tail weight of a distribution the concept nevertheless plays an important role in both descriptive and inferential statistics. This has led researchers to propose alternative measures. Two common expressions to measure kurtosis are $\beta_2 = \mu_4 / \mu_2^2$ (Pearson's measure) and $\gamma_2 = \beta_2 - 3$ (Fisher's measure). We will discuss, in this article the problem of peakedness for symmetric and asymmetric distributions.

2. W-PEAKEDNESS

Wu (2002) defined w-Peakedness at point t, for a continuous random variable x wit density function f(x), having k modes as,

$$w(t) = \frac{E(x-t)^4}{\left[E(x-t)\right]^2}\left[2f(t) - f\left((t-\sigma)/k\right) - f\left((t+\sigma)/k\right)\right] \qquad 1$$

It is worth to note that $w(t)$ will be positive at peak point, negative at velley point and zero at flat point. The w-Peakedness at $t = \mu$ can be simplified as

$$w(t) = \frac{\mu_4}{\sigma^4}\left[\int_{\mu-\sigma}^{\mu+\sigma} \left|f'(x)\right| dx\right] \qquad 2$$

In next section we describe the distributional shape of symmetrical and asymmetrical distribution.

3. W-PEAKEDNESS: A SIMULATION STUDY

Table 1 and table 2 display theoretical expressions for (β_2) and $(w(\mu))$ for symmetric and asymmetric distributions considered in this study. In this section w-Peakedness measures have been evaluated for the symmetric distributions which satisfy the condition $f(\mu-\sigma) = f(\mu+\sigma)$ and for few asymmetric distributions, as listed in Table 1 and Table 2. All simulations are based on 100 samples, each of size 10, from listed distributions (To compare the measures (β_2) and $(w(\mu))$ were evaluated using the information given in Tables 1 and 2 and their performances were compared. Table 1 and 2 display both measures and it can be noted that generally kurtosis is free of parameters- more specifically kurtosis does not depend on the standard deviation of the distribution. Further note that w-Peakedness possesses an inverse relationship with the standard deviation of the respective distribution.

Statistical software Minitab® was used to generate random sample and to perform the simulation study. Macros for estimation of kurtosis and w-Peakedness for some distributions are listed in this section. Interested readers may contact authors to obtain macros for other distributions to perform similar computations. Table 3 summarizes the output while running MACRO1. Both β_2 and $w(\mu)$ are constant for Uniform distribution. Note that a and b, as shown in Table 3 are the MLEs of parameters of Uniform distribution. Similar results for other distributions are summarized in Tables 4 to 6.

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*****************************************************************************************
MACRO1
                    Kurtosis & w-Peakedness for UNIFORM DISTRIBUTION

Name c11 'mean' c12 'stdev' c13 'a' and c14 'b' c15 'second central moment' c16 'Fourth central moment' c17
'kurtosis' c18 'w-peakedness'
Let k1=-5
Let k2=5
Base 10000.
Random 100 c1-c10;
Uniform k1 k2.
Rmean c1-c10 c11.
Rstdev c1-c10 c12.
Let c13 =  c11-(sqrt(3))*c12
Let c14 =  c11+(sqrt(3))*c12
Let c15 =  ((c14-c13)**2)/12
Let c16 =  ((c14-c13)**4)/80
Let c17 = (c16/((c15)**2))
Let c20 = 1/(c14-c13)
Let c18 = c17*(2*c20-c20-c20)
end
*****************************************************************************************
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*****************************************************************************************

MACRO2
                    Kurtosis & w-Peakedness for Normal distribution

Name k3 'pi' c11 'mean' c12 'stdev' c13 'second central moment' c14 'Fourth central moment' c15 'kurtosis' c16
'w-peakednes'
Let k1=0
Let k2=1
Let k3=3.142857
Base 10000.
Random 100 c1-c10;
normal k1 k2.
Rmean c1-c10 c11.
Rstdev c1-c10 c12.
Let c13 = (c12**2)
Let c14 = 3*(c13**2)
Let c15 =  c14/(c13**2)
Let c16 = ((6/(sqrt(2*pi)))*(1-(1/sqrt(Expo(1)))))*(1/c12)
end
*****************************************************************************************
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****************************************************************************************
MACRO3
                    Kurtosis & w-Peakedness for Exponential distribution  α = 1
Name c11 'mean' c12 'stdev' c13 'Alpha' c14 'kurtosis' c15 'w-peakednes'
Let k1=1
Base 10000.
Random 100 c1-c10;
Exponential k1.
Rmean c1-c10 c11.
Rstdev c1-c10 c12.
Let c13= 1/c11
Let c14 = (3+6/c13)
Let c15 =9*(2/(expo(1))-(1/((expo(1))**2)))*(1/c12)
End
****************************************************************************************
```

```
****************************************************************************************
MACRO4
                    Kurtosis & w-Peakedness for Logistic distribution

Name c11 'mean' c12 'stdev' c13 'w-peakedness' k1 'Location parameter' k2 'Scale parameter'
Let k1=2
Let k2=5
Let k3=3.142857
Let k4 = sqrt(3)
Base 10000.
Random 100 c1-c10;
Logistic k1 k2.
Rmean c1-c10 c11.
Rstdev c1-c10 c12.
Let        c13        =        (4.2*(k3/k4))*(0.5-(expo(-k3/k4))/(1+((expo(-k3/k4))**2))-(expo(-
k3/k4))/(1+((expo(k3/k4))**2)))*(1/c12)
end
****************************************************************************************
```

Distributions	PDF	Kurtosis β_2	$w(\mu)$
Uniform	$f(x) = \frac{1}{b-a}; a < x < b$	$\frac{9}{5}$	$\frac{9}{\sigma}$
Normal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \; -\infty < X < \infty$	3	$\frac{6}{\sqrt{2\pi}}\left[1 - \frac{1}{\sqrt{e}}\right] \cdot \frac{1}{\sigma}$
T-Distribution	$g(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu}\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}; -\infty \le t \le \infty$	$\frac{3\nu^2}{(\nu-2)(\nu-4)}$	$\frac{6\nu\Gamma\left(\frac{\nu+1}{2}\right)}{(\nu-4)\sqrt{\pi\nu}}\left[1 - \left(\frac{\nu-2}{\nu-1}\right)^{\left(\frac{\nu+1}{2}\right)}\right]\frac{1}{\sigma}$
Hat Distribution	$f(x:\beta) = \frac{-1}{\beta^2}(\lvert x \rvert - \beta) : -\beta \le x \le \beta$	$\frac{12}{5}$	$\frac{4}{5} \cdot \frac{1}{\sigma}$
Double Exponential	$f(x) = \frac{1}{2}\lambda e^{-\lambda \lvert x \rvert} \quad -\infty \le x \le \infty$	6	$6\sqrt{2}\left(1 - e^{-\sqrt{2}}\right)\frac{1}{\sigma}$

Table 1: Comparison of Kurtosis and w-Peakedness for selected symmetrical distributions

Distributions	Probability Density Functions	β_2	$w(\mu)$
Gamma	$f(x) = \frac{1}{\Gamma\alpha\,\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}};\; x > 0$	$3 + \frac{6}{\alpha}$	$\left(3+\frac{6}{\alpha}\right)\frac{\alpha^{\alpha-\frac{1}{2}} e^{-\alpha}}{\Gamma\alpha}\left[2 - \left(1 - \frac{1}{\sqrt{\alpha}}\right)^{\alpha-1} e^{\sqrt{\alpha}} - \left(1 + \frac{1}{\sqrt{\alpha}}\right)^{\alpha-1} e^{-\sqrt{\alpha}}\right]\frac{1}{\sigma}$
Exponential	$f(x) = \frac{1}{\beta} x e^{-\frac{x}{\beta}};\; x > 0$	9	$9\left(\frac{2}{e} - \frac{1}{e^2}\right) \cdot \frac{1}{\sigma}$
Chi Square	$f(x) = \frac{1}{\Gamma\frac{n}{2} 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}};\; x > 0$	$3 + \frac{12}{n}$	$\left(3+\frac{12}{n}\right)\frac{\frac{n^{\frac{1}{2}(n-1)}}{2} e^{-\frac{n}{2}}}{\Gamma\alpha}\left[2 - \left(1 - \frac{1}{\sqrt{\frac{n}{2}}}\right)^{\frac{n}{2}-1} e^{\sqrt{\frac{n}{2}}} - \left(1 + \frac{1}{\sqrt{\frac{n}{2}}}\right)^{\frac{n}{2}-1} e^{-\sqrt{\frac{n}{2}}}\right]\frac{1}{\sigma}$
Logistic	$f(x) = \frac{e^{\left(-\frac{x-\alpha}{\beta}\right)}}{\beta\left[1 + e^{\left(-\frac{x-\alpha}{\beta}\right)}\right]^2} - \infty < x < \infty$	4.2	$\frac{4.2\pi}{\sqrt{3}}\left[\frac{1}{2} - \frac{e^{-\frac{\pi}{\sqrt{3}}}}{\left[1 + e^{-\frac{\pi}{\sqrt{3}}}\right]^2} - \frac{e^{\frac{\pi}{\sqrt{3}}}}{\left[1 + e^{\frac{\pi}{\sqrt{3}}}\right]^2}\right]\frac{1}{\sigma}$

Table 2: Comparison of Kurtosis and w-Peakedness for selected Asymmetrical distributions

No	1	…..	10	Mean	Stdev	a	b	II	IV	B2	w-Peakedness
1	-2.98		-4.48	-1.39	2.09	-5.02	2.24	4.39	34.64	1.8	0
2	4.64		1.34	0.59	2.45	-3.66	4.83	6.01	64.95	1.8	0
3	2.55		-1.15	-0.25	3.38	-6.09	5.6	11.4	233.77	1.8	0
.	-3.3		1.13	0.58	3.01	-4.64	5.8	9.09	148.65	1.8	0
.	-2.26		-2.47	-0.38	3.27	-6.05	5.29	10.71	206.63	1.8	0
.	-0.05		3.37	-0.29	2.7	-4.96	4.38	7.28	95.34	1.8	0
.	1.02		0.91	1.17	1.41	-1.28	3.61	1.99	7.15	1.8	0
.	4.45		-4.91	-0.49	3.23	-6.09	5.11	10.45	196.41	1.8	0
.	0.24		4.11	0.78	3.06	-4.53	6.09	9.39	158.69	1.8	0
.	-1.05		-4.45	0.54	2.97	-4.61	5.69	8.84	140.8	1.8	0
.	-0.56		0.65	0.15	2.35	-3.93	4.23	5.54	55.23	1.8	0
.	-0.17		0.43	0.23	2.61	-4.29	4.75	6.81	83.38	1.8	0
100	0.7		-4.41	1.38	2.94	-3.71	6.46	8.62	133.83	1.8	0

Table 3 Kurtosis and w-Peakedness for Uniform distribution with a = -5 and b = 5. [Note that Est_a and Est_b are MLEs of a and b and II and IV are second fourth central moments, respectively].

						Central Moment			
No	**1**	**..**	**10**	**mean**	**stdev**	**II**	**IV**	**kurtosis**	**w-peakedness**
1	0.52		-0.8	0.15	0.62	0.39	0.45	3	1.517
2	-0.68		-1.35	-0.57	0.91	0.83	2.08	3	1.0317
3	-1.78		-0.61	-0.18	1.13	1.27	4.87	3	0.8343
	-0.04		-1.34	0.2	0.75	0.56	0.94	3	1.2587
	0.14		0.29	0.13	0.9	0.82	2.01	3	1.0409
	0.59		0.74	-0.16	0.75	0.56	0.93	3	1.2616
	0.55		0.85	0.47	1.17	1.37	5.67	3	0.8032
	-2.42		-0.73	0.17	1.29	1.67	8.37	3	0.7286
100	-2.79		-0.7	-0.43	1.35	1.83	10.04	3	0.6963

Table 4 Kurtosis and w-Peakedness for Standard Normal distribution

S. No	1	…..	10	mean	stdev	Alpha	kurtosis	w-Peakedness
1	0.22585		0.37997	0.50809	0.38969	1.96814	6.0486	13.867
2	3.32784		0.71738	1.12441	1.03514	0.88935	9.7465	5.2204
3	1.40518		2.11177	0.97098	0.98946	1.02989	8.8259	5.4614
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
98	0.59764		1.10333	0.89196	0.83034	1.12113	8.3517	6.5079
99	0.79204		3.90175	1.31661	1.24574	0.75953	10.8996	4.3378
100	1.37128		1.72231	0.8874	0.52428	1.12688	8.3244	10.3072

Table 5 Kurtosis and w-Peakedness for Exponential distribution with $\alpha = 1$

S. No	1	…..	10	mean	Stdev	kurtosis	w-peakedness
1	-4.8642		-1.8583	-1.5631	5.5149	4.2	0.465814
2	18.4566		2.2394	4.24815	7.2486	4.2	0.354402
3	7.6186		11.9139	1.63796	9.0357	4.2	0.284308
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
98	0.9944		5.5011	2.49174	6.3725	4.2	0.403129
99	2.9444		21.4067	5.22783	8.7387	4.2	0.293971
100	7.3927		9.6275	2.67189	5.5194	4.2	0.465437

Table 6 Kurtosis and w-Peakedness for Logistic distribution with $\alpha = 2$ and $\beta = 5$

4. CONCLUSION

In this article we worked on some special continuous probability distribution and found properties related to the shape and spread of the distribution for different parameters. Two methods were employed- moment ratio method and w-Peakedness. Wu (2002) explores the results for some distribution. We have extended his results to other distributions. It is observed that kurtosis, based on moment ratio, gives a numerical value for Kurtosis but w-Peakedness varies as does standard deviation. The w-Peakedness could be proved helpful in many real life situations such as stock values, earthquakes, sales and demand etc.The w-Peakedness method could also be generalized to bivariate continuous distributions.

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