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Finite Difference Method with Dirichlet Problems of 2D Laplace's Equation in Elliptic Domain

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Abstract- **In this study finite difference method (FDM) is used with Dirichlet boundary conditions on rectangular domain to solve the 2D Laplace equation. The chosen body is elliptical, which is discretized into square grids. The finite difference method is applied for numerical differentiation of the observed example of rectangular domain with Dirichlet boundary conditions. The obtained numerical results are compared with analytical solution. The obtained results show the efficiency of the FDM and settled with the obtained exact solution. The study objective is to check the accuracy of FDM for the numerical solutions of elliptical bodies of 2D Laplace equations. The study contributes to find the heat (temperature) distribution inside a regular rectangular elliptical discretized body.**

Keywords: Dirichlet Boundary Conditions, Finite Difference Method, Laplace Equation, Elliptic Domain.

I. Introduction

The Laplace partial differential equation in two independent variables have important applications in engineering and science, like fluid flow, electricity and steady heat conduction. In engineering and science mostly deals with variables like and to discuss space with timealso as independent variable for a modeled physical problem is considering as dependent variable. Engineers and scientist investigate the actual partial differential equations (PDE's) that given the investigated physical problem. Many numerical methods are invented in 20th century to solve Elliptic partial differential equations (EPDE's). Physical problems like sound, heat, electrodynamics, fluid flow, elasticity etc. are formulated mathematically by Partial differential equations (PDE's). The Neumann and Dirichlet boundary conditions are mostly applied to obtain the solution of 2D Laplace equation. M.L. Dhumal and S.B. Kiwne [1] used Neumann and Dirichlet boundary conditions to obtain the solution of Laplace equation. The approximate solution of two dimensional Laplace equation using Dirichlet conditions is also discussed by Parag V. Patil and J.S.V.R. Krishna Prasad [2].

Laplace equation is used to solve Cauchy problem by Qian et al [3]. The solution of Laplace equation with simple boundary conditions studied by Morales et al [4]. Lesnic *et al* [5] did work for the solution of Cauchy problem to the Laplace equation using an iterative boundary element method.

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Li *et al* [6] have studied Laplace's equation on elliptic domains by using Dirichlet conditions. Laplace equation in circular domains with circular hole by using Neumann problems of Laplace's equation in circular domains with circular holes have studied by Lee *et al* [7]. Lee *et al* concluded that the method of field equation's is an effective method to solve the Neumann problems. Smith G.D. [8], Ames W.F. [9], Lapidus Land Pinder G.F. [10] and Greenspan D. and Parter S.V. [11] studied the Finite difference methods (FDM) for partial differential equations.

Many numerical methods are invented in 20th century to solve Elliptic partial differential equations. Since 1900 the applications of FDM for PDE's have been known. To solve the elliptic interface problems finite difference method is an accurate method studied by J. Thomas [12]. In 1960 the mesh based methods finite difference method and finite element method was used for numerical solutions of ODE's and PDE's. Jensen [13] worked with fully arbitrary meshes by using FDM. FDM's and FEM's are more suitable for regular meshes. Perrone and Kaos [14] worked on irregular meshes by using two dimensional FDM. P.G. Martinsson [15] discussed for variable coefficient elliptic partial differential equations discretized by composite spectral collocation method. For solving irregular domains by FEM is a relatively time consuming. Ames [16] worked to solve PDE's in irregular domains by FDM. On irregular 2D domain Orovio et al [17] studied the spectral method to solve reaction-diffusion equation. This paper used finite difference method to get the discrete numerical approximations for the derivative. Finite difference method is used here to discretize the domain into uniform grids.

In engineering elliptic partial differential equations used to describe steady-state boundary value problems. For the approximate solution of elliptic partial differential equations (EPDE's), the given partial differential equation is converted into an algebraic difference equation. In this paper we used finite difference method to determine potential in rectangular domain using Dirichlet boundary conditions.

II. Problem Formulation

$$\nabla^2 T^* = \frac{\partial^2 T^*}{\partial x^2} + \frac{\partial^2 T^*}{\partial y^2} = 0 \_\_\_\_\_\_ (1)$$

Is the elliptic partial differential equation (steady state) with two spatial dimensions, such that

$0 \le x \le a \quad and \quad 0 \le y \le b \quad where \;\; a = b = 1$

$T^*(x^*, y^*)$ is the steady state potential distribution in the given domain. The Dirichlet boundary conditions on three sides are homogeneous and on one side non-homogeneous are shown in Figure 1:

$$T^*(x,0) = 0, \quad T^*(x,b) = 75$$
$$T^*(0,y) = 0, \quad T^*(a,y) = 0$$
$$P(0) = 0 \;\; and \;\; P(1) = 1$$

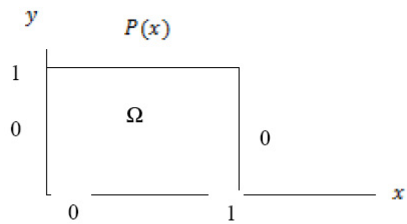


Figure 1. Rectangular domain of the assumed problem

The rectangular domain Ω is divided into finite number of square components. The division is such that each of the line and node of the field is shared with the connected elements other than the sides of the boundaries. The nodes and lines numbering are shown in Figure 2, as follows:

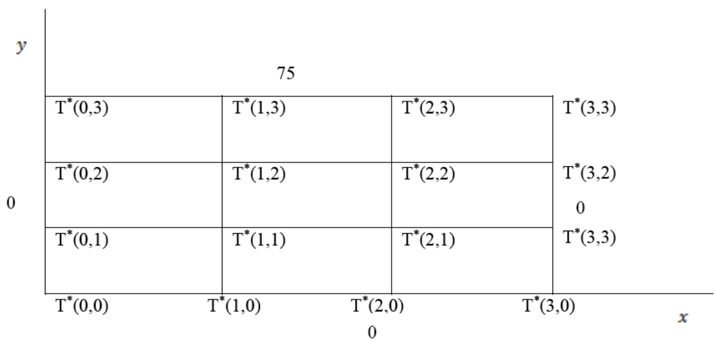


Figure 2. Potential distribution discretized region

III. The Finite Difference Methods (FDM)

FDM is a simple and easiest technique to numerical solutions of elliptic partial differential equations. In this problem approximated all the derivatives using finite differences. The discretization of the region in the directions $x^* \, and \, y^*$ of with a change of $\Delta^* x \, and \, \Delta^* y$ such that $\Delta^* x = \Delta^* y = h$ The discretized scheme is shown in Figure 3 below.

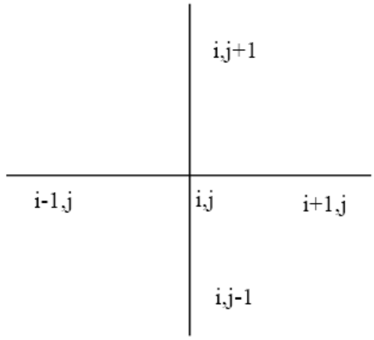


Figure 3. The discretized scheme in 2D

The equation to find the temperature at the particular nodes is

$$T^*_{i+1,j} + T^*_{i-1,j} + T^*_{i,j+1} + T^*_{i,j-1} - 4T^*_{i,j} = 0 \_\_\_\_\_ (2)$$

i. Derivation

Let us consider a Laplace Equation in two dimensional space on a rectangular shape like

$$T^*_{xx} + T^*_{yy} = 0 \_\_\_\_\_ (3)$$

With the conditions $0 \le x \le a \;\; and \;\; 0 \le y \le b \quad a = b = 1$

The Dirichlet boundary conditions are

$$T^*(x,0) = 0, \quad T^*(x,1) = 75; 0 \le x \le 1$$
$$T^*(0,y) = 0, \quad T^*(1,y) = 0; \quad 0 \le y \le 1$$
$$P(0) = 0 \;\; and \;\; pP(1) = 1$$

The grids are uniform in both $x \, and \, y$ directions. The objective is to find the approximate solution at the grid points only, that is $T^*_{ij} \approx T^*(x_i, y_j)$. The finite difference approximation to the partial derivatives at the grid point (x_i, y_j) are:

$$T^*_{xx} \approx \frac{T^*_{i-1,j} - 2T^*_{i,j} + T^*_{i+1,j}}{(\Delta x)^2} \_\_\_\_\_\_\_\_\_\_ (4)$$

$$T^*_{yy} \approx \frac{T^*_{i,j-1} - 2T^*_{i,j} + T^*_{i,j+1}}{(\Delta^* y)^2} \_\_\_\_\_\_\_\_\_\_ (5)$$

Plugging these approximations into the Laplace equation at the point, (x^*_i, y^*_j) we get

$$\frac{T^*_{i-1,j} - 2T^*_{i,j} + T^*_{i+1,j}}{(\Delta^* x)^2} + \frac{T^*_{i,j-1} - 2T^*_{i,j} + T^*_{i,j+1}}{(\Delta^* y)^2} = 0 \_\_\_\_\_\_(6)$$

$$where\,(\Delta^* x)^2 = (\Delta^* y)^2 = h^2$$

$$T^*_{i+1,j} + T^*_{i-1,j} + T^*_{i,j+1} + T^*_{i,j-1} - 4T^*_{i,j} = 0 \_\_\_\_\_(7)$$

Equation (7) is a discrete equation holds at every grid point (x_i, y_j) not on the boundary that is i = 1,2,3, - - - - - , *N and j = 1,2,3 - - - - -, N means (N-1) (N-1) equations*

The boundary conditions are:

$$T^*_{0,j} = 0, T^*_{N,j} = 0; 0 \le j \le N$$

$$T^*_{i,0} = 0, T^*_{i,N} = P(x_i),\ 0 \le x \le N$$

Note that the only interior points will be the unknowns. The equation for the temperature distribution at a particular node is:

$$T^*_{i,j} = \frac{T^*_{i+1,j} + T^*_{i-1,j} + T^*_{i,j+1} + T^*_{i,j-1}}{4} \_\_\_\_\_\_(8)$$

The temperature at the four sides is given, at all the internal points the temperature is assumed. Here this study used the Gauss-Seidel iterative method for solving the system of equations. All the points, which have equal steps horizontally and vertically, the potential is distributed by the finite difference equation (8).

The gauss-Seidel iterative process for the numerical solution of the assumed problem is, shown in Table I:

TABLE I
NUMERICAL RESULTS

Iterations	$T^*_{1,1}$	$T^*_{2,1}$	$T^*_{1,2}$	$T^*_{2,2}$
01	0	0	18.75	18.75
02	4.6875	4.6875	23.4375	23.4375
03	7.03125	7.03125	25.78125	25.78125
04	8.203125	8.203125	26.953125	26.953125
05	8.7890625	8.7890625	27.5390625	27.5390625
06	9.08203125	9.08203125	27.83203125	27.83203125
07	9.228515625	9.228515625	27.978515625	27.978515625
08	9.3017578125	9.3017578125	28.0517578125	28.0517578125
09	9.3383789062	9.3383789062	28.0883789062	28.0883789062
10	9.3566894531	9.3566894531	28.1066894516	28.1066894516

The stopping criterion for the iterations is:
So at the 9th iteration

$$\left|\varepsilon_{i,j}\right| = \left|\frac{T^*{}_{i,j}\,present - T^*{}_{i,j}\,previous}{T^*{}_{i,j}\,present}\right|$$

$$\left|\varepsilon_{i,j}\right| = \left|\frac{9.3383789062 - 9.3017578125}{9.3383789062}\right| * 100 = 0.3921\% \approx 0.4\%$$

At the 10th iteration

$$\left|\varepsilon_{i,j}\right| = \left|\frac{9.4566894531 - 9.3383789062}{9.3566894531}\right| * 100 = 0.0196\% \approx 0.02\%$$

this is negligible so we stop and this solution is appropriate and reliable approximate solution.

IV. Analytical Solution

The two dimensions Laplacian equation with the boundary conditions are:

$$\nabla^2 T = 0 \qquad (9)$$

BC: $T^*(0,y) = 0\ ,\ \ T^*(a,y) = 0$
$T^*(x,0) = 0\ ,\ \ T^*(x,b) = 75$

Separation of variables is used to reduce PDE to ODEs. In the method of variable separation, it tried to find the solution in the form of product later on replaced the solution into Laplacian equation. The constant of separation is introduced by λ. Here only certain values of λ are allowable. The determined solution of Laplacian equation is only satisfied the boundary conditions at λ < 0.
$T^*(x,y) = X(x)Y(y)$ these solutions have

$$X(x) = C_1 \cos \alpha x^* + C_2 \sin \alpha x^* \qquad (10)$$

using boudary conditions

$$X(x) = C_n \sin(\frac{n\pi}{a})x^*\ ;\alpha = \frac{n\pi}{a} \qquad (11)$$

$$Y(y) = A_n e^{\frac{n\pi}{a}y^*} + B_n e^{-\frac{n\pi}{a}y^*} \qquad (12)$$

The product solution of Laplace equation with the given boundary conditions is shown below and represented in Table II:

$$75 = \sum_{n=1}^{\infty} 2A_n \sin(\frac{n\pi}{a})x \sinh\ n\pi \_\_\_\_\_\_\_\_(13)$$

$$\int_0^a \sin(\frac{n\pi}{a})x \sin(\frac{m\pi}{a})x\,dx = \{0, n \neq m \ and\ \frac{a}{2}, n = m$$

finally we get

$$T(x,y) = \frac{300}{\pi} \sum_{k-1}^{\infty} \frac{\sin(n\pi/a)x \sinh(n\pi/a)y}{n \sinh(n\pi)} \_\_\_\_\_\_\_\_\_\_(14)$$

when n is odd that is $n = 2k-1,\ k \in z$

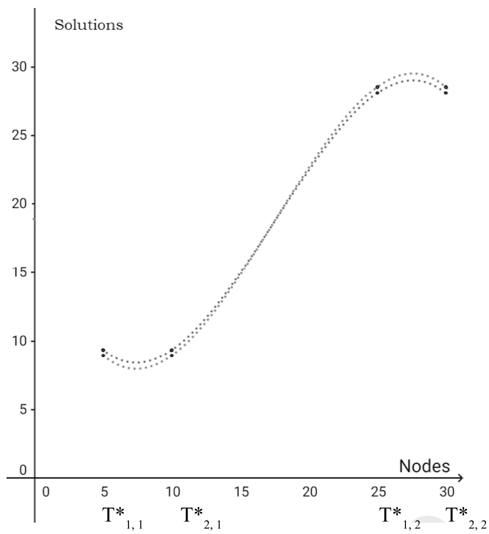
TABLE II
EXACT SOLUTION

Nodes	Exact Solution
$T_{1,1}$	8.94613
$T_{2,1}$	8.94613
$T_{1,2}$	28.5539
$T_{2,2}$	28.5539

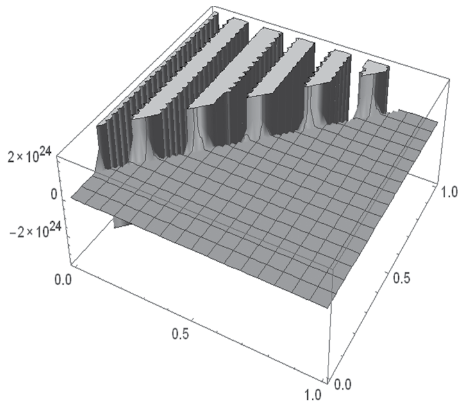
This is the exact solution of 2D-Laplace equation obtained by using variable separation with Dirichlet boundary conditions.

A. Graphical Representation of the FDM Solution and Exact Solution

Following graph a and b are the representation of the FDM solution:



Graph a: Comparison of Numerical FDM and Exact Result



Group b: Potential Distribution 3D Representation

V. Conclusion and Discussion

This study focused on the software uses in varies domains to obtain solutions numerically by using numerical methods particularly Finite Difference method (FDM) to solve 2D Laplace equations with Dirichlet boundary conditions. The graph and the tables I and table II results shown the good agreement of the exact and numerical solution obtained by FDM. Finite difference method is actual an average discretized domain method so it is more appropriate method as compare to other numerical methods for potential distribution. The results of table I showed that for the prediction of potential distribution in the regular rectangle domains Finite difference technique is superior in both competence and accuracy. To understand other quantities like potential distributions on irregular domains or flow and velocity distribution in varies geometries are the research areas in future for the researchers.

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