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The vibration spectrum of two Euler–Bernoulli beams coupled via a dissipative joint

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The asymptotic estimation of the vibration spectrum for a system of two identical Euler–Bernoulli beams coupled via each of the four standard types of linear dissipative joint has been solved for the case when one beam is clamped and the other beam is free at the outer ends. Here, we generalize those results and solve the problem for all 40 combinations of energy-conserving end conditions. We provide both asymptotic and numerical results, and we compare the various systems with an eye toward determining which configurations lead to asymptotically equivalent vibration spectra.

1. Introduction

The design of large or complex structures — bridges, airplanes, robots, buildings, machinery, etc. — entails the joining or coupling of smaller, simpler components, which often can be modeled as beams, plates, or shells. These couplings may include active or passive damping mechanisms for the damping of unwanted vibrations. Successful design requires a knowledge of the system's vibration spectrum, i.e., the set of its natural frequencies of vibration.

There are four standard linear models for describing the vibration of beams — the Euler–Bernoulli, Rayleigh, shear, and Timoshenko beams. Of these, the Euler–Bernoulli is the simplest, with each of the others incorporating one or more physical effects neglected by the Euler–Bernoulli model. Despite the better accuracy of these latter models, the Euler–Bernoulli is accurate enough to be the model of choice for a multitude of physical applications. In addition, the most commonly utilized models for plates and shells are those based on the same assumptions as those governing the Euler–Bernoulli beam. Indeed, given its simplicity and applicability, the Euler-Bernoulli beam may be thought of as the most universal element in structural dynamics.

MSC2010: 74H10, 74H15.
Keywords: vibration, eigenfrequency, Euler–Bernoulli beam, dissipative.

In this paper, we consider the vibration of a system consisting of two identical Euler–Bernoulli beams, coupled end to end by each of four standard types of dissipative joint, and satisfying any of the standard energy-conserving boundary conditions at each end. Our intent is to estimate and classify the vibration spectrum for all 40 possible configurations (four joint conditions and ten sets of end conditions.)

The problem of serially connected Euler–Bernoulli beams seems first to have been treated in [Chen et al. 1987], while the specific problem here was solved in [Chen et al. 1989] for the case involving a so-called type I joint with clamped-free end conditions. The authors employ an asymptotic method in order to compute the spectrum analytically, and provide numerical results for comparison. In addition, they provide physical models for the four joint types (three being special cases, involving only some of the *damping parameters*), and present some early experimental results. In [Chen et al. 1988], the authors provide more experimental results and, after smoothing this data, show good agreement with the results from [Chen et al. 1989].

Krantz and Paulsen [1991] generalize to a great extent the asymptotic results in [Chen et al. 1989]. They again treat the case with clamped-free end conditions, but they consider all four types of joints. In addition, they allow for an arbitrary number of beams of arbitrary length! Finally, in [Chen and Zhou 1990], an alternate solution of the problem in [Chen et al. 1989] is provided, using the elegant asymptotic *wave propagation method* (WPM) of Keller and Rubinow [1960].

In this paper, we consider the case of two identical Euler–Bernoulli beams subject to any of the four types of joint conditions, as given in [Chen et al. 1989], and we generalize by considering all possible combinations of energy-conserving end conditions. We employ WPM in order to derive analytic/asymptotic results, and the Legendre–Tau spectral method for numerical comparisons. These are the first numerical results that we know of for Euler–Bernoulli systems with types II, III and IV joints, and the first asymptotic results for systems without clamped-free end conditions. The asymptotic results allow us easily to compare the vibration spectra for all 40 configurations, permitting us to categorize them, in order to see which configurations may be equivalent insofar as they lead to identical vibration spectra.

This paper is organized as follows: In Section 2, we present the problem and, in Section 3, it is recast in dimensionless form; WPM is applied and the asymptotic results are presented in Section 4, with a brief discussion of the results in Section 5. In Section 6, the numerical results and comparisons are presented.

2. The problem

As mentioned, we consider the problem of two identical Euler–Bernoulli beams, connected by any of the four standard dissipative joints, as presented in [Chen et al. 1989]. We have, then, an Euler–Bernoulli beam equation satisfied along each beam:

$$\begin{aligned} mw_{1tt} + EIw_{1xxxx} &= 0, \quad -L < x < 0, \; t > 0, \\ mw_{2tt} + EIw_{2xxxx} &= 0, \quad\quad 0 < x < L, \; t > 0. \end{aligned}$$

Here, $w_j(x,t)$, $j = 1, 2$, is the transverse displacement along beam j, E is the constant Young's modulus, I is the constant (vertical) moment of inertia, and m is the constant linear mass density.

In addition, we have the joint conditions:

Type I:

$$\begin{aligned} M_2(0,t) &= M_1(0,t), \\ V_2(0,t) &= V_1(0,t), \\ w_{2t}(0,t) - w_{1t}(0,t) &= k_1^2 V_1(0,t) + c_1 M_1(0,t), \\ w_{2xt}(0,t) - w_{1xt}(0,t) &= c_2 V_1(0,t) - k_2^2 M_1(0,t); \end{aligned}$$

Type II:

$$\begin{aligned} w_2(0,t) &= w_1(0,t), \\ M_2(0,t) &= M_1(0,t), \\ V_2(0,t) - V_1(0,t) &= k_1^2 w_{1x}(0,t) + c_1 M_1(0,t), \\ w_{2xt}(0,t) - w_{1xt}(0,t) &= c_2 w_{1t}(0,t) - k_2^2 M_1(0,t); \end{aligned}$$

Type III:

$$\begin{aligned} w_2(0,t) &= w_1(0,t), \\ w_{2x}(0,t) &= w_{1x}(0,t), \\ V_2(0,t) - V_1(0,t) &= k_1^2 w_{1t}(0,t) + c_1 w_{1xt}(0,t), \\ M_2(0,t) - M_1(0,t) &= c_2 w_{1t}(0,t) - k_2^2 w_{1xt}(0,t); \end{aligned}$$

Type IV:

$$\begin{aligned} w_{2x}(0,t) &= w_{1x}(0,t), \\ V_2(0,t) &= V_1(0,t), \\ w_{2t}(0,t) - w_{1t}(0,t) &= k_1^2 V_1(0,t) + c_1 w_{1xt}(0,t), \\ M_2(0,t) - M_1(0,t) &= c_2 V_1(0,t) - k_2^2 w_{1xt}(0,t); \end{aligned}$$

where $M_j(x,t)$ is the bending moment, and $V_j(x,t)$ the shear force, along beam j. The *damping constants* k_1^2, k_2^2, c_1 and c_2 ensure dissipation of energy so long as

$$k_1^2 + k_2^2 > 0 \quad \text{and} \quad k_1^2\alpha^2 + k_2^2\beta^2 + (c_1 - c_2)\alpha\beta > 0 \quad \forall \alpha, \beta \in \mathbb{R}$$

[Chen and Zhou 1990]. For the sake of convenience, we assume throughout the paper that $k_1 \neq 0$ (corresponding to "type a" joints in [Krantz and Paulsen 1991]) and that $k_1^2 k_2^2 + c_1 c_2 > 0$. (It is easy to show that $k_1^2 k_2^2 + c_1 c_2 \geq 0$, with equality if and only if $c_1 = -c_2 = \pm k_1 k_2$.)

Finally, at the left end of the first beam, we have one of the energy-conserving boundary conditions

$$\begin{aligned} \text{clamped (C):} \quad & w_1(-L,t) = w_{1x}(-L,t) = 0, \\ \text{simply supported (S):} \quad & w_1(-L,t) = w_{1xx}(-L,t) = 0, \\ \text{roller supported (R):} \quad & w_{1x}(-L,t) = w_{1xxx}(-L,t) = 0, \\ \text{free (F):} \quad & w_{1xx}(-L,t) = w_{1xxx}(-L,t) = 0, \end{aligned}$$

and similarly at the right end of the second beam. Thus, we have the following ten combinations of boundary conditions to consider:

C-C, C-S, C-R, C-F, S-S, S-R, S-F, R-R, R-F, F-F.

We note here that, in order for a *joint* to exist, at least one of the variables w (or w_t), w_x (or w_{xt}), M or V must be discontinuous. In addition, at most one of each pair of *conjugate variables* (w and V, w_x and M) can be discontinuous. Thus, types I–IV do, indeed, represent the most general situation for linear joints [Pilkey 1969].

3. Dimensionless form

We first separate variables,

$$w_j(x,t) = e^{-i\xi^2 t} v_j(x), \quad j = 1, 2,$$

and introduce the new variables

$$y = \frac{x}{L}, \quad u_j(y) = \frac{v_j(x)}{L}, \quad j = 1, 2.$$

Also, in order to apply WPM, we let $y \to -y$ along the second beam, as it is convenient to have both beams on the same y-interval. The resulting dimensionless ODEs are

$$u_j^{(4)}(y) - k^4 u_j(y) = 0, \quad -1 < y < 0, \ j = 1, 2, \tag{1}$$

where

$$k^2 = \sqrt{\frac{m}{EI}} L^2 \xi^2.$$

The new joint conditions are:

Type I:

$$\begin{gathered} u_2''(0) - u_1''(0) = 0, \\ u_2'''(0) + u_1'''(0) = 0, \\ ik^2[u_2(0) - u_1(0)] - p_{11}u_1'''(0) - q_{11}u_1''(0) = 0, \\ ik^2[u_2'(0) + u_2'(0)] - p_{12}u_1''(0) + q_{12}u_1'''(0) = 0; \end{gathered}$$

Type II:

$$\begin{aligned} u_2(0) - u_1(0) &= 0, \\ u_2''(0) - u_1''(0) &= 0, \\ u_2'''(0) + u_1'''(0) + ik^2 p_{21} u_1(0) + q_{21} u_1''(0) &= 0, \\ ik^2[u_2'(0) + u_1'(0)] - p_{22} u_1''(0) + ik^2 q_{22} u_1(0) &= 0; \end{aligned}$$

Type III:

$$\begin{aligned} u_2(0) - u_1(0) &= 0, \\ u_2'(0) + u_1'(0) &= 0, \\ u_2''(0) - u_1''(0) + ik^2 p_{31} u_1'(0) - ik^2 q_{31} u_1(0) &= 0, \\ u_2'''(0) + u_1'''(0) + ik^2 p_{32} u_1(0) + ik^2 q_{32} u_1'(0) &= 0; \end{aligned}$$

Type IV:

$$\begin{aligned} u_2'(0) + u_1'(0) &= 0, \\ u_2'''(0) + u_1'''(0) &= 0, \\ ik^2[u_2(0) - u_1(0)] - p_{41} u_1'''(0) - ik^2 q_{41} u_1'(0) &= 0, \\ u_2''(0) - u_1''(0) + ik^2 p_{42} u_1'(0) - q_{42} u_1(0) &= 0. \end{aligned}$$

Here, the constants p_{ij} and q_{ij}, where $i = 1, 2, 3, 4,\ j = 1, 2$, are given by:

Type I:

$$p_{11} = \frac{k_1^2 \sqrt{mEI}}{L}, \quad p_{12} = k_2^2 L \sqrt{mEI}, \quad q_{11} = c_1 \sqrt{mEI}, \quad q_{12} = c_2 \sqrt{mEI};$$

Type II:

$$p_{21} = \frac{k_1^2 L}{\sqrt{mEI}}, \quad p_{22} = k_2^2 L \sqrt{mEI}, \quad q_{21} = c_1 L, \quad q_{22} = c_2 L;$$

Type III:

$$p_{31} = \frac{k_1^2}{L\sqrt{mEI}}, \quad p_{32} = \frac{k_2^2 L}{\sqrt{mEI}}, \quad q_{31} = \frac{c_1}{\sqrt{mEI}}, \quad q_{32} = \frac{c_2}{\sqrt{mEI}};$$

Type IV:

$$p_{41} = \frac{k_1^2 \sqrt{mEI}}{L}, \quad p_{42} = \frac{k_2^2}{L\sqrt{mEI}}, \quad q_{41} = \frac{c_1}{L}, \quad q_{42} = \frac{c_2}{L}.$$

Note that $k_1^2 \alpha^2 + k_2^2 \beta^2 + (c_1 - c_2)\alpha\beta \geq 0$ if and only if

$$p_{j1}\alpha^2 + p_{j2}\beta^2 + (q_{j1} - q_{j2})\alpha\beta \geq 0, \quad j = 1, 2, 3, 4.$$

For $j = 1, 2$, the new boundary conditions are

$$\begin{aligned} \text{C:}\ \ & u_j(-1) = u_j'(-1) = 0, \\ \text{S:}\ \ & u_j(-1) = u_j''(-1) = 0, \\ \text{R:}\ \ & u_j'(-1) = u_j'''(-1) = 0, \\ \text{F:}\ \ & u_j''(-1) = u_j'''(-1) = 0. \end{aligned}$$

4. Asymptotic estimation of vibration frequencies by WPM

Applying WPM to the problem is identical to writing the general solutions of the ODEs (1) as

$$\begin{aligned} u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} &= \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{ikx} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{-ikx} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{kx} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} e^{-k(x+1)} \\ &= \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} e^{ik(x+1)} + \begin{bmatrix} B_3 \\ B_4 \end{bmatrix} e^{-ik(x+1)} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{kx} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} e^{-k(x+1)}, \end{aligned} \tag{2}$$

applying the joint conditions to the first expression in (2) and the boundary conditions to the second expression in (2). Here, we follow Chen and Zhou [1990] and stipulate that $\mathrm{Re}(k) \geq 0$ (else, we just replace k by $-k$). Applying the boundary conditions, neglecting the terms of $\mathbb{O}(e^{-k})$, and eliminating D_1 and D_2 leads to

$$\begin{bmatrix} A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} B_3 \\ B_4 \end{bmatrix} = R_2 \begin{bmatrix} B_3 \\ B_4 \end{bmatrix}. \tag{3}$$

Here, a and b depend on the boundary conditions, as follows:

$$a = b = i : \text{ C-C, C-F, F-F,}$$

$$a = b = 1 : \text{ R-R,} \quad a = b = -1 : \text{ S-S,} \quad a = -1, b = 1 : \text{ S-R,}$$

$$a = i, b = 1 : \text{C-R,} \quad a = i, b = -1 : \text{C-S,} \quad a = 1, b = i : \text{R-F,} \quad a = -1, b = i : \text{S-F,}$$

where, e.g., C-F signifies that the first beam is clamped at the left end and the second beam is free at the right end.

Next, we apply the joint conditions, again neglecting terms of $\mathbb{O}(e^{-k})$, and eliminate C_1 and C_2. The result is a relationship of the form

$$M_1(k) \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = M_2(k) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \tag{4}$$

where each matrix M_j is 2×2. Solving for $\left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$, we have

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = M_1^{-1}(k) M_2(k) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = R_1(k) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \tag{5}$$

for which we find it more convenient to write

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \frac{1}{\det M_1(k)} R_1'(k) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}. \tag{6}$$

For the sake of completeness, we provide R_1' and $\det M_1$ for each of the four joints:

Type I:

$$\det M_1 = 2(1+i)p_{11}k^2 + i(p_{11}p_{12} + q_{11}q_{12} + 8)k + 2(-1+i)p_{12} = t_{11},$$

$$R_1' = \begin{bmatrix} u+v & w+z \\ w-z & u-v \end{bmatrix} = T_{11},$$

where

$$u = 2ip_{11}k^2 - (p_{11}p_{12} + q_{11}q_{12})k - 2ip_{12}, \quad v = 2(q_{11} - q_{12})k,$$

$$w = 2p_{11}k^2 + 8ik - 2p_{12}, \quad z = 2(q_{11} + q_{12})k.$$

Type II:

$$\det M_1 = 8ik^2 + 2(-1+i)(p_{21} + p_{22} + q_{21} + q_{22})k - 2(p_{21}p_{22} + q_{21}q_{22}),$$

$$R_1' = \begin{bmatrix} -v & 2u \\ 2u & -v \end{bmatrix},$$

where

$$u = 4ik^2 + [-(p_{21} + p_{22}) + i(q_{21} + q_{22})]k,$$

$$v = 2[i(p_{21} + p_{22}) - (q_{21} + q_{22})]k - 2(p_{21}p_{22} + q_{21}q_{22}).$$

Type III:

$$\det M_1 = 2(1+i)p_{31}k^2 + i(p_{31}p_{32} + q_{31}q_{32})k + 2(-1+i)p_{32} = t_{31},$$

$$R_1' = \begin{bmatrix} u+v & w-z \\ w+z & u-v \end{bmatrix} = T_{31},$$

where

$$u = 2ip_{31}k^2 - (p_{31}p_{32} + q_{31}q_{32})k - 2ip_{32}, \quad v = -2(q_{31} - q_{32})k,$$

$$w = 2p_{31}k^2 + 8ik - p_{32}, \quad z = 2(q_{31} + q_{32})k.$$

Type IV:

$$\det M_1 = -2i(p_{41}p_{42} + q_{41}q_{42})k^2 + 2(1-i)(p_{41} + p_{42} + q_{41} + q_{42})k + 8,$$

$$R_1' = \begin{bmatrix} -u & -2v \\ -2v & -u \end{bmatrix},$$

where

$$u = 2i(p_{41}p_{42} + q_{41}q_{42})k^2 + 2[-(p_{41} + p_{42}) + i(q_{41} + q_{42})]k,$$

$$v = [i(p_{41} + p_{42}) - (q_{41} + q_{42})]k - 4.$$

Now, we also see from the general solution (2) that

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} e^{ik}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_3 \\ B_4 \end{bmatrix} e^{-ik}. \tag{7}$$

Combining (3), (5), and (7) then gives us

$$\begin{aligned} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = R_1(k) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} &= R_1(k)e^{ik} \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} \\ &= R_1(k)e^{ik} R_2 \begin{bmatrix} B_3 \\ B_4 \end{bmatrix} = R_1(k)e^{ik} R_2 e^{ik} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \end{aligned} \tag{8}$$

Thus, we are to find those values of k for which

$$\det [e^{2ik} R_1(k) R_2 - I] = 0$$

or

$$\det \big[R_1' R_2 - e^{-2ik} (\det M_1(k)) I\big] = \det [R_1' R_2 - \lambda I] = 0.$$

Thus, we need only compute the eigenvalues of $R_1'(k)R_2$.

It is easy to see that, if we identify $p_{11} \to p_{31},\ p_{12} \to p_{32},\ q_{11} \to q_{31},\ q_{12} \to q_{32}$, we have $t_{31} = t_{11}$ and $T_{31} = (T_{11})^T$. Thus, $R_{11}' R_2$ and $R_{13}' R_2$ will have the same eigenvalues, and the spectra for the types I and III joints are identical for each R_2; i.e., given any set of end/boundary conditions, the types I and III joints have identical spectra, *asymptotically*. This generalizes the result in [Krantz and Paulsen 1991] where they show that these spectra are identical in the case of C-F end conditions.

Continuing the analysis, in each case the matrix $R_1'(k)R_2$ will have two eigenvalues, $\lambda_1(k)$ and $\lambda_2(k)$. It is easy to show, as in [Chen and Zhou 1990], that these eigenvalues are distinct. It follows that, in each case, there will be two streams or branches of frequencies, satisfying

$$\lambda_j(k) = e^{-2ik} \det M_1(k) \quad \text{or} \quad e^{-2ik} = \frac{\lambda_j(k)}{\det M_1(k)}, \quad j = 1, 2, \tag{9}$$

where each $\lambda_j(k)$ and $\det M_1(k)$ are quadratic polynomials in k. Thus, as (9) is unwieldy, we follow [Chen and Zhou 1990] and we use the first-degree Taylor approximation

$$\frac{a_1 k^2 + b_1 k + c_1}{a_2 k^2 + b_2 k + c_2} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2^2} \frac{1}{k} + \mathbb{O}\Big(\frac{1}{k^2}\Big). \tag{10}$$

Applying (10) to (9) yields an equation of the form

$$e^{-2ik} = d_1\Big(1 - d_2 \frac{1}{k}\Big) + \mathbb{O}\Big(\frac{1}{k^2}\Big), \quad |d_1| = 1, \tag{11}$$

and, taking the (complex) log of (11) and using the Taylor approximation

$$\ln\Big(1 - d_2\frac{1}{k} + \mathbb{O}\Big(\frac{1}{k^2}\Big)\Big) = -d_2\frac{1}{k} + \mathbb{O}\Big(\frac{1}{k^2}\Big),$$

we have

$$-2ik = -d_2\frac{1}{k} + i(\arg d_1 - 2n\pi) + \mathbb{O}\Big(\frac{1}{k^2}\Big), \quad n = 0, 1, 2, \ldots. \tag{12}$$

We note here that the choice of $-2n\pi$ is based on our earlier assumption that $\mathrm{Re}(k) \geq 0$. We rewrite (12) as a quadratic equation, realizing that multiplying by k will add an extraneous root of $\mathbb{O}(1/k)$, and after also employing the Taylor approximation

$$\sqrt{1+\epsilon} = 1 + \tfrac{1}{2}\epsilon + \mathbb{O}(\epsilon^2),$$

we arrive at

$$-ik^2 = -d_2 - \big(\tfrac{1}{2}\arg d_1 - n\pi\big)^2 i + \mathbb{O}\Big(\frac{1}{k}\Big).$$

Here, we provide the expressions for $-ik^2$ for all 40 cases:

<u>Type I:</u>

<u>$a = b$ (C-C, C-F, F-F, S-S, R-R, F-F):</u>

$$-ik^2 = -\frac{p_{11}p_{12} + q_{11}q_{12}}{2p_{11}} - \big(\tfrac{1}{2}\arg a - n\pi\big)^2 i,$$
$$-ik^2 = -\frac{4}{p_{11}} - \big(\tfrac{1}{2}\arg(-a) - n\pi\big)^2 i;$$

<u>$a = -b = -1$ (S-R):</u>

$$-ik^2 = -(4p_{11})^{-1}[p_{11}p_{12} + q_{11}q_{12} + 8 - 2\sqrt{2}(q_{11} - q_{12})i] - \big(\tfrac{1}{8}\pi - n\pi\big)^2 i,$$
$$-ik^2 = -(4p_{11})^{-1}[p_{11}p_{12} + q_{11}q_{12} + 8 + 2\sqrt{2}(q_{11} - q_{12})i] - \big(\tfrac{5}{8}\pi - n\pi\big)^2 i;$$

<u>$a = 1, b = i$ (R-F):</u>

$$-ik^2 = -(4\sqrt{3}p_{11})^{-1}[(\sqrt{3}+1)(p_{11}p_{12}+q_{11}q_{12})+4i(q_{11}-q_{12})+8(\sqrt{3}-1)] - \big(\tfrac{1}{12}\pi - n\pi\big)^2 i,$$
$$-ik^2 = -(4\sqrt{3}p_{11})^{-1}[(\sqrt{3}-1)(p_{11}p_{12}+q_{11}q_{12})-4i(q_{11}-q_{12})+8(\sqrt{3}+1)] - \big(\tfrac{5}{12}\pi - n\pi\big)^2 i;$$

$a=-1,\ b=i$ (S-F):

$$-ik^2=-(4\sqrt{3}p_{11})^{-1}[(\sqrt{3}+1)(p_{11}p_{12}+q_{11}q_{12})-4i(q_{11}-q_{12})+8(\sqrt{3}-1)] -\left(\tfrac{1}{3}\pi-n\pi\right)^2 i,$$

$$-ik^2=-(4\sqrt{3}p_{11})^{-1}[(\sqrt{3}-1)(p_{11}p_{12}+q_{11}q_{12})+4i(q_{11}-q_{12})+8(\sqrt{3}+1)] -\left(\tfrac{2}{3}\pi-n\pi\right)^2 i;$$

$a=i,\ b=1$ (C-R):

$$-ik^2=-(4\sqrt{3}p_{11})^{-1}[(\sqrt{3}+1)(p_{11}p_{12}+q_{11}q_{12})-4i(q_{11}-q_{12})+8(\sqrt{3}-1)] -\left(\tfrac{1}{12}\pi-n\pi\right)^2 i,$$

$$-ik^2=-(4\sqrt{3}p_{11})^{-1}[(\sqrt{3}-1)(p_{11}p_{12}+q_{11}q_{12})+4i(q_{11}-q_{12})+8(\sqrt{3}+1)] -\left(\tfrac{5}{12}\pi-n\pi\right)^2 i;$$

$a=i,\ b=-1$ (C-S):

$$-ik^2=-(4\sqrt{3}p_{11})^{-1}[(\sqrt{3}+1)(p_{11}p_{12}+q_{11}q_{12})+4i(q_{11}-q_{12})+8(\sqrt{3}-1)] -\left(\tfrac{1}{3}\pi-n\pi\right)^2 i,$$

$$-ik^2=-(4\sqrt{3}p_{11})^{-1}[(\sqrt{3}-1)(p_{11}p_{12}+q_{11}q_{12})-4i(q_{11}-q_{12})+8(\sqrt{3}+1)] -\left(\tfrac{2}{3}\pi-n\pi\right)^2 i.$$

Type III: This is the same as type I, with $p_{1j}\to p_{3j},\ q_{1j}\to q_{3j},\ j=1,2$.

Type II:

$a=b$ (C-C, C-F, S-S, R-R, F-F):

$$-ik^2=-\tfrac{1}{2}[p_{21}+p_{22}+i(q_{21}+q_{22})]-\left(\tfrac{1}{2}\arg a-n\pi\right)^2 i,$$

$$-ik^2=-\left(\tfrac{1}{2}\arg(-a)-n\pi\right)^2 i;$$

$a=-b=-1$ (S-R):

$$-ik^2=-\tfrac{1}{4}[p_{21}+p_{22}+i(q_{21}+q_{22})]-\left(\tfrac{1}{4}\pi-n\pi\right)^2 i,$$

$$-ik^2=-\tfrac{1}{4}[p_{21}+p_{22}+i(q_{21}+q_{22})]-\left(\tfrac{3}{4}\pi-n\pi\right)^2 i;$$

$a=i,\ b=1$ (C-R); $a=1,\ b=i$ (R-F):

$$-ik^2=-\tfrac{1}{8}(2+\sqrt{2})[p_{21}+p_{22}+i(q_{21}+q_{22})]-\left(\tfrac{1}{8}\pi-n\pi\right)^2 i,$$

$$-ik^2=-\tfrac{1}{8}(2-\sqrt{2})[p_{21}+p_{22}+i(q_{21}+q_{22})]-\left(\tfrac{5}{8}\pi-n\pi\right)^2 i;$$

$a = i$, $b = -1$ (C-S); $a = -1$, $b = i$ (S-F):

$$-ik^2 = -\tfrac{1}{8}(2+\sqrt{2})[p_{21}+p_{22}+i(q_{21}+q_{22})] - \left(\tfrac{3}{8}\pi - n\pi\right)^2 i,$$
$$-ik^2 = -\tfrac{1}{8}(2-\sqrt{2})[p_{21}+p_{22}+i(q_{21}+q_{22})] - \left(\tfrac{7}{8}\pi - n\pi\right)^2 i.$$

Type IV:

$a = b$ (C-C, C-F, S-S, R-R, F-F):

$$-ik^2 = -2\frac{p_{41}+p_{42}+i(q_{41}+q_{42})}{p_{41}p_{42}+q_{41}q_{42}} - \left(\tfrac{1}{2}\arg a - n\pi\right)^2 i,$$
$$-ik^2 = -\left(\tfrac{1}{2}\arg a - n\pi\right)^2 i;$$

$a \neq b$ (C-S, C-R, S-R, S-F, R-F):

$$-ik^2 = -\frac{p_{41}+p_{42}+i(q_{41}+q_{42})}{p_{41}p_{42}+q_{41}q_{42}} - \left(\tfrac{1}{2}\arg a - n\pi\right)^2 i,$$
$$-ik^2 = -\frac{p_{41}+p_{42}+i(q_{41}+q_{42})}{p_{41}p_{42}+q_{41}q_{42}} - \left(\tfrac{1}{2}\arg b - n\pi\right)^2 i.$$

5. Discussion of asymptotic results

Again, we begin by noting that, for each set of end conditions, the type I and type III joints are asymptotically equivalent. This agrees with what is found in [Krantz and Paulsen 1991] for C-F end conditions.

We see also that, for many choices of the end conditions, the damping rates for the type II and type IV joints are asymptotically equivalent. Specifically, for those cases satisfying $a = b$, there is an asymptotically undamped branch, while, for the other branch, we need only choose our damping constants so that

$$p_{21} = \frac{4p_{41}}{p_{41}p_{42}+q_{41}q_{42}}, \qquad \text{etc.}$$

We have a similar equivalence for the case $a = -b = -1$ (S-R).

It is of particular interest that, in so many cases, for each type of joint, a term of the form $q_{j1} - q_{j2}$ or $q_{j1} + q_{j2}$ appears in $\mathrm{Im}(-ik^2)$. Thus, there are examples where the q_{j1} and q_{j2} affect the "frequency part" of the eigenfrequencies. Indeed, a term of this form appears in all cases except for those where there is a type I or type III joint and end conditions satisfying $a = b$. Thus, this behavior would not have been encountered in [Chen and Zhou 1990]. These terms *are* encountered in [Krantz and Paulsen 1991]; however, they seem to be discarded.

More specifically, in computing the damping rates, [Krantz and Paulsen 1991] arrives at a correct term similar to

$$p_{j1} + p_{j2} + i(q_{j1} + q_{j2}),$$

and then arrives at the, again correct, damping rate of

$$-\operatorname{Re}[p_{j1} + p_{j2} + i(q_{j1} + q_{j2})].$$

However, the $i(q_{j1} + q_{j2})$ part is then dropped from consideration; although, as we shall see below, these efforts do show up in the numerical results. This does, however, seem to be an easy fix for Krantz and Paulsen [1991, p. 399].

6. Numerical results and comparisons

We have applied the Legendre–Tau spectral method to the problem. The problem is recast so that each beam has domain $-1 \le x \le 1$, after which we approximate u_1 and u_2 by

$$u_1(x) = \sum_{n=0}^{N} a_n P_n(x), \quad u_2(x) = \sum_{n=0}^{N} b_n P_n(x), \tag{13}$$

where P_n is the Legendre polynomial of degree n [Gottlieb and Orszag 1977]. Computations were performed within MATLAB, and also using Fortran 90 on a laptop. Computations at $N = 40$ and $N = 42$ show that all results in the table below converge to at least five decimal places.

In each table, we present the first 20 eigenfrequencies. We note here that, although we have only negative imaginary parts in our asymptotic results, in fact the conjugate of each eigenfrequency also is an eigenfrequency. In the following example, we list only those with positive imaginary parts.

In our first example, we compare numerical results for a type I and a type III joint, with C-F end conditions, for $q_{j1} = q_{j2} = 0$, and for various values of $p_{11} = p_{31}$ and $p_{12} = p_{32}$. The results appear in Tables 1–3.

The purpose here is threefold — to compare the numerical results for type I and type III joints (remembering that we have shown them to be asymptotically equivalent), to compare the numerical and asymptotic results, of course, and to see what happens when we vary the "dominant" damping parameters, p_{ij}.

For Table 1, we have taken $p_{11} = p_{31} = p_{12} = p_{32} = 1$. The first thing we must point out is the very close match between the type I and type III numerical results. We shall see similar behavior in the remaining results examining types I and III (Tables 2–4).

We also are surprised to see such a good match between the numerical and asymptotic results at this low end of the spectrum. Indeed, from the second eigenfrequency on, it is clear that the numerical spectrum already has split into the expected two branches or streams.

For Table 2, we have let $p_{11} = p_{31} = 2$ and $p_{12} = p_{32} = 0.5$. Again, we have a very close match between types I and III, and a close match between the numerical and asymptotic results.

type I numerical		type III numerical		WPM	
Re	Im	Re	Im	Re	Im
−0.49507	0.77898	−0.49507	0.77898		
−0.53772	5.5070	−0.53772	5.5070	−0.5	5.5517
−3.6704	20.562	−3.6704	20.562	−4.0	22.207
−0.49979	30.205	−0.49979	30.205	−0.5	30.226
−3.8734	60.711	−3.8734	60.711	−4.0	61.685
−0.49981	74.625	−0.49981	74.625	−0.5	74.639
−3.9307	120.20	−3.9307	120.20	−4.0	120.90
−0.49989	138.78	−0.49989	138.78	−0.5	138.79
−3.9561	199.31	−3.9561	199.31	−4.0	199.86
−0.49992	222.67	−0.49992	222.67	−0.5	222.68
−3.9697	298.10	−3.9697	298.10	−4.0	298.56
−0.49995	326.31	−0.49995	326.31	−0.5	326.31
−3.9778	416.61	−3.9777	416.61	−4.0	416.99
−0.49996	449.68	−0.49997	449.68	−0.5	449.68
−3.9830	554.83	−3.9829	554.83	−4.0	555.17
−0.49997	592.79	−0.49999	592.79	−0.5	592.79
−3.9866	712.79	−3.9866	712.79	−4.0	713.08
−0.49997	755.64	−0.49987	755.64	−0.5	755.64
−3.9889	890.47	−3.9901	890.47	−4.0	890.73
−0.50000	938.22	−4.9987	938.22	−0.5	938.23

Table 1. Types I and III joints, C-F end conditions, with $p_{11} = p_{12} = 1,\ q_{11} = q_{12} = 0$.

For Table 3, we have $p_{11} = p_{31} = 0.5$ and $p_{12} = p_{32} = 2$. Here, once more, the match for types I and III is very close. Meanwhile, the convergence of the numerical to the asymptotic results is somewhat slower than in the previous two tables, especially for the branch with real part equaling -8. Indeed, this slower but smooth convergence is seen quite clearly in Figure 1, where we have plotted the data from Table 3.

For Table 4, we continue to consider types I and III joints and C-F end conditions, with $p_{11} = p_{31} = p_{12} = p_{32} = 1$ but with $q_{11} = q_{31} = 0.5$ and $q_{12} = q_{32} = 0.7$. Once again, the types I and III results are an excellent match. In addition, the smooth convergence of the numerical to the asymptotic results is similar to that in the previous example, and can be seen clearly in Figure 2.

Given the excellent agreement between the type I and type III numerical results, we are curious as to "how equivalent" they actually are. We have tried to compare the determinant equations for the exact solutions, but so far we have had no luck.

type I numerical		type III numerical		WPM	
Re	Im	Re	Im	Re	Im
−1.7301	1.1041	−1.7300	1.1041		
−0.28489	5.5602	−0.28489	5.5602	−0.25	5.5517
−1.9703	21.818	−1.9702	21.818	−2.0	22.207
−0.25005	30.221	−0.25005	30.220	−0.25	30.226
−1.9856	61.556	−1.9856	61.445	−2.0	61.685
−0.24998	75.635	−0.24998	74.635	−0.25	74.639
−1.9917	120.73	−1.9917	120.72	−2.0	120.90
−0.24999	138.79	−0.24999	138.79	−0.25	138.79
−1.9946	199.72	−1.9946	199.72	−2.0	199.86
−0.24999	222.68	−0.24999	222.68	−0.25	222.68
−1.9963	298.44	−1.9963	298.44	−2.0	298.56
−0.24999	326.31	−0.24999	326.31	−0.25	326.13
−1.9973	416.90	−1.9973	416.90	−2.0	418.99
−0.25000	449.68	−0.25000	449.68	−0.25	449.68
−1.9979	555.08	−1.9979	555.08	−2.0	555.17
−0.25000	592.79	−0.24999	592.79	−0.25	592.79
−1.9983	713.00	−1.9983	713.00	−2.0	713.08
−0.25000	755.64	−0.25004	755.64	−0.25	755.64
−1.9989	890.67	−1.9984	890.67	−2.0	890.73
−0.25000	938.23	−0.25003	938.23	−0.25	938.23

Table 2. Types I and III joints, C-F end conditions, with $p_{11} = 2$, $p_{12} = 0.5$, $q_{11} = q_{12} = 0$.

For Table 5, we consider a type II joint with C-F end conditions. The purpose here is to investigate the behavior of the "undamped" branch, the contribution of q_{21} and q_{22} to the imaginary parts of the eigenfrequencies, and, of course, again to compare the numerical and asymptotic results.

Here, we let $p_{21} = p_{22} = 1$. The first two columns give the numerical results, and the next two columns the asymptotic results for the case where $q_{21} = q_{22} = 0$. We see here that the numerical real parts for the "undamped" branch are very small and, in most cases, are negative, as expected. For those that are not negative (the fifth, seventh and thirteenth eigenfrequencies), we assume that it is due to the numerical approximation. In addition, the match between the numerical and asymptotic results is again quite good, even as early as the second eigenfrequency. This can also be seen clearly in Figure 3, where we have plotted these results.

The last four columns are arranged as are the first four, but here we have let $q_{21} = 0.5$ and $q_{22} = 0.7$. We note that the effect of these values on the imaginary

type I numerical		type III numerical		WPM	
Re	Im	Re	Im	Re	Im
−0.41842	0.79717	−0.41842	0.79717		
−1.0353	5.3641	−1.0353	5.3641	−1.0	5.5517
−3.8189	17.042	−3.8189	17.042	−8.0	22.207
−0.99732	30.143	−0.99732	30.143	−1.0	30.226
−6.6452	57.777	−6.6452	57.777	−8.0	61.685
−0.99849	74.584	−0.99849	74.584	−1.0	74.639
−7.3547	118.07	−7.3547	118.07	−8.0	120.90
−0.99909	138.75	−0.99909	138.75	−1.0	138.79
−7.6154	197.65	−7.6154	197.65	−8.0	199.86
−0.99940	222.65	−0.99940	222.65	−1.0	222.68
−7.7423	296.75	−7.7423	296.75	−8.0	298.56
−0.99957	326.29	−0.99957	326.29	−1.0	326.13
−7.8144	415.45	−7.8144	415.46	−8.0	418.99
−0.99968	449.66	−0.99969	449.66	−1.0	449.68
−7.8598	553.83	−7.8595	553.83	−8.0	555.17
−0.99977	592.77	−0.99980	592.77	−1.0	592.79
−7.8901	711.90	−7.8905	711.91	−8.0	713.08
−0.99976	755.62	−0.99947	755.62	−1.0	755.64
−7.9105	889.67	−7.9140	889.68	−8.0	890.73
−0.99981	938.20	−0.99924	938.22	−1.0	938.23

Table 3. Types I and III joints, C-F end conditions, with $p_{11} = 0.5$, $p_{12} = 2$, $q_{11} = q_{12} = 0$.

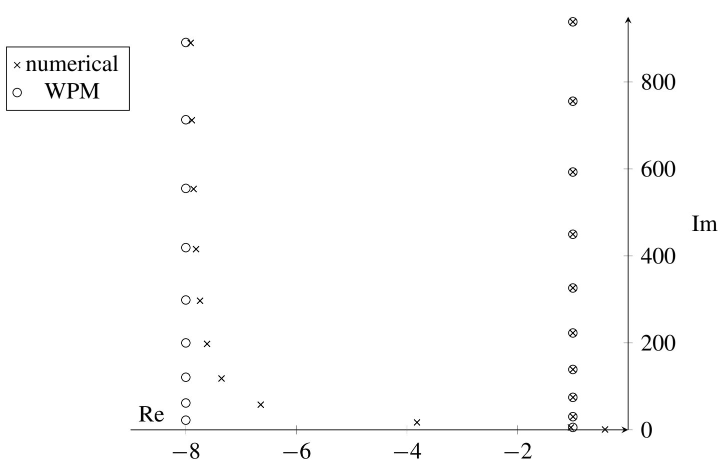

Figure 1. Plot of the vibration frequencies from Table 3.

type I numerical		type III numerical		WPM	
Re	Im	Re	Im	Re	Im
−0.45378	0.76895	−0.45378	0.76895		
−0.95917	5.2515	−0.95917	5.2515	−1.35	5.5517
−3.6112	17.067	−3.6112	17.067	−8.0	22.207
−1.0664	29.951	−1.0664	29.951	−1.35	30.226
−6.5928	57.645	−6.5928	57.645	−8.0	61.685
−1.1599	74.377	−1.1599	74.377	−1.35	74.639
−7.3698	117.99	−7.3698	117.99	−8.0	120.90
−1.2226	138.56	−1.2226	138.56	−1.35	138.79
−7.6396	197.61	−7.6396	197.61	−8.0	199.86
−1.2611	222.48	−1.2611	222.48	−1.35	222.68
−7.7645	296.72	−7.7645	296.72	−8.0	298.56
−1.2853	326.14	−1.2853	325.14	−1.35	326.13
−7.8331	415.44	−7.8331	415.44	−8.0	418.99
−1.3012	449.53	−1.3012	449.53	−1.35	449.68
−7.8751	553.82	−7.8751	553.82	−8.0	555.17
−1.3119	592.65	−1.3120	592.65	−1.35	592.79
−7.9026	711.89	−7.9028	711.89	−8.0	713.08
−1.3197	755.51	−1.3196	755.52	−1.35	755.64
−7.9225	889.67	−7.9222	889.67	−8.0	890.73
−1.3253	938.12	−1.3251	938.11	−1.35	938.23

Table 4. Types I and III joints, C-F end conditions, with $p_{11} = p_{12} = 1$, $q_{11} = 0.5$, $q_{12} = 0.7$.

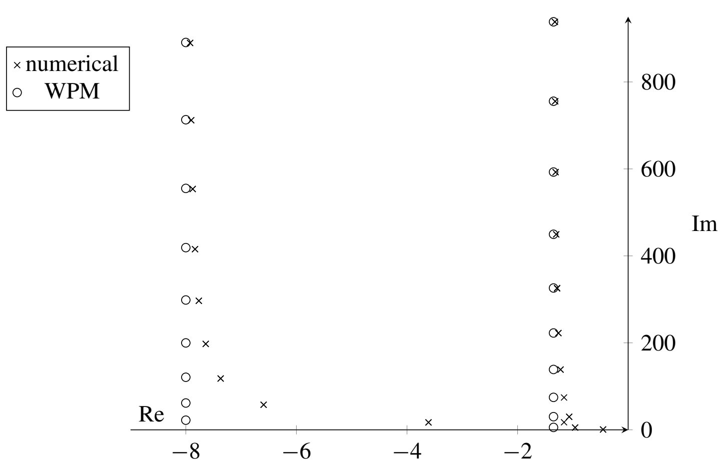

Figure 2. Plot of the vibration frequencies from Table 4.

type II joint							
$q_{11} = q_{12} = 0$				$q_{11} = 0.5, q_{12} = 0.7$			
numerical		WPM		numerical		WPM	
Re	Im	Re	Im	Re	Im	Re	Im
-0.21945	0.82672	0.0	0.61685	-0.16897	0.72184	0.0	0.61685
-1.0236	5.5617	-1.0	5.5517	-0.95445	6.1571	-1.0	6.1577
$-7.6 \cdot 10^{-4}$	15.424	0.0	15.421	$-6.6 \cdot 10^{-4}$	15.424	0.0	15.421
-1.0025	30.234	-1.0	30.226	-0.98966	30.834	-1.0	30.826
$1.4 \cdot 10^{-6}$	49.964	0.0	49.965	$-1.3 \cdot 10^{-6}$	49.965	0.0	49.965
-1.0011	74.642	-1.0	74.639	-0.99584	75.242	-1.0	75.239
$3.4 \cdot 10^{-10}$	104.25	0.0	104.25	$1.1 \cdot 10^{-9}$	104.25	0.0	104.25
-1.0006	138.70	-1.0	138.79	-0.99777	139.39	-1.0	139.39
$-1.3 \cdot 10^{-8}$	178.27	0.0	178.27	$-1.1 \cdot 10^{-8}$	178.27	0.0	178.27
-1.0004	222.68	-1.0	222.68	-0.99861	223.28	-1.0	223.28
$-9.5 \cdot 10^{-8}$	272.03	0.0	272.03	$-7.5 \cdot 10^{-8}$	272.03	0.0	272.03
-1.0003	326.31	-1.0	326.31	-0.99905	326.91	-1.0	326.91
$2.2 \cdot 10^{-8}$	385.53	0.0	385.53	$3.8 \cdot 10^{-7}$	385.53	0.0	385.53
-1.0002	449.68	-1.0	449.68	-0.99930	450.28	-1.0	450.28
$-5.3 \cdot 10^{-8}$	518.77	0.0	518.77	$-5.4 \cdot 10^{-7}$	518.77	0.0	518.77
-1.0001	592.79	-1.0	592.79	-0.99947	593.39	-1.0	593.39
$-8.1 \cdot 10^{-6}$	671.75	0.0	671.75	$-7.6 \cdot 10^{-6}$	671.75	0.0	671.75
-1.0002	755.64	-1.0	755.64	-0.99970	756.24	-1.0	756.24
$-8.5 \cdot 10^{-6}$	844.47	0.0	844.47	$-2.2 \cdot 10^{-5}$	844.48	0.0	844.47
-1.0002	938.23	-1.0	938.23	-0.99972	938.84	-1.0	938.83

Table 5. Type II joint, C-F end conditions, with $p_{11} = p_{12} = 1$, $q_{11} = q_{12} = 0$, and $p_{11} = p_{12} = 1$, $q_{11} = 0.5$, $q_{12} = 0.7$.

parts of the eigenfrequencies of the "damped" branch should be

$$\frac{q_{21} + q_{22}}{2} = 0.6 \tag{14}$$

and, indeed, this is what we see in the numerical results. Here, again, and in Figure 4, we see a strong match between the numerical and asymptotic results.

Table 6 is arranged exactly as Table 5, but here we consider, instead, a type IV joint, with $p_{41} = p_{42} = 1$. As before, $q_{41} = q_{42} = 0$ for the first four columns, while $q_{41} = 0.5$ and $q_{42} = 0.7$ for the last four. Once more, we provide the first twenty eigenfrequencies. For the $q_{41} = q_{42} = 0$ results, the asymptotic results occur in pairs with equal imaginary parts, and we can see from the table and from Figure 5, where these data are plotted, that the numerical results are approaching the same behavior asymptotically. For the case $q_{41} = 0.5$, $q_{42} = 0.7$, we again see the effect of nonzero

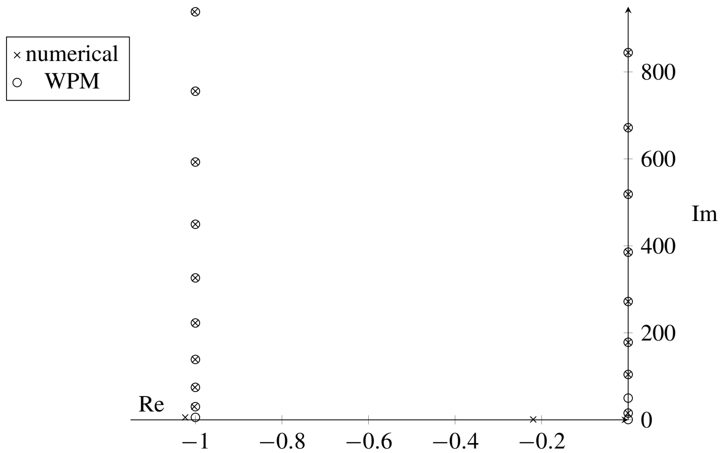

Figure 3. Plot of the vibration frequencies from Table 5 for the case $q_{11} = q_{12} = 0$.

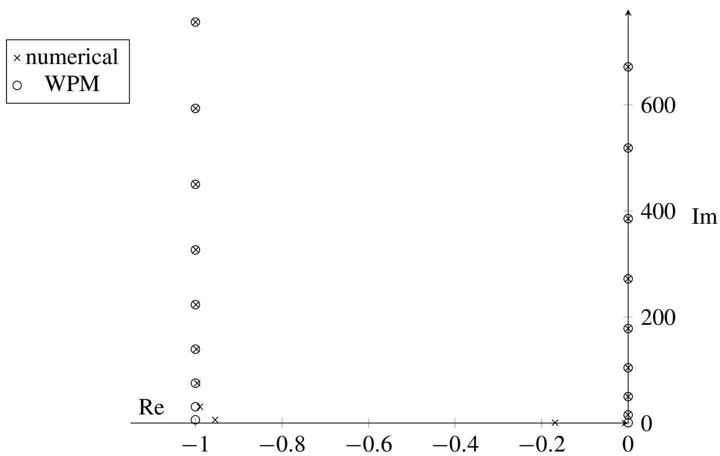

Figure 4. Plot of the vibration frequencies from Table 5 for the case $q_{11} = 0.5$, $q_{12} = 0.7$.

q-values on the imaginary part of the "damped" branch. Here, the effect is

$$\frac{q_{41} + q_{42}}{p_{41} p_{42} + q_{41} q_{42}} = 1.7778. \tag{15}$$

We plot these results in Figure 6, where, although it is difficult to see the effects of the nonzero q-values, we can see, again, a very good match between the asymptotic and numerical results.

We realize that the damping parameters we have used may not be physically realistic. Indeed, in other work, we have seen that, for realistic data, the convergence

type IV joint							
$q_{41} = q_{42} = 0$				$q_{41} = 0.5, q_{42} = 0.7$			
numerical		WPM		numerical		WPM	
Re	Im	Re	Im	Re	Im	Re	Im
$3.3 \cdot 10^{-16}$	0.0000			$7.8 \cdot 10^{-16}$	0.0000		
−0.46155	0.53410			−0.26096	0.45158		
$-4.2 \cdot 10^{-2}$	5.5592	0.0	5.5517	$-2.3 \cdot 10^{-2}$	5.5606	0.0	5.5517
−4.0964	6.6697	−4.0	5.5517	−2.3443	7.6742	−2.96	7.3295
$-1.2 \cdot 10^{-4}$	30.226	0.0	30.226	$-7.0 \cdot 10^{-5}$	30.226	0.0	30.226
−4.1652	30.395	−4.0	30.226	−2.8528	32.136	−2.96	32.004
$-1.2 \cdot 10^{-4}$	74.639	0.0	74.639	$-1.5 \cdot 10^{-7}$	74.639	0.0	74.639
−4.0704	74.698	−4.0	74.639	−2.9217	76.474	−2.96	76.417
$-2.1 \cdot 10^{-10}$	138.79	0.0	138.79	$2.3 \cdot 10^{-9}$	138.79	0.0	138.79
−4.0382	138.82	−4.0	138.79	−2.9415	140.60	−2.96	140.57
$-1.4 \cdot 10^{-8}$	222.68	0.0	222.68	$-2.4 \cdot 10^{-10}$	222.68	0.0	222.68
−4.0239	222.70	−4.0	222.68	−2.9498	224.48	−2.96	224.46
$-3.5 \cdot 10^{-8}$	326.31	0.0	326.31	$-5.0 \cdot 10^{-8}$	326.31	0.0	326.31
−4.0163	326.33	−4.0	326.31	−2.9540	328.10	−2.96	328.09
$-8.7 \cdot 10^{-8}$	449.68	0.0	449.68	$-2.7 \cdot 10^{-7}$	449.68	0.0	449.68
−4.0118	449.69	−4.0	449.68	−2.9565	451.47	−2.96	451.46
$-3.2 \cdot 10^{-7}$	592.79	0.0	592.79	$1.2 \cdot 10^{-7}$	592.79	0.0	592.79
−4.0090	592.80	−4.0	592.79	−2.9581	594.58	−2.96	594.57
$1.8 \cdot 10^{-6}$	755.64	0.0	755.64	$3.6 \cdot 10^{-6}$	755.64	0.0	755.64
−4.0071	755.65	−4.0	755.64	−2.9592	757.42	−2.96	757.42

Table 6. Type IV joint, C-F end conditions, with $p_{11} = p_{12} = 1$, $q_{11} = q_{12} = 0$, and $p_{11} = p_{12} = 1$, $q_{11} = 0.5$, $q_{12} = 0.7$.

of the numerical to the asymptotic results sometimes takes much longer. However, we have not been able to find realistic parameters in the literature. In particular, in the two papers which give experimental results [Chen et al. 1988; 1989], the physical parameters have not been determined, and the comparison with the asymptotic results is based instead on a very clever use of the patterns that result from looking at various differences between the eigenfrequencies.

Finally, we should mention that, in order to utilize the *wave propagation method* in its current form, it is necessary that the possible wave speeds are the same along each beam, thus the assumption here and in the references that each of the physical parameters m, E, and I is the same for each beam. We can generalize a bit, given that the wave speeds actually depend only on the ratio EI/m, so we need only have the ratio be the same for each beam. Once this condition is not met, however, the problem becomes far more difficult — indeed, we have found nothing in the literature regarding an asymptotic analysis of this problem.

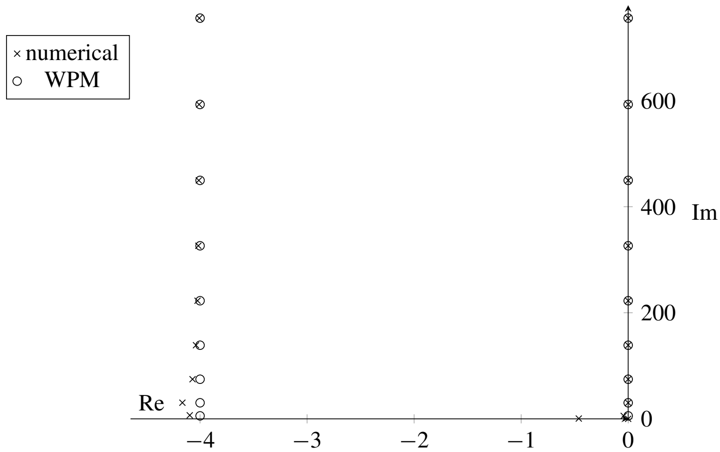

Figure 5. Plot of the vibration frequencies from Table 6 for the case $q_{41} = q_{42} = 0$.

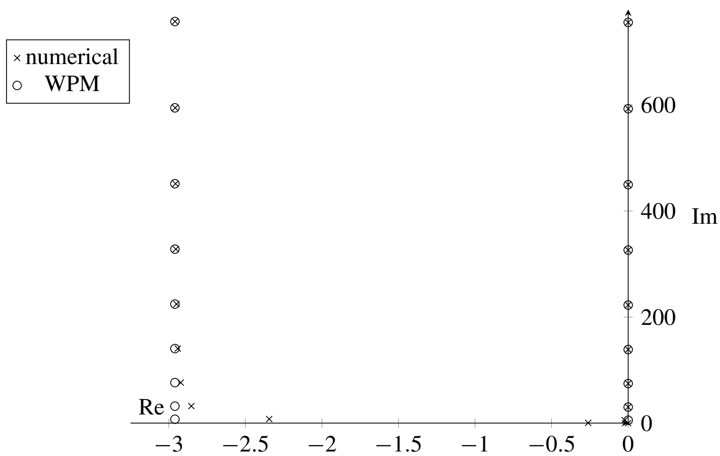

Figure 6. Plot of the vibration frequencies from Table 6 for the case $q_{41} = 0.5$, $q_{42} = 0.7$.

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