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Functional integration on spaces of connections

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Functional Integration on Spaces of Connections

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Abstract

Let G be a compact connected Lie group and $P \to M$ a smooth principal G-bundle. Let a 'cylinder function' on the space $\mathcal{A}$ of smooth connections on P be a continuous function of the holonomies of A along finitely many piecewise smoothly immersed curves in M, and let a generalized measure on $\mathcal{A}$ be a bounded linear functional on cylinder functions. We construct a generalized measure on the space of connections that extends the uniform measure of Ashtekar, Lewandowski and Baez to the smooth case, and prove it is invariant under all automorphisms of P, not necessarily the identity on the base space M. Using 'spin networks' we construct explicit functions spanning the corresponding Hilbert space $L^2(\mathcal{A}/\mathcal{G})$, where $\mathcal{G}$ is the group of gauge transformations.

1   Introduction

Integrals over spaces of connections play an important role in modern gauge theory, but as these spaces are infinite-dimensional, it is often difficult to make heuristic computations involving such integrals rigorous. Suppose one has a smooth principal G-bundle $P \to M$, and let $\mathcal{A}$ be the space of smooth connections on M. Then $\mathcal{A}$ is an affine space, and becomes a vector space after an arbitrary choice of some point as origin, so initially it may be tempting to integrate functions using some sort of 'Lebesgue measure' on $\mathcal{A}$. Unfortunately, various theorems [10] indicate that there are no well-behaved translation-invariant measures on an infinite-dimensional vector space.

One might then restrict ones ambition to integrating 'cylinder functions' and certain limits thereof. A cylinder function on $\mathcal{A}$ is one that depends on finitely many coordinates, that is, one of the form

$$F(A) = f(\ell_1(A), \ldots, \ell_n(A))$$

---

where $\ell_i \colon \mathcal{A} \to \mathbb{R}$ are continuous linear functionals and $f \colon \mathbb{R}^n \to \mathbb{C}$ is bounded and continuous. To integrate these all one needs is a 'cylinder measure'; the theory of these is well-developed and widely used in probability theory, quantum mechanics and quantum field theory [10, 13].

However, in gauge theory the fact that $\mathcal{A}$ is an affine space is in many ways less important than the fact that the group $\mathcal{G}$ of gauge transformations acts on it. For example, one is often interested in integrating gauge-invariant functions on $\mathcal{A}$. This amounts to doing integrals on $\mathcal{A}/\mathcal{G}$, an infinite-dimensional space which is not an affine space, nor even a manifold, but a kind of stratified space. In applications to physics [8, 11, 12, 15] one is often interested in integrating 'Wilson loops', gauge-invariant functions of the form

$$ F(A) = \operatorname{tr}(\operatorname{T} \exp \int_\gamma A) $$

where γ is a smooth loop in M, $\operatorname{T} \exp \int_\gamma A$ denotes the holonomy of A around γ, and the trace is taken in some finite-dimensional representation of G. Wilson loops are typically not easy to approximate by cylinder functions unless G is abelian, so it is difficult to integrate them against cylinder measures.

Motivated by Rovelli and Smolin's work [17] on the loop representation of quantum gravity, Ashtekar and Isham [1] introduced a nonlinear version of the cylinder measure idea which is specially adapted to gauge theory. Taking advantage of subsequent reworkings, we may describe their idea as follows. One first *redefines* a 'cylinder function' on $\mathcal{A}$ to be one of the form

$$ F(A) = f(\operatorname{T} \exp \int_{c_1} A, \ldots, \operatorname{T} \exp \int_{c_n} A) \tag{1} $$

where c_i are piecewise smooth paths in M, the holonomy $\operatorname{T} \exp \int_{c_i} A$ of the connection A along c_i is identified with an element of G by means of a trivialization of P over the endpoints of c_i, and $f \colon G^n \to \mathbb{C}$ is continuous. Taking the completion of this algebra in the sup norm

$$ \|F\|_\infty = \sup_{A \in \mathcal{A}} |F(A)|, $$

one obtains a C*-algebra of bounded continuous functions on $\mathcal{A}$. Then one defines a 'generalized measure' μ on $\mathcal{A}$ to be a bounded linear functional on this C*-algebra. Using the Gelfand-Naimark spectral theory, this C*-algebra can be identified with the C*-algebra of all continuous functions on a compact space $\overline{\mathcal{A}}$ of which $\mathcal{A}$ is a dense subset. Elements of $\overline{\mathcal{A}}$ are called 'generalized connections', and the holonomy of one of these generalized connections along a piecewise smooth path is still well-defined. By the Riesz-Markov theorem, generalized measures on $\mathcal{A}$ can be identified with finite regular Borel measures on $\overline{\mathcal{A}}$.

One might hope for some generalized measure on $\mathcal{A}$ to serve as a substitute for the nonexistent 'Lebesgue measure' on $\mathcal{A}$. At the bare minimum one would like this

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generalized measure to be invariant under all automorphisms of the bundle P — e.g., gauge transformations and lifts of diffeomorphisms of M. In a search for something along these lines, Ashtekar and Lewandowski [2] discovered that the study of generalized measures becomes more manageable when one works with cylinder functions defined using piecewise analytic paths. Technically, the difficulty with piecewise smooth paths is that they can intersect in very complicated ways — even in a Cantor set. Piecewise analytic paths, on the other hand, can only intersect in an infinite set if they overlap for some closed interval. This turns out to greatly simplify matters.

After further work by Ashtekar, Lewandowski and Baez [3, 5, 6, 7, 14], the theory of generalized measures in the analytic context now looks as follows. One assumes M is a real-analytic manifold, G is a compact connected Lie group, and $P \to M$ is a smooth principal G-bundle. One works only with cylinder functions for which the paths c_i are piecewise real-analytic. Letting $\text{Fun}_\omega(\mathcal{A})$ denote the completion of this space of cylinder functions in the sup norm, one then defines a 'generalized measure' to be a bounded linear functional on $\text{Fun}_\omega(\mathcal{A})$.

The results of Ashtekar and Isham still hold: $\text{Fun}_\omega(\mathcal{A})$ is isomorphic to the C*-algebra of continuous functions on a space $\overline{\mathcal{A}}$ containing $\mathcal{A}$ as a dense subset, and generalized measures on $\mathcal{A}$ are the same as finite regular Borel measures on $\overline{\mathcal{A}}$. In the analytic context, however, it is not too hard to construct a canonical generalized measure on $\mathcal{A}$, the 'uniform' generalized measure. This generalized measure is invariant under all automorphisms of the bundle P that act on the base manifold M as real-analytic diffeomorphisms. The uniform generalized measure is not the only one invariant under all these automorphisms. In fact, many such generalized measures exist, and they can be constructed and — in a rather abstract sense — classified using the notion of an 'embedded graph'. An embedded graph C is a finite set of analytic paths $c_i \colon [0,1] \to M$ that are 1-1, embeddings when restricted to $(0,1)$, and nonintersecting except possibly at their endpoints. These paths are called the 'edges' of the graph. One can reduce the study of holonomies along any finite set of real-analytic paths to the case of a graph, because given any such set of paths, one can write them as finite products of the edges of some graph (and their inverses). Given an embedded graph C with n edges, and trivializing P over the endpoints of all the edges, a generalized measure μ on $\mathcal{A}$ determines a measure μ_C on G^n by

$$\int_{G^n} f(g_1, \ldots, g_n) d\mu_C = \mu(F)$$

where F is related to f by equation (1). The measures μ_C for all embedded graphs C determine the generalized measure μ, and the uniform generalized measure on $\mathcal{A}$ is the unique one for which all the measures μ_C are normalized Haar measure on some product of copies of G.

It is natural to wonder whether these results depend crucially on the use of analytic paths. This is not a question of merely technical interest. One might argue that the analyticity assumptions are not so bad, since every paracompact smooth manifold admits a real-analytic structure, which is unique up to smooth diffeomorphism

[16, 20]. However, in applications to topological quantum field theory and the loop representation of quantum gravity, diffeomorphism-invariance plays a key role, and real-analytic diffeomorphisms do behave very differently from smooth ones. After all, a real-analytic diffeomorphism of a connected manifold is completely determined by its restriction to an arbitrarily small neighborhood. To see how this impinges on questions of real physical interest, it is interesting to read the recent work of Ashtekar, Lewandowski, Marolf, Mourão and Thiemann on diffeomorphism-invariant gauge theories [4].

The goal of this paper is to treat the case where M is merely *smooth*. We work with cylinder functions on $\mathcal{A}$ for which the paths c_i are 'curves', that is, *piecewise smoothly immersed*, and we let $\mathrm{Fun}(\mathcal{A})$ denote the completion of the space of these cylinder functions in the sup norm. For us, a generalized measure will be a continuous linear functional on $\mathrm{Fun}(\mathcal{A})$. Note that if M is real-analytic then $\mathrm{Fun}_\omega(\mathcal{A}) \subseteq \mathrm{Fun}(\mathcal{A})$, so any of our generalized measures restricts to a generalized measure as defined in the real-analytic context. In particular, we construct a generalized measure that is invariant under all automorphisms of the bundle P, and which restricts to the uniform generalized measure when M is real-analytic. We again call this the 'uniform' generalized measure.

In fact, this uniform generalized measure was already constructed by Ashtekar and Lewandowski [2] in the case $G = \mathrm{U}(1)$, using special properties of abelian Lie groups. From this point of view, the advance of the present paper consists of being able to handle nonabelian groups. But our work also establishes a framework for handling other generalized measures in the smooth context.

The main ideas behind this framework are as follows. In analogy with the analytic case, for every family of curves $C = \{c_1, \ldots, c_n\}$ a generalized measure μ on $\mathcal{A}$ determines a measure μ_C on G^n by

$$\int_{G^n} f(g_1, \ldots, g_n) d\mu_C = \mu(F)$$

where F is related to f by equation (1). The goal is thus to reconstruct a generalized measure μ starting from such a measure μ_C for every family C. Of course, some conditions must hold for such a collection of measures μ_C to come from a generalized measure on $\mathcal{A}$. In particular, not all n-tuples of elements of G can be simultaneously attained as the holonomies of some fixed connection along the curves in C, but only those lying in some subset $\mathcal{A}_C \subseteq G^n$. To come from a generalized measure, for each family C the measure μ_C will need to be supported on this 'attainable subset' $\mathcal{A}_C$, so we need a good understanding of this subset. In particular, in contrast to the analytic case, we cannot reduce the problem to considering nice families such as embedded graphs for which $\mathcal{A}_C = G^n$.

The reason why $\mathcal{A}_C$ may not be all of G^n is that there may be relations among the holonomies along the curves in the family C. These relations occur when the curves overlap for some open interval, so we need to introduce a notion of a 'type'

of possible overlap. Due to the complicated ways curves can intersect in the smooth context, a given type may occur infinitely often in a family C; for a simple example see Figure 1.
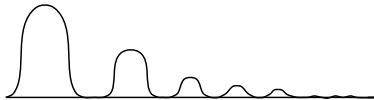

Figure 1: A family of curves with a type occurring infinitely often

The goal of Section 2 is to describe the possible holonomies of a family of curves. This is done first for especially well-behaved families of curves called 'tassels'. Roughly, a tassel based on a point $p \in M$ is a family of curves for which, when it is restricted to any neighborhood of p, the same types of overlap still occur. This self-similarity forces $\mathcal{A}_C$ to be a subgroup of G^n, in fact a subgroup easily presented in terms of the types of overlap occurring in C. Then we introduce the notion of a 'web'. This is a family W of curves that can be written as a union of tassels $W^1, \ldots, W^k$, sufficiently separated so that $\mathcal{A}_W = \mathcal{A}_{W^1} \times \cdots \times \mathcal{A}_{W^k}$. We will show that the holonomies along any family can be expressed in terms of the holonomies along some web, thus giving an algebraic description of the possible holonomies. The proofs of these facts, that is of Propositions 1 and 2, can safely be skipped by a reader looking for an initial overview of the results of this paper.

In fact, if the family one started with consisted of the edges of an embedded graph, the tassels this construction would produce would be the restriction of the edges to each cell in a cell decomposition dual to the graph. Thus each tassel would contain one vertex p of the graph, and would in fact be based at p. One should therefore think of a web as a generalization of a finite graph, and a tassel based at p as a generalization of a neighborhood of a vertex p.

In Section 3 we give a criterion for a collection of measures μ_W, one for each web W, to arise from a generalized measure μ on $\mathcal{A}$. We also show that μ is uniquely determined by the measures μ_W, so that we have a tool for constructing generalized measures. In Section 4 we apply this tool to construct the uniform generalized measure.

In recent work on the loop representation of quantum gravity, 'spin network states' play an important role [4, 5, 18, 19]. These have already been dealt with rigorously in the analytic context, and in Section 5 we describe how they work in the smooth context. The basic idea is as follows. Using the uniform generalized measure ν on $\mathcal{A}$, one can define a Hilbert space $L^2(\mathcal{A})$ by completing $\text{Fun}(\mathcal{A})$ in the norm associated to the inner product

$$\langle F, G \rangle = \nu(\overline{F} G).$$

The group $\mathcal{G}$ of gauge transformations acts on $\mathcal{A}$, and this gives rise to a unitary representation of $\mathcal{G}$ on $L^2(\mathcal{A})$. We define $L^2(\mathcal{A}/\mathcal{G})$ to be the subspace of $\mathcal{G}$-invariant vectors in $L^2(\mathcal{A})$. The 'spin network states' form a very explicit 'local' orthonormal

basis of $L^2(\mathcal{A}/\mathcal{G})$, which is to say an orthonormal basis of the subspace associated to each web W. In the analytic context, they are formed using embedded graphs whose edges are labeled with irreducible unitary representations of G, and whose vertices are labeled with intertwining operators from the tensor product of the representations labeling the 'incoming' edges, to the tensor product of the representations labeling 'outgoing' edges. In the smooth context spin networks are formed using webs equipped with similar, but more subtle, representation-theoretic data. An embedded graph is a special case of a web, and in this case our spin network states reduce to the spin network states as defined in the analytic context. However, it is not yet clear whether the spin networks can be combined in a simple fashion to give an orthonormal basis of all of $L^2(\mathcal{A}/\mathcal{G})$ simultaneously, as in the analytic case.

2 Webs

Fix a connected compact Lie group G, a smooth (paracompact) N-dimensional manifold M, and a smooth principal G-bundle $P \to M$. By a *curve* we mean a piecewise smooth map from a finite closed interval of $\mathbb{R}$ to M that is an immersion on each piece. Two curves are considered *equivalent* if one is the composition of the other with a positive diffeomorphism between their domains. A *family* of curves is a finite set of curves with a chosen ordering $c_1, \ldots, c_n$. If C is such a family, let $\mathrm{Range}(C)$ be the union of the ranges of the individual curves.

If $c_1 : [a, b] \to M$ and $c_2 : [c, d] \to M$ are two curves such that $c_1(a) = c_2(d)$, we can form the *product* $c_1 c_2$ by gluing them together at this common point. Of course this is defined only up to equivalence of curves. It is exactly like the product in the fundamental groupoid, except that we do not identify homotopic curves. It is still associative, however, and there is a category whose objects are points in M and whose morphisms (other than identity morphisms) are equivalence classes of curves. Define the *inverse c^{-1}* of a curve c to be c reparametrized by an order-reversing map, again up to equivalence. This is not truly an inverse for the product, but merely a contravariant functor.

If every curve in the family C is equivalent to a (finite) product of curves in the family D and their inverses, we say that C *depends* on D. We say that a collection of families of curves $C^1, \ldots, C^k$ is *independent* if when $i \neq j$, any curve in the family C^i intersects any curve in the family C^j, if at all, only at their endpoints, and there is a neighborhood of each such intersection point whose intersection with $\mathrm{Range}(C^i \cup C^j)$ is an embedded interval. Obviously even if two families are not independent, one may not depend on the other.

The above definitions are motivated by considering holonomies of connections along these curves. The map from curves to holonomies given by such a connection sends product to product and inverse to inverse. If one family of curves depends on another, one can compute the holonomy of a connection along all the curves in the first from the same information about the second. If two families are independent,

knowing the holonomies along one family tells one nothing about the holonomies along the other.

A *subcurve* of a curve c is a curve equivalent to the restriction of c to a subinterval of its domain. The *restriction* of a family C to a closed set $K \subset M$ is the family gotten by restricting each c_i to each interval of $c_i^{-1}[K]$. A point $p \in \text{Range}(C)$ is a *regular point* if it is not the image of an endpoint or nondifferentiable point of C, and there is a neighborhood of it whose intersection with $\text{Range}(C)$ is an embedded interval.

A family of curves C is *parametrized consistently* if each curve is parametrized so that $c_i(t) = c_j(s)$ implies $t = s$. Thus each of the curves is actually an embedding, and each point p in the range of the family is associated to a unique value of the parameter, which we call $t(p)$. If a family $\{c_1, \ldots, c_n\}$ is parametrized consistently and p is a point in $\text{Range}(C)$, define the *type* of a regular point p, τ_p, to be the Lie subgroup of G^n consisting of all n-tuples $(g_1, \ldots, g_n)$ such that for some $g \in G$ we have $g_i = g$ if p lies on c_i, and $g_i = 1$ otherwise. This gives a canonical isomorphism between any type and G.

A fundamental concept in all that follows is that of a 'tassel'. A family of curves T is a *tassel based on* $p \in \text{Range}(T)$ if:

(a) $\text{Range}(T)$ lies in a contractible open subset of M

(b) T can be consistently parametrized in such a way that $c_i(0) = p$ is the left endpoint of every curve c_i

(c) Two curves in T that intersect at a point other than p intersect at a point other than p in every neighborhood of p

(d) Any type which occurs at some point in $\text{Range}(T)$ occurs in every neighborhood of p.

One may visualize the curves of the tassel as radiating outwards from the base p. See Figure 2 for an example.
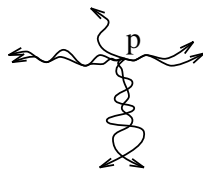

Figure 2: A tassel based at p

Finally, a *web* W is a finite independent collection of tassels $W^1, \ldots, W^k$ such that no tassel contains the base of another. We frequently apply concepts defined for families of curves to webs without comment, using the fact that the web W has an associated family $W^1 \cup \cdots \cup W^k$. For example, we say that a family depends on a

web W if it depends on the family $W^1 \cup \cdots \cup W^k$. Our first main result about webs is:

Proposition 1 *Any family of curves C depends on a web W.*

We begin with a technical lemma.

Lemma 1 *Let C be a family of smooth curves $c_1, \ldots, c_n$.*

(a) *The preimage of any point in M under any c_i is finite.*

(b) *Every point $p \in \operatorname{Range}(C)$ has a contractible open neighborhood O admitting coordinates $x_1, \ldots x_N$ such that for each i, $dx_1(c_i(t))/dt \neq 0$ on $c_i^{-1}[O]$.*

(c) *Given any point $p \in \operatorname{Range}(C)$ and any open neighborhood U of p, there is an open subneighborhood N of p such that for each i, $c_i^{-1}[N]$ is a finite union of intervals, each containing a point of $c_i^{-1}[p]$.*

(d) *The set of regular points is open and dense in $\operatorname{Range}(C)$.*

(e) *Given any point $p \in \operatorname{Range}(C)$ and any open neighborhood U of p, there is an open subneighborhood N with the properties in part (c) such that every point of $\operatorname{Range}(C)$ lying on the boundary of N is a regular point.*

Proof.

(a) If not, the preimage would have an accumulation point, and at that point c_i would not be an immersion.

(b) We can choose an open neighborhood U about p with coordinates $x_1, \ldots, x_N$ such that for all i we have $dx_1(c_i(t))/dt \neq 0$ at all of the finitely many points in the preimage of p under c_i. Each such point has an open interval around it such that $dx_1(c_i(t))/dt \neq 0$ throughout that interval. The union of the images of the complements of these intervals is a compact set $K \subseteq M$ not containing p. It follows that any contractible open neighborhood O of p contained in $U - K$ has the desired properties.

(c) Choose a coordinate patch O around p as in part (b) of this lemma, and consider the hyperplane through p on which x_1 is constant. The points of intersection of $\operatorname{Range}(C)$ with this hyperplane are all transverse, so they have no accumulation points. Thus a small open neighborhood of p in the hyperplane only intersects $\operatorname{Range}(C)$ at p. Shrinking this neighborhood to a sufficiently small subneighborhood, its product with a sufficiently small open interval in the x_1 axis is an open neighborhood N of p that only intersects each c_i in finitely many embedded open intervals containing p. This choice of N has the desired properties. See Figure 3 for an illustration.
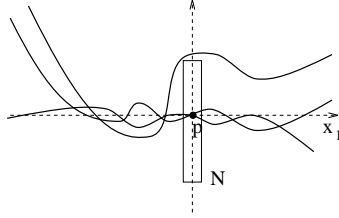
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Figure 3: Choosing the neighborhood N of p

(d) Consider any neighborhood U of a point $p \in \text{Range}(C)$. For each point in U, consider the total number of points in the preimages of all the curves c_i. By part (a), we can pick a point $p_0 \in \text{Range}(C)$ for which this number is minimal. We will show p_0 is regular.

Choose a subneighborhood N of U as in part (c) of this lemma, and small enough that each c_i is 1-1 on each component of the preimage of N. Each $r \in N \cap \text{Range}(C)$ has at most one preimage point in each of the intervals comprising $c_i^{-1}[N]$, and since p_0 was minimal and has exactly one preimage point in each of these intervals by part (c), r must have exactly one in each. Thus the images of these intervals in N must coincide, and hence $N \cap \text{Range}(C)$ must be an embedded open interval in M. Since p_0 cannot be the image of an endpoint, p_0 is regular.

(e) Choose N as in the proof of part (c). Recall that $dx_1(c_i(t))/dt \neq 0$ for all i and all t in the preimage of p under c_i. If we choose the interval (a, b) in the x_1 axis used to define N sufficiently small, each curve c_i intersects the boundary of N only at the planes where x_1 equals a or b. Moreover, choosing this interval sufficiently small guarantees that the intersection points are transversal. By part (d), it follows that we can choose a and b such that these intersection points are all regular points. $\qquad \square$

Proof of Proposition 1. Let C be a family of curves; we may assume all the curves in C are smooth, since any family depends on a family of smooth curves. By Lemma 1(b) and the compactness of $\text{Range}(C)$, we can cover $\text{Range}(C)$ with open sets $O_1, \ldots, O_m$ such that each O_l is contractible and admits coordinates as in the lemma.

We claim that each $p \in \text{Range}(C)$ has an open neighborhood N_p with the following properties:

(i) N_p is contained in every O_l containing p.

(ii) $c_i^{-1}[\overline{N}_p]$ is a finite union of intervals.

(iii) C restricted to $\overline{N}_p$ depends on a tassel based at p.

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(iv) the points of $\mathrm{Range}(C)$ lying on the boundary of N_p are all regular points.

To see this, we first tentatively take as N_p the subneighborhood given by Lemma 1(e) of the intersection of the O_l's containing p. Then $c_i^{-1}[\overline{N}_p]$ is a finite union of closed intervals, and the points of $\mathrm{Range}(C)$ lying on the boundary of N_p are all regular points. Use as coordinates on N_p any of the coordinates on the O_l's containing p; without loss of generality we assume that $x_1(p) = 0$. By Lemma 1(b) the restriction of C to $\overline{N}_p$ is a family consistently parametrized by the coordinate x_1, and by Lemma 1(e) each curve c in this family has 0 in its domain and $c(0) = p$. Take each curve in this family, and if its domain $[a, b]$ contains 0 in its interior, replace it with the two subcurves given by its restriction to $[0, b]$ and the inverse of its restriction to $[a, 0]$. Denote the resulting family by C_p. We can parametrize each curve in C_p by $|x_1|$, and then not only will C_p be consistently parametrized, but also each curve c in C_p will have $c(0) = p$ as its left endpoint.

Note that the family C restricted to $\overline{N}_p$ depends on the family C_p. Thus N_p has all the properties claimed except that C_p might not be a tassel. By the previous paragraph, and since $\mathrm{Range}(C_p)$ lies in some contractible open set O_l, the only way C_p can fail to be a tassel is by violating conditions (c) or (d) in the definition of a tassel.

To get condition (c) to hold, choose a neighborhood of p in N_p small enough that any two curves which intersect do so arbitrarily close to p, and choose a subneighborhood as in Lemma 1(e). Use this subneighborhood as a new choice of N_p, and restrict C_p to the new $\overline{N}_p$. This leaves us with a neighborhood N_p with all the properties claimed except that C_p might violate condition (d) in the definition of a tassel. To get condition (d) to hold, note that for each type τ occurring in C_p, either τ occurs at a sequence of points approaching p, or it does not. If it does not, choose a neighborhood of p in N_p which excludes all points of type τ, and choose a subneighborhood as in Lemma 1(e). Use this subneighborhood as a new choice of N_p, and restrict the original family C_p to the new $\overline{N}_p$, obtaining a new family C_p. Since there were only finitely many distinct types in the original C_p, and the new C_p has fewer types, this process must end. Now C_p is a tassel and N_p is a neighborhood of p having all the properties claimed. See Figure 4 for an example of this construction.

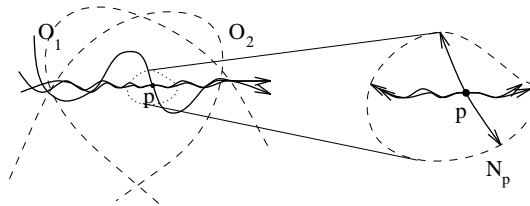Figure 4: Forming a tassel in a neighborhood of p

Now cover $\mathrm{Range}(C)$ with finitely many of these open sets N_p. Call them $N_1, \ldots, N_k$, call the associated tassels $W^1, \ldots, W^k$, and call the points at which they are based

$p_1, \ldots p_k$. We claim that if $p_j \in N_i$ for some $i \neq j$, then $N_i \cup N_j$ is still a neighborhood of p_i with properties (i-iv). Properties (ii) and (iv) are clear. For (i), note that $N_i \cup N_j$ is contained in any O_l containing p_i, because N_i is and $p_j \in N_i \subseteq O_l$ so N_j is as well. For (iii), coordinatize $N_i \cup N_j$ using the coordinates on some O_l containing p_i, and construct a family as before, breaking the restriction of each curve in C to $\overline{N}_i \cup \overline{N}_j$ into two subcurves with $x_1 \geq 0$ and $x_1 \leq 0$ if necessary, and parametrizing them consistently by the value of $|x_1|$. To see that this family is a tassel, the only nontrivial thing to check is condition (d). Notice that any type occurring in this family in $\overline{N}_i$ corresponds to a type occurring in W^i, and therefore occurs arbitrarily close to p_i. Any type occurring in the family in $\overline{N}_j - \overline{N}_i$ corresponds to a type occurring in W^j, and thus occurs arbitrarily close to p_j. But then it also occurs in N_i, and thus arbitrarily close to p_i. See Figure 5 for an illustration. Here bold curves are in W^j, light curves are in W^i, and medium weight curves are in their union.
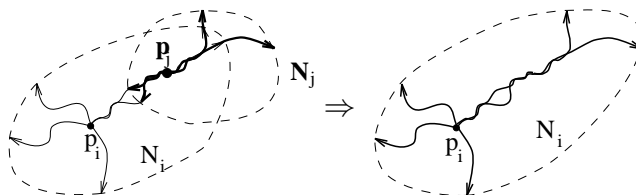

Figure 5: The union is a tassel based at p_i

Using this fact, we can replace the N_i by unions thereof until no N_i contains p_j for $i \neq j$, and succeed in covering Range(C) with open neighborhoods $N_1, \ldots, N_k$ of points $p_1, \ldots, p_k$ such that (i-iv) hold and such that $p_i \in N_j$ only for $i = j$.

To prove the theorem, we will now shrink each N_i to a smaller neighborhood of p_i while maintaining properties (i-iv), so that the resulting neighborhoods no longer intersect, but their closures still cover Range(C). When we have done this, the restriction of C to the closure of each neighborhood will depend on a tassel by (iii). Moreover, these tassels will form a web by (iv). Finally, C will depend on this web by (ii).

To shrink the N_i in this way, first replace each N_j for $j > 1$ with $N_j - \overline{N}_1$, leaving N_1 the same. Properties (i) and (iv) for the original N_j's easily imply those properties for the new N_j's. Property (ii) holds because the new N_j has only finitely many boundary points. As for property (iii), recall that $p_i \in N_j$ only for $i = j$. the only way W^j restricted to the new $\overline{N}_j$ could fail to be a tassel is by having a component that does not pass through p_j; but this could only happen if a curve of W^1 had been a subcurve of W^j, in which case $p_1 \in N_j$.

Next replace N_j with $N_j - \overline{N_2}$ for each $j > 2$, and so on. When we are done, we find that C depends on the modified $W^1, \ldots, W^k$, which are all tassels, contain neighborhoods of the p_i's, and intersect only at boundary points. Since all boundary points are boundary points of the original N_j's, they are regular points of C, and

therefore satisfy the condition for boundary points of a web. We thus obtain tassels $W^1, \ldots, W^k$ forming a web on which C depends. $\qquad\square$

Let $\mathcal{A}$ be the space of smooth connections on P, equipped with its C^∞ topology. Given a curve $c\colon [a, b] \to M$, let $\mathcal{A}_c$ be the set of functions

$$\theta\colon P_{c(a)} \to P_{c(b)}$$

compatible with the right action of G on P:

$$\theta(xg) = \theta(x)g.$$

Given $A \in \mathcal{A}$, the holonomy $\mathrm{T}\exp\int_c A$ of A along c is an element of this set $\mathcal{A}_c$. Of course, a trivialization of P at the endpoints of c allows us to identify $\mathcal{A}_c$ with G, and this gives $\mathcal{A}_c$ the structure of a smooth manifold in a manner independent of the trivialization. Note also that $\mathcal{A}_c$ and the holonomy $\mathrm{T}\exp\int_c A \in \mathcal{A}_c$ only depend on the equivalence class of c.

More generally, if $C = \{c_i\colon 1 \leq i \leq n\}$ is a family, let

$$\mathcal{A}_C \subseteq \prod_{i=1}^n \mathcal{A}_{c_i}$$

be the subspace consisting of all elements of the form

$$\left(\mathrm{T}\exp\int_{c_1} A, \ldots, \mathrm{T}\exp\int_{c_n} A\right)$$

for some connection $A \in \mathcal{A}$. We call $\mathcal{A}_C$ the space of *connections on C*, and give it the subspace topology. If W is a web consisting of tassels $W^1, \ldots, W^k$, we define $\mathcal{A}_W$ to be $\mathcal{A}_{W^1 \cup \ldots \cup W^k}$, and again call this the space of connections on W.

Note that the map $p_C\colon \mathcal{A} \to \mathcal{A}_C$ given by

$$p_C(A) = \left(\mathrm{T}\exp\int_{c_1} A, \ldots, \mathrm{T}\exp\int_{c_n} A\right)$$

is continuous and onto. Furthermore, if the product of curves $c_1 c_2$ exists there is a smooth map

$$\begin{aligned}
\mathcal{A}_{c_1} \times \mathcal{A}_{c_2} &\to \mathcal{A}_{c_1 c_2} \\
(\theta_1, \theta_2) &\mapsto \theta_1 \theta_2.
\end{aligned}$$

There is also for any curve c a smooth map

$$\begin{aligned}
\mathcal{A}_c &\to \mathcal{A}_{c^{-1}} \\
\theta &\mapsto \theta^{-1}.
\end{aligned}$$

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Thus if C depends on D, a particular choice of a way to write each curve in C as a product of curves in D and their inverses gives a smooth map

$$p_{CD} \colon \mathcal{A}_D \to \mathcal{A}_C.$$

Note that

$$p_{CD} p_D = p_C.$$

Since p_D is onto, it follows that p_{CD} is independent of how we write curves in C in terms of curves in D. Since p_C is onto, it also follows that p_{CD} is onto.

Now suppose μ is a finite Borel measure on $\mathcal{A}$. Then for any family C, μ pushes forward by the map p_C to a finite Borel measure μ_C on $\mathcal{A}_C$. The collection $\{\mu_C\}$ satisfies an obvious consistency condition: whenever C depends on D, the measure μ_D pushes forward by the map p_{CD} to the measure μ_C. Not all collections of measures $\{\mu_C\}$ satisfying this consistency condition arise from finite Borel measures on $\mathcal{A}$ in this way, but as we shall see in Section 3, if such a collection satisfies a certain uniform bound, it arises from a generalized measure on $\mathcal{A}$. This is essentially how we construct the uniform generalized measure. However, it is easier in practice to construct generalized measures from collections $\{\mu_W\}$ where W ranges over all webs, rather than all families. Proposition 1 is one of the results we need for this, since it allows us to express any family in terms of a web. The second result we need is a good description of $\mathcal{A}_W$ when W is a web.

If T is a tassel consisting of the curves $\{c_i \colon 1 \le i \le n\}$, then we let G_T be the smallest closed subgroup of G^n containing all the types occurring in T, which of course is a Lie group. Then we have:

Proposition 2 *If $T = \{c_i \colon 1 \le i \le n\}$ is a tassel, and we fix a trivialization of P over the endpoints of the curves c_i to identify $\mathcal{A}_T$ with a subset of G^n, then we have $\mathcal{A}_T = G_T$. If W is a web consisting of tassels $W^1, \ldots, W^k$, then $\mathcal{A}_W = \mathcal{A}_{W^1} \times \cdots \times \mathcal{A}_{W^k}$.*

Proof. First suppose T is a tassel. Since $\mathrm{Range}(T)$ is contained in a contractible U, we can trivialize P over $\mathrm{Range}(T)$, and by a suitable gauge transformation we choose this trivialization so that it agrees with the specified trivialization of P over the endpoints of the curves c_i. This allows us to treat the holonomy of a connection along any of these curves from any point $c_i(s)$ to any point $c_i(t)$ as an element of G. It also allows us to treat a connection on $P|U$ as a $\mathrm{Lie}(G)$-valued one-form.

We claim that given finitely many disjoint neighborhoods $N_\alpha \subseteq U$ intersecting $\mathrm{Range}(T)$ in open intervals I_α which contain no endpoints or nondifferentiable points of curves in T, there is a connection A_0 on $P|U$ whose holonomy along I_α is g_α for any $g_\alpha \in G$. To see this, map I_α to G smoothly so that a neighborhood of its left endpoint gets sent to 1 and a neighborhood of its right endpoint gets sent to g_α. Pull the derivative back to I_α, extend it to a smooth $\mathrm{Lie}(G)$-valued 1-form on N_α which is trivial in a neighborhood O_α of the two endpoints, and multiply it by a smooth function which is 1 on I_α and 0 near the boundary of N_α outside of O_α. This gives

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a connection on $P|N_\alpha$ whose holonomy along I_α is g_α. Defining A_0 this way on each N_α and setting $A_0 = 0$ outside $N = \bigcup N_\alpha$ proves the claim.

Notice if one of these intervals I_α is of type τ, $t[I_\alpha] \subset [a, b]$, and $[a, b]$ is in the domain of every c_i, then the sequence of holonomies along the c_i restricted to $[a, b]$ can be made to be any element of τ by the above procedure.

Consider any element of G_T and write it as $\prod_{i=1}^n g_i$ where each g_i is in τ_i, a type occurring in $\mathrm{Range}(T)$. By the definition of tassel, we can choose a decreasing sequence (with respect to the parameter t) of regular points p_i of type τ_i, for $i = 1, \ldots, n$, such that each $t(p_i)$ is in the interior of the domain of every curve in C. Choose nonintersecting neighborhoods N_i, and construct a connection A_0 on $P|U$ which is trivial outside the N_i and with holonomy $\prod_{i=1}^n g_i$. Thus every element of G_T is the holonomy of some connection A_0 on $P|U$. Moreover, since $\mathrm{Range}(T)$ is closed, we can find a connection $A \in \mathcal{A}$ on all of P which equals A_0 on $\mathrm{Range}(T)$, and thus has the same holonomy along each curve c_i. It follows that $G_T \subseteq \mathcal{A}_T$.

On the other hand, consider the map $\mathcal{C} : \mathbb{R}^+ \to \bigoplus_n TM$ sending each t to $\bigoplus_n (c_i(t), c_i'(t))$. If $c_i(t)$ is not defined, use $(q_i, 0)$, where q_i is the right endpoint of c_i, and if $c_i'(t)$ is not defined, use $(c_i(t), 0)$. This is continuous except at finitely many points, namely endpoints or points of nondifferentiability of any c_i. Since A gives a $\mathrm{Lie}(G)$-valued one-form, we can interpret it as a map $A : \bigoplus_n TM \to \bigoplus_n \mathrm{Lie}(G)$, so that $A \circ \mathcal{C} : \mathbb{R}^+ \to \bigoplus_n \mathrm{Lie}(G)$ is continuous except at finitely many points.

The set of t such that q is regular for all q with $t(q) = t$ is open dense, by Lemma 1(d). For such a t, $A \circ \mathcal{C}$ is a sum of elements in the Lie algebras of the types occurring with parameter value t, and thus is in $\mathrm{Lie}(G_T)$. By continuity the range of $A \circ \mathcal{C}$ is in $\mathrm{Lie}(G_T)$ except for finitely many points. But $p_T(A)$ is the endpoint of a curve in G^n starting at the identity and having derivative $A \circ \mathcal{C}(t)$ at t. Since this curve lies entirely in G_T, $p_T(\mathcal{A}) \in G_T$, so $\mathcal{A}_T \subseteq G_T$.

Now suppose W is a web consisting of tassels $W^1, \ldots, W^k$. Clearly $\mathcal{A}_W \subseteq \mathcal{A}_{W^1} \times \mathcal{A}_{W^k}$, so we need merely prove the reverse inclusion. In fact, we shall fix a trivialization of P over the subset of M consisting of all the endpoints of the curves in W, so as to identify each space $\mathcal{A}_{W^j}$ with G_{W^j}, and we shall show that given $(g_1, \cdots, g_k) \in G_{W^1} \times \cdots G_{W^k}$, there is a connection $A \in \mathcal{A}$ with $p_{W^j}(A) = g_j$ for all i. Let U_j be a contractible neighborhood containing the tassel W^j. We can choose a trivialization of P over each U_j which agrees with the above trivialization over all the endpoints of the curves in W. Moreover, we can choose these trivializations so that they agree in a small neighborhood O of all these endpoints. The construction above then gives for each tassel W^j a connection A_j on $P|U_j$ such that A_j has the desired holonomies along all curves in W^j. Moreover, given any neighborhood V_j of $\mathrm{Range}(W^j)$ with $\overline{V}_j \subset U_j$, we can choose A_j such that A_j vanishes in V_j except in an arbitrarily small neighborhood N_j of finitely many points in the interiors of the curves in W^j. (Here we use the trivialization of $P|U_j$ to think of A_j as a $\mathrm{Lie}(G)$-valued 1-form.) If we choose the V_j's small enough that $V_i \cap V_j \subseteq O$ for $i \neq j$, and choose the N_j's small enough that $V_i \cap V_j \cap N_k = \emptyset$ for all $i \neq j$, then the connections A_j agree on all

the overlaps $V_i \cap V_j$ so there exists a connection A_0 on $P|\bigcup V_i$ having the desired holonomies on all curves in every tassel W^j. Since $\text{Range}(W) \subset \bigcup V_j$ is closed there exists a connection $A \in \mathcal{A}$ that equals A_0 over $\text{Range}(W)$, so $p_{W^j}(A) = g_j$ for all j as desired. $\qquad \square$

3 Generalized Measures

Let $\text{Fun}_0(\mathcal{A})$ be the algebra of *cylinder functions* on $\mathcal{A}$, that is functions of the form

$$F(A) = f(p_C(A))$$

where C is some family of curves and $f \colon \mathcal{A}_C \to \mathbb{C}$ is continuous. Let $\text{Fun}(\mathcal{A})$ be the completion of $\text{Fun}_0(\mathcal{A})$ in the sup norm. We define a *generalized measure* on $\mathcal{A}$ to be a continuous linear functional on $\text{Fun}(\mathcal{A})$. Given a generalized measure μ on $\mathcal{A}$, for any family (or web) C we can define a bounded linear functional $(p_C)_*\mu$ on the algebra of continuous functions on $\mathcal{A}_C$ by:

$$((p_C)_*\mu)(f) = \mu(f \circ p_C).$$

By the Riesz-Markov theorem, such a bounded linear functional is just a finite regular Borel measure on $\mathcal{A}$. (Henceforth when we write simply 'measure' we shall always mean a finite regular Borel measure.)

In short, a generalized measure on $\mathcal{A}$ determines a collection of measures on the spaces $\mathcal{A}_C$ for all families C, and in fact such a collection satisfying certain conditions uniquely determines a generalized measure. In light of Propositions 1 and 2, however, it is natural to translate this into the language of webs.

Theorem 1 *Given a generalized measure μ on $\mathcal{A}$ and setting $\mu_W = (p_W)_*\mu$ for any web W, the collection $\{\mu_W\}$ is:*

(a) *Consistent: if the web W depends on the web X then $(p_{WX})_*\mu_X = \mu_W$.*

(b) *Uniformly bounded: the linear functionals $\mu_W \colon C(\mathcal{A}_W) \to \mathbb{C}$ are uniformly bounded as W ranges over all webs.*

Conversely, given any such consistent and uniformly bounded collection $\{\mu_W\}$ of measures on the spaces $\mathcal{A}_W$, there exists a unique generalized measure μ on $\mathcal{A}$ for which $(p_W)_*\mu = \mu_W$ for all webs W.

Proof. It is clear that given a generalized measure μ on $\mathcal{A}$, the measures $\mu_W = (p_W)_*\mu$ are consistent, and are uniformly bounded by the norm of μ.

For the converse, suppose $\{\mu_W\}$ is a collection of measures on the space $\mathcal{A}_W$ satisfying (a) and (b). First we define a linear functional μ_0 on $\text{Fun}_0(\mathcal{A})$ as follows. For any $F \in \text{Fun}_0(\mathcal{A})$, choose a family C and let $f_C \colon \mathcal{A}_C \to \mathbb{C}$ be a continuous function

with $F = f_C \circ p_C$. By Proposition 1, there is a web W upon which C depends, so defining $f = f_C \circ p_{CW}$, we have $F = f \circ p_W$. Now define

$$\mu_0(F) = \mu_W(f).$$

Of course, we need to check that μ_0 is well-defined and linear. Suppose that $f' \colon \mathcal{A}_{W'} \to \mathbb{C}$ is also continuous and $F = f' \circ p_{W'}$. By Proposition 1 again choose X upon which $W \cup W'$, and hence W and W', depend. Then by (a)

$$\mu_W = (p_{WX})_* \mu_X, \qquad \mu_{W'} = (p_{W'W})_* \mu_X.$$

Also, since $f \circ p_W = f' \circ p_{W'}$ we have $f \circ p_{WX} \circ p_X = f \circ p_{W'X} \circ p_W$, but since p_X is onto this implies

$$f \circ p_{WX} = f' \circ p_{W'X}.$$

Thus we have

$$
\begin{aligned}
\mu_W(f) &= ((p_{WX})_* \mu_X)(W) \\
&= \mu_X(f \circ p_{WX}) \\
&= \mu_X(f' \circ p_{W'X}) \\
&= ((p_{W'X})_* \mu_X)(f') \\
&= \mu_{W'}(f')
\end{aligned}
$$

so μ_0 is well-defined. The linearity of μ_0 then follows from the linearity of each of the μ_W's.

By (b) we can choose $M > 0$ such that $\|\mu_W\| < M$ for all W, and this implies that $|\mu_0(F)|$ for all $F \in \mathrm{Fun}_0(\mathcal{A})$. Since $\mathrm{Fun}_0(\mathcal{A})$ is dense in $\mathrm{Fun}(\mathcal{A})$, μ_0 extends uniquely to a bounded linear functional μ on $\mathrm{Fun}(\mathcal{A})$. By construction, $(p_W)_* \mu = \mu_W$ for all W. The uniqueness of μ with this property also follows from the fact that $\mathrm{Fun}_0(\mathcal{A})$ is dense in $\mathrm{Fun}(\mathcal{A})$. $\qquad \square$

In fact, generalized measures on $\mathcal{A}$ are the same thing as measures on the projective limit $\overline{\mathcal{A}}$ of the spaces $\mathcal{A}_C$, where the families C are ordered by dependence. In these terms, Proposition 1 says that webs are cofinal in the set of all families, and Theorem 1 is seen as a special case of a very general result, namely that a measure on a projective limit of spaces can be constructed from a consistent and uniformly bounded collection of measures on any cofinal set of these spaces. Ashtekar and Lewandowski have given a clear exposition of this approach in the analytic context [3], but here we chose to prove everything 'from scratch.'

Elements of $\overline{\mathcal{A}}$ may be called *generalized connections* on P. Abstractly, $\overline{\mathcal{A}}$ is simply the Gelfand spectrum of the C*-algebra $\mathrm{Fun}(\mathcal{A})$. The space $\mathcal{A}$ of smooth connections on P naturally maps into $\overline{\mathcal{A}}$ in a one-to-one and continuous way, and the image is dense in $\overline{\mathcal{A}}$. Thus generalized connections may be regarded as limits of smooth connections.

4 The Uniform Measure

In this section we construct a generalized measure ν on $\mathcal{A}$ which we call the 'uniform measure'. Theorem 1 suggests that we do this by choosing for each web W a measure ν_W in some canonical way. In the special case of a web consisting of a single tassel T, fixing a trivialization over the endpoints lets us think of ν_T as a measure on G_T. Since G_T is a compact Lie group, an obvious choice is Haar measure on G_T. For more general webs it is natural to use a product of Haar measures. This in fact gives a generalized measure.

Theorem 2 *There exists a unique generalized measure ν on $\mathcal{A}$ such that ν_T is Haar measure on G_T for any tassel T and any choice of trivialization of the endpoints, and $\nu_W = \nu_{W^1} \times \cdots \times \nu_{W^k}$ for any web W consisting of tassels $W^1, \ldots, W^k$.*

Proof. We first must prove that ν_T, for a tassel T based at a point p, is independent of the choice of trivialization. A change in the trivialization would effectively replace the holonomy $g \in G$ of a given connection along c_i by $h_p g_i h_i$, where h_p and h_i are elements of G expressing the change of trivialization at the point p and the right endpoint of c_i respectively. Thus G_T gets sent to $\vec{h}_l G_T \vec{h}_r$, where $\vec{h}_l = (h_p, \ldots, h_p)$ and $\vec{h}_r = (h_1, \ldots, h_n)$, and $h_i = h_j$ if c_i and c_j have the same right endpoint.

Now consider any point q in $\text{Range}(T)$. The set of t such that all points in $\text{Range}(T)$ with parameter value t are regular is open and dense, so there are such $t < t(q)$ and $t > t(q)$ arbitrarily close to $t(q)$. For $t < t(q)$ sufficiently close, every curve that goes through q goes through exactly one of the regular points with parameter value t, and none shares a regular point with a curve that does not go through q. Thus the group generated by their types includes points in G^n with a g in the ith entry if c_i goes through q and 1 if it does not. Likewise, taking $t > t(q)$ and small enough, we can find in the group generated by the types elements in G^n with a g in the ith entry if c_i goes through q and does not end there, and a 1 otherwise. Putting these together, we see that G_T contains every element of G^n with a g in the ith entry if c_i has q as its endpoint and a 1 otherwise. Thus $\vec{h}_r$ is an element of G_T. Likewise $\vec{h}_l \in G_T$. But since Haar measure on a Lie group is invariant under left and right multiplication, it gets sent to itself under the map $x \mapsto \vec{h}_l x \vec{h}_r$. Thus the assignment of measures to tassels, and hence to webs, is independent of the choice of trivialization, and therefore well-defined.

Now, to check condition (a) of Theorem 1, first consider a tassel T based on p, and let $W = \{W^1, \ldots, W^k\}$ be a web on which T depends. We will show that $\nu_T = (p_{TW})_* \nu_W$ in four cases. These are illustrated in Figure 6, where the curves of T are represented in bold and the curves of W are represented in medium weight.

(i) *W consists of a single tassel W^1 based at $p_1 = p$.* Since each curve in W^1 has p as a left endpoint, and each curve in T contains p only as its left endpoint, every curve in T is a curve in W^1. Thus writing $G_T \subseteq G^n$ and $G_{W^1} \subseteq G^{n_1}$

in the standard way, $p_{TW^1}: G_{W^1} \to G_T$ sends $(g_1, \ldots, g_{n_1})$ to $(g_{k_1}, \ldots, g_{k_n})$ for some integers $1 \le k_j \le n_1$. In particular, p_{TW^1} is an onto homomorphism from G_{W^1} to G_T. The image of Haar measure under an onto homomorphism is Haar measure again, so $\nu_T = (p_{TW^1})_* \nu_{W^1} = (p_{TW})_* \nu_W$.

(ii) *W consists of a single tassel W^1, and $p_1 \ne p$.* Let W' be the set of curves in W^1 which contain p, and W'' be the set of curves which do not. Then since every curve in T contains p only as its left endpoint, every curve in T can be written either as c_i^{-1} for $c_i \in W'$ or as $c_j c_i^{-1}$ for $c_i \in W'$, $c_j \in W''$. Clearly $(p_{TW^1})_* \nu_{W^1}$ is a probability measure on G_T, so it suffices to show that $(p_{TW^1})_* \nu_{W^1}$ is invariant under right multiplication by elements of G_T. Equivalently, since G_T is generated by the types in T, we must show that $(p_{TW^1})_* \nu_{W^1}$ is invariant under right multiplication by any element of τ, for τ a type of T. For this, choose a point $q \in \text{Range}(T)$ with $\tau_q = \tau$ and with $t(q)$ small enough that q is not on any curve of W''. We can identify τ with G by the canonical isomorphism, and hence with τ', the type of q in W'. Then we have, for $g \in \tau$ and $h \in G_{W^1}$

$$p_{TW^1}(h)g = p_{TW^1}(g^{-1}h)$$

so

$$(p_{TW^1})_*(\nu_{W^1})g = (p_{TW^1})_*(g^{-1}\nu_{W^1}) = (p_{TW^1})_*(\nu_{W^1}).$$

(iii) *Each W^i contains a curve containing p that is a subcurve of a curve in T.* If there is a j with $p_j = p$, then since in a web no tassel is based on a point of intersection with other tassels, there is only one W^j in W, and we are in situation (i). So assume $p_j \ne p$ for all j.

Suppose c in T is product of curves including one in W^j and one in W^i, the one in W^j being the one which contains p. Then p_j and p_i lie on c. Since W^i contains a subcurve of some curve c' in T containing p_j and p, we have that c intersects c' at a point $p_j \ne p$. Thus they intersect infinitely many times, arbitrarily near p, and thus the curves in W^i and W^j intersect infinitely many times. Since this is impossible in a web, we conclude that each curve c in T depends on one W^i. Further, a curve depending on W^j cannot intersect a curve depending on W^i for $j \ne i$ (except at p), because then they would intersect infinitely many times, arbitrarily near p, and so would curves in W^j and W^i .

Thus the W^j's separate T into families of curves $T^1, \ldots, T^k$ that intersect only at p, with each T^j depending on W^j. But then each T^j is a tassel based at p. Any type of T is a type of some T^j, and commutes with all types of all other T^i. Thus $G_T = G_{T^1} \times \cdots \times G_{T^k}$, and $\nu_T = \nu_{T^1} \times \cdots \times \nu_{T^k}$ (Of course, $\{T^j\}$ is not a web, because they all intersect at their bases). Thus p_{TW} can be written as a product of maps $p_{T^j W^j}$, so it suffices to show that $(p_{T^j W^j})_* \nu_{W^j} = \nu_{T^j}$ in order to conclude that $\nu_{T^1} \times \cdots \times \nu_{T^k} = (p_{TW})_* \nu_W$. But T^j is a single tassel depending on the single tassel W^j, so by (i) and (ii) we have $\nu_{T^j} = p_{T^j W^j} \nu_{W^j}$.

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(iv) *W is arbitrary.* Let W_0 be the set of all W^j which contain a curve containing p that is a subcurve of a curve in T, and let W_1 be the set of all other W^j. Let C_0 be T restricted to Range(W_0), and C_1 be T restricted to Range(W_1). Since every curve in C_0 contains p, every curve in C_1 does not, and every curve in T contains p exactly once, it follows that every curve in T is either a curve in C_0 or a product of a curve in C_1 and a curve in C_0. Now C_1 depends on W_1, so by trivializing P over a neighborhood containing T the set $p_{C_1 W_1}(\mathcal{A}_{W_1})$ may be thought of a subset of a product of copies of G. Since this subset consists of products of types in T, it is contained in G_T, so $(p_{C_1 W_1})_* \nu_{W_1}$ may be viewed as a probability measure μ on G_T. Since in this interpretation $p_{T W_0 \cup W_1}(x_0, x_1) = p_{C_1 W_1}(x_1) \cdot p_{C_0 W_0}(x_0)$, the measure $(p_{TW})_* \nu_W$ is the convolution of μ and $(p_{C_0 W_0})_* \nu_{W_0}$.

Now C_0 is a tassel based on p, and $G_{C_0} = G_T$, because every type occurring in T occurs arbitrarily close to p, and hence in C_0. So by (iii) $(p_{C_0 W_0})_* \nu_{W_0} = \nu_T$. But it is well known that the convolution of a probability measure on a group with Haar measure is again Haar measure, so $(p_{TW})_* \nu_W = \nu_T$.
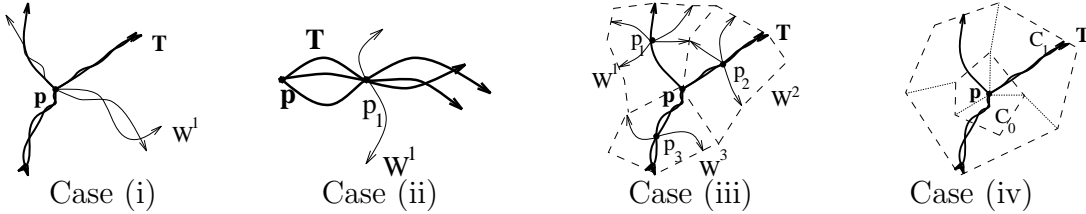

Case (i)   Case (ii)   Case (iii)   Case (iv)

Figure 6: Four cases of writing a tassel in terms of a web

To finish checking condition (a) of Theorem 1, we suppose that $W = \{W^1, \ldots, W^l\}$ is a web depending on the web $X = \{X^1, \ldots, X^k\}$, and show that $(p_{WX})_* \nu_X = \nu_W$.

To see this, note that any X^i can be divided into equivalence classes $X^i_1, \ldots, X^i_{n_i}$ of curves which are parallel at p_i, and that curves from different equivalence classes do not intersect except at p_i (this is essentially the argument in point (iv) above). Thus every type of X^i is a type of some X^i_j, and commutes with all types of any other $X^i_{j'}$. In particular $G_{X^i} = G_{X^i_1} \times \cdots \times G_{X^i_{n_i}}$ and $\nu_X = \nu_{X^1_1} \times \cdots \times \nu_{X^1_{n_1}} \times \nu_{X^2_1} \times \cdots \times \nu_{X^k_{n_k}}$.

By (i-iv), it suffices to show that $(p_{WX})_* \nu_X$ assigns an independent measure to each W^m, and by the above it suffices to show that curves in different W^m's do not depend under p_{WX} on curves in the same X^i_j. This is clear, because if they did then p_i would be in the range of both of these W^m's, but no neighborhood of it could be an interval because the curves from the two different tassels would be parallel at p_i.

Condition (b) of Theorem 1 is immediate. Each ν_W is a probability measure, so as a linear functional it has norm 1. $\qquad \square$

We call this generalized measure ν the *uniform generalized measure.* This generalized measure has a number of important properties. First notice that the group

Aut(P) of automorphisms of the bundle P acts on the space $\mathcal{A}$, and thus acts as automorphisms of the C*-algebra Fun($\mathcal{A}$) via

$$(gF)(A) = F(g^{-1}A).$$

As a consequence it acts dually on the space Fun($\mathcal{A}$)* of generalized measures on $\mathcal{A}$. We shall show that ν is invariant under this action. Moreover, since a generalized measure μ on $\mathcal{A}$ is equivalent to a measure on $\overline{\mathcal{A}}$, it is natural to speak of μ being a *probability measure* if $\mu(1) = 1$ and for all $F \in$ Fun($\mathcal{A}$), $F \geq 0$ implies $\mu(F) \geq 0$. Borrowing some terminology from C*-algebra theory, we also say that a probability measure μ is *faithful* if $F \geq 0$ and $\mu(F) = 0$ imply $F = 0$ for all $F \in$ Fun($\mathcal{A}$).

Corollary 1 *The uniform generalized measure ν is a faithful probability measure, invariant under the action of* Aut(P).

Proof. To see that ν is a faithful probability measure it suffices to check that ν_W is a faithful probability measure for each web W. For this, in turn, it suffices to check it for a tassel, and Haar measure is clearly a faithful probability measure. To see that ν is invariant, note that every step in its construction was manifestly invariant except the choice of trivialization, and we showed that ν was independent of that. $\qquad\square$

5 Spin Networks

Since ν is a faithful probability measure, we may define $L^2(\mathcal{A})$ as the Hilbert space completion of the space Fun($\mathcal{A}$) with the inner product $\langle f, g \rangle = \nu(\overline{f}g)$. Equivalently, we could set $L^2(\mathcal{A}) = L^2(\overline{\mathcal{A}}, d\nu)$. Since ν is invariant under Aut(P), there is a unitary representation of Aut(P) on $L^2(\mathcal{A})$, and thus a unitary representation of the subgroup $\mathcal{G} \subseteq$ Aut(P) consisting of gauge transformations. We define $L^2(\mathcal{A}/\mathcal{G})$ to be the closed subspace consisting of vectors in $L^2(\mathcal{A})$ invariant under the action of $\mathcal{G}$. In this section we describe an explicit set of functions on $\mathcal{A}$ spanning the Hilbert space $L^2(\mathcal{A}/\mathcal{G})$; by analogy with the analytic case [9] we call these 'spin networks'.

Given any family C, let $L^2(\mathcal{A}_C)$ be the Hilbert space of square-integrable functions on $\mathcal{A}_C$ with respect to the measure ν_C. Of course, the map $f \mapsto f \circ p_C$ from Fun($\mathcal{A}_C$) to Fun$_0(\mathcal{A})$ extends to an isometry of $L^2(\mathcal{A}_C)$ into $L^2(\mathcal{A})$, and the union of the images of these isometries over all families C is dense in $L^2(\mathcal{A})$. In fact if C depends on D then the embedding of $L^2(\mathcal{A}_C)$ in $L^2(\mathcal{A})$ factors through that of $L^2(\mathcal{A}_D)$, so the union of the images of $L^2(\mathcal{A}_W)$ over all webs W is also dense. In keeping with the philosophy of this paper, one can try to understand $L^2(\mathcal{A})$ by understanding $L^2(\mathcal{A}_W)$ for all webs W.

If $W = \{W^1, \ldots, W^k\}$ is a web, then $L^2(\mathcal{A}_W)$ is fairly simple to describe. Fixing a trivialization of P over the endpoints of the curves, $L^2(\mathcal{A}_W) \cong L^2(G_{W^1}) \otimes \cdots \otimes L^2(G_{W^k})$. Note however that this isomorphism changes when we change the trivialization. Understanding how it changes is a part of what we need to describe the gauge invariant subspace.

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For each family C, the group $\mathcal{G}$ acts on $\mathcal{A}_C$. The quotient of $\mathcal{G}$ by the subgroup which acts trivially on $\mathcal{A}_C$ is a finite-dimensional Lie group $\mathcal{G}_C$, which is actually the product over all endpoints q of curves in C of the groups $\mathcal{G}_q$ of gauge transformations of the fibers P_q. Fixing a trivialization of P_q gives an isomorphism between $\mathcal{G}_q$ and G, so we can think of $\mathcal{G}_C$ as a product of copies of G. The action of $\mathcal{G}$ on $\mathcal{A}_C$ gives a unitary representation on $L^2(\mathcal{A}_C)$, and when C depends on D the natural embedding $L^2(\mathcal{A}_C) \hookrightarrow L^2(\mathcal{A}_D)$ is an intertwining operator. Let $L^2(\mathcal{A}_C/\mathcal{G}_C)$ be the subspace of $\mathcal{G}_C$-invariant vectors in $L^2(\mathcal{A}_C)$. As before, $L^2(\mathcal{A}_C/\mathcal{G}_C)$ embeds into $L^2(\mathcal{A}_D/\mathcal{G}_D)$ if C depends on D, they both embed into $L^2(\mathcal{A}/\mathcal{G})$ in a consistent fashion, and the image of all such embeddings is dense in $L^2(\mathcal{A}/\mathcal{G})$ as C ranges over all families, or all webs. We will construct an orthonormal basis of $L^2(\mathcal{A}_W/\mathcal{G}_W)$ for each web W. The resulting set of vectors will thus give a set spanning $L^2(\mathcal{A}/\mathcal{G})$.

To do this, we need an understanding of the action of $\mathcal{G}_W$ on $L^2(\mathcal{A}_W)$. We begin by considering the action of $\mathcal{G}_T$ on $\mathcal{A}_T$ when T is a tassel. If T is a tassel based at p, then $\mathcal{G}_p$ is the group G, with action inherited from the left action of G_T on $L^2(G_T)$ by the map $g \mapsto (g, \ldots, g) \in G_T$. If q is any right endpoint of curves in T, then $\mathcal{G}_q$ is the group G, with action inherited from the right action of G_T on $L^2(G_T)$ by the map $g \mapsto (g_1, \ldots, g_n) \in G_T$, where g_i equals g in every entry corresponding to a curve with endpoint q, and equals 1 otherwise (see the proof of Theorem 2).

More precisely, the Peter-Weyl theorem states that $L^2(G_T)$ as a left and right G_T-module decomposes as

$$\bigoplus_{\lambda \in \Lambda_{G_T}} R_\lambda \otimes R_\lambda^\dagger,$$

where Λ_{G_T} is the set of all isomorphism classes of irreducible unitary representations of G_T, R_λ is an element of the isomorphism class λ as a left representation, and $R_\lambda^\dagger$ is the dual space of R_λ, as a right representation. If p is the base of T, and H_p is the subgroup of G_T consisting of all $(g, \ldots, g) \in G^n$, then the action of $\mathcal{G}_p$ on $L^2(\mathcal{A}_T) \cong \bigoplus_{\lambda \in \Lambda_{G_T}} R_\lambda \otimes R_\lambda^\dagger$ is the left action of $H_p \subset G_T$. Likewise if H_q is the subgroup of G_T consisting of all $(g_1, \ldots, g_n) \in G^n$ with $g_i = g$ if the ith curve in T has q as its right endpoint, and $g_i = 1$ otherwise, then the action of $\mathcal{G}_q$ is the right action of $H_q \in G_T$ on $\bigoplus_{\lambda \in \Lambda_{G_T}} R_\lambda \otimes R_\lambda^\dagger$.

If W is a web, we can write

$$L^2(\mathcal{A}_W) \cong \bigotimes_{j=1}^{k} \bigoplus_{\lambda_j \in \Lambda_j} R_{\lambda_j} \otimes R_{\lambda_j}^\dagger,$$

where Λ_j is shorthand for $\Lambda_{G_W^j}$. The action of the gauge group will be the same, except if a point q is the right endpoint of more than one tassel, in which case it is the right endpoint of two tassels, say W^j and W^i. In this case $\mathcal{G}_q \cong G$ acts on $L^2(G_{W^j}) \otimes L^2(G_{W^i})$ by the tensor product of the actions on each individually. Invariant vectors under this action come from invariant elements of $R_{\lambda_j}^\dagger \otimes R_{\lambda_i}^\dagger$ for

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some choice of λ_j and λ_i. Since the actions of different groups $\mathcal{G}_q$ commute, we can decompose each $R_{\lambda_j}^\dagger$ into an orthogonal direct sum of tensor products, over every q an endpoint for W^j, of irreducible unitary right representations of $\mathcal{G}_q$.

To construct actual $\mathcal{G}_W$-invariant elements of $L^2(\mathcal{A}_W)$, recall that the Peter-Weyl isomorphism is given by sending the element $v \otimes w \in R_\lambda \otimes R_\lambda^\dagger$ to the function $f(g) = (w, gv)$ for $g \in G_C$, $(\cdot, \cdot)$ being the usual pairing of a vector space with its dual, but multiplied by the square root of $\dim(R)$ to make the isomorphism unitary. So choose a representation $\lambda_j \in \Lambda_j$ for $1 \le j \le k$, choose an H_{p_j}-invariant vector $\vec{v}_j$ in R_{λ_j}, choose a term in the direct sum decomposition of each $R_{\lambda_j}^\dagger$, such that the representation assigned to each endpoint q which is only an endpoint for W^j is assigned the trivial representation and the representations assigned to a q which is an endpoint for W^i and W^j respectively are dual representations. Also choose a vector $\vec{w}_q$ in the trivial representation chosen for each q bounding one tassel, and an invariant element $\vec{w}_q$ of the representation $V \otimes V^*$ chosen for each q bounding two tassels. Notice that $\bigotimes_{j=1}^k g_j \vec{v}_j$ for $g_j \in G_{W^j}$ is an element of $\bigotimes_{j=1}^k R_{\lambda_j}$, and that $\bigotimes_q \vec{w}_q$, the product being over all endpoints q, is after reordering appropriately an element of $\bigotimes_{j=1}^k R_{\lambda_j}^\dagger$, and thus they can be paired (by the rescaled pairing) to get a number, which we call $f_{\{\vec{v}_j\},\{\vec{w}_q\}}(g_1, \ldots, g_k)$. We call such a function a *spin network*.

Theorem 3

(a) $f_{\{\vec{v}_j\},\{\vec{w}_q\}}$ is invariant under gauge transformations, and therefore in particular does not depend on the choice of trivialization at endpoints.

(b) the function $f_{\{\vec{v}_j\},\{\vec{w}_q\}}$ is in $L^2(\mathcal{A}_W/\mathcal{G}_W)$, and furthermore

$$\langle f_{\{\vec{v}_j\},\{\vec{w}_q\}}, f_{\{\vec{v}_j'\},\{\vec{w}_q'\}} \rangle = \prod_{j,q} \langle \vec{v}_j, \vec{v}_j' \rangle \langle \vec{w}_q, \vec{w}_q' \rangle.$$

(c) Choosing $\vec{v}_j$ from an orthonormal basis of the subspace of R_{λ_j} of $\mathcal{G}_{p_j}$-invariant vectors, and choosing fixed unit vectors w_p for each term in the direct sum decomposition of $R_{\lambda_j}^\dagger$, we get an orthonormal basis for $L^2(\mathcal{A}_W/\mathcal{G}_W)$.

Proof.

(a) The vectors $\{\vec{v}_j\}$ and $\{\vec{w}_q\}$ are invariant under gauge transformations, so f is.

(b) That f is in $L^2(\mathcal{A}_W/\mathcal{G}_W)$ follows from the previous point and the formula for the inner product, which is simply the statement that the Peter-Weyl isomorphism is a Hilbert space isomorphism.

(c) They are clearly orthonormal, and they certainly span the space of spin networks. But by the Peter-Weyl theorem, every $\mathcal{G}_W$-invariant element of $L^2(\mathcal{A}_W)$

is spanned by those of the form $\bigotimes_{j=1}^{k}(\vec{w}_j, g_j\vec{v}_j)$, with $\vec{v}_j$ and $\vec{w}_j$ invariant elements of R_{λ_j} and $R_{\lambda_j}^{\dagger}$ for some λ_j. Since such $\bigotimes_j \vec{w}_j$ are certainly spanned by all the tensor products $\bigotimes_p \vec{w}_p$ used to construct spin-networks, it is clear that the spin networks span $L^2(\mathcal{A}_W/\mathcal{G}_W)$. $\qquad\square$

Acknowledgements

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