Identification of Nonlinear Systems: Volterra Series Simplification

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Traditional measurement of multimedia systems, e.g. linear impulse response and transfer function, are sufficient but not faultless. For these methods the pure linear system is considered and nonlinearities, which are usually included in real systems, are disregarded. One of the ways to describe and analyze a nonlinear system is by using Volterra Series representation. However, this representation uses an enormous number of coefficients. In this work a simplification of this method is proposed and an experiment with an audio amplifier is shown.

Keywords: Volterra Series, nonlinear, system, identification, audio.

1 Introduction

As the nonlinear properties of the analyzed multimedia/audio system are unknown, the system is considered as a black box. For such a system, only input and output are observable. This black box is time invariant which means that the properties of the black box do not depend explicitly on time. Signal $y(t)$ is the system's response at the output to an input signal $x(t)$. Any given input $x_i(t)$ produces a unique output $y_i(t)$. Considering a nonlinear system, not only one input $x(t)$ can produce the same output $y(t)$. However, the converse is not true, i.e., there is a unique response $y(t)$ to input $x(t)$. The black box with its properties can be represented as shown in Fig. 1, where the symbol H_n is called a *Volterra operator*. This Volterra series theory was introduced in [1] and later used in electro-acoustics with maximum length sequence excitation [2].

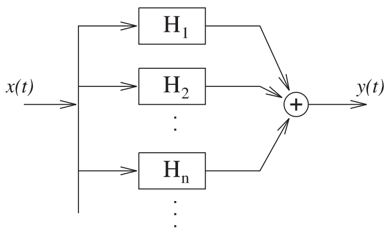

Fig. 1: Schematic representation of a Volterra series model

The relation between the output and the input can be expressed in the form given by the total sum

$$y(t) = \sum_n \mathbf{H_n}[x(t)] \tag{1}$$

in which

$$\mathbf{H_n}[x(t)] = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)\, x(t-\tau_1)\ldots x(t-\tau_n)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n \tag{2}$$

represents n-dimensional convolution of the input signal $x(t)$ and n-dimensional *Volterra kernel* $h_n(\tau_1,\ldots, \tau_n)$. The symbol $\mathbf{H_n}$ represents the n-th order *Volterra operator*. If the total *Volterra series* sum is itemized into the sum of the separated convolutions, the relation between the input and the output will be:

$$\begin{aligned} y(t) = & \int_{-\infty}^{\infty} h_1(\tau_1)\, x(t-\tau_1)\mathrm{d}\tau_1 \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_3(\tau_1,\tau_2,\tau_3)\, x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3 \\ & \vdots \\ & + \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)\, x(t-\tau_1)\ldots x(t-\tau_n)\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n \end{aligned} \tag{3}$$

2 First-order Volterra systems

For this section only a causal, stable and LTI (linear time invariant) *first-order Volterra system* will be considered. This can be expressed as

$$y(t) = \mathbf{H_1}[x(t)] \tag{4}$$

which can be expanded by using Volterra operator $\mathbf{H_1}$ in the form

$$y(t) = \int_{-\infty}^{\infty} h_1(\tau)\, x(t-\tau)\mathrm{d}\tau. \tag{5}$$

This equation represents a simple one-dimensional convolution, which determines a pure linear system. The *first-order Volterra system* is in general a linear system, in which the *first-order Volterra kernel* $h_1(t)$ is called the impulse response of the system. This impulse response can be obtained by Dirac impulse excitation $\delta(t)$, from

$$h_1(t) = \mathbf{H_1}[\delta(t)] \tag{6}$$

3 The second-order Volterra system

Since the LTI system keeps rules of linear combination, the response to a linear combination of input signals equals a linear combination of the outputs. The second-order system does not keep the rules of linear combination, but the rules of bilinear combination. The response to a linear combination of input signals equals a bilinear combination of the output signals. Let us take into consideration a causal, stable, second-order system, which is defined by

$$y(t) = \mathbf{H_2}[x(t)]\,. \tag{7}$$

Operator $\mathbf{H_2}$ is called a *second-order Volterra operator*. This operator is expressed by formula Eq. (2)

$$\mathbf{H_2}[x(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2. \tag{8}$$

The function $h_2(\tau_1, \tau_2)$ is called a second-order Volterra kernel. Generally, this function needs not to be axis-symmetric by axis $h_2(\tau, \tau)$, but for reasons of definiteness it would be better to consider this function as axis-symmetric by axis $h_2^*(\tau_1, \tau_2)$. The symmetrization can be done by

$$h_2(\tau_1,\tau_2) = \frac{1}{2}\left[h_2^*(\tau_1,\tau_2) + h_2^*(\tau_2,\tau_1)\right]. \tag{9}$$

From now on, the only axis-symmetric kernels will be considered. This can be represented as

$$h_2(\tau_1,\tau_2) = h_2(\tau_2,\tau_1). \tag{10}$$

As is known from the theory of linear systems and as is described in Eq. (6), the impulse response of a first-order system (linear system) can be obtained as a response to a Dirac impulse.

Let us take into consideration the signal $x(t) = \delta(t)$, which is brought into the input of a second-order system. The output is given by

$$\begin{aligned} y(t) &= \mathbf{H_2}[\delta(t)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)\, \delta(t-\tau_1)\, \delta(t-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \qquad (11) \\ &= h_2(t,t). \end{aligned}$$

The response to the Dirac Impulse does not determine the second-order system, but represents just a slice through the axis of *second-order Volterra kernel* (see Fig. 2).

Let the input signal $x(t)$ be given by the sum of two signals $x_1(t)+x_2(t)$. The response to such a signal is given by

$$\begin{aligned} y(t) &= \mathbf{H_2}[x(t)] = \mathbf{H_2}[x_1(t) + x_2(t)] \\ &= \mathbf{H_2}\{x_1(t), x_1(t)\} + 2\mathbf{H_2}\{x_1(t), x_2(t)\} + \mathbf{H_2}\{x_2(t), x_2(t)\} \qquad (12) \\ &= \mathbf{H_2}[x_1(t)] + 2\mathbf{H_2}\{x_1(t), x_1(t)\} + \mathbf{H_2}[x_2(t)], \end{aligned}$$

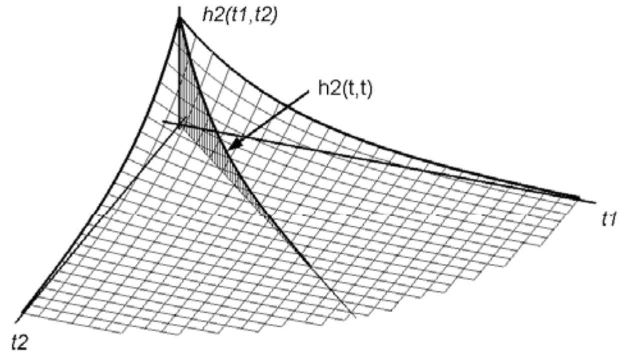

Fig. 2: Example of second-order Volterra kernel

where $\mathbf{H_2}\{\bullet\}$ is a *bilinear Volterra operator*, which is defined by

$$\mathbf{H_2}\{x_1(t), x_2(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)\, x_1(t-\tau_1)\, x_2(t-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2. \tag{13}$$

Thence

$$\mathbf{H_2}\{x_1(t), x_1(t)\} = \mathbf{H_2}[x_1(t)], \tag{14}$$

thus a *bilinear Volterra operator* applied to two same signals is simply speaking a *second-order Volterra operator*.

Generally, any higher-order system can be considered, but the complexity increases as the order of the system increases. A representation of the higher order is also more difficult to imagine as the dimension increases in size. The analysis consisting of finding all other kernels is based on finding the higher-order kernel and then recursively on finding lower--order kernels.

4 A simplified model

Using the whole Volterra model introduces many difficulties into both identifying and reconstructing a nonlinear system. Since the n-th Volterra kernel is a function of n variables, the model which represents the system has to contain many coefficients necessary to determine the system. This section describes a simplified model, which reduces the number of coefficients required for a Volterra series representation. The first simplification replaces the n-th Volterra kernel by its symmetric representation. The second-order Volterra kernel will be reduced to

$$h_2(\tau_1,\tau_2) = h_2'(\tau_1)\cdot h_2'(\tau_2). \tag{15}$$

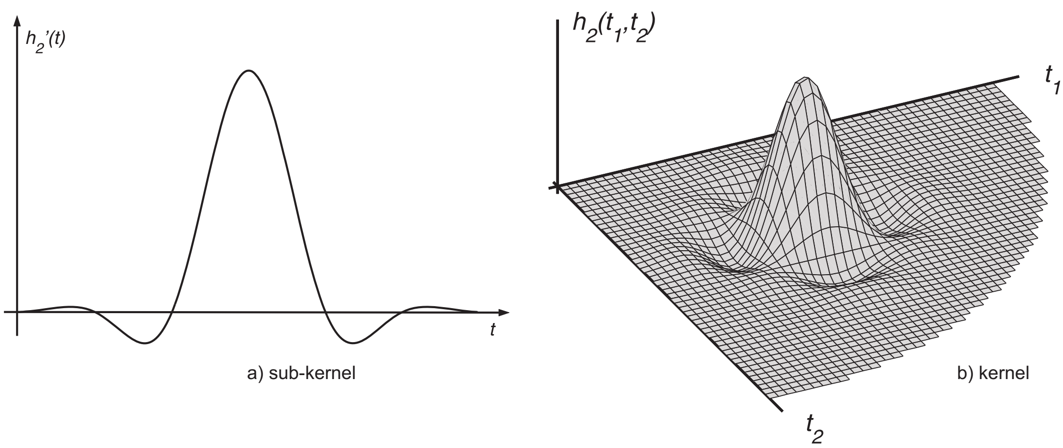

Fig. 3: A demonstration of kernel simplification

This is demonstrated in Fig. 3, which shows the sub-kernel $h_2'(\tau)$ and kernel $h_2(\tau_1,\tau_2)$.

Generally for higher Volterra kernels it stands that

$$h_n(\tau_1,\tau_2,\ldots,\tau_n)=\prod_n h_n'(\tau). \tag{16}$$

The output signal of the second-order system is in Eq. (17)

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2'(\tau_1)\, h_2'(\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \\ &= \int_{-\infty}^{\infty} h_2'(\tau_1)\, x(t-\tau_1)\mathrm{d}\tau_1 \cdot \int_{-\infty}^{\infty} h_2'(\tau_2)\, x(t-\tau_2)\mathrm{d}\tau_2 \\ &= \left[\int_{-\infty}^{\infty} h_2'(\tau)\, x(t-\tau)\mathrm{d}\tau\right]^2 . \end{aligned} \tag{17}$$

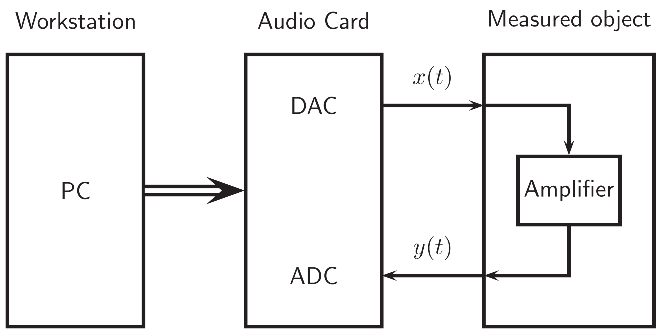

Fig. 4: Scheme of measured system (amplifier)

The scheme from Fig. 1 can be simplified by applying the simplifications described above. The simplified Volterra model is not able to determine all the nonlinearities in the same manner as the full Volterra model [1], but it will be shown that, in some cases, such as analysis of an amplifier in weakly nonlinear mode, the simplified model is sufficiently precise.

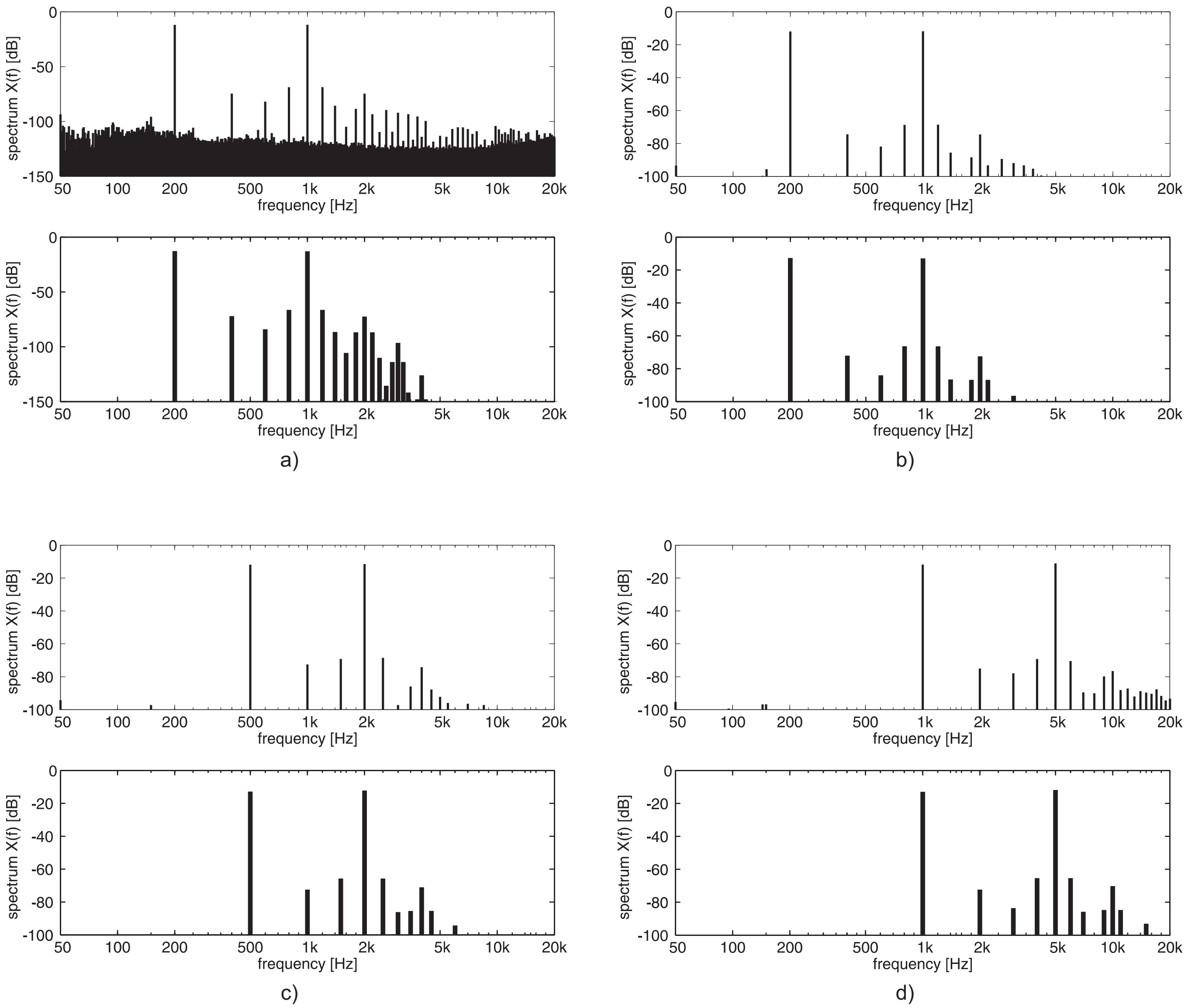

Fig. 5: Comparison of responses to a) 200 Hz and 1 kHz tones, up to –150 dB; b) 200 Hz and 1 kHz tones, up to –100 dB; c) 500 Hz and 2 kHz tones, up to –100 dB; d) 1 kHz and 5 kHz tones, up to –100 dB; SONY SDP-300 – above, model – below

Then, the output $y(t)$ is then given by

$$y(t) = \int_{-\infty}^{\infty} h_1'(\tau)\, x(t-\tau)\mathrm{d}\tau + \left[\int_{-\infty}^{\infty} h_2'(\tau)\, x(t-\tau)\mathrm{d}\tau\right]^2 \vdots + \left[\int_{-\infty}^{\infty} h_N'(\tau)\, x(t-\tau)\mathrm{d}\tau\right]^N \tag{18}$$

which can be rewritten into a shortened form

$$y(t) = \sum_{n=1}^{N} \left[\int_{-\infty}^{\infty} h_n'(\tau)\, x(t-\tau)\mathrm{d}\tau\right]^n . \tag{19}$$

5 Measuring non-linear audio systems

The simplification of Volterra kernels described above has been tested on a real audio system with nonlinear behavior. The method described above gives sufficiently precise results with respect to a weak nonlinear mode. If the higher kernels are too feeble, i.e. if the nonlinearity is weak, it is better to use the simplest model, as the higher kernels are near the level of noise. The simplified method for determining the sub-kernels has been verified on surround processor SONY SDP-E300, used in amplifier mode. The measurement scheme for identifying of Volterra sub-kernels is shown in Fig. 4. A simple method using a workstation with an audio card has been used to generate and record input and output signals.

To verify the simplified Volterra model a comparison between an audio amplifier and the Volterra model was performed. The input signal consisting of two sinusoids was put into both the audio amplifier and the model. The output spectrum of the two models was compared. The results are shown in Fig. 5.

6 Conclusion

The method for identifying nonlinear systems using a simplified Volterra Series representation has been presented and tested on a real (low-cost) audio system. The results of the nonlinear model are in some cases (weak nonlinearities) very similar to the real system. In cases of more complex nonlinearities the model gives worse results, and the simplification is not appropriate for use. The simplification of kernels gives better results in systems with weak nonlinearities, which can be found in multimedia systems such as amplifiers, loudspeakers, etc.

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References

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