A Mathematical Approach for Evaluation of Surface Topography Parameters

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The probability characteristics of surface topography parameters described by the composition of the deterministic component and the homogeneous random normal field were analysed. Formulae for the calculation of the mathematical expectation of the R_{as} parameter and the evaluation of its variance are given.

Keywords: mathematical approach, surface topography, deterministic component.

1 Introduction

Nominally flat surfaces are widely used in practice. These can be mathematically described by the composition of the deterministic and random components of irregularities in the given Cartesian coordinate system:

$$h(x, y) = S(x, y) + \zeta(x, y) \tag{1}$$

where $S(x, y)$ is the deterministic function of the surface (x, y) coordinates and $\zeta(x, y)$ is the homogeneous random normal field. The parameters of surface irregularities measured over the whole surface (the topographic parameters) characterize more positively the functional properties of the surface than the $\boldsymbol{h}$ profile parameters. Surface deviations are functions of two coordinates (x, y) and therefore the profile evaluation gives incomplete information about the surface.

2 Surface topography parameter measurement

For surface topography parameter measurement it is necessary to determine the actual value of the parameter and to know the accuracy of the measurements. In this case the analogue mean value is taken for the actual value of the parameter, and[s] the measurement error is determined by the systematic and random components. The series of the surface topographic parameters can be represented as an averaging operator of the generalized transformation $G\{h(x, y)\}$ of the surface coordinates on the given rectangular area of the surface $L_1 \times L_2$ with sides L_1 and L_2 [1]:

$$P_s = \frac{1}{L_1 L_2} \int_0^{L_1} \int_0^{L_2} G\{h(x, y)\} \, dx \, dy \tag{2}$$

Since there is a random component on the measured surface, the topographic parameter measured is a random value, which is characterized by the mathematical expectation $E(P_s)$ and the variance $D(P_s)$. Therefore, one of the problems in measuring the topographic parameter is the determination of its probability characteristics, i.e., the mathematical expectation $E(P_s)$ and the variance $D(P_s)$. It is known that the mathematical expectation of the parameters given by equation (2) can be derived by integration of the mathematical expectation $E(G)$ of the transformation $G\{h(x, y)\}$ in equation (2) by the x and y variables. As an example of the application $E(G)$ of the transformation $G\{h(x, y)\}$ in equation (2) by the x and y variables. As an example of the application of measuring methods of the topographic parameters the $P_s = R_{as}$ parameter is used. This is the arithmetic mean deviation of the surface coordinates of the mean plane.

$$E(R_{as}) = \frac{1}{L_1 L_2} \int_0^{L_1} \int_0^{L_2} E\{|h(x, y)|\} \, dx \, dy \tag{3}$$

where $|h(x, y)|$ is the absolute value of the $h(x, y)$ surface coordinate. This expression can also be extended for the $h(x, y)$ surface:

$$E\{h(x, y)\} = \left(\frac{2}{\Pi}\right)^{1/2} \sigma \exp\left[-\frac{\{S(x, y)\}^2}{2\sigma^2}\right] + S(x, y)\, \Phi\left\{\frac{S(x, y)}{\sigma 2^{1/2}}\right\} \tag{4}$$

where $\phi(z) = \frac{2}{\Pi^{1/2}} \int_0^z \exp\left(-t^2\right) dt$ is the Laplace function.

The generalized transformation $G\{h(x, y)\}$ is the random field which has the correlation function [3] defined as

$$K_G(x_1, x_2, y_1, y_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G(h_1) G(h_2) f(h_1, h_2) \, dh_1 \, dh_2 + \\ - E\{G(h_1)\}\{G(h_2)\}, \tag{5}$$

where $h_1 = h(x_1, y_1)$, $h_2 = h(x_2, y_2)$ are the coordinates of the surface at the (x_1, y_1) and (x_2, y_2) points, $f(h_1, h_2)$ distribution density expands into a series in terms of Hermite polynomials. The correlation function (5) can be represented as [3]:

$$K_G(x_1, x_2, y_1, y_2) = \sum_{n=1}^{+\infty} C_n(x_1, y_1) C_n(x_1 + \tau_1, y_1 + \tau_2) \frac{\{\rho(\tau_1, \tau_2)\}}{n!} \tag{6}$$

$$C_n(x_1, y_1) = \frac{1}{(2\Pi)^{1/2}} \int_{-\infty}^{+\infty} G(\sigma h) H_n\left(h - \frac{S}{\sigma}\right) \exp\left\{-\frac{(h - S/\sigma)^2}{2}\right\} dh \tag{7}$$

$$\tau_1 = x_2 - x_1$$

$$\tau_2 = y_2 - y_1$$

σ is the r.m.s. deviation of the random component $\zeta(x, y)$ and $\rho(\tau_1, \tau_2)$ is the correlation coefficient of the random components $\zeta(x, y)$, $h = h(x, y)$, $S = S(x, y)$. For the generalized transformation $G = \{h(x, y)\} = |h(x, y)|$, which determines the R_{as} parameter, the coefficients C_n in equation (7) are written as follows after the transformation:

$$C_1(x_1, y_1) = \left(\frac{2}{\Pi}\right)^{1/2} S(x_1, x_2) \exp\left[-\frac{\{S(x_1, y_1)\}^2}{2\sigma}\right] + \sigma\Phi\left\{\frac{S(x_1, y_1)}{\sigma 2^{1/2}}\right\}$$

$$C_n(x_1, y_1) = \left(\frac{2}{\Pi}\right)^{1/2} \sigma \exp\left[-\frac{\{S(x, y)\}^2}{2\sigma^2}\right] \times \left[H_{n-2}\left\{-\frac{S(x_1, y_1)}{\sigma}\right\} + \frac{S(x_1, y_1)}{\sigma} H_{n-1}\left\{-\frac{S(x_1, y_1)}{\sigma}\right\}\right] \tag{8}$$

where $H_n(z)$ are Hermite polynomials. The variance of the parameter P_S is determined by integration of the correlation function $K_G(x_1, x_2, y_1, y_2)$ of the generalized transformation $G\{h(x, y)\}$ using variables x_1, x_2, y_1, y_2:

$$D(P_S) = \frac{1}{L_1^2 L_2^2} \int_0^{L_1}\int_0^{L_1}\int_0^{L_2} K_G(x_1, x_2, y_1, y_2)\, dx_1\, dx_2\, dy_1\, dy_2. \tag{9}$$

Calculation of the integral in equation (9) involves considerable difficulties; therefore, in the general case, formula (9) in not suitable for calculations. However, having $S(x, y) \geq \sigma$ and $S(x, y) \leq \sigma$ offers simplified evaluations of the variance. Thus, having $S(x, y) \geq \sigma$ from equation (8),

$$C_1 \approx \sigma$$
$$C_n \approx 0 \qquad n = 2, 3, \ldots \tag{10}$$

The correlation function of the transformation $G\{h(x, y)\}$ from equation (6) is reported approximately as

$$K_G(\tau_1, \tau_2) \approx \sigma^2 \rho(\tau_1, \tau_2). \tag{11}$$

By using $S(x, y) \leq \sigma$ from equation (8) we obtain

$$C_1 \approx 0$$
$$C_n \approx \left(\frac{2}{\Pi}\right)^{1/2} \sigma H_{n-2}(0).$$

The correlation function of the transformation $G\{h(x, y)\}$ from equation (6) is

$$K_G(\tau_1, \tau_2) \approx \sum_{n=2}^{+\infty} C_n^2 \frac{\{\rho(\tau_1, \tau_2)\}^n}{n!}. \tag{12}$$

In both cases the correlation function $K_G(\tau_1, \tau_2)$ is a function of only two variables τ_1 and τ_2. Thus, for the above-mentioned approximations, $D(P_S)$ can be calculated by using the formula:

$$D(P_S) = \frac{4}{L_1 L_2} \int_0^{L_1}\int_0^{L_2} \left(1 - \frac{\tau_1}{L_1}\right)\left(1 - \frac{\tau_2}{L_2}\right) K_G(\tau_1, \tau_2)\, d\tau_1\, d\tau_2. \tag{13}$$

The following notation is now used:

$$S_{K_G} = \frac{1}{K_G(0,0)} \int_0^{\infty}\int_0^{\infty} K_G(\tau_1, \tau_2)\, d\tau_1\, d\tau_2 \tag{14}$$

with $L_1 \times L_2 \geq S_{K_G}$ equation (13) may be written approximately:

$$D(P_S) \approx \frac{4}{L_1 L_2} K_G(0,0) S_{K_G}. \tag{15}$$

$K_G(0,0)$ is the variance of $G\{h(x, y)\}$. It is not possible to carry out analogue measurements of the topographic parameter P_S. Therefore the analogue-discrete and discrete methods are the only ones that can be used to measure the topographic parameters. With analogue-discrete measurements the averaging operator is:

$$\hat{P}_S = \frac{1}{N}\frac{1}{L_2} \sum_{i=1}^{N_1} \int_0^{L_2} G\{h(i\Delta_1, y)\} dy \tag{16}$$

where Δ_1 is the sampling in the sampling interval between the profiles and N_1 is the number of profiles. The evaluation of the parameter in equation (2) in the general case is therefore shifted together with the task of determining the parameter probability characteristics, i.e., for its mathematical expectation and variance it is necessary to determine the shift in the value of the evaluation of equation (16) with respect to that of equation (2). We can regard the probability characteristics of the evaluation (16) for that particular transformation as

$$G\{h(i\Delta_1, y)\} = |h(i\Delta_1, y)|. \tag{17}$$

$\hat{R}_{as}$ represents the analogue-discrete evaluation of the topographic parameter R_{as}. The mathematical expectation of the transformation (17) will be determined by the expression in equation (4) with $S(x, y) = S(i\Delta_1, y)$. The mathematical expectation of the $\hat{R}_{as}$ parameter is identically:

$$E(\hat{R}_{as}) = \frac{1}{N_1}\frac{1}{L_2} \sum_{i=1}^{N_1} \int_0^{L_2} E\{|h(i\Delta_1, y)|\} dy. \tag{18}$$

The integrals in equation (18) are not taken into account, but if the deterministic component $S(i\Delta_1, y)$ is linearized with a small error using the ***y*** argument on the terminal number of intervals the length of which is Δ_2 then it is possible to obtain an exact expression for the evaluation $E(\hat{R}_{as})$ of the topographic parameter R_{as}. For this, we expand the deterministic component $S(i\Delta_1, y)$ in its Taylor series by using the y argument up to the linear terms at the point $j = 1, 2, \ldots, n$:

$$S(i\Delta_1, y) \approx S(i\Delta_1, j\Delta_2) + \frac{\partial S(i\Delta_1, j\Delta_2)}{\partial y}(y - \Delta_2). \tag{19}$$

By substituting equation (19) into equation (18) and taking into account equation (4), we obtain:

$$E(\hat{R}_{as}) = \frac{1}{N_1}\frac{1}{L_2} \sum_{i=1}^{N_1}\sum_{j=1}^{n} \int_{(j-1)\Delta_2}^{j\Delta_2} \Im$$

$$\Im = \left[\left(\frac{2}{\Pi}\right)^{1/2} \sigma \exp\left\{-\frac{(\alpha_{ij} y + \beta_{ij})^2}{2\sigma^2}\right\} + (\alpha_{ij} y + \beta_{ij})\, \mathbf{\Phi}\left(\frac{\alpha_{ij} y + \beta_{ij}}{2\sigma^{1/2}}\right)\right] dy \tag{20}$$

where

$$\alpha_{ij} = \frac{\partial S(i\Delta_1, j\Delta_2)}{\partial y}$$
$$\beta_{ij} = S(i\Delta_1, j\Delta_2) - \frac{\partial S(i\Delta_1, j\Delta_2)}{\partial y} j\Delta_2. \tag{21}$$

The integrals in both components of expression (20) can be tabulated [4]. Then after transformation we obtain:

$$E\left(\hat{R}_{as}\right)=\frac{1}{N_1}\frac{1}{L_2}\sum_{i=1}^{N_1}\sum_{j=1}^{n}\frac{1}{2C}\left\{\left(\frac{A_{ij}C}{\alpha_{ij}}-\frac{1}{4A_{ij}\alpha_{ij}}+2CD_{ij}\right)\Phi\left(A_{ij}C\right)+\Gamma\right\} \tag{22}$$

$$\Gamma=\left(\frac{B_{ij}C}{\alpha_{ij}}-\frac{1}{4B_{ij}\alpha_{ij}}+2CD_{ij}\right)\Phi\left(B_{ij}C\right)+\frac{A_{ij}}{\alpha_{ij}\Pi^{1/2}}\exp\left(-C^2A_{ij}^2\right)+$$
$$+\frac{B_{ij}}{\alpha_{ij}\Pi^{1/2}}\exp\left(-C^2B_{ij}^2\right)$$

where

$$A_{ij}=\alpha_{ij}\Delta_1+\beta_{ij}$$
$$B_{ij}=\alpha_{ij}(j-1)\Delta_1+\beta_{ij}$$
$$C=\frac{1}{2\sigma^{1/2}}$$
$$D_{ij}=\frac{2\sigma^2}{\Pi^{1/2}\alpha_{ii}}$$

The $\Phi(z)$ and $\exp(-z^2)$ and the standard calculation program for the $\Phi(z)$ and $\exp(-z^2)$ functions can be used to calculate $E(\hat{R}_{as})$ on the computer. Formula (22) shows that the mathematical expectation of the parameter $\hat{R}_{as}$ depends on the characteristic of the $\partial S(i\Delta_1)/\partial y$, the deterministic component $S(i\Delta_1, y)$ and the random component σ^2. With the definite relationship between the deterministic and the random components, an approximate evaluation of $E(\hat{R}_{as})$ can be obtained.

Thus, having $S(x,y)\le\sigma$ and linearizing equation (4) we obtain:

$$E\left(\hat{R}_{as}\right)\approx\frac{1}{N_1}\frac{1}{L_2}\sum_{i=1}^{N_1}\int_0^{L_2}\left|S\left(i\Delta_1,y\right)\right|\mathrm{d}y. \tag{23}$$

By using $S(x,y)\le\sigma$ and linearizing equation (4):

$$E\left(\left|h(x,y)\right|\right)\approx\left(\frac{2}{\Pi}\right)^{1/2}\sigma+\frac{\left\{S(x,y)\right\}^2}{(2\Pi)^{1/2}\sigma}. \tag{24}$$

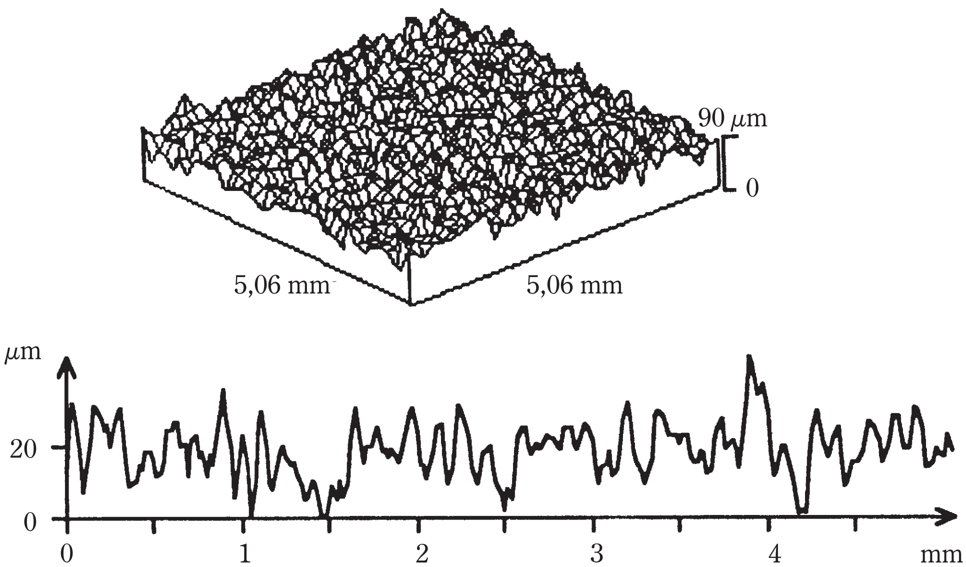


Fig. 1: Representation of surface roughness measurement

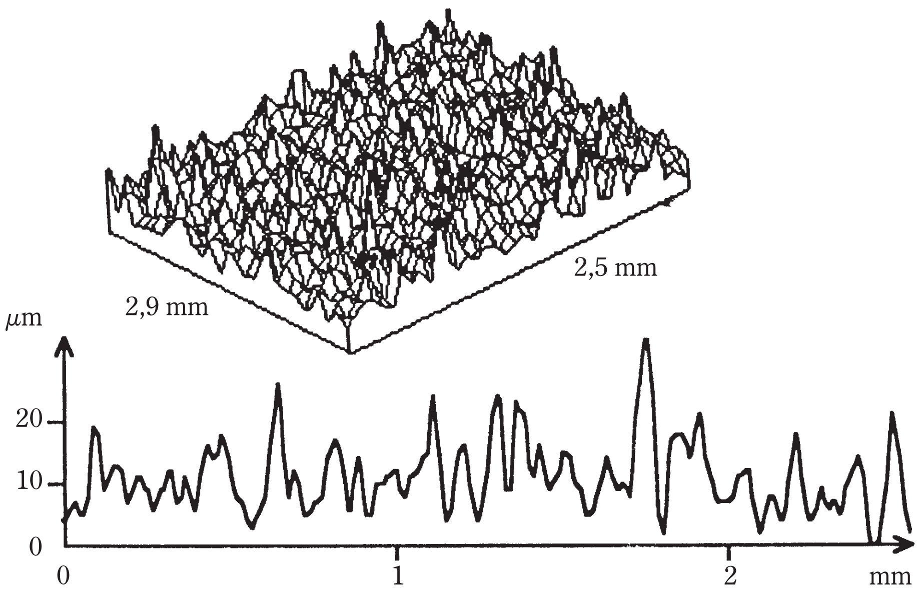
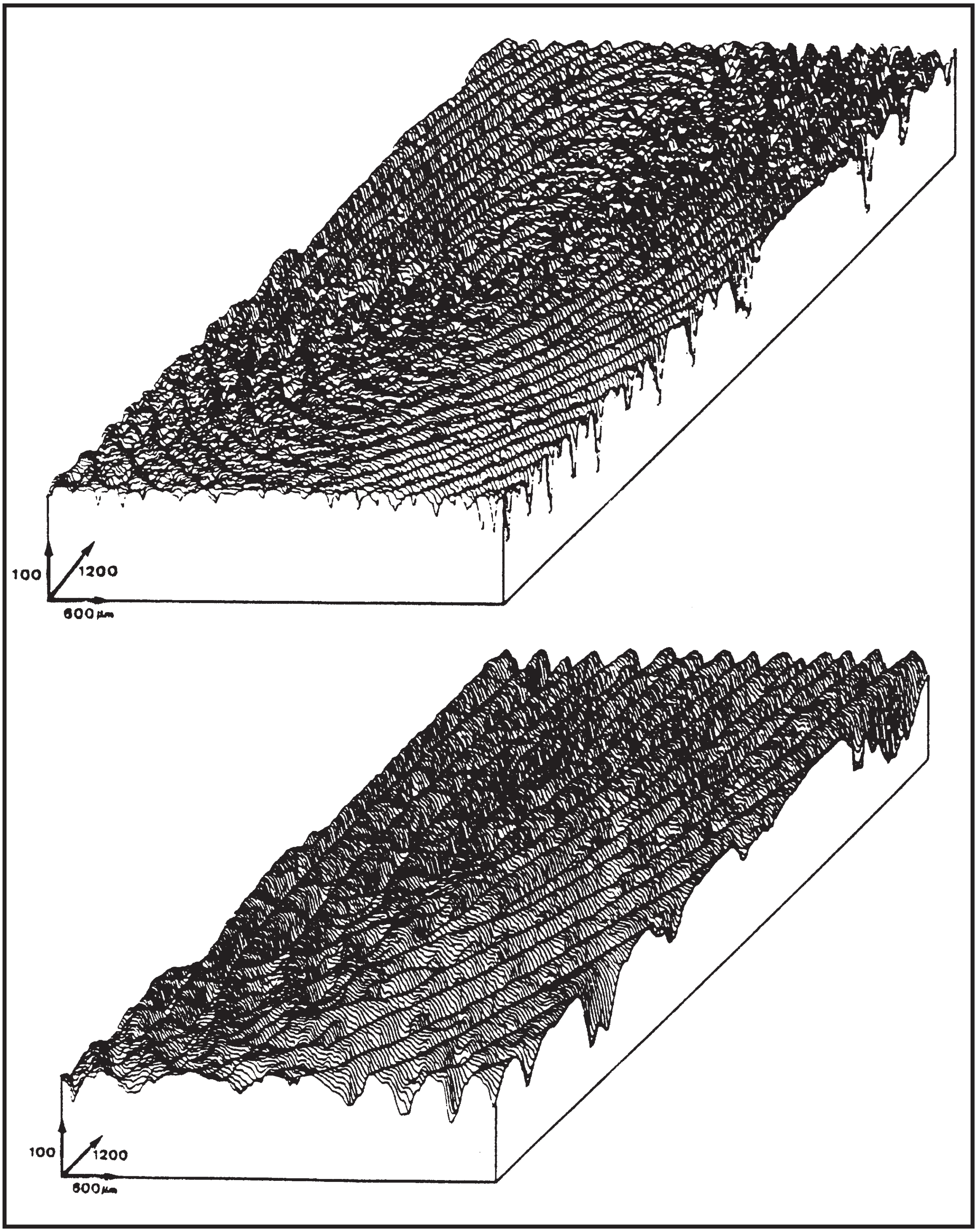


Fig. 2: Representation of surface roughness measurement



Figs. 3 and 4: A 3D topographical image

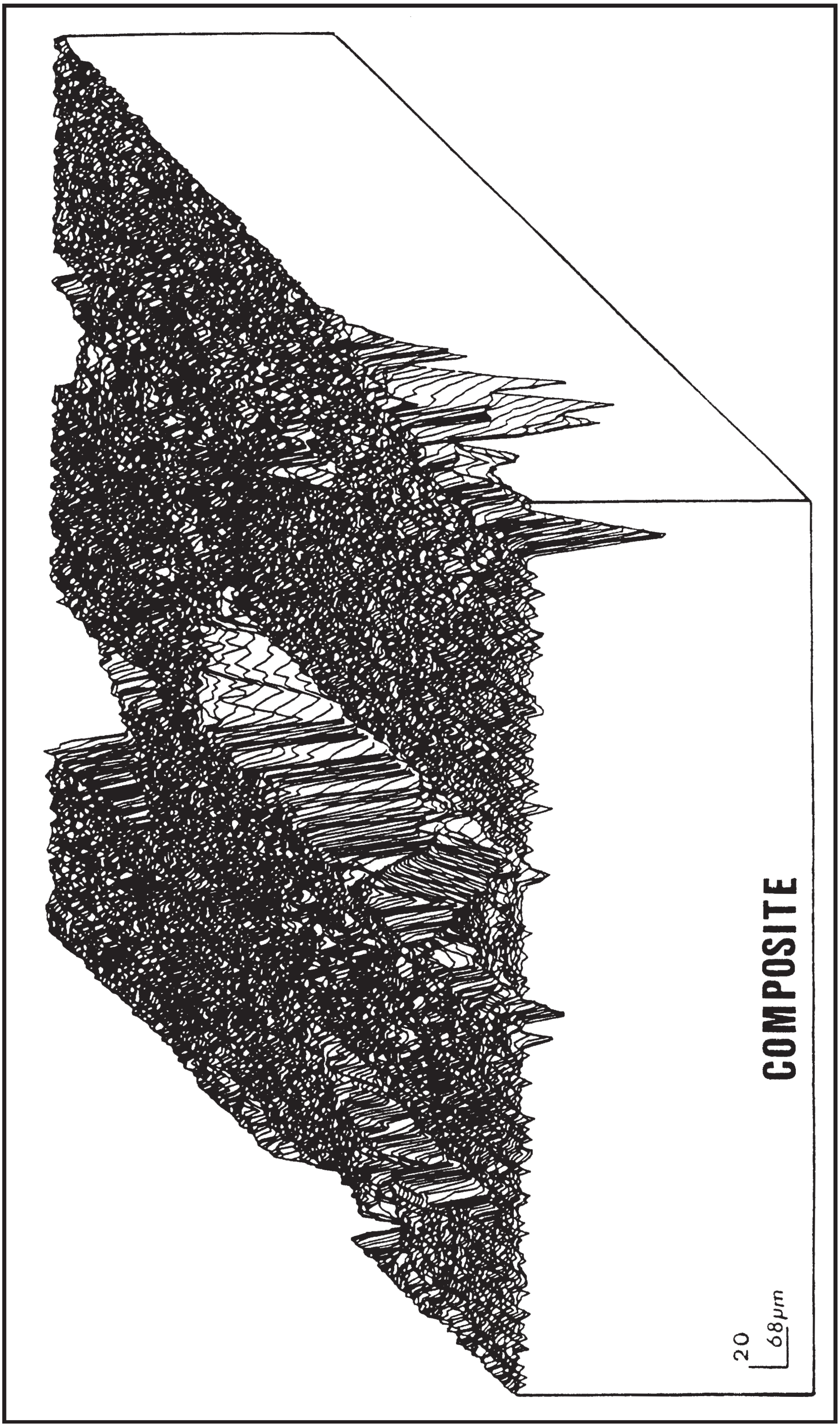


Fig. 5: Representation of 3D roughness test in a large scale

Then

$$E\left(\hat{R}_{as}\right) \approx \left(\frac{2}{\Pi}\right)^{1/2} \sigma + \frac{1}{L_1 L_2 \sigma (2\Pi)^{1/2}} \int_0^{L_1} \int_0^{L_2} \left\{S\left(x, y\right)\right\}^2 \mathrm{d}x\,\mathrm{d}y. \quad (25)$$

To analyse the $E(\hat{R}_{as})$ dependence on S/σ and to determine what relationship with S/σ is needed in order to make formulae (23) and (25) applicable to the determination of the mathematical expectation $E(\hat{R}_{as})$, calculations were carried out using formulae (22), (23) and (25). The mathematical models of the composition surface and the experimental calculation of the parameter

$$\hat{R}_{as} = \sum_{i=1}^{N_1} \sum_{j=1}^{N_i} \left|h_{ij}\right| \quad (26)$$

where h_{ij} is the deviation of the $h(x, y)$ surface from the mean plane at the discrete points. The random component was modelled with the help of the random number generator. The deterministic component was designated by formula (19).

3 Discussion

- If the deterministic component is a piecewise linear function and the random component is a homogeneous normal field, then the mathematical expectation of parameter $\hat{R}_{as}$ increases with increasing S/σ.
- If $S/\sigma \prec 1$, when $E(\hat{R}_{as})$ is calculated the influence of the deterministic component can be ignored and the error is less than 10 %.
- When $S/\sigma \prec 3$, formula (25) can be used to calculate $E(\hat{R}_{as})$ with an error of less than 10 %.
- If $S/\sigma \succ 4$, $E(\hat{R}_{as})$ can be evaluated with an error of less than 10 % by using the deterministic component (formula 23).
- Formula (22) holds for the whole range of S/σ changes.
- Full agreement with the theoretical expression can be demonstrated by the experimental calculations using formula (26).
- Some optical profilometers based on these principles are shown in figures (1–6) to measure the statistical parameters of rough surfaces. The contrast is found to be related to surface roughness when the length of coherence of the light [are] is comparable in magnitude.

4 Conclusion

The theoretical dependence of the mathematical expectation of parameter $\hat{R}_{as}$ has been determined for the case when the deterministic component $S(x, y)$ can be linearized on the separate sections. The dependence of the evaluation bias in parameter $\hat{R}_{as}$ on the relationship between the random and deterministic components has been determined.

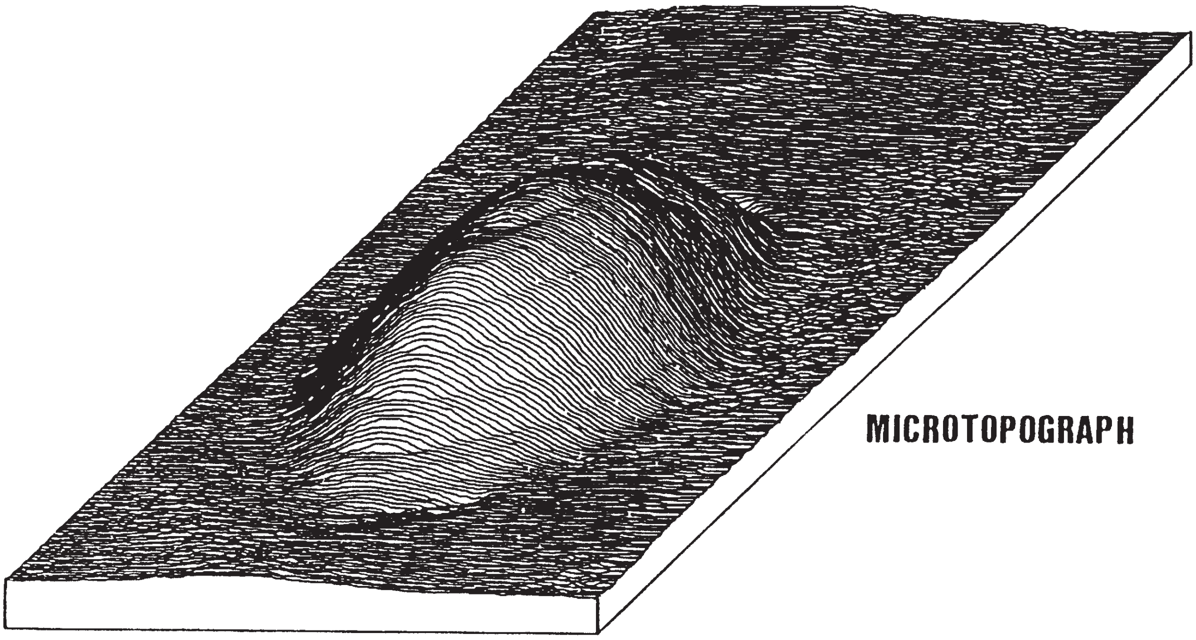


Fig. 6: Microtopograph of a rough surface

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