On $U_q(\mathfrak{sl}_2)$-actions on the Quantum Plane

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Abstract

To give the complete list of $U_q(\mathfrak{sl}_2)$-actions of the quantum plane, we first obtain the structure of quantum plane automorphisms. Then we introduce some special symbolic matrices to classify the series of actions using the weights. There are uncountably many isomorphism classes of the symmetries. We give the classical limit of the above actions.

Keywords: quantum universal enveloping algebra, Hopf algebra, Verma module, representation, composition series, projection, weight.

We present and classify $U_q(\mathfrak{sl}_2)$-actions on the quantum plane [1]. The general form of an automorphism of the quantum plane [5] allows us to use the notion of weight. To classify the actions we introduce a pair of symbolic matrices, which label the presence of nonzero weight vectors. Finally, we present the classical limit of the obtained actions.

The definitions of a Hopf algebra H and H-action, the quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ (determined by its generators k, k^{-1}, e, f), and other notations can be found in [3]. The quantum plane is a unital algebra $\mathbb{C}_q[x,y]$ generated by x, y and the relation $yx = qxy$, and we assume that $0 < q < 1$.

The notation $\mathbb{C}_q[x,y]_i$ for the i-th homogeneous component of $\mathbb{C}_q[x,y]$, being the linear span of the monomials $x^m y^n$ with $m + n = i$, is used. Denote by $(p)_i$ the i-th homogeneous component of a polynomial $p \in \mathbb{C}_q[x,y]$, that is the projection of p onto $\mathbb{C}_q[x,y]_i$ parallel to the direct sum of all other homogeneous components of $\mathbb{C}_q[x,y]$. Denote by $\mathbb{C}[x]$ and $\mathbb{C}[y]$ the linear spans of $\{x^n | n \geq 0\}$ and $\{y^n | n \geq 0\}$, respectively. The direct sum decompositions $\mathbb{C}_q[x,y] = \mathbb{C}[x] \oplus y\mathbb{C}_q[x,y] = \mathbb{C}[y] \oplus x\mathbb{C}_q[x,y]$ is obvious. Let $(P)_x$ be a projection of a polynomial $P \in \mathbb{C}_q[x,y]$ to $\mathbb{C}[x]$ parallel to $y\mathbb{C}_q[x,y]$.

Proposition 1 *Let Ψ be an automorphism of $\mathbb{C}_q[x,y]$, then there exist nonzero constants α, β such that [5]*

$$\Psi\colon x \mapsto \alpha x, \qquad y \mapsto \beta y. \tag{1}$$

For any $U_q(\mathfrak{sl}_2)$-action on $\mathbb{C}_q[x,y]$, we associate a 2×3 matrix, to be referred to as a *full action matrix*

$$M \stackrel{def}{=} \left\| \begin{matrix} k(x) & k(y) \\ e(x) & e(y) \\ f(x) & f(y) \end{matrix} \right\|. \tag{2}$$

An extension of $U_q(\mathfrak{sl}_2)$-action from the generators to $\mathbb{C}_q[x,y]$ is given by $(ab)u \stackrel{def}{=} a(bu), a(uv) \stackrel{def}{=} \Sigma_i (a_i' u) \cdot (a_i'' v), a, b \in U_q(\mathfrak{sl}_2), u, v \in \mathbb{C}_q[x,y]$ together with the natural compatibility conditions [3].

We have from (1) that the action of k is determined by its action Ψ on x and y given by a 1×2 matrix M_k

$$M_k \stackrel{def}{=} \|k(x), k(y)\| = \|\alpha x, \beta y\|, \tag{3}$$

where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. This allows us to introduce the weight of $x^n y^m \in \mathbb{C}_q[x,y]$ as $\mathbf{wt}(x^n y^m) = \alpha^n \beta^m$. Another submatrix of M is

$$M_{ef} \stackrel{def}{=} \left\| \begin{matrix} e(x) & e(y) \\ f(x) & f(y) \end{matrix} \right\|. \tag{4}$$

We call M_k and M_{ef} an *action k-matrix* and an *action ef-matrix*, respectively.

Each entry of M is a weight vector (by (3) and (1)), and all the nonzero monomials which constitute a specific entry have the same weight. We use the notation

$$\mathbf{wt}(M) \stackrel{def}{=} \begin{pmatrix} \mathbf{wt}(k(x)) & \mathbf{wt}(k(y)) \\ \mathbf{wt}(e(x)) & \mathbf{wt}(e(y)) \\ \mathbf{wt}(f(x)) & \mathbf{wt}(\mathbf{f}(y)) \end{pmatrix} \tag{5}$$

$$\bowtie \begin{pmatrix} \mathbf{wt}(x) & \mathbf{wt}(y) \\ q^2\mathbf{wt}(x) & q^2\mathbf{wt}(y) \\ q^{-2}\mathbf{wt}(x) & q^{-2}\mathbf{wt}(y) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ q^2\alpha & q^2\beta \\ q^{-2}\alpha & q^{-2}\beta \end{pmatrix},$$

where the matrix relation $\bowtie$ is treated as a set of elementwise equalities if they are applicable, that is, when the corresponding entry of M is nonzero (hence admits a well-defined weight).

Denote by $(M)_i$ the i-th homogeneous component of M which, if nonzero, admits a well-defined weight. Introduce the constants a_0, b_0, c_0, $d_0 \in \mathbb{C}$ such that the zero degree component of the full action matrix is

$$(M)_0 = \begin{pmatrix} 0 & 0 \\ a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}_0 . \tag{6}$$

We keep the subscript 0 to the matrix in the r.h.s. to emphasize the origin of this matrix as the 0-th homogeneous component of M. Weights of nonzero projections of (weight) entries of M should have the same weight, then

$$\mathbf{wt}\left((M)_0\right) \bowtie \begin{pmatrix} 0 & 0 \\ q^2\alpha & q^2\beta \\ q^{-2}\alpha & q^{-2}\beta \end{pmatrix}_0 . \tag{7}$$

All the entries of $(M)_0$ are constants (6), and so

$$\mathbf{wt}\left((M)_0\right) \bowtie \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}_0 . \tag{8}$$

Let us use $(M_{ef})_i$ to construct a symbolic matrix $\left(\overset{\star}{M}_{ef}\right)_i$ whose entries are symbols $\mathbf{0}$ or $\star$ in such a way: a nonzero entry of $(M_{ef})_i$ is replaced by $\star$, while a zero entry is replaced by the symbol $\mathbf{0}$. For 0-th components the specific relations involved in (7) imply that each column of $\left(\overset{\star}{M}_{ef}\right)_0$ should contain at least one $\mathbf{0}$, therefore we have 9 possibilities.

Apply e and f to $yx = qxy$ using (3) to get

$$ye(x) - q\beta e(x)y = qxe(y) - \alpha e(y)x, \tag{9}$$

$$f(x)y - q^{-1}\beta^{-1}yf(x) = q^{-1}f(y)x - \alpha^{-1}xf(y). \tag{10}$$

We project (9)-(10) to $\mathbb{C}_q[x,y]_1$ and obtain $a_0(1-q\beta)y = b_0(q-\alpha)x$, $d_0\left(1-q\alpha^{-1}\right)x = c_0\left(q-\beta^{-1}\right)y$, which gives

$$a_0(1-q\beta) = b_0(q-\alpha) = d_0\left(1-q\alpha^{-1}\right) = c_0\left(q-\beta^{-1}\right) = 0. \tag{11}$$

Due to (11), weight constants α and β are

$$1)\quad a_0 \neq 0 \Longrightarrow \beta = q^{-1}, \tag{12}$$

$$2)\quad b_0 \neq 0 \Longrightarrow \alpha = q, \tag{13}$$

$$3)\quad c_0 \neq 0 \Longrightarrow \beta = q^{-1}, \tag{14}$$

$$4)\quad d_0 \neq 0 \Longrightarrow \alpha = q. \tag{15}$$

We compare this to (7)–(8) and deduce that the symbolic matrices containing two $\star$'s should be excluded. Using (7) and (12)–(15) we conclude that the position of $\star$ in the remaining symbolic matrices determines the associated weight constants

$$\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 \Longrightarrow \alpha = q^{-2}, \quad \beta = q^{-1}, \tag{16}$$

$$\begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 \Longrightarrow \alpha = q, \quad \beta = q^{-2}, \tag{17}$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} \end{pmatrix}_0 \Longrightarrow \alpha = q^2, \quad \beta = q^{-1}, \tag{18}$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_0 \Longrightarrow \alpha = q, \quad \beta = q^2, \tag{19}$$

and the matrix $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0$ does not determine any weight constants.

In the 1-st homogeneous component we have $\mathbf{wt}(e(x)) = q^2\mathbf{wt}(x) \neq \mathbf{wt}(x)$ (because $0 < q < 1$), which implies $(e(x))_1 = a_1 y$, and similarly we obtain

$$(M_{ef})_1 = \begin{pmatrix} a_1 y & b_1 x \\ c_1 y & d_1 x \end{pmatrix}_1 , \tag{20}$$

where $a_1, b_1, c_1, d_1 \in \mathbb{C}$. So we introduce a symbolic matrix $\left(\overset{\star}{M}_{ef}\right)_1$ as above. The relations between weights similar to (7) give

$$\mathbf{wt}\left((M_{ef})_1\right) \bowtie \begin{pmatrix} q^2\alpha & q^2\beta \\ q^{-2}\alpha & q^{-2}\beta \end{pmatrix}_1 \bowtie \begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix}_1 . \tag{21}$$

As a consequence we have that every row and every column of $\left(\overset{\star}{M}_{ef}\right)_1$ should contain at least one $\mathbf{0}$. We project (9)-(10) to $\mathbb{C}_q[x,y]_2$ and get

$$a_1(1-q\beta)y^2 = b_1(q-\alpha)x^2, \tag{22}$$

$$d_1\left(1-q\alpha^{-1}\right)x^2 = c_1\left(q-\beta^{-1}\right)y^2, \tag{23}$$

whence $a_1(1-q\beta) = b_1(q-\alpha) = d_1\left(1-q\alpha^{-1}\right) = c_1\left(q-\beta^{-1}\right) = 0$. So we obtain

$$1)\quad a_1 \neq 0 \Longrightarrow \beta = q^{-1}, \tag{24}$$

$$2)\quad b_1 \neq 0 \Longrightarrow \alpha = q, \tag{25}$$

$$3)\quad c_1 \neq 0 \Longrightarrow \beta = q^{-1}, \tag{26}$$

$$4)\quad d_1 \neq 0 \Longrightarrow \alpha = q. \tag{27}$$

The symbolic matrix $\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_1$ can be discarded from the list of symbolic matrices with at least one $\mathbf{0}$ at every row or column, because of (21), (24)–(27). For other symbolic matrices with the above property we have

$$\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1 \Longrightarrow \alpha = q^{-3}, \quad \beta = q^{-1}, \qquad (28)$$

$$\begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1 \Longrightarrow \alpha = q, \quad \beta = q^{-1}, \qquad (29)$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} \end{pmatrix}_1 \Longrightarrow \alpha = q, \quad \beta = q^{-1}, \qquad (30)$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_1 \Longrightarrow \alpha = q, \quad \beta = q^{3}, \qquad (31)$$

$$\begin{pmatrix} \mathbf{0} & \star \\ \star & \mathbf{0} \end{pmatrix}_1 \Longrightarrow \alpha = q, \quad \beta = q^{-1}, \qquad (32)$$

and the matrix $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1$ does not determine the weight constants.

Let us introduce a table of families of $U_q(\mathfrak{sl}_2)$-actions, each family is labeled by two symbolic matrices $\left(\overset{\star}{M}_{ef}\right)_0$, $\left(\overset{\star}{M}_{ef}\right)_1$, and we call it a $\left[\left(\overset{\star}{M}_{ef}\right)_0 ; \left(\overset{\star}{M}_{ef}\right)_1\right]$-series.

Note that the series labeled with pairs of nonzero symbolic matrices at both positions are empty, because each such matrix determines a pair of specific weight constants α and β (16)–(19) which fails to coincide to any pair of such constants associated to the set of nonzero symbolic matrices at the second position (28)–(32). Also the series with zero symbolic matrix at the first position and symbolic matrices containing only one $\star$ at the second position are empty. In this way we get 24 "empty" $\left[\left(\overset{\star}{M}_{ef}\right)_0 ; \left(\overset{\star}{M}_{ef}\right)_1\right]$-series.

Let us turn to "non-empty" series and begin with the case in which the action ef-matrix is zero.

Theorem 2 *The* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of four $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane given by*

$$k(x) = \pm x, \quad k(y) = \pm y, \qquad (33)$$

$$e(x) = e(y) = f(x) = f(y) = 0, \qquad (34)$$

which are pairwise non-isomorphic.

The next theorem describes the well-known symmetry [6, 7].

Theorem 3 *The* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \star \\ \star & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of a one-parameter ($\tau \in \mathbb{C} \setminus \{0\}$) family of $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane*

$$k(x) = qx, \quad k(y) = q^{-1}y, \qquad (35)$$

$$e(x) = 0, \quad e(y) = \tau x, \qquad (36)$$

$$f(x) = \tau^{-1}y, \quad f(y) = 0. \qquad (37)$$

All these structures are isomorphic, in particular to the action as above with $\tau = 1$.

The essential claim here which is not covered by [6, 7], is that no higher (> 1) degree terms could appear in the expressions for $e(x)$, $e(y)$, $f(x)$, $f(y)$ in (36) and (37). This can be proved by a routine computation which relies upon our assumption $0 < q < 1$.

Consider the symmetries whose symbolic matrix $\left(\overset{\star}{M}_{ef}\right)_0$ contains one $\star$.

Theorem 4 *The* $\left[\begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of a one-parameter ($b_0 \in \mathbb{C} \setminus \{0\}$) family of $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane*

$$k(x) = qx, \quad k(y) = q^{-2}y, \qquad (38)$$

$$e(x) = 0, \quad e(y) = b_0, \qquad (39)$$

$$f(x) = b_0^{-1}xy, \quad f(y) = -qb_0^{-1}y^2. \qquad (40)$$

All these structures are isomorphic, in particular to the action as above with $b_0 = 1$.

Theorem 5 *The* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of a one-parameter ($c_0 \in \mathbb{C} \setminus \{0\}$) family of $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane*

$$k(x) = q^2x, \quad k(y) = q^{-1}y, \qquad (41)$$

$$e(x) = -qc_0^{-1}x^2, \quad e(y) = c_0^{-1}xy, \qquad (42)$$

$$f(x) = c_0, \quad f(y) = 0. \qquad (43)$$

All these structures are isomorphic, in particular to the action as above with $c_0 = 1$.

Theorem 6 *The* $\left[\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of a three-parameter ($a_0 \in \mathbb{C} \setminus \{0\}$, $s, t \in \mathbb{C}$) family of $U_q(\mathfrak{sl}_2)$-actions on the quantum plane*

$$k(x) = q^{-2}x, \quad k(y) = q^{-1}y, \qquad (44)$$

$$e(x) = a_0, \quad e(y) = 0, \qquad (45)$$

$$f(x) = -qa_0^{-1}x^2 + ty^4, \quad f(y) = -qa_0^{-1}xy + sy^3. \qquad (46)$$

The generic domain $\{(a_0, s, t) \,|\, s \neq 0, t \neq 0\}$ *with respect to the parameters splits into uncountably many disjoint subsets*

$$\{(a_0, s, t) \,|\, s \neq 0, t \neq 0, \varphi = const\},$$

where $\varphi = \dfrac{t}{a_0 s^2}$. *Each of these subsets corresponds to an isomorphism class of* $U_q(\mathfrak{sl}_2)$*-module algebra structures. Additionally there exist three more isomorphism classes which correspond to the subsets*

$$\begin{aligned} &\{(a_0, s, t) \,|\, s \neq 0, t = 0\}, \\ &\{(a_0, s, t) \,|\, s = 0, t \neq 0\}, \\ &\{(a_0, s, t) \,|\, s = 0, t = 0\}. \end{aligned} \tag{47}$$

The specific form of weights for x and y discards primordially all but finitely many terms (monomials) that could appear in the expressions for $e(x)$, $e(y)$, $f(x)$, $f(y)$ in (45) and (46). Thus it becomes much easier to establish the latter relations than to do this for the corresponding relations in the previous theorems. To prove the rest of the claims, one needs to guess the explicit form of the required isomorphisms.

Theorem 7 *The* $\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$*-series consists of a three-parameter (*$d_0 \in \mathbb{C} \setminus \{0\}$, $s, t \in \mathbb{C}$*) family of* $U_q(\mathfrak{sl}_2)$*-actions on the quantum plane*

$$k(x) = qx, \qquad k(y) = q^2 y, \tag{48}$$

$$e(x) = -q d_0^{-1} xy + s x^3, \quad e(y) = -q d_0^{-1} y^2 + t x^4, \tag{49}$$

$$f(x) = 0, \qquad f(y) = d_0, \tag{50}$$

Here we have the domain $\{(d_0, s, t) \,|\, s \neq 0, t \neq 0\}$ *which splits into the disjoint subsets* $\{(d_0, s, t) \,|\, s \neq 0, t \neq 0, \varphi = const\}$ *with* $\varphi = \dfrac{t}{d_0 s^2}$. *This uncountable family of subsets is in one-to-one correspondence to isomorphism classes of* $U_q(\mathfrak{sl}_2)$*-module algebra structures. In addition, one also has three more isomorphism classes which are labelled by the subsets* $\{(d_0, s, t) \,|\, s \neq 0, t = 0\}$, $\{(d_0, s, t) \,|\, s = 0, t \neq 0\}$, $\{(d_0, s, t) \,|\, s = 0, t = 0\}$.

Remark 8 *The* $U_q(\mathfrak{sl}_2)$*-symmetries on* $\mathbb{C}_q[x, y]$ *picked from different series are nonisomorphic, and the actions of* k *in different series are different.*

Remark 9 *There are no* $U_q(\mathfrak{sl}_2)$*-symmetries on* $\mathbb{C}_q[x, y]$ *other than those presented in the above theorems, because the assumptions exhaust all admissible forms for the components* $(M_{ef})_0$, $(M_{ef})_1$ *of the action* ef*-matrix.*

The associated classical limit actions of the Lie algebra $\mathfrak{sl}_2$ (here it is the Lie algebra generated by e, f, h subject to the relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$) on $C[x, y]$ by differentiations is derived from the quantum action via substituting $k = q^h$ with subsequent formal passage to the limit as $q \to 1$.

We present all quantum and classical actions in Table 1. Note that there exist more $\mathfrak{sl}_2$-actions on $C[x, y]$ by differentiations (see, e.g. [8]) than one can see in Table 1. It follows from our results that the rest of the classical actions admit no quantum counterparts. On the other hand, among the quantum actions listed in the first row of Table 1, the only one to which the above classical limit procedure is applicable, is the action with $k(x) = x$, $k(y) = y$. The remaining three actions of this series admit no classical limit in the above sense.

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Table 1:

Symbolic matrices	**U_q-module algebra structures**	**Classical limit $\mathfrak{sl}_2$-actions by differentiations**
$\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$	$k(x) = \pm x$, $k(y) = \pm y$, $e(x) = e(y) = 0$, $f(x) = f(y) = 0$,	$h(x) = 0,\ h(y) = 0$, $e(x) = e(y) = 0$, $f(x) = f(y) = 0$,
$\left[\begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$	$k(x) = qx$, $k(y) = q^{-2}y$, $e(x) = 0,\ e(y) = b_0$, $f(x) = b_0^{-1}xy$, $f(y) = -qb_0^{-1}y^2$	$h(x) = x$, $h(y) = -2y$, $e(x) = 0,\ e(y) = b_0$, $f(x) = b_0^{-1}xy$, $f(y) = -b_0^{-1}y^2$
$\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$	$k(x) = q^2x$, $k(y) = q^{-1}y$, $e(x) = -qc_0^{-1}x^2$, $e(y) = c_0^{-1}xy$, $f(x) = c_0,\ f(y) = 0$,	$h(x) = 2x$, $h(y) = -y$ $e(x) = -c_0^{-1}x^2$, $e(y) = c_0^{-1}xy$, $f(x) = c_0,\ f(y) = 0$.
$\left[\begin{pmatrix} \star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$	$k(x) = q^{-2}x$, $k(y) = q^{-1}y$, $e(x) = a_0,\ e(y) = 0$, $f(x) = -qa_0^{-1}x^2 + ty^4$, $f(y) = -qa_0^{-1}xy + sy^3$.	$h(x) = -2x$, $h(y) = -y$, $e(x) = a_0,\ e(y) = 0$, $f(x) = -a_0^{-1}x^2 + ty^4$, $f(y) = -a_0^{-1}xy + sy^3$.
$\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \star \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_1\right]$	$k(x) = qx,\ k(y) = q^2y$, $e(x) = -qd_0^{-1}xy + sx^3$, $e(y) = -qd_0^{-1}y^2 + tx^4$, $f(x) = 0$, $f(y) = d_0$,	$h(x) = x,\ h(y) = 2y$, $e(x) = -d_0^{-1}xy + sx^3$, $e(y) = -d_0^{-1}y^2 + tx^4$, $f(x) = 0$, $f(y) = d_0$,
$\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_0 ; \begin{pmatrix} \mathbf{0} & \star \\ \star & \mathbf{0} \end{pmatrix}_1\right]$	$k(x) = qx$, $k(y) = q^{-1}y$, $e(x) = 0,\ e(y) = \tau x$, $f(x) = \tau^{-1}y$, $f(y) = 0$,	$h(x) = x,\ h(y) = -y$, $e(x) = 0$, $e(y) = \tau x$, $f(x) = \tau^{-1}y$, $f(y) = 0$.

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